

Genus distributions of graphs under self-edge-amalgamations

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Abstract

We investigate the well-known problem of counting graph imbeddings on all oriented surfaces with a focus on graphs that are obtained by pasting together two *root-edges* of another base graph. We require that the *partitioned genus distribution* of the base graph with respect to these *root-edges* be known and that both *root-edges* have two 2-valent endpoints. We derive general formulas for calculating the genus distributions of graphs that can be obtained either by *self-co-amalgamating* or by *self-contra-amalgamating* a base graph whose *partitioned genus distribution* is already known. We see how these general formulas provide a unified approach to calculating genus distributions of many new graph families, such as *co-pasted* and *contra-pasted closed chains* of copies of the triangular prism graph, as well as graph families like circular and Möbius ladders with previously known solutions to the genus distribution problem.

Keywords: Graph, genus distribution, self-edge-amalgamation.

Math. Subj. Class.: 05C10

1 Introduction

Prior research on the problem of counting oriented imbeddings of graphs on all 2-manifolds has yielded closed formulas, recursions for generating tabular data, and asymptotic lower bounds for various families of graphs. Our first installment [22] demonstrated a method for calculating *partitioned genus distributions* of arbitrarily long edge-linked chains obtained by iteratively edge-amalgamating copies of one or more base graphs of known *partitioned genus distribution*. In this installment, we develop two formulas that use the *partitioned genus distribution* of a graph to calculate the genus distribution of the graph obtained through *self-edge-amalgamation* of its two *roots*. Thus, combined with the results derived

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in the first installment, we can first obtain a recursion for the genus distributions of an infinite family of *open chains* of edge-amalgamated copies of a base graph, and then apply the two formulas derived in this paper to obtain genus distributions of the corresponding one or two infinite families of *closed chains*. In this manner, one can calculate the genus distribution for arbitrarily large graphs. This paper is self-contained, though we strongly recommend reading [22] before reading this paper.

Precedents for work on counting imbeddings on various orientable and non-orientable surfaces include [2], [3], [4], [10], [12], [16], [17], [18], [20], [21], [24], [25], [26], [27], [28], [29], and [30] amongst many others. Some recent endeavors in this context have resulted in a series of related works in [11], [14], [7], [8], [9] and [23]. Prior work on counting graph imbeddings in a minimum-genus surface includes [1], [5], [6], and [15]. We briefly review some of the terminology introduced in [22].

We define a **double-edge-rooted graph** as a graph that has two distinct edges designated as **root-edges**, or more simply, as **roots**. In this paper, we require that a root-edge have two 2-valent endpoints. We use the notation (G, e, f) to mean that G is double-edge-rooted, with edges e and f serving as root-edges. We can abbreviate (G, e, f) as G where it is clear from context that a double-edge-rooted graph is intended. A closed walk traced just inside the boundary of a face of an embedding is referred to as a **face-boundary walk**. We abbreviate face-boundary walk as **fb-walk**. We define a **strand**, with respect to a root-edge e (**e-strand** for short), to be an open subwalk of an fb-walk that runs between any two occurrences of the endpoints of e , such that it has in its interior neither an occurrence of e nor of the endpoints of e .

The remainder of this section reviews some concepts and terminology introduced in the first installment of this paper, followed by a formal description of the *self-edge-amalgamation* operation.

First-order sub-partials

For each imbedding of a double-edge-rooted graph, there are two possibilities for each of its root-edges. Either,

- two distinct fb-walks are incident on it;
- or the same fb-walk is incident on both sides of it.

Accordingly, we use the mnemonic d for “different” and s for “same”, and we categorize each imbedding of the double-edge-rooted graph (G, e, f) into the four types: dd , ds , sd , or ss . The first letter in each of these types is for root-edge e and the second for the root-edge f .

The genus distribution of a double-edge-rooted graph (G, e, f) is further partitioned into the **double-root partials** $dd_i(G, e, f)$, $ds_i(G, e, f)$, $sd_i(G, e, f)$, $ss_i(G, e, f)$. These are the numbers of cellular imbeddings of (G, e, f) on the surface S_i having types dd , ds , sd , or ss , respectively. For the purpose of our derivations, these partials are refined into the following **first-order sub-partials**:

- $dd_i^0(G, e, f)$, $ds_i^0(G, e, f)$, $sd_i^0(G, e, f)$ and $ss_i^0(G, e, f)$ are the numbers of imbeddings of G on surface S_i of types dd , ds , sd and ss , respectively, such that none of the fb-walks incident on e is incident on f .
- $dd'_i(G, e, f)$, $ds'_i(G, e, f)$, and $sd'_i(G, e, f)$ are the numbers of imbeddings of G on surface S_i of types dd , ds and sd , respectively, such that exactly one fb-walk incident on e is also incident on f .

- $dd''_i(G, e, f)$ is the number of imbeddings of G on surface S_i of type dd such that both fb-walks incident on e are also incident on f .
- $ss^1_i(G, e, f)$ and $ss^2_i(G, e, f)$ partition the number of imbeddings on surface S_i of type ss which are not of sub-type ss^0 . The sub-partial ss^1 counts the cases where exactly one e -strand contains both occurrences of f , while the sub-partial ss^2 counts the cases where each e -strand contains an occurrence of f .

It follows from these definitions that each double-root partial is the sum of its first-order sub-partials. Moreover,

$$g_i(G) = dd_i(G) + ds_i(G) + sd_i(G) + ss_i(G)$$

Second-order sub-partials

The four first-order sub-partials dd' , dd'' , ds' and sd' are further refined into second-order sub-partials. To define these, we imagine assigning arbitrary orientations to root-edges e and f of the graph (G, e, f) . These assigned orientations are referred to as **pasting-orientations** of root-edges. Given an oriented imbedding of (G, e, f) , as we walk along the oriented root-edge e towards its head, the left side of the edge is labeled 1 and the right side is labeled 2. Whereas when we walk along root-edge f towards its head, the left side is labeled 3 and the right side is labeled 4. This is illustrated in Figure 1.

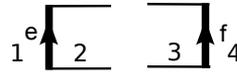


Figure 1: Model for second-order sub-partials with labelled edge-sides.

Accordingly, the second-order partials are described as follows:

- By definition, a dd' -type imbedding has exactly one fb-walk incident on both root-edges. However, the fb-walk incident on both root-edges may combine either the faces 1 and 4, the faces 2 and 3, the faces 1 and 3, or the faces 2 and 4. Accordingly, we define $\overrightarrow{dd'}_i(G, e, f)$, $\overleftarrow{dd'}_i(G, e, f)$, $\overrightarrow{dd''}_i(G, e, f)$ and $\overleftarrow{dd''}_i(G, e, f)$ to be the numbers of imbeddings of type dd'_i such that (1,4), (2,3), (1,3) and (2,4) are the respective pairs of sides that occur on the same fb-walk, as shown in the top row of Figure 2.
- $\overrightarrow{dd''}_i(G, e, f)$ is the number of imbeddings of type dd''_i such that the pair of sides (1,4) occurs on one of the fb-walks incident on the root-edges e and f and the pair of sides (2,3) occurs on the other. Also, $\overleftarrow{dd''}_i(G, e, f)$ is defined similarly, but with (1,3) and (2,4) as pairs of sides occurring on the two fb-walks incident on root-edges e and f .
- $\overrightarrow{ds'}_i(G, e, f)$ and $\overleftarrow{ds'}_i(G, e, f)$ are the numbers of imbeddings of type ds' such that the three sides 1,3,4 (in the former case) and the three sides 2,3,4 (in the latter case) all occur on the same fb-walk.
- $\overrightarrow{sd'}_i(G, e, f)$ and $\overleftarrow{sd'}_i(G, e, f)$ are the numbers of imbeddings of type sd' such that the three sides 1,2,4 (in the former case) and the three sides 1,2,3 (in the latter case) all occur in the same fb-walk.

Clearly, each first-order sub-partial is the sum of its second-order sub-partial. A *partitioned genus distribution* of a graph is its genus distribution partitioned into sub-partial.

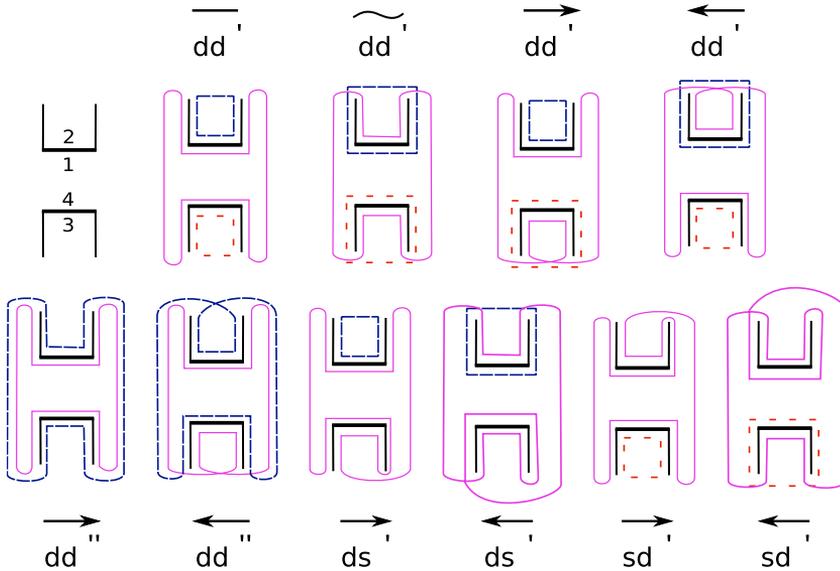


Figure 2: Modeling second-order sub-partial.

Self-edge-amalgamation

The *self-edge-amalgamation* of a graph is an operation whereby two root-edges of a double-edge-rooted graph are pasted together. Amongst many other classes, one can obtain families of closed chains of graphs, including circular ladders and Möbius ladders, by applying self-edge-amalgamation to families of open chains, such as the closed-end ladders. Informally, we use *pastings* to refer to any kind of amalgamation. Where it is clear from context that a self-edge-amalgamation is intended, we may use the terminology *self-amalgamation*. We denote this operation by a monadic operator acting on a double-edge-rooted graph operand (G, e, f) :

$$W = *_{ef}(G, e, f)$$

The self-edge-amalgamation of a double-edge-rooted graph (G, e, f) produces a graph W obtained from G by identifying edges e and f . We fix the pasting-orientations on the root-edges by arbitrarily orienting e and f . Accordingly, the edge-ends of e and f at the tail are e^- and f^- , while the ones at the head are e^+ and f^+ . Now edges e and f can be pasted in two different ways. One way of pasting, called *co-self-amalgamation*, identifies the edge-end e^- with f^- and the edge-end e^+ with f^+ . The other way of pasting, called *contra-self-amalgamation*, pairs the edge-end e^- with f^+ and the edge-end e^+ with f^- . These two ways of self-edge-amalgamating a graph produces graphs which may be non-isomorphic, as seen later in this paper.

REMARK Assignment of labels 1 through 4 to the edge-sides of root-edges is also relative to these same pasting-orientations.

We defined *open chains* in [22] as graphs formed by iteratively pasting copies of smaller graphs along root-edges. Graphs that are obtained from a self-amalgamation of double-edge-rooted open chains are referred to as *closed chains*. Depending on which type of pasting is used, these may be classified as *co-pasted* or *contra-pasted closed chains*.

We work under the assumption that we already have the partitioned genus distribution of the graph that we wish to self-edge-amalgamate. For smaller graphs, this can be done easily by using the Heffter-Edmonds algorithm [13]. For large graphs, some new ways of finding partitioned genus distributions are presented in [11], [14] and [22].

2 Productions for self-edge-amalgamation

Let x_i be any double-root sub-partial. Then a *production* is used to represent the ways in which an imbedding of a double-edge-rooted graph (G, e, f) of type x on surface S_i self-edge-amalgamates on the root-edges e and f to give various types of imbeddings of the resulting graph W . Formally, we write

$$x_i(G, e, f) \longrightarrow g_{k_1}(W) + g_{k_2}(W) + g_{k_3}(W) + g_{k_4}(W)$$

where k_1, k_2, k_3, k_4 are (not necessarily distinct) integer-valued functions of i . This can be read as follows:

An imbedding of (G, e, f) of type x on surface S_i self-amalgamates on the root-edges e and f to give four imbeddings of the graph W on the surfaces $S_{k_1}, S_{k_2}, S_{k_3}$ and S_{k_4} .

The left-hand side of the production is known as the *production head* and the right-hand side of the production as the *production body*.

REMARK The number of terms in the production body could be larger if the degrees of the endpoints of root-edges were larger.

The production, as we have defined it, does not specify whether the self-amalgamation is a co-self-paste or a contra-self-paste. However, as we shall see, for an application that seeks to find the genus distribution of a self-amalgamated graph, the system of productions will consistently refer to only one of the two types of self-pasting. While considering the self-edge-amalgamation for an imbedding on an oriented surface and modeling it using a production, it is important to maintain a sense of orientation of the strands. Each imbedding of a self-edge-amalgamated graph $W = *_{ef}(G, e, f)$ induces a unique imbedding of the graph G , such that the rotation system of W is consistent with the rotation system of G .

Theorem 2.1. *Let (G, e, f) be a double-edge-rooted graph, where both root-edges have two 2-valent endpoints. Then the following productions apply to all scenarios of co-self-paste and contra-self-paste in which no fb-walk of the imbedding of G is incident on both root-edges e and f :*

$$dd_i^0(G) \longrightarrow 2g_{i+1}(W) + 2g_{i+2}(W) \tag{2.1}$$

$$ds_i^0(G) \longrightarrow 4g_{i+1}(W) \tag{2.2}$$

$$sd_i^0(G) \longrightarrow 4g_{i+1}(W) \tag{2.3}$$

$$ss_i^0(G) \longrightarrow 4g_{i+1}(W) \tag{2.4}$$

Proof. The proof of Production (2.1) follows by face-tracing of the fb-walks incident on both root-edges of the graph G . We examine in closer detail the recombination of strands caused by a self-edge-amalgamation where the imbedding of graph G is of type dd^0 . The production shown in the upper half of Figure 3 depicts the case of a co-self-paste while the drawing in the lower half shows a contra-self-paste on the same root-edges. In both cases we get the same results.

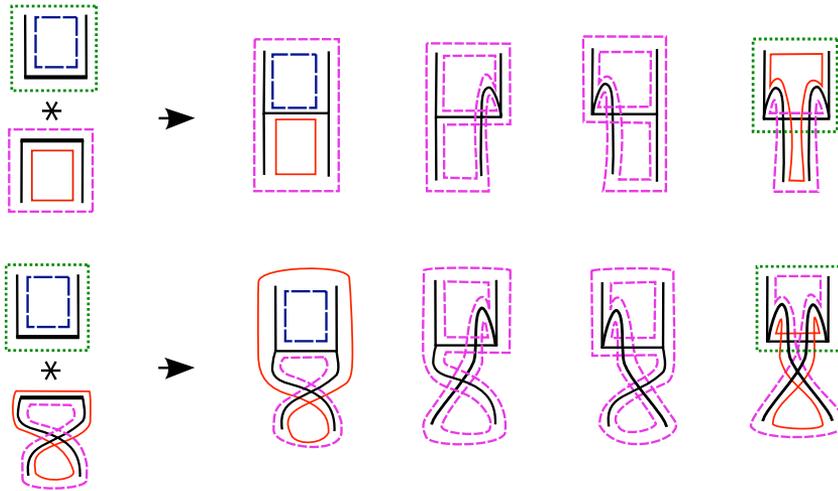


Figure 3: Productions for co-self-pasting and contra-self-pasting a dd^0 -type imbedding of (G, e, f) .

The self-amalgamation produces two fewer vertices and one less edge in each of the four resultant graph imbeddings. The first and last imbeddings shown for each production have two fb-walks merging as a consequence of self-amalgamation. In the second and third imbeddings, all four fb-walks that are incident on the two root-edges merge into a single fb-walk. Applying the Euler polyhedral equation, we see that in the former case the decrease of a single face results in a genus increment of 1, while in the latter case the decrease of three fb-walks results in a genus increment of 2. The proofs for the remaining productions are similar and also follow by face-tracing. For the sake of brevity, we leave these to the reader. \square

For imbeddings in which one or two fb-walks are incident on both root-edges, the productions for co-self-pasting and contra-self-pasting may differ. In particular, for dd' and dd'' we get different results for both ways of pasting.

Theorem 2.2. *Let (G, e, f) be a double-edge-rooted graph, where both root-edges have two 2-valent endpoints. Then the following productions describe all cases of co-self-pasting for imbeddings of G of type dd' :*

$$\overline{dd'}_i(G) \longrightarrow g_i(W) + 3g_{i+1}(W) \tag{2.5}$$

$$\widetilde{dd'}_i(G) \longrightarrow g_i(W) + 3g_{i+1}(W) \tag{2.6}$$

$$\overrightarrow{dd'}_i(G) \longrightarrow 4g_{i+1}(W) \tag{2.7}$$

$$\overleftarrow{dd'}_i(G) \longrightarrow 4g_{i+1}(W) \tag{2.8}$$

Furthermore, the following productions describe all cases of co-self-pasting for imbeddings of G of type dd'' :

$$\overrightarrow{dd''}_i(G) \longrightarrow 4g_i(W) \tag{2.9}$$

$$\overleftarrow{dd''}_i(G) \longrightarrow 2g_i(W) + 2g_{i+1}(W) \tag{2.10}$$

Proof. For illustration of Production (2.5), which describes the effects of co-self-pasting a $\overrightarrow{dd'}$ -type imbedding of G , we refer to the upper half of Figure 4. We observe that the first imbedding of graph W stands out from the other three, in that there is a net increase of one fb-walk as the fb-walk incident on both roots of the imbedding of G breaks into two fb-walks during self-pasting. This does not occur in the other three cases, where there is a net decrease of one fb-walk in the resulting imbedding. The former results in an unchanged genus of the resultant graph imbedding, while the latter results in a genus increment of one. This accounts for Production (2.5).

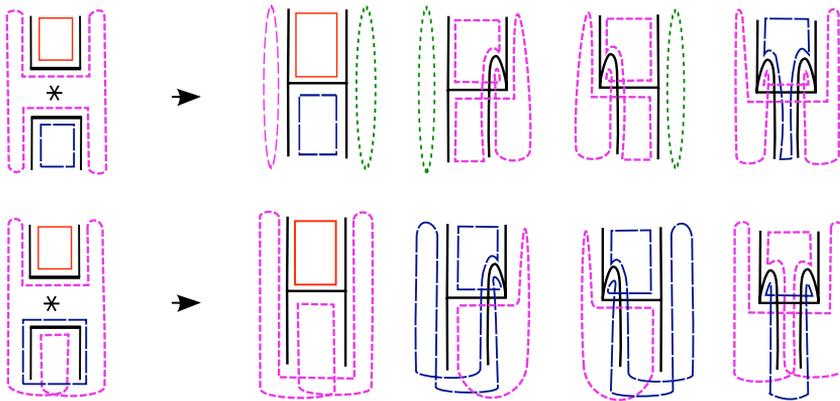


Figure 4: Productions for co-self-pasting $\overrightarrow{dd'}$ -type and $\overrightarrow{dd''}$ -type imbeddings of (G, e, f) .

For Production (2.7), the illustration in the bottom half of Figure 4 shows that all four imbeddings resulting from the self-pasting of G end up with one less face, thereby warranting a genus increment of 1 for the resulting imbeddings. Proofs for Productions (2.6) and (2.8) are similar to the proofs for (2.5) and (2.7), respectively, and we leave them to the reader.

Productions (2.9) and (2.10) for co-self-pasting a $\overrightarrow{dd''}$ - or a $\overleftarrow{dd''}$ -type imbedding of G can also be derived by using the same technique. We include Figure 5 for aiding the reader with the proof of Production (2.10), and omit the proof of Production (2.9). \square

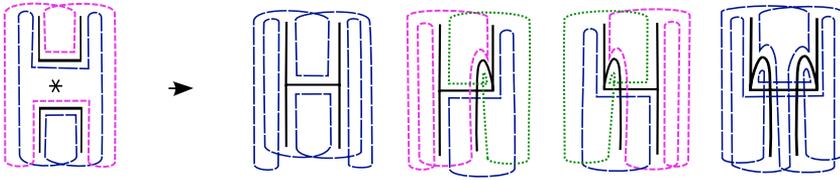


Figure 5: Production for co-self-pasting a $\overleftarrow{dd''}$ -type imbedding of (G, e, f) .

Theorem 2.3. *Let (G, e, f) be a double-edge-rooted graph, where both root-edges have two 2-valent endpoints. Then the following productions apply for contra-self-pasting all dd' -type imbeddings of G :*

$$\overline{dd'}_i(G) \longrightarrow 4g_{i+1}(W) \tag{2.11}$$

$$\widetilde{dd'}_i(G) \longrightarrow 4g_{i+1}(W) \tag{2.12}$$

$$\overrightarrow{dd'}_i(G) \longrightarrow g_i(W) + 3g_{i+1}(W) \tag{2.13}$$

$$\overleftarrow{dd'}_i(G) \longrightarrow g_i(W) + 3g_{i+1}(W) \tag{2.14}$$

Furthermore, the following productions apply for contra-self-pasting all dd'' -type imbeddings of G :

$$\overrightarrow{dd''}_i(G) \longrightarrow 2g_i(W) + 2g_{i+1}(W) \tag{2.15}$$

$$\overleftarrow{dd''}_i(G) \longrightarrow 4g_i(W) \tag{2.16}$$

Proof. Figure 6 illustrates Production (2.12) in the upper half and Production (2.14) in the bottom half.

For Production (2.12), in all four imbeddings resulting from the contra-self-pasting, the three fb-walks incident on the root-edges of the graph G break into strands that merge into two fb-walks, as shown in Figure 6. While, for Production (2.14), this happens for only three of the resulting imbeddings of graph W . For the remaining imbedding, the fb-walks break into strands that recombine to give four distinct fb-walks. Proofs of Productions (2.11) and (2.13) are similar and left to the reader.

Similarly, applying contra-self-pasting to the root-edges of a $\overrightarrow{dd''}$ -type imbedding of (G, e, f) results in two imbeddings of W where all the fb-walks incident on the roots merge into a single fb-walk, and two imbeddings where they break into strands that recombine into three distinct fb-walks, as shown in Figure 7. This proves Production (2.15). Proof for Production (2.16) is similar and we leave it to the reader. \square

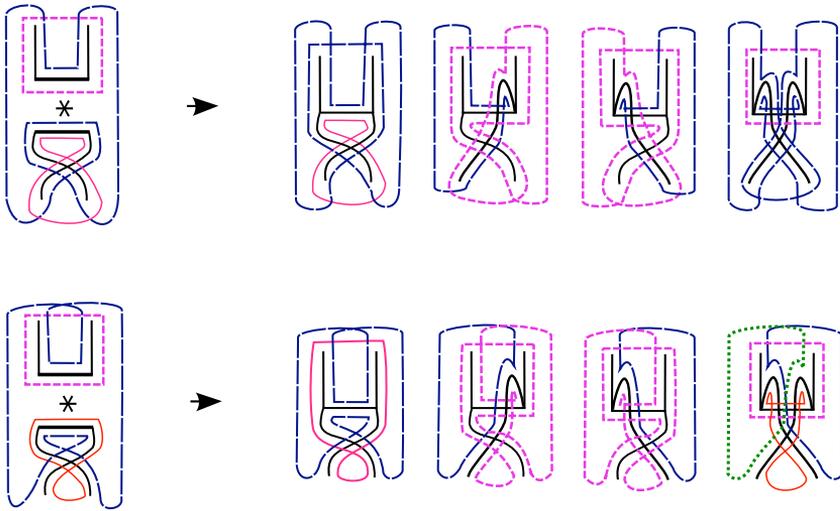


Figure 6: Productions for contra-self-pasting \overleftarrow{dd}' -type and \overleftarrow{dd}' -type imbeddings of (G, e, f) .

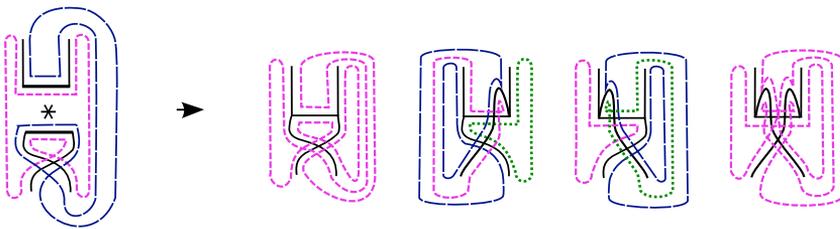


Figure 7: Production for contra-self-pasting \overrightarrow{dd}'' -type imbeddings of (G, e, f) .

Theorem 2.4. *Let (G, e, f) be a double-edge-rooted graph, where both roots have two 2-valent endpoints. Then the following productions cover all scenarios for co-self-pasting and contra-self-pasting, where the imbedding of G is of type ds' or sd' :*

$$ds'_i(G) \longrightarrow 2g_i(W) + 2g_{i+1}(W) \tag{2.17}$$

$$sd'_i(G) \longrightarrow 2g_i(W) + 2g_{i+1}(W) \tag{2.18}$$

Proof. Due to the symmetry of the models $\overrightarrow{ds'}$ and $\overrightarrow{sd'}$, and of $\overleftarrow{ds'}$ and $\overleftarrow{sd'}$, we need only provide the proof for one of the two Productions (2.17) and (2.18). Figure 8 illustrates the proof for the co-self-pasted and contra-self-pasted $\overrightarrow{ds'}$ -type imbedding of G . We get two imbeddings with a genus increment of 0 and two with a genus increment of 1. We leave the proof for a $\overleftarrow{ds'}$ -type imbedding of G to the reader. The reader will observe that the production body for the imbedding type $\overleftarrow{ds'}$ is identical to the production body for the imbedding type $\overrightarrow{ds'}$. This is true for both co-self-pasting and contra-self-pasting operations. This completes the proof of Production (2.17). \square

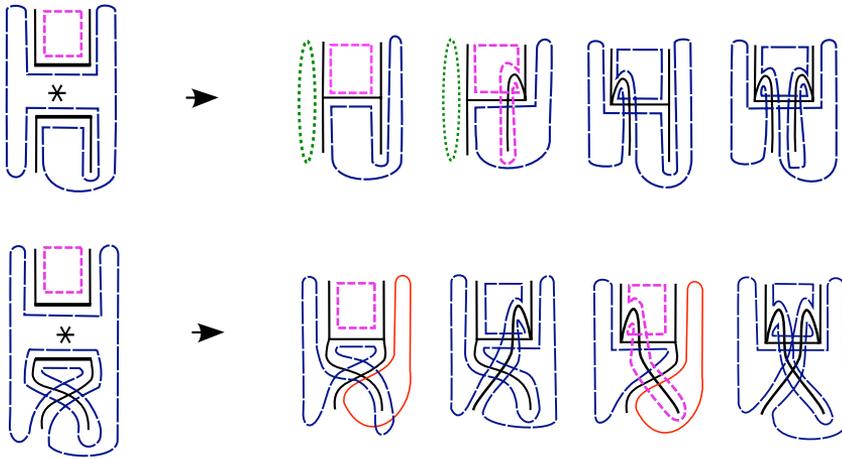


Figure 8: Productions for co-pasting and contra-pasting a $\overrightarrow{ds'}$ -type imbedding of (G, e, d) .

Theorem 2.5. *Let (G, e, f) be a double-edge-rooted graph, where both roots have two 2-valent endpoints. Then the following productions apply for co-self-pasting or contra-self-pasting an ss^1 or ss^2 -type imbedding of G :*

$$ss^1_i(G) \longrightarrow 4g_i(W) \tag{2.19}$$

$$ss^2_i(G) \longrightarrow 3g_i(W) + g_{i-1}(W) \tag{2.20}$$

Proof. When the imbedding of a double-edge-rooted graph G is of type ss^1 , a self-amalgamation on the root-edges breaks the single fb-walk incident on both roots into strands that recombine to give two fb-walks in all the corresponding imbeddings of the self-amalgamated graph W . The additional face balances out the decrease of two vertices and one edge to retain the same genus in each of the four resulting imbeddings of W . This holds for co-self-pasting as well as contra-self-pasting as evident from Figure 9.

We leave the proof of Production 2.20 for the self-amalgamation of an ss^2 -type imbedding to the reader. □

REMARK As it turns out, the productions for second-order sub-partial types of dd' and dd'' are the only ones that disagree for a co-self-paste and a contra-self-paste. Moreover, the results for the co- and contra-self-amalgamation for both dd' - and dd'' -type imbeddings are symmetric in the sense that the production body for co-self-pasting a $\overrightarrow{dd''}$ -type (or a $\overleftarrow{dd''}$ -type) imbedding of G is the same as the production body for contra-self-pasting a $\overleftarrow{dd''}$ -type (or a $\overrightarrow{dd''}$ -type) imbedding of G . Likewise, for the pairs of $\overrightarrow{dd'}$ -, $\overleftarrow{dd'}$ -types of imbeddings of G which show a symmetry with the $\overrightarrow{dd'}$ -, $\overleftarrow{dd'}$ -types of imbeddings of G .

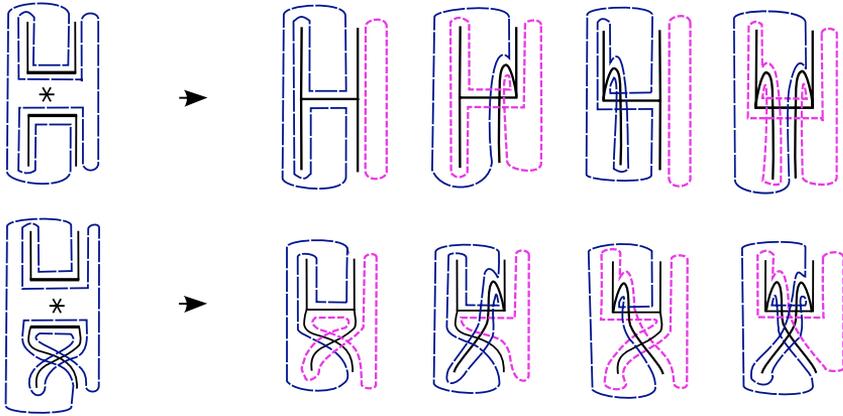


Figure 9: Productions for co-self-pasting and contra-self-pasting an ss^1 -type imbedding of (G, e, f) .

Theorem 2.6. *Let W be the graph formed by co-self-pasting of (G, e, f) . Then,*

$$\begin{aligned}
 g_i(W) = & 2dd_{i-2}^0(G) + 2dd_{i-1}^0(G) + 3\overleftarrow{dd'}_{i-1}(G) + 3\overrightarrow{dd'}_{i-1}(G) + 4\overrightarrow{dd'}_{i-1}(G) \\
 & + 4\overleftarrow{dd'}_{i-1}(G) + 2\overleftarrow{dd''}_{i-1}(G) + 4ds_{i-1}^0(G) + 2ds'_{i-1}(G) + 4sd_{i-1}^0(G) \\
 & + 2sd'_{i-1}(G) + 4ss_{i-1}^0(G) + \overleftarrow{dd'}_i(G) + \overrightarrow{dd'}_i(G) + 4\overrightarrow{dd''}_i(G) + 2\overleftarrow{dd''}_i(G) \\
 & + 2ds'_i(G) + 2sd'_i(G) + 4ss_i^1(G) + 3ss_i^2(G) + ss_{i+1}^2(G) \quad (2.21)
 \end{aligned}$$

Proof. The Production (2.1):

$$dd_i^0(G) \longrightarrow 2g_{i+1}(W) + 2g_{i+2}(W)$$

indicates that each dd^0 -type imbedding of (G, e, f) on the surface S_i when self-amalgamated, induces two imbeddings of W on S_{i+1} and two on the surface S_{i+2} . These contributions account for the first two terms $2dd_{i-2}^0(G) + 2dd_{i-1}^0(G)$ on the right-hand side of Equation (2.21). Taking into account all the contributions made by productions listed in Theorems 2.1–2.5, the result follows. \square

Theorem 2.7. *Let W be the graph formed by contra-self-pasting of (G, e, f) . Then,*

$$\begin{aligned}
 g_i(W) = & 2dd_{i-2}^0(G) + 2dd_{i-1}^0(G) + 3\overrightarrow{dd'}_{i-1}(G) + 3\overleftarrow{dd'}_{i-1}(G) + 4\overleftarrow{dd'}_{i-1}(G) \\
 & + 4\overrightarrow{dd'}_{i-1}(G) + 2\overrightarrow{dd''}_{i-1}(G) + 4ds_{i-1}^0(G) + 2ds'_{i-1}(G) + 4sd_{i-1}^0(G) \\
 & + 2sd'_{i-1}(G) + 4ss_{i-1}^0(G) + \overrightarrow{dd'}_i(G) + \overleftarrow{dd'}_i(G) + 4\overleftarrow{dd''}_i(G) + 2\overrightarrow{dd''}_i(G) \\
 & + 2ds'_i(G) + 2sd'_i(G) + 4ss_i^1(G) + 3ss_i^2(G) + ss_{i+1}^2(G) \quad (2.22)
 \end{aligned}$$

Proof. The proof for Equation (2.22) is obvious from Equation (2.21) and our earlier remarks on the symmetries of the productions for dd' and dd'' second-order sub-partial types. \square

Thus, depending on whether one plans on forming a closed chain through a co-self-amalgamation or through a contra-self-amalgamation, the genus distribution of the closed chain is calculated by using Theorems 2.6 or 2.7, respectively. For instance, using the partitioned genus distributions calculated for closed-end ladders in [22], we can now use Theorems 2.6 and 2.7 to obtain partitioned genus-distribution for both circular and Möbius ladders.

3 Application: Revisiting circular ladders and Möbius ladders

The genus distributions of circular ladders and Möbius ladders were first derived by [19]. §4 of [22] shows how calculation of the *double-root* genus distributions of closed-end ladders is reducible to a routine recursion. This in turn reduces the derivation of the genus distributions of circular and Möbius ladders, in turn, to a routine substitution into an equation.

Let L_n be the closed-end ladder with 2 end-rungs and n interior rungs, as shown in Figure 10.

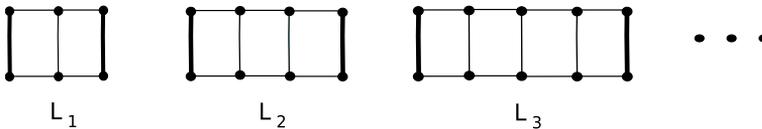


Figure 10: Closed-end ladders L_n .

Let CL_n denote the circular ladder with n rungs as illustrated in Figure 11. We observe that co-self-pasting the closed-end ladder L_n on the root-edges yields CL_{n+1} . Similarly,

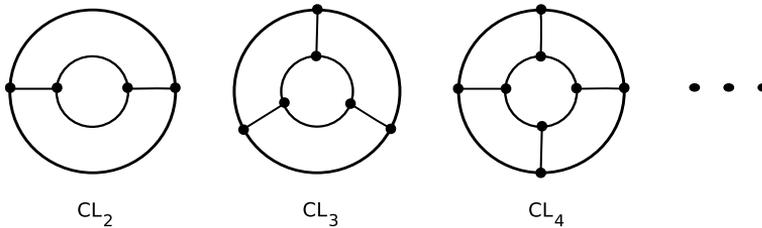


Figure 11: Circular ladders CL_n .

we denote a Möbius ladder with n rungs by ML_n as shown in Figure 12. It can be observed that contra-self-pasting the closed-end ladder L_n on the root-edges yields ML_{n+1} .

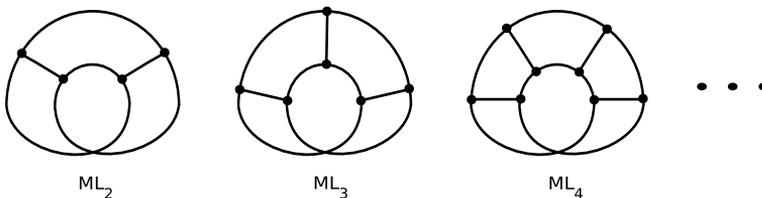


Figure 12: Möbius ladders ML_n .

We give a small example that demonstrates how to calculate genus distributions of CL_4 and ML_4 from the partitioned genus distribution of L_3 . We begin by reproducing in Table 1 the partitioned genus distribution of the closed-end ladder L_3 originally derived in [22].

Table 1: Double-root partials of L_3 .

L_3							
k	0	1	2	k	0	1	2
dd_k^0	6	0	0	\overrightarrow{ds}'_k	0	2	0
\overleftarrow{dd}'_k	1	6	0	\overleftarrow{ds}'_k	0	2	0
\widetilde{dd}'_k	1	6	0	sd_k^0	0	4	0
\overrightarrow{dd}'_k	0	6	0	\overrightarrow{sd}'_k	0	2	0
\overleftarrow{dd}'_k	0	6	0	\overleftarrow{sd}'_k	0	2	0
\overrightarrow{dd}''_k	0	0	0	ss_k^0	0	0	0
\overleftarrow{dd}''_k	0	0	0	ss_k^1	0	0	8
ds_k^0	0	4	0	ss_k^2	0	0	8

Simply plugging the values from Table 1 into Equation (2.21) yields the genus distributions of circular ladder CL_4 .

$$g_0(CL_4) = \overleftarrow{dd}'_0(L_3) + \widetilde{dd}'_0(L_3) = 1 + 1 = 2$$

$$\begin{aligned} g_1(CL_4) &= 2dd_0^0(L_3) + 3\overleftarrow{dd}'_0(L_3) + 3\widetilde{dd}'_0(L_3) + \overleftarrow{dd}'_1(L_3) + \widetilde{dd}'_1(L_3) + 2ds'_1(L_3) \\ &\quad + 2sd_1^0(L_3) + ss_2^2(L_3) \\ &= 2 \times 6 + 3 \times 1 + 3 \times 1 + 6 + 6 + 2 \times 4 + 2 \times 4 + 8 \\ &= 54 \end{aligned}$$

$$\begin{aligned} g_2(CL_4) &= 2dd_0^0(L_3) + 3\overleftarrow{dd}'_1(L_3) + 3\widetilde{dd}'_1(L_3) + \overrightarrow{dd}'_1(L_3) + \overleftarrow{dd}'_1(L_3) + 4ds_1^0(L_3) \\ &\quad + 2ds'_1(L_3) + 4sd_1^0(L_3) + 2sd'_1(L_3) + 4ss_2^1(L_3) + 3ss_2^2(L_3) \\ &= 2 \times 6 + 3 \times 6 + 3 \times 6 + 4 \times 6 + 4 \times 6 + 4 \times 4 + 2 \times 4 + 4 \times 4 \\ &\quad + 2 \times 4 + 4 \times 8 + 3 \times 8 \\ &= 200 \end{aligned}$$

$$g_3(CL_4) = 0$$

Whereas, plugging the values from Table 1 into Equation (2.22) produces the genus distribution of the Möbius ladder ML_4 :

$$\begin{aligned}
 g_0(ML_4) &= 0 \\
 g_1(ML_4) &= 2dd_0^0(L_3) + 4\overleftarrow{dd}'_0(L_3) + 4\widetilde{dd}'_0(L_3) + \overrightarrow{dd}'_1(L_3) + \overleftarrow{dd}'_1(L_3) + 2ds'_1(L_3) \\
 &\quad + 2sd'_1(L_3) + ss_2^2(L_3) \\
 &= 2 \times 6 + 4 \times 1 + 4 \times 1 + 6 + 6 + 2 \times 4 + 2 \times 4 + 8 \\
 &= 56 \\
 g_2(ML_4) &= 2dd_1^0(L_3) + 3\overrightarrow{dd}'_1(L_3) + 3\overleftarrow{dd}'_1(L_3) + 4\overleftarrow{dd}'_1(L_3) + 4\widetilde{dd}'_1(L_3) \\
 &\quad + 4ds_1^0(L_3) + 2ds'_1(L_3) + 4sd_1^0(L_3) + 2sd'_1(L_3) + 4ss_2^1(L_3) + 3ss_2^2(L_3) \\
 &= 2 \times 6 + 3 \times 6 + 3 \times 6 + 4 \times 6 + 4 \times 6 + 4 \times 4 + 2 \times 4 + 4 \times 4 + 2 \times 4 \\
 &\quad + 4 \times 8 + 3 \times 8 \\
 &= 200 \\
 g_3(ML_4) &= 0
 \end{aligned}$$

4 Application: Closed chains of copies of a triangular prism graph

As an example of two entirely new calculations of genus distributions of closed chains, we consider closed chains of copies of the triangular prism graph. Figure 13 shows a triangular prism graph at the left, where two of its edges are trisected and their middle-thirds are designated as root-edges. We show the root-edges darker by convention.

Let Δ_G denote the double-edge-rooted triangular prism graph. Figure 13 shows some small double-edge-rooted open chains of copies of the graph Δ_G . We denote an open chain consisting of n copies of Δ_G by $\mathcal{P}r_n$.

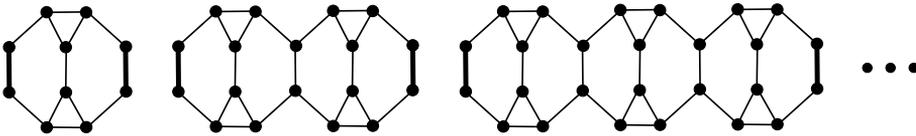


Figure 13: Open chains of copies of a triangular prism graph.

We begin the derivations for closed chains of copies of Δ_G by first producing the partitioned genus distribution of the open chains $\mathcal{P}r_1$ and $\mathcal{P}r_2$ in Table 2. These have been derived using the methods in [22].

Table 2: Genus distributions of the open chains $\mathcal{P}r_1$ and $\mathcal{P}r_2$ of 1 and 2 copies of Δ_G , respectively.

$\mathcal{P}r_n$	$\mathcal{P}r_1$			$\mathcal{P}r_2$				
k	0	1	2	0	1	2	3	4
dd_k^0	0	0	0	6	176	704	0	0
\overrightarrow{dd}'_k	1	0	0	1	46	400	0	0
\widetilde{dd}'_k	1	0	0	1	46	400	0	0
\overrightarrow{dd}''_k	0	0	0	0	22	256	0	0
\overleftarrow{dd}''_k	0	0	0	0	22	256	0	0
\overrightarrow{dd}'''_k	0	12	0	0	0	48	0	0
\overleftarrow{dd}'''_k	0	0	0	0	0	48	0	0
ds_k^0	0	4	0	0	44	800	384	0
\overrightarrow{ds}'_k	0	4	0	0	6	272	960	0
\overleftarrow{ds}'_k	0	4	0	0	6	272	960	0
sd_k^0	0	4	0	0	44	800	384	0
\overrightarrow{sd}'_k	0	4	0	0	6	272	960	0
\overleftarrow{sd}'_k	0	4	0	0	6	272	960	0
ss_k^0	0	0	0	0	0	320	1888	0
ss_k^1	0	0	24	0	0	72	1664	2304
ss_k^2	0	2	0	0	0	8	288	0
g_k	2	38	24	8	424	5200	8448	2304

Let $\mathcal{C}P r_n$ be the co-self-amalgamated closed chain of n copies of Δ_G , as shown in Figure 14. We plug the values from Table 2 into Equation (2.21) to calculate genus distributions for $\mathcal{C}P r_1$ and $\mathcal{C}P r_2$.

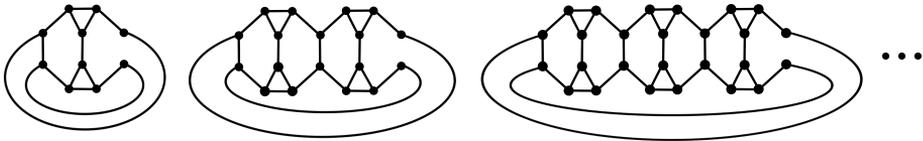


Figure 14: Co-pasted closed chains of copies of a triangular prism graph.

We illustrate this as follows:

$$\begin{aligned}
 g_0(\mathcal{CPr}_1) &= \overline{dd'}_0(\mathcal{Pr}_1) + \widetilde{dd'}_0(\mathcal{Pr}_1) + 4\overrightarrow{dd''}_0(\mathcal{Pr}_1) + 2\overleftarrow{dd''}_0(\mathcal{Pr}_1) + 2ds'_0(\mathcal{Pr}_1) \\
 &\quad + 2sd'_0(\mathcal{Pr}_1) + 4ss_0^1(\mathcal{Pr}_1) + 3ss_0^2(\mathcal{Pr}_1) + ss_1^2(\mathcal{Pr}_1) \\
 &= 1 + 1 + 4 \times 0 + 2 \times 0 + 2 \times 0 + 2 \times 0 + 4 \times 0 + 3 \times 0 + 2 \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 g_1(\mathcal{CPr}_1) &= 2dd_0^0(\mathcal{Pr}_1) + 3\overline{dd'}_0(\mathcal{Pr}_1) + 3\widetilde{dd'}_0(\mathcal{Pr}_1) + 4\overrightarrow{dd''}_0(\mathcal{Pr}_1) + 4\overleftarrow{dd''}_0(\mathcal{Pr}_1) \\
 &\quad + 2\overleftarrow{dd''}_0(\mathcal{Pr}_1) + 4ds_0^0(\mathcal{Pr}_1) + 2ds'_0(\mathcal{Pr}_1) + 4sd_0^0(\mathcal{Pr}_1) + 2sd'_0(\mathcal{Pr}_1) \\
 &\quad + 4ss_0^0(\mathcal{Pr}_1) + \overline{dd'}_1(\mathcal{Pr}_1) + \widetilde{dd'}_1(\mathcal{Pr}_1) + 4\overrightarrow{dd''}_1(\mathcal{Pr}_1) + 2\overleftarrow{dd''}_1(\mathcal{Pr}_1) \\
 &\quad + 2ds'_1(\mathcal{Pr}_1) + 2sd'_1(\mathcal{Pr}_1) + 4ss_1^1(\mathcal{Pr}_1) + 3ss_1^2(\mathcal{Pr}_1) + ss_2^2(\mathcal{Pr}_1) \\
 &= 2 \times 0 + 3 \times 1 + 3 \times 1 + 4 \times 0 + 4 \times 0 + 2 \times 0 + 4 \times 0 + 2 \times 0 \\
 &\quad + 4 \times 0 + 2 \times 0 + 4 \times 0 + 0 + 0 + 4 \times 12 + 2 \times 0 + 2 \times 8 + 2 \times 8 \\
 &\quad + 4 \times 0 + 3 \times 2 + 0 \\
 &= 92
 \end{aligned}$$

$$\begin{aligned}
 g_2(\mathcal{CPr}_1) &= 2dd_0^0(\mathcal{Pr}_1) + 2dd_1^0(\mathcal{Pr}_1) + 3\overline{dd'}_1(\mathcal{Pr}_1) + 3\widetilde{dd'}_1(\mathcal{Pr}_1) + 4\overrightarrow{dd''}_1(\mathcal{Pr}_1) \\
 &\quad + 4\overleftarrow{dd''}_1(\mathcal{Pr}_1) + 2\overleftarrow{dd''}_1(\mathcal{Pr}_1) + 4ds_1^0(\mathcal{Pr}_1) + 2ds'_1(\mathcal{Pr}_1) + 4sd_1^0(\mathcal{Pr}_1) \\
 &\quad + 2sd'_1(\mathcal{Pr}_1) + 4ss_1^0(\mathcal{Pr}_1) + \overline{dd'}_2(\mathcal{Pr}_1) + \widetilde{dd'}_2(\mathcal{Pr}_1) + 4\overrightarrow{dd''}_2(\mathcal{Pr}_1) \\
 &\quad + 2\overleftarrow{dd''}_2(\mathcal{Pr}_1) + 2ds'_2(\mathcal{Pr}_1) + 2sd'_2(\mathcal{Pr}_1) + 4ss_2^1(\mathcal{Pr}_1) + 3ss_2^2(\mathcal{Pr}_1) \\
 &\quad + ss_3^2(\mathcal{Pr}_1) \\
 &= 2 \times 0 + 2 \times 0 + 3 \times 0 + 3 \times 0 + 4 \times 0 + 4 \times 0 + 2 \times 0 + 4 \times 4 + 2 \times 8 \\
 &\quad + 4 \times 4 + 2 \times 8 + 4 \times 0 + 0 + 0 + 4 \times 0 + 2 \times 0 + 2 \times 0 + 2 \times 0 \\
 &\quad + 4 \times 24 + 3 \times 0 + 0 \\
 &= 160
 \end{aligned}$$

$$\begin{aligned}
 g_0(\mathcal{CPr}_2) &= \overline{dd'}_0(\mathcal{Pr}_2) + \widetilde{dd'}_0(\mathcal{Pr}_2) + 4\overrightarrow{dd''}_0(\mathcal{Pr}_2) + 2\overleftarrow{dd''}_0(\mathcal{Pr}_2) + 2ds'_0(\mathcal{Pr}_2) \\
 &\quad + 2sd'_0(\mathcal{Pr}_2) + 4ss_0^1(\mathcal{Pr}_2) + 3ss_0^2(\mathcal{Pr}_2) + ss_1^2(\mathcal{Pr}_2) \\
 &= 1 + 1 + 4 \times 0 + 2 \times 0 + 2 \times 0 + 2 \times 0 + 4 \times 0 + 3 \times 0 + 0 \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 g_1(\mathcal{CPr}_2) &= 2dd_0^0(\mathcal{Pr}_2) + 3\overline{dd'}_0(\mathcal{Pr}_2) + 3\widetilde{dd'}_0(\mathcal{Pr}_2) + 4\overrightarrow{dd''}_0(\mathcal{Pr}_2) + 4\overleftarrow{dd''}_0(\mathcal{Pr}_2) \\
 &\quad + 2\overleftarrow{dd''}_0(\mathcal{Pr}_2) + 4ds_0^0(\mathcal{Pr}_2) + 2ds'_0(\mathcal{Pr}_2) + 4sd_0^0(\mathcal{Pr}_2) + 2sd'_0(\mathcal{Pr}_2) \\
 &\quad + 4ss_0^0(\mathcal{Pr}_2) + \overline{dd'}_1(\mathcal{Pr}_2) + \widetilde{dd'}_1(\mathcal{Pr}_2) + 4\overrightarrow{dd''}_1(\mathcal{Pr}_2) + 2\overleftarrow{dd''}_1(\mathcal{Pr}_2) \\
 &\quad + 2ds'_1(\mathcal{Pr}_2) + 2sd'_1(\mathcal{Pr}_2) + 4ss_1^1(\mathcal{Pr}_2) + 3ss_1^2(\mathcal{Pr}_2) + ss_2^2(\mathcal{Pr}_2) \\
 &= 2 \times 6 + 3 \times 1 + 3 \times 1 + 4 \times 0 + 4 \times 0 + 2 \times 0 + 4 \times 0 + 2 \times 0 + 4 \times 0 \\
 &\quad + 2 \times 0 + 4 \times 0 + 46 + 46 + 4 \times 0 + 2 \times 0 + 2 \times 12 + 2 \times 12 + 4 \times 0 \\
 &\quad + 3 \times 0 + 8 \\
 &= 166
 \end{aligned}$$

$$\begin{aligned}
 g_2(\mathcal{CP}r_2) &= 2dd_0^0(\mathcal{P}r_2) + 2dd_1^0(\mathcal{P}r_2) + 3\overleftarrow{dd}'_1(\mathcal{P}r_2) + 3\widetilde{dd}'_1(\mathcal{P}r_2) + 4\overrightarrow{dd}'_1(\mathcal{P}r_2) \\
 &\quad + 4\overleftarrow{dd}''_1(\mathcal{P}r_2) + 2\overleftarrow{dd}'''_1(\mathcal{P}r_2) + 4ds_1^0(\mathcal{P}r_2) + 2ds'_1(\mathcal{P}r_2) + 4sd_1^0(\mathcal{P}r_2) \\
 &\quad + 2sd'_1(\mathcal{P}r_2) + 4ss_1^0(\mathcal{P}r_2) + \overline{dd}'_2(\mathcal{P}r_2) + \widetilde{dd}'_2(\mathcal{P}r_2) + 4\overrightarrow{dd}''_2(\mathcal{P}r_2) \\
 &\quad + 2\overleftarrow{dd}'''_2(\mathcal{P}r_2) + 2ds_2^0(\mathcal{P}r_2) + 2ds'_2(\mathcal{P}r_2) + 4ss_2^1(\mathcal{P}r_2) + 3ss_2^2(\mathcal{P}r_2) \\
 &\quad + ss_3^2(\mathcal{P}r_2) \\
 &= 2 \times 6 + 2 \times 176 + 3 \times 46 + 3 \times 46 + 4 \times 22 + 4 \times 22 + 2 \times 0 + 4 \times 44 \\
 &\quad + 2 \times 12 + 4 \times 44 + 2 \times 12 + 4 \times 0 + 400 + 400 + 4 \times 48 + 2 \times 48 \\
 &\quad + 2 \times 544 + 2 \times 544 + 4 \times 72 + 3 \times 8 + 288 \\
 &= 5080
 \end{aligned}$$

$$\begin{aligned}
 g_3(\mathcal{CP}r_2) &= 2dd_1^0(\mathcal{P}r_2) + 2dd_2^0(\mathcal{P}r_2) + 3\overline{dd}'_2(\mathcal{P}r_2) + 3\widetilde{dd}'_2(\mathcal{P}r_2) + 4\overrightarrow{dd}''_2(\mathcal{P}r_2) \\
 &\quad + 4\overleftarrow{dd}'''_2(\mathcal{P}r_2) + 2\overleftarrow{dd}''''_2(\mathcal{P}r_2) + 4ds_2^0(\mathcal{P}r_2) + 2ds'_2(\mathcal{P}r_2) + 4sd_2^0(\mathcal{P}r_2) \\
 &\quad + 2sd'_2(\mathcal{P}r_2) + 4ss_2^0(\mathcal{P}r_2) + \overline{dd}'_3(\mathcal{P}r_2) + \widetilde{dd}'_3(\mathcal{P}r_2) + 4\overrightarrow{dd}''_3(\mathcal{P}r_2) \\
 &\quad + 2\overleftarrow{dd}'''_3(\mathcal{P}r_2) + 2ds_3^0(\mathcal{P}r_2) + 2ds'_3(\mathcal{P}r_2) + 4ss_3^1(\mathcal{P}r_2) + 3ss_3^2(\mathcal{P}r_2) \\
 &\quad + ss_4^2(\mathcal{P}r_2) \\
 &= 2 \times 176 + 2 \times 704 + 3 \times 400 + 3 \times 400 + 4 \times 256 + 4 \times 256 + 2 \times 48 \\
 &\quad + 4 \times 800 + 2 \times 544 + 4 \times 800 + 2 \times 544 + 4 \times 320 + 0 + 0 + 4 \times 0 \\
 &\quad + 2 \times 0 + 2 \times 1920 + 2 \times 1920 + 4 \times 1664 + 3 \times 288 + 0 \\
 &= 31360
 \end{aligned}$$

$$\begin{aligned}
 g_4(\mathcal{CP}r_2) &= 2dd_2^0(\mathcal{P}r_2) + 2dd_3^0(\mathcal{P}r_2) + 3\overline{dd}'_3(\mathcal{P}r_2) + 3\widetilde{dd}'_3(\mathcal{P}r_2) + 4\overrightarrow{dd}''_3(\mathcal{P}r_2) \\
 &\quad + 4\overleftarrow{dd}'''_3(\mathcal{P}r_2) + 2\overleftarrow{dd}''''_3(\mathcal{P}r_2) + 4ds_3^0(\mathcal{P}r_2) + 2ds'_3(\mathcal{P}r_2) + 4sd_3^0(\mathcal{P}r_2) \\
 &\quad + 2sd'_3(\mathcal{P}r_2) + 4ss_3^0(\mathcal{P}r_2) + \overline{dd}'_4(\mathcal{P}r_2) + \widetilde{dd}'_4(\mathcal{P}r_2) + 4\overrightarrow{dd}''_4(\mathcal{P}r_2) \\
 &\quad + 2\overleftarrow{dd}'''_4(\mathcal{P}r_2) + 2ds_4^0(\mathcal{P}r_2) + 2ds'_4(\mathcal{P}r_2) + 4ss_4^1(\mathcal{P}r_2) + 3ss_4^2(\mathcal{P}r_2) \\
 &\quad + ss_5^2(\mathcal{P}r_2) \\
 &= 2 \times 704 + 2 \times 0 + 3 \times 0 + 3 \times 0 + 4 \times 0 + 4 \times 0 + 2 \times 0 + 4 \times 384 \\
 &\quad + 2 \times 1920 + 4 \times 384 + 2 \times 1920 + 4 \times 1888 + 0 + 0 + 4 \times 0 + 2 \times 0 \\
 &\quad + 2 \times 0 + 2 \times 0 + 4 \times 2304 + 3 \times 0 + 0 \\
 &= 28928
 \end{aligned}$$

Let $\mathcal{KP}r_n$ be the contra-self-amalgamated closed chain of n copies of Δ_G , as illustrated in Figure 15. We calculate genus distributions for contra-self-amalgamated closed chains $\mathcal{KP}r_1$ and $\mathcal{KP}r_2$ by substituting values from Table 2 into Equation (2.22) as follows:

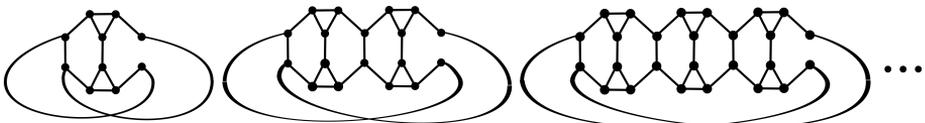


Figure 15: Contra-pasted closed chains of copies of a triangular prism graph.

$$\begin{aligned}
g_0(\mathcal{KPr}_1) &= \overrightarrow{dd'}_0(\mathcal{Pr}_1) + \overleftarrow{dd'}_0(\mathcal{Pr}_1) + 4\overleftarrow{dd''}_0(\mathcal{Pr}_1) + 2\overrightarrow{dd''}_0(\mathcal{Pr}_1) + 2ds'_0(\mathcal{Pr}_1) \\
&\quad + 2sd'_0(\mathcal{Pr}_1) + 4ss_0^1(\mathcal{Pr}_1) + 3ss_0^2(\mathcal{Pr}_1) + ss_1^2(\mathcal{Pr}_1) \\
&= 0 + 0 + 4 \times 0 + 2 \times 0 + 2 \times 0 + 2 \times 0 + 4 \times 0 + 3 \times 0 + 2 \\
&= 2
\end{aligned}$$

$$\begin{aligned}
g_1(\mathcal{KPr}_1) &= 2dd_0^0(\mathcal{Pr}_1) + 3\overrightarrow{dd'}_0(\mathcal{Pr}_1) + 3\overleftarrow{dd'}_0(\mathcal{Pr}_1) + 4\overrightarrow{dd''}_0(\mathcal{Pr}_1) + 4\overleftarrow{dd''}_0(\mathcal{Pr}_1) \\
&\quad + 2\overrightarrow{dd''}_0(\mathcal{Pr}_1) + 4ds_0^0(\mathcal{Pr}_1) + 2ds'_0(\mathcal{Pr}_1) + 4sd_0^0(\mathcal{Pr}_1) + 2sd'_0(\mathcal{Pr}_1) \\
&\quad + 4ss_0^0(\mathcal{Pr}_1) + \overrightarrow{dd'}_1(\mathcal{Pr}_1) + \overleftarrow{dd'}_1(\mathcal{Pr}_1) + 4\overleftarrow{dd''}_1(\mathcal{Pr}_1) + 2\overrightarrow{dd''}_1(\mathcal{Pr}_1) \\
&\quad + 2ds'_1(\mathcal{Pr}_1) + 2sd'_1(\mathcal{Pr}_1) + 4ss_1^1(\mathcal{Pr}_1) + 3ss_1^2(\mathcal{Pr}_1) + ss_2^2(\mathcal{Pr}_1) \\
&= 2 \times 0 + 3 \times 0 + 3 \times 0 + 4 \times 1 + 4 \times 1 + 2 \times 0 + 4 \times 0 + 2 \times 0 + 4 \times 0 \\
&\quad + 2 \times 0 + 4 \times 0 + 0 + 0 + 4 \times 0 + 2 \times 12 + 2 \times 8 + 2 \times 8 + 4 \times 0 + 3 \times 2 \\
&\quad + 0 \\
&= 70
\end{aligned}$$

$$\begin{aligned}
g_2(\mathcal{KPr}_1) &= 2dd_0^0(\mathcal{Pr}_1) + 2dd_1^0(\mathcal{Pr}_1) + 3\overrightarrow{dd'}_1(\mathcal{Pr}_1) + 3\overleftarrow{dd'}_1(\mathcal{Pr}_1) + 4\overrightarrow{dd''}_1(\mathcal{Pr}_1) \\
&\quad + 4\overleftarrow{dd''}_1(\mathcal{Pr}_1) + 2\overrightarrow{dd''}_1(\mathcal{Pr}_1) + 4ds_1^0(\mathcal{Pr}_1) + 2ds'_1(\mathcal{Pr}_1) + 4sd_1^0(\mathcal{Pr}_1) \\
&\quad + 2sd'_1(\mathcal{Pr}_1) + 4ss_1^0(\mathcal{Pr}_1) + \overrightarrow{dd'}_2(\mathcal{Pr}_1) + \overleftarrow{dd'}_2(\mathcal{Pr}_1) + 4\overleftarrow{dd''}_2(\mathcal{Pr}_1) \\
&\quad + 2\overrightarrow{dd''}_2(\mathcal{Pr}_1) + 2ds'_2(\mathcal{Pr}_1) + 2sd'_2(\mathcal{Pr}_1) + 4ss_2^1(\mathcal{Pr}_1) + 3ss_2^2(\mathcal{Pr}_1) \\
&\quad + ss_3^2(\mathcal{Pr}_1) \\
&= 2 \times 0 + 2 \times 0 + 3 \times 0 + 3 \times 0 + 4 \times 0 + 4 \times 0 + 2 \times 12 + 4 \times 4 + 2 \times 8 \\
&\quad + 4 \times 4 + 2 \times 8 + 4 \times 0 + 0 + 0 + 4 \times 0 + 2 \times 0 + 2 \times 0 + 2 \times 0 \\
&\quad + 4 \times 24 + 3 \times 0 + 0 \\
&= 184
\end{aligned}$$

$$\begin{aligned}
g_0(\mathcal{KPr}_2) &= \overrightarrow{dd'}_0(\mathcal{Pr}_2) + \overleftarrow{dd'}_0(\mathcal{Pr}_2) + 4\overleftarrow{dd''}_0(\mathcal{Pr}_2) + 2\overrightarrow{dd''}_0(\mathcal{Pr}_2) + 2ds'_0(\mathcal{Pr}_2) \\
&\quad + 2sd'_0(\mathcal{Pr}_2) + 4ss_0^1(\mathcal{Pr}_2) + 3ss_0^2(\mathcal{Pr}_2) + ss_1^2(\mathcal{Pr}_2) \\
&= 0 + 0 + 4 \times 0 + 2 \times 0 + 2 \times 0 + 2 \times 0 + 4 \times 0 + 3 \times 0 + 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
g_1(\mathcal{KPr}_2) &= 2dd_0^0(\mathcal{Pr}_2) + 3\overrightarrow{dd'}_0(\mathcal{Pr}_2) + 3\overleftarrow{dd'}_0(\mathcal{Pr}_2) + 4\overrightarrow{dd''}_0(\mathcal{Pr}_2) + 4\overleftarrow{dd''}_0(\mathcal{Pr}_2) \\
&\quad + 2\overrightarrow{dd''}_0(\mathcal{Pr}_2) + 4ds_0^0(\mathcal{Pr}_2) + 2ds'_0(\mathcal{Pr}_2) + 4sd_0^0(\mathcal{Pr}_2) + 2sd'_0(\mathcal{Pr}_2) \\
&\quad + 4ss_0^0(\mathcal{Pr}_2) + \overrightarrow{dd'}_1(\mathcal{Pr}_2) + \overleftarrow{dd'}_1(\mathcal{Pr}_2) + 4\overleftarrow{dd''}_1(\mathcal{Pr}_2) + 2\overrightarrow{dd''}_1(\mathcal{Pr}_2) \\
&\quad + 2ds'_1(\mathcal{Pr}_2) + 2sd'_1(\mathcal{Pr}_2) + 4ss_1^1(\mathcal{Pr}_2) + 3ss_1^2(\mathcal{Pr}_2) + ss_2^2(\mathcal{Pr}_2) \\
&= 2 \times 6 + 3 \times 0 + 3 \times 0 + 4 \times 1 + 4 \times 1 + 2 \times 0 + 4 \times 0 + 2 \times 0 + 4 \times 0 \\
&\quad + 2 \times 0 + 4 \times 0 + 22 + 22 + 4 \times 0 + 2 \times 0 + 2 \times 12 + 2 \times 12 + 4 \times 0 \\
&\quad + 3 \times 0 + 8 \\
&= 120
\end{aligned}$$

$$\begin{aligned}
 g_2(\mathcal{KPr}_2) &= 2dd_0^0(\mathcal{Pr}_2) + 2dd_1^0(\mathcal{Pr}_2) + 3\overrightarrow{dd}'_1(\mathcal{Pr}_2) + 3\overleftarrow{dd}'_1(\mathcal{Pr}_2) + 4\overline{dd}'_1(\mathcal{Pr}_2) \\
 &\quad + 4\widetilde{dd}'_1(\mathcal{Pr}_2) + 2\overline{dd}''_1(\mathcal{Pr}_2) + 4ds_1^0(\mathcal{Pr}_2) + 2ds'_1(\mathcal{Pr}_2) + 4sd_1^0(\mathcal{Pr}_2) \\
 &\quad + 2sd'_1(\mathcal{Pr}_2) + 4ss_1^0(\mathcal{Pr}_2) + \overrightarrow{dd}''_2(\mathcal{Pr}_2) + \overleftarrow{dd}''_2(\mathcal{Pr}_2) + 4\overline{dd}''_2(\mathcal{Pr}_2) \\
 &\quad + 2\overline{dd}''_2(\mathcal{Pr}_2) + 2ds_2(\mathcal{Pr}_2) + 2sd'_2(\mathcal{Pr}_2) + 4ss_2^1(\mathcal{Pr}_2) + 3ss_2^2(\mathcal{Pr}_2) \\
 &\quad + ss_3^2(\mathcal{Pr}_2) \\
 &= 2 \times 6 + 2 \times 176 + 3 \times 22 + 3 \times 22 + 4 \times 46 + 4 \times 46 + 2 \times 0 + 4 \times 44 \\
 &\quad + 2 \times 12 + 4 \times 44 + 2 \times 12 + 4 \times 0 + 256 + 256 + 4 \times 48 + 2 \times 48 \\
 &\quad + 2 \times 544 + 2 \times 544 + 4 \times 72 + 3 \times 8 + 288 \\
 &= 4840
 \end{aligned}$$

$$\begin{aligned}
 g_3(\mathcal{KPr}_2) &= 2dd_1^0(\mathcal{Pr}_2) + 2dd_2^0(\mathcal{Pr}_2) + 3\overrightarrow{dd}'_2(\mathcal{Pr}_2) + 3\overleftarrow{dd}'_2(\mathcal{Pr}_2) + 4\overline{dd}'_2(\mathcal{Pr}_2) \\
 &\quad + 4\widetilde{dd}'_2(\mathcal{Pr}_2) + 2\overline{dd}''_2(\mathcal{Pr}_2) + 4ds_2^0(\mathcal{Pr}_2) + 2ds'_2(\mathcal{Pr}_2) + 4sd_2^0(\mathcal{Pr}_2) \\
 &\quad + 2sd'_2(\mathcal{Pr}_2) + 4ss_2^0(\mathcal{Pr}_2) + \overrightarrow{dd}''_3(\mathcal{Pr}_2) + \overleftarrow{dd}''_3(\mathcal{Pr}_2) + 4\overline{dd}''_3(\mathcal{Pr}_2) \\
 &\quad + 2\overline{dd}''_3(\mathcal{Pr}_2) + 2ds_3(\mathcal{Pr}_2) + 2sd'_3(\mathcal{Pr}_2) + 4ss_3^1(\mathcal{Pr}_2) + 3ss_3^2(\mathcal{Pr}_2) \\
 &\quad + ss_4^2(\mathcal{Pr}_2) \\
 &= 2 \times 176 + 2 \times 704 + 3 \times 256 + 3 \times 256 + 4 \times 400 + 4 \times 400 + 2 \times 48 \\
 &\quad + 4 \times 800 + 2 \times 544 + 4 \times 800 + 2 \times 544 + 4 \times 320 + 0 + 0 + 4 \times 0 \\
 &\quad + 2 \times 0 + 2 \times 1920 + 2 \times 1920 + 4 \times 1664 + 3 \times 288 + 0 \\
 &= 31648
 \end{aligned}$$

$$\begin{aligned}
 g_4(\mathcal{KPr}_2) &= 2dd_2^0(\mathcal{Pr}_2) + 2dd_3^0(\mathcal{Pr}_2) + 3\overrightarrow{dd}'_3(\mathcal{Pr}_2) + 3\overleftarrow{dd}'_3(\mathcal{Pr}_2) + 4\overline{dd}'_3(\mathcal{Pr}_2) \\
 &\quad + 4\widetilde{dd}'_3(\mathcal{Pr}_2) + 2\overline{dd}''_3(\mathcal{Pr}_2) + 4ds_3^0(\mathcal{Pr}_2) + 2ds'_3(\mathcal{Pr}_2) + 4sd_3^0(\mathcal{Pr}_2) \\
 &\quad + 2sd'_3(\mathcal{Pr}_2) + 4ss_3^0(\mathcal{Pr}_2) + \overrightarrow{dd}''_4(\mathcal{Pr}_2) + \overleftarrow{dd}''_4(\mathcal{Pr}_2) + 4\overline{dd}''_4(\mathcal{Pr}_2) \\
 &\quad + 2\overline{dd}''_4(\mathcal{Pr}_2) + 2ds_4(\mathcal{Pr}_2) + 2sd'_4(\mathcal{Pr}_2) + 4ss_4^1(\mathcal{Pr}_2) + 3ss_4^2(\mathcal{Pr}_2) \\
 &\quad + ss_5^2(\mathcal{Pr}_2) \\
 &= 2 \times 704 + 2 \times 0 + 3 \times 0 + 3 \times 0 + 4 \times 0 + 4 \times 0 + 2 \times 0 + 4 \times 384 \\
 &\quad + 2 \times 1920 + 4 \times 384 + 2 \times 1920 + 4 \times 1888 + 0 + 0 + 4 \times 0 \\
 &\quad + 2 \times 0 + 2 \times 0 + 2 \times 0 + 4 \times 2304 + 3 \times 0 + 0 \\
 &= 28928
 \end{aligned}$$

In a similar manner, we can compute the genus distributions for \mathcal{CPr}_n and \mathcal{KPr}_n for higher values of n . We conclude this section by giving the partitioned genus distribution of \mathcal{Pr}_3 in Table 3 and listing the genus distributions of \mathcal{CPr}_3 and \mathcal{KPr}_3 obtained similarly.

Table 3: Genus distributions of the open chain $\mathcal{P}r_3$ of 3 copies of Δ_G .

$\mathcal{P}r_n$	$\mathcal{P}r_3$						
	k	0	1	2	3	4	5
dd_k^0	30	2080	40000	211840	67584	0	0
\overleftarrow{dd}'_k	1	82	2608	27104	38400	0	0
\widetilde{dd}'_k	1	82	2608	27104	38400	0	0
\overrightarrow{dd}'_k	0	46	2176	25376	38400	0	0
\overleftarrow{dd}'_k	0	46	2176	25376	38400	0	0
\overrightarrow{dd}''_k	0	0	0	288	3456	0	0
\overleftarrow{dd}''_k	0	0	0	288	3456	0	0
ds_k^0	0	212	11744	149568	336896	36864	0
\overrightarrow{ds}'_k	0	6	640	17664	113792	92160	0
\overleftarrow{ds}'_k	0	6	640	17664	113792	92160	0
sd_k^0	0	212	11744	149568	336896	36864	0
\overrightarrow{sd}'_k	0	6	640	17664	113792	92160	0
\overleftarrow{sd}'_k	0	6	640	17664	113792	92160	0
ss_k^0	0	0	1496	61728	393728	417792	0
ss_k^1	0	0	72	6144	112000	411648	221184
ss_k^2	0	0	0	32	3456	0	0
g_k	32	2784	77184	755072	1866240	1271808	221184

$$\begin{aligned}
 g_0(\mathcal{CPr}_3) &= 2 & g_1(\mathcal{CPr}_3) &= 278 & g_2(\mathcal{CPr}_3) &= 17480 \\
 g_3(\mathcal{CPr}_3) &= 447648 & g_4(\mathcal{CPr}_3) &= 3920896 & g_5(\mathcal{CPr}_3) &= 8667648 \\
 g_6(\mathcal{CPr}_3) &= 3723264
 \end{aligned}$$

$$\begin{aligned}
 g_0(\mathcal{KPr}_3) &= 0 & g_1(\mathcal{KPr}_3) &= 208 & g_2(\mathcal{KPr}_3) &= 16688 \\
 g_3(\mathcal{KPr}_3) &= 445056 & g_4(\mathcal{KPr}_3) &= 3924352 & g_5(\mathcal{KPr}_3) &= 8667648 \\
 g_6(\mathcal{KPr}_3) &= 3723264
 \end{aligned}$$

5 Conclusions

This paper derives closed formulas for calculating the genus distributions of graphs resulting from co-self-amalgamation and contra-self-amalgamation of any double-edge-rooted graph (G, e, f) , where both the edges e and f have two 2-valent endpoints and the double-root partitioned genus distribution of (G, e, f) is known. This enables one to make genus distribution calculations for edge-amalgamated closed chains corresponding to many infinite families of open chains. To illustrate this, we demonstrated how the genus distributions of the circular ladder CL_4 and the Möbius ladder ML_4 can be calculated from the partitioned genus distribution of the closed-end ladder L_3 . We also showed genus distribution calculations of small co-self-pasted and contra-self-pasted closed chains of copies of the triangular prism graph. In this manner, calculating genus distribution of arbitrarily large closed chains consisting of identical copies of the same graph as well as well interleaved copies of different graphs becomes tractible.

It would be interesting to know if open as well as closed chains of base graphs with strongly unimodal genus distributions are unimodal. The results in this paper formulate the genus distribution of closed chains as a linear combination of sub-partials of the corresponding open chains. In turn, each of these sub-partials of an open chain is calculated as a linear combination of convolutions of sub-partials of its constituent subgraphs. Although convolutions of strongly unimodal sequences are strongly unimodal, linear combinations of unimodal sequences need not be unimodal, unless the modes are sufficiently close together. This seems quite complicated, due to the large number of sub-partials. Nevertheless, the analysis of the closed-formulas presented in this paper may prove to be useful for establishing structural results related to genus distributions such as the unimodality conjecture. In particular, if the unimodality conjecture is false, the results could help construct a counterexample to disprove the conjecture. Further opportunities for research also include self-edge-amalgamation on root-edges with endpoints of higher-valence.

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