The existence of square non-integer Heffter arrays

Nicholas J. Cavenagh

Department of Mathematics, The University of Waikato,
Private Bag 3105, Hamilton 3240, New Zealand

Jeffery H. Dinitz

Department of Mathematics and Statistics, University of Vermont,
Burlington, VT 05405, USA

Diane M. Donovan

School of Mathematics and Physics, The University of Queensland,
Queensland 4072, Australia

Emine Şule Yazıcı

Department of Mathematics, Koç University, Istanbul, Turkey

Abstract

A Heffter array $H(n; k)$ is an $n \times n$ matrix such that each row and column contains k filled cells, each row and column sum is divisible by $2nk + 1$ and either x or $-x$ appears in the array for each integer $1 \leqslant x \leqslant nk$. Heffter arrays are useful for embedding the graph K_{2nk+1} on an orientable surface. An integer Heffter array is one in which each row and column sum is 0. Necessary and sufficient conditions (on n and k) for the existence of an integer Heffter array $H(n; k)$ were verified by Archdeacon, Dinitz, Donovan and Yazıcı (2015) and Dinitz and Wanless (2017). In this paper we consider square Heffter arrays that are not necessarily integer. We show that such Heffter arrays exist whenever $3 \leqslant k < n$.

Keywords: Heffter arrays, biembedding cycle systems.

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E-mail addresses: nickc@waikato.ac.nz (Nicholas J. Cavenagh), jeff.dinitz@uvm.edu (Jeffery H. Dinitz), dmd@maths.uq.edu.au (Diane M. Donovan), eyazici@ku.edu.tr (Emine Şule Yazıcı)

1   Introduction

A Heffter array $H(m, n; s, t)$ is an $m \times n$ matrix of integers such that:

(1) each row contains s filled cells and each column contains t filled cells;

(2) the elements in every row and column sum to 0 in $\mathbb{Z}_{2ms+1}$; and

(3) for each integer $1 \leqslant x \leqslant ms$, either x or $-x$ appears in the array.

If the Heffter array is square, then $m = n$ and necessarily $s = t$. We denote such Heffter arrays by $H(n; k)$, where each row and each column contains k filled cells. A Heffter array is called an *integer* Heffter array if Condition (2) in the definition of a Heffter array above is strengthened so that the elements in every row and every column sum to zero in $\mathbb{Z}$.

Archdeacon, in [1], was the first to define and study a Heffter array $H(m, n; s, t)$. He showed that a Heffter array with a pair of special orderings can be used to construct an embedding of the complete graph K_{2ms+1} on a surface. This connection is formalised in the following theorem. For definitions of simple and compatible orderings refer to [1].

Theorem 1.1 ([1]). *Given a Heffter array $H(m, n; s, t)$ with compatible orderings ω_r of the symbols in the rows of the array and ω_c on the symbols in the columns of the array, then there exists an embedding of K_{2ms+1} such that every edge is on a face of size s and a face of size t. Moreover, if ω_r and ω_c are both simple, then all faces are simple cycles.*

The embedding of K_{2ms+1} given in Theorem 1.1 provides a connection with the embedding of cycle systems. A t-cycle system on n points is a decomposition of the edges of K_n into t-cycles. A t-cycle system C on K_n is cyclic if there is a labeling of the vertex set of K_n with the elements of $\mathbb{Z}_n$ such that the permutation $x \to x + 1$ preserves the cycles of C. A biembedding of an s-cycle system and a t-cycle system is a face 2-colorable topological embedding of the complete graph K_{2ms+1} in which one color class is comprised of the cycles in the s-cycle system and the other class contains the cycles in the t-cycle system, see for instance [4, 5, 6, 8, 9, 10, 11] for further details.

A number of papers have appeared on the construction of Heffter arrays, $H(m, n; s, t)$. The case where the array contained no empty cells was studied in [2], with results summarised in Theorem 1.2.

Theorem 1.2 ([2]). *There is an $H(m, n; n, m)$ for all $m, n \geqslant 3$ and an integer Heffter array $H(m, n; n, m)$ exists if and only if $m, n \geqslant 3$ and $mn \equiv 0, 3 \pmod 4$.*

The papers [3, 7] focused on square integer Heffter arrays $H(n; k)$ and verified their existence for all admissible orders. This result is summarized in the following theorem.

Theorem 1.3 ([3, 7]). *There exists an integer $H(n; k)$ if and only if $3 \leqslant k \leqslant n$ and $nk \equiv 0, 3 \pmod 4$.*

Table 1 lists the possible cases and cites the article which verifies existence of square integer Heffter arrays, where DNE represents a value that does not exist. In these cases we will verify existence for the non-integer Heffter arrays $H(n; k)$. The main result of this paper is the following.

Theorem 1.4. *There exists an $H(n; k)$ if and only if $3 \leqslant k \leqslant n$.*

Table 1: Existence results for square integer Heffter arrays $H(n; k)$.

$n \setminus k$	0	1	2	3
0	[3]	[3, 7]	[3]	[3]
1	[3]	DNE	DNE	[3]
2	[3]	DNE	[3]	DNE
3	[3]	[3, 7]	DNE	DNE

Table 2: Cases for non-integer Heffter arrays $H(n; k)$.

	Case A	Case B	Case C	Case D	Case E
k	2 (mod 4)	3 (mod 4)	3 (mod 4)	1 (mod 4)	1 (mod 4)
n	1, 3 (mod 4)	3 (mod 4)	2 (mod 4)	1 (mod 4)	2 (mod 4)

From Theorem 1.2 above, the case $n = k$ has been solved, so we henceforth assume that $n > k$. The cases that need to be addressed are set out in Table 2. Cases A, B, C, D and E are solved by Theorems 3.2, 4.2, 5.2, 6.2 and 7.2, respectively, thus proving Theorem 1.4.

In this paper the rows and columns of a square $n \times n$ array are always indexed by the elements of $\{1, 2, \ldots, n\}$. Unless otherwise stated, when working modulo n, replace 0 by n, so we use the symbols $1, 2, \ldots, n$ instead of $0, 1, \ldots, n-1$. While rows and columns are calculated modulo an integer, entries are always expressed as non-zero integers. Throughout this paper $A[r, c] = x$ denotes the occurrence of symbol x in cell (r, c) of array A.

By $A \pm z$ we refer to the array obtained by replacing $A[r, c]$ by $A[r, c] + z$ (if $A[r, c] > 0$) and $A[r, c] - z$ (if $A[r, c] < 0$). If each row and each column of A contains the same number of positive and negative numbers, then $A \pm z$ has the same row and column sums as A. In this case we say A is *shiftable*. The *support* of an array A is defined to be the set containing the absolute value of the elements contained in A. If A is an array with support S and z a nonnegative integer, then $A \pm z$ has support $S + z$.

2   Increasing k from base cases

For each of the cases set out in Table 2 our overall strategy is to generate a base case $H(n; k)$ where k takes the smallest possible value and then increase k by multiples of 4, adjoining 4 additional entries to each row and column. In this section we outline various tools to enable this process. To this end, we introduce the following definitions.

We associate the cells of an $n \times n$ array with the complete bipartite graph $K_{n,n}$ where partite sets are denoted $\{a_i \mid i = 1, 2, \ldots, n\}$ and $\{b_j \mid j = 1, 2, \ldots, n\}$ and the edge $\{a_i, b_j\}$ corresponds to the cell (i, j). We say that in an $n \times n$ array a set of cells S forms a *2-factor* if the corresponding set of edges in the graph $K_{n,n}$ forms a spanning 2-regular graph and forms a *Hamilton cycle* if the corresponding set of edges forms a single cycle of length $2n$.

For each $d \in \{0, 1, \ldots, n - 1\}$, we define the *diagonal* D_d to be the set of cells of the form $(r + d, r)$, $1 \leqslant r \leqslant n$ (evaluated modulo n). Observe that the cells $D_i \cup D_j$ form a Hamilton cycle whenever $j - i$ is coprime to n.

Lemma 2.1. *Let S_1 and S_2 be two disjoint sets of cells in an $n \times n$ array which each form*

Hamilton cycles. The cells of $S_1 \cup S_2$ can be filled with the elements of $\{1, 2, \ldots, 4n\}$ so that each row and column sum is equal to $8n + 2$.

Proof. Let the cells of S_1 and S_2 be $\{e_i \mid 1 \leqslant i \leqslant 2n\}$ and $\{f_i \mid 1 \leqslant i \leqslant 2n\}$, respectively, where:

- Cells e_i and e_{i+1} are in the same row (column) whenever i is odd (respectively, even);
- Cells f_i and f_{i+1} are in the same row (column) whenever i is odd (respectively, even);
- Cells e_1 and f_1 are in the same row.

Place 1 in cell e_1, $4n$ in cell f_1 and:

- $2n - 2i + 1$ in cell e_{2i+1}, where $1 \leqslant i \leqslant n - 1$; $2n + 2i - 1$ in cell e_{2i} where $1 \leqslant i \leqslant n$;
- $2n + 2i$ in cell f_{2i+1} where $1 \leqslant i \leqslant n - 1$; $2n - 2i + 2$ in cell f_{2i} where $1 \leqslant i \leqslant n$.

The entries in cells e_1, e_2, f_1 and f_2 add to $1 + (2n + 1) + 4n + 2n = 8n + 2$. For every other row, there are two cells from S_1 with entries adding to $4n + 2$ and two cells from S_2 with entries adding to $4n$. For every column, there are two cells from S_1 adding to $4n$ and two cells from S_2 adding to $4n + 2$. See the example below.   □

We demonstrate Lemma 2.1 below when $n = 9$. The elements of S_1 are underlined.

$$S_1 \cup S_2$$

$\underline{1}$	$\underline{19}$	18			36			
34	$\underline{17}$	$\underline{21}$			2			
		$\underline{15}$	$\underline{23}$	10	26			
4			$\underline{13}$	$\underline{25}$				32
	14			$\underline{11}$	$\underline{27}$		22	
		20			$\underline{9}$	$\underline{29}$	16	
			30			$\underline{7}$	$\underline{31}$	6
	24				12		$\underline{5}$	$\underline{33}$
$\underline{35}$			8	28				$\underline{3}$

The following theorem will be crucial in Cases A and D.

Theorem 2.2. *Let $H(n; k)$ be a Heffter array such that each row and column sums to $2nk + 1$. Suppose there exist Hamilton cycles H_1 and H_2 disjoint to each other and to the filled cells of $H(n; k)$. Then there exists an $H(n; k + 4)$ Heffter array with row and column sums equal to $2n(k + 4) + 1$, where the filled cells are precisely the filled cells of $H(n; k)$, H_1 and H_2.*

Proof. Let A_0 represent the $H(n; k)$ and negate each element so that each row and column has sum equal to $-(2nk + 1)$. From Lemma 2.1, there exists an array A_1' on the cells of H_1 and H_2 such that each row and column sum is equal to $8n + 2$; add $n(k + 4) - (4n) = nk$ to

each element of A_1' to create a new array A_1 that has support $\{nk+1, nk+2, \ldots, n(k+4)\}$. Note that in A_1 each row and column sum is equal to $8n + 2 + 4nk$. Let A be the union of A_0 with A_1. The resulting array A has support $\{1, 2, \ldots, n(k+4)\}$, with $k+4$ filled cells in each row and column. Finally, each row and column sum of A is

$$-(2nk+1) + (8n+2) + 4(nk) = 2n(k+4) + 1,$$

as desired. □

The following lemma generalizes Theorem 2.2 from [7], and is used in Cases B and C.

Lemma 2.3. *Let S_1 and S_2 be two disjoint sets of cells in an $n \times n$ array which each form Hamilton cycles. Then for any positive integers t and $s > t + 2n$, the cells of $S_1 \cup S_2$ can be filled with elements to make a shiftable array with support $\{s+i, t+i \mid 1 \leqslant i \leqslant 2n\}$ so that the four elements in each row and each column sum to 0.*

Proof. Let the sets of cells of S_1 and S_2 be $\{e_i \mid 1 \leqslant i \leqslant 2n\}$ and $\{f_i \mid 1 \leqslant i \leqslant 2n\}$, respectively, where:

- Cells e_i and e_{i+1} are in the same row (column) whenever i is odd (respectively, even);
- Cells f_i and f_{i+1} are in the same row (column) whenever i is odd (respectively, even);
- Cells e_1 and f_1 are in the same row.

Place:

- $s + 2n$ in cell e_1 and $-(t + 2n)$ in cell f_1, with sum $s - t$;
- $s + 2i$ in cell e_{2i+1} and $-(t + 2i)$ in cell f_{2i+1}, with sum $s - t$, for $1 \leqslant i \leqslant n - 1$,
- $-(s + 2i - 1)$ in cell e_{2i} and $t + 2i - 1$ in cell f_{2i}, with sum $t - s$, for $1 \leqslant i \leqslant n$.

It now follows that the row sums are 0. Using similar arguments it can be seen that the columns also sum to 0. Observe that there are two positive and two negative integers in each row and column; thus the array is shiftable. □

The proof of the following lemma is similar to the proof of Lemma 2.3; we use this in Case E.

Lemma 2.4. *Let n be even. Let S_1 and S_2 be two disjoint sets of cells in an $n \times n$ array which each form 2-factors that are the union of two n-cycles. Then for any positive integers s, t, u and v where $s > t + n$, $t > u + n$ and $u > v + n$, the cells of $S_1 \cup S_2$ can be filled with elements to make a shiftable array with support $\{s+i, t+i, u+i, v+i \mid 1 \leqslant i \leqslant n\}$ so that the four elements in each row and column sum to 0.*

Proof. Let C_i, C_i' be the cycles of the 2-factor S_i, $i \in \{1, 2\}$, where C_1 and C_1' share a row and C_2 and C_2' share a row. Let the sets of cells of C_1, C_1', C_2 and C_2' be $\{e_i \mid 1 \leqslant i \leqslant n\}$, $\{f_i \mid 1 \leqslant i \leqslant n\}$, $\{g_i \mid 1 \leqslant i \leqslant n\}$ and $\{h_i \mid 1 \leqslant i \leqslant n\}$, respectively, where:

- Cells e_i and e_{i+1} are in the same row (column) whenever i is odd (respectively, even);

- Cells f_i and f_{i+1} are in the same row (column) whenever i is odd (respectively, even);

- Cells g_i and g_{i+1} are in the same row (column) whenever i is odd (respectively, even);

- Cells h_i and h_{i+1} are in the same row (column) whenever i is odd (respectively, even);

- Cells e_1 and g_1 are in the same row; cells f_1 and h_1 are in the same row.

Place:

- $s + n$ in cell e_1, $-(t + n)$ in cell g_1 and $u + n$ in cell f_1, $-(v + n)$ in cell h_1;

- $s + 2i$ in cell e_{2i+1} and $-(t + 2i)$ in cell g_{2i+1}, for $1 \leqslant i \leqslant n/2 - 1$;

- $-(s + 2i - 1)$ in cell e_{2i} and $t + 2i - 1$ in cell g_{2i}, for $1 \leqslant i \leqslant n/2$;

- $u + 2i$ in cell f_{2i+1} and $-(v + 2i)$ in cell h_{2i+1}, for $1 \leqslant i \leqslant n/2 - 1$;

- $-(u + 2i - 1)$ in cell f_{2i} and $v + 2i - 1$ in cell h_{2i}, for $1 \leqslant i \leqslant n/2$.

It now follows that the row sums are 0. Using similar arguments it can be seen that the columns also sum to 0.   □

3   Case A: $k \equiv 2 \pmod 4$

In this section we construct a Heffter array $H(n; k)$, for $n \equiv 1, 3 \pmod 4$ and $k \equiv 2 \pmod 4$, where $k < n$. Row and column sums will always equal $2nk + 1$. We start with an example of our construction.

$H(15; 6)$

6									−4	89	81	1	8	
	12									−88	83	87	85	2
86		18									−82	77	3	79
73	80		24									−76	71	9
15	67	74		30									−70	65
59	21	61	68		36									−64
−58	53	27	55	62		42								
	−52	47	33	49	56		48							
		−46	41	39	43	50		54						
			−40	35	45	37	44		60					
				−34	29	51	31	38		66				
					−28	23	57	25	32		72			
						−22	17	63	19	26		78		
							−16	11	69	13	20		84	
								−10	5	75	7	14		90

Lemma 3.1. *For $n \equiv 1, 3 \pmod 4$, $n \geqslant 7$ and $k = 6$ there exists a Heffter array $H(n; 6)$.*

Proof. We remind the reader that rows and columns are calculated modulo n but the array entries are not. The array $A = A[r, c]$ is defined as follows, where $1 \leqslant i \leqslant n$:

$$A[i, i] = 6i, \qquad\qquad\qquad A[i + 2, i] = 6n + 2 - 6i,$$
$$A[i + 1, n - 2 + i] = 6n + 1 - 6i, \qquad A[i + 2, n - 2 + i] = 6i - 3,$$
$$A[i, n - 5 + i] = 6n + 5 - 6i, \qquad A[i + 1, n - 5 + i] = -6n - 4 + 6i.$$

Then the support of A is $\{1, 2, \ldots, 6n\}$. The sets of elements in rows 1, 2 and i, $3 \leqslant i \leqslant n$, are, respectively:

$$\{6, 8, 1, 6n - 9, 6n - 1, -4\},$$
$$\{12, 2, 6n - 5, 6n - 3, 6n - 7, -(6n - 2)\},$$
$$\{6i, 6n + 2 - 6(i - 2), 6n + 1 - 6(i - 1), 6(i - 2) - 3,$$
$$6n + 5 - 6i, -6n - 4 + 6(i - 1)\}.$$

Thus in each case the sum of elements in a row is $12n + 1$.

The set of elements in column i, $1 \leqslant i \leqslant n - 5$ is:

$$\{6i, 6n + 2 - 6i, 6n + 1 - 6(i + 2), 6(i + 2) - 3,$$
$$6n + 5 - 6(i + 5), -6n - 4 + 6(i + 5)\}.$$

The set of elements in columns $n - 4$, $n - 3$, $n - 2$, $n - 1$ and n are, respectively:

$$\{6n - 24, 26, 13, 6n - 15, 6n - 1, -(6n - 2)\},$$
$$\{6n - 18, 20, 7, 6n - 9, 6n - 7, -(6n - 8)\},$$
$$\{6n - 12, 14, 1, 6n - 3, 6n - 13, -(6n - 14)\},$$
$$\{6n - 6, 8, 6n - 5, 3, 6n - 19, -(6n - 20)\},$$
$$\{6n, 2, 6n - 11, 9, 6n - 25, -(6n - 26)\}.$$

Thus in each case the sum of elements in a column is $12n + 1$. $\qquad\square$

Theorem 3.2. *There exists a Heffter array $H(n; k)$ for all $n \equiv 1, 3 \pmod 4$ and $k \equiv 2 \pmod 4$, where $n > k \geqslant 6$.*

Proof. Let $k = 4p + 6$. Then $4p + 6 \leqslant n - 1$ so $p \leqslant (n - 7)/4$. We have solved the case $p = 0$ in Lemma 3.1 so we may assume $p \geqslant 1$. Observe that the Heffter array given in the proof of that lemma uses only elements in diagonals D_0, D_2, D_3, D_4, D_5 and D_6 and so does not intersect the diagonals $D_7, D_8, \ldots, D_{n-1}$. We can apply Theorem 2.2 recursively, where the diagonals $D_7, D_8, \ldots, D_{6+2p-1}, D_{6+2p}$ can be paired to give sets of cells S_1 and $D_{6+2p+1}, D_{6+2p+2}, \ldots, D_{6+4p-1}, D_{6+4p}$ paired to give sets of cells S_2. The result is a Heffter array $H(n; k)$ with constant row and column sum $2nk + 1$ whenever $k \equiv 2 \pmod 4$, $n \equiv 1, 3 \pmod 4$ and $n > k \geqslant 6$. $\qquad\square$

4   Case B: $k \equiv 3 \pmod 4$ and $n \equiv 3 \pmod 4$

In this section we construct a Heffter array $H(n; k)$ where $n = 4m + 3$, $k = 4p + 3$ and $k < n$.

We will begin with $k = 3$. We first assume that $m \geqslant 4$ and construct an $n \times n$ array which is the concatenation of three smaller arrays, $A_0 = A_0[r,c]$, of dimension $(4m-7) \times (4m-7)$, $A_1 = A_1[r,c]$ of dimension 7×7 and C of dimension 3×3, each containing 3 filled cells per row and column. So we see that $n = 4m+3$. The sum of the rows and columns in A_0 and A_1 will be 0, while the sum of the rows and columns in C will be $2nk + 1$.

We begin with an example of the main construction of this section.

$$H(19; 3)$$

16	−48	32																
17	27		−44															
−33		−14		47														
	21		15		−36													
		−18		−13		31												
			29		−9		−20											
				−34		−12		46										
					45		−10	−35										
						−19	30	−11										
									−25	24	1							
									22	−50		28						
									3		37		−40					
										26		23		−49				
											−38		42		−4			
												−51		8	43			
													−2	41	−39			
																5	57	53
																54	6	55
																56	52	7

Lemma 4.1. *For $n \equiv 3 \pmod 4$ and $n \geqslant 7$ there exists a Heffter array $H(n;3)$.*

Proof. Let $n = 4m + 3$. We first assume that $m \geqslant 4$ and so $n \geqslant 19$. The small cases will be dealt with at the end of the proof. Let A'_0 be:

$$
\begin{aligned}
A'_0[2i-1, 2i] &= 8m+1-i, & 1 &\leqslant i \leqslant m, \\
A'_0[2i, 2i-1] &= -(8m+i), & 1 &\leqslant i \leqslant m, \\
A'_0[2i, 2i+1] &= 12m-i, & 1 &\leqslant i \leqslant m-1, \\
A'_0[2i+1, 2i] &= -(4m+1+i), & 1 &\leqslant i \leqslant m-1, \\
A'_0[2m-2+2i, 2m-1+2i] &= 5m+i, & 1 &\leqslant i \leqslant m-3, \\
A'_0[2m-1+2i, 2m-2+2i] &= -(11m+1-i), & 1 &\leqslant i \leqslant m-3,
\end{aligned}
$$

$$A_0'[2m - 1 + 2i, 2m + 2i] = 9m + i, \qquad 1 \leqslant i \leqslant m - 4,$$
$$A_0'[2m + 2i, 2m - 1 + 2i] = -(7m + 1 - i), \quad 1 \leqslant i \leqslant m - 4,$$
$$A_0'[i + 1, i + 1] = -(4m - 1 - i), \quad 1 \leqslant i \leqslant 2m - 2,$$
$$A_0'[2m + i, 2m + i] = 2m - i, \qquad 1 \leqslant i \leqslant 2m - 8,$$
$$A_0'[2m, 2m] = 4m - 1, \quad A_0'[1, 4m - 7] = -12m,$$
$$A_0'[1, 1] = 4m, \qquad A_0'[4m - 7, 1] = 4m + 1,$$
$$A_0'[4m - 7, 4m - 7] = 6m + 3.$$

We illustrate A_0' in the case $m = 4$:

$$A_0'(m = 4)$$

16	32							−48
−33	−14	47						
	−18	−13	31					
		−34	−12	46				
			−19	−11	30			
				−35	−10	45		
					−20	−9	29	
						−36	15	21
17							−44	27

First observe that the array A_0' is a $(4m - 7) \times (4m - 7)$ array that has 3 filled cells in each row and column.

To confirm that the row and columns sums are 0, note that this array was constructed by taking the first $4m - 8$ rows and columns of the integer Heffter array $H(4m; 3)$ given in [3], then placing entry $-12m$ in cell $(1, 4m - 7)$, entry $4m + 1$ in cell $(4m - 7, 1)$ and entry $6m + 3$ in cell $(4m - 7, 4m - 7)$. Thus we need only check the sum of row $4m - 7$ which is $(4m + 1) + (6m + 3) - (10m + 4) = 0$ and the sum of column $4m - 7$ which is $-12m + (6m - 3) + (6m + 3) = 0$. Hence each row and column in the array A_0' sums to zero.

Although not necessary for the case $k = 3$, for larger values of k (see the following theorem) we map the rows and columns of A_0' so that the filled cells are a subset of the union of diagonals $D_0 \cup D_1 \cup D_2 \cup D_{n-2} \cup D_{n-1}$. This can be done by applying the mapping

$$i \mapsto \begin{cases} 2i - 1, & \text{when } 1 \leqslant i \leqslant 2m - 3, \\ 8m - 2i - 12, & \text{when } 2m - 2 \leqslant i \leqslant 4m - 7, \end{cases}$$

to the rows and column of A_0'. This does not change the row and column sum and the support is still

$$\{1, 2, \ldots, 4m + 3\} \setminus \{1, 2, \ldots, 7, 2m, 6m - 2, \ldots, 6m + 2, 6m + 4,$$
$$10m - 3, \ldots, 10m + 3, 12m + 1, 12m + 2, \ldots, 12m + 9\}.$$

We call this rearranged array A_0; see $H(19; 3)$ above for A_0 when $m = 4$.

Next, let A_1 be:

$$A_1$$

$-(6m+1)$	$6m$	1				
$6m-2$	$-(12m+2)$		$6m+4$			
3		$10m-3$		$-10m$		
	$6m+2$		$6m-1$		$-(12m+1)$	
		$-(10m-2)$		$10m+2$		-4
			$-(12m+3)$		$2m$	$10m+3$
				-2	$10m+1$	$-(10m-1)$

It is easy to check that this array has row and column sum 0 and support

$$\{1, 2, 3, 4, 2m, 6m-2, \ldots, 6m+2, 6m+4,$$
$$10m-3, \ldots, 10m+3, 12m+1, 12m+2, 12m+3\}.$$

We place A_1 on the intersection of row and column sets $\{4m-6, 4m-5, \ldots, 4m\}$.

Finally, place the block C on the intersection of the row and column sets $\{4m+1, 4m+2, 4m+3\}$. Observe that the rows and columns sum to $2nk+1$. It is convenient to express C in terms of n and k, as it will be part of more general constructions in the next theorems. However, if $k = 3$, observe that $\{12m+4, 12m+5, \ldots, 12m+9\} = \{nk-5, nk-4, \ldots, nk\}$.

$$C$$

5	nk	$nk-4$
$nk-3$	6	$nk-2$
$nk-1$	$nk-5$	7

Let A be the concatenation of A_0, A_1 and C to obtain an $H(4m+3; 3)$ Heffter array for all $m \geqslant 4$. An $H(7; 3)$ is given in the Appendix. When $n = 11$ or 15, concatenating the array $B(n)$ given in the Appendix for these size with C will produce an $H(n; 3)$ with similar properties and support. □

We now consider the case when $k > 3$; we apply the techniques developed in Section 2.

Theorem 4.2. *There exists a Heffter array $H(n; k)$ for all $n \equiv 3 \pmod 4$ and $k \equiv 3 \pmod 4$, where $n > k \geqslant 3$.*

Proof. Given Lemma 4.1 we only need to address the case $k = 4p + 3$ where $1 \leqslant p < m$. Since $k \leqslant n - 4$, $p \leqslant (n-7)/4$. First observe that the filled cells of $H(n; 3)$ given in the previous lemma are a subset the union of diagonals $D_0, D_1, D_2, D_{n-2}, D_{n-1}$ with support $\{1, 2, \ldots, 12m+3\} \cup \{nk-5, nk-4, \ldots, nk\}$. We next identify p disjoint Hamilton cycles, that are also disjoint from diagonals $D_0, D_1, D_2, D_{n-2}, D_{n-1}$, by pairing the remaining $(n-5)$ diagonals. Then we apply Lemma 2.3, contributing $\{12m+4, 12m+5, \ldots, nk-6\}$ to the support. The result is a Heffter array $H(n; k)$ with each row and column sum equal to 0 except for the final three rows and columns which sum to $2nk+1$. □

5 Case C: $k \equiv 3 \pmod 4$ and $n \equiv 2 \pmod 4$

We construct a Heffter array $H(n; k)$ where $n = 4m + 6$ and $k = 4p + 3$ for $k < n$.

As in the previous section, we will begin with $k = 3$. Firstly we will assume that $m \geqslant 7$ and construct an $n \times n$ array which is the concatenation of several smaller arrays. Here we use four smaller arrays, $A_0 = A_0[r, c]$ of size $(4m - 13) \times (4m - 13)$, $A_1 = A_1[r, c]$ of size 13×13, $A_2 = A_2[r, c]$ of size 3×3 and $A_3 = A_3[r, c]$ of size 3×3, each containing 3 filled cells per row and column. The sum of the rows and columns in A_0, A_1 and A_2 is 0, while the sum of the rows and columns in A_3 is $2nk + 1$.

Lemma 5.1. *For $n \equiv 2 \pmod 4$ and $n \geqslant 34$ there exists a Heffter array $H(n; 3)$.*

Proof. Let $n = 4m + 6$ and $m \geqslant 7$. Let A_0' be:

$$
\begin{aligned}
A_0'[2i - 1, 2i] &= 8m + 1 - i, & 1 &\leqslant i \leqslant m, \\
A_0'[2i, 2i - 1] &= -(8m + i), & 1 &\leqslant i \leqslant m, \\
A_0'[2i, 2i + 1] &= 12m - i, & 1 &\leqslant i \leqslant m - 1, \\
A_0'[2i + 1, 2i] &= -(4m + 1 + i), & 1 &\leqslant i \leqslant m - 1, \\
A_0'[2m - 2 + 2i, 2m - 1 + 2i] &= 5m + i, & 1 &\leqslant i \leqslant m - 6, \\
A_0'[2m - 1 + 2i, 2m - 2 + 2i] &= -(11m + 1 - i), & 1 &\leqslant i \leqslant m - 6, \\
A_0'[2m - 1 + 2i, 2m + 2i] &= 9m + i, & 1 &\leqslant i \leqslant m - 7, \\
A_0'[2m + 2i, 2m - 1 + 2i] &= -(7m + 1 - i), & 1 &\leqslant i \leqslant m - 7, \\
A_0'[i + 1, i + 1] &= -(4m - 1 - i), & 1 &\leqslant i \leqslant 2m - 2, \\
A_0'[2m + i, 2m + i] &= 2m - i, & 1 &\leqslant i \leqslant 2m - 14, \\
\end{aligned}
$$
$$
A_0'[2m, 2m] = 4m - 1, \quad A_0'[1, 4m - 13] = -(12m),
$$
$$
A_0'[1, 1] = 4m, \quad A_0'[4m - 13, 1] = 4m + 1,
$$
$$
A_0'[4m - 13, 4m - 13] = 6m + 6.
$$

We illustrate A_0' when $m = 7$:

$$A_0'(m = 7)$$

28	56													−84
−57	−26	83												
	−30	−25	55											
		−58	−24	82										
			−31	−23	54									
				−59	−22	81								
					−32	−21	53							
						−60	−20	80						
							−33	−19	52					
								−61	−18	79				
									−34	−17	51			
										−62	−16	78		
											−35	−15	50	
											−63	27	36
	29												−77	48

Observe that the array A_0' is a $(4m - 13) \times (4m - 13)$ array that has 3 filled cells in each row and column. Similarly to the previous case, this array was constructed by taking the first $4m - 14$ rows and columns of the integer Heffter array $H(4m; 3)$ given in [3], then placing entry $-12m$ in cell $(1, 4m - 13)$, entry $4m + 1$ in cell $(4m - 13, 1)$ and entry $6m + 6$ in cell $(4m - 13, 4m - 13)$. Thus we need only check the sum of row $4m - 13$ which is $(4m + 1) + (6m + 6) - (10m + 7) = 0$ and the sum of column $4m - 13$ which is $-12m + (6m - 6) + (6m + 6) = 0$. Hence all rows and columns in the array A_0' sum to zero. We apply the same mapping as in the proof of Lemma 4.1 to the rows and columns so that all non-empty cells are a subset of the diagonals $D_0, D_1, D_2, D_{n-2}, D_{n-1}$. Let A_0 be the resultant array. The support for A_0 is

$$\{1, 2, \ldots, 12m + 8\} \setminus \{1, 2, \ldots, 13, 2m, 6m - 5, 6m - 4, \ldots, 6m + 5, 6m + 7,$$
$$10m - 6, 10m - 5, \ldots, 10m + 6, 12m + 1, 12m + 2, \ldots, 12m + 18\}.$$

We now arrange the 57 missing symbols into a 13×13 array A_1 and two 3×3 arrays A_2 and A_3, where A_2 and A_3 ares shown below and A_1 is given at the end of the Appendix.

A_2

-8	$nk - 2$	$-(nk - 10)$
$-(nk - 9)$	-9	nk
$nk - 1$	$-(nk - 11)$	-10

A_3

11	$nk - 3$	$nk - 7$
$nk - 6$	12	$nk - 5$
$nk - 4$	$nk - 8$	13

The support for A_1 is

$$\{1, 2, \ldots, 7, 2m, 6m - 5, 6m - 4, \ldots, 6m + 5, 6m + 7,$$
$$10m - 6, 10m - 5, \ldots, 10m + 6, 12m + 1, 12m + 2, \ldots, 12m + 6\}.$$

Finally we give A_2 and A_3 with support

$$\{8, 9, 10, 11, 12, 13\} \cup \{nk - 11, nk - 10, \ldots, nk\}.$$

In the case $k = 3$, observe that $\{nk - 11, nk - 10, \ldots, nk\} = \{12m + 7, 12m + 8, \ldots, 12m + 18\}$.

The concatenation of A_0, A_1, A_2 and A_3 gives a $H(4m+6; 3)$ Heffter array for $m \geqslant 7$. $\square$

Theorem 5.2. *There exists a Heffter array $H(n; k)$ for all $n \equiv 2 \pmod 4$ and $k \equiv 3 \pmod 4$, where $n > k \geqslant 3$.*

Proof. An $H(6; 3)$ is given in the Appendix. Otherwise let $n = 4m + 6$ where $m \geqslant 1$. When $1 \leqslant m \leqslant 5$ concatenate the $B(4m+6)$ given in the Appendix with the array C given in Case B to get an $H(4m + 6; 3)$. Observe that when $1 \leqslant m \leqslant 4$ the entries are only on diagonals $D_0, D_1, D_2, D_{n-2}, D_{n-1}$ as before. When $n = 26$ the entries are on diagonals $D_0, D_1, D_2, D_{24}, D_{25}$ and D_9. We pair up the rest of the diagonals as $\{D_{2i}, D_{2i+1}\}$ for $5 \leqslant i \leqslant 11$, and $\{D_3, D_4\}, \{D_5, D_6\}, \{D_7, D_8\}$ to get the required Hamilton cycles. When $m = 6$, $n = 30$; an $H(30; 3)$ is given in the Appendix. Two Hamilton cycles H and K are also given as a reference on the array. These Hamilton cycles together with $H(30; 3)$ only have entries on diagonals $D_0, D_1, D_2, D_3, D_{27}, D_{28}, D_{29}, D_{18}$ and D_{11}. For the rest of the diagonals, pair them up as $\{D_4, D_5\}, \{D_6, D_7\}, \{D_8, D_9\}, \{D_{12}, D_{13}\}$,

$\{D_{14}, D_{15}\}$, $\{D_{16}, D_{17}\}$ and $\{D_{2i-1}, D_{2i}\}$ for $10 \leqslant i \leqslant 13$ to get the necessary Hamilton cycles.

When $m \geqslant 7$, a $H(n; 3)$ exists by Lemma 5.1, then the proof follows as in Lemma 4.1.

□

6   Case D: $k \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 4$

In this section we construct a Heffter array $H(n; k)$ where $n = 4m + 1$ and $k = 4p + 1$, with $k < n$ and hence $m \geqslant 2$. The case $H(9; 5)$ is given in the Appendix and so henceforth we assume $m \geqslant 3$.

We begin with $k = 5$ construct an $n \times n$ array for which the sum of each row and column is $2nk + 1 = 40m + 11$. First we give an example $H(17; 5)$ of our construction. (Note a Hamilton cycle H has been included for the case $k > 5$.)

$$H(17; 5)$$

85	-27		11				50		H	H						52
68	84	-28		12			35		H	H						
	67	83	-29		13				H	H					37	
		66	82	-30		14		39		H	H					
			65	81	-31		15		41		H	H				
				64	80	-32				43			H		16	H
					63	79	-33		17			45			H	H
H	H					62	78	-34	18			47				
-20	H	H					61	60			21			49		
	H	H						77	59	-19		3				51
36			H	H					76	58	-22		23			
	38			H	H					75	57	-4		5		
		40		H	H		25				74	56	-24			
		42			H	H						73	55	-6		7
2			44			H	H						72	54	-1	
	9			46			H	53						71	H	-8
H		10			48	-26	H								70	69

Lemma 6.1. *For $n \equiv 1 \pmod 4$, $n \geqslant 13$ and $k = 5$ there exists a Heffter array $H(n; 5)$.*

Proof. Let $n = 4m + 1$ for $m \geqslant 3$. In every case here row and column sums will equal $40m + 11$.

We give the general construction of A below.

$$A[4m - 1, 4m] = -1, \qquad A[4m - 1, 1] = 2,$$
$$A[4m, 2] = 2m + 1, \qquad A[4m + 1, 3] = 2m + 2,$$
$$A[2m - 2, 4m] = 4m, \qquad A[2m - 1, 2m + 2] = 4m + 1,$$
$$A[2m, 2m + 3] = 4m + 2, \qquad A[2m + 2, 2m + 3] = -(4m + 3),$$
$$A[2m + 1, 1] = -(4m + 4), \qquad A[1, 2m] = 12m + 2,$$

$$A[4m+1, 2m+1] = -(6m+2), \quad A[4m-3, 2m+1] = 6m+1,$$
$$A[2m, 2m+2] = -(8m+2), \qquad A[2, 2m+1] = 8m+3,$$
$$A[3, 4m] = 8m+5, \qquad A[1, 4m+1] = 12m+4,$$
$$A[4m, 2m+2] = 12m+5, \qquad A[4m+1, 4m+1] = 16m+5.$$

$$
\begin{aligned}
A[i, i+3] &= 2m+i+2, & 1 \leqslant i \leqslant 2m-3, & \quad (6.1)\\
A[i+1, i] &= 16m+5-i, & 1 \leqslant i \leqslant 2m, & \quad (6.2)\\
A[i, i] &= 20m+6-i, & 1 \leqslant i \leqslant 2m, & \quad (6.3)\\
A[2m-i, 2m+1-i] &= -(8m+2-i), & 1 \leqslant i \leqslant 2m-1, & \quad (6.4)\\
A[2m+3-i, 4m+2-i] &= 12m+5-2i, & 1 \leqslant i \leqslant 2m-1, & \quad (6.5)\\
A[2m+i, 2m+i] &= 14m+5-i, & 1 \leqslant i \leqslant 2m-1, & \quad (6.6)\\
A[2m+1+i, 2m+i] &= 18m+6-i, & 1 \leqslant i \leqslant 2m, & \quad (6.7)\\
A[4m+2-i, 2m-i] &= 12m+2-2i, & 1 \leqslant i \leqslant 2m-1, & \quad (6.8)\\
A[2m+2i, 2m+2i+3] &= 2i+1, & 1 \leqslant i \leqslant m-1, & \quad (6.9)\\
A[2m+2i+2, 2m+2i+3] &= -(2i+2), & 1 \leqslant i \leqslant m-1, & \quad (6.10)\\
A[2m+2i-1, 2m+2i+2] &= 4m+2i+3, & 1 \leqslant i \leqslant m-2, & \quad (6.11)\\
A[2m+2i+1, 2m+2i+2] &= -(4m+2i+4), & 1 \leqslant i \leqslant m-2. & \quad (6.12)
\end{aligned}
$$

We note that the support of A contains:

- $3, 4, \ldots, 2m$ by (6.9) and (6.10),
- $2m+3, 2m+4, \ldots, 4m-1$ by (6.1),
- $4m+5, 4m+6, \ldots, 6m$ by (6.11) and (6.12),
- $6m+3, 6m+4, \ldots, 8m+1$ by (6.4),
- $8m+4, 8m+6, 8m+7, \ldots, 12m, 12m+1, 12m+3$ by (6.5) and (6.8),
- $12m+6, 12m+7, \ldots, 16m+4$ by (6.6) and (6.2),
- $16m+6, 16m+7, \ldots, 20m+5$ by (6.7) and (6.3).

It follows that the support of A is $\{1, 2, \ldots, 20m+5\}$ as required.

To verify the sum of each row and column is $40m+11$ we begin by noting that, respectively, (6.1), (6.2), (6.3), (6.4) and (6.5) give the sum for row r (where $4 \leqslant r \leqslant 2m-3$) and (6.1), (6.2), (6.3), (6.4) and (6.8) give the sum for column c (where $4 \leqslant c \leqslant 2m-1$) as:

$$(2m+r+2)+(16m+6-r)+(20m+6-r)-(6m+r+2)+(8m-1+2r) = 40m+11$$

and

$$(2m+c-1)+(16m+5-c)+(20m+6-c)-(6m+1+c)+(8m+2+2c) = 40m+11.$$

For row $2m+r$, where $3 \leqslant r \leqslant 2m-4$ (or $r = 2m-2$) and column $2m+c$, where $4 \leqslant c \leqslant 2m-1$, we argue as follows.

Respectively (6.6), (6.7) and (6.8) give a partial sum of $40m + 10$ for row r and (6.5), (6.6) and (6.7) give a partial sum of $40m + 12$ for column c:

$$(14m + 5 - r) + (18m + 7 - r) + (8m - 2 + 2r) = 40m + 10, \qquad (6.13)$$
$$(8m + 1 + 2c) + (14m + 5 - c) + (18m + 6 - c) = 40m + 12. \qquad (6.14)$$

Next (6.9) and (6.10) imply that if r is even, row $2m + r$ contain the entries $r + 1$ and $-r$ giving a partial sum of 1. If r is odd, (6.11) and (6.12) imply that row $2m + r$ contains the entries $4m + r + 4$ and $-(4m + r + 3)$ also giving a partial sum of 1. Adding this to the partial sum in (6.13) we get an overall row sum of $40m + 11$. Then (6.9) and (6.10) imply that if c is odd, column $2m + c$ contains the entries $c - 2$ and $-(c - 1)$ giving a partial sum of -1. If c is even, (6.11) and (6.12) imply that column $2m + c$ contains the entries $4m + c + 1$ and $-(4m + c + 2)$ also giving a partial sum of -1. Adding this to the partial sum in (6.14) we get a column sum of $40m + 11$.

The remaining rows and columns can be checked individually to complete the proof that all rows and columns sum to $40m + 11$. Thus we have the required $H(4m + 1; 5)$. $\square$

Next, with care, we add up to $2(m - 2)$ Hamilton cycles to obtain an $H(4m + 1; 4p + 5)$ where $p \leqslant m - 2$.

Theorem 6.2. *For* $n \equiv 1 \pmod 4$ *and* $k \equiv 1 \pmod 4$ *there exists a Heffter array* $H(n; k)$*, where* $k < n$.

Proof. Let $n = 4m + 1$. The Heffter array $H(9; 5)$ is given in the Appendix. Otherwise, an $H(n; 5)$ exists by Lemma 6.1. This was labeled A in the proof of that lemma. Observe that the occupied cells of A are a subset of the union of diagonals

$$\mathcal{D} := D_{n-3} \cup D_{n-2} \cup D_{n-1} \cup D_0 \cup D_1 \cup D_4 \cup D_{2m-4}$$
$$\cup D_{2m-2} \cup D_{2m-1} \cup D_{2m} \cup D_{2m+2}.$$

For each $m \geqslant 3$, the following set of cells is a subset of $\mathcal{D}$, that does *not* intersect A and forms a Hamilton cycle H:

$$\{(i, 2m + 1 + i), (i, 2m + 2 + i) \mid 1 \leqslant i \leqslant 2m - 3\}$$
$$\cup \{(2m - 2, 4m - 1), (2m - 2, 4m + 1)\}$$
$$\cup \{(2m - 1, 4m + 1), (2m - 1, 4m), (4m, 4m), (4m, 2m + 1)\}$$
$$\cup \{(i, i - 2m + 1), (i, i - 2m + 2) \mid 2m \leqslant i \leqslant 4m - 1\}$$
$$\cup \{(4m + 1, 1), (4m + 1, 2m + 2)\}.$$

(See the array $H(17; 5)$ above for an example.) Thus there exists $4m + 1 - 11 = 4(m - 2) - 2$ diagonals that do not intersect $A \cup H$ and so it is possible to construct $2m - 5$ disjoint Hamilton cycles by pairing empty diagonals that are either distance 1 or 2 apart.

For $m \geqslant 6$ a possible pairing of diagonals is:

$$\{D_2, D_3\};$$
$$\{D_{2m-5}, D_{2m-3}\};$$
$$\{D_{5+2i}, D_{6+2i}\}, \ 0 \leqslant i \leqslant m - 6;$$
$$\{D_{2m+1}, D_{2m+3}\};$$
$$\{D_{2m+4+2i}, D_{2m+5+2i}\}, \ 0 \leqslant i \leqslant m - 4.$$

When $m = 4$ we pair the diagonals as $\{D_2, D_3\}; \{D_9, D_{11}\}; \{D_{12}, D_{13}\}$ and when $m = 5$ we pair the diagonals as $\{D_2, D_3\}; \{D_5, D_7\}; \{D_{11}, D_{13}\}; \{D_{14}, D_{15}\};$ $\{D_{16}, D_{17}\}$.

Together with H this gives a total of $2m - 4$ Hamilton cycles. Thus applying Theorem 2.2 recursively, we can form a Heffter array for each k such that $k \equiv 1 \pmod 4$ and $k \leqslant n - 4$. □

7 Case E: $k \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$

In this section we construct a Heffter array $H(n; k)$ where $n = 4m + 2$ and $k = 4p + 1$, where $k < n$.

We demonstrate the following construction in the case $m = 4$ with an example of an $H(18; 5)$; the cycles H and K will be needed later for the case $k > 5$.

$H(18;5)$

2					51	20	−38	−35									
−40	4				53	19	−36										
−42	−28	6				37	27										
39	−44	−29	8				26										
41	25	−46	−30	10													
	43	24	−48	−31	12												
		45	23	−50	−32	14											
			47	22	−52	−33	16										
				49	21	−54	−34	18									
									1	81	K	H	69	−60	K	90	H
									H	3	80	K	H	70	−61	K	89
									88	H	5	79	K	H	71	−62	K
									72	87	H	7	78	K	H	K	−63
									−55	K	86	H	9	77	K	H	64
									K	−56	K	85	H	11	76	65	H
									75	66	−57	K	84	H	13	H	K
									K	H	67	−58	K	83	H	15	74
									H	K	H	68	−59	K	82	73	17

Lemma 7.1. *For $n \equiv 2 \pmod 4$, $n \geqslant 6$ there exists a Heffter array $H(n; 5)$.*

Proof. Let $n = 4m + 2$. The case $H(n; k) = H(6; 5)$ is given in the Appendix and so henceforth we assume $m \geqslant 2$. Our Heffter array will be the concatenation of an array A_0 in the first $2m + 1$ rows and columns and an array A_1 in the last $2m + 1$ rows and columns. The row and column sums of A_0 will be 0 and the row and column sums of A_1 will be $2nk + 1$.

In our definition of A_0, rows and columns are calculated modulo $2m + 1$ rather than n. We begin by defining a $(2m + 1) \times (2m + 1)$ array A_0' which has support

$$\{2, 4, 5, \ldots, 4m + 2\} \cup \{4m + 3, 4m + 4, \ldots, 12m + 6\}$$

and for which each row sums to 0 and each column sums to 0 except for columns 1 and $2m + 1$, which sum to $-(2m + 1)$ and $2m + 1$, respectively. Then we swap the entry $-(8m + 4)$ in cell $(2, 1)$ with the entry $-(8m + 8)$ in cell $(2, 2m + 1)$ and swap the entry $6m + 2$ in cell $(4, 1)$ with the entry $8m + 7$ in cell $(4, 2m + 1)$. The result will be an array A_0 defined on row and column set $\{1, 2, \ldots, 2m + 1\}$ with row and column sums equal to 0.

To this end, for $1 \leqslant i \leqslant 2m + 1$ let

$$A_0'[i, i] = 2i; \qquad A_0'[3 - i, 2m + 1 - i] = 4m + 2 + i;$$
$$A_0'[2 + i, 1 + i] = -(6m + 3 + i); \qquad A_0'[2 + i, i - 2] = 8m + 3 + 2i;$$
$$A_0'[i, i - 2] = -(8m + 4 + 2i).$$

Now let

$$A_0[r, c] = \begin{cases} -(8m + 8), & (r, c) = (2, 1), \\ -(8m + 4), & (r, c) = (2, 2m + 1), \\ 8m + 7, & (r, c) = (4, 1), \\ 6m + 2, & (r, c) = (4, 2m + 1), \\ A_0'[r, c], & \text{otherwise.} \end{cases}$$

The case $m = 4$ is illustrated in the example $H(18; 5)$ given above. It will be useful to note that the non-empty cells of this $(2m + 1) \times (2m + 1)$ array A_0 are a subset of the union of diagonals

$$\mathcal{D}_0 := D_0 \cup D_1 \cup D_2 \cup D_3 \cup D_4.$$

Next, we define a $(2m + 1) \times (2m + 1)$ array A_1, on row and column set $\{2m + 2, 2m + 3, \ldots, 4m + 2\}$ with support $\{1, 3, 4, \ldots, 4m + 1\} \cup \{12m + 7, 12m + 8, \ldots, 16m + 8\} \cup \{kn - 4m - 1, kn - 4m, \ldots, kn\}$. For the case $k = 5$, observe that $\{kn - 4m - 1, kn - 4m, \ldots, kn\} = \{16m + 9, 16m + 10, \ldots, 20m + 10\}$. Thus when $k = 5$ the support of $A_0 \cup A_1$ is $\{1, 2, \ldots, 20m + 10\}$ as required. Each row and column of A_1 will sum to $2nk + 1$. We first give A_1 for the cases $m = 2$ and $m = 3$ separately; then we present the general formula.

$$A_1(m = 2, n = 10)$$

1	$10k - 4$	-31	$10k - 2$	37
39	3	$10k$	-34	$10k - 7$
-33	40	5	$10k - 8$	$10k - 3$
$10k - 1$	$10k - 6$	36	7	-35
$10k - 5$	-32	$10k - 9$	38	9

$$A_1 \ (m = 3, n = 14)$$

1	$14k$			-43	$14k - 7$	50
$14k - 1$	3	51	$14k - 8$		-44	
	-45	5	$14k - 2$	55		$14k - 12$
		$14k - 3$	7	$14k - 11$	54	-46
56		-47		9	$14k - 13$	$14k - 4$
$14k - 6$	53		-48	$14k - 9$	11	
-49	$14k - 10$	$14k - 5$	52			13

Otherwise $m \geqslant 4$ and we define A_1 as follows. The rows and columns are defined modulo $2m + 1$ rather than modulo n. To construct the overall Heffter array, the array A_1 is then shifted by adding $2m + 1$ (as an integer) to each row and column.

$$A_1[4, 1] = 14m + 8; \qquad A_1[5, 2m + 1] = 14m + 9; \qquad (7.1)$$
$$A_1[6, 2m] = 16m + 8; \qquad A_1[2m - 1, 1] = kn - 4m + 1; \qquad (7.2)$$
$$A_1[2m, 2m + 1] = kn - 4m; \qquad A_1[2m + 1, 2m] = kn - 4m - 1; \qquad (7.3)$$
$$A_1[i, i] = 2i - 1, \qquad 1 \leqslant i \leqslant 2m + 1; \qquad (7.4)$$
$$A_1[i, 2m - 1 + i] = kn + 1 - i, \qquad 1 \leqslant i \leqslant 2m + 1; \qquad (7.5)$$
$$A_1[i, i + 1] = kn - 2m - i, \qquad 1 \leqslant i \leqslant 2m - 2; \qquad (7.6)$$
$$A_1[4 + i, i] = -(12m + 6 + i), \qquad 1 \leqslant i \leqslant 2m + 1; \qquad (7.7)$$
$$A_1[6 + i, 1 + i] = 14m + 9 + i, \qquad 1 \leqslant i \leqslant 2m - 2. \qquad (7.8)$$

We note that the support for A_1 is the union of the sets:

- $\{1, 3, 5, 6, \ldots, 4m + 1\}$ (by (7.4)),
- $\{12m + 7, 12m + 8, \ldots, 14m + 7\}$ (by (7.7)),
- $\{14m + 8, 14m + 9\}$ (by (7.1)),
- $\{14m + 10, \ldots, 16m + 7\}$ (by (7.8)),
- $\{16m + 8\}$ (by (7.2)),
- $\{kn - 4m - 1, kn - 4m, kn - 4m + 1\}$ (by (7.2), (7.3)),
- $\{kn - 4m + 2, \ldots, kn - 2m - 1\}$ (by (7.6)) and
- $\{kn - 2m, \ldots, kn\}$ (by (7.5)).

We next check the row and column sums. For row r in the range 7 to $2m + 1$, (7.5) and (7.6) give a partial sum of

$$(kn + 1 - r) + (kn - 2m - r) = 2kn - 2m + 1 - 2r,$$

while (7.7) and (7.8) give a partial sum of

$$(-12m - 6 - (r - 4)) + (14m + 9 + (r - 6)) = 2m + 1.$$

Now combined with (7.4) the sum of these rows is

$$(2nk - 2m + 1 - 2r) + (2m + 1) + (2r - 1) = 2nk + 1,$$

as required.

For column c in the range 2 to $2m - 1$, (7.5) and (7.6) give a partial sum of

$$(kn + 1 - (c + 2)) + (kn - 2m - (c - 1)) = 2kn - 2m - 2c,$$

while (7.7) and (7.8) give a partial sum of

$$(-12m - 6 - c) + (14m + 9 + (c - 1)) = 2m + 2.$$

Now combined with (7.4) the sum of these columns is

$$(2nk - 2m - 2c) + (2m + 2) + (2c - 1) = 2nk + 1.$$

The sum of the remaining rows and columns can be calculated individually and overall the rows and columns of A_1 sum to $2nk + 1$ as required. Thus the concatenation of A_0 with A_1 gives an $H(4m + 2; 5)$. □

Theorem 7.2. *For $n \equiv 2 \pmod 4$, $n \geqslant 6$ and $k \equiv 1 \pmod 4$ there exists a Heffter array $H(n; k)$, where $k < n$.*

Proof. Let $n = 4m + 2$ and $k = 4p + 1$. A $H(n; 5)$ exists by Lemma 7.1. Otherwise, $k \geqslant 9$ and $m \geqslant 2$. We take the array $A = A_0 \cup A_1$ from the previous lemma.

We will construct $m - 2$ cycles of length n (that is, on $2(2m + 1) = n$ cells) in the upper left-hand (A_0) and lower right-hand (A_1) quadrants, and m further cycles of length n in each of the remaining quadrants. Together these form $2m - 2$ disjoint 2-factors.

From the proof of Lemma 7.1, within A_0 there are $2m + 1 - 5 = 2(m - 2)$ empty diagonals, which we take in pairs to obtain $m - 2$ cycles of length n. Next take the intersection of the last $2m + 1$ rows and columns, this is the $(2m + 1) \times (2m + 1)$ subarray that contains A_1. We will refer to diagonals within that subarray only, recalling that the rows and columns are calculated modulo $2m + 1$. We aim to find $m - 2$ cycles of length n from this subarray. The case $m = 2$ is trivial and for the case $m = 3$, observe that the empty cells of A_1 in the previous lemma form a cycle of length 14. Otherwise $m \geqslant 4$ and the array A_1 occupies diagonals $D_0, D_1, D_2, D_3, D_4, D_5, D_7, D_{2m-2}$ and D_{2m}.

Next take the following cells

$$\begin{aligned}
H = \big(&\{(i + 1, i), (2m - 2 + i, i) \mid 1 \leqslant i \leqslant 2m + 1\} \setminus \\
&\{(2m - 1, 1), (2m + 1, 2m)\} \big) \cup \{(2m - 1, 2m), (2m + 1, 1)\} \\
K = \big(&\{(3 + i, i), (7 + i, i) \mid 1 \leqslant i \leqslant 2m + 1\} \setminus \\
&\{(4, 1), (6, 2m)\} \big) \cup \{(4, 2m), (6, 1)\}.
\end{aligned}$$

In the example $H(18; 5)$ above these cycles are shown in cells marked by H and K, respectively. Observe that H and K form two cycles of length $4m + 2$ disjoint from A_1 but are a subset of

$$D_1 \cup D_{2m-2} \cup D_{2m} \cup D_3 \cup D_7 \cup D_5.$$

Thus there exists $2m + 1 - 9 = 2(m - 4)$ diagonals that do not intersect $A_1 \cup H \cup K$. For $m \geqslant 6$, we can thus form $m - 4$ cycles of length n by taking pairs of diagonals:

$$\begin{aligned}
&\{D_6, D_8\}; \\
&\{D_{2m-3}, D_{2m-1}\}; \\
&\{D_{9+2i}, D_{10+2i}\}, \ 0 \leqslant i \leqslant m - 7,
\end{aligned}$$

and when $m = 5$, $2m + 1 = 11$ so we get one cycle of length n by taking the diagonals D_6 and D_9. Thus with H and K we have $m - 2$ cycles of length n that are disjoint from A_1 and each other; together these form $m - 2$ 2-factors, each consisting of two cycles of length n.

We can create a further m cycles of length n in each of the remaining quadrants, as these cells are all empty. Altogether we have $2m - 2$ disjoint 2-factors. Thus by Lemma 2.4,

we can fill $4(p - 1)$ cells in each row and column with support $\{16m + 9, 16m + 10, \ldots,$ $kn - 4m - 2\}$ without changing the row and column sums, where $k = 4p + 1$. Thus there exists an $H(4m + 2; 4p + 1)$ Heffter array for each $m \geqslant 2$ and $p \leqslant m$.   □

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Appendix

Case B

$H(7;3)$

15	−13	−2				
−11	14		−3			
−4		−8	12			
	−1	10	−9			
				5	21	17
				18	6	19
				20	16	7

$B(11)$

−1	18	−17					
24	−2		−22				
−23		−3	26				
	−16	20	−4				
				19	−8	−11	
				−9	21		−12
				−10		25	−15
					−13	−14	27

$B(15)$

1	−36	35									
−34	−3		37								
33			−22	−11							
	39	−21			−18						
		−14			−12	26					
			−15	−17			32				
				28			−19	−9			
					30	−10			−20		
						−16			24	−8	
							−13	38			−25
								−29		31	−2
									−4	−23	27

Case C

$H(6;3)$

-1	-16				17
-11		-4			15
12		-9	-3		
	-2		10	-8	
		13	-7	-6	
	18			14	5

$B(10)$

1	22	-23				
17	2		-19			
-18			15	3		
	-24	14			10	
		9			11	-20
			4	-16		12
				13	-21	8

$B(14)$

-34	-1	35							
-2	24		-22						
36		-32	-4						
	-23	-3	26						
			-20	28	-8				
			30	-9	-21				
			-10	-19	29				
						-11	27	-16	
						25	-12		-13
						-14		-17	31
							-15	33	-18

$B(18)$

1	21	-22												
-36	-4		40											
35			-12	-23										
	-17	33			-16									
		-11			42	-31								
			-28	41			-13							
				-18			43	-25						
					-26	45			-19					
						-14			34	-20				
							-30	-2			32			
								27		-24	-3			
									-15	44	-29			
												-8	46	-38
												-39	-9	48
												47	-37	-10

$B(22)$

1	−36	35																	
−30	−4		34																
29		−3	−26																
40	−2		−38																
	−33		54	−21															
		−31	48		−17														
			−22		42	−20													
				−16	53		−37												
				−32		−15	47												
					−25	39		−14											
						−19		−27	46										
							52	−24	−28										
							−23	41	−18										
										−11	55	−44							
										−45	−12	57							
										56	−43	−13							
													−8	58	−50				
													−51	−9	60				
													59	−49	−10				

B(26)

−64	1	63	−49																		
	65	−16	−49	−50																	
−3		53	−4	70																	
		−66		−22	69																
	−47		68	−48	−20																
			−18	59	−41																
				−52	−10	62															
				−17	51	−34															
67				−39	−28																
					−29	61	−32														
					−42	−30	72														
					71	−31	−40														
							−2	60	−58												
							23	15	−38												
							−21	−33	54												
							−27	−8	35												
									43	14	−57										
									24	13	−37										
									−46	−9											
										−26	44	12	55	−19	45						
											25	−36	11	−56							

$H(30, 3)$

1	61	K																									K	H	H
−62	H	−3	80																									H	K
−77	K	−12	H	−71																									K
83	−21	H	H	H	81																								
K	−49	−60	75	H	−26																								
H	H	K	K	H	−59																								
	63	K	−4	K	79	H																							
	H	−22	K	−57	−20	78																							
		−58	H	K	74	−50	H																						
			K	−24	K	−34	82	K																					
				H	−48	H	−35	73	K																				
76					H	K	−38	K	−36	H																			
					H	K	−40	K	−47	−37	H																		
							K	84	K	−72	70	K																	
								K	2	−19	46	H																	
								H	−27	39	−64	K																	
								K	25	−18	69	−17	H																
									K	−51	8	9	45	K															
									K	33	56	−52	−16	−65	68	K													
										K	−28	K	−29	10	−42	44	K												
											K	32	H	−15	−66	67	K												
												K	55	11	−14	23	−54	H											
													K	−53	31	43	−13	H	H										
														K	K	H	H	K	K										
K															5	kn − 3	kn − 5	H	kn − 4										
H	K														kn	6	kn − 1	H	kn − 2										
H	H	K													kn − 3	kn − 5	K	K	7										

A_1

1	10m − 4	−(10m − 3)									
6m − 5	2	−(6m − 3)									
−(6m − 4)		−(6m + 7)	12m + 3								
	−(10m − 2)	10m + 4		−6							
	−7			10m + 1	−(10m − 6)						
		12m + 4	−(6m + 5)		−(6m − 1)						
			−(6m − 2)		−(6m + 3)	12m + 1					
				−(10m − 5)	−5		10m				
				10m − 1			−(10m + 3)	4			
					12m + 2	−6m			−(6m + 2)		
						−(6m + 1)			12m + 5	−(6m + 4)	
							3	10m + 2	12m + 6		−(10m + 5)
							−(10m + 6)			12m + 6	−2m

Cases D and E

$H(9;5)$

45	36	20					−18	8
−16	24	43	34					6
	44	35	22	7		−17		
		5	42	−15	33		26	
9			−10	32	41	19		
	1			40	−2	21	31	
		−12			23	30	39	11
25			3		−4	38		29
28	−14			27			13	37

$H(6;5)$

1	2	3		−25	19
5	6	16	4		30
23	7	9	8	14	
11		15	12	10	13
	24	18	17	29	−27
21	22		20	−28	26

$H(13;9)$

65	−21	H	9		38		K	K			H	40
52	64	−22	H	10		27		K	K			H
H	51	63	−23	H	11				K	K	29	
		50	62	−24	H	H		31		K	12	K
	H	H	49	61	−25		13		33		K	K
K	K		H	48	60	H	−26	14		35		
−16	K	K			47	46	H	H	17		37	
		K	K		H	59	45	−15	H	3		39
28	H		K	K		19	58	44	−18	H		
	30			K	K		H	57	43	−4	H	5
2		32		H	K	K			56	42	−1	H
H	7		34			K	41	H		55	K	−6
K		8		36		−20	K		H	H	54	53