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Nambu–Jona-Lasinio model from QCD

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The NJL model [1–3] proves to be effective in description of low-energy hadron physics. The model starts with effective chiral invariant Lagrangian

$$\frac{G_1}{2}\left(\bar\psi\tau^b\gamma_5\psi\,\bar\psi\tau^b\gamma_5\psi\,-\,\bar\psi\,\psi\,\bar\psi\,\psi\right), \qquad (1)$$

where ψ is the light quark doublet $(u,\ d)$. This interaction is non-renormalizable, so one is forced to introduce an ultraviolet cut-off Λ. Thus we have at least two arbitrary parameters

$$G_1\,; \quad \Lambda_1\,;$$

to be adjusted by comparison with real physics. It comes out that after such adjustment (and similar procedure for the vector sector and for the s-quark terms) we obtain satisfactory description of light mesons and their low-energy interactions.

However, the problem how to calculate the parameters G_i and Λ_i from the fundamental QCD was not solved for a long time. The main problem here is to find a method to obtain effective interactions from fundamental gauge interactions, *e.g.* QCD.

There are also non-local variants of the NJL model, in which one introduces a form-factor $F(q_i)$ into the effective interaction of the type (1) instead of a cut-off Λ. In this case again there was no regular method to obtain this function F and one has to make an arbitrary assumption for the choice.

Our goal is to formulate a regular approach, which allows to obtain a unique solution for the form-factors and other necessary quantities of the effective interactions. In particular we apply this approach to the NJL effective interaction.

The approach is based on the Bogoliubov compensation principle [4,5].

The main principle of the approach is to check if an effective interaction could be generated in a chosen variant of a renormalizable theory.

In previous works [6–12] the Bogoliubov compensation principle was applied to studies of spontaneous generation of effective non-local interactions in renormalizable gauge theories. In view of this one performs an “add and subtract” procedure for the effective interaction with a form-factor. Then one assumes the presence of the effective interaction in the interaction Lagrangian and the same term with the opposite sign is assigned to the newly defined free Lagrangian.

The QCD Lagrangian with two light quarks is (u and d)

$$L = \sum_{k=1}^{2}\left(\frac{\imath}{2}\Big(\bar\psi_k\gamma_\mu\partial_\mu\psi_k - \partial_\mu\bar\psi_k\gamma_\mu\psi_k\Big) - m_0\bar\psi_k\psi_k + g_s\bar\psi_k\gamma_\mu t^a A^a_\mu\psi_k\right) - \frac{1}{4}\Big(F^a_{\mu\nu}F^a_{\mu\nu}\Big)\,. \tag{2}$$

Let us assume that a non-local NJL interaction is spontaneously generated in this theory. We use the Bogoliubov "add and subtract" procedure to check the assumption. We have

$$\begin{aligned} L &= L_0 \;+\; L_{int}\,,\\ L_0 &= \frac{\imath}{2}\Big(\bar\psi\gamma_\mu\partial_\mu\psi - \partial_\mu\bar\psi\gamma_\mu\psi\Big) - m_0\bar\psi\,\psi + \frac{G_1}{2}\Big(\bar\psi\tau^b\gamma_5\psi\,\bar\psi\tau^b\gamma_5\psi \;-\; \bar\psi\,\psi\,\bar\psi\,\psi\Big)\\ &+ \frac{G_2}{2}\Big(\bar\psi\tau^b\gamma_\mu\psi\,\bar\psi\tau^b\gamma_\mu\psi + \bar\psi\tau^b\gamma_5\gamma_\mu\psi\bar\psi\tau^b\gamma_5\gamma_\mu\psi\Big) - \frac{1}{4}\,F^a_{0\,\mu\nu}F^a_{0\,\mu\nu}\,, \end{aligned} \tag{3}$$

$$\begin{aligned} L_{int} &= g_s\,\bar\psi\gamma_\mu t^a A^a_\mu\psi \;-\; \frac{G_1}{2}\Big(\bar\psi\tau^b\gamma_5\psi\,\bar\psi\tau^b\gamma_5\psi - \bar\psi\,\psi\,\bar\psi\,\psi\Big)\\ &- \frac{G_2}{2}\Big(\bar\psi\tau^b\gamma_\mu\psi\,\bar\psi\tau^b\gamma_\mu\psi + \bar\psi\tau^b\gamma_5\gamma_\mu\psi\bar\psi\tau^b\gamma_5\gamma_\mu\psi\Big)\\ &- \frac{1}{4}\left(F^a_{\mu\nu}F^a_{\mu\nu} - F^a_{0\,\mu\nu}F^a_{0\,\mu\nu}\right)\,. \end{aligned} \tag{4}$$

Here the notation *e.g.* $\frac{G_1}{2}\bar\psi\,\psi\,\bar\psi\,\psi$ means the corresponding non-local vertex in the momentum space

$$\imath(2\pi)^4 G_1\,\bar u^a(p)\,u_a(q)\,\bar u^b(k)\,u_b(t)\,F(p,q,k,t)\,\delta(p+q+k+t)\,, \tag{5}$$

where $F(p,q,k,t)$ is a form-factor, $p,\ q,\ k,\ t$ are respectively incoming momenta and a, b are isotopic indices of corresponding quarks.

Let us consider expression (3) as the new **free** Lagrangian L_0, whereas expression (4) is the new **interaction** Lagrangian L_{int}. The compensation equation demands fully connected four-fermion vertices, following from Lagrangian L_0, to be zero. The equation has evidently

1. a perturbative trivial solution $G_i = 0$;
2. but it might also have a non-perturbative non-trivial solution, which we shall look for.

In the first approximation we use the following assumptions.

1. Loop numbers 0, 1, 2. For one-loop case only a trivial solution exists.
2. Procedure of linearizing over form-factor, which leads to linear integral equations.
3. Intermediate UV cut-off Λ, results not depending on the value of this cut-off.
4. IR cut-off at the lower limit of integration by momentum squared q^2 at value m^2.

5. Only the first two terms of the $1/N$ expansion ($N = 3$).
6. We look for a solution with the following simple dependence on all four variables:

$$F(p_1, p_2, p_3, p_4) = F\left(\frac{p_1^2 + p_2^2 + p_3^2 + p_4^2}{2}\right) . \tag{6}$$

Then we come to the following integral equation (see [8])

$$\begin{aligned}
F_1(x) = A + \frac{3\,G_2}{8\pi^2}\left(2\Lambda^2 + x\log\frac{x}{\Lambda^2} - \frac{3}{2}x - \frac{\mu^2}{2x}\right) - \frac{(G_1^2 + 6G_1G_2)N}{32\,\pi^4} \times \\
\Bigg(\frac{1}{6x}\int_\mu^x (y^2 - 3\mu^2)F_1(y)\,dy + \frac{3}{2}\int_\mu^x yF_1(y)\,dy + \frac{x^2 - 3\mu^2}{6}\int_x^\infty \frac{F_1(y)}{y}dy + \\
\log x\int_\mu^x yF_1(y)\,dy + x\log x\int_\mu^x F_1(y)\,dy + \int_x^\infty y\log y\,F_1(y)\,dy + \\
x\int_x^\infty \left(\log y + \frac{3}{2}\right)F_1(y)dy + \left(2\Lambda^2 - \frac{3}{2}x\right)\int_\mu^\infty F_1(y)\,dy - \frac{3}{2}\int_\mu^\infty yF_1(y)\,dy - \\
\log\Lambda^2\left(\int_\mu^\infty yF_1(y)\,dy + x\int_\mu^\infty F_1(y)\,dy\right)\Bigg); \; \mu = m_0^2;\; x = p^2;\; y = q^2; \\
A = \frac{G_1N\Lambda^2}{2\pi^2}\left(1 + \frac{1}{4N} - \frac{G_1N}{2\pi^2}\left(1 + \frac{1}{2N}\right)\int_\mu^\infty F_1(y)\,dy\right) .
\end{aligned} \tag{7}$$

The equation has the following solution decreasing at infinity

$$\begin{aligned}
F_1(z) = C_1\,G_{06}^{40}\left(z|1, \frac{1}{2}, \frac{1}{2}, 0, a, b\right) + C_2\,G_{06}^{40}\left(z|1, \frac{1}{2}, b, a, \frac{1}{2}, 0, \right) \\
+ C_3\,G_{06}^{40}\left(z|1, 0, b, a, \frac{1}{2}, \frac{1}{2}\right) , \\
a = -\frac{1 - \sqrt{1 - 64u_0}}{4} , \qquad b = -\frac{1 + \sqrt{1 - 64u_0}}{4} ,
\end{aligned} \tag{8}$$

where $x = p^2$, $y = q^2$ are respectively external momentum squared and inegration momentum squared,

$$G_{pq}^{mn}\left(z\,|^{a_1, \ldots, a_p}_{b_1, \ldots, b_q}\right)$$

is a Meijer G-function [13],

$$\beta = \frac{(G_1^2 + 6G_1G_2)N}{16\pi^4} , \quad z = \frac{\beta x^2}{2^6} , \quad u_0 = \frac{\beta\mu^2}{64} , \quad F_1(u_0) = 1 .$$

The constants C_i are defined by the boundary conditions

$$\frac{3\,G_2}{8\pi^2} - \frac{\beta}{2}\int_{m_0^2}^\infty F_1(y)\,dy = 0 , \quad \int_{m_0^2}^\infty y\,F_1(y)\,dy = 0 , \quad \int_{m_0^2}^\infty y^2\,F_1(y)\,dy = 0 . \tag{9}$$

These conditions and the condition $A = 0$ lead to the cancellation of all terms in equation (7) being proportional to Λ^2 and $\log\,\Lambda^2$. So we have the unique solution. The values of the parameter u_0 and the ratio of two constants G_i are also fixed

$$u_0 = 1.92 \cdot 10^{-8} \simeq 2 \cdot 10^{-8} , \quad G_1 = \frac{6}{13}\,G_2 . \tag{10}$$

We would draw attention to a natural appearance of a small quantity u_0. So G_1 and G_2 are both defined in terms of m_0.

Thus we have the unique non-trivial solution of the compensation equation, which contains no additional parameters. It is important that the solution exists only for positive G_2 and due to (10) for positive G_1 as well.

Now we have the non-trivial solution, which lead to the following effective Lagrangian

$$\begin{aligned} L = & \frac{\imath}{2}\Big(\bar\psi\gamma_\mu\partial_\mu\psi - \partial_\mu\bar\psi\gamma_\mu\psi\Big) - \frac{1}{4}\, F^a_{\mu\nu}F^a_{\mu\nu} - m_0\bar\psi\,\psi \\ & + g_s\,\bar\psi\gamma_\mu t^a A^a_\mu\psi - \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} \\ & - \frac{G_1}{2}\Big(\bar\psi\tau^b\gamma_5\psi\bar\psi\tau^b\gamma_5\psi - \bar\psi\psi\bar\psi\psi\Big) \\ & - \frac{G_2}{2}\Big(\bar\psi\tau^b\gamma_\mu\psi\bar\psi\tau^b\gamma_\mu\psi + \bar\psi\tau^b\gamma_5\gamma_\mu\psi\bar\psi\tau^b\gamma_5\gamma_\mu\psi\Big)\,. \end{aligned} \tag{11}$$

Here $g_s^2/4\,\pi = \alpha_s(q^2)$ is the running constant depending on the momentum variable. We need this constant in the low-momenta region. We assume that in this region $\alpha_s(q^2)$ may be approximated by its average value α_s. The possible range of values of α_s is from 0.40 up to 0.75.

Thus we come to the effective non-local NJL interaction which we use to obtain the description of low-energy hadron physics [7,8,11]. In this way we obtain expressions for all quantities under study.

Analysis shows that the optimal set of low-energy parameters corresponds to $\alpha_s = 0.67$ and $m_0 = 20.3\,\text{MeV}$. We present a set of calculated parameters for these conditions including the quark condensate,the parameters of the σ-meson as well as the parameters of ρ and a_1-mesons:

$$\begin{aligned} \alpha_s &= 0.673\,; \qquad m_0 \;=\; 20.3\,MeV\,; \\ m_\pi &= 135\,MeV\,; \quad m_\sigma \;=\; 492\,MeV\,; \quad \Gamma_\sigma \;=\; 574\,MeV \\ f_\pi &= 93\,MeV\,; \quad m \;=\; 295\,MeV\,; \quad <\bar q\, q>= -\,(222\,MeV)^3\,; \\ G_1 &= \frac{1}{(244\,MeV)^2}\,; \qquad g \;=\; 3.16\,. \\ M_\rho &= 926.3\,MeV(771.1 \pm 0.9); \quad \Gamma_\rho \;=\; 159.5\,MeV(149.2 \pm 0.7); \\ M_{a_1} &= 1174.8\,MeV(1230 \pm 40); \quad \Gamma_{a_1} \;=\; 350\,MeV(250 - 600); \\ \Gamma(a_1 \to \sigma\,\pi)/\Gamma_{a_1} &= 0.23\,(0.188 \pm 0.043). \end{aligned}$$

The upper line here is our input, while all other quantities are calculated from these two fundamental parameters. The overall accuracy may be estimated to be on the order of 10 – 15%. The worst accuracy occurs in the value of M_ρ (20%). It seems that the vectors and the axials need further study.

Important result: average value of $\alpha_s \simeq 0.67$ **agrees with calculated low-energy** α_s **[9]. So we have consistent description of low-energy hadron physics with only one dimensional parameter, e.g.** m_0 **or** f_π**.**

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