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Modeling the QCD Vacuum

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Abstract. A central issue in quantum field theory and in particular QCD is to find the physical vacuum state. Point form quantum field theory provides a useful setting in which to model the physical vacuum state. In this note the defining equations and elementary properties of the physical vacuum are discussed in the context of the point form. A simple model is presented which illustrates some of the general ideas.

In point form relativistic quantum mechanics [1] all interactions are in the four-momentum operator P^μ and Lorentz transformations are kinematic. The equations that express the relativistic content of a point form theory are

$$[P_\mu, P_\nu] = 0 \tag{1}$$

$$U_\Lambda P_\mu U_\Lambda^{-1} = (\Lambda^{-1})^\nu_\mu P_\nu, \tag{2}$$

where U_Λ is the unitary operator representing the Lorentz transformation Λ on some model Hilbert or generalized Fock space.

Given a four-momentum operator, the goal is to solve the eigenvalue problem

$$P^\mu|\Psi_p> = p^\mu|\Psi_p> \tag{3}$$

and from this get the physical vacuum, bound and scattering states.

There are several ways of generating four-momentum operators P^μ that satisfy the point form equations. One, called the Bakamjian-Thomas method [2], is relevant for finite degree of freedom systems. The other, integrating free fields over the forward hyperboloid [3], is of primary interest in this paper.

If a four-momentum operator is constructed that satisfies the above point form equations, solving the vacuum problem means finding a vector $|\Omega>$ in a suitable space such that

$$P^\mu|\Omega> = 0 \tag{4}$$

$$U_\Lambda|\Omega> = |\Omega>. \tag{5}$$

Here it should be noted that, unlike the situation in nonrelativistic quantum mechanics, where eigenvalues of a Hamiltonian are only defined up to a constant, and only energy differences are observable, it is not possible to add constants to the four-momentum operator and still maintain Lorentz covariance; that is, if

$P_\mu \rightarrow P'_\mu = P_\mu + c_\mu I$, where the c's are constants and I is the identity operator, then P'_μ will not satisfy Eq.(2).

Just as a Hamiltonian can be written as the sum of free and interacting Hamiltonians, so also the four-momentum operator can be written as the sum of free and interacting four-momentum operators. If there are no interactions and $P^\mu = P^\mu_{fr}$, then the well known solution to the vacuum problem is the Fock vacuum. If an interaction is added, so that $P^\mu = P^\mu_{fr} + \alpha P^\mu_I$, then the vacuum solution, Eq.(4) must reduce to the Fock vacuum when the bare coupling constant $\alpha = 0$. Conversely, since it is not possible to add constants to the four-momentum operator, the solution to the vacuum problem, Eq.(4) entails the possibility of fine tuning the coupling constant α. A simple model of such a possibility is given in the following paragraphs.

To investigate the vacuum structure it suffices to analyze only the zero component of Eq.(4), for if $|\Omega>$ is Lorentz invariant, Eq.(5), it follows that

$$\begin{aligned} U_\Lambda P^0|\Omega> &= U_\Lambda P^0 U_\Lambda^{-1} U_\Lambda|\Omega> \qquad (6)\\ &= ((\Lambda^0_0)^{-1}P^0 + (\Lambda^0_i)^{-1}P^i)|\Omega> \\ &= (\Lambda^0_i)^{-1}P^i|\Omega> \\ &= 0, \qquad (7)\end{aligned}$$

which implies that the momentum operator acting on the physical vacuum also gives zero, as required. Thus, in the following we will look only at the ground state eigenvalue problem for the energy operator P^0.

The purpose of this contribution to the Bled Workshop is to look at a single mode "approximation" to the energy operator of a full infinite degree of freedom system. Thus, let a_i, b_i, c_k denote respectively bare fermion, antifermion, and boson annihilation operators where the indices include both space-time (four-velocity $v = \frac{p}{m}$ and spin projections) and internal variables such as charge or isospin. Then the free four-momentum operator can be written as

$$P^\mu(fr) := m \sum \int dv v^\mu (a^\dagger_i a_i + b^\dagger_i b_i + \kappa c^\dagger_k c_k), \qquad (8)$$

where $dv := \frac{d^3v}{v_0}$ is the Lorentz invariant measure in four-velocity space, κ is a dimensionless relative bare boson mass parameter and m is a constant with the dimensions of mass; its value is determined by relating a physical mass such as the nucleon mass to the dimensionless eigenvalue of the corresponding stable particle. Because of the transformation properties of the creation and annihilation operators inherited from the one particle states, the free four-momentum operator, as defined in Eq.(8), satisfies the point form equations (1) and (2).

Interactions are obtained by integrating vertices, products of free fields, over the forward hyperboloid[3][4]. The fundamental vertex is the trilinear vertex, which is bilinear in fermion-antifermion creation and annihilation operators, and linear in boson creation and annihilation operators. That is, such vertices have the general form $V \sim (a^\dagger + b)(a + b^\dagger)(c + c^\dagger) = (a^\dagger a + b b^\dagger + a^\dagger b^\dagger + b a)(c + c^\dagger)$ so,

as shown in reference [4] the interacting four-momentum operator for trilinear vertices can be written as

$$P^{\mu}(I) = \alpha \sum \int d\nu (\mathcal{A}(X_k^{\mu}) c_k + \mathcal{A}(X_k^{\mu})^{\dagger} c_k^{\dagger}), \tag{9}$$

where $\mathcal{A}(X_k^{\mu}) := (a_{i_1}^{\dagger}, b_{i_1})(X_k^{\mu})_{i_1 i_2}(a_{i_2}, b_{i_2}^{\dagger})^{T}$, and the X's depend on the type of fermionic-bosonic coupling.

The zeroth component of the eigenvector equation, Eq.(3), is

$$\begin{aligned} P_F^0(fr) + \sum \int d\nu (\kappa \nu^0 c_k^{\dagger} c_k \\ + \alpha \mathcal{A}(X_k^0) c_k + \alpha \mathcal{A}(X_k^0)^{\dagger} c_k^{\dagger})) |\Psi_\lambda> = \lambda |\Psi_\lambda> \end{aligned} \tag{10}$$

To distinguish between the continuum energy operator in Eq.(10) and its finite approximation, the fundamental operator to be diagonalized is (a Hamiltonian) denoted by H, made out of creation and annihilation operators with a finite number of modes, whose form mimics Eq.(10):

$$H = \sum e_i (a_i^{\dagger} a_i + b_i^{\dagger} b_i + \kappa c_k^{\dagger} c_k) + \alpha \sum \mathcal{A}(X_k) c_k + \mathcal{A}(X_k^{\dagger}) c_k^{\dagger} \tag{11}$$

$$= \sum e_i + \sum e_i (a_i^{\dagger} a_i - b_i b_i^{\dagger} + \kappa c_i^{\dagger} c_i) + \alpha (\mathcal{A}(X_i) c_i + \mathcal{A}(X_i^{\dagger}) c_i^{\dagger}) \tag{12}$$

$$= \sum e_i + \mathcal{A}(E) + \kappa \sum e_i c_i^{\dagger} c_i + \alpha \sum (\mathcal{A}(X_i) c_i + \mathcal{A}(X_i^{\dagger}) c_i^{\dagger}), \tag{13}$$

$$E := diag(e_1, e_2, ..., e_N, -e_1, -e_2, ..., -e_N), \tag{14}$$

where the discrete "energy" $e_i = \sqrt{1 + \nu_i^2}$

Reference [4] shows that for a large choice of the X's, the ground state for the Hamiltonian in Eq.(13) goes as $-|constant|\alpha$ for $\alpha \gg 1$. Therefore there is no ground state solution equal to zero other than the free field solution.

One possibility is to add boson selfcoupling interactions. Consider a simple one mode Hamiltonian model, in which the selfcoupling is generated by the quartic anharmonic oscillator:

$$\begin{aligned} H &= \frac{1}{2}(x^2 + p^2) + \alpha^2 x^4 \\ &= c^{\dagger} c + \alpha^2 (c + c^{\dagger})^4 \end{aligned} \tag{15}$$

References [4] and [5] show how the boson Lie algebra is given as the contraction limit of a compact Lie algebra (of the group U(2)) whose Hamiltonian is

$$H_M = J_1 + \alpha^2 (\tilde{J}_+ + \tilde{J}_-)^4 \tag{16}$$

$$= \frac{M - J_z}{2} + \alpha^2 \rho^4 J_x^4 \tag{17}$$

$$= \frac{M + J_x}{2} + \frac{\alpha^2}{M^2} J_z^4, \tag{18}$$

and in the contraction limit, in which the Lie algebra contraction parameter $\rho \to 0$ as $M \to \infty$, such that $\rho M^2 = 1$, the eigenvalues of H_M, Eq.(18) converge to the

eigenvalues of H, Eq.(15). In Eq.(17) the Lie algebra basis of U(2) has been written in a U(1)xSU(2) basis, and in Eq.(18), the contraction parameter has been eliminated by writing $\rho = \frac{1}{M^2}$. The goal is to numerically find the lowest eigenvalues for fixed coupling as M, the U(2) irrep label, gets large. Using an SU(2) Lie algebra automorphism to interchange the x and z generators generates a tridiagonal matrix in the basis given in Eq.(18). Reference [4] shows that the true eigenvalue is approached for M about 100.

Next consider a one mode system coupling fermions and bosons, with no quartic boson selfcoupling. The Hamiltonians are now

$$H = 1 + (a^\dagger a - bb^\dagger) + c^\dagger c + \alpha(\mathcal{A}(X)c + \mathcal{A}(X^\dagger)c^\dagger) \tag{19}$$

$$= 1 + \mathcal{A}(E) + c^\dagger c + \alpha(\mathcal{A}(X)c + \mathcal{A}(X^\dagger)c^\dagger; \tag{20}$$

$$H_M = 1 + \mathcal{A}(E) + J_1 + \alpha(\mathcal{A}(X)J_- + \mathcal{A}(X^\dagger)J_+) \tag{21}$$

$$E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{22}$$

For one mode the fermion space is two dimensional ($|0>$ and $a^\dagger b^\dagger|0>$) and the boson space is M+1 dimensional. When H_M is diagonalized, the lowest eigenvalue linearly decreases with respect to the bare coupling constant α, for $\alpha \gg 1$. Reference [4] shows this behavior of the ground state holds even for many mode systems.

Finally, if the boson selfcoupling term, the anharmonic term in Eq.(17) is added to the Hamiltonian, Eq.(21), the result is a model of trilinear coupling with a quartic boson selfcoupling, a simple "QCD" one mode model:

$$H_M^{QCD} = 1 + \mathcal{A}(E) + J_1 + \alpha(\mathcal{A}(X)J_- + \mathcal{A}(X^\dagger)J_+) + \alpha^2(J_- + J_+)^4; \tag{23}$$

the lowest eigenvalue for small values of M, as a function of the bare coupling parameter have been numerically calculated. The ground state eigenvalue as a function of the bare coupling parameter starts at zero, becomes negative and then rises, passing through zero; if such behavior presists in the large mode limit, this raises the possibility of fine tuning the bare coupling parameter.

References

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