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16 The γ^a Matrices, $\tilde{\gamma}^a$ Matrices and Generators of Lorentz Rotations in Clifford Space — Determining in the *Spin-charge-family* Theory Spins, Charges and Families of Fermions — in $(3+1)$-dimensional Space ⋆

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Abstract. In the *spin-charge-family* theory there are in d-dimensional space 2^d Clifford vectors, describing internal degrees of freedom of fermions — their families and family members. Due to two kinds of the Clifford algebra objects, defined in this theory as γ^a and $\tilde{\gamma}^a$ [2–7], each vector carries two kinds of indices. Operators $\gamma^a\,\gamma^b$ determine in $d=(3+1)$ space the spin and all the charges of quarks and leptons, $\tilde{\gamma}^a\,\tilde{\gamma}^b$ determine families of quarks and leptons. In this contribution basis in $d=(3+1)$ Clifford space is chosen in a way that the matrix representation of the γ^a matrices and of the generators of the Lorentz transformations in internal space $S^{ab}=\frac{i}{4}(\gamma^a\gamma^b-\gamma^b\gamma^a)$ coincide for each family quantum number, determined with $\tilde{S}^{ab}=\frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b-\tilde{\gamma}^b\tilde{\gamma}^a)$, with Dirac matrices. We do not take here into account the second quantization requirements [?], which reduce the number of states from 2^d to $2^{\frac{d}{2}-1}$ families of $2^{\frac{d}{2}-1}$ family members each, but this is the case for $d=2(2n+1)$, since in the *spin-charge-family* theory $d>4$.

Povzetek. V teoriji *spinov-nabojev-družin* je v d-razsežnem prostoru 2^d Cliffordovih vektorjev, ki opisujejo notranje prostostne stopnje fermionov, to je njihove družine in člane družin. Ker imamo dve vrsti Cliffordovih objektov, ki so v tej teoriji definirani kot γ^a in $\tilde{\gamma}^a$ [2–7], ima vsak vektor dve vrsti indeksov. Operatorji $S^{ab}=\frac{i}{4}(\gamma^a\gamma^b-\gamma^b\gamma^a)$ določajo v $d=(3+1)$-razsežnem prostoru spin in vse naboje kvarkov in leptonov, $\tilde{S}^{ab}=\frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b-\tilde{\gamma}^b\tilde{\gamma}^a)$ pa kvantna števila njihovih družin. V tem prispevku je baza v $d=(3+1)$ Cliffordovem prostoru izbrana tako, da matrične upodobitve operatorjev γ^a in generatorjev Lorentzovih transformacij S^{ab} v notranjem prostoru sovpadajo z Diracovimi matrikami za vsako družinsko kvantno število, določeno s $\tilde{S}^{ab}$. V prispevku ne upoštevamo zahtev druge kvantizacije [8], ki zmanjšajo število stanj z 2^d na $2^{\frac{d}{2}-1}$ družin s po $2^{\frac{d}{2}-1}$ člani. Vendar velja v teoriji *spinov-nabojev-družin* to le za $d=2(2n+1)$, kjer je $d>4$.

⋆ This contribution is written to help readers of the Bled proceedings and participants at future Bled Workshops "What Comes Beyond the Standard Models" to understand the difference between the Dirac γ^a matrices and the $\tilde{\gamma}^a$ matrices, which are all defined in 2^d space and used in the *spin-charge-family* theory to describe families and family members [2–7].

Keywords:Dirac matices, Clifford algebra,Kaluza-Klein theories, Higher dimensional spaces, Beyond the standard model, Lepton and quark families
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16.1 Introduction

In the *spin-charge-family* theory there are in d-dimensional space two kinds of operators, γ^a and $\tilde{\gamma}^a$, which operate on 2^d Clifford vectors, describing internal degrees of freedom of fermions; $\tilde{\gamma}^a$ determine family quantum numbers, γ^a determine family members. Due to these two kinds of the Clifford algebra objects each vector carries two kinds of indexes [2–7]. Operators $\frac{i}{2}\gamma^a\,\gamma^b$ determine in $d = (3+1)$ space the spin and all the charges of quarks and leptons, $\frac{i}{2}\tilde{\gamma}^a\,\tilde{\gamma}^b$ determine families of quarks and leptons.

Here only basis in $d = (3+1)$ Clifford space is discussed, which in the *spin-charge-family* theory is only a part of $d = (13+1)$. The basis is chosen in a way that the matrix representation of the γ^a matrices and of the generators of the Lorentz transformations in internal space $S^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$ coincide for each family quantum number, determined with $\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)$, with Dirac matrices.

This contribution is written to help the reader of the proceedings of Bled workshops "What comes beyond the standard models" to realize the differences between the Dirac matrices (operators) γ^a and the operators $\tilde{\gamma}^a$ [2].

We do not take here into account the second quantization requirements [8], which reduce the number of states from 2^d to $2^{\frac{d}{2}-1}$ families of $2^{\frac{d}{2}-1}$ family members each, since these requirements concern the states in $d = 2(2n+1)$, and not at all the particular subspace, in our case $d = (3+1)$.

We use in this contribution 2^d vectors in Clifford space, expressible with γ^a with the properties

$$\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}\,. \tag{16.1}$$

A general vector can correspondingly be written as

$$\mathbf{B} = \sum_{k=0}^{d} a_{a_1 a_2 \ldots a_k}\, \gamma^{a_1}\gamma^{a_2}\ldots\gamma^{a_k}|\psi_{oc}>, \quad a_i \le a_{i+1}\,, \tag{16.2}$$

where $|\psi_o>$ is the vacuum state. We arrange these vectors as products of nilpotents and projectors

$$\begin{aligned} \stackrel{ab}{(k)}:&= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,,\\ \stackrel{ab}{[k]}:&= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \end{aligned} \tag{16.3}$$

where $k^2 = \eta^{aa}\eta^{bb}$, their Hermitian conjugate values are

$$\stackrel{ab}{(k)}^{\dagger} = \eta^{aa}\stackrel{ab}{(-k)}, \quad \stackrel{ab}{[k]}^{\dagger} = \stackrel{ab}{[k]}\,, \tag{16.4}$$

and that they all are eigenstates of the Cartan subalgebra of the generators of the Lorentz transformations $S^{ab} = \frac{i}{4}(\gamma^a\gamma^b = \gamma^{ab}\gamma^a)$ in this internal space

$$S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\;d}, \tag{16.5}$$

with the eigenvalues

$$S^{ab} \stackrel{ab}{(k)} = \frac{1}{2}k \stackrel{ab}{(k)}, \quad S^{ab} \stackrel{ab}{[k]} = \frac{1}{2}k \stackrel{ab}{[k]} \;. \tag{16.6}$$

We find in this Clifford algebra space two kinds of the Clifford algebra objects, besides γ^a also $\tilde{\gamma}^a$ [2–7], which anticommute with γ^a

$$\begin{aligned} &\{\gamma^a, \tilde{\gamma^b}\}_+ = 0\,, \\ &\{\gamma^a, \tilde{\gamma^b}\}_+ = I \;\; 2\eta^{ab}, \quad \text{for} \quad a, b \quad \in \{0, 1, 2, 3, 5, \cdots, d\}, \end{aligned} \tag{16.7}$$

for any d, even or odd. I is the unit element in the Clifford algebra. One of the authors (N.S.M.B.) recognized these two possibilities in Grassmann space [2]. But one can as well as understand the appearance of the two kinds of the Clifford algebra object by recognizing

$$\begin{aligned} \gamma^a\, \mathbf{B}\, |\psi_o>: &= (\, a_0\, \gamma^a \; + a_{a_1}\, \gamma^a \gamma^{a_1} + a_{a_1 a_2}\, \gamma^a \gamma^{a_1}\gamma^{a_2} + \cdots + \\ &\quad a_{a_1 \cdots a_d}\, \gamma^a \gamma^{a_1} \cdots \gamma^{a_d}\,)\, |\psi_{oc}>, \\ \tilde{\gamma}^a\, \mathbf{B}\, |\psi_o>: &= (\, i\, a_0 \gamma^a - i\, a_{a_1} \gamma^{a_1}\, \gamma^a + i\, a_{a_1 a_2} \gamma^{a_1}\gamma^{a_2}\, \gamma^a + \cdots + \\ &\quad i\,(-1)^d\, a_{a_1 \cdots a_d} \gamma^{a_1} \cdots \gamma^{a_d}\, \gamma^a\,)\, |\psi_o> \;. \end{aligned} \tag{16.8}$$

The nilpotents and projectors oof Eq. (16.3) are the eigenstates also of the generators of the Cartan subalgebra

$$\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\;d}, \tag{16.9}$$

with the eigenvalues

$$\tilde{S}^{ab} \stackrel{ab}{(k)} = \frac{k}{2} \stackrel{ab}{(k)}, \quad \tilde{S}^{ab} \stackrel{ab}{[k]} = -\frac{k}{2} \stackrel{ab}{[k]} \;. \tag{16.10}$$

One finds the relations

$$\begin{aligned} &\gamma^a \stackrel{ab}{(k)} = \eta^{aa} \stackrel{ab}{[-k]}, \; \gamma^b \stackrel{ab}{(k)} = -ik \stackrel{ab}{[-k]}, \; \gamma^a \stackrel{ab}{[k]} = \stackrel{ab}{(-k)}, \; \gamma^b \stackrel{ab}{[k]} = -ik\eta^{aa} \stackrel{ab}{(-k)}, \\ &\tilde{\gamma^a} \stackrel{ab}{(k)} = -i\eta^{aa} \stackrel{ab}{[k]}, \; \tilde{\gamma^b} \stackrel{ab}{(k)} = -k \stackrel{ab}{[k]}, \; \tilde{\gamma^a} \stackrel{ab}{[k]} = \; i \stackrel{ab}{(k)}, \; \tilde{\gamma^b} \stackrel{ab}{[k]} = -k\eta^{aa} \stackrel{ab}{(k)} \;. \end{aligned} \tag{16.11}$$

We discuss in what follows the representations of the operators γ^a, $\tilde{\gamma}^a$, S^{ab} and $\tilde{S}^{ab}$ only in $d = (3 + 1)$.

In Ref. [8], as well as in this proceedings, the second quantization in Clifford and in Grassmann space is discussed. There the restrictions on the choices of products of nilpotents and projectors, which can be recognized as independent states in the Clifford space, and yet allow the second quantization, is analyzed. The restrictions reduce, as noticed above, the number of states from 2^d to $2^{\frac{d}{2}-1}$ families with $2^{\frac{d}{2}-1}$ family members each. All the states of this contribution appear as a part of states (included as factors) already in $d = (5+1)$.

In what follows we shall notpay attention on these limitations. We only present matrices of the operators γ^a, $\tilde{\gamma}^a$, S^{ab} and $\tilde{S}^{ab}$ for all possible states.

16.2 Basis in $d = (3+1)$

There are $2^4 = 16$ basic states in $d = (3+1)$. We make a choice of products of nilpotents and projectors, which are eigenstates of the Cartan subalgebra operators as presented in Eqs. (16.6, 16.10). The family members are reachable by S^{ab}, or by γ^a representing twice two vectors of definite handedness $\Gamma^{(d)}$ in $d = (3+1)$

$$\Gamma^{(d)} := (i)^{d/2} \prod_a (\sqrt{\eta^{aa}}\gamma^a), \quad \text{if} \quad d = 2n. \tag{16.12}$$

Each vector carries also the family handedness

$$\tilde{\Gamma}^{(d)} := (i)^{d/2} \prod_a (\sqrt{\eta^{aa}}\tilde{\gamma}^a), \quad \text{if} \quad d = 2n. \tag{16.13}$$

In what follows we first define the basic states and then represent all the operators — γ^a, S^{ab}, $\tilde{\gamma}^a$, $\tilde{S}^{ab}$, $\Gamma^{(d)}$ ($= -4iS^{03}S^{12}$ in $d = 4$), $\tilde{\Gamma}^{(d)}$ ($= -4i\tilde{S}^{03}\tilde{S}^{12}$ in $d = 4$) — as 16×16 matrices in this basis. We see that the operators have a 4×4 diagonal or off diagonal or partly diagonal and partly off diagonal substructure.

Let us start with the definition of the basic states, presented in Table 16.1.

As seen in Table 16.1 γ^a change handedness. S^{ab}, which do not belong to Cartan subalgebra, generate all the states of one representation of particular handedness, Eq. (16.12), and particular family quantum number. $\tilde{S}^{ab}$, which do not belong to Cartan subalgebra, transform a family member of one family into the same family member of another family, $\tilde{\gamma}^a$ change the family quantum number as well as the handedness $\tilde{\Gamma}^{(3+1)}$, Eq. (16.13).

Dirac matrices γ^a and S^{ab} do not distinguish among the families, they "see" all the families in the same way and correspondingly "see" only four states — instead of 4× four states. The operators γ^a and S^{ab} are correspondingly 4×4 matrices.

Let us define, to simplify the notation, the unit 4×4 submatrix and the submatrix with all the matrix elements equal to zero as follows

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{16.14}$$

We also use (2×2) Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{16.15}$$

$d = 4$	ψ_i	$\gamma^0\,\psi_i$	$\gamma^1\,\psi_i$	$\gamma^2\,\psi_i$	$\gamma^3\,\psi_i$	$\tilde{\gamma}^0\,\psi_i$	$\tilde{\gamma}^1\,\psi_i$	$\tilde{\gamma}^2\,\psi_i$	$\tilde{\gamma}^3\,\psi_i$	S^{03}	S^{12}	$\tilde{S}^{03}$	$\tilde{S}^{12}$	Γ^{3+1}	$\tilde{\Gamma}^{3+1}$
ψ^1_1	$(+i)(+)$	ψ^1_3	ψ^1_4	$i\psi^1_4$	ψ^1_3	$-i\psi^2_1$	$-i\psi^3_1$	ψ^3_1	$-i\psi^2_1$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	1	1
ψ^1_2	$[-i][-]$	ψ^1_4	ψ^1_3	$-i\psi^1_3$	$-\psi^1_4$	$i\psi^2_2$	$i\psi^3_2$	$-\psi^3_2$	$i\psi^2_2$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	1	1
ψ^1_3	$[-i](+)$	ψ^1_1	$-\psi^1_2$	$-i\psi^1_2$	$-\psi^1_1$	$i\psi^2_3$	$i\psi^3_3$	$-\psi^3_3$	$i\psi^2_3$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	-1	1
ψ^1_4	$(+i)[-]$	ψ^1_2	$-\psi^1_1$	$i\psi^1_1$	ψ^1_2	$-i\psi^2_4$	$-i\psi^3_4$	ψ^3_4	$-i\psi^2_4$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	-1	1
ψ^2_1	$[+i](+)$	ψ^2_3	$-\psi^2_4$	$-i\psi^2_4$	ψ^2_3	$i\psi^1_1$	$i\psi^4_1$	$-\psi^4_1$	$-i\psi^1_1$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	1	-1
ψ^2_2	$(-i)[-]$	ψ^2_4	$-\psi^2_3$	$i\psi^2_3$	$-\psi^2_4$	$-i\psi^1_2$	$-i\psi^4_2$	ψ^4_2	$i\psi^1_2$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	1	-1
ψ^2_3	$(-i)(+)$	ψ^2_1	ψ^2_2	$i\psi^2_2$	$-\psi^2_1$	$-i\psi^1_3$	$-i\psi^4_3$	ψ^4_3	$i\psi^1_3$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	-1	-1
ψ^2_4	$[+i][-]$	ψ^2_2	ψ^2_1	$-i\psi^2_1$	ψ^2_2	$i\psi^1_4$	$i\psi^4_4$	$-\psi^4_4$	$-i\psi^1_4$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	-1	-1
ψ^3_1	$(+i)[+]$	ψ^3_3	$-\psi^3_4$	$-i\psi^3_4$	ψ^3_3	$i\psi^4_1$	$-i\psi^1_1$	$-\psi^1_1$	$i\psi^4_1$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	1	-1
ψ^3_2	$[-i](-)$	ψ^3_4	$-\psi^3_3$	$i\psi^3_3$	$-\psi^3_4$	$-i\psi^4_2$	$i\psi^1_2$	ψ^1_2	$-i\psi^4_2$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	1	-1
ψ^3_3	$[-i][+]$	ψ^3_1	ψ^3_2	$i\psi^3_2$	$-\psi^3_1$	$-i\psi^4_3$	$i\psi^1_3$	ψ^1_3	$-i\psi^4_3$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	-1	-1
ψ^3_4	$(+i)(-)$	ψ^3_2	ψ^3_1	$-i\psi^3_1$	ψ^3_2	$i\psi^4_4$	$-i\psi^1_4$	$-\psi^1_4$	$i\psi^4_4$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	-1	-1
ψ^4_1	$[+i][+]$	ψ^4_3	ψ^4_4	$i\psi^4_4$	ψ^4_3	$-i\psi^3_1$	$i\psi^2_1$	ψ^2_1	$i\psi^3_1$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	1	1
ψ^4_2	$(-i)(-)$	ψ^4_4	ψ^4_3	$-i\psi^4_3$	$-\psi^4_4$	$i\psi^3_2$	$-i\psi^2_2$	$-\psi^2_2$	$-i\psi^3_2$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	1	1
ψ^4_3	$(-i)[+]$	ψ^4_1	$-\psi^4_2$	$-i\psi^4_2$	$-\psi^4_1$	$i\psi^3_3$	$-i\psi^2_3$	$-\psi^2_3$	$-i\psi^3_3$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	-1	1
ψ^4_4	$[+i](-)$	ψ^4_2	$-\psi^4_1$	$i\psi^4_1$	ψ^4_2	$-i\psi^3_4$	$i\psi_8$	ψ^2_4	$i\psi^3_4$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	-1	1

Table 16.1. In this table $2^d = 16$ vectors, describing internal space of fermions in $d = (3+1)$, are presented. Each vector carries the family member quantum number — determined by S^{03} and S^{12}, Eqs. (16.6) — and the family quantum number — determined by $\tilde{S}^{03}$ and $\tilde{S}^{12}$, Eq. (16.10).

Looking in Table 16.1 one easily finds the matrix representations for γ^0, γ^1, γ^2 and γ^3

$$\gamma^0 = \begin{pmatrix} \begin{smallmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{smallmatrix} \end{pmatrix}, \tag{16.16}$$

$$\gamma^1 = \begin{pmatrix} \begin{smallmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{smallmatrix} \end{pmatrix}, \tag{16.17}$$

$$\gamma^2 = \begin{pmatrix} \begin{smallmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{smallmatrix} & 0 & 0 & 0 \\ \mathbf{0} & \begin{smallmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{smallmatrix} \end{pmatrix}, \tag{16.18}$$

$$\gamma^3 = \begin{pmatrix} \begin{smallmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{smallmatrix} \end{pmatrix}. \tag{16.19}$$

One sees as well the 4×4 substructure along the diagonal of 16×16 matrices.

The representations of the $\tilde{\gamma}^a$, these do not appear in the Dirac case, manifest the off diagonal structure as follows

$$\tilde{\gamma}^0 = \begin{pmatrix} \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} & \mathbf{0} \end{pmatrix}, \tag{16.20}$$

$$\tilde{\gamma}^1 = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} \\ \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{16.21}$$

$$\tilde{\gamma}^2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\sigma^3 & 0 \\ 0 & \sigma^3 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{smallmatrix} \\ \begin{smallmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} -\sigma^3 & 0 \\ 0 & \sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{16.22}$$

$$\tilde{\gamma}^3 = \begin{pmatrix} \mathbf{0} & \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} & \mathbf{0} \end{pmatrix}. \tag{16.23}$$

Matrices S^{ab} have again the 4×4 substructure along the diagonal structure, as expected, manifesting the repetition of the Dirac 4×4 matrices, since the Dirac S^{ab} do not distinguish among families.

$$S^{01} = \begin{pmatrix} \begin{smallmatrix} \frac{i}{2}\sigma^1 & 0 \\ 0 & -\frac{i}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} -\frac{i}{2}\sigma^1 & 0 \\ 0 & \frac{i}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\frac{i}{2}\sigma^1 & 0 \\ 0 & \frac{i}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^1 & 0 \\ 0 & -\frac{i}{2}\sigma^1 \end{smallmatrix} \end{pmatrix}, \tag{16.24}$$

$$S^{02} = \begin{pmatrix} \begin{smallmatrix} -\frac{i}{2}\sigma^2 & 0 \\ 0 & \frac{i}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^2 & 0 \\ 0 & -\frac{i}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^2 & 0 \\ 0 & -\frac{i}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\frac{i}{2}\sigma^2 & 0 \\ 0 & \frac{i}{2}\sigma^2 \end{smallmatrix} \end{pmatrix}, \tag{16.25}$$

$$S^{03} = \begin{pmatrix} \begin{smallmatrix} \frac{i}{2}\sigma^3 & 0 \\ 0 & -\frac{i}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^3 & 0 \\ 0 & -\frac{i}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^3 & 0 \\ 0 & -\frac{i}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^3 & 0 \\ 0 & -\frac{i}{2}\sigma^3 \end{smallmatrix} \end{pmatrix}, \tag{16.26}$$

$$S^{12} = \begin{pmatrix} \begin{smallmatrix} \frac{1}{2}\sigma^3 & 0 \\ 0 & \frac{1}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} \frac{1}{2}\sigma^3 & 0 \\ 0 & \frac{1}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{1}{2}\sigma^3 & 0 \\ 0 & \frac{1}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{1}{2}\sigma^3 & 0 \\ 0 & \frac{1}{2}\sigma^3 \end{smallmatrix} \end{pmatrix}, \tag{16.27}$$

$$S^{13} = \begin{pmatrix} \begin{smallmatrix} \frac{1}{2}\sigma^2 & 0 \\ 0 & \frac{1}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} -\frac{1}{2}\sigma^2 & 0 \\ 0 & -\frac{1}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\frac{1}{2}\sigma^2 & 0 \\ 0 & -\frac{1}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{1}{2}\sigma^2 & 0 \\ 0 & \frac{1}{2}\sigma^2 \end{smallmatrix} \end{pmatrix}, \tag{16.28}$$

$$S^{23} = \begin{pmatrix} \begin{smallmatrix} \frac{1}{2}\sigma^1 & 0 \\ 0 & \frac{1}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} -\frac{1}{2}\sigma^1 & 0 \\ 0 & -\frac{1}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\frac{1}{2}\sigma^1 & 0 \\ 0 & -\frac{1}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{1}{2}\sigma^1 & 0 \\ 0 & \frac{1}{2}\sigma^1 \end{smallmatrix} \end{pmatrix}. \tag{16.29}$$

$$\Gamma^{3+1} = -4iS^{03}S^{12} = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \end{pmatrix}. \tag{16.30}$$

The operators $\tilde{S}^{ab}$ have again off diagonal 4×4 substructure, except $\tilde{S}^{03}$ and $\tilde{S}^{12}$, which are diagonal.

$$\tilde{S}^{01} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{i}{2}\mathbf{1} \\ \mathbf{0} & \mathbf{0} & -\frac{i}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ -\frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{16.31}$$

$$\tilde{S}^{02} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ -\frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{16.32}$$

$$\tilde{S}^{03} = \begin{pmatrix} \frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{i}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{i}{2}\mathbf{1} \end{pmatrix}, \tag{16.33}$$

$$\tilde{S}^{12} = \begin{pmatrix} \frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{2}\mathbf{1} \end{pmatrix}, \tag{16.34}$$

$$\tilde{S}^{13} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{i}{2}\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \frac{i}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{16.35}$$

$$\tilde{S}^{23} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{2}\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ -\frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{16.36}$$

$$\tilde{\Gamma}^{3+1} = -4i\tilde{S}^{03}\tilde{S}^{12} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}. \tag{16.37}$$

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