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ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc.imfm.si
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 4 (2011) 95–109

The total weak discrepancy of a partially ordered set

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Received 13 June 2010, accepted 13 October 2010, published online 15 March 2011

Abstract

We define the total weak discrepancy of a poset P as the minimum nonnegative integer k for which there exists a function $f : V \to \mathbf{Z}$ satisfying (i) if $a \prec b$ then $f(a) + 1 \leq f(b)$ and (ii) $\sum |f(a) - f(b)| \leq k$, where the sum is taken over all unordered pairs $\{a, b\}$ of incomparable elements. If we allow k and f to take real values, we call the minimum k the fractional total weak discrepancy of P. These concepts are related to the notions of weak and fractional weak discrepancy, where (ii) must hold not for the sum but for each individual pair of incomparable elements of P. We prove that, unlike the latter, the total weak and fractional total weak discrepancy of P are always the same, and we give a polynomial-time algorithm to find their common value. We use linear programming duality and complementary slackness to obtain this result.

Keywords: Posets, weak discrepancy, fractional weak discrepancy, total linear discrepancy.

Math. Subj. Class.: 06A06, 06A07, 90B10

Dedication

The third author was fortunate to have Mike Albertson as her instructor when she was a high school student at the Hampshire College Summer Studies in Mathematics program. In the years since he continued to be a mentor and friend. All three of us have benefited from his part in organizing the CoNE conferences, his example as a prominent researcher at a liberal arts college, and his many other contributions to discrete mathematics.

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1 Introduction

In this paper we will consider irreflexive posets $P = (V, \prec)$, and write $x \parallel y$ when elements x and y in V are incomparable. We begin with some background on weak and fractional weak discrepancy.

1.1 Background and examples

A poset $P = (V, \prec)$ is a *weak order* if there exists a real-valued function $f : V \to \mathbf{R}$ so that $x \prec y$ if and only if $f(x) < f(y)$. We can think of such a function as assigning a rank to each element of P in such a way that respects the ordering $\prec$ and gives incomparable elements equal rank. Sometimes it is desirable to rank the elements of a poset that is not a weak order. For example, a poset could represent a set V of employees partially ordered by their value to a company and the function value $f(v)$ could represent employee v's salary. We want such a ranking function to satisfy two "fairness" conditions: first that a more valuable employee receives a significantly higher salary, and second that we seek to minimize the largest discrepancy in salaries between incomparable employees. Additional motivating examples can be found in [12]. These fairness conditions are made more formal in the following definition.

Definition 1.1. The *fractional weak discrepancy* of a poset $P = (V, \prec)$, denoted by $wd_F(P)$, is the minimum nonnegative real number k for which there exists a function $f : V \to \mathbf{R}$ satisfying

(i) if $a \prec b$ then $f(a) + 1 \leq f(b)$ ("up" constraints)

(ii) if $a \parallel b$ then $|f(a) - f(b)| \leq k$. ("side" constraints)

For this and similar definitions, we will call a function f that achieves the minimum value of k an *optimal labeling* of P.

Fractional weak discrepancy was first defined in [8] and studied further in [7, 9, 10, 11]. The integer version of the problem, where each function value $f(v)$ must be an integer, was introduced in [13] as the *weakness* of a poset, and studied further as *weak discrepancy* in [4, 12].

Definition 1.2. The *weak discrepancy* of a poset $P = (V, \prec)$, denoted by $wd(P)$, is the minimum nonnegative integer k for which there exists a function $f : V \to \mathbf{Z}$ satisfying conditions (i) and (ii) of Definition 1.1.

In [8], the authors used linear programming duality to prove that $wd_F(P)$ is always rational and that the weak discrepancy of a poset is always its fractional weak discrepancy rounded up to the next integer, $wd(P) = \lceil wd_F(P) \rceil$. Moreover, taking the ceilings (or floors) of an optimal labeling for the fractional problem gives an optimal labeling for the integer problem. The linear programming approach shows that weak and fractional weak discrepancy can be computed in polynomial time.

For example, the poset Z shown in Figure 1 with optimal labelings has $wd_F(Z) = 4/3$ and $wd(Z) = 2$. Optimality follows from results in [8].

Returning to the salary example, we can take a different approach to evaluating the fairness of the ranking function f by minimizing the *average*, or equivalently, the *total* discrepancy in salaries $f(v)$ between incomparable employees v rather than the largest discrepancy. Definitions 1.3 and 1.4 formalize this notion in its fractional and integer versions.

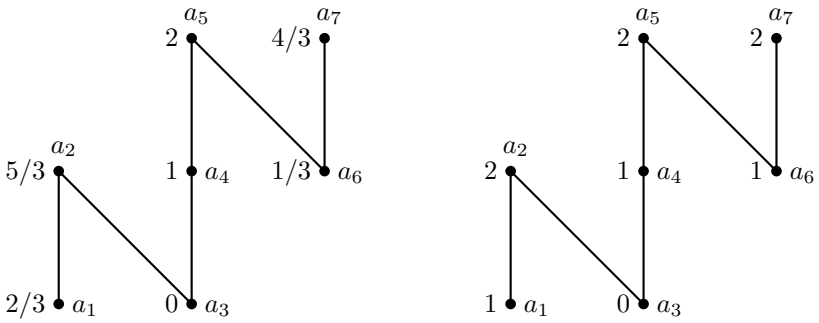

Figure 1: The poset Z with optimal fractional weak (left) and optimal weak (right) labelings. Here $wd_F(Z) = 4/3$ and $wd(Z) = 2$.

Definition 1.3. The *fractional total weak discrepancy* of a poset $P = (V, \prec)$, denoted by $tw_F(P)$, is the minimum nonnegative real number k for which there exists a function $f : V \to \mathbf{R}$ satisfying

(i) if $a \prec b$ then $f(a) + 1 \leq f(b)$

(ii) $\sum |f(a) - f(b)| \leq k$, where the sum is taken over all unordered pairs $\{a, b\}$ of incomparable elements $a \parallel b$.

Definition 1.4. The *total weak discrepancy* of a poset $P = (V, \prec)$, denoted by $tw(P)$, is the minimum nonnegative integer k for which there exists a function $f : V \to \mathbf{Z}$ satisfying conditions (i) and (ii) of Definition 1.3.

Thus we have four variants of weak discrepancy, as shown in Table 1. Each of these variants represents a different problem, but we will prove in Theorem 3.1 that the two problems in the "sum" column of the table are essentially the same.

labels	$\min \max_{a \parallel b} \lvert f(a) - f(b) \rvert$	$\min \sum_{a \parallel b} \lvert f(a) - f(b) \rvert$
integers	weak discrepancy $wd(P)$	total weak discrepancy $tw(P)$
reals	fractional weak discr. $wd_F(P)$	fractional total weak discr. $tw_F(P)$

Table 1: Four variants of weak discrepancy.

An optimal labeling for fractional weak discrepancy need not be optimal for fractional total weak discrepancy. An example is given by the poset Z in Figure 1. The fractional labeling on the left is optimal for $wd_F(P)$ and gives the value $\sum |f(a) - f(b)| = 28/3$. But the integer labeling on the right gives $\sum |f(a) - f(b)| = 9$, so $tw_F(Z) \leq tw(Z) \leq 9$. In fact, in Example 3.3 we conclude that $tw_F(Z) = tw(Z) = 9$.

Also, an optimal labeling for weak discrepancy need not be optimal for total weak discrepancy. While the optimal labeling for $wd(Z)$ is optimal for $tw(Z)$, this is not the case for the poset Y in Figure 2. The labeling on the left is optimal for weak discrepancy but not for total weak discrepancy. In fact, there is no labeling of Y that is simultaneously

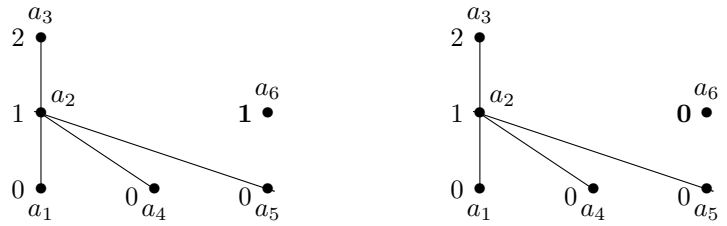

Figure 2: The poset Y with optimal weak (left) and total weak (right) labelings. The labels are identical except for the point a_6. Here $wd_F(Y) = wd(Y) = 1$ and $tw(Y) = 3$. The optimal labeling for weak discrepancy is not optimal for total weak discrepancy.

optimal for the weak and total weak discrepancy problems. In Section 2, we use Y to illustrate our linear programming approach to finding total weak discrepancy.

1.2 Comparability invariants

We discuss comparability invariance results for several forms of discrepancy. Recall that a property of a poset P is a *comparability invariant* if it is shared by all posets with the same comparability graph as P. The *linear discrepancy* of a poset, defined formally in [12], is equivalent to the weak discrepancy with the additional condition in Definition 1.2 that the labeling function be injective. Similarly, total linear discrepancy, studied in [2] and [5], is equivalent to total weak discrepancy with an injective labeling function.

It was shown in [4] that weak discrepancy is a comparability invariant and in [12] that linear discrepancy is also a comparability invariant. Indeed, the linear discrepancy of a poset is equal to the bandwidth of its incomparability graph [3]. In contrast, the posets P and Q in Figure 3 show that total weak discrepancy is not a comparability invariant. The reader can check that P and Q have the same comparability graph but different values of tw. Indeed, the labelings shown are optimal for total weak discrepancy, with $tw(P) = 2$ and $tw(Q) = 3$. Likewise, total linear discrepancy is not a comparability invariant. Using the results of [2] and [5], it is easy to check that the total linear discrepancy of P is 8, while that of Q is 7.

In particular, because total weak discrepancy and total linear discrepancy are not comparability invariants, there can be no result analogous to the bandwidth result for these types of discrepancy.

2 Fractional total weak discrepancy and linear programming

For the remainder of this article we let $P = (V, \prec)$ be a poset with at least one incomparable pair of elements. Let $V = \{a_1, a_2, \ldots, a_n\}$. Choosing a labeling function f on V corresponds to choosing real numbers $x_1, x_2, \ldots, x_n$, where $x_i = f(a_i)$. Thus we can express the fractional total weak discrepancy $tw_F(P)$ as the solution to the following optimization problem T. The decision variables for T are $x_1, x_2, \ldots, x_n$ and a real number k.

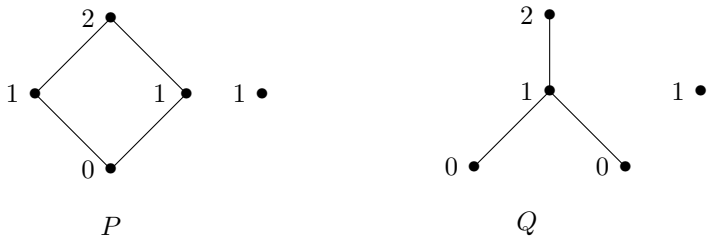

Figure 3: Two posets with the same comparability graph but with different total weak discrepancies. Here $tw(P) = 2$ but $tw(Q) = 3$.

Problem T : $\min k$ subject to

$$\begin{array}{cc} x_i - x_j \leq -1 & \text{when } a_i \prec a_j \\ \sum |x_i - x_j| \leq k & \text{over all pairs } \{a_i, a_j\} \text{ with } a_i \parallel a_j \\ x_i,\ k & \text{unrestricted in sign} \end{array} \tag{2.1}$$

Any choice of the variables x_i, k that satisfies the constraints must have $k \geq 0$, but it will be convenient to leave the sign of k unrestricted when we convert T to a linear programming problem and take its dual. Also, if there is an optimal solution to T then there is one where each $x_i \geq 0$, since any translation of the x_i without changing k also satisfies the constraints.

For the remainder of this paper, we will use the poset Y in Figure 2 as a running example to illustrate our results and methods. We begin in Example 2.1, where we compare the labelings in Figure 2 and start to formulate the linear program for Y. We ignore "up" constraints that are implied by transitivity and will prove in Proposition 2.5 that this is permissible.

Example 2.1. Let the elements of the poset Y be $a_1, \ldots, a_6$ as shown in Figure 2. The constraints $a_i \prec a_j$ (ignoring those implied by transitivity) correspond to the (i, j) pairs

$$(1, 2), (2, 3), (4, 2), (5, 2).$$

The incomparabilities can be written as $a_i \parallel a_j,\ i < j$, corresponding to the (i, j) pairs

$$(1, 4), (1, 5), (1, 6), (2, 6), (3, 6), (4, 5), (4, 6), (5, 6).$$

The labeling on the left in Figure 2 satisfies the constraints in Definition 1.3 for $k \geq 4$ while the labeling on the right satisfies them for $k \geq 3$. Thus the labeling on the left cannot be optimal for total weak discrepancy, and $tw_F(Y) \leq tw(Y) \leq 3$. We will verify that $tw_F(Y) = tw(Y) = 3$ in Example 2.8. □

We will show that the (fractional) total weak discrepancy problem T is equivalent to a linear programming problem, P_T, which we will think of as the *primal* problem in a primal-dual pair. The decision variables for P_T are $x_1, x_2, \ldots, x_n$ together with real numbers k_{ij}, indexed by the unordered pairs of incomparable elements of V. The constraints show

that each $k_{ij} \geq 0$ in any feasible solution but it will be convenient to leave their signs unrestricted.

Problem P_T : $\min \sum_{\substack{a_i \| a_j \\ i<j}} k_{ij}$ subject to

$$\begin{aligned} x_i - x_j &\leq -1 && \text{when } a_i \prec a_j \\ x_i - x_j - k_{ij} &\leq 0 \\ x_j - x_i - k_{ij} &\leq 0 && \text{when } a_i \parallel a_j,\ i<j \\ x_i,\ k_{ij} & && \text{unrestricted in sign} \end{aligned} \tag{2.2}$$

Analogous to Definition 1.1, we call $x_i - x_j \leq -1$ in problem P_T an "up" constraint and $x_i - x_j - k_{ij} \leq 0$ a "side" constraint. There are two side constraints for each pair (i, j) with $i < j$.

Proposition 2.2. *Problems T and P_T are equivalent. Thus the fractional total weak discrepancy $tw_F(P)$ of a poset P is the optimal value of the objective function of the linear program P_T.*

Proof. We will use vector notation to describe the solutions of T and P_T, writing $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{k} = (\ldots, k_{ij}, \ldots)^T$ for given values of the variables x_i, k_{ij}.

Let $\{\mathbf{x}, k\}$ be a feasible solution for T, and let $k_{ij} = |x_i - x_j|$ for all pairs (i, j) with $a_i \parallel a_j, i < j$. Then $\{\mathbf{x}, \mathbf{k}\}$ is a feasible solution for P_T and $\sum k_{ij} \leq k$. Moreover, if $\{\mathbf{x}, k\}$ is optimal for T then $\sum k_{ij} = k$, since otherwise we could reduce k in (2.1) to the value of $\sum k_{ij}$. We will show $\{\mathbf{x}, \mathbf{k}\}$ is optimal for P_T. If not, there is some $\{\mathbf{y}, \mathbf{l}\}$ that is feasible for P_T and has $l = \sum l_{ij} < k$. Then $\{\mathbf{y}, l\}$ is feasible for T, which contradicts the optimality of $\{\mathbf{x}, k\}$. So $\{\mathbf{x}, \mathbf{k}\}$ is optimal for P_T and the objective functions for the two problems have the same optimal value.

Conversely, let $\{\mathbf{x}, \mathbf{k}\}$ be feasible for P_T and let $k = \sum k_{ij}$. Then $\{\mathbf{x}, k\}$ is feasible for T. Suppose $\{\mathbf{x}, \mathbf{k}\}$ is optimal for P_T. If $\{\mathbf{x}, k\}$ is not optimal for T, then there is some feasible $\{\mathbf{y}, l\}$ with $l < k$. There is then a corresponding $\{\mathbf{y}, \mathbf{l}\}$ that is feasible for P_T with objective value $\sum l_{ij} \leq l < k$, a contradiction. So $\{\mathbf{x}, k\}$ is optimal for T and the two objective functions again have the same optimal value. □

It is helpful to write P_T in matrix form. We will need notation to describe the up and side constraints. Recall that $n = |V|$. Let $U = \{(a, b) : a \prec b\}$ and let $m = |U|$, the number of up constraints in P_T. Let $S = \{(a, b) : a \parallel b\}$. Since we are considering ordered pairs here, $|S|$ is even, $|S| = 2p$. Then p is the number of pairs of side constraints. When forming the matrices we list the up constraints and the pairs of side constraints lexicographically, as in Example 2.1. We then list the two side constraints in each pair consecutively.

Let $\mathbf{e_m} = (1, \ldots, 1)^T \in \mathbf{R}^m$. Let $\mathbf{x} \in \mathbf{R}^n$ and $\mathbf{k} \in \mathbf{R}^p$ be as in the proof of Proposition 2.2. Let A be the $m \times n$ coefficient matrix for $\mathbf{x}$ in the up constraints of P_T. Let B be the $2p \times n$ coefficient matrix for $\mathbf{x}$ in the side constraints. Let $-H$ be the $2p \times p$ coefficient matrix for $\mathbf{k}$ in the side constraints. Each column of H corresponds to a pair of side constraints and is all zero except for two consecutive ones, corresponding to the two constraints in the pair.

Then we can write the primal problem as follows, with $n + p$ variables and $m + 2p$ constraints:

$$\begin{pmatrix} A & O \\ B & -H \end{pmatrix} = \left(\begin{array}{cccccc|cccccccc}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right)$$

Figure 4: The coefficient matrix in problem P_T for the poset Y of Figure 2.

$$\begin{aligned} P_T : \min \mathbf{e_p} \cdot \mathbf{k} \\ A\mathbf{x} \leq -\mathbf{e_m} \\ B\mathbf{x} - H\mathbf{k} \leq \mathbf{0}_{2p} \\ \mathbf{x} \in \mathbf{R}^n, \mathbf{k} \in \mathbf{R}^p \text{ unrestricted} \end{aligned} \tag{2.3}$$

Example 2.3. We return to the example Y in Figure 2, again ignoring up constraints implied by transitivity. We have $n = 6, m = 4$ and $p = 8$. The matrix A is 4×6, B is 16×6, and H is 16×8. The constraint matrix for P_T is then 20×14 and is shown in Figure 4. □

We will now formulate the *dual* D_T. We temporarily define dual variables u'_{ij}, s'_{ij} and corresponding vectors $\mathbf{u'} \in \mathbf{R}^m, \mathbf{s'} \in \mathbf{R}^{2p}$, but will then rewrite the problem with new variables. Here the u'_{ij} correspond to the up constraints $a_i \prec a_j$ of the primal P_T, and the s'_{ij} correspond to the side constraints $a_i \parallel a_j$. The dual problem is

$$\begin{aligned} D_T : \max -\mathbf{e_m} \cdot \mathbf{u'} \\ A^T\mathbf{u'} + B^T\mathbf{s'} = \mathbf{0_n} \\ -H^T\mathbf{s'} = \mathbf{e}_p \\ \mathbf{u'} \leq \mathbf{0_m}, \mathbf{s'} \leq \mathbf{0_{2p}} \end{aligned} \tag{2.4}$$

It is convenient to set $\mathbf{u} = -\mathbf{u'}, \mathbf{s} = -\mathbf{s'}$ and rewrite the dual problem as

$$\begin{aligned} D_T : \max \mathbf{e_m} \cdot \mathbf{u} \\ A^T\mathbf{u} + B^T\mathbf{s} = \mathbf{0_n} \\ H^T\mathbf{s} = \mathbf{e}_p \\ \mathbf{u} \geq \mathbf{0_m}, \mathbf{s} \geq \mathbf{0_{2p}} \end{aligned} \tag{2.5}$$

We can interpret the dual (2.5) as a circulation problem in a network with additional constraints. Let $\vec{G}_P = (V, E)$ be the digraph with $E = U \cup S$. That is, there is an arc (a, b) for each of the m up constraints $a \prec b$ and for each of the $2p$ side constraints $a \parallel b$. The dual variables $\mathbf{u}, \mathbf{s}$ can be thought of as flows along the arcs of $\vec{G}_P$. The objective in the dual is to maximize the sum of the flows along the "up arcs".

Each row of A^T and of B^T corresponds to a node a of $\vec{G}_P$ (i.e., an element of V) and contains only the values $0, \pm 1$. The columns correspond to the arcs of $\vec{G}_P$, and the value is 1 for each arc leaving a and -1 for each arc entering a. So the first constraint in (2.5) states that flow is conserved at each node. Each of the p rows of H^T corresponds to a pair of "side arcs" $(a, b), (b, a)$ with $a \parallel b$. The second constraint states that the sum of the flows along the arcs in each such pair equals one. This is illustrated in Example 2.4.

Example 2.4. Consider the poset Y and the matrices in Example 2.3. The first row of $A^T\mathbf{u} + B^T\mathbf{s} = \mathbf{0_n}$ is

$$u_{12} + s_{14} - s_{41} + s_{15} - s_{51} + s_{16} - s_{61} = 0,$$

which says that the sum of the flows leaving node a_1 equals the sum of the flows entering it. One up arc and three side arcs leave a_1 and three side arcs enter it.

The first row of $H^T\mathbf{s} = \mathbf{e}_p$ is

$$s_{14} + s_{41} = 1,$$

which says that the flows along the pair of side arcs between nodes a_1 and a_4 add up to one. □

Since this circulation problem is the dual of the original problem P_T, an optimal solution also gives the value of the fractional total weak discrepancy $tw_F(P)$. That is, $tw_F(P)$ is the maximum sum of the flows along the up arcs of $\vec{G}_P$ taken over all circulations where the sum of the flows on *each pair* of side arcs equals one. By contrast, the fractional weak discrepancy $wd_F(P)$ is the maximum sum of the flows along the up arcs when the sum of the flows along *all* the side arcs equals one [8].

As Proposition 2.5 shows, when forming the digraph $\vec{G}_P$ we can safely ignore the up arcs implied by transitivity. That is, we take as up arcs the elements of $U_1 = \{(a, b) : a \prec\cdot b\}$, where $a \prec\cdot b$ means that b *covers* a, i.e., $a \prec b$ but there is no $c \in V$ with $a \prec c \prec b$. This justifies the way we formulated the up constraints in Example 2.1 and the matrix A in Example 2.3.

Proposition 2.5. *In any optimal solution to D_T, the flow is zero along each arc of $U \backslash U_1$. Thus, without loss of generality, we only need to consider up arcs (a, b) where $a \prec\cdot b$.*

Proof. Consider any feasible solution to D_T with flow $h > 0$ along an arc $(a, b) \in U \backslash U_1$ where $a \prec c \prec b$ for some node c. Set the flow to zero on (a, b) and add h to the flows on (a, c) and (c, b), leaving all other flows unchanged. The new flows satisfy the constraints in (2.5) with a larger value of the objective function, so the original solution was not optimal. □

We can get a great deal of insight into the nature of optimal solutions to the primal and dual problems by using the complementary slackness principle of linear programming. Recall that an inequality constraint is *binding* if its slack/surplus is zero, i.e., the solution satisfies the constraint with equality.

Complementary Slackness Principle (e.g., see [6]): Consider a pair of feasible solutions to a linear programming problem and its dual. Both solutions are optimal if and only if (i) whenever a primal variable is positive then the corresponding dual constraint is binding and (ii) whenever a dual variable is positive then the corresponding primal constraint is binding.

For the pair of problems in (2.3), (2.5) this gives the following result.

Proposition 2.6. Let $\{\mathbf{x}, \mathbf{k}\}$ be a feasible solution for P_T and let $\{\mathbf{u}, \mathbf{s}\}$ be a feasible solution for D_T. These solutions are optimal if and only if the following two conditions hold.

(a) Let $a_i \prec a_j$ and $u_{ij} > 0$. Then $x_j = x_i + 1$.

(b) Let $a_i \parallel a_j$. If $s_{ij} > 0$, then $x_i \geq x_j$.

Proof. Each dual constraint is an equation, so the complementary slackness conditions are always satisfied for the primal variables in any pair of feasible solutions. Each dual variable u_{ij} corresponds to an ordered pair (i, j) with $a_i \prec a_j$ and to the primal constraint $x_i - x_j \leq -1$. Each dual variable s_{ij} corresponds to (i, j) with $a_i \parallel a_j$ and to $x_i - x_j - k_{ij} \leq 0$.

Thus the complementary slackness conditions for the two problems reduce to the conditions

$$u_{ij}(x_i - x_j + 1) = 0, \qquad s_{ij}(x_i - x_j - k_{ij}) = 0.$$

Since $k_{ij} \geq 0$ in any feasible solution to the primal, the result follows from the Complementary Slackness Principle. □

So at optimality, if an up flow u_{ij} in the circulation (dual) problem is positive then the labels in the primal problem increase by exactly one, i.e., $x_j = x_i + 1$. If a side flow is positive then the labels are nonincreasing. It immediately follows that if the side flows in both directions between two nodes are positive then the node labels are equal. In particular, the following corollary holds.

Corollary 2.7. At optimality, if $s_{ij}, s_{ji} > 0$ then $x_i = x_j$.

Example 2.8. Figure 5 displays the poset Y with elements labeled as shown on the right in Figure 2, together with a feasible assignment of arc flows (only positive side arc flows are shown). In Example 2.1, we used the node labels to show that $tw_F(Y) \leq tw(Y) \leq 3$. By duality, the arc flows show that $tw_F(Y) \geq 3$, the sum of the up flows. This verifies that the label and flow assignments are optimal and that $tw_F(Y) = tw(Y) = 3$. Observe that these assignments have the properties of Proposition 2.6 and Corollary 2.7. □

3 Fractional total weak discrepancy equals total weak discrepancy

We can now prove that $tw_F(P)$ is always an integer and thus equals $tw(P)$. This is true even though there are optimal labelings, i.e., optimal solutions to P_T, that are not integer labelings (adding any constant to an optimal labeling gives another optimal labeling). In general, the coefficient matrix $\left(\begin{smallmatrix} A & O \\ B & -H \end{smallmatrix}\right)$ is not totally unimodular since it contains the submatrix $\left(\begin{smallmatrix} 1 & -1 \\ -1 & -1 \end{smallmatrix}\right)$. For instance, in Example 2.3 we can form this submatrix from the first column and first two rows of B and $-H$.

We will proceed as follows. First, we will use linear programming to find an optimal solution to the dual problem D_T. We will use this solution, which is a circulation, to assign integer node labels $x_i = f(a_i)$ that are feasible for the primal P_T and satisfy the complementary slackness conditions. It will then follow from Proposition 2.6 that this label assignment f is optimal.

Theorem 3.1. *The fractional total weak discrepancy $tw_F(P)$ of any poset P can always be achieved with integer labels. Thus $tw_F(P)$ is an integer and equals $tw(P)$.*

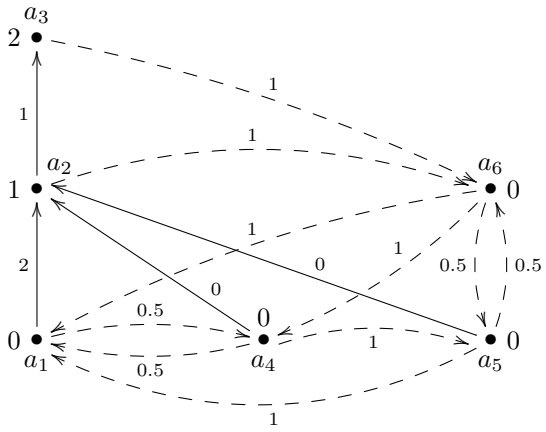

Figure 5: The poset Y with node labels and arc flows. Side flows are shown with dashes and only side arcs with positive flow are shown. Example 2.8 shows the labels and flows are optimal and that $tw_F(Y) = tw(Y) = 3$.

Proof. We first show the dual problem D_T has an optimal solution. It has a feasible solution, since we can set all the up flows equal to 0 and all the side flows equal to $1/2$. Each feasible solution is a circulation in the digraph $\vec{G}_P = (V, E)$ formed from $P = (V, \prec)$ and can be decomposed into at most $|E|$ cycle flows (e.g., see [1]). Since each side flow lies in $[0, 1]$, the flow in each cycle is bounded by one and so the sum of the up flows is bounded by $|E|^2$. Thus the dual problem is bounded and has an optimal solution $\{\mathbf{u}, \mathbf{s}\}$. As a result, the primal is also feasible (this can also be shown using a linear extension of P).

Without loss of generality, we can assume the poset P is inseparable, i.e., it is not the lexicographic sum of nonempty subsets. If it were separable, we could partition P into inseparable components [13] and apply the argument that follows to each component.

We now present a polynomial-time algorithm for defining a labeling f of the elements of P that is optimal for the primal P_T and has *integer* values. We choose a minimal element $a_1 \in V$ and set the interval $I(a_1) = [0, 0]$. The algorithm begins by initializing the remaining ranges $I(a) = [l(a), r(a)]$ into which the labels will fall, and then progressively narrows these ranges. After no more narrowing is possible, we will define the labels by $f(a) = l(a)$ (or alternatively, by $f(a) = r(a)$), for all $a \in V$. In particular, if $I(a)$ is a singleton at any stage, then $f(a)$ will be this value. Our initialization implies $f(a_1) = 0$, but by translating all the values we can set $f(a_1)$ arbitrarily. The algorithm is based on the k-weak leveling algorithm of [13], which runs in polynomial time.

Recall that Propositions 2.5 and 2.6 and Corollary 2.7 tell us how any optimal values x_i for the primal problem must relate to each other. We use these relations to initialize the ranges as follows, so as to guarantee that each $x_i \in I(a_i)$. Figure 6 shows how to make the initial assignment for the poset Y that appears in Figures 2 and 5.

- $I(a_1) = [0, 0]$.
- If $a_1 \prec a_i$, using Propositions 2.5 and 2.6(a) we set

$$I(a_i) = \begin{cases} [1,1], & \text{if } u_{1i} > 0 \\ [1,\infty), & \text{if } u_{1i} = 0. \end{cases} \tag{3.1}$$

- If $a_1 \parallel a_i$, using Proposition 2.6(b) and Corollary 2.7 we set

$$I(a_i) = \begin{cases} (-\infty,0], & \text{if } s_{1i} = 1,\ s_{i1} = 0 \\ [0,\infty), & \text{if } s_{1i} = 0,\ s_{i1} = 1 \\ [0,0], & \text{if } s_{1i},\ s_{i1} > 0. \end{cases} \tag{3.2}$$

Let M be the $(n-1) \times (n-1)$ matrix with rows and columns indexed by the elements of V other than a_1, and having ones down the diagonal and zeroes elsewhere. We denote the entries of M by M_{ab}. We will use this matrix to keep track of which $I(a)$ need to be narrowed. Since M is symmetric we could only consider entries above or below the main diagonal, but that would complicate the notation.

We repeatedly apply the following steps to the ranges $I(a) = [l(a), r(a)]$, again using Proposition 2.6 and Corollary 2.7 to maintain complementary slackness for the relations considered at each point. We only apply steps that narrow the ranges, and continue until no further narrowing is possible.

Choose distinct elements a and b in $V \backslash a_1$ with $M_{ab} = 0$. If one or more of the following situations occurs when we consider these elements as a_i, a_j in some order, then we take the steps indicated. If $a \parallel b$ and both side flows between them are positive, we apply (3.5) both as written and with a_i, a_j interchanged. Figure 7 illustrates the effect of several of these steps when we begin with the initial ranges shown in Figure 6.

- If $a_i \prec a_j$ and

$$\begin{aligned} l(a_j) \le l(a_i), &\quad \text{then increase } \ l(a_j) \text{ to } l(a_i) + 1 \\ r(a_i) \ge r(a_j), &\quad \text{then decrease } r(a_i) \text{ to } r(a_j) - 1. \end{aligned} \tag{3.3}$$

- If $a_i \prec a_j,\ u_{ij} > 0$ and

$$\begin{aligned} l(a_i) < l(a_j) - 1, &\quad \text{then increase } \ l(a_i) \text{ to } l(a_j) - 1 \\ r(a_j) > r(a_i) + 1, &\quad \text{then decrease } r(a_j) \text{ to } r(a_i) + 1. \end{aligned} \tag{3.4}$$

- If $a_i \parallel a_j,\ s_{ij} > 0$, and

$$\begin{aligned} l(a_i) < l(a_j), &\quad \text{then increase } \ l(a_i) \text{ to } l(a_j) \\ r(a_i) < r(a_j), &\quad \text{then decrease } r(a_j) \text{ to } r(a_i). \end{aligned} \tag{3.5}$$

Reasoning again that the conditions in Proposition 2.6 and Corollary 2.7 are necessary, we conclude that after a range $I(a)$ is narrowed, it still must contain $f(a)$ for any optimal labeling f for which $f(a_1) = 0$. Since the primal P_T is feasible, the ranges are nonempty (i.e., $l(a) \le r(a)$ in all cases).

Now set $M_{ab} = M_{ba} = 1$. If $I(a)$ (respectively, $I(b)$) is bounded at the end of this step and was narrowed during it, set $M_{ac} = M_{ca} = 0$ ($M_{bc} = M_{cb} = 0$) for all $c \neq a, b$. Note

that in these steps, we will never increase an endpoint to ∞ or decrease it to $-\infty$, and so at most one endpoint of each range can be $\pm\infty$.

Now repeat the procedure if possible. Once all entries of M equal one, no further narrowing steps can be executed. We then set $f(a) = l(a)$ for each $a \in V$. Since ones in M can only be changed back to zeroes for bounded intervals that have been narrowed, only a finite number of changes are possible and the algorithm must terminate. We will show below that the ranges are all bounded after the last step, so that f is well-defined.

After every step, the endpoint of each range is either an integer or $\pm\infty$. The construction shows that once further narrowing is impossible, the complementary slackness conditions must hold for all up and side relations involving elements whose ranges are bounded. So it remains to prove that after the last step, the ranges are all bounded. Then since the conditions in Proposition 2.6 are sufficient, we can conclude the integer labeling f is optimal.

Case (a). For a contradiction, suppose that some range $I(a_i) = (-\infty, r(a_i)]$ after the last step. Then $-\infty$ must have been the left endpoint of this range initially, so $a_1 \parallel a_i$ and $s_{1i} = 1, s_{i1} = 0$. Again using the decomposition of a circulation into cycle flows, we see there must be a directed cycle in $\vec{G}_P$ that contains (a_1, a_i) and has positive flow $0 < h \leq 1$ along each of its arcs. Let the next arc in the cycle be (a_i, a_j), for some a_j. This is either an up arc with $u_{ij} \geq h > 0$ or a side arc with $s_{ij} \geq h > 0$. Since no further narrowing is possible, conditions (3.4) and (3.5) show that $l(a_j) = -\infty$. Continuing around the cycle until we reach a_1, we arrive at a contradiction since $l(a_1) = 0$. Thus each range is bounded below.

Case (b). For a contradiction, suppose some $I(a_i) = [l(a_i), \infty)$ after the last step. We partition V into sets $L = \{y \in V : r(y) < \infty\}$ and $R = \{z \in V : r(z) = \infty\}$. These are nonempty, since $a_1 \in L, a_i \in R$. Now take arbitrary elements $y \in L$, $z \in R$. We will prove that $y \prec z$ and then use our assumption that P is inseparable to reach a contradiction.

Suppose $z \prec y$. Either $r(y)$ was finite initially or became finite at some step when $I(y)$ was narrowed. In either case, we would have applied (3.3) to (z, y) when $r(y)$ was finite and thus made $r(z)$ finite. So $z \not\prec y$.

Now suppose $y \parallel z$. If the flow from y to z is positive, applying (3.5) with $a_i = y, a_j = z$ when $r(y) < \infty$ would have made $r(z) < \infty$, so this cannot occur. Thus the flow from z to y equals one. We can reason similarly to Case (a) to conclude there is a directed cycle containing (z, y) with positive flow h along each of its arcs. Let the next arc in the cycle be (y, a_j) for some a_j. By (3.4) and (3.5), $r(a_j) < \infty$ so $a_j \in L$. Continuing around the cycle until we arrive at z, we contradict $z \in R$.

Thus $y \prec z$ for all $y \in L$ and $z \in R$. But this is impossible since P is inseparable. So each range is bounded above.

So after the last step of the algorithm, each range is a bounded interval. We choose the label $f(a)$ to be either the left or right endpoint of the range, making the same choice for each a. The resulting labeling consists of integers and is feasible for the primal P_T. By complementary slackness, the labeling is optimal. The reasoning in [13] shows the algorithm runs in $O(|V|^4)$ time. □

Example 3.2. Figure 6 shows the initial ranges that the algorithm prescribes for the node labels of the poset Y in Examples 2.1 and 2.8, based on the solution given for the dual in

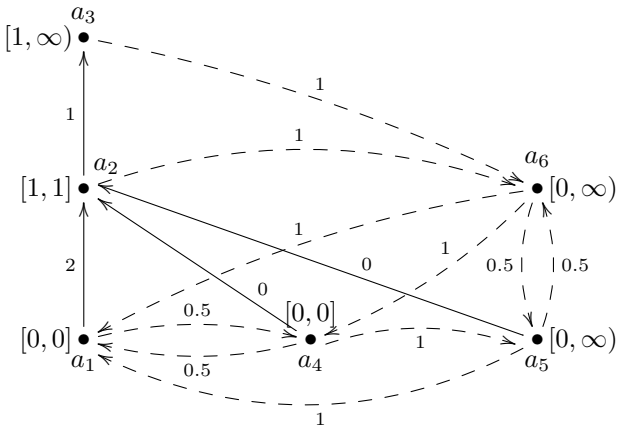

Figure 6: The poset Y with initial ranges for node labels set by choosing a_1 as a minimal element and applying (3.1), (3.2) to the optimal arc flows of Figure 5.

Figure 5. Figure 7 shows the ranges after applying the algorithm to the pairs $\{a_2, a_i\}, i = 3, \ldots, 6$. For example, we apply (3.3) to change $I(a_3)$ from $[1, \infty)$ to $[2, \infty)$ and then apply (3.4) to change it again to $[2, 2]$. Similarly, we apply (3.3) to narrow $I(a_5)$ to $[0, 0]$ and apply (3.5) to narrow $I(a_6)$ to $[0, 1]$. At a later stage, we consider $\{a_5, a_6\}$ and narrow $I(a_6)$ further to $[0, 0]$. The optimal labeling this produces is the one on the right side of Figure 2. □

Example 3.3. In Example 3.2 the ranges all reduce to singletons, but this is not always the case. We do not show the steps for the poset Z of Figure 1, but some of the ranges reduce to intervals with positive length. In particular, $I(a_3) = [-1, 0], I(a_4) = [0, 1]$, and $I(a_5) = [1, 2]$. Choosing $f(a) = r(a)$ in all cases produces the integer labeling on the right side of Figure 1. So this labeling is optimal for both weak and total weak discrepancy, and $tw(Z) = 9$. We could also have reached this conclusion by finding a feasible assignment of flows for which the sum of the up flows equals 9. □

4 Open questions

In the examples presented here and others we have studied, all basic feasible solutions to the dual problem D_T (not only the optimal ones) have had integer flows on the "up" arcs and flows equal to either 0, 1, or $1/2$ on the "side" arcs. Feasible solutions with other flows have turned out to be non-basic, i.e., not extreme points of the constraint polyhedron. We ask whether what we have observed is true in general.

Must every basic feasible solution to the dual problem be a circulation with integer up flows and side flows equal to $0, 1,$ *or* $1/2$?

The labeling functions that are optimal for the total linear discrepancy of a poset are characterized in [2] and [5]. We can pose a similar problem for total weak discrepancy.

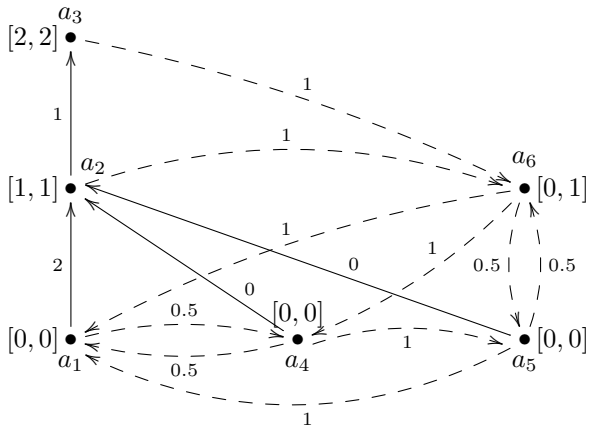

Figure 7: The poset Y with ranges after applying (3.3), (3.4), (3.5) to the pairs $\{a_2, a_i\}, i = 3, \ldots, 6$. When the algorithm terminates, the ranges reduce to the labels on the right side of Figure 2.

Which labelings of a poset are optimal for total weak discrepancy?

In [7]–[11], we find the range of the fractional weak discrepancy function for semiorders, interval orders, and split semiorders. We can ask analogous questions for total weak discrepancy.

What is the range of the total weak discrepancy function for various classes of posets?

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