



Mathematical aspects of phase rotation ambiguities in partial wave analyses

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Abstract. The observables in a single-channel 2-body scattering problem remain invariant once the amplitude is multiplied by an overall energy- and angle-dependent phase. This invariance is known as the continuum ambiguity. Also, mostly in truncated partial wave analyses (TPWAs), discrete ambiguities originating from complex conjugation of roots are known to occur. In this note, it is shown that the general continuum ambiguity mixes partial waves and that for scalar particles, discrete ambiguities are just a subset of continuum ambiguities with a specific phase. A numerical method is outlined briefly, which can determine the relevant connecting phases.

1 Introduction

We assume the well-known partial wave decomposition of the amplitude $A(W, \theta)$ for a $2 \rightarrow 2$ -scattering process of spinless particles

$$A(W, \theta) = \sum_{\ell=0}^{\infty} (2\ell + 1) A_{\ell}(W) P_{\ell}(\cos \theta). \quad (1)$$

The *data* out of which partial waves shall be extracted are given by the differential cross section, which is (ignoring phase-space factors)

$$\sigma_0(W, \theta) = |A(W, \theta)|^2. \quad (2)$$

Making a *complete experiment analysis* [1] for this simple example, we see that the cross section constrains the amplitude to a circle for each energy and angle: $|A(W, \theta)| = \sqrt{\sigma_0(W, \theta)}$. Thus, one energy- and angle-dependent phase is in principle unknown when based on data alone. The other side of the medal in this case is given by the fact that the amplitude itself can be rotated by an arbitrary energy- and angle-dependent phase and the cross section does not change. This invariance is known as the *continuum ambiguity* [2]:

$$A(W, \theta) \rightarrow \tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta). \quad (3)$$

Another concept known in the literature on partial wave analyses is that of so-called *discrete ambiguities* [2–4]. Suppose the full amplitude $A(W, \theta)$ can be split

into a product of a linear-factor of the angular variable, for instance $\cos \theta$, and a remainder-amplitude $\hat{A}(W, \theta)$ [3]:

$$A(W, \theta) = \hat{A}(W, \theta) (\cos \theta - \alpha). \quad (4)$$

This is generally the case whenever the amplitude is a polynomial (i.e. the series (1) is truncated), but it may also be possible for infinite partial wave models. Then, it is seen quickly that the cross section (2) is invariant under complex conjugation of the root α , which causes the discrete ambiguity

$$\alpha \longrightarrow \alpha^*. \quad (5)$$

Figure 1 shows a schematic illustration of the meaning of the terms *continuum*-vs. *discrete ambiguities*. In this proceeding, the purely mathematical mechanisms (3) and (5) are investigated. Of course, constraints from physics may reduce the amount of ambiguity encountered. For instance, unitarity is a very powerful constraint which, for elastic scatterings, leaves only one remaining non-trivial so-called *Crichton*-ambiguity [5]. This is believed to be true independent of any truncation-order L of the partial wave expansion [2]. However, in energy-regimes where the scattering becomes inelastic, so-called *islands of ambiguity* are known to exist [6].

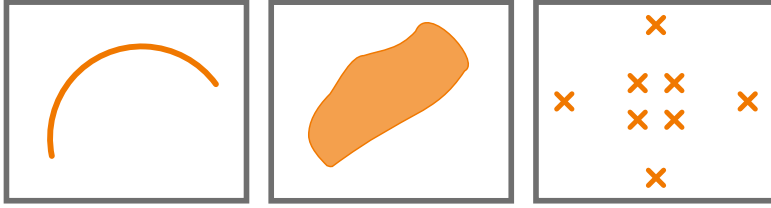


Fig. 1. Three schematic pictures are shown in order to distinguish the terms *discrete*- and *continuum* ambiguities. The grey colored box depicts in each case the higher-dimensional parameter-space composed by the partial wave amplitudes, be it for infinite partial wave models, or for truncated ones.

Left: One-dimensional (for instance circular) arcs can be traced out by continuum ambiguity transformations, both for infinite and truncated models.

Center: Connected continua in amplitude space, containing an infinite number of points with identical cross section, can be generated by use of angle-dependent rotations (3) (however, only in case the partial wave series goes to infinity). The connected patches are also called *islands of ambiguity* [2,6].

Right: Discrete ambiguities refer to cases where the cross section is the same for discretely located points in amplitude space. These ambiguities are most prominent in TPWAs [2,4]. However, two-fold discrete ambiguities can also appear for infinite partial wave models, once elastic unitarity is valid [2].

These figures have been published in reference [8].

Although here we focus just on the scalar example, ambiguities have become a topic of interest in the quest for so-called *complete experiments* in reactions with spin, for instance photoproduction of pseudoscalar mesons [1,7].

This proceeding is a briefer version of the more detailed publication [8]. The arXiv-reference [9] also treats very similar issues, as does the contribution of Alfred Švarc to these proceedings.

2 The effect of continuum ambiguity transformations on partial wave decompositions

We let the general transformation (3) act on $A(W, \theta)$ and assume a partial wave decomposition for the original as well as the rotated amplitude

$$\begin{aligned} A(W, \theta) &\longrightarrow \tilde{A}(W, \theta) = e^{i\Phi(W, \theta)} A(W, \theta) = e^{i\Phi(W, \theta)} \sum_{\ell=0}^{\infty} (2\ell+1) A_{\ell}(W) P_{\ell}(\cos \theta) \\ &\equiv \sum_{\ell=0}^{\infty} (2\ell+1) \tilde{A}_{\ell}(W) P_{\ell}(\cos \theta). \end{aligned} \quad (6)$$

Out of the infinitely many possibilities to parametrize the angular dependence of the phase-rotation, the convenient choice of a Legendre-series is employed

$$e^{i\Phi(W, \theta)} = \sum_{k=0}^{\infty} L_k(W) P_k(\cos \theta). \quad (7)$$

In case this form of the rotation is inserted into the partial wave projection integrals of the general rotated waves \tilde{A}_{ℓ} (cf. equation (6)), the following *mixing formula* emerges [10]

$$\tilde{A}_{\ell}(W) = \sum_{k=0}^{\infty} L_k(W) \sum_{m=|k-\ell|}^{k+\ell} \langle k, 0; \ell, 0 | m, 0 \rangle^2 A_m(W). \quad (8)$$

Here, $\langle j_1, m_1; j_2, m_2 | J, M \rangle$ is just a usual Glebsch-Gordan coefficient.

Some more remarks should be made on the formula (8): first of all, although it's derivation is not difficult, this author has (at least up to this point) not found this expression in the literature, at least in this particular form. However, mixing-phenomena have been pointed out for πN -scattering [11] and for photoproduction [12].

Secondly, it can be seen quickly from the mixing formula that for angle-independent phases, i.e. when only the coefficient L_0 survives in the parametrization (7) of the rotation-functions, partial waves do not mix. Rather, in this case each partial wave is multiplied by $L_0(W) = e^{i\Phi(W)}$. However, once the phase $\Phi(W, \theta)$ carries even a weak angle-dependence, the expansion (7) directly becomes infinite and thus introduces contributions to an infinite partial wave set via the mixing-formula. There may be (a lot of) cases where the series (7) converges quickly and in these instances, it is safe to truncate the infinite equation-system (8) at some point.

It has to be stated that the mixing under very general continuum ambiguity transformations may lead to the mis-identification of resonance quantum numbers (reference [9] illustrates this fact on a toy-model example).

3 Discrete ambiguities as continuum ambiguity transformations

In case of a polynomial-amplitude, i.e. a truncation of the infinite series (1) at some finite cutoff L , the amplitude decomposes into a product of linear factors [4]

$$A(W, \theta) = \sum_{\ell=0}^L (2\ell + 1) A_{\ell}(W) P_{\ell}(\cos \theta) \equiv \lambda \prod_{i=1}^L (\cos \theta - \alpha_i), \quad (9)$$

with a complex normalization proportional to the highest wave $\lambda \propto A_L(W)$. In case of a TPWA, one energy-dependent overall phase has to be fixed. This could be done, for instance, by choosing λ real and positive: $\lambda = |\lambda|$. Sometimes it is also customary to fix the phase of the S -wave.

Gersten [4] showed that discrete ambiguities in the TPWA can occur in case subsets of the roots $\{\alpha_i\}$ are complex conjugated. All combinatorial possibilities can be parametrized by a set of mappings π_p , the number of which rises exponentially with L :

$$\pi_p(\alpha_i) := \begin{cases} \alpha_i & , \mu_i(p) = 0 \\ \alpha_i^* & , \mu_i(p) = 1 \end{cases}, \quad p = \sum_{i=1}^L \mu_i(p) 2^{(i-1)}, \quad p = 0, \dots, (2^L - 1). \quad (10)$$

In case these maps are applied, they yield a set of 2^L polynomial-amplitudes, which all have identical cross section:

$$A^{(p)}(W, \theta) = \lambda \prod_{i=1}^L (\cos \theta - \pi_p[\alpha_i]) \equiv \sum_{\ell=0}^L (2\ell + 1) A_{\ell}^{(p)}(W) P_{\ell}(\cos \theta). \quad (11)$$

Since σ_0 is invariant under the discrete Gersten-ambiguities, these transformations can effectively only be rotations (because of $|A| = \sqrt{\sigma_0}$). More precisely, because one overall phase is fixed for all partial waves, discrete ambiguities can only be angle-dependent rotations. The corresponding rotation-functions are just fractions of two polynomial amplitudes

$$e^{i\varphi_p(W, \theta)} = \frac{A^{(p)}(W, \theta)}{A(W, \theta)} = \frac{(\cos \theta - \pi_p[\alpha_1]) \dots (\cos \theta - \pi_p[\alpha_L])}{(\cos \theta - \alpha_1) \dots (\cos \theta - \alpha_L)}. \quad (12)$$

Therefore, discrete ambiguities mix partial waves, just as the general continuum ambiguities do. Furthermore, the expression on the right-hand-side of (12) is explicitly an infinite series in $\cos \theta$. Thus, one may expect an infinite tower of rotated partial waves \tilde{A}_{ℓ} to be non-vanishing upon consideration of the mixing-formula (8). However, in this case of course the rotation fine-tunes exact cancellations in the results of the mixing for all higher partial waves $\tilde{A}_{\ell > L}$.

Furthermore, Gersten [4] claims (without proof) that the root-conjugations *exhaust all possibilities* for discrete ambiguities of the TPWA. We have to state that we believe him.

The remainder of this proceeding is used to outline a numerical method that is *orthogonal* to the Gersten-formalism, but which can also substantiate this claim.

4 Functional minimizations show exhaustiveness of Gersten-ambiguities

We use the notation $x = \cos \theta$, introduce the complex rotation function $F(W, x) := e^{i\Phi(W, x)}$ and from now on drop the explicit energy W . The proposed numerical method assumes a truncated full amplitude $A(x)$ as a known input. Then, all possible functions $F(x)$ are scanned numerically for only those that satisfy the following two conditions:

(I) The complex solution-function $F(x)$ has to have modulus 1 for each value of x .

$$|F(x)|^2 = 1, \quad \forall x \in [-1, 1]. \quad (13)$$

(II) The rotated amplitude $\tilde{A}(x)$, coming out of an amplitude $A(x)$ truncated at L , has to be truncated as well, i.e.

$$\tilde{A}_{L+k} = 0, \quad \forall k = 1, \dots, \infty. \quad (14)$$

Formally, this *scanning*-procedure can be implemented by minimizing a suitably defined *functional* of $F(x)$:

$$\begin{aligned} \mathcal{W}[F(x)] := & \sum_x \left(\operatorname{Re}[F(x)]^2 + \operatorname{Im}[F(x)]^2 - 1 \right)^2 \\ & + \operatorname{Im} \left[\frac{1}{2} \int_{-1}^{+1} dx F(x) A(x) \right]^2 + \sum_{k \geq 1} \left\{ \operatorname{Re} \left[\frac{1}{2} \int_{-1}^{+1} dx F(x) A(x) P_{L+k}(x) \right]^2 \right. \\ & \left. + \operatorname{Im} \left[\frac{1}{2} \int_{-1}^{+1} dx F(x) A(x) P_{L+k}(x) \right]^2 \right\} \longrightarrow \min. \end{aligned} \quad (15)$$

Here, the first term ensures the unimodularity of $F(x)$ (i.e. condition (I)), the second fixes a phase-convention on the S -wave \tilde{A}_0 and the big sum over k sets all higher partial wave of the rotated amplitude to zero.

It has to be clear that for practical numerical applications, the sums over k and x have to be finite, i.e. the former is cut off and the latter is defined on a grid of x -values. Also, a general function $F(x)$ is defined by an infinite amount of real degrees of freedom, which has to be made finite as well.

This can be achieved for instance by using a finite Legendre-expansion, i.e. a truncated version of equation (7) (with possibly large cutoff \mathcal{L}_{cut}), or by discretizing $F(x)$ on a finite grid of points $\{x_n\} \in [-1, 1]$. More details on the numerical minimizations can be found in reference [8].

The only non-redundant solutions of this procedure are, in the end, the Gersten-rotation functions (12). Figures 2 to 5 illustrate this fact for the simple toy-model [8] (partial waves given in arbitrary units):

$$\begin{aligned} A(x) &= \sum_{\ell=0}^2 (2\ell+1) A_\ell P_\ell(x) = A_0 + 3A_1 P_1(x) + 5A_2 P_2(x) \\ &= 5 + 3(0.4 + 0.3i)x + \frac{5}{2}(0.02 + 0.01i)(3x^2 - 1). \end{aligned} \quad (16)$$

$$\underline{e^{i\varphi_0(x)}}$$

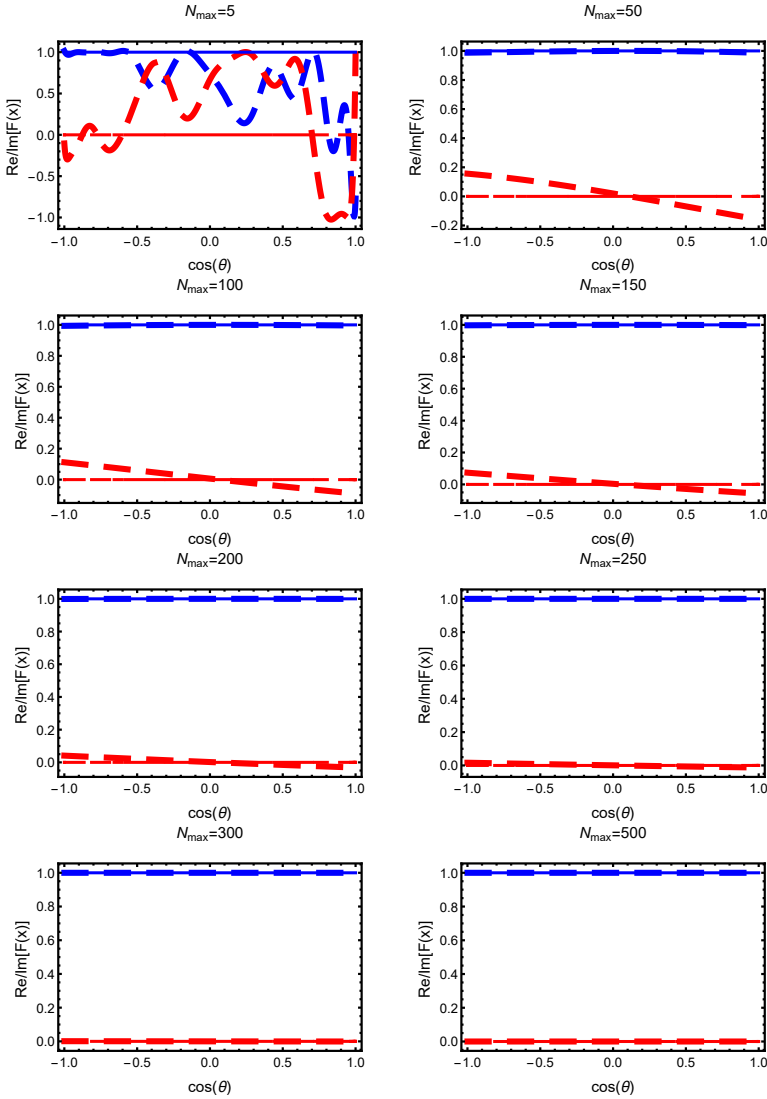


Fig. 2. (Color online) The convergence of the functional minimization procedure is illustrated in these plots. For the discrete ambiguities $e^{i\varphi_0(x)}$ and $e^{i\varphi_1(x)}$ of the toy-model (16), two randomly drawn initial functions have been chosen from an applied ensemble of initial conditions in the search. These initial conditions then converged to these two respective Gersten-rotations. Results are shown for different values of the maximal number of iterations N_{\max} of the minimizer, as indicated. Numbers range from $N_{\max} = 5$ up to $N_{\max} = 500$. In all plots, the real- and imaginary parts of the precise Gersten-ambiguity are drawn as blue and red solid lines. The results of the functional minimizations up to N_{\max} are drawn as thick dashed lines, having the same color-coding for real- and imaginary parts (color online). These figures have already been published in reference [8].

$$\underline{e^{i\varphi_1(x)}}$$

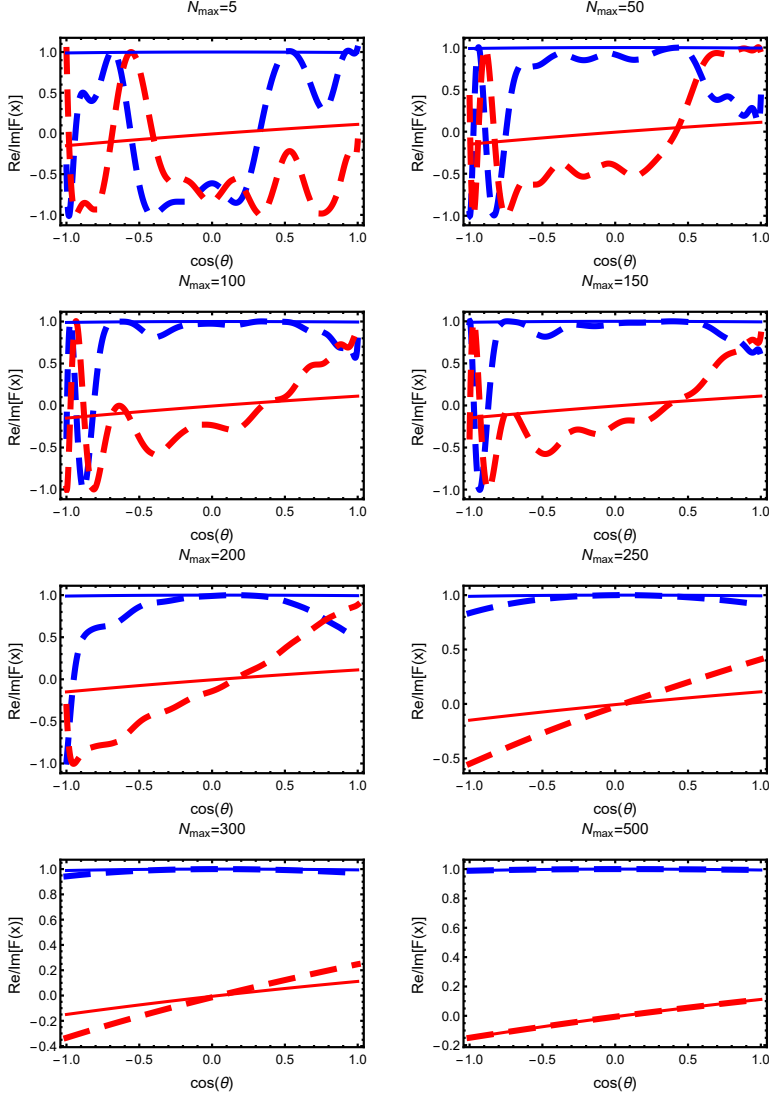


Fig. 3. These plots are the continuation of Figure 2. The convergence of the numerical minimization of the functional (15) is shown for the phase $e^{i\varphi_1(x)}$, which generates discrete ambiguities of the toy-model (16).

$$\underline{e^{i\varphi_2(x)}}$$

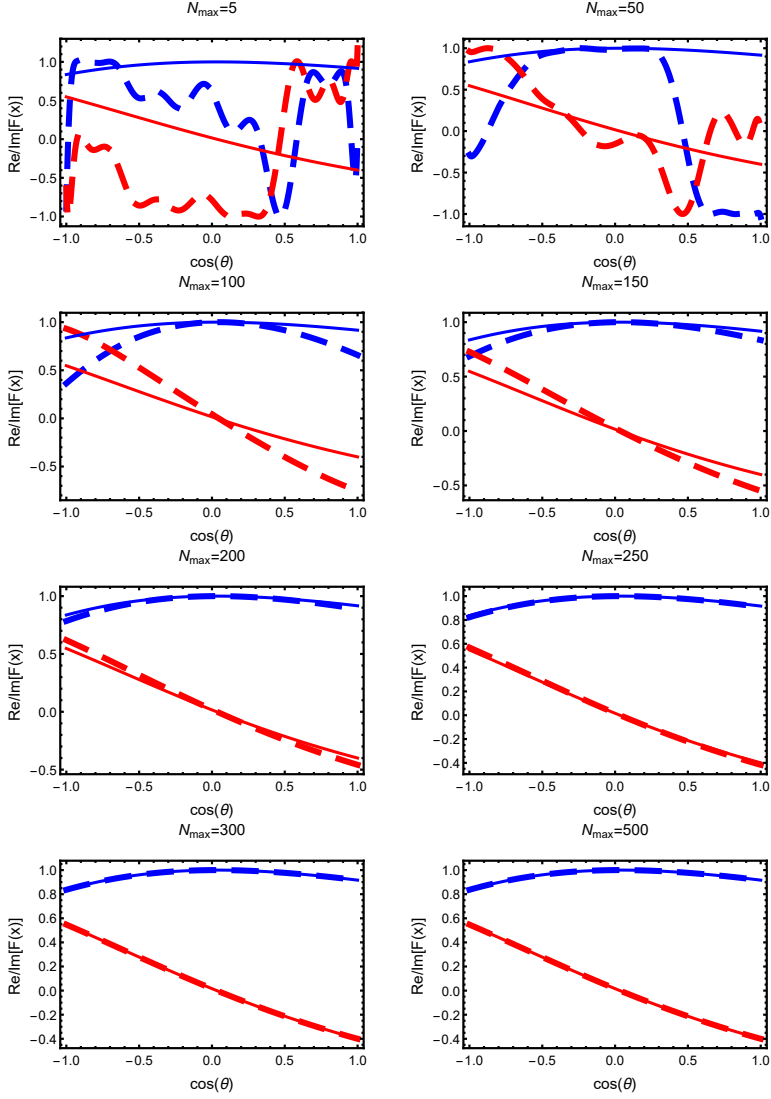


Fig. 4. Same as before, but for the phase $e^{i\varphi_2(x)}$.

$$\underline{e^{i\varphi_3(x)}}$$

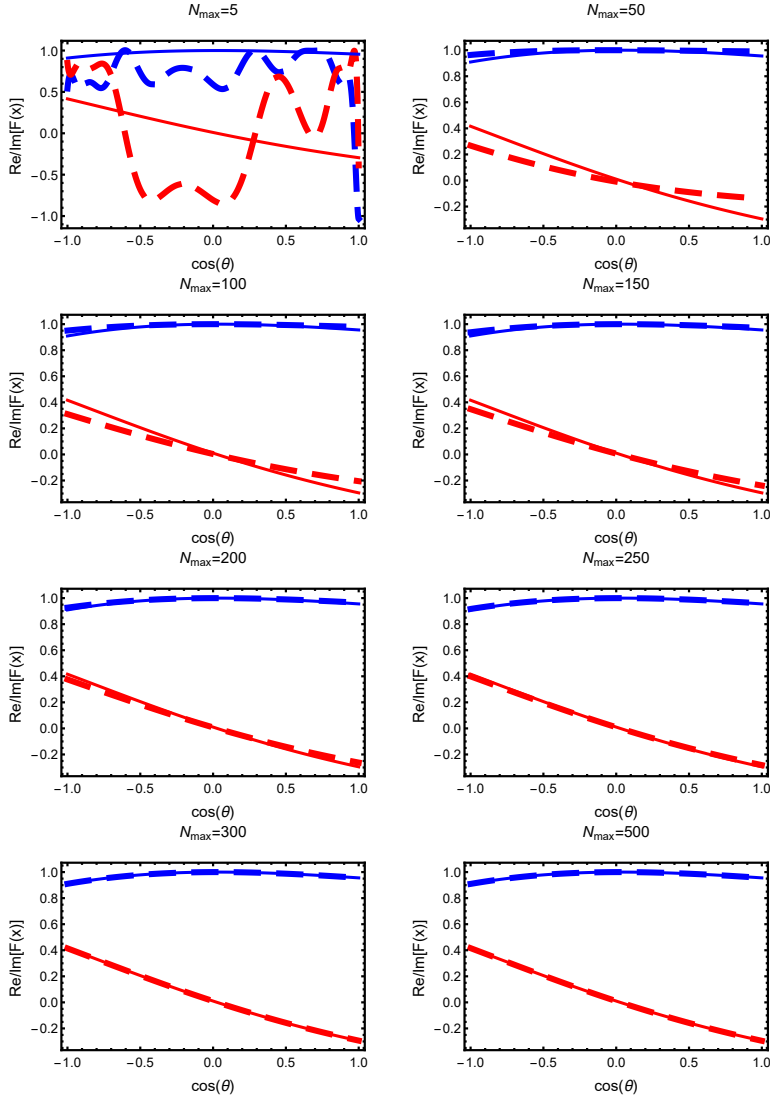


Fig. 5. Same as before, but for the phase $e^{i\varphi_3(x)}$.

This model is truncated at $L = 2$. Thus it has two roots (α_1, α_2) and $2^2 = 4$ Gersten-ambiguities. The latter are generated by four phase-rotation functions: $e^{i\varphi_0(x)} = 1$, $e^{i\varphi_1(x)}$, $e^{i\varphi_2(x)}$ and $e^{i\varphi_3(x)}$. Figures 2 to 5 demonstrate the convergence-process of the functional minimization towards a particular Gersten-rotation, for very general initial functions. The fact that always one of the four Gersten-rotations is found is *independent* of the choice of the initial function.

5 Conclusions & Outlook

We have seen that general continuum ambiguity transformations, as well as discrete Gersten-ambiguities, are in the end manifestations of the same thing: angle-dependent phase-rotations. Therefore, they both mix partial waves.

The rotations belonging to the Gersten-symmetries have the following defining property: they are the only rotations which, if applied to an original truncated model, leave the truncation order L untouched. In order to demonstrate this fact, a (possibly) new numerical method has been outlined capable of determining all continuum ambiguity transformations satisfying pre-defined constraints.

A possible further avenue of research may consist off the generalization of these findings to reactions with spin, for instance pseudoscalar meson photoproduction. Here, the massive amount of new polarization data gathered over the last years have renewed interest in questions of the uniqueness of partial wave decompositions. However, once one transitions to the case with spin, some open issues still exist, as have already been discussed during the workshop.

Acknowledgments

The author (again, as in 2015) wishes to thank the organizers for the hospitality, as well as for providing a very relaxed and friendly atmosphere during the workshop.

This particular Bled-workshop takes a special place in this author's biography, since after 4 months of battle with a very bad knee-injury, the participation in the workshop marked one of the first careful steps back into the world. Furthermore, the wonderful nature and environment of Bled itself turned out to be instrumental on the way of healing. By making the room on the ground floor of the Villa Plemelj available, the organizers have provided a key to make participation possible at all, and the author wishes to express deep gratitude for that. The author's wife also wishes to thank the organizers for the possibility to stay in Bled, as well as the nice hikes she made with the other participant's spouses. In fact, one early morning she was very brave and made a balloon ride over the lake of Bled. This author decided to include one of her aerial photographs into the proceeding.

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10. One just has to use the re-expansion

$$\begin{aligned}
 P_k(x)P_\ell(x) &= \sum_{m=|k-\ell|}^{k+\ell} \begin{pmatrix} k & \ell & m \\ 0 & 0 & 0 \end{pmatrix}^2 (2m+1) P_m(x) \\
 &= \sum_{m=|k-\ell|}^{k+\ell} \langle k, 0; \ell, 0 | m, 0 \rangle^2 P_m(x).
 \end{aligned} \tag{17}$$

For the first equality, see the reference:

W. J. Thompson, *Angular Momentum*, John Wiley & Sons (2008).

The second equality uses a well-known relation between 3j-symbols and Clebsch-Gordan coefficients.

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