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The $2A$-Majorana representations of the Harada-Norton group

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Abstract

We show that all $2A$-Majorana representations of the Harada-Norton group F_5 have the same shape. If $\mathcal{R}$ is such a representation, we determine, using the theory of association schemes, the dimension and the irreducible constituents of the linear span U of the Majorana axes. Finally, we prove that, if $\mathcal{R}$ is based on the (unique) embedding of F_5 in the Monster, U is closed under the algebra product.

Keywords: Majorana representations, association schemes, Monster algebra, Harada-Norton group.

Math. Subj. Class.: 20D08, 20C34, 05E30, 17B69.

1 Introduction

Let $(W, \cdot)$ be a real commutative algebra endowed with a scalar product $(\,,\,)_W$ and denote with $Aut(W)$ the group of algebra automorphisms of W that preserve the scalar product. We shall assume that, for every $u, v, w \in W$,

(M1) $(\,,\,)_W$ is *associative* , that is $(u \cdot v, w) = (u, v \cdot w)$,

(M2) the *Norton Inequality*, $(u \cdot u, v \cdot v) \geq (u \cdot v, u \cdot v)$, holds.

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Recall that a *Majorana axis* of W (see [10, Definition 8.6.1] or, equivalently, [9, p. 2423]) is a vector $a \in W$ such that

(M3) a has length 1,

(M4) the adjoint endomorphism $ad(a)$, induced by multiplication by a on the $\mathbb{R}$-vector space W, is semisimple with spectrum contained in $\{1, 0, 2^{-2}, 2^{-5}\}$,

(M5) a spans linearly the eigenspace relative to the eigenvalue 1 of $ad(a)$,

(M6) the linear transformation $a^\tau : W \to W$, that inverts the eigenvectors of $ad(a)$ relative to 2^{-5} and centralises the other eigenvectors, preserves the algebra product,

(M7) the linear transformation $a^\sigma : C_W(a^\tau) \to C_W(a^\tau)$, that inverts the eigenvectors of $ad(a)$ relative to 2^{-2} and centralises the other eigenvectors contained in $C_W(a^\tau)$, preserves the restriction to $C_W(a^\tau)$ of the algebra product.

Denote with $\mathcal{A}$ the set of Majorana axes of W. If $a \in \mathcal{A}$, the map a^τ is called a *Majorana involution* corresponding to a. Note that, by (M1) and (M4), W decomposes into an orthogonal sum of $ad(a)$-eigenspaces, hence (M6) actually implies that every Majorana involution is an element of $Aut(W)$. Let

$$\tau : \mathcal{A} \to Aut(W)$$

be the map $a \mapsto a^\tau$. Note that $\mathcal{A}$ is invariant under $Aut(W)$ and, for $a \in \mathcal{A}$ and $\delta \in Aut(W)$, we have

$$(a^\delta)^\tau = \delta^{-1} a^\tau \delta,$$

so that the set $\mathcal{A}^\tau$ of Majorana involutions is invariant under conjugation by elements of $Aut(W)$.

The fundamental examples of Majorana involutions are given by the $2A_M$-involutions (i.e. those centralised by the double cover of the Baby Monster) of the Monster group M acting on the 196884-dimensional Conway-Norton-Griess algebra W_M. A key result, in this context, is the Norton-Sakuma Theorem, that classifies and describes the *Norton-Sakuma algebras*, i.e. the algebras that are generated by a pair of Majorana axes [19] (see also [9, Section 2.6]). By S. Sakuma's classification, every Norton-Sakuma algebra is isomorphic to a subalgebra of W_M generated by a pair of Majorana axes a_0, a_1 corresponding via τ to $2A_M$-involutions in M. In [17] S. Norton proved that the latter algebras (hence all Norton-Sakuma algebras) fall into nine isomorphism types, labelled $1A$, $2A$, $2B$, $3A$, $3C$, $4A$, $4B$, $5A$, and $6A$, accordingly to the conjugacy class in the Monster of the element $a_0^\tau a_1^\tau$. Further, Norton produced, for each type, a basis (the *Norton basis*), the relative structure constants and the Gram matrix. Table 1 (which is an extract from Table 3 in [9]) summarises the results from the Norton-Sakuma Theorem we need for this paper: more precisely, for each pair of distinct Majorana axes a_0, a_1, we give the Norton basis of the algebra generated by a_0 and a_1, and the relevant (for this paper) scalar products (with the same scaling as in [9]):

Here, for $\rho := a_0^\tau a_1^\tau$ in each Norton-Sakuma algebra,

- $a_{-1} := a_1^{\rho^{-1}}$, $a_{-2} := a_0^{\rho^{-1}}$, $a_2 := a_0^\rho$, $a_3 := a_1^\rho$, in particular they are Majorana axes.

Table 1: Norton bases and relevant scalar products for the Norton-Sakuma algebras.

Type	Norton basis	Scalar Products
$2A$	a_0, a_1, a_ρ	$(a_0, a_1)_W = \frac{1}{2^3}$
$2B$	a_0, a_1	$(a_0, a_1)_W = 0$
$3A$	a_0, a_1, a_{-1}, u_ρ	$(a_0, a_1)_W = \frac{13}{2^8}$
$3C$	a_0, a_1, a_{-1}	$(a_0, a_1)_W = \frac{1}{2^6}$
$4A$	$a_0, a_1, a_{-1}, a_2, v_\rho$	$(a_0, a_1)_W = \frac{1}{2^5}$
$4B$	$a_0, a_1, a_{-1}, a_2, a_{\rho^2}$	$(a_0, a_1)_W = \frac{1}{2^6}$
$5A$	$a_0, a_1, a_{-1}, a_2, a_{-2}, w_\rho$	$(a_0, a_1)_W = \frac{3}{2^7}$
$6A$	$a_0, a_1, a_{-1}, a_2, a_{-2}, a_3, a_{\rho^3}, u_{\rho^2}$	$(a_0, a_1)_W = \frac{5}{2^8}$

- The vectors u_ρ, v_ρ, $\pm w_\rho$, resp. u_{ρ^2}, appearing in the algebras of type $3A$, $4A$, $5A$, resp. $6A$, are called $3A$-, $4A$-, $5A$-, resp. $3A$-, *axes* and, in each Norton-Sakuma algebra, they are defined as follows,

$$\begin{aligned}
u_\rho &:= \tfrac{2^6}{3^3 5}(2a_0 + 2a_1 + a_{-1}) - \tfrac{2^{11}}{3^3 5} a_0 \cdot a_1, \\
v_\rho &:= a_0 + a_1 + \tfrac{1}{3}(a_{-1} + a_{-1}) - \tfrac{2^6}{3} a_0 \cdot a_1, \\
w_\rho &:= -\tfrac{1}{2^7}(3a_0 + 3a_1 - a_{-1} - a_{-1} - a_{-2}) + a_0 \cdot a_1, \\
u_{\rho^2} &:= \tfrac{2^6}{3^3 5}(2a_0 + 2a_{-1} + a_{-2}) - \tfrac{2^{11}}{3^3 5} a_0 \cdot a_{-1}.
\end{aligned}$$

  The indexing with powers of ρ is justified by the fact that, in the action of M on W_M, for $3 \leq N \leq 5$, the NA-axes are essentially determined (up to the sign in the $5A$-case) by the cyclic groups $\langle \rho \rangle$ in M of order N (see [9, p. 2450]). It is not clear if that property follows from Axioms (M1)-(M7), therefore axiom (M8)(b) was added in [3] in the definition of Majorana representations.

- The vectors a_ρ, a_{ρ^2}, resp. a_{ρ^3} appearing in the algebras of type $2A$-, $4B$-, resp. $6A$ are further Majorana axes. As above, the indexing is suggested by the action of M on W_M since, in that case, whenever a_0 and a_1 generate a subalgebra of type $2A$, the product $\rho = a_0^\tau a_1^\tau$ is the Majorana involution corresponding to a_ρ. As in the previous paragraph, that property will be axiomatised in (M8)(a). Finally, by the Norton-Sakuma Theorem (see [9, Lemma 2.20 (iv) and (v)]), a_0 and a_2 (resp. a_0 and a_3) generate a subalgebra of type $2A$ in the algebra of type $4B$ (resp. $6A$) and, for $i \in \{2, 3\}$, by the definition of a_i, the product $a_0^\tau a_i^\tau$ is equal to ρ^i.

The Norton-Sakuma Theorem inspired the definition of Majorana representations, introduced by A. A. Ivanov in [10] in order to provide an axiomtic framework for studying the actions of $2A_M$-generated subgroups of M on W_M.

Let G be a finite group, $\mathcal{T}$ a G-invariant set of involutions generating G,

$$\phi\colon G \to Aut(W)$$

a faithful representation of G on W, and

$$\psi\colon \mathcal{T} \to \mathcal{A}$$

be an injective map such that for every $g \in G$ and $t \in \mathcal{T}$,

$$(t^{\psi})^{\tau} := t^{\phi} \tag{1.1}$$

and

$$(t^{\psi})^{g^{\phi}} = (g^{-1}tg)^{\psi}. \tag{1.2}$$

The quintet

$$\mathcal{R} := (G, \mathcal{T}, W, \phi, \psi)$$

is called a *Majorana representation* (or, to put evidence on the set $\mathcal{T}$, a $\mathcal{T}$-*Majorana representation*) of G, if $\mathcal{R}$ satisfies the following condition (see [3, Axiom **M8**]):

(M8) (a) For t_1 and t_2 in $\mathcal{T}$, the Norton-Sakuma algebra generated by t_1^{ψ} and t_2^{ψ} has type $2A$ if and only if $t_1t_2 \in \mathcal{T}$.

(b) Suppose that t_1, t_2, t_3, and t_4 are elements of $\mathcal{T}$ such that $t_1t_2 = t_3t_4$ and the subalgebras generated by t_1^{ψ}, t_2^{ψ} and t_3^{ψ}, t_4^{ψ} have both type $3A$, $4A$, or $5A$. Then $u_{(t_1t_2)^{\phi}} = u_{(t_3t_4)^{\phi}}$, $v_{(t_1t_2)^{\phi}} = v_{(t_3t_4)^{\phi}}$, or $w_{(t_1t_2)^{\phi}} = w_{(t_3t_4)^{\phi}}$, respectively.

Axiom (M8)(a) and Norton-Sakuma Theorem (see [9, Lemma 2.20]) imply that,

> if t_1^{ψ} and t_2^{ψ} generate a Norton-Sakuma subalgebra of W of type $2A$, $4B$, or $6A$, then t_1t_2, $(t_1t_2)^2$, or $(t_1t_3)^3$ belongs to $\mathcal{T}$, and $(t_1t_2)^{\psi}$, $((t_1t_2)^2)^{\psi}$, or $((t_1t_3)^3)^{\psi}$ coincides with $a_{(t_1t_2)^{\phi}}$, $a_{((t_1t_2)^2)^{\phi}}$, or $a_{((t_1t_2)^3)^{\phi}}$, respectively.

An immediate consequence of that definition is that, given a Majorana representation

$$\mathcal{R} := (G, \mathcal{T}, W, \phi, \psi)$$

of a group G and a nonempty subset $\mathcal{T}_0$ of $\mathcal{T}$, such that $\mathcal{T}_0$ is $\langle \mathcal{T}_0 \rangle$-invariant, the quintet

$$\mathcal{R}_{\langle \mathcal{T}_0 \rangle} := (\langle \mathcal{T}_0 \rangle, \mathcal{T}_0, W, \phi|_{\langle \mathcal{T}_0 \rangle}, \psi|_{\mathcal{T}_0}) \tag{1.3}$$

is a $\mathcal{T}_0$-Majorana representation of $\langle \mathcal{T}_0 \rangle$. Further, if we replace W with the subalgebra $W_{\mathcal{T}_0}$ generated by the set of Majorana axes $\mathcal{T}_0^{\psi}$ in the quintet (1.3), we still have a Majorana representation of $\langle \mathcal{T}_0 \rangle$ provided $\langle \mathcal{T}_0 \rangle$ acts nontrivially on $W_{\mathcal{T}_0}$ (which is the case, e.g., when $\langle \mathcal{T}_0 \rangle$ has trivial centre). In particular, if ϵ is an embedding of a group H in M and H^{ϵ} is generated by a subset $\mathcal{T}$ of $2A_M$, then H inherits a $(\mathcal{T} \cap H^{\epsilon})^{\epsilon^{-1}}$-Majorana representation $\mathcal{R}_{\epsilon}$ obtained by composing ϵ with the restriction of $\mathcal{R}_M$ to H^{ϵ}. In that case, the Majorana representation $\mathcal{R}_{\epsilon}$ of H is said to be *based on the embedding* ϵ. In this paper, whenever a Majorana representation of a group G is based on an embedding ϵ in the Monster, we shall always identify G with G^{ϵ}.

For a pair (a, b) of elements in W, denote the subalgebra they generate with $\langle\langle a, b \rangle\rangle$. Let $\mathcal{R}$ be as above, the *shape* of $\mathcal{R}$ is a function $sh_{\mathcal{R}}$ from the set of the nondiagonal orbitals of G on $\mathcal{T}$ to the set of types of the Norton-Sakuma algebras so that

1. $sh_{\mathcal{R}}((t,s)^G) = NX$ if and only if ts has order N and the algebra $\langle\langle t^{\psi}, s^{\psi}\rangle\rangle$ is a Norton-Sakuma algebra of type NX.

2. $sh_{\mathcal{R}}$ must respect the embeddings of the algebras:

$$2A \hookrightarrow 4B, 2A \hookrightarrow 6A, 2B \hookrightarrow 4A, 3A \hookrightarrow 6A$$

in the sense that, for $t, r_1, r_2 \in \mathcal{T}$, if $\langle\langle t^{\psi}\rangle\rangle < \langle\langle t^{\psi}, r_1^{\psi}\rangle\rangle < \langle\langle t^{\psi}, r_2^{\psi}\rangle\rangle$, then

$$(sh_{\mathcal{R}}((t,r_1)^G),\ sh_{\mathcal{R}}((t,r_2)^G)) \in \{(2A,4B),(2A,6A),(2B,4A),(3A,6A)\}.$$

Remark: Clearly, if $\mathcal{T}_0$ is a $\langle\mathcal{T}_0\rangle$-invariant nonempty subset of $\mathcal{T}$, the shape of $\mathcal{R}_{\langle\mathcal{T}_0\rangle}$ is the restriction of $sh_{\mathcal{R}}$ to $\mathcal{T}_0 \times \mathcal{T}_0$.

Majorana representations of several groups have already been investigated (see [9, 11, 12, 13, 14, 5, 3, 6]).

In this paper we study the $2A$-Majorana representations of the Harada-Norton group F_5, where $2A$ is the set of the involutions of F_5 whose centraliser is $(2HS)\cdot 2$, the double cover of the Higman-Sims group extended by its outer automorphism group of order 2. We shall show that every $2A$-Majorana representation of F_5 has the same shape as the Majorana representations of F_5 based on its embedding into M as the subgroup generated by the set of involutions in $2A_M$ that centralise an element of type $5A$ (here $2A = 2A_M \cap F_5$, see [4]). By [18, Theorem 21], that one is the unique embedding of F_5 into M (up to conjugation in M), hence, since F_5 is transitive on $2A$, there is (up to conjugation in M) only one Majorana representation of F_5 based on an embedding in M. We prove the following result.

Theorem 1.1. *Let W be as above and $\mathcal{R} := (F_5, 2A, W, \phi, \psi)$ be a $2A$-Majorana representation of F_5 on W. Then*

(i) $\mathcal{R}$ has the shape given in Table 3;

(ii) The $\mathbb{R}$-linear span $\langle 2A^{\psi}\rangle$ of $2A^{\psi}$ has dimension $18\,316$*;*

(iii) $\langle 2A^{\psi}\rangle$ is the direct sum of three irreducible $\mathbb{R}[F_5]$-submodules of dimensions 1, 8910 *and* 9405*, respectively;*

(iv) if $\mathcal{R}$ is based on the embedding of F_5 in M, then $W_{2A} = \langle 2A^{\psi}\rangle$.

Unless explicitly stated, for the remainder of this paper we shall stick to the notations introduced in this section. We shall also set $\mathcal{T} := 2A$.

2 The First Eigenmatrix

By [4, p. 166], we have $|\mathcal{T}| = 1539000$, and it seems hard, at present, to perform a direct computation of the dimension of the linear span of $\mathcal{T}^{\psi}$. We therefore apply the theory of association schemes as in [14] and [6] to reduce ourselves to a more manageable situation. The first step is to compute the first eigenmatrix of the association scheme relative to the permutation action of F_5 on $\mathcal{T}$ (see [1, pp. 59-60]). For that purpose, we need to recover some information about the action F_5 induces by conjugation on $\mathcal{T}$.

Let $n := |\mathcal{T}|$ and let $t_1, \ldots, t_n$ be the distinct elements of $\mathcal{T}$, so that

$$\mathcal{B} := (t_1, \ldots, t_n)$$

is an ordered basis for the complex permutation module V of F_5 on $\mathcal{T}$. With respect to $\mathcal{B}$, we identify $End_{\mathbb{C}}(V)$ with the set of $n \times n$ matrices with complex entries. Let $T_0, \ldots, T_8$ be the orbitals of F_5 on $\mathcal{T}$ and, for every $k \in \{0, \ldots, 8\}$, let A_k be the *adjacency matrix* associated to the orbital T_k, that is

$$(A_k)_{ij} = \begin{cases} 1 & \text{if the pair } (t_i, t_j) \text{ is in } T_k \\ 0 & \text{otherwise.} \end{cases}$$

By [1, Theorem 1.3], the 9-tuple $(A_0, \ldots, A_8)$ is a basis for the *centralizer algebra*

$$\mathcal{C} := End_{\mathbb{C}[F_5]}(V).$$

For $i, j, k \in \{0, \ldots, 8\}$, let p_{ij}^k be the number of elements z in $\mathcal{T}$ such that for a fixed pair (x, y) in T_k we have $(x, z) \in T_i$ and $(z, y) \in T_j$. By definition, the p_{ij}^k's are all non negative integers and, by [1, §2.2], they are the structure constants of $\mathcal{C}$ relative to the basis $(A_0, \ldots, A_8)$, that is

$$A_i A_j = \sum_{k=0}^{8} p_{ij}^k A_k. \tag{2.1}$$

The matrix B_i of size 9 whose j, k entry is p_{ij}^k is called *ith intersection matrix*. Clearly, B_i^t is the matrix associated to the endomorphism induced by A_i on $\mathcal{C}$ via left multiplication with respect to the basis $(A_0, \ldots, A_8)$, in particular B_i has the same eigenvalues as A_i. By [8, Lemma 2.18.1*(ii)*] we may choose the indexes of the orbitals $T_0, \ldots, T_8$ in such a way that T_0 is the diagonal orbital (hence B_0 is the identity matrix), T_1 is the non-diagonal orbital of smallest size, and the first intersection matrix B_1 is as follows:

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1408 & 53 & 32 & 18 & 4 & 2 & 0 & 0 & 0 \\ 0 & 50 & 0 & 2 & 12 & 0 & 2 & 0 & 0 \\ 0 & 450 & 32 & 100 & 32 & 50 & 32 & 0 & 0 \\ 0 & 350 & 672 & 112 & 160 & 100 & 92 & 160 & 0 \\ 0 & 504 & 0 & 504 & 288 & 356 & 312 & 320 & 0 \\ 0 & 0 & 672 & 672 & 552 & 650 & 720 & 640 & 1280 \\ 0 & 0 & 0 & 0 & 360 & 250 & 240 & 288 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 128 \end{pmatrix}.$$

By [1, Theorem 3.1], we have that V decomposes into the direct sum

$$V = V_0 \oplus \ldots \oplus V_8 \tag{2.2}$$

of nine irreducible $\mathbb{C}[F_5]$-submodules. Since F_5 is transitive on $\mathcal{T}$, the subspace linearly spanned by the sum of all elements of $\mathcal{T}$ is the unique trivial submodule of V. As usual, we shall denote it by V_0. Since the action of F_5 on $\mathcal{T}$ is multiplicity free (see [8, Lemma 2.18.1.(*ii*)]), the V_j's are minimal common eigenspaces for the adjacency matrices A_i. It follows that there is a complex invertible matrix D that simultaneously diagonalises the matrices A_i's. By the definition of the adjacency matrices, we have that, for each i, the sums (say k_i) of the entries in each row of the matrices A_i are constant, whence V_0 is a k_i-eigenspace for A_i, for each i.

For $i, j \in \{0, \ldots, 8\}$, let p_{ij} be the eigenvalue of A_j on V_i. The 9×9 matrix $P := (p_{ij})$ is called the *first eigenmatrix* of the association scheme $(\mathcal{T}, \{T_0, \ldots, T_8\})$.

Lemma 2.1. *With the above notations,*

$$P = \begin{pmatrix} 1 & 1408 & 2200 & 35200 & 123200 & 354816 & 739200 & 277200 & 5775 \\ 1 & 128 & 200 & 0 & 1600 & -2304 & 0 & 0 & 375 \\ 1 & 28 & -50 & -50 & -100 & 396 & -750 & 450 & 75 \\ 1 & 16 & 4 & -56 & -136 & -288 & 504 & 0 & -45 \\ 1 & -32 & 40 & -80 & 80 & 576 & -240 & -360 & 15 \\ 1 & -47 & -50 & 250 & 350 & -504 & 0 & 0 & 0 \\ 1 & -112 & 300 & 1000 & -2200 & -864 & -1800 & 3600 & 75 \\ 1 & 208 & -50 & 2200 & -2800 & 2016 & 4200 & -6300 & 525 \\ 1 & 208 & 100 & 1000 & 1400 & 2016 & -4200 & 0 & -525 \end{pmatrix}.$$

Proof. Note that, since A_0 is the identity matrix, $p_{i0} = 1$ for all i's. Straightforward computation shows that the eigenvalues of B_1 are 1408, 128, 28, 16, -32, -47, -112, 208, and 208, giving the first two columns of P. Set

$$(\lambda_0, \ldots, \lambda_8) = (1408, 128, 28, 16, -32, -47, -112, 208, 208).$$

For each $h \in \{0, \ldots, 8\}$, let $\mathcal{S}_h$ be the linear system

$$(B_1 - \lambda_h Id)\, {}^t(1, \lambda_h, x_2, \ldots, x_8) = 0 \tag{2.3}$$

in the indeterminates $x_2, \ldots, x_8$. Taking $i = 1$ in Equation (2.1) and multipling each term by D on the right and by D^{-1} on the left, we get

$$(D^{-1}A_1D)(D^{-1}A_jD) = \sum_{h=0}^{8} p_{1j}^{h}(D^{-1}A_hD). \tag{2.4}$$

Since the matrices $D^{-1}A_hD$ are diagonal with eigenvalues p_{kh} on the common eigenspaces V_k, for each $k \in \{0, \ldots, 8\}$, from Equation (2.4) we obtain that the relation

$$\lambda_k p_{kj} = \sum_{h=0}^{8} p_{1j}^{h} p_{kh} \tag{2.5}$$

holds for every $k \in \{0, \ldots, 8\}$. Note that the second member is the jth entry of the vector $B_1\, {}^t(1, \lambda_k, p_{k2}, \ldots, p_{k8})$, therefore Equation (2.5) implies that the 9-tuple

$$(1, \lambda_k, p_{k2}, \ldots, p_{k8})$$

is an eigenvector for B_1 relative to the eigenvalue λ_k, for every $k \in \{0, \ldots, 8\}$. Since, for $k \neq 7, 8$, the eigenvalue λ_k has multiplicity 1, it follows that the first seven rows of the matrix P can be obtained computing the unique solution $(p_{k2}, \ldots, p_{k8})$ of the system $\mathcal{S}_k$ for each $k \in \{1, \ldots, 6\}$.

We are now left with the last two rows of the matrix P, corresponding to the eigenvalue 208 of B_1. The set of solutions of the system $\mathcal{S}_7$,

$$(B_1 - 208 Id)\, {}^t(1, 208, x_2, \ldots, x_8) = 0,$$

is

$$\{(25 - \frac{x}{7}, 1600 + \frac{8x}{7}, -700 - 4x, 2016, 8x, -3150 - 6x, x) \mid \text{ where } x \in \mathbb{R}\}.$$

Therefore, for suitable $x, y \in \mathbb{R}$, we can write the last two rows of the matrix P as follows

$$1, 208, 25 - \frac{x}{7}, 1600 + \frac{8x}{7}, -700 - 4x, 2016, 8x, -3150 - 6x, x$$

$$1, 208, 25 - \frac{y}{7}, 1600 + \frac{8y}{7}, -700 - 4y, 2016, 8y, -3150 - 6y, y.$$

Set $m_i = \dim_{\mathbb{R}}(V_i)$. Then $m_0 = 1$ and, for $1 \leq i \leq 6$, m_i can be computed from the rows of P using the following formula (see [1, Theorem 4.1]):

$$m_i = \frac{n}{\sum_{j=0}^{8} k_j^{-1} p_{ij}^2}$$

from which we get $m_1 = 16929$, $m_2 = 267520$, $m_3 = 653125$, $m_4 = 365750$, $m_5 = 214016$, $m_6 = 8910$, whence

$$m_7 + m_8 = n - \sum_{i=0}^{6} m_i = 12749.$$

Comparing that value with the decomposition of the permutation module of F_5 on $\mathcal{T}$ into irreducible submodules given in [8, Lemma 2.18.1.(*ii*)], we obtain that, modulo interchanging the indices 7 and 8,

$$m_7 = 3344 \text{ and } m_8 = 9405.$$

By the Column Orthogonality Relation of the first eigenmatrix,

$$\sum_{k=0}^{8} m_k p_{ki} p_{kj} = n k_i \delta_{ij}$$

(see [1, Theorem 3.5]), applied with $(i, j) = (0, 8)$ and $(i, j) = (8, 8)$, we get the quadratic system

$$\begin{cases} 3344x + 9405y = -3182025 \\ 3344x^2 + 9405y^2 = 3513943125 \end{cases}$$

whose solutions are

$$(x, y) = (525, -525) \text{ or } (x, y) = (1575/61, 62475/61).$$

By [2, Theorem 3.5(b)], the matrices A_i's are symmetric, since, by [4], the Frobenius-Schur indices of the irreducible constituents of the permutation character of F_5 on $\mathcal{T}$ is $+1$ (and the action is multiplicity free). Thus, recalling that the p_{ij}^k's are all non negative integers, in order to determine which of the two solutions is the right one, we may use the formula

$$p_{ij}^h = \frac{1}{n k_h} tr(A_i A_j A_h) \tag{2.6}$$

(see [1, Theorem 3.6(ii)]). Since the trace is invariant by matrix conjugation, $tr(A_i A_j A_h)$ can be obtained by multiplying, entry-wise, the ith, jth, and hth columns of the matrix P and adding the entries of the resulting column. In that way, we get that the entries p_{2j}^k are integers only in the case when $(x, y) = (525, -525)$. □

3 The shape

We continue with the notations of the last section. The next lemma recalls some known facts about conjugacy classes in M and F_5 (see [16, 15]). For the remainder of this paper let H be the centraliser in M of an A_5-subgroup of type $(2A, 3A, 5A)$. By [15], we have that $H \cong A_{12}$ and we may w.l.o.g. assume that F_5 centralizes a $5A$-element in that A_5-subgroup, in particular $H \leq F_5$.

Lemma 3.1. *Denoting the conjugacy classes of M and F_5 as in [4], the correspondences between the conjugacy classes of the elements of order less or equal to 6 in M, F_5 and H are as in Table 2.*

Table 2: Correspondences between the conjugacy classes of the elements of order at most 6 in M, F_5, and H.

Conj. class in M	$2A$	$2B$	$3A$	$4A$	$4B$	$5A$	$6A$
Conj. class in F_5	$2A$	$2B$	$3A$	$4A$	$4B$	$5A$	$6A$
Cycle type in H	$2^2, 2^6$	2^4	$3, 3^2, 3^4$	4^2, $4^2 \cdot 2^2$	$4 \cdot 2$, $4 \cdot 2^2$	$5, 5^2$	$3 \cdot 2^2$, $6 \cdot 2^3$, $6^2, 3^2 \cdot 2^2$

Let $(t_1, \ldots, t_n)$ be as in the previous section. For $i, j \in \{1, \ldots n\}$, set

$$\gamma_{ij} := (t_i^\psi, t_j^\psi)_W.$$

Lemma 3.2. *If (t_i, t_j) and (t_h, t_k) belong to the same orbital of F_5 on $\mathcal{T}$, then $\gamma_{ij} = \gamma_{hk}$.*

Proof. That follows immediately from Equation (1.2) and the definition of γ_{ij}. □

Thus, we can set, for $k \in \{0, \ldots, 8\}$ and $(t, s) \in T_k$,

$$\gamma_k := (t^\psi, s^\psi)_W. \tag{3.1}$$

Lemma 3.3. *For every $x \in \{2^2, 3, 4 \cdot 2, 2^4, 5\}$ there are pairs of involutions of type 2^2 in A_{12} such that their product has cycle type x. Every element of cycle type $4^2 \cdot 2^2$ in A_{12} is the product of two elements of cycle type 2^6.*

Proof. That is an elementary computation (note that two elements of cycle type 2^6 whose product has cycle type $4^2 \cdot 2^2$ are explicitely given in the proof of Lemma 3.4). □

Lemma 3.4. *With the above notations, for every $k \in \{0, \ldots, 8\}$ and $(t, s) \in T_k$, the scalar products γ_k's are given in Table 3.*

Proof. The first two columns of Table 3 follow from Lemma 2.1. The correspondence that associates to each orbital T_k of F_5 on $\mathcal{T}$ the F_5-conjugacy class x_k of the products ts, where $(t, s) \in T_k$, has been determined by Segev in [20], giving the third column.

Table 3: Valencies, shapes, and scalar products related to the orbitals of F_5 on the set of its $2A$-involutions.

k	$\|t^{C_{F_5}(s)}\|$	$(st)^{F_5}$	$sh_{\mathcal{R}}(T_k)$	γ_k
0	1	1	–	1
1	1408	$5A$	$5A$	$3/2^7$
2	2200	$2A$	$2A$	$1/2^3$
3	35200	$3A$	$3A$	$13/2^8$
4	123200	$4B$	$4B$	$1/2^6$
5	354816	$5E$	$5A$	$3/2^7$
6	739200	$6A$	$6A$	$5/2^8$
7	277200	$4A$	$4A$	$1/2^5$
8	5775	$2B$	$2B$	0

Assume $sh_{\mathcal{R}}(T_k) = NX$, where $N \in \{1, \ldots, 6\}$ and $X \in \{A, B, C\}$. By the definition of shape, for $(t, s) \in T_k$, we have that $|st| = N$. In particular, for k equal to 1, 5 and 6, we have that $sh_{\mathcal{R}}(T_k)$ is equal to $5A$, $5A$, and $6A$, respectively.

Let $k \in \{2, 3, 4, 8\}$. By the second and third rows of Table 2 and Lemma 3.3 there are involutions s and t of cycle type 2^2 in $\mathcal{T} \cap H$ such that $st \in x_k$, whence, by the first and third columns of Table 3,

$$(s, t) \in T_k \cap (H \times H).$$

By the remark in the introduction, we have that

$$sh_{\mathcal{R}}(T_k) = sh_{\mathcal{R}_H}((s, t)^H),$$

whence Lemma 8 and Table 10 in [6] give the entry in the fourth column corresponding to k.

Assume now $k = 7$. Choose the elements

$$s = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12) \text{ and } t = (1, 3)(2, 4)(5, 7)(6, 9)(8, 11)(10, 12)$$

in H. Then st has cycle type $4^2{\cdot}2^2$. By Table 2, s and t are contained in $\mathcal{T}$ and $(st)^{F_5} = 4A$, hence, by the third column of Table 3, $(s, t) \in T_7$ and, by the Norton-Sakuma Theorem, $sh_{\mathcal{R}}(T_7) \in \{4A, 4B\}$. By Equation (1.2),

$$(t^{\psi})^{(ts)^{\phi}} = (t^{ts})^{\psi} = (t^s)^{\psi},$$

so we have that t^{ψ} and $(t^s)^{\psi}$ are contained in the subalgebra generated by t^{ψ} and s^{ψ}, which is $\langle s, t\rangle$-invariant. Since tt^s has cycle type 2^4, by Table 2 it belongs to the class $2B$ of F_5, whence, by the third column of Table 3, $(t, t^s) \in T_8$ and the subalgebra generated by $t^{\psi}, (t^s)^{\psi}$ is of type $2B$, by the previous paragraph. By the second condition of the definition of the shape, $sh_{\mathcal{R}}(T_7) = 4A$.

Finally, the last column follows from Table 1. □

4 Closure

Lemma 4.1. *Suppose that $\mathcal{R}$ is based on the embedding of F_5 in M. Then*

$$\langle \mathcal{T}^\psi \rangle = W_\mathcal{T}.$$

Proof. Let H be the subgroup of F_5 isomorphic to A_{12} defined as in the previous section. Let t, s be distinct elements of $\mathcal{T}$, set $\rho = (ts)^\phi$ and let N be the order of ρ. Let U be the Norton-Sakuma algebra generated by t^ψ and s^ψ, and let NX be its type. By Table 1, if NX is contained in $\{2A, 2B, 4B\}$, then U is linearly spanned by elements in $\mathcal{T}^\psi$, otherwise, by Lemma 3.4, $NX \in \{3A, 4A, 5A, 6A\}$ and U has a basis all of whose elements but the NX-axis are Majorana axes. Therefore, with the notations of Table 1, we may assume that $NX \in \{3A, 4A, 5A, 6A\}$ and show that, in all those cases, the NX-axes u_ρ, v_ρ, w_ρ, u_{ρ^2} are contained in $\langle \mathcal{T}^\psi \rangle$.

If ts has order $3, 4$, or 5, then, by Lemma 3.1, there is $g \in F_5$, depending on ts, such that ts is an element of cycle type respectively 3, $4^2 \cdot 2^2$, and 5 in H^g. By Lemma 3.3, there are elements t' and s' of cycle type 2^2 or 2^6 in H^g such that $ts = t's'$. By Lemma 3.1, $(t')^\psi$ and $(s')^\psi$ generate a Norton-Sakuma algebra of the same type as U, thus, by Axiom (M8)(b), we have that $u_\rho = u_{(t's')^\phi}$, $v_\rho = v_{(t's')^\phi}$, and $w_\rho = w_{(t's')^\phi}$, respectively.

Assume $NX = 3A$. By [3, Corollary 3.2], $u_{(t's')^\phi}$ is a linear combination of elements of $(\mathcal{T} \cap H^g)^\psi$ and we are done.

Similarly, assume $NX = 4A$ (resp. $NX = 5A$). By [3], second formula in the abstract, or Section 6 (resp. Lemma 5.1), we have that $v_{(t's')^\phi}$ (resp. $w_{(t's')^\phi}$) is a linear combination of elements in $(\mathcal{T} \cap H^g)^\psi$ and $3A$-axes, and we are done by the previous case.

Finally assume $NX = 6A$. Then, by the remarks after Table 1, u_{ρ^2} is a $3A$-axis and again we are done by the $3A$ case. $\square$

Note that in the previous proof we require that $\mathcal{R}$ is based on the embedding of F_5 in M only to deal with the case $4A$, all the other cases following from results of [3] that depend only on the shape of that representation of A_{12}.

5 Proof of Theorem 1.1

The first claim of Theorem 1.1 follows from Lemma 3.4 and the last is the content of Lemma 4.1. To prove the second and the third claims, let

$$\Gamma = (\gamma_{ij})$$

be the Gram matrix of $(\,,\,)_W$ associated to the n-tuple $(t_1^\psi, \ldots, t_n^\psi)$. By an elementary result on Euclidean spaces, we have that

$$\mathrm{rank}(\Gamma) = \dim_\mathbb{R}(\langle \mathrm{t}^\psi \mid \mathrm{t} \in \mathcal{T} \rangle). \tag{5.1}$$

Since $T_0, \ldots, T_8$ is a partition of $\mathcal{T} \times \mathcal{T}$ and, by Equation (3.1), $\gamma_k = \gamma_{ij}$, for $(t_i, t_j) \in T_k$, we have that

$$\Gamma = \sum_{k=0}^{8} \gamma_k A_k. \tag{5.2}$$

Let D be as in Section 2. From Equation (5.2) we get:

$$\overline{\Gamma} := D^{-1}\Gamma D = \sum_{k=0}^{8} \gamma_k D^{-1} A_k D, \tag{5.3}$$

where all the matrices $\overline{\Gamma}$, and $\overline{A}_k := D^{-1}A_k D$ for $k \in \{0, \dots, 8\}$, are diagonal. Now, clearly, the rank of Γ is equal to the rank of $\overline{\Gamma}$, hence (being $\overline{\Gamma}$ diagonal) to the number of nonzero entries of $\overline{\Gamma}$. By Lemma 3.4 (Table 3), Equation (5.3) becomes

$$\overline{\Gamma} = \overline{A}_0 + \frac{3}{2^7}\overline{A}_1 + \frac{1}{8}\overline{A}_2 + \frac{13}{2^8}\overline{A}_3 + \frac{1}{2^6}\overline{A}_4 + \frac{3}{2^7}\overline{A}_5 + \frac{5}{2^8}\overline{A}_6 + \frac{1}{2^5}\overline{A}_7 + 0\overline{A}_8,$$

which, by Lemma 2.1, gives the eigenvalues

$$70875/2, 0, 0, 0, 0, 0, 875/8, 0, 225/4$$

of $\overline{\Gamma}$ on the subspaces $V_0, \dots, V_8$, respectively. Hence

$$\dim_{\mathbb{R}}(\langle \mathcal{T}^{\psi} \rangle) = m_0 + m_6 + m_8 = 1 + 9405 + 8910 = 18\,316.$$

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