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The clone cover*

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Abstract

Each finite graph on n vertices determines a special $(n-1)$-fold covering graph that we call the *clone cover*. Several equivalent definitions and basic properties about this remarkable construction are presented. In particular, we show that for $k \geq 2$, the clone cover of a k-connected graph is k-connected, the clone cover of a planar graph is planar and the clone cover of a hamiltonian graph is hamiltonian. As for symmetry properties, in most cases we also understand the structure of the automorphism groups of these covers. A particularly nice property is that every automorphism of the base graph lifts to an automorphism of its clone cover. We also show that the covering projection from the clone cover onto its corresponding 2-connected base graph is never a regular covering, except when the base graph is a cycle.

Keywords: Covering projection, canonical cover, regular cover, automorphisms.

Math. Subj. Class.: 57M10, 20B25, 05E18, 05C75, 05C76

*Dedicated to Dragan Marušič at the occasion of his 60th birthday.

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1 Introduction

Some coverings are "natural" or *canonical* in the sense that they are determined by the graph itself. A typical example is the universal cover, which is a tree, usually, an infinite tree [6]. Another such example is the so-called canonical double cover, or the Kronecker cover. It can be described as the tensor product of the graph in question by K_2 [5]. And there is also the trivial cover with the identity mapping as the covering projection.

In this paper we describe another canonical cover, an $(n-1)$-fold covering of a graph on n vertices, called the *clone cover*.[1] The clone cover of a graph X will be denoted by $\mathrm{Clone}(X)$. We present four equivalent definitions and several basic properties of this canonical covering graph. An application in mathematical chemistry can be found in [3]. As usual in the theory of covering graphs we will always assume that X is connected.

First, we study graph-theoretical properties such as connectedness, genus, and hamiltonicity. It turns out that the clone cover of a planar graph is also planar, the clone cover of a k-connected graph for $k \geq 2$ is also k-connected, and the clone cover of a hamiltonian graph is also hamiltonian. All these properties are far from being guaranteed for general covering graphs. In the second part of the paper we study automorphisms of such covers. Each automorphism of a graph X lifts to an automorphism of $\mathrm{Clone}(X)$, and moreover, the automorphism group of X embeds isomorphically in the automorphism group of $\mathrm{Clone}(X)$. This also is not true for general coverings. In most cases the covering projection $\mathrm{Clone}(X) \to X$ is irregular, with trivial group of covering transformations. Finally, there is a natural quotient projection $\mathrm{Clone}(X) \to X$, called contraction, that is different from the covering projection. This enables us to determine the full automorphism group of $\mathrm{Clone}(X)$ for certain classes of 2-connected graphs X.

2 Preliminaries

In this section we review some basic definitions and elementary properties of covering graphs. The most frequent descriptions and constructions of coverings use *voltage graphs*. These were first introduced by Gross and Tucker and popularized in their classic text [4]. In this paper a slightly different but equivalent approach is taken, following [9]. There are two differences in the approaches. While [4] requires a choice of directions of edges in the base graph, the approach in [9] mantains the base graph as completely undirected. The other advantage of [9] is that the base graph may also be a pregraph, that is, a graph with pending semi-edges. Pregraphs, however, will not be used in this paper.

A *graph* X is a quadruple $X = (V, S, i, r)$ where V is a finite set of *vertices*, S is a finite set of *arcs*, i is a mapping $S \to V$, specifying the *initial vertex* of each arc, while the reversal involution $r : S \to S$ is an involution without fixed points. The *terminal vertex* of an arc is then specified by the mapping $t : S \to V$, $t(s) = i(r(s))$. An arc s and its reverse $r(s)$ form an *edge* with *endvertices* $i(s)$ and $i(r(s))$. Two vertices are *adjacent* if they are the endvertices of a common edge. If every edge of the graph has two distinct endvertices and no two edges have the same endvertices, the graph is *simple*. We consider only simple graphs, with at least one edge, to avoid trivialities.

We will use the following notation. The set of vertices of a graph X will be denoted by $V(X)$, the set of its arcs by $S(X)$, and the set of its edges by $E(X)$. In a simple graph every edge is uniquely determined by its endvertices. Therefore we will denote an edge with endvertices u and v by $\{u, v\}$. An arc with initial vertex u and terminal vertex v will

[1] Note that this construction was previously called TheCover.

be denoted by $[u, v]$, or more briefly by uv. The set of vertices, adjacent to a vertex v of X, will be denoted by $N(v)$.

Let X and Y be graphs. A mapping $p : Y \to X$ that takes vertices to vertices and arcs to arcs is called a *homomorphism* if $p(i(s)) = i(p(s))$ and $p(r(s)) = r(p(s))$ for every $s \in S$. A surjective homomorphism $p : Y \to X$ is called a *covering projection* if the set $N(v)$ is mapped bijectively onto the set $N(p(v))$, for each vertex $v \in V(Y)$. The graph X is usually referred to as the *base graph* and Y as the *covering graph*. We call $p^{-1}(u)$ the *fiber* over $u \in V(X)$, and $p^{-1}(s)$ the *fiber* over $s \in S(X)$. We will assume that X is connected. This implies that all the fibers are of the same size.

By [4, Theorem 2.4.5], every covering graph can be obtained as follows. Let L be a finite (labeling) set and X a finite connected simple graph. Let $\tau \colon X \to \mathrm{Sym}(L)$ be a *permutation voltage assignment* on X, defined by $\tau(s) \in \mathrm{Sym}(L)$ for each arc s in X, and satisfying the condition $\tau(r(s)) = \tau^{-1}(s)$. The graph X together with the assignment τ is called a *permutation voltage graph* (X, τ). To a permutation voltage graph (X, τ) we associate a *derived graph* $Y = \mathrm{Cov}_\tau(X)$, with vertex set $V(Y) = V(X) \times L$, arc set $S(Y) = S(X) \times L$, and mappings i, r satisfying

$$\begin{aligned} i(s,j) &= (i(s),j) \quad \text{for any } (s,j) \in S(Y) \text{ and} \\ r(s,j) &= (r(s),j^{\tau(s)}) \quad \text{for any } (s,j) \in S(Y). \end{aligned}$$

In other words, with each arc $uv \in S(X)$ and each $j \in L$ we associate the arc $(uv, j) = [(u, j), (v, j^{\tau(u,v)})]$ from $(u, j) \in V(Y)$ to $(v, j^{\tau(uv)}) \in V(Y)$. Note that its reverse arc is $(vu, j^{\tau(uv)}) = [(v, j^{\tau(uv)}), (u, j)]$, from $(v, j^{\tau(uv)})$ to (u, j). Hence these opposite arcs form an edge "over" the edge $\{u, v\}$. Therefore the graph Y is a covering graph over the base graph X, with the *natural* covering projection $p : Y \to X$ taking a vertex $(u, j) \in V(Y)$ to $u \in V(X)$ and an arc (uv, j) in Y to the arc uv in X.

3 Constructions

Let X be a connected graph on $n \geq 2$ vertices. We begin by constructing a canonical n-fold covering graph of X, where we use $V(X)$ as the labeling set L. Let $\tau \colon X \to \mathrm{Sym}(V(X))$ be a permutation voltage assignment on X defined by the transposition

$$\tau(uv) = (u, v) \in \mathrm{Sym}(V(X))$$

for each arc uv in X. The associated covering graph is denoted by $\mathrm{Cov}(X)$. The vertex set of $\mathrm{Cov}(X)$ is $V(X) \times V(X)$ while $E(X) \times V(X)$ is the edge set. The edge set can be naturally partitioned into three subsets, namely, the subset of

- diagonal edges $\{(u, u), (v, v)\}$,
- connecting edges $\{(u, v), (v, u)\}$, $u \neq v$, and
- inner edges $\{(u, w), (v, w)\}$, $w \neq u, v$.

We call this partition the *fundamental edge partition*. The three different types of edges in the fiber over one edge are shown in Figure 1.

Example 3.1. Let us consider the graph $K_{2,3}$. The voltage assignment in Figure 2 determines the 5-fold covering graph $\mathrm{Cov}(K_{2,3})$ in Figure 3.

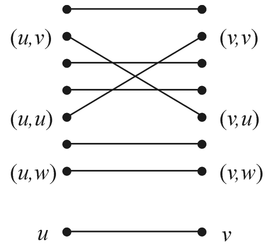


Figure 1: The lift of an edge.

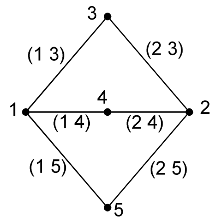


Figure 2: A voltage graph $K_{2,3}$ for Cov$(K_{2,3})$.

3.1 The construction of $\mathrm{Clone}(X)$ via Cov(X)

Let X be a connected graph on $n \geq 2$ vertices. The following proposition carves $\mathrm{Clone}(X)$ out of the auxiliary graph Cov(X).

Proposition 3.2. Cov(X) *is a disjoint union of two covering graphs of* X*. One is isomorphic to* Cov$_{\mathrm{id}}(X) = X$.

Proof. The subgraph of Cov(X) induced by the diagonal vertices $\{(v, v),\ v \in X\}$ (and diagonal edges) is isomorphic to X, and the restriction of the covering projection to it gives Cov$_{\mathrm{id}}(X)$. □

The subgraph of Cov(X) that does not contain the diagonal vertices is called the *clone cover* and denoted by $\mathrm{Clone}(X)$. Clearly, $\mathrm{Clone}(X)$ is an $(n-1)$-fold covering graph over X. We call the subgraph of $\mathrm{Clone}(X)$, spanned by the vertices $\{(v, i); v \in V(X)\backslash i\}$, the i-th *layer* of $\mathrm{Clone}(X)$.

Example 3.3. Let X be the cycle on n vertices for $n \geq 3$. Then $\mathrm{Clone}(X)$ is the cycle on $n(n-1)$ vertices.

(a) (b)

Figure 3: Cov$(K_{2,3})$ has two components: (a) Cov$_{\mathrm{id}}(K_{2,3})$, (b) $\mathrm{Clone}(K_{2,3})$.

3.2 A direct permutation voltage graph construction

This construction depends on the choice of the base vertex b of X. The permutation voltages are taken from $\mathrm{Sym}(V(X) - \{b\})$. They are defined as follows. The permutation voltages on arcs incident with the vertex b are equal to the identity while the voltages of arcs uv not involving b are, as before, equal to the transposition (u, v). The corresponding covering graph is denoted, for the time being, by $\mathrm{Clone}_d(X, b)$.

3.3 Combinatorial Construction

Let X be a connected graph on n vertices. We define the graph $\mathrm{Clone}_c(X)$ as follows. The vertex set W of $\mathrm{Clone}_c(X)$ consists of all $n(n-1)$ pairs of vertices $(u, v) \in V(X) \times V(X)$ with $u \neq v$. There are two sets of edges. Each edge $\{u, v\}$ from X gives rise to the edge $\{(u, v), (v, u)\}$ in $\mathrm{Clone}_c(X)$ (these will correspond to the connecting edges). For each $w \in V(X)$, different from u and v, we get in total $(n - 2)$ (inner) edges $\{(u, w), (v, w)\}$. It is not hard to show that the projection from W to $V(X)$ defined by $(u, v) \mapsto u$ is an $(n - 1)$-fold covering projection.

3.4 Graphical Construction

Let X_v denote the graph X with vertex v removed. The graph $\mathrm{Clone}_g(X)$ is obtained from the collection of n *vertex-deleted subgraphs* $X_v = X - v$ by adding, for each edge $\{u, v\}$ of X, an edge joining the vertex u in X_v to the vertex v in X_u; see Figure 4.

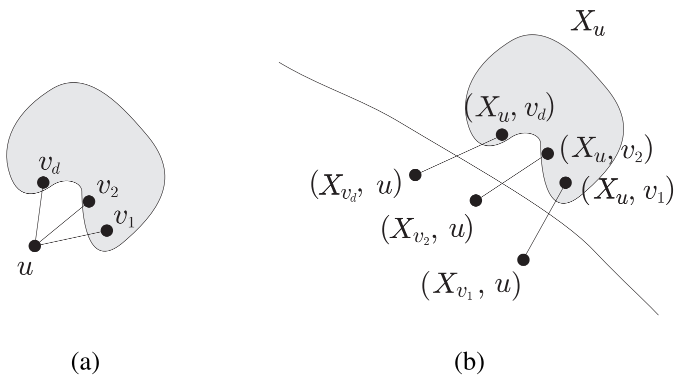


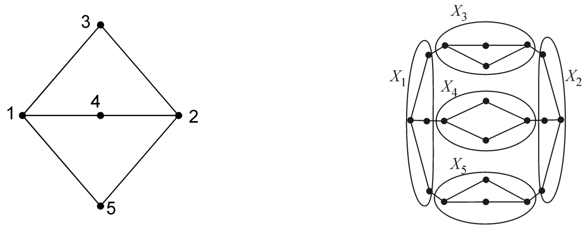
Figure 4: (a) The graph X and one of its vertices u. $\mathrm{Clone}_g(X)$ is obtained in such a way that each vertex u is replaced by a vertex deleted subgraph X_u. In (b), this is shown for the vertex u.

The edges of $\mathrm{Clone}_g(X)$ can be naturally partitioned (or colored) into two classes: the edges belonging to each vertex-deleted subgraph X_v, and the connecting edges. Each edge of X lifts to one connecting edge and $(n - 2)$ original edges.

Example 3.4. Figure 5 shows the graphical construction of $\mathrm{Clone}(K_{2,3})$.

Proposition 3.5. *Let X be a 2-connected graph. Then X is a minor of* $\mathrm{Clone}_g(X)$.

Proof. If X is 2-connected, then for every vertex $u \in V(X)$ the vertex-deleted subgraph X_u is connected. If for each u we contract the edges of the copy of X_u from $\mathrm{Clone}_g(X)$,



(a) $K_{2,3}$ (b) $\mathrm{Clone}(K_{2,3})$.

Figure 5: The graphical construction of $\mathrm{Clone}(K_{2,3})$.

then this copy of X_u is contracted to a single vertex and the resulting graph is isomorphic to X. Hence X is a minor of $\mathrm{Clone}_g(X)$. □

3.5 Equivalence of the four constructions

Here we prove that the above four definitions are equivalent.

Theorem 3.6. *The covers* $\mathrm{Clone}(X), \mathrm{Clone}_d(X,b), \mathrm{Clone}_c(X)$, *and* $\mathrm{Clone}_g(X)$ *are isomorphic.*

Proof. It is easy to see that $\mathrm{Clone}(X)$ and $\mathrm{Clone}_c(X)$ are isomorphic since they have the same vertex set and the same edge set. Also the mapping that sends the vertex (X_u, v) of $\mathrm{Clone}_g(X)$ to the vertex (v, u) in $\mathrm{Clone}(X)$ is an isomorphism.

To finish the proof we show that $\mathrm{Clone}_c(X)$ and $\mathrm{Clone}_d(X,b)$ are isomorphic. Define the mapping $\varphi : V(\mathrm{Clone}_c(X)) \to V(\mathrm{Clone}_d(X,b))$ by

$$\varphi(u,v) = \begin{cases} (u,v) & \text{if } v \neq b, \\ (u,u) & \text{if } v = b. \end{cases}$$

This is obviously bijective. The edges of the form $\{(u,b),(v,b)\}$ are mapped to the edges of the form $\{(u,u),(v,v)\}$, while all other edges of $\mathrm{Clone}_c(X)$ are mapped to the edges with the same labels in $\mathrm{Clone}_d(X,b)$. This shows that φ is an isomorphism. In particular, the choice of the vertex b in $\mathrm{Clone}_d(X) = \mathrm{Clone}_d(X,b)$ is irrelevant. □

3.6 Lifts of cycles

Recall from general theory [4] that any voltage assignment can be naturally extended from arcs to walks by successively multiplying voltages of arcs encountered along the walk. The voltage of a walk actually tells how this walk lifts to the corresponding covering graph. We are particularly interested in how a given cycle of the base graph lifts. Clearly, a cycle lifts to a collection of cycles.

Theorem 3.7. [4, Theorem 2.4.3] *Consider a covering projection $p\colon \tilde{X} \to X$ arising from a permutation voltage assignment in S_n on X. If C is a cycle of length k in X whose voltage has cycle structure $(c_1, \dots, c_n)$, then the preimage of C in the derived graph has $c_1 + \cdots + c_n$ components, consisting of exactly c_j cycles of length kj, for $j = 1, \dots, n$.*

Let X be a graph on n vertices, with the voltage assignment $\tau(uv) = (u,v) \in S(V(X))$ for each arc uv in X. Recall that this assignment gives rise to the covering graph $\mathrm{Cov}(X)$ which consists of an isomorphic copy of X, and $\mathrm{Clone}(X)$. In this particular setting it is easy to see that the voltage of a directed cycle $C = v_0v_1 \ldots v_mv_0$ in X, rooted at v_0, is then

$$(v_m, v_{m-1}, \ldots, v_1)(v_0) \in \mathrm{Sym}(V(X)).$$

The following proposition is therefore a direct consequence of Theorem 3.7 .

Proposition 3.8. *Let X be a connected graph. A k-cycle in X based at u lifts in* $\mathrm{Clone}(X)$ *to one "long" cycle of length $k(k-1)$ based at (u, v), where $v \neq u$ is any vertex in the cycle, and $n - k$ "short" cycles of length k based at (u, v) where v is any vertex not in the cycle.*

Corollary 3.9. *Let X be a connected graph on n vertices. If X contains a cycle of length $k < n$ then also* $\mathrm{Clone}(X)$ *contains a cycle of length k.*

4 Graph-theoretical properties

A natural problem to consider is the impact of a given graph invariant of a graph such as girth, connectivity or diameter, on its clone cover. Some invariants are easy to determine. For instance, girth is a well-known graph invariant measuring the lengh of the shortest cycle in a graph. Any connected graph that is not a cycle has the same girth as its clone cover by Corollary 3.9, and the girth of the clone cover of a cycle on n vertices is $n(n-1)$ by Proposition 3.8. In this section some other graph invarants are studied.

4.1 Connectivity

The graph $\mathrm{Clone}(X)$ can be connected or disconnected, with an easy test for connectivity. Recall that a *block* of a graph X is a maximal connected subgraph of X without a cut-vertex. If X contains no cut-vertex, then X itself is called a block.

Theorem 4.1. *Let X be a connected graph. Then* $\mathrm{Clone}(X)$ *is connected if and only if X is a block. Moreover, if X is k-connected, where $k \geq 2$, then* $\mathrm{Clone}(X)$ *is also k-connected.*

Proof. In this proof we will use the graphical construction of Clone. Suppose X has a cut-vertex v, and let the vertices v_1 and v_2 be in different blocks of X. Then the vertices (X_{v_1}, v_2) and (X_{v_2}, v_1) are in different components of $\mathrm{Clone}(X)$, since every path between them would pass through X_v, and in X_v there is no edge between the vertices of the blocks of v_1 and v_2. Therefore $\mathrm{Clone}(X)$ is not connected.

If X is a block that is not 2-conneced, then it is isomorphic to the complete graph on two vertices. So $\mathrm{Clone}(X)$ is isomorphic to X and hence connected.

Suppose now that X is k-connected, where $k \geq 2$. We will prove that $\mathrm{Clone}(X)$ is k-connected (and therefore also connected). By the global version of Menger's theorem, it is enough to prove that for any two distinct vertices in $\mathrm{Clone}(X)$ there exist k internally disjoint paths between them. Note that each of the subgraphs X_u of $\mathrm{Clone}(X)$ is connected since X is 2-connected.

We use the following notation. Let $P = u_1u_2 \ldots u_t$ be a path in X. Then $P(u_i, \ldots, u_j)$ denotes the part of P between the vertices u_i and u_j for $1 \leq i \leq j \leq t$. Let $u \in V(X)$ be distinct from $u_1, u_2, \ldots, u_t$. By $\tilde{P}_u(u_1, u_2, \ldots, u_t)$, or more briefly, by $\tilde{P}_u$, we denote the path $(X_u, u_1)(X_u, u_2) \ldots (X_u, u_t)$ in $\mathrm{Clone}(X)$ that is contained in X_u. We denote a path in $\mathrm{Clone}(X)$ between the vertices (X_{u_1}, u_2) and (X_{u_t}, u_{t-1}) of the form

$$(X_{u_1}, u_2)(X_{u_2}, u_1) \quad \dots \quad (X_{u_2}, u_3)(X_{u_3}, u_2)\dots$$
$$(X_{u_{t-2}}, u_{t-1})(X_{u_{t-1}}, u_{t-2})\dots(X_{u_{t-1}}, u_t)(X_{u_t}, u_{t-1})$$

by $\tilde{P}(u_1, u_2, \dots, u_t)$, or more briefly, by $\tilde{P}$. The walks in the same copy of any vertex deleted subgraph can be arbitrary paths.

Let (X_u, v) and (X_w, z) be two distinct vertices of $\mathrm{Clone}(X)$. We now construct k internally disjoint paths between them. We distinguish two cases.

Case 1. Suppose $u = w$. Let $P_1, \dots, P_k$ be k internally disjoint paths between v and z in X. If none of them contains u, we have k disjoint paths between (X_u, v) and (X_u, z) in $\mathrm{Clone}(X)$ which are all contained in X_u. If one of the paths, say P_1, contains u, we have only $k-1$ internally disjoint paths contained in X_u. We now construct another path that will also use other vertex-deleted subgraphs. Let $P = P_1 = vu_1 \dots u_t u u_{t+1} \dots u_s z$. Let Q be a path between u and u_{t+1} in X that does not contain u_t, and let R be a path between u_t and u in X that does not contain u_{t+1}. Since X is 2-connected, such paths exist. Then

$$\tilde{P}_u(v, u_1, \dots, u_t)(X_{u_t}, u)\tilde{Q}_{u_t}(X_{u_{t+1}}, u_t)\tilde{R}_{u_{t+1}}(X_u, u_{t+1})\tilde{P}_u(u_{t+1} \dots, u_t, z)$$

is a path between (X_u, v) and (X_u, z) that is internally disjoint from each of $P_2, \dots, P_k$.

Case 2. Suppose $u \neq w$. Let $P^i = uu_1^i \dots u_{t_i}^i, w$ be k internally disjoint paths between u and w in X. By Dirac's Fan Lemma (see, for example, [10, Theorem 4.2.23]), there exist internally disjoint paths from v to $u_1^1, \dots, u_1^k$ in X, say $Q^1, \dots, Q^k$. Similarly, there exist internally disjoint paths from $u_{t_1}^1, \dots, u_{t_k}^k$ to z in X, say $R^1, \dots, R^k$. Suppose first that u is not contained in any of the paths Q^i or R^i. Then for $i = 1, \dots, k$ the paths

$$S^i = \tilde{Q}_u^i \tilde{P}^i(u, u_1^i, \dots, u_{t_i}^i, w)\tilde{R}_w^i$$

are k internally disjoint paths between (X_u, v) and (X_w, z); see the top path in Figure 6.

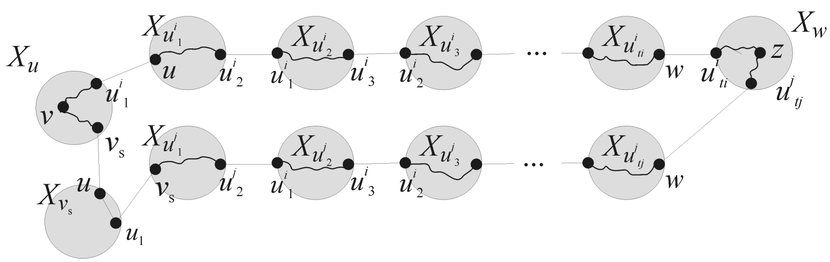


Figure 6: Two of internally disjoint paths in $\mathrm{Clone}(X)$.

Suppose now that u belongs to Q^j for some j. Let $Q = Q^j = vv_1 \dots v_s uu_1^j$. If v_s belongs to P^j, we just interchange the roles of u_1^j and v_s and construct S^j as before. Otherwise, we may assume that v_s does not belong to any of the paths P^i for $i \neq j$. We can do this since if for some i the vertex v_s belongs to P^i, so P^i is of the form $uu_1^i \dots v_s \dots u_{t_i}^i w$, then we may replace P^i by the path $uv_s \dots u_{t_i}^i w$ that is also internally disjoint from the other $k-1$ paths between u and w in X. We define

$$S^j = \tilde{Q}_u^j(v, v_1, \dots, v_s)(X_{v_s}, u)(X_{v_s}, u_1^j)(X_{u_1^j}, v_s)\dots(X_{u_1^j}, u_2^j)\tilde{P}^i(u_2^j, \dots, u_{t_j}^j, w)\tilde{R}_w^j.$$

Note that the subgraph X_{v_s} is not used by any of the other paths S^i for $i \neq j$, so S^j is internally disjoint with them; see the bottom path in Figure 6. If u belongs to some R^ℓ, we modify S^ℓ in a similar way as above. Again, we have k internally disjoint paths between (X_u, v) and (X_w, z). □

If X is connected, every block of X is either a maximal 2-connected subgraph or a bridge. Different blocks can have at most one vertex in common, which is then a cut-vertex of X. Therefore every edge lies in a unique block, and X is the union of its blocks. We denote by $X_1 \oplus_v X_2$, or just $X_1 \oplus X_2$, the union of two graphs with a common vertex v. We denote by $X_1 \sqcup X_2$ the disjoint union of two graphs.

Lemma 4.2. *Let $X = B \oplus_v C$ be composed of two blocks B and C with a common vertex v. Let $\{u_1, u_2, \ldots, u_p, v\}$ be the vertex set of B and let $\{w_1, w_2, \ldots, w_q, v\}$ be the vertex set of C. Then* $\mathrm{Clone}(X)$ *is isomorphic to the following graph:*

$$(\mathrm{Clone}(B) \oplus_{(v,u_1)} C \oplus_{(v,u_2)} C \oplus \cdots \oplus_{(v,u_p)} C) \sqcup \\ (\mathrm{Clone}(C) \oplus_{(v,w_1)} B \oplus_{(v,w_2)} B \oplus \cdots \oplus_{(v,w_q)} B).$$

Proof. As in the proof of Theorem 4.1 we see that $\mathrm{Clone}(B)$ and $\mathrm{Clone}(C)$ are connected and lie in different components of $\mathrm{Clone}(X)$. The claim follows from the fact that every vertex deleted subgraph X_{u_i} contains a copy of C, and every vertex deleted subgraph X_{w_i} contains a copy of B. □

Corollary 4.3. *Let X be a connected graph and let $B_1, B_2, \ldots, B_k$ be the blocks of X. In other words, $X = B_1 \oplus B_2 \oplus \cdots \oplus B_k$. Let C_i consist of the blocks of X different from i, for $i = 1, \ldots, k$. Then* $\mathrm{Clone}(X)$ *is isomorphic to the following graph (with blocks attached at appropriate vertices):*

$$(\mathrm{Clone}(B_1) \oplus_{i=1}^{k-1} (C_1)) \sqcup (\mathrm{Clone}(B_2) \oplus_{i=1}^{k-1} (C_2)) \sqcup \cdots \sqcup (\mathrm{Clone}(B_k) \oplus_{i=1}^{k-1} (C_k)).$$

In particular, the number of components of $\mathrm{Clone}(X)$ *is equal to the number of blocks of X.*

4.2 Bipartiteness

Any covering graph of a bipartite graph is obviously bipartite. The graph $\mathrm{Clone}(C_n)$ is the cycle $C_{n(n-1)}$, hence it is bipartite. It turns out that odd cycles are the only non-bipartite graphs for which the clone cover is bipartite.

Proposition 4.4. *Let X be a graph that is not a cycle. Then* $\mathrm{Clone}(X)$ *is bipartite if and only if X is bipartite.*

Proof. Suppose X is not bipartite, and let C be an odd cycle in X as short as possible. Since X is not a cycle, there exists a vertex v of X not in C. In $\mathrm{Clone}(X)$ there is a copy of C in the layer corresponding to v, thus also $\mathrm{Clone}(X)$ is not bipartite.

Conversely, suppose that $\mathrm{Clone}(X)$ is not bipartite. Then it contains an odd cycle. By Proposition 3.8, this must come from a cycle of the same odd length in X (since $k(k-1)$ is even for all k), and therefore X is not bipartite. □

4.3 Hamiltonicity

Although the cycle structure of the clone covers is fairly well understood, no complete characterization of hamiltonian clone covers is known. However, there is a simple sufficient condition for the base graph to have a hamiltonian clone cover. By Proposition 3.8, a Hamilton cycle (of length n) in the base graph X lifts to one cycle of length $n(n-1)$ in $\mathrm{Clone}(X)$. The cycle of length $n(n-1)$ is a Hamilton cycle in $\mathrm{Clone}(X)$. We record this formally.

Theorem 4.5. *Let X be a hamiltonian graph. Then* $\mathrm{Clone}(X)$ *is hamiltonian.*

We only have a partial converse of this result. However, no examples are known of a 2-connected non-hamiltonian graph for which the clone cover is hamiltonian.

Proposition 4.6. *Let X be a non-hamiltonian graph of minimal degree at most three. Then* $\mathrm{Clone}(X)$ *is also non-hamiltonian.*

Proof. Let v be a vertex of degree at most three in X, and suppose that $\mathrm{Clone}(X)$ has a Hamilton cycle H. Denote by H_v the subgraph of H restricted to X_v. Since X_v is connected to the rest of the graph by at most three edges, H_v forms a Hamilton path in X_v. Denote the vertices of degree 1 of this path by (X_v, u) and (X_v, w). Then u and w are neighbors of v in X. By adding edges $\{v, u\}$ and $\{v, w\}$ to the projection of H_v on X, we obtain a Hamilton cycle in X. A contradiction. □

Proposition 4.7. *Let X be a graph and let $v \in V(X)$ be a vertex of degree k such that $X \backslash N(v)$ has more than k components (X is not hamiltonian). Then* $\mathrm{Clone}(X)$ *is also not hamiltonian.*

Proof. Let $N(v) = \{v_1, v_2, \ldots, v_k\}$ and let $U = \{(v, v_1), (v, v_2), \ldots, (v, v_k)\}$. Then $\mathrm{Clone}(X) \backslash U$ has more than k components since $X_v \backslash U$ has at least k components and is not connected to the rest of the graph. Therefore $\mathrm{Clone}(X)$ is not hamiltonian. □

4.4 Planarity

Recall the graphical construction of $\mathrm{Clone}(X)$: the graph $\mathrm{Clone}(X)$ can be obtained from the graph X by "replacing" each vertex v by $X_v = X - v$. Using this fact, we make the following observations.

Theorem 4.8. *Let X be a graph. Then* $\mathrm{Clone}(X)$ *is planar if and only if X is planar.*

Proof. Let X be a planar graph and let Y be a planar embedding of X. We choose an orientation of the plane. Let u be a vertex of X and let Y^u be an embedding of X such that u lies on the outer face with the cyclic order of the neighbors of each vertex reversed with regard to Y. Then the order of the neighbors of u along the outer face of $Y^u - u$ is the same as the order of the neighbors of u in Y. Therefore it is possible to replace u in Y by the graph $Y^u - u$, and connect each of the neighbors of u in Y by the corresponding neighbor of u in Y^u such that this replacement yields a plane graph again. Doing this for each vertex of Y we obtain a planar embedding of $\mathrm{Clone}(X)$.

Conversely, if X is not planar, then it contains a copy of $K_{3,3}$ or K_5 as a minor. If X is 2-connected, then it is a minor of $\mathrm{Clone}(X)$ by Proposition 3.5, and therefore also $\mathrm{Clone}(X)$ contains a copy of $K_{3,3}$ or K_5 as a minor. If X is not 2-connected, then $\mathrm{Clone}(X)$ contains each block of X as a subgraph by Corollary 4.3. In this case, at least one block of X is not planar and therefore also $\mathrm{Clone}(X)$ is not planar. □

Similarly we can give an upper bound on the genus of $\mathrm{Clone}(X)$ in terms of genus of X.

Theorem 4.9. *Let X be a graph on n vertices. The following bound holds for the genus γ:*

$$\gamma(\mathrm{Clone}(X)) \leq (n+1)\gamma(X).$$

The same inequality holds for the non-orientable genus.

Proof. Let m denote the number of edges of X and let $g = \gamma(X)$. Let f denote the number of faces in the genus embedding of X, and let f_i be the number od faces of length i, for $i \geq 3$. Suppose first that all the faces of X are cycles. Recall that the voltage of a cycle of length k is a cycle of length $k-1$ in $\mathrm{Sym}(V(X)) \cong S_n$ fixing $n-k+1$ symbols. By Proposition 3.8, such a cycle lifts to one cycle of length $k(k-1)$ and $n-k$ cycles of length k in $\mathrm{Clone}(X)$.

Take an embedding of $\mathrm{Clone}(X))$ in which the cyclic order of the edges around each vertex is the same as in the genus embedding of X. Then a face of X lifts to a face of $\mathrm{Clone}(X))$. Denote by n', m', f', g' the number of vertices, number of edges, number of faces, and the genus of this embedding of $\mathrm{Clone}(X))$, respectively. Then $n' = (n-1)n$, $m' = (n-1)m$, and

$$f' = \sum_{i\geq 3}(n-i+1)f_i = \sum_{i\geq 3}(n+1)f_i - \sum_{i\geq 3} if_i = (n+1)f - 2m.$$

Now we can compute g':

$$\begin{aligned} g' &= (2+m'-n'-f')/2 = (2+m(n-1)-n(n-1)-f(n+1)+2m)/2 \\ &= (2n+2+m(n+1)-n(n+1)-f(n+1))/2 = (n+1)g. \end{aligned}$$

If a face of X of length k is not a cycle, it lifts to more than $n-k+1$ faces, which makes the genus of such an embedding of $\mathrm{Clone}(X)$ even smaller than $(n+1)g$. In any case we have $\gamma(\mathrm{Clone}(X)) \leq (n+1)g$.

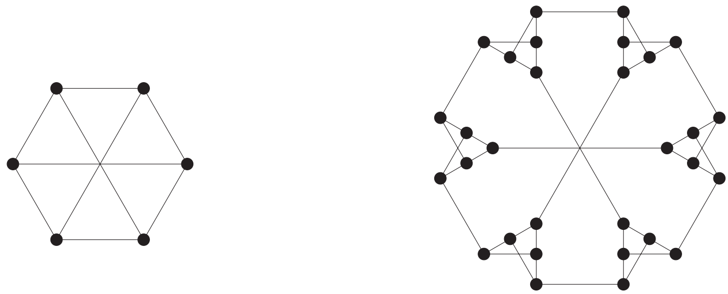
The same reasoning holds also for the nonorientable embeddings. □

Example 4.10. The genus of $K_{3,3}$, which is a graph on 6 vertices, is 1. By Theorem 4.9, the genus of $\mathrm{Clone}(K_{3,3})$ is at most 7; see Figure 7.

5 Algebraic properties

A reasonable assumption when studying algebraic properties of covering graphs, or indeed graphs in general, is to restrict our considerations to connected covering graphs – which in our case translates to requiring that the base graphs are at least 2-connected, in view of Theorem 4.1 and the assumption that the base graph is not K_2.

There are two different kinds of automorphisms of a covering graph: the ones that are lifts of some automorphism of the base graph, and the ones that are not. Along these lines we consider certain structural properties of the automorphism group of $\mathrm{Clone}(X)$, edge- and vertex-transitivity of $\mathrm{Clone}(X)$, and regularity of the covering projection $\mathrm{Clone}(X) \to X$. Automorphisms that respect the fundamental edge partition, see Subsection 5.2 below, will play a significant role in this context.



(a) (b)

Figure 7: (a) The graph $K_{3,3}$, (b) $\mathrm{Clone}(K_{3,3})$.

5.1 Lifts of automorphisms along the covering projection

Certain automorphisms of a covering graph can be studied in terms of automorphisms of the base graph. Such automorphisms arise as lifts of automorphisms, a concept we shall now define. Let $p : \tilde{X} \to X$ be a covering projection of graphs, and let f be an automorphism of X. We say that f *lifts* if there exists an automorphism $\tilde{f}$ of $\tilde{X}$, a *lift* of f, such that the following diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & X \end{array}$$

is commutative; in other words, $f \circ p = p \circ \tilde{f}$. Observe that a lift of an automorphism maps bijectively fibers over vertices (resp. edges) to fibers over vertices (resp. edges), in particular, any lift of f maps the fiber over a vertex $u \in V(X)$ to the fiber over the vertex $f(u)$.

Suppose that all the elements of a subgroup $G \leq$ Aut(X) have a lift. Then the lifts of all automorphisms from G form a subgroup of Aut$(\tilde{X})$ which we denote by $\tilde{G}$. In particular, the lift of the trivial group is known as the group of *covering transformations* and is denoted by CT(p). Further, if G lifts, then there exists an epimorphism $p_G : \tilde{G} \to G$ with CT(p) as its kernel. Hence $\tilde{G}$ is an extension of CT(p) by G, and the set of all lifts of a given $f \in$ Aut(X) is a coset of CT(p) in $\tilde{G}$. As a final opening remark in this section, recall from general theory that CT(p) acts semiregularly on the covering graph $\tilde{X}$ whenever $\tilde{X}$ is connected; that is, CT(p) acts without fixed points both on vertices and on arcs of $\tilde{X}$. Moreover, each lift is uniquely determined by the mapping of a single vertex. For a background on lifting automorphisms in terms of voltages we refer the reader to [8].

In general, not every automorphism of the base graph X lifts. This is not the case with the natural covering projection $p\colon \mathrm{Clone}(X) \to X$, $p = \mathrm{pr}_1\colon (u, v) \mapsto u$. To this end let us introduce, for each automorphism f of the graph X, a mapping $\bar{f}\colon \mathrm{Clone}(X) \to \mathrm{Clone}(X)$ which we call the *diagonal* mapping, defined by

$$\bar{f}\colon (u, i) \mapsto (f(u), f(i)).$$

Theorem 5.1. *Let f be an automorphism of X. Then the map $\bar{f}$ is an automorphism of*

$\mathrm{Clone}(X)$ *and is a lift of* f.

Proof. Obviously, $\bar{f}$ is well defined since $i \neq u$ implies $f(i) \neq f(u)$, and moreover, it is bijective on the vertex set of $\mathrm{Clone}(X)$.

We will show that $\bar{f}$ maps arcs to arcs, so it is an automorphism. Let uv be an arc in X, and let a be an arc in $\mathrm{Clone}(X)$ from (u,i) to $(v, i^{(u,v)})$. The vertex (u,i) is mapped by $\bar{f}$ to $(f(u), f(i))$. Since $f(i^{(u,v)}) = f(i)^{(f(u),f(v))}$ and the vertex $(v, i^{(u,v)})$ is mapped by $\bar{f}$ to the vertex $(f(v), f(i^{(u,v)}) = (f(v), f(i)^{(f(u),f(v))})$, it follows that $f(a)$ is an arc in $\mathrm{Clone}(X)$, as required.

Let (u,i) be a vertex from $\mathrm{Clone}(X)$. Then $f \circ p(u,i) = f(u) = p(f(u), f(i)) = p \circ \bar{f}(u,i)$. Let a be an arc in $\mathrm{Clone}(X)$ from (u,i) to $(v, i^{(u,v)})$. Then $f \circ p(a) = f([u,v]) = [f(u), f(v)] = p([(f(u), f(i)), (f(v)f(i)^{(u,v)})]) = p \circ \bar{f}(a)$. Thus, $\bar{f}$ is an automorphism and clearly a lift of f.

□

In view of the above theorem, $\mathrm{Clone}(X)$ inherits all the symmetries of X in the sense that there is a natural injection of $\mathrm{Aut}(X)$ into $\mathrm{Aut}(\mathrm{Clone}(X))$ taking $f \mapsto \bar{f}$. This injection is actually a group homomorphism, as we shall see shortly. For convenience we denote by $A = \mathrm{Aut}(X)$ the full automorphism group of X and $\tilde{A}$ its lift.

Proposition 5.2. *The set* $\bar{A}$ *of all diagonal mappings is a subgroup of* $\mathrm{Aut}(\mathrm{Clone}(X))$, *isomorphic to* A, *and* $\tilde{A} = \mathrm{CT}(p) \rtimes \bar{A}$.

Proof. The set $\bar{A}$ is clearly a complete system of coset representatives of $\mathrm{CT}(p)$ within $\tilde{A}$ since for each $f \in A$ there is only one diagonal mapping. Moreover, from the definition of the diagonal mapping it easily follows that $\overline{fg} = \bar{f}\bar{g}$. Hence $\bar{A}$ is a complement to $\mathrm{CT}(p)$ within $\tilde{A}$, and so $\tilde{A} = \mathrm{CT}(p) \rtimes \bar{A}$, as required. □

From now we shall be explicitly assuming that the base graph X is 2-connected, as already anticipated at the beginning of this section. The simplest base graphs of this kind are the cycles.

Proposition 5.3. *Let* $p\colon \mathrm{Clone}(C_n) \to C_n$ *be the covering projection for the* n*-cycle* C_n, *for* $n \geq 3$. *Then* $\mathrm{CT}(p)$ *is isomorphic to* $\mathbb{Z}_{n-1}$, *and* $\mathrm{Aut}(\mathrm{Clone}(C_n)) \cong D_{n(n-1)} \cong \mathbb{Z}_{n-1} \rtimes D_n$.

Proof. Recall that $\mathrm{Clone}(C_n) = C_{n(n-1)}$ which has the automorphism group isomorphic to the dihedral group $D_{n(n-1)}$ of order $2n(n-1)$. The subgroup of $\mathrm{Aut}(\mathrm{Clone}(C_n))$ generated by the n-step rotation of $C_{n(n-1)}$ is isomorphic to $\mathbb{Z}_{n-1}$. It is easy to see that each of these automorphisms is a covering transformation. Since $A = \mathrm{Aut}(C_n)$ is isomorphic to the dihedral group of order $2n$, it follows that $\tilde{A} = \mathrm{CT}(p) \rtimes \bar{A}$ has order $2n(n-1)$. Hence $\mathrm{Aut}(\mathrm{Clone}(C_n)) = \tilde{A}$. □

$\mathrm{Clone}(C_n)$ appears to be rather special, since every vertex deleted subgraph of a cycle is acyclic. In all other cases where the clone cover is connected, the group of covering transformations is trivial, and this fact has strong impact on symmetry properties of Clone.

Theorem 5.4. *Let* X *be a* 2*-connected graph that is not a cycle. Then* $\mathrm{CT}(p)$ *is trivial, and hence the lifted group is equal to the group of diagonal mappings – that is,* $\tilde{A} = \bar{A}$.

Proof. From the assumptions it easily follows that in X there exist two distinct vertices, connected with three internally disjoint paths P, Q, R. Let C be the cycle formed by the paths P and Q, and let C' be the cycle formed by P and R. Let k and k' be the lengths of C and C', respectively. Then C lifts to one $k(k-1)$ cycle and to $n-k$ cycles of length k, while C' lifts to one $k'(k'-1)$ cycle and $n-k'$ cycles of length k', where $n = |V(X)|$. Denote the $k(k-1)$ cycle over C by $\tilde{C}$, and the $k'(k'-1)$ cycle over C' by $\tilde{C}'$.

Let now $f \in \mathrm{CT}(p)$ be a covering transformation. Since f permutes the cycles over C we have $f(\tilde{C}) = \tilde{C}$. Similarly, $f(\tilde{C}') = \tilde{C}'$. Denote the union of C and C' by $Y = C \cup C'$, and let $\tilde{Y}$ be the connected component of the preimage $p^{-1}(Y)$ containing $\tilde{C} \cup \tilde{C}'$. Note that $\tilde{Y} = \mathrm{Clone}(Y)$. Moreover, $f(\tilde{Y}) = \tilde{Y}$. Further, the restriction of f to $\tilde{Y}$ is a covering transformation of the projection $\mathrm{Clone}(Y) \to Y$. Since $\mathrm{CT}(p)$ acts without fixed points, it is enough to show that the group of covering transformations of the projection $\mathrm{Clone}(Y) \to Y$ is trivial.

To formally prove the above assertion we shall actually prove that the group of covering transformations of the auxiliary covering $\bar{p}\colon \mathrm{Cov}(Y) \to Y$ (which is isomorphic to that of $\mathrm{Clone}(Y) \to Y$) must be trivial.

Let $V(Y) = \{u_0, u_1 \ldots, u_s, x_{s+1}, \ldots, x_{k-1}, y_{s+1}, \ldots, y_{k'-1}\}$ be the vertex set of $Y = C \cup C'$, and let $C = u_0 u_1 \ldots u_s x_{s+1} \ldots x_{k-1} u_0$ and $C' = u_0 u_1 \ldots u_s y_{s+1} \ldots y_{k'-1} u_0$ be the corresponding directed cycles, rooted at u_0. The first cycle has voltage $\alpha = (x_{k-1}, \ldots, x_{s+1}, u_s, \ldots, u_1)$ while $\beta = (y_{k'-1}, \ldots, y_{s+1}, u_s, \ldots, u_1)$ is the voltage of the second one. From general theory [8, Corollary 7.3] it easily follows that the group of covering transformations of a connected cover given by permutation voltages is isomorphic to the centralizer in the symmetric group of the subgroup generated by the voltages of all closed walks at a chosen vertex. Hence in our case $\mathrm{CT}(\bar{p})$ is isomorphic to the centralizer of α and β in $\mathrm{Sym}(V(Y))$. Let τ commute with both α and β. If we represent these permutations graphically as a colored digraph on the vertex set $V(Y)$, then τ corresponds to a color- and direction-preserving automorphism of this digraph. From the structure of the above colored 'permutation digraph', it is now immediate that τ must be trivial, as required. □

Recall that a cover is *regular* if the fiber-preserving automorphisms act transitively on each fiber. The three canonical covers mentioned in the introduction, namely the universal cover, the Kronecker cover, and the trivial cover, are all regular covers for all base graphs. However, the clone cover is in most cases an irregular covering.

Theorem 5.5. *Let X be a 2-connected graph. Then $\mathrm{Clone}(X) \to X$ is not a regular covering projection unless $X = C_n$.*

Proof. By Theorem 5.4 we know that the group of covering transformations is trivial, except when $X = C_n$. This completes the proof. □

In particular, the graph $\mathrm{Clone}(K_n)$, $n \geq 4$, is an irregular cover of K_n.

5.2 Automorphisms that respect the fundamental edge partition

We will say that an automorphism of $\mathrm{Clone}(X)$ *respects the fundamental edge partition* if it takes inner edges to inner edges, and connecting edges to connecting edges. From the graphical construction of $\mathrm{Clone}(X)$ it follows that inner edges can be naturally partitioned into layers – which are nothing but the vertex deleted subgraphs of X. Consequently, the property of preserving the edge partition is equivalent to requiring that layers are mapped to layers, at least when $\mathrm{Clone}(X)$ is connected.

Proposition 5.6. *Let X be a 2-connected graph. An automorphism of $\mathrm{Clone}(X)$ respects the fundamental edge partition if and only if it maps layers to layers.*

Proof. Let f be an automorphism of $\mathrm{Clone}(X)$. Suppose f maps layers to layers. Then it maps inner edges to inner edges. Hence it must also map connecting edges to connecting edges, and must therefore respect the fundamental edge partition.

Conversely, suppose that f respects the fundamental edge partition. Then it maps inner edges to inner edges. Since X is 2-connected, every layer of $\mathrm{Clone}(X)$, which is just a vertex-deleted subgraph of X, is connected. Take two vertices (u, v) and (w, v) from the same layer of $\mathrm{Clone}(X)$. Then there exists a path between them, consisting only of inner edges. But then also $f(u, v)$ and $f(w, v)$ are connected by a path consisting only of inner edges. Therefore $f(u, v)$ and $f(w, v)$ are in the same layer of $\mathrm{Clone}(X)$. This shows that f takes layers to layers. □

Note that all automorphisms that respect the fundamental edge partition form a subgroup in $\mathrm{Aut}(\mathrm{Clone}(X))$ which we denote $\mathcal{E}$. We are now going to explicitly describe the structure of this group whenever $\mathrm{Clone}(X)$ is connected. To start with, note that there is a natural mapping $\mathrm{contr}\colon \mathrm{Clone}(X) \to X$, called *contraction*, defined by collapsing each vertex-deleted subgraph to its corresponding vertex v. To put it differently, contraction is in fact the projection $\mathrm{contr} = \mathrm{pr}_2\colon (u, v) \mapsto v$ onto the second coordinate – in contrast with the covering projection which is the projection onto the first coordinate. Let now f and $\hat{f}$ be automorphisms of X and $\mathrm{Clone}(X)$, respectively, such that the following diagram

$$\begin{array}{ccc} \mathrm{Clone}(X) & \xrightarrow{\hat{f}} & \mathrm{Clone}(X) \\ \mathrm{contr}\big\downarrow & & \big\downarrow\mathrm{contr} \\ X & \xrightarrow{f} & X \end{array}$$

is commutative. We then say that f *lifts* and that $\hat{f}$ *projects along the contraction*. In a similar fashion we speak about lifting and projecting groups. In view of Proposition 5.6 we have the following obvious characterization of the fundamental edge partition preserving subgroup.

Proposition 5.7. *Let X be a 2-connected graph. Then $\mathcal{E}$ is precisely the subgroup of automorphisms of $\mathrm{Clone}(X)$ that projects along the contraction. Moreover, the contraction induces a group homomorphism $\mathcal{E} \to \mathrm{Aut}(X)$.*

We have already remarked that $\mathrm{Clone}(C_n)$ is rather special for several reasons. Apart from the fact that its group of covering transformations is not trivial, it is also true that it does not respect the fundamental edge partition, in view of the next result and Theorem 5.2.

Theorem 5.8. *Let X be a 2-connected graph and let $A = \mathrm{Aut}(X)$. Then the maximal subgroup in the lifted group $\tilde{A}$ that respects the fundamental edge partition is the group $\bar{A}$ of diagonal mappings. In particular, the induced homomorphism $\mathcal{E} \to A$ is surjective.*

Proof. Choose an arc uv in X and its corresponding unique connecting arc $[(u,v),(v,u)]$ in $\mathrm{Clone}(X)$. If $\tilde{f} \in \tilde{A}$ preserves the fundamental edge partition, then it must map $[(u,v),(v,u)]$ to the unique connecting arc over $[f(u),f(v)]$, that is, to $[(f(u),f(v)),(f(v),f(u))]$. It follows that $\tilde{f}(u,v) = (f(u),f(v)) = \bar{f}(u,v)$, where $\bar{f}$ is the diagonal mapping. Since the covering graph is connected, a lift of $f \in \mathrm{Aut}(X)$ is uniquely determined by the mapping of a single vertex. Hence $\tilde{f} = \bar{f}$.

The final statement obviously holds since each $f \in A$ lifts to $\bar{f} \in \mathcal{E}$. This completes the proof. □

In order to identify the group $\mathcal{E}$, we need another definition. For a vertex $v \in V(X)$, let $A_{N(v)} \leq A_v$ denote the subgroup in the stabilizer of v fixing all vertices in the neighborhood $N(v)$ point-wise. For each $f \in A_{N(v)}$ let

$$f^\sharp(x,i) = \begin{cases} (x,i) & \text{if } i \neq v, \\ (f(x),v) & \text{if } i = v. \end{cases}$$

Clearly, $f^\sharp$ is an automorphism of $\mathrm{Clone}(X)$ that preserves the fundamental edge partition; its projection along the contraction is the identity automorphism of X, but $f^\sharp$ does not project along the covering projection; see below. Let $A^\sharp_{N(v)} = \{f^\sharp \mid f \in A_{N(v)}\}$. Note that $A^\sharp_{N(v)} \cong A_{N(v)}$. We are now ready to identify the fundamental edge partition preserving subgroup $\mathcal{E}$.

Theorem 5.9. *Let X be a 2-connected graph. Denote its automorphism group $Aut(X)$ by A, and let $\bar{A}$ be the group of diagonal mappings of* $\mathrm{Clone}(X)$*. Then $\mathcal{E}$ is the internal semi-direct product*

$$\Pi_{v\in V(X)} A^\sharp_{N(v)} \rtimes \bar{A}.$$

Proof. For each vertex $v \in V = V(X)$, a typical element of the Cartesian product of groups $\Pi_{v\in V} A^\sharp_{N(v)}$ has the form $\Pi_{v\in V} f^\sharp_v$, where $f_v \in A_{N(v)}$. Note further that $\Pi_{v\in V} A^\sharp_{N(v)}$ indeed exists as a group of automorphisms of $\mathrm{Clone}(X)$. Since each automorphism $f^\sharp_v$ projects to the identity along the contraction, we know that $\Pi_{v\in V} A^\sharp_{N(v)}$ is contained in the kernel $\mathcal{K}$ of the homomorphism $\mathcal{E} \to A$ induced by the contraction.

Conversely, let $\hat{f} \in \mathcal{E}$ be in the kernel $\mathcal{K}$. Then $\hat{f}$ fixes each layer set-wise, and its restriction to the v-th layer induces an automorphism f_v of the vertex deleted subgraph X_v. It follows that $\hat{f}$ can be written as the product $\hat{f} = \Pi_{v\in V} f^\sharp_v$, with commuting factors. Hence $\hat{f}$ is an element of the Cartesian product of subgroups $A^\sharp_{N(v)}$, $v \in V$, and so the kernel is precisely this group.

Since the homomorphism $\mathcal{E} \to A$ is surjective, $\mathcal{E}$ is an extension of its kernel $\mathcal{K}$ by A. Observe that the diagonal mapping $\bar{f} \in \bar{A}$ projects to f both along the covering projection and along the contraction. This means that the group $\bar{A}$ is a system of coset representatives of $\mathcal{K}$, and so $\mathcal{E}$ is a semi-direct product of $\Pi_{v\in V} A^\sharp_{N(v)}$ by $\bar{A}$. □

5.3 The full automorphism group of Clone

Let X be a 2-connected graph. We have seen that $A = \mathrm{Aut}(X)$ embeds in $\mathrm{Aut}(\mathrm{Clone}(X))$ as the group of diagonal mappings $\bar{A}$. Now $\mathrm{Clone}(X)$ may have other automorphisms, for two reasons.

Firstly, the group of covering transformations is not always trivial; this happens only when X is a cycle, and in that case the lifted group is in fact the full automorphism group of $\mathrm{Clone}(X)$. Secondly, the fundamental edge partition preserving subgroup $\mathcal{E}$ may be larger than $\bar{A}$. In view of Theorem 5.9, the latter happens if and only if there exists a nontrivial automorphism of X fixing a vertex and its neighboring vertices point-wise. Example 5.10 provides an instance of such a case.

Example 5.10. The automorphism group of $K_{2,3}$ is $S_2 \times S_3$ and has 12 elements. On the other hand, $\mathrm{Aut}(\mathrm{Clone}(K_{2,3}))$ has additional automorphisms of order two, and has 96 elements; see Figure 3. In fact, the full automorphism group is equal to $\mathcal{E}$, which is by Theorem 5.9 isomorphic to $\mathbb{Z}_2^3 \rtimes (S_3 \times \mathbb{Z}_2)$.

The case when when the full automorphism group of $\mathrm{Clone}(X)$ is equal to $\mathcal{E}$ is particularly interesting, since then the group $\mathrm{Aut}(\mathrm{Clone}(X))$ can be determined using Theorem 5.9. Moreover, the fact that every automorphism respects the fundamental edge-partition has strong impact on the transitivity properties of $\mathrm{Clone}(X)$.

Theorem 5.11. *Let X be a 2-connected graph such that every automorphism of $\mathrm{Clone}(X)$ respects the fundamental edge partition. Then the following hold.*

(a) $\mathrm{Clone}(X)$ *is not edge-transitive.*

(b) $\mathrm{Clone}(X)$ *is not vertex-transitive unless $X = K_n$, for some $n \geq 4$.*

Proof. If every automorphism of $\mathrm{Clone}(X)$ respects the fundamental edge partition, then no inner edge can be mapped by an automorphism to a connecting edge, and vice versa. Therefore $\mathrm{Clone}(X)$ is not edge-transitive. Also in this case, $\mathrm{Clone}(X)$ can be vertex-transitive only if all the vertex-deleted subgraphs of X are vertex-transitive, and isomorphic to each other. This only happens when X is a complete graph on more than 3 vertices. □

In certain cases it is easy to show that any automorphism of $\mathrm{Clone}(X)$ preserves the fundamental edge partition.

Proposition 5.12. *Let X be a 2-connected graph of girth g such that each edge of each vertex deleted subgraph is contained in a cycle of length at most $2g - 1$. Then each automorphism of* $\mathrm{Clone}(X)$ *preserves the fundamental edge partition.*

Proof. Since a layer naturally corresponds to a vertex deleted subgraph of X, each inner edge of $\mathrm{Clone}(X)$ belongs to a cycle of length at most $2g - 1$. On the other hand, no connecting edge belongs to such a cycle. The conclusion follows. □

There are several families of graphs which fulfill the conditions of Proposition 5.12. We state the following without proof.

Proposition 5.13.

- *The complete graph K_n for $n \geq 4$ has girth 3. Each vertex deleted subgraph of K_n is isomorphic to K_{n-1}; any edge of a complete graph on at least 3 vertices is contained in a 3-cycle.*

- *A hypercube Q_n for $n \geq 3$ has girth 4 and every edge in any vertex-deleted subgraph of Q_n lies in a 4-cycle.*
- *A cartesian product of 2-connected graphs has girth at most 4 and every edge in any vertex-deleted subgraph of such a graph lies in a 4-cycle.*
- *A generalized Petersen graph $G(n, 2)$ for $n > 8$ has girth 5 and every edge in any vertex-deleted subgraph lies in a 5-cycle or an 8-cycle.*

Moreover, the above families of graphs are all 2-connected.

Since several other generalized Petersen graphs also fulfill the conditions of Proposition 5.12, it would be interesting to characterize them. Such a characterization has to take into account the girth of every single member of the family of generalized Petersen graphs, which, in turn, was calculated in [1]. On the other hand, many interesting families of 2-connected graphs do not satisfy the condition of Proposition 5.12, yet their clone covers still have the required property, for instance, the wheel graphs W_n, $n \geq 6$. In contrast, the clone cover of a cycle is different: the fundamental edge partition preserving subgroup has index $n-1$ in the automorphism group of $\mathrm{Clone}(C_n)$. We believe that this is the only exception.

Conjecture 5.14. *Let X be a 2-connected graph that is not a cycle. Then any automorphism of $\mathrm{Clone}(X)$ preserves the fundamental edge partition.*

6 Concluding remarks

Note that $\mathrm{Clone}(K_n)$ has yet another nice description, namely, as the line graph of the first subdivision of K_n. Along these lines, $\mathrm{Clone}(K_n)$ was considered in [7], where it was shown that with few exceptions, $\mathrm{Clone}(K_n)$ is the only m-sheeted covering graph of K_n, for $m \leq n-1$, such that the full automorphism group of K_n has a lift.

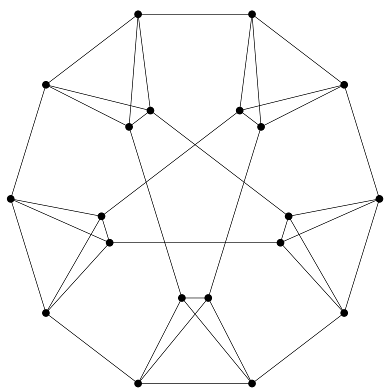


Figure 8: $\mathrm{Clone}(K_5)$ is not a regular cover. It is vertex- but not edge-transitive.

On the other hand, $\mathrm{Clone}(K_5)$ was studied in connection with a graph-theoretical interpretation of the Jahn-Teller effect [2]. In order to clarify the role of degenerate eigenvalues (that is, eigenvalues having higher multiplicities) in the Jahn-Teller distortion, graphs with symmetry group S_5 were sought [3]. It turned out that both K_5 and $\mathrm{Clone}(K_5)$ have their automorphism group isomorphic to S_5. Our paper gives, among other results, a theoretical background for this result. As noted in Proposition 5.13, the conditions of Proposition 5.12 are satisfied by K_n for $n \geq 4$. Hence according to Theorem 5.9, Aut(K_n) is isomorphic to Aut$(\mathrm{Clone}(K_n))$. Figure 8 depicts the graph $\mathrm{Clone}(K_5)$, which was used in [3].

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