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On geometric trilateral-free (n_3) configurations

Michael W. Raney
Georgetown University, Washington, DC, USA

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Abstract

This note presents the first known examples of a geometric trilateral-free (23_3) configuration and a geometric trilateral-free (27_3) configuration. The (27_3) configuration is also pentalateral-free.

Keywords: Configurations, trilaterals.

Math. Subj. Class.: 05B30, 51E30, 05C38

1 Introduction

A *(combinatorial)* (n_3) *configuration* is an incidence structure consisting of n distinct points and n distinct lines for which each point lies on exactly three lines, each line is incident with exactly three points, and any two points are incident with at most one common line. If an (n_3) configuration may be depicted in the Euclidean plane using points and (straight) lines, it is said to be *geometric*. As observed in [5] (pg. 17–18), it is evident that every geometric (n_3) configuration is combinatorial, but the converse of this statement does not hold.

Adopting the terminology from [2], we say that a *g-lateral* in a configuration is a cyclically ordered set $\{p_0, l_0, p_1, l_1, \ldots, l_{g-2}, p_{g-1}, l_{g-1}\}$ of pairwise distinct points p_i and pairwise distinct lines such that p_i is incident with l_{i-1} and l_i for each $i \in \mathbb{Z}_g$. Hence a 3-lateral is a trilateral, or triangle, a 4-lateral is a quadrilateral, and a 5-lateral is a pentalateral, according to the previously established nomenclature. A configuration is *g-lateral-free*, for a particular $g \in \{3, 4, 5\}$, if no g-lateral exists within the configuration.

Several recent papers (see [1], [3]) have examined triangle-free (n_3) configurations. The smallest example of a triangle-free configuration is the Cremona-Richmond (15_3) configuration. A theorem mentioned in [5] (Theorem 5.4.3, pg. 333) states the following:

Theorem 1.1. *For every $n \geq 15$ except $n = 16$ and possibly $n = 23$ and $n = 27$, there are geometric trilateral-free (n_3) configurations.*

E-mail address: mwr23@georgetown.edu (Michael W. Raney)

In this note we provide new examples of a geometric, triangle-free (23_3) configuration and a geometric, triangle-free (27_3) configuration, so that this theorem may now be modifed:

Theorem 1.2. *For every $n \geq 15$ except $n = 16$, there are geometric trilateral-free (n_3) configurations.*

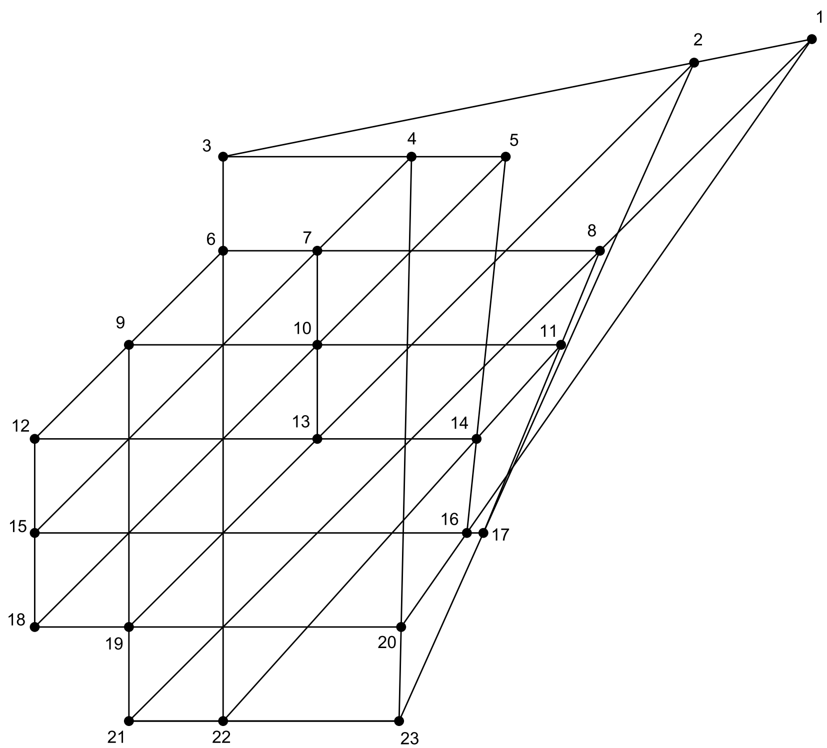
Additionally, the (27_3) configuration is also pentalateral-free. It serves as the smallest known example of a geometric (n_3) configuration that is both 3-lateral-free and 5-lateral-free; the formerly smallest known example of such a configuration is a (51_3) configuration [2].

2 The examples

Configuration tables and diagrams of both of these new configurations $\mathcal{C}_1$ and $\mathcal{C}_2$ are provided below, and rational coordinates for their geometric realizations are given. Verification that the former configuration is trilateral-free, and that the latter configuration is trilateral-free and pentalateral-free, has been conducted using *Mathematica*.

2.1 $\mathcal{C}_1$, a geometric triangle-free (23_3) configuration

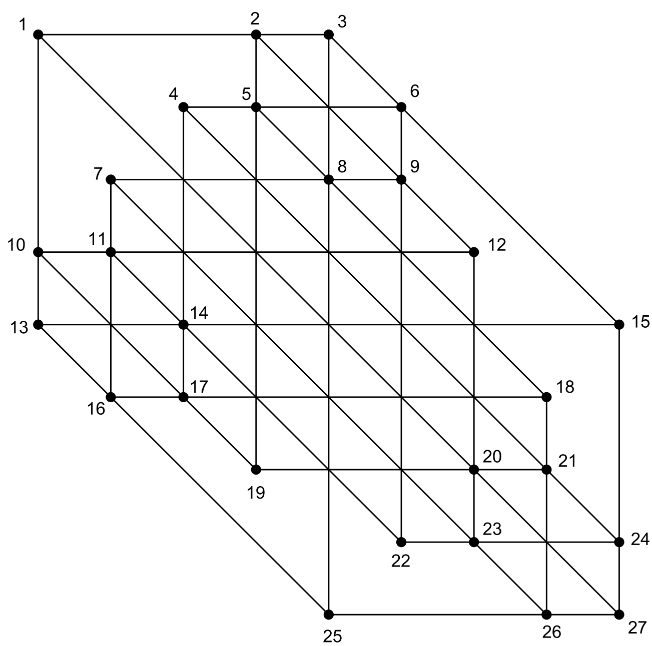
$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 8 & 9 & 9 & 11 & 12 & 12 & 15 & 18 & 21 \\ 2 & 8 & 16 & 13 & 17 & 4 & 6 & 7 & 20 & 10 & 14 & 7 & 9 & 10 & 11 & 10 & 19 & 14 & 13 & 15 & 16 & 19 & 22 \\ 3 & 21 & 20 & 19 & 23 & 5 & 22 & 15 & 23 & 18 & 16 & 8 & 12 & 13 & 17 & 11 & 21 & 22 & 14 & 18 & 17 & 20 & 23 \end{pmatrix}$$



$Point$	$Coordinates$	$Point$	$Coordinates$	$Point$	$Coordinates$
1	$(33/4, 29/4)$	2	$(7,7)$	3	$(2,6)$
4	$(4,6)$	5	$(5,6)$	6	$(2,5)$
7	$(3,5)$	8	$(6,5)$	9	$(1,4)$
10	$(3,4)$	11	$(542/97, 4)$	12	$(0,3)$
13	$(3,3)$	14	$(455/97, 3)$	15	$(0,2)$
16	$(445/97, 2)$	17	$(462/97, 2)$	18	$(0,1)$
19	$(1,1)$	20	$(1132/291, 1)$	21	$(1,0)$
22	$(2,0)$	23	$(1876/485, 0)$		

2.2 $\mathcal{C}_2$, a geometric triangle-free, pentalateral-free (27_3) configuration

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 4 & 5 & 6 & 7 & 7 & 7 & 10 & 10 & 11 & 12 & 13 & 13 & 15 & 16 & 18 & 19 & 22 & 25 \\ 2 & 10 & 20 & 5 & 9 & 6 & 8 & 5 & 14 & 21 & 8 & 9 & 8 & 11 & 23 & 11 & 17 & 14 & 20 & 14 & 16 & 24 & 17 & 21 & 20 & 23 & 26 \\ 3 & 13 & 27 & 19 & 12 & 15 & 25 & 6 & 17 & 24 & 18 & 22 & 9 & 16 & 26 & 12 & 19 & 22 & 23 & 15 & 25 & 27 & 18 & 26 & 21 & 24 & 27 \end{pmatrix}$$



$Point$	$Coordinates$	$Point$	$Coordinates$	$Point$	$Coordinates$
1	$(0,8)$	2	$(3,8)$	3	$(4,8)$
4	$(2,7)$	5	$(3,7)$	6	$(5,7)$
7	$(1,6)$	8	$(4,6)$	9	$(5,6)$
10	$(0,5)$	11	$(1,5)$	12	$(6,5)$
13	$(0,4)$	14	$(2,4)$	15	$(8,4)$
16	$(1,3)$	17	$(2,3)$	18	$(7,3)$
19	$(3,2)$	20	$(6,2)$	21	$(7,2)$
22	$(5,1)$	23	$(6,1)$	24	$(8,1)$
25	$(4,0)$	26	$(7,0)$	27	$(8,0)$

3 Motivation for the results

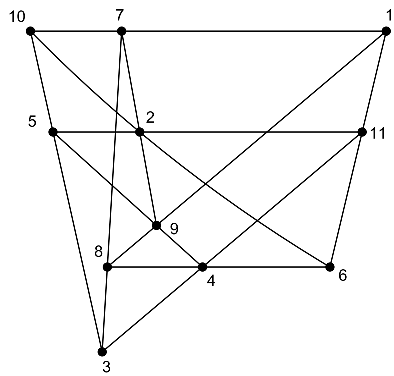
Both $\mathcal{C}_1$ and $\mathcal{C}_2$ have arisen serendipitously in conjunction with the author's study of *magic* (n_3) configurations. An (n_3) configuration is said to be magic if it is possible to assign the integers $\{1, 2, \ldots, n\}$ as labels for its n points, where each integer is used exactly once, in such a manner that the sum of the point labels along each line of the configuration is always the same magic constant, M. Since each point of the configuration is involved in three such sums, we see that

$$\begin{aligned} nM &= 3\sum_{i=1}^{n} i = 3\frac{n(n+1)}{2} \\ M &= \frac{3}{2}(n+1) \end{aligned}$$

Hence n must be odd (and at least 7) for a magic configuration to be possible. The smallest example of a magic configuration turns out to one of the 31 (11_3) configurations. Its combinatorial table is

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \\ 6 & 7 & 8 & 5 & 6 & 7 & 4 & 5 & 7 & 5 & 6 \\ 11 & 10 & 9 & 11 & 10 & 9 & 11 & 10 & 8 & 9 & 8 \end{pmatrix}$$

This configuration is $(11_3)_{17}$, according to the (11_3) configuration labeling scheme initiated in [4] and referenced in [6],[7].



Magic (n_3) configurations have not, to the author's knowledge, been previously considered in the literature on configurations, although other magic configurations such as magic stars have been studied [8].

$\mathcal{C}_1$ is dual to the magic (23_3) configuration

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 6 & 6 & 6 & 7 & 7 & 7 & 8 & 9 \\ 13 & 14 & 15 & 11 & 12 & 15 & 10 & 15 & 16 & 12 & 13 & 14 & 8 & 9 & 14 & 9 & 10 & 13 & 8 & 10 & 11 & 12 & 11 \\ 22 & 21 & 20 & 23 & 22 & 19 & 23 & 18 & 17 & 20 & 19 & 18 & 23 & 22 & 17 & 21 & 20 & 17 & 21 & 19 & 18 & 16 & 16 \end{pmatrix}$$

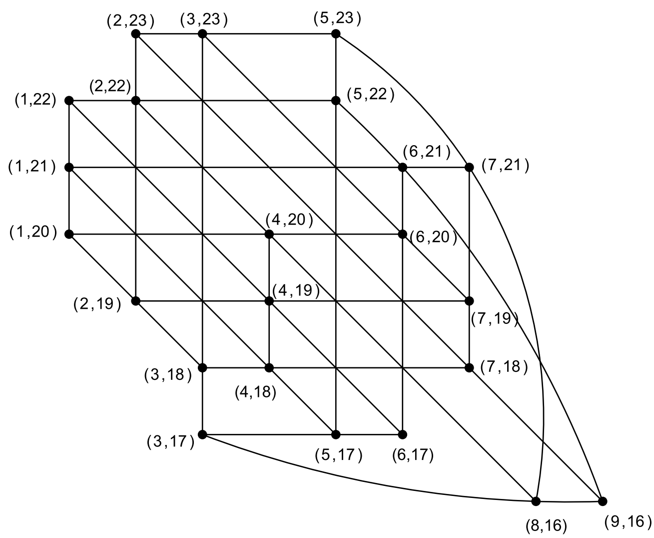
with magic constant $\frac{3}{2}(23+1) = 36$. Also, $\mathcal{C}_2$ is dual to the magic (27_3) configuration

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 6 & 6 & 6 & 7 & 7 & 7 & 8 & 8 & 8 & 9 & 9 & 9 \\ 14 & 17 & 18 & 15 & 16 & 18 & 13 & 16 & 17 & 11 & 12 & 17 & 10 & 12 & 18 & 10 & 11 & 16 & 11 & 14 & 15 & 12 & 13 & 15 & 10 & 13 & 14 \\ 27 & 24 & 23 & 25 & 24 & 22 & 26 & 23 & 22 & 27 & 26 & 21 & 27 & 25 & 19 & 26 & 25 & 20 & 24 & 21 & 20 & 22 & 21 & 19 & 23 & 20 & 19 \end{pmatrix}$$

with magic constant $\frac{3}{2}(27+1) = 42$. This means that in each case there exists an isomorphism between $\mathcal{C}_i$ and its dual that interchanges the roles of points and lines while preserving incidence structure. We say that the dual configuration of a magic configuration is *comagic*; hence $\mathcal{C}_1$ and $\mathcal{C}_2$ are comagic. So for both $\mathcal{C}_1$ and $\mathcal{C}_2$ it is possible to label its lines in such a manner that the sum of the labels of the three lines incident to any point of the configuration is always the same magic constant, again $\frac{3}{2}(n+1)$.

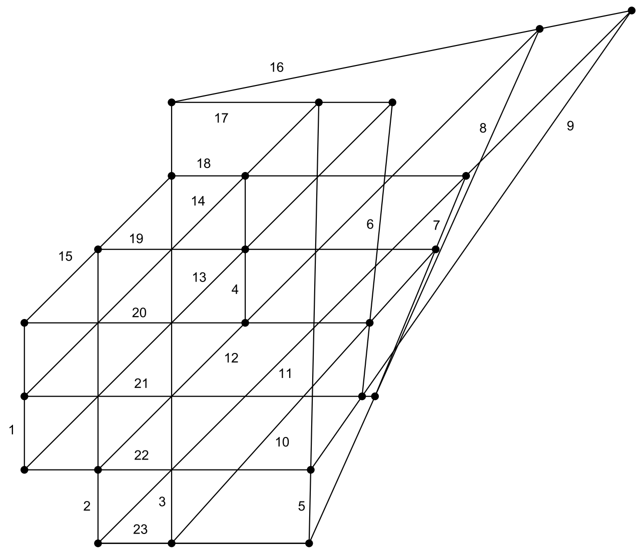
It turns out that a diagram associated with a comagic configuration may be conveniently constructed. Suppose that $(x_1\ \ x_2\ \ x_3)^T$ is a line in the configuration table of the original magic configuration, where $x_1 < x_2 < x_3$. It follows that the point $(x_1, x_2, x_3) \in \mathbb{R}^3$ lies in the plane $\{(x, y, z) \in \mathbb{R}^3 \colon x + y + z = \frac{3}{2}(n+1)\}$. After plotting each corresponding point in this plane, for $k = 1, \ldots, n$ we connect three points with an arc (labeled k) if the three points share k as a coordinate. We thereby produce a diagram within the plane $x + y + z = \frac{3}{2}(n+1)$.

Next, we project the diagram onto the xz-plane by simply eliminating the y-coordinate. No information about the configuration is lost when doing this, since for any point we may recapture $x_2 = \frac{3}{2}(n+1) - x_1 - x_3$. Below is a diagram for $\mathcal{C}_1$ achieved in this fashion with each (x_1, x_3) point indicated.



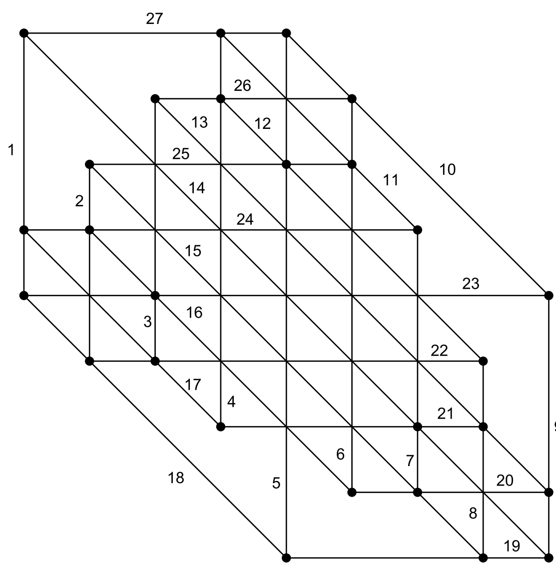
Observe that this diagram has three nonlinear arcs. After some algebraic manipulation involving shifting seven of the 23 points, we find that it is possible to recast the diagram so that all of the arcs indeed are straight lines. After rescaling the points (via the transformation $(x, z) \mapsto (x-1, 23-z)$) we arrive at the geometric realization for $\mathcal{C}_1$ provided in Section 2.1.

We again depict the diagram for $\mathcal{C}_1$, this time with its associated magic line labeling.



When undergoing this process for the (27_3) configuration, we discover pleasantly that no shifting of arcs is required. This is a consequence of each line $(x_1 \;\; x_2 \;\; x_3)^T$ satisfying the conditions $1 \leq x_1 \leq 9$, $10 \leq x_2 \leq 18$, and $19 \leq x_3 \leq 27$. After lopping off the x_2-coordinates and rescaling the resulting points (via the transformation $(x, z) \mapsto (x - 1, z - 19)$) we arrive at the geometric realization for $\mathcal{C}_2$ provided in Section 2.2.

We display the diagram of $\mathcal{C}_2$ again with its associated magic line labeling.



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