# Optimal velocity profile planning considering velocity, acceleration and jerk constraints

**Martina Loknar, Gregor Klančar, Sašo Blažič**

*Faculty of Electrical Engineering, Tržaška cesta 25, SI-1000 Ljubljana, Slovenia*
*E-mail: martina.loknar@fe.uni-lj.si*

## Abstract

*The aim of this study is to determine time-minimal velocity profile for a wheeled mobile robot whose movement complies with velocity, acceleration and jerk constraints and is restricted to an arbitrary predefined path. The proposed two-step algorithm first enables the determination of the velocity profile that respects the speed and acceleration constraints and in the second step additionally applies the limitation of the jerk.*

## 1 Introduction

Optimal control theory describes the application of forces to a system for the purpose of maximizing some measure of performance or minimizing a cost function [1]. It is an extension of the calculus of variations and this mathematical optimization method is largely due to the work of [2]. Using Pontryagin's maximum principle, in [3] the authors demonstrated that all time-optimal motions of the mobile platform with two independently driven wheels occur for controls that are at each instant on the upper or lower limit.

Mechanical systems, including biological systems, have maximum allowances related to various dynamic variables, above which the components of the system may begin to fail. Excess of jerk thresholds is associated with mechanical wear, tool life repercussion and adverse effect on the actuator performance, degradation of machine junctions, and in biological systems tear of ligaments and muscles or breakage of bones [4], [5]. Minimizing jerk is therefore beneficial in reducing stress and wear of the mechanical components, extending machine or tool life, reducing position tracking errors, minimizing the excitation of vibrations in general, enabling better quality finishes in machining tasks and ensuring that the motor is able to provide the requested current fast enough. Mechanical and robotics engineers have recognized the benefits of jerk minimization and therefore prefer to design jerk-limited profiles [6].

Authors in [7] investigated time optimal two-stage path planing under kinematic and dynamic constraints and obtained the shortest path and a time optimal velocity profile. A path planning technique to minimise the time of reaching the end point in desired direction and with desired velocity is presented in [8]. In [9] the authors suggested a methodology to generate minimum time optimal velocity profiles for a vehicle with prescribed acceleration limits along a specified path. Similarly, a method for minimum-time velocity planning with velocity, acceleration and jerk constraints was proposed, based on a sequence of linear programming feasibility checks, depending on motion constraints and generic boundary conditions [10].

This paper outlines a new approach of determining time-minimal optimal velocity profile for a wheeled mobile robot on a predefined path. We believe that we have found an innovative solution that is easy to implement and computationally undemanding with coherent course of calculation that relies heavily on analytical expressions of given physical quantities. The first step of the algorithm ensures that the resulting velocity profile complies with speed and acceleration constraints. The additional jerk constraints are considered in the second step of the algorithm, where the acceleration discontinuities are eliminated by smoothing the velocity profile from the first step.

## 2 Problem formulation

Let the motion of a particle along a three times continuously differentiable plane curve $\mathcal{C}$ be described as a function of time $t \in [0, t_f]$ by the position vector $\mathbf{r}(t)$ measured from a given fixed origin. Velocity $\mathbf{v}(t)$, acceleration $\mathbf{a}(t)$ and jerk $\mathbf{j}(t)$ vectors can be expressed in the tangential-normal form as:

$$\mathbf{v}(t) = v(t) \cdot \hat{\mathbf{T}} \tag{1a}$$

$$\mathbf{a}(t) = a_T(t) \cdot \hat{\mathbf{T}} + a_R(t) \cdot \hat{\mathbf{N}} \tag{1b}$$

$$\mathbf{j}(t) = j_T(t) \cdot \hat{\mathbf{T}} + j_R(t) \cdot \hat{\mathbf{N}}, \tag{1c}$$

where $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ are the unit tangent vector and the unit normal vector, respectively:

$$\hat{\mathbf{T}}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}, \quad \hat{\mathbf{N}}(t) = \frac{\dot{\hat{\mathbf{T}}}(t)}{\|\dot{\hat{\mathbf{T}}}(t)\|}. \tag{2}$$

Given a feasible path from initial to final point, the optimization problem is to find the velocity profile $v(t)$ that reaches the end of the path in minimum time in a way that none of the velocity, acceleration or jerk constraints from

Eqs. $(3a, 3b, 3c)$ are violated:

$$0 \leq \|\mathbf{v}(t)\| \leq v_{MAX}; \quad \forall t \in [0, t_f], \quad (3a)$$

$$\frac{a_T^2(t)}{a_{T_{MAX}}^2} + \frac{a_R^2(t)}{a_{R_{MAX}}^2} \leq 1; \quad \forall t \in [0, t_f], \quad (3b)$$

$$\frac{j_T^2(t)}{j_{T_{MAX}}^2} + \frac{j_R^2(t)}{j_{R_{MAX}}^2} \leq 1; \quad \forall t \in [0, t_f]. \quad (3c)$$

We defined the acceleration and jerk constraints in a similar way as [8].

## 3 Optimal velocity profile algorithm

The predefined path $\mathcal{C}$ in the two-dimensional space is given as $\mathbf{s_p}(u) = [x_p(u), y_p(u)]^{\mathrm{T}}$ with parameter $u \in [u_0, u_f]$. The parametrized velocity is:

$$\mathbf{v_p}(u) = \frac{d\mathbf{s_p}(u)}{du} = \left[x_p'(u), y_p'(u)\right]^{\mathrm{T}} \quad (4)$$

with the magnitude:

$$v_p(u) = \|\mathbf{v_p}(u)\| = \sqrt{(x_p'(u))^2 + (y_p'(u))^2}. \quad (5)$$

The orientation equals the four-quadrant inverse tangent of the quotient of Cartesian components of the translational velocity:

$$\phi_p(u) = \operatorname{atan2}(y_p'(u), x_p'(u)), \quad (6)$$

from which follows the expression for the magnitude of parametrized angular velocity:

$$\omega_p(u) = \frac{d\phi_p(u)}{du} = \frac{x_p'(u) \cdot y_p''(u) - x_p''(u) \cdot y_p'(u)}{x_p'^2(u) + y_p'^2(u)}. \quad (7)$$

The curvature $\kappa_p(u)$ is:

$$\kappa_p(u) = \frac{d\phi_p(u)}{dl} = \frac{d\phi_p(u)}{du} \cdot \frac{du}{dl} = \frac{x_p'(u) \cdot y_p''(u) - y_p'(u) \cdot x_p''(u)}{(x_p'(u)^2 + y_p'(u)^2)^{3/2}}, \quad (8)$$

where $l$ is the arc length parameter of a curve.

### 3.1 Step 1: Complying with velocity and acceleration constraints

In order to apply actual velocity and acceleration restrictions, the two operating physical quantities ought to be expressed as functions of time.

The robot's movement on a path can be described by monotonously increasing parameter $u$ of the curve or by time $t$; the former measuring the position on a trajectory and the latter the time at which a certain position on a path is reached. We present the dependence of $u$ on $t$ by schedule $u(t)$ and as a result the parametrized functions $v_p$, $\omega_p$, $\kappa_p$ defined in Eq. (5, 7, 8) become composite functions $v_p(u(t))$, $\omega_p(u(t))$ and $\kappa_p(u(t))$. Applying the chain rule allows us to calculate the magnitudes $v(t)$ and $\omega(t)$:

$$v(t) = \left\|\frac{d\mathbf{s_p}(u(t))}{du} \cdot \frac{du}{dt}\right\| = v_p(u(t)) \cdot \dot{u}(t), \quad (9a)$$

$$\omega(t) = \frac{d\phi_p(u(t))}{du} \cdot \frac{du}{dt} = \omega_p(u(t)) \cdot \dot{u}(t). \quad (9b)$$

Expressing the velocity and angular velocity solely as functions of time reveals that the time dependant velocities differ from the corresponding parametrized ones by a factor of $\dot{u}(t)$. Obtaining the desired velocity profile thus requires calculating the schedule $u = u(t)$ and its time derivative $\dot{u}(t)$. The curvature, unlike the velocity or angular velocity, does not depend on the parametrization of the curve:

$$\kappa(t) = \kappa_p(u(t)). \quad (10)$$

Acceleration vector is the derivative of velocity vector from Eq. (1a):

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \dot{v}(t) \cdot \hat{\mathbf{T}} + v(t) \cdot \dot{\hat{\mathbf{T}}}. \quad (11)$$

The time derivatives of unit tangential and unit normal vector can be expressed from the Frenet–Serret formulas:

$$\dot{\hat{\mathbf{T}}}(t) = \kappa(t) \cdot v(t) \cdot \hat{\mathbf{N}}(t), \quad (12a)$$

$$\dot{\hat{\mathbf{N}}}(t) = -\kappa(t) \cdot v(t) \cdot \hat{\mathbf{T}}. \quad (12b)$$

Using the equality from Eq. (12a) we find that:

$$\mathbf{a}(t) = \dot{v}(t) \cdot \hat{\mathbf{T}}(t) + \kappa(t) \cdot v^2(t) \cdot \hat{\mathbf{N}}(t). \quad (13)$$

The expressions for the normal and tangential components with respect to time follow from Eqs. (1a, 9a, 13):

$$a_T(t) = v_p'(u(t)) \cdot \dot{u}(t) + v_p(u(t)) \cdot \ddot{u}(t), \quad (14a)$$

$$a_R(t) = \kappa_p(u(t)) \cdot v_p^2(u(t)) \cdot \dot{u}^2(t). \quad (14b)$$

The proposed method of incorporating speed and acceleration constraints in the calculation of velocity profile, indirectly determined by schedules $u(t)$ and $\dot{u}(t)$, begins with the identification of $N$ points on the curve, the so-called turning points, where the curvature reaches the local maximum. The value of parameter $u$ in the $i-$th turning point ($i = \{1, \ldots, N\}$) is denoted as $u_{TP_i}$. The speed in these points reaches local minimum, the tangential acceleration $a_T(t)$ is therefore zero and radial acceleration $a_R(t)$ is maximal. From Eq. (14b) it follows:

$$\dot{u}_{TPi} = \sqrt{\frac{a_{R_{MAX}}}{v_p^2(u_{TPi}) \cdot \kappa_p(u_{TPi})}}. \quad (15)$$

It is also possible to implement the initial and final speed requirements by treating the initial and final point of the trajectory similarly as the turning points and using Eq. (9a). For values of $\dot{u}$ we get:

$$\dot{u}_{TP_0} = \frac{v|_{t=0}}{v_p(u_{TP_0})}, \quad \dot{u}_{TP_{N+1}} = \frac{v|_{t=t_f}}{v_p(u_{TP_{N+1}})}, \quad (16)$$

where the initial and the final point are denoted as $TP_0$ and $TP_{N+1}$, respectively.

To get the complete velocity profile $v(t)$ of the mobile robot, the values of $u$ and $\dot{u}$ have to be determined also in between the turning points. Our realization of the proposed method for this calculation stems from knowing the fixed values of $u_{TP_i}$ and $u\dot{_{TP_i}}$ for $i \in \{0, 1, \ldots, N+$

$1\}$ and the analytical formula for $\ddot{u}$ as a function of $u$ and $\dot{u}$ that follows from Eqs. (14a, 14b, 3b, 5):

$$\ddot{u}(t) = \pm\, a_{T_{MAX}} \sqrt{\frac{1}{x_p'^2 + y_p'^2} - \frac{(x_p'^2 + y_p'^2)\kappa_p^2}{a_{R_{MAX}}^2} \dot{u}^4(t)} - \frac{x_p' x_p'' + y_p' y_p''}{x_p'^2 + y_p'^2} \dot{u}^2(t), \tag{17}$$

where all the quantities with the index $p$ depend directly on $u$. The basic idea of the algorithm that returns schedules $u(t)$ and $\dot{u}(t)$ is to find the solution to the initial value problem by applying a numerical method of solving ordinary differential equations; or more specifically: to integrate backward and forward in time around each turning point where the initial conditions $u_{TP_i}$ and $\dot{u}_{TP_i}$ are set in order to determine the discrete values of $t$, $u$ and $\dot{u}$. According to Euler's method (the simplest explicit iterative method) the values $u_{k+1}$ and $\dot{u}_{k+1}$ in the $(k+1)$-th step of the calculation are:

$$\dot{u}_{k+1} = \dot{u}_k \pm \ddot{u}_k \cdot T_s, \tag{18a}$$

$$u_{k+1} = u_k \pm \dot{u}_k \cdot T_s, \tag{18b}$$

where $T_s$ is the sampling time and $u_k$, $\dot{u}_k$, $\ddot{u}_k$ the values calculated in the $k$-th step and perform as the current initial values and/or slopes with negative or positive sign (backward/forward integration). The sought-after schedule $\dot{u}(u(t))$ is defined by the minimum of the separate profiles around the turning points as shown in Figure (1).

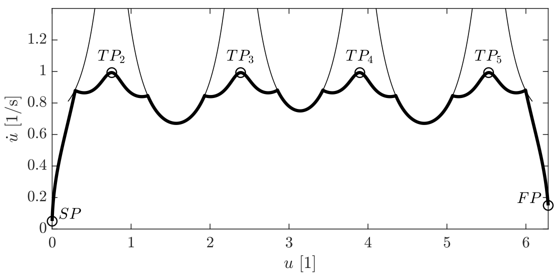


Figure 1: Resulting $\dot{u}(u(t))$ profile (bold line) on a path $\mathbf{s_p}(u) = (\cos(u), \sin(2u)), u \in [0, 2\pi]$ is determined by the separate profiles $\dot{u}(u(t))$ around the $TP_i$ (thin lines).

### 3.2 Step 2: Complying with jerk constraints

The resulting velocity profile of the first step of the calculation is infeasible for an actual implementation on a robot due to the sudden changes of acceleration. To apply jerk constraints to the existing velocity profile the expression from Eq. (1c) can be written more specifically by differentiating acceleration vector in Eq. (13) using Eqs. (12a, 12b):

$$\mathbf{j}(t) = \frac{d\mathbf{a}(t)}{dt} = \left(\ddot{u}(t) - v^3(t) \cdot \kappa^2(t)\right) \hat{\mathbf{T}}(t) + \frac{1}{v(t)} \cdot \left(\frac{d}{dt} v^3(t) \cdot \kappa(t)\right) \hat{\mathbf{N}}(t). \tag{19}$$

From Eqs. (9a, 19) we get the following expressions for the tangential component of the jerk:

$$\begin{aligned} j_T(t) &= v_p''(u(t)) \cdot \dot{u}^3(t) + 3v_p' \dot{u}(t) \cdot \ddot{u}(t) \\ &\quad - v_p^3(u(t)) \cdot \dot{u}^3(t) \cdot \kappa_p^2(u(t)) \\ &\quad + v_p(u(t)) \cdot \dddot{u}(t), \end{aligned} \tag{20}$$

and its radial component:

$$\begin{aligned} j_R(t) &= 3v_p'(u(t)) \cdot v_p(u(t)) \cdot \dot{u}^3(t) \cdot \kappa_p(u(t)) + \\ &\quad + 3v_p^2(u(t)) \cdot \kappa_p(u(t)) \cdot \dot{u}(t) \cdot \ddot{u}(t) + \\ &\quad + v_p^2(u(t)) \cdot \dot{u}^3(t) \cdot \kappa_p'(u(t)). \end{aligned} \tag{21}$$

The third time derivative of parameter $u(t)$ with implemented jerk constraints follows from Eq. (20):

$$\dddot{u} = \frac{1}{v_p} \cdot \left(j_T(t) - v_p'' \dot{u}^3 - 3v_p' \dot{u}\ddot{u} + v_p^3 \dot{u}^3 \kappa_p^2\right). \tag{22}$$

Using Eq. (22) and Eq. (3c) to determine the value of $j_T(t)$, the aim of the second step of calculation is to smooth the intervals in the original velocity profile that contain points with abrupt changes of acceleration. Similarly as in the Eq. (18a, 18b), forward Euler integration is applied, yet with an additional calculation in the $(k+1)$-th step:

$$\ddot{u}_{k+1} = \ddot{u}_k + \dddot{u}_k \cdot T_s. \tag{23}$$

To determine the adequate initial value of Euler method, let us first introduce $u_{CP_\ell}$ and its corresponding time derivative $\dot{u}_{CP_\ell}$, $\ell \in \{1, \ldots, M\}$, as the value of $u$ and $\dot{u}$ in the critical points on the curve, where the acceleration is discontinuous. Two guesses for the initial value of $u$ for each critical point $CP_\ell$, $u_{L_\ell}$ and $u_{H_\ell}$, are then selected according to the following restriction:

$$u_{CP_{\ell-1}} < u_{L_\ell} < u_{H_\ell} < u_{CP_\ell}. \tag{24}$$

Euler integration from either $u_{H_\ell}$ and $u_{L_\ell}$ onward produces a smooth extension to the original profile $\dot{u}(u)$; in the former case the extension forms an obtuse angle at point of intersection with the original profile (Figure (2), left) and in the latter case the extension never reconnects back onto the original profile (Figure (2), middle). The correct initial value for the forward Euler method is found by bisecting the interval $[u_{L_\ell}, u_{H_\ell}]$. The solution is a range of values of $u$ and $\dot{u}$ that does not introduce additional discontinuities when inserted into the existing $\dot{u}(u)$ profile (Figure 2, right).

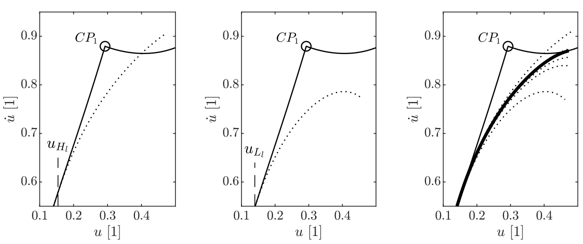


Figure 2: A display of iteration steps of bisection for determination of a smooth $\dot{u}(u(t))$ profile that eliminates acceleration discontinuities.

# 4 Results

In order to demonstrate our proposed method, the problem is defined as follows: Compute the optimal velocity profile that will result in the shortest travelling time for a path $x_p(u) = \cos(u)$, $y_p(u) = \sin(2u)$ where $u \in [0, 2\pi]$ with the following restrictions: $v_{MAX} = 1.5\,\mathrm{m/s}$, $a_{T_{MAX}} = 2\,\mathrm{m/s^2}$, $a_{R_{MAX}} = 4\,\mathrm{m/s^2}$, $j_{T_{MAX}} = 10\,\mathrm{m/s^3}$ and $j_{R_{MAX}} = 10\,\mathrm{m/s^3}$.

Figure (3) shows $\dot{u}(u)$ dependence after the first ($\dot{u}_I(u)$) and the second step ($\dot{u}_{II}(u)$) of the calculation of the optimal velocity profile. The velocity profile $v_I(t)$ in Figure (4) respects speed and acceleration restrictions, but the final velocity profile $v_{II}(t)$ complies also with the jerk constraints. Figure (5) shows the temporal dependence of normalized values of speed, acceleration and jerk restrictions and proves optimality because at any given moment on the curve one of these dynamical quantities reaches its maximum.

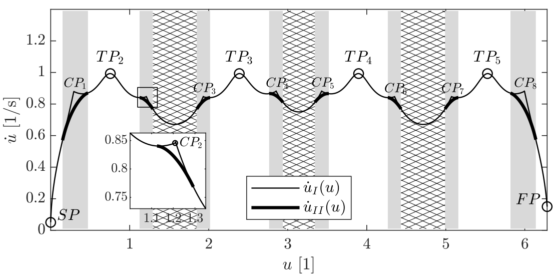


Figure 3: $\dot{u}_I(u)$ and $\dot{u}_{II}(u)$ with identified $TP_i$ and $CP_i$. The hatched areas represent speed restriction intervals, shaded areas jerk restriction intervals and areas with the white background acceleration restriction intervals.

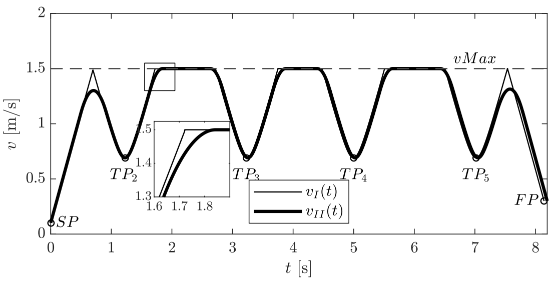


Figure 4: $v_I(t)$ and $v_{II}(t)$ profiles.

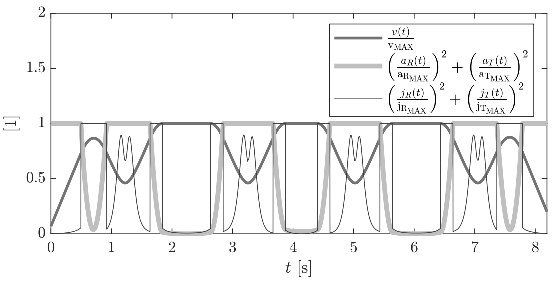


Figure 5: Normalized dynamical restrictions over time.

# 5 Conclusion

The results of this study support the proposed idea of generating an optimal velocity profile for a wheeled mobile robot that moves on an arbitrary path as in every point in time of the movement the robot has either maximum allowed speed, acceleration or jerk.

The present study has only investigated and limited jerk at the switch-overs from the intervals of restricted speed movement to the intervals of restricted acceleration movement and vice versa, where the infinite delta impulses of the jerk were expected. Generally speaking, the non-compliance to the jerk limitations could also appear in other points or intervals of the path if only $j_{T_{MAX}}$ and $j_{R_{MAX}}$ were set low enough. We are now in the process of investigating this problem with significantly greater complexity.


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