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On the eigenvalues of complete bipartite signed graphs*

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Abstract

Let $\Gamma = (G, \sigma)$ be a signed graph, where σ is the sign function on the edges of G. The adjacency matrix of Γ is defined canonically. Let $(K_{p,q}, \sigma)$, $p \leq q$, be a complete bipartite signed graph with bipartition (U_p, V_q), where $U_p = \{u_1, u_2, \dots, u_p\}$ and $V_q = \{v_1, v_2, \dots, v_q\}$. Let $(K_{p,q}, \sigma)[U_r \cup V_s]$, $r \leq p$ and $s \leq q$, be an induced signed subgraph on minimum vertices $r+s$, which contains all negative edges of the signed graph $(K_{p,q}, \sigma)$. In this paper, we show that the nullity of the signed graph $(K_{p,q}, \sigma)$ is at least $p+q-2k-2$, where $k = \min(r, s)$. The spectrum of a complete bipartite signed graph whose negative edges induce either a disjoint complete bipartite subgraphs or a path is determined. Finally, we obtain the spectrum of a complete bipartite signed graph whose negative edges (positive edges) induce a regular subgraph H. It turns out that there is a relationship between the eigenvalues of this complete bipartite signed graph and the non-negative eigenvalues of H.

Keywords: Signed graph, adjacency matrix, nullity, spectrum of complete bipartite graph.

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1 Introduction

A signed graph (or briefly sigraph) Γ is an ordered pair (G, σ), where $G = (V(G), E(G))$ is a graph (called the underlying graph), and $\sigma\colon E(G) \longrightarrow \{-1, 1\}$ is a sign function defined on the edge set of G. A signed graph is all-positive (resp. all-negative) if all of its edges are positive (resp. negative) and is denoted by $\Gamma = (G, +)$ (resp. $\Gamma = (G, -)$). The sign of a cycle in a signed graph is the product of the signs of its edges. A signed cycle is said to be positive (resp. negative) if its sign is positive (resp. negative). A signed graph is said to be balanced if none of its cycles is negative, otherwise unbalanced.

Let $A(G) = (a_{ij})$ be the adjacency matrix of G. The adjacency matrix of a signed graph $\Gamma = (G, \sigma)$ is a square matrix $A(\Gamma) = A(G, \sigma) = \left(a^{\sigma}_{ij}\right)$, where $a^{\sigma}_{ij} = \sigma\left(v_i v_j\right) a_{ij}$. For a matrix Z, the characteristic polynomial $|xI - Z|$ will be denoted by $\phi(Z, x)$. If Γ is a signed graph, we use $\phi(\Gamma, x)$ instead of $\phi(A(\Gamma), x)$. The eigenvalues of $A(\Gamma)$ are the eigenvalues of the signed graph Γ. The set of all distinct eigenvalues of Γ along with their multiplicities is called the spectrum of Γ. If the distinct eigenvalues of Γ are $\mu_1 > \cdots > \mu_k$, and their multiplicities are $m\left(\mu_1\right), \ldots, m\left(\mu_k\right)$, then we write

$$\operatorname{Spec}(\Gamma) = \begin{pmatrix} \mu_1 & \ldots & \mu_k \\ m\left(\mu_1\right) & \ldots & m\left(\mu_k\right) \end{pmatrix}.$$

The nullity of a signed graph Γ is the multiplicity of the eigenvalue 0 in its spectrum. It is denoted by $\eta(\Gamma)$.

Two signed graphs $\Gamma_1 = (G_1, \sigma_1)$ and $\Gamma_2 = (G_2, \sigma_2)$ are isomorphic if there is a graph isomorphism $f\colon G_1 \to G_2$ that preserves signs of the edges. If $\theta\colon V(G) \to \{+1, -1\}$ is the switching function, then switching of the signed graph $\Gamma = (G, \sigma)$ by θ means changing σ to σ^{θ} defined by

$$\sigma^{\theta}(uv) = \theta(u)\sigma(uv)\theta(v).$$

For more information about switching and recent work on signed graphs, we refer to [1, 3, 4, 5, 6, 8, 11, 12, 13, 14, 15, 16, 17].

We note that the sign function for the signed subgraph is the restriction of the original function. For $\Gamma = (G, \sigma)$ and $X \subseteq V(G)$, $\Gamma[X]$ denotes the signed subgraph induced by X, while $\Gamma - X = \Gamma[V(G) \backslash X]$. Sometimes, we also write $\Gamma - \Gamma[X]$ instead of $\Gamma - X$. Let (G, K^-) (resp. (G, K^+)) be the signed graph whose negative edges (resp. positive edges) induce a subgraph K. As usual, K_n denotes the complete graph of order n. The complete bipartite graph with two parts $U_p = \{u_1, u_2, \ldots, u_p\}$ and $V_q = \{v_1, v_2, \ldots, v_q\}$ as a partition of its vertex set is denoted by $K_{p,q}$. Also, P_n denotes the path on n vertices. $J_{r\times s}$ denotes an all-one matrix of size $r \times s$ and $O_{r\times s}$ denotes an all-zero matrix of size $r \times s$.

In the recent years, several researchers have shown interest in signed graphs, including complete bipartite signed graphs and complete signed graphs, which have a variety of applications, see [1, 2] and the references therein.

The remainder of the paper is organized as follows. In Section 2, we give some preliminary results which will be used in the sequel. In Section 3, we show that the nullity of $(K_{p,q}, \sigma)$ is at least $p + q - 2k - 2$, where $k = \min(r, s)$ and $(K_{p,q}, \sigma)[U_r \cup V_s]$, $r \leq p$ and $s \leq q$, is an induced signed subgraph on minimum vertices $r + s$, which contain all negative edges of the signed graph $(K_{p,q}, \sigma)$. In Section 4, we determine the spectrum of a complete bipartite signed graph whose negative edges (positive edges) induce (i) disjoint complete bipartite subgraphs and (ii) a path. In Section 5, we determine the spectrum of a complete bipartite signed graph whose negative edges (positive edges) induce an regular subgraph H. Also, we obtain a relation between the eigenvalues of this complete bipartite signed graph and the non-negative eigenvalues of H. For definitions and notations of graphs, we refer to [10].

2 Preliminaries

Consider $\mu_1, \mu_2, \dots, \mu_n$ as the eigenvalues of the signed graph Γ. If for each i there exists some j such that $\mu_i + \mu_j = 0$, then we say that the spectrum is symmetric with respect to 0. It is well known that an unsigned graph which contains at least one edge is bipartite if and only if its spectrum considered as a set of points on the real axis is symmetric with respect to the origin. There exist nonbipartite signed graphs with this property as can be seen in [14]. The following result can be seen in [7].

Lemma 2.1 ([7, Theorem 2.1])**.** *Let Γ be a signed graph of order n. Then the following statements are equivalent.*

(i) *Spectrum of Γ is symmetric about the origin,*

(ii) $\phi(\Gamma, x) = x^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k c_{2k} x^{n-2k}$, *where c_{2k} are non negative integers for all $k = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$,*

(iii) *Γ and $-\Gamma$ are cospectral, where $-\Gamma$ is the signed graph obtained by negating sign of each edge of Γ.*

Consider the matrix M having the block form as follows.

$$M = \begin{pmatrix} A & \beta & \cdots & \beta & \beta \\ \beta^\top & B & \cdots & C & C \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \beta^\top & C & \cdots & B & C \\ \beta^\top & C & \cdots & C & B \end{pmatrix} \tag{2.1}$$

where $A \in \mathbb{R}^{t\times t}$, $\beta \in \mathbb{R}^{t\times s}$ and $B, C \in \mathbb{R}^{s\times s}$, such that $n = t + cs$, with c being the number of copies of B. The spectrum of this matrix can be obtained as the union of the spectrum of smaller matrices using the following technique given in [9]. In the statement of the following result, $\mathrm{Spec}^{(k)}(Z)$ denotes the multi-set formed by k copies of the spectrum of Z, denoted by $\mathrm{Spec}(Z)$.

Lemma 2.2. *Let M be a matrix of the form given in (2.1) with $c \geq 1$ copies of the block B. Then*

(i) $\mathrm{Spec}(B - C) \subseteq \mathrm{Spec}(M)$ *with multiplicity $c - 1$,*

(ii) $\mathrm{Spec}(M) \setminus \mathrm{Spec}^{(c-1)}(B - C) = \mathrm{Spec}(M')$ *is the set of the remaining $t + s$ eigenvalues of M, where*

$$M' = \begin{pmatrix} A & \sqrt{c}\cdot\beta \\ \sqrt{c}\cdot\beta^\top & B + (c-1)C \end{pmatrix}.$$

The next two results are concerned with the spectrum of special 2×2 block matrices.

Lemma 2.3. *Let* $X = \begin{pmatrix} O_{p\times p} & A_{p\times q} \\ A^\top_{q\times p} & O_{q\times q} \end{pmatrix}$ *be a real symmetric matrix of order* $p+q$, $q \geq p$. *Then*

(i) $m(0) \geq q-p$,

(ii) $\pm\sqrt{\mu} \in \operatorname{Spec}(X)$, *where* μ *is an eigenvalue of a positive semidefinite square matrix* $A_{p\times q}A^\top_{q\times p}$.

Proof. By the Schur complement formula, the determinant of a 2×2 block matrix is given by

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \left|D\right|\left|A - BD^{-1}C\right|,$$

where A and D are square blocks and D is nonsingular. So, we have

$$\begin{aligned}\phi(X,x) = \begin{vmatrix} xI_p & -A_{p\times q} \\ -A^\top_{q\times p} & xI_q \end{vmatrix} &= x^q \left|(xI_p) - A_{p\times q}\left(xI_q\right)^{-1} A^\top_{q\times p}\right| \\ &= x^q \left|\frac{1}{x}\left(x^2 I_p - A_{p\times q}A^\top_{q\times p}\right)\right| \\ &= x^{q-p}\phi\left(A_{p\times q}A^\top_{q\times p}, x^2\right).\end{aligned}$$

This completes the proof. □

Corollary 2.4. *Let* $X = \begin{pmatrix} O_{p\times p} & A_{p\times p} \\ A_{p\times p} & O_{p\times p} \end{pmatrix}$ *be a real symmetric matrix of order* $2p$. *Then* $\pm\mu \in \operatorname{Spec}(X)$, *where* μ *is an eigenvalue of the square matrix* $A_{p\times p}$.

We conclude this section with the following remark.

Remark 2.5. Let $(K_{p,q},\sigma)$ be a complete bipartite signed graph with bipartition (U_p, V_q), where $U_p = \{u_1, u_2, \ldots, u_p\}$ and $V_q = \{v_1, v_2, \ldots, v_q\}$. Then with a suitable labelling of the vertices of $(K_{p,q},\sigma)$, its adjacency matrix is given by

$$A(K_{p,q},\sigma) = \begin{pmatrix} O_{p\times p} & B_{p\times q} \\ B^\top_{q\times p} & O_{q\times q} \end{pmatrix}.$$

In view of Lemma 2.3, we observe that the spectrum of $(K_{p,q},\sigma)$ is related with the spectrum of the matrix $B_{p\times q}B^\top_{q\times p}$. Thus from here onwards, we focus on the matrix $B_{p\times q}$ and we call it as the spectral block of the adjacency matrix of the signed graph $(K_{p,q},\sigma)$.

3 Nullity of the signed graph $(K_{p,q},\sigma)$

In this section, we obtain a lower bound for the nullity of $\Gamma = (K_{p,q},\sigma)$ for any sign function σ, subject to the given condition.

Theorem 3.1. *Let* $(K_{p,q},\sigma)$, $p \leq q$, *be a complete bipartite signed graph and let* $(K_{p,q},\sigma)[U_r \cup V_s]$, $r \leq p$ *and* $s \leq q$, *be its induced signed subgraph on minimum vertices* $r+s$, *which contains all negative edges of the signed graph* $(K_{p,q},\sigma)$. *Then* $\eta((K_{p,q},\sigma)) \geq p+q-2k-2$, *where* $k = \min(r,s)$.

Proof. Note that the order of $(K_{p,q},\sigma)[U_r \cup V_s]$ is $r+s$. With a suitable labelling of the vertices of $(K_{p,q},\sigma)$, the adjacency matrix is given by

$$A(K_{p,q},\sigma) = \begin{pmatrix} O_{p\times p} & B_{p\times q} \\ B^\top_{q\times p} & O_{q\times q} \end{pmatrix},$$

where $B_{p\times q}$ is the spectral block of the adjacency matrix of the signed graph $(K_{p,q},\sigma)$. By Lemma 2.3, we get

$$\phi(A(K_{p,q},\sigma),x) = x^{q-p}\phi\left(B_{p\times q}B^\top_{q\times p}, x^2\right). \tag{3.1}$$

Without loss of generality, we may assume that $r < p$ and $s < q$. As $(K_{p,q},\sigma)[U_r \cup V_s]$ is an induced signed subgraph on minimum vertices $r+s$, which contain all negative edges of the signed graph $(K_{p,q},\sigma)$, we have

$$B_{p\times q} = \begin{pmatrix} X_{r\times s} & J_{r\times q-s} \\ J_{p-r\times s} & J_{p-r\times q-s} \end{pmatrix},$$

where $X_{r\times s}$ is the spectral block of the adjacency matrix of the signed graph $(K_{p,q},\sigma)[U_r \cup V_s]$. The transpose of a 2×2 block matrix is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^\top = \begin{pmatrix} A^\top & C^\top \\ B^\top & D^\top \end{pmatrix}.$$

Together with the fact that $J_{m\times n}J_{n\times m} = nJ_{m\times m}$, this yields

$$\begin{aligned} B_{p\times q}B^\top_{q\times p} &= \begin{pmatrix} X_{r\times s} & J_{r\times q-s} \\ J_{p-r\times s} & J_{p-r\times q-s} \end{pmatrix} \times \begin{pmatrix} X^\top_{s\times r} & J_{s\times p-r} \\ J_{q-s\times r} & J_{q-s\times p-r} \end{pmatrix} \\ &= \begin{pmatrix} X_{r\times s}X^\top_{s\times r} + (q-s)J_{r\times r} & X_{r\times s}J_{s\times p-r} + (q-s)J_{r\times p-r} \\ J_{p-r\times s}X^\top_{s\times r} + (q-s)J_{p-r\times r} & sJ_{p-r\times p-r} + (q-s)J_{p-r\times p-r} \end{pmatrix} \\ &= \begin{pmatrix} X_{r\times s}X^\top_{s\times r} + (q-s)J_{r\times r} & X_{r\times s}J_{s\times p-r} + (q-s)J_{r\times p-r} \\ J_{p-r\times s}X^\top_{s\times r} + (q-s)J_{p-r\times r} & qJ_{p-r\times p-r} \end{pmatrix}. \end{aligned}$$

Now, it is easy to see that $X_{r\times s}J_{s\times 1} + (q-s)J_{r\times 1} = Y + (q-s)J_{r\times 1}$, where Y is the column vector of the row sums of the matrix $X_{r\times s}$. Let $Z = [Y + (q-s)J_{r\times 1}\; Y + (q-s)J_{r\times 1} \;\cdots\; Y + (q-s)J_{r\times 1}] \in \mathbb{R}^{r\times p-r}$ be a matrix of order $r \times p-r$. Then, we have

$$B_{p\times q}B^\top_{q\times p} = \begin{pmatrix} X_{r\times s}X^\top_{s\times r} + (q-s)J_{r\times r} & Z \\ Z^\top & qJ_{p-r\times p-r} \end{pmatrix}. \tag{3.2}$$

The matrix $B_{p\times q}B^\top_{q\times p}$ has a special kind of symmetry. Taking $A = X_{r\times s}X^\top_{s\times r} + (q-s)J_{r\times r}$, $\beta = Y + (q-s)J_{r\times 1}$, $B = [q]$ and $C = [q]$ in (2.1), from Lemma 2.2, we get $\mathrm{Spec}^{p-r-1}(B-C) = \mathrm{Spec}^{p-r-1}([0]) \subseteq \mathrm{Spec}(B_{p\times q}B^\top_{q\times p})$. Again by Equation (3.1), Equation (3.2) and Lemma 2.2, we obtain

$$\phi(A(K_{p,q},\sigma),x) = x^{\alpha}\phi\left(Z_1, x^2\right), \tag{3.3}$$

where $\alpha = q+p-2r-2$ and

$$Z_1 = \begin{pmatrix} X_{r\times s}X^\top_{s\times r} + (q-s)J_{r\times r} & \sqrt{p-r}(Y + (q-s)J_{r\times 1}) \\ \sqrt{p-r}(Y + (q-s)J_{r\times 1})^\top & q(p-r) \end{pmatrix}.$$

Also, we have

$$B_{q\times p}^{\top}B_{p\times q}=\begin{pmatrix} X_{s\times r}^{\top}X_{r\times s}+(p-r)J_{s\times s} & X_{s\times r}^{\top}J_{r\times q-s}+(p-r)J_{s\times q-s}\\ J_{q-s\times r}X_{r\times s}+(p-r)J_{q-s\times s} & pJ_{q-s\times q-s}\end{pmatrix}.$$

Now, $X_{s\times r}^{\top}J_{r\times 1}+(p-r)J_{s\times 1}=Y'+(p-r)J_{s\times 1}$, where Y' is the column vector of the column sums of the matrix $X_{r\times s}$. Let $Z'=[Y'+(p-r)J_{s\times 1}\ \ Y'+(p-r)J_{s\times 1}\ \ \cdots\ \ Y'+(p-r)J_{s\times 1}]\in\mathbb{R}^{s\times q-s}$ be a matrix of order $s\times q-s$. Then,

$$B_{q\times p}^{\top}B_{p\times q}=\begin{pmatrix} X_{s\times r}^{\top}X_{r\times s}+(p-r)J_{s\times s} & Z'\\ Z'^{\top} & pJ_{q-s\times q-s}\end{pmatrix}. \tag{3.4}$$

Taking $A=X_{s\times r}^{\top}X_{r\times s}+(p-r)J_{s\times s}$, $\beta=Y'+(p-r)J_{s\times 1}$, $B=[p]$ and $C=[p]$ in (2.1), from Lemma 2.2, we get $\text{Spec}^{q-s-1}(B-C)=\text{Spec}^{q-s-1}([0])\subseteq\text{Spec}(B_{p\times q}^{\top}B_{p\times q})$. Note that the eigenvalues of $B_{q\times p}^{\top}B_{p\times q}$ are given by the eigenvalues of $B_{p\times q}B_{q\times p}^{\top}$, together with the eigenvalue 0 of multiplicity $q-p$. Therefore, by Equation (3.1), Equation (3.4) and Lemma 2.2, we obtain

$$\phi(A(K_{p,q},\sigma),x)=x^{\zeta}\phi\left(Z_2,x^2\right), \tag{3.5}$$

where $\zeta=q+p-2s-2$ and

$$Z_2=\begin{pmatrix} X_{s\times r}^{\top}X_{r\times s}+(p-r)J_{s\times s} & \sqrt{q-s}(Y'+(p-r)J_{s\times 1})\\ \sqrt{q-s}(Y'+(p-r)J_{s\times 1})^{\top} & p(q-s)\end{pmatrix}.$$

Hence the result follows by Equation (3.3) and Equation (3.5). □

As $(K_{p,q},\sigma)$ is a bipartite signed graph, therefore its spectrum is symmetric about the origin. Thus, the following is an immediate consequence of Theorem 3.1 and Lemma 2.1.

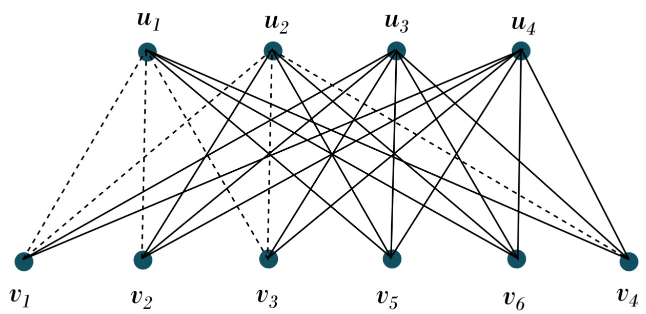

Figure 1: The signed graph $(K_{4,6},\sigma)$.

Corollary 3.2. *Let $(K_{p,q},\sigma)$, $p\le q$, be a complete bipartite signed graph and let $(K_{p,q},\sigma)[U_r\cup V_s]$, $r\le p$ and $s\le q$, be its induced subgraph on minimum vertices $r+s$, which contains all positive edges of the signed graph $(K_{p,q},\sigma)$. Then $\eta((K_{p,q},\sigma))\ge p+q-2k-2$, where $k=\min(r,s)$.*

We end this section with an example which shows that the lower bound for the nullity, given in Theorem 3.1, of $(K_{p,q},\sigma)$ is best possible.

Example 3.3. Consider the complete bipartite signed graph $(K_{4,6},\sigma)$ as shown in Figure 1. Plain lines denote the positive edges and dashed lines denote the negative edges. It contains an induced signed subgraph $(K_{4,6},\sigma)[U_2,V_4]$ on 6 vertices which contains all negative edges of $(K_{4,6},\sigma)$. Here, we have $p=4$, $q=6$ and $k=2$. Therefore, by Theorem 3.1, $\eta((K_{4,6},\sigma))\geq 4$. The spectral block of the adjacency matrix of the induced signed subgraph $(K_{4,6},\sigma)[U_2,V_4]$ is given as

$$X_{2\times 4}=\begin{pmatrix} -1 & -1 & -1 & 1\\ -1 & 1 & -1 & -1\end{pmatrix}.$$

Therefore, by Equation (3.3), we get

$$\phi(A(K_{4,6},\sigma),x)=x^4\phi\left(\begin{pmatrix} 6 & 2 & 0\\ 2 & 6 & 0\\ 0 & 0 & 12\end{pmatrix},x^2\right).$$

Thus, it is easy to see that

$$\operatorname{Spec}((K_{4,6},\sigma))=\begin{pmatrix} 2\sqrt{3} & 2\sqrt{2} & 2 & 0 & -2 & -2\sqrt{2} & -2\sqrt{3}\\ 1 & 1 & 1 & 4 & 1 & 1 & 1\end{pmatrix}.$$

4 Spectrum of $(K_{p,q},\sigma)$ when negative edges induce either a disjoint complete bipartite subgraphs or a path

We begin this section with the computation of the spectrum of the complete bipartite signed graph $\left(K_{p,q},K^-_{r,s}\right)$ whose negative edges induce a subgraph $K_{r,s}$.

Theorem 4.1. *Let $\left(K_{p,q},K^-_{r,s}\right)$, $p\leq q$, $r\leq p$ and $s\leq q$, be a complete bipartite signed graph whose negative edges induce a subgraph $K_{r,s}$ of order $r+s$. Then the spectrum of $\left(K_{p,q},K^-_{r,s}\right)$ is given as*

$$\operatorname{Spec}(\left(K_{p,q},K^-_{r,s}\right))=\begin{pmatrix} \mu_1 & \mu_2 & 0 & -\mu_2 & -\mu_1\\ 1 & 1 & p+q-4 & 1 & 1\end{pmatrix},$$

where

$$\mu_1,\mu_2=\sqrt{\frac{pq\pm\sqrt{p^2q^2-16rs(p-r)(q-s)}}{2}}.$$

Proof. By Equation (3.3), we have

$$\phi\left(\left(K_{p,q},K^-_{r,s}\right),x\right)=x^\alpha\phi\left(\begin{pmatrix} X_{r\times s}X^\top_{s\times r}+(q-s)J_{r\times r} & \sqrt{p-r}\beta\\ \sqrt{p-r}\beta^\top & q(p-r)\end{pmatrix},x^2\right), \quad (4.1)$$

where $\alpha=q+p-2r-2$, $\beta=Y+(q-s)J_{r\times 1}$, and Y is the column vector of the row sums of spectral block $X_{r\times s}$ of the adjacency matrix of an induced signed subgraph $K_{r,s}$, whose all edges are negative. Clearly, $X_{r\times s}X^\top_{s\times r}+(q-s)J_{r\times r}=-J_{r\times s}\times -J_{s\times r}+(q-s)J_{r\times r}=qJ_{r\times r}$ and $Y+(q-s)J_{r\times 1}=(q-2s)J_{r\times 1}$. Therefore, Equation (4.1) takes the form

$$\phi\left(\left(K_{p,q},K^-_{r,s}\right),x\right)=x^\alpha\phi\left(\begin{pmatrix} qJ_{r\times r} & \sqrt{p-r}(q-2s)J_{r\times 1}\\ \sqrt{p-r}(q-2s)J_{1\times r} & q(p-r)\end{pmatrix},x^2\right). \quad (4.2)$$

For $p \neq r$, it can be easily seen that the real symmetric matrix

$$Z_1 = \begin{pmatrix} qJ_{r\times r} & \sqrt{p-r}(q-2s)J_{r\times 1} \\ \sqrt{p-r}(q-2s)J_{1\times r} & q(p-r) \end{pmatrix}$$

has rank 2. Now, let x_1 and x_2 be the non zero eigenvalues of Z_1. We have

$$x_1 + x_2 = tr(Z_1) = rq + q(p-r) = pq. \tag{4.3}$$

Also,

$$x_1^2 + x_2^2 = tr(Z_1^2) = p^2q^2 - 16rs(p-r)(q-s). \tag{4.4}$$

Equations (4.3) and (4.4), imply that

$$x_1,\ x_2 = \frac{pq \pm \sqrt{p^2q^2 - 16rs(p-r)(q-s)}}{2}. \tag{4.5}$$

Thus, Equation (4.2) yields that

$$\phi\left(\left(K_{p,q}, K_{r,s}^-\right), x\right) = x^{p+q-4}(x^4 - (x_1+x_2)x^2 + x_1x_2),$$

where x_1 and x_2 are given in Equation (4.5). This proves the result. □

As a consequence, we compute the spectrum of a complete bipartite signed graph whose positive edges induce a complete bipartite subgraph.

Corollary 4.2. *Let $\left(K_{p,q}, K_{r,s}^+\right)$, $p \leq q$, $r \leq p$ and $s \leq q$, be a complete bipartite signed graph whose positive edges induce a subgraph $K_{r,s}$ of order $r+s$. Then the spectrum of $\left(K_{p,q}, K_{r,s}^+\right)$ is given as*

$$\mathrm{Spec}(\left(K_{p,q}, K_{r,s}^+\right)) = \begin{pmatrix} \mu_1 & \mu_2 & 0 & -\mu_2 & -\mu_1 \\ 1 & 1 & p+q-4 & 1 & 1 \end{pmatrix},$$

where

$$\mu_1, \mu_2 = \sqrt{\frac{pq \pm \sqrt{p^2q^2 - 16rs(p-r)(q-s)}}{2}}.$$

Now, we obtain the characteristic polynomial of the complete bipartite signed graph $(K_{p,q}, \sigma)$ whose negative edges form the disjoint subgraphs $K_{r,s}$ of different orders.

Theorem 4.3. *Let $(K_{p,q}, \sigma)$, $p \leq q$, be a complete bipartite signed graph whose negative edges induce disjoint complete bipartite subgraphs of orders $r_1 + s_1$, $r_2 + s_2$, …, $r_k + s_k$ such that $\sum_{i=1}^k r_i = r$, $\sum_{i=1}^k s_i = s$, $r \leq p$ and $s \leq q$. Then the characteristic polynomial of $(K_{p,q}, \sigma)$ is given as*

$$\phi\left(\left(K_{p,q}, \sigma\right), x\right) = x^{p+q-2k-2}\phi\left(Z', x^2\right),$$

where

$$Z' = \begin{pmatrix} r_1c_{11} & r_2c_{12} & \cdots & r_kc_{1k} & c(q-2s_1) \\ r_1c_{21} & r_2c_{22} & \cdots & r_kc_{2k} & c(q-2s_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_1c_{k1} & r_2c_{k2} & \cdots & r_kc_{kk} & c(q-2s_k) \\ r_1c(q-2s_1) & r_2c(q-2s_2) & \cdots & r_kc(q-2s_k) & q(p-r) \end{pmatrix}$$

is a matrix of order $k+1$, $c = \sqrt{p-r}$, $c_{ij} = q$ if $i = j$ and $c_{ij} = q - 2s_i - 2s_j$ otherwise.

Proof. Consider the matrix given in Equation (3.3)

$$Z_1 = \begin{pmatrix} X_{r\times s}X^{\top}_{s\times r} + (q-s)J_{r\times r} & \sqrt{p-r}(Y + (q-s)J_{r\times 1}) \\ \sqrt{p-r}(Y + (q-s)J_{r\times 1})^{\top} & q(p-r) \end{pmatrix}, \tag{4.6}$$

where Y is the column vector of the row sums of the spectral block $X_{r\times s}$ of the adjacency matrix of an induced signed subgraph of $(K_{p,q},\sigma)$ which contains all its negative edges. Hence, with a suitable relabelling of vertices of the induced signed subgraph, we have

$$X_{r\times s} = \begin{pmatrix} -J_{r_1\times s_1} & J_{r_1\times s_2} & \cdots & J_{r_1\times s_k} \\ J_{r_2\times s_1} & -J_{r_2\times s_2} & \cdots & J_{r_2\times s_k} \\ \vdots & \vdots & \ddots & \vdots \\ J_{r_k\times s_1} & J_{r_k\times s_2} & \cdots & -J_{r_k\times s_k} \end{pmatrix},$$

where $J_{r_i\times s_i}$ is the spectral block of the adjacency matrix of the complete bipartite subgraph K_{r_i,s_i}, $i=1,2,\ldots,k$, $\sum_{i=1}^{k} r_i = r$ and $\sum_{i=1}^{k} s_i = s$. Now, it is easy to obtain

$$X_{r\times s}X^{\top}_{s\times r} = \begin{pmatrix} b_{11}J_{r_1\times r_1} & b_{12}J_{r_1\times r_2} & \cdots & b_{1k}J_{r_1\times r_k} \\ b_{21}J_{r_2\times r_1} & b_{22}J_{r_2\times r_2} & \cdots & b_{2k}J_{r_2\times r_k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1}J_{r_k\times r_1} & b_{k2}J_{r_k\times r_2} & \cdots & b_{kk}J_{r_k\times r_k} \end{pmatrix},$$

where, $b_{ij} = \sum_{i=1}^{k} s_i = s$ if $i=j$ and $b_{ij} = s - 2s_i - 2s_j$, otherwise. As Y is the column vector of the row sums of the spectral block $X_{r\times s}$, therefore the matrix Z_1 given in (4.6) takes the form

$$Z_1 = \begin{pmatrix} c_{11}J_{r_1\times r_1} & c_{12}J_{r_1\times r_2} & \cdots & c_{1k}J_{r_1\times r_k} & c(q-2s_1)J_{r_1\times 1} \\ c_{21}J_{r_2\times r_1} & c_{22}J_{r_2\times r_2} & \cdots & c_{2k}J_{r_2\times r_k} & c(q-2s_2)J_{r_2\times 1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{k1}J_{r_k\times r_1} & c_{k2}J_{r_k\times r_2} & \cdots & c_{kk}J_{r_k\times r_k} & c(q-2s_k)J_{r_k\times 1} \\ c(q-2s_1)J_{1\times r_1} & c(q-2s_2)J_{1\times r_2} & \cdots & c(q-2s_k)J_{1\times r_k} & q(p-r)J_{1\times 1} \end{pmatrix},$$

where $c = \sqrt{p-r}$, $c_{ij} = q$ if $i=j$ and $c_{ij} = q - 2s_i - 2s_j$ otherwise. Clearly, the matrix Z_1 has equitable quotient matrix Z', where

$$Z' = \begin{pmatrix} r_1c_{11} & r_2c_{12} & \cdots & r_kc_{1k} & c(q-2s_1) \\ r_1c_{21} & r_2c_{22} & \cdots & r_kc_{2k} & c(q-2s_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_1c_{k1} & r_2c_{k2} & \cdots & r_kc_{kk} & c(q-2s_k) \\ r_1c(q-2s_1) & r_2c(q-2s_2) & \cdots & r_kc(q-2s_k) & q(p-r) \end{pmatrix}.$$

Now by [18, Theorem 3.1], $\mathrm{Spec}(Z_1) = \mathrm{Spec}(Z') \cup \begin{pmatrix} 0 \\ r-k \end{pmatrix}$, where Z' is equitable quotient matrix of Z_1 and is given as

$$Z' = \begin{pmatrix} r_1c_{11} & r_2c_{12} & \cdots & r_kc_{1k} & c(q-2s_1) \\ r_1c_{21} & r_2c_{22} & \cdots & r_kc_{2k} & c(q-2s_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_1c_{k1} & r_2c_{k2} & \cdots & r_kc_{kk} & c(q-2s_k) \\ r_1c(q-2s_1) & r_2c(q-2s_2) & \cdots & r_kc(q-2s_k) & q(p-r) \end{pmatrix},$$

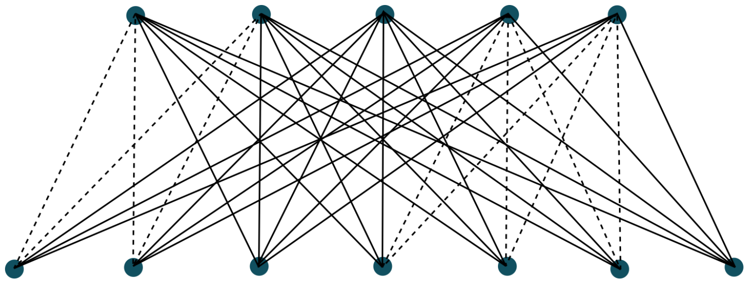

Figure 2: Signed graph whose negative edges induce two disjoint complete bipartite subgraphs.

where $c = \sqrt{p-r}$, $c_{ij} = q$ if $i = j$ and $c_{ij} = q - 2s_i - 2s_j$ otherwise. As $\mathrm{Spec}(Z_1) = \mathrm{Spec}(Z') \cup \begin{pmatrix} 0 \\ r-k \end{pmatrix}$, therefore the result follows by Equation (3.3) and Equation (4.6). □

An application of Theorem 4.3 can be seen in the following example.

Example 4.4. Consider the complete bipartite signed graph $(K_{5,7}, \sigma)$ as shown in Figure 2. Here, we have $p = 5$, $q = 7$, $r_1 = 2$, $s_1 = 2$, $r_2 = 2$, $s_2 = 3$, $r = r_1 + r_2 = 4$ and $s = s_1 + s_2 = 5$. Therefore, by Theorem 4.3, we get

$$\phi(A(K_{5,7}, \sigma), x) = x^6 \phi\left(\begin{pmatrix} 14 & -6 & 3 \\ -6 & 14 & 1 \\ 6 & 2 & 7 \end{pmatrix}, x^2\right).$$

Thus, it is easy to see that

$$\mathrm{Spec}((K_{5,7}, \sigma)) = \begin{pmatrix} 4.50 & 3.37 & 1.82 & 0 & -1.82 & -3.37 & -4.50 \\ 1 & 1 & 1 & 6 & 1 & 1 & 1 \end{pmatrix}.$$

The next corollary follows from Lemma 2.1 and Theorem 4.3 which gives the spectrum of a complete bipartite signed graph whose positive edges induce the disjoint complete bipartite subgraphs of different orders.

Corollary 4.5. *Let $(K_{p,q}, \sigma)$, $p \leq q$, be a complete bipartite signed graph whose positive edges induce disjoint complete bipartite subgraphs of orders $r_1 + s_1$, $r_2 + s_2$, ..., $r_k + s_k$ such that $\sum_{i=1}^{k} r_i = r$, $\sum_{i=1}^{k} s_i = s$, $r \leq p$ and $s \leq q$. Then the characteristic polynomial of $(K_{p,q}, \sigma)$ is given as*

$$\phi\left((K_{p,q}, \sigma), x\right) = x^{p+q-2k-2} \phi\left(Z', x^2\right),$$

where

$$Z' = \begin{pmatrix} r_1 c_{11} & r_2 c_{12} & \cdots & r_k c_{1k} & c(q-2s_1) \\ r_1 c_{21} & r_2 c_{22} & \cdots & r_k c_{2k} & c(q-2s_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_1 c_{k1} & r_2 c_{k2} & \cdots & r_k c_{kk} & c(q-2s_k) \\ r_1 c(q-2s_1) & r_2 c(q-2s_2) & \cdots & r_k c(q-2s_k) & q(p-r) \end{pmatrix}$$

is a matrix of order $k+1$, $c = \sqrt{p-r}$, $c_{ij} = q$ if $i = j$ and $c_{ij} = q - 2s_i - 2s_j$ otherwise.

We conclude this section with the following result whose proof can be obtained in a similar way as in Theorem 4.3.

Theorem 4.6. (i) *Let* $\left(K_{p,q}, P_{2r}^{-}\right)$, $p \leq q$ *and* $r \geq 1$, *be a complete bipartite signed graph whose negative edges induce a path on* $2r$ *vertices. Then the characteristic polynomial of* $\left(K_{p,q}, P_{2r}^{-}\right)$ *is given as*

$$\phi\left(\left(K_{p,q}, P_{2r}^{-}\right), x\right) = x^{p+q-2r-2}\phi\left(Z', x^2\right),$$

where

$$Z' = \begin{pmatrix} q & q-2 & q-6 & q-6 & \cdots & q-6 & c(q-2) \\ q-2 & q & q-4 & q-8 & \cdots & q-8 & c(q-4) \\ q-6 & q-4 & q & q-4 & \ddots & \vdots & \vdots \\ q-6 & q-8 & q-4 & q & \ddots & q-8 & c(q-4) \\ \vdots & \vdots & \ddots & \ddots & \ddots & q-4 & c(q-4) \\ q-6 & q-8 & \cdots & q-8 & q-4 & q & c(q-4) \\ c(q-2) & c(q-4) & \cdots & c(q-4) & c(q-4) & c(q-4) & q(p-r-1) \end{pmatrix}$$

is a positive semidefinite matrix of order $r+1$ *and* $c = \sqrt{p-r-1}$.
(ii) *Let* $\left(K_{p,q}, P_{2r+1}^{-}\right)$, $p \leq q$ *and* $r \geq 1$, *be a complete bipartite signed graph whose negative edges induce a path on* $2r+1$ *vertices with both pendent vertices of the path* P_{2r+1} *in* U_p. *Then the characteristic polynomial of* $\left(K_{p,q}, P_{2r+1}^{-}\right)$ *is given as*

$$\phi\left(\left(K_{p,q}, P_{2r+1}^{-}\right), x\right) = x^{p+q-2r-4}\phi\left(Z', x^2\right),$$

where

$$Z' = \begin{pmatrix} q & q-2 & q-6 & q-6 & \cdots & q-6 & q-4 & c(q-2) \\ q-2 & q & q-4 & q-8 & \cdots & q-8 & q-6 & c(q-4) \\ q-6 & q-4 & q & q-4 & \ddots & \vdots & \vdots & \vdots \\ q-6 & q-8 & q-4 & q & \ddots & q-8 & q-6 & c(q-4) \\ \vdots & \vdots & \ddots & \ddots & \ddots & q-4 & q-6 & c(q-4) \\ q-6 & q-8 & \cdots & q-8 & q-4 & q & q-2 & c(q-4) \\ q-4 & q-6 & \cdots & q-6 & q-6 & q-2 & q & c(q-2) \\ c(q-2) & c(q-4) & \cdots & c(q-4) & c(q-4) & c(q-4) & c(q-2) & q(p-r) \end{pmatrix}$$

is a positive semidefinite matrix of order $r+2$ *and* $c = \sqrt{p-r}$.
(iii) *Let* $\left(K_{p,q}, P_{2r+1}^{-}\right)$, $p \leq q$ *and* $r \geq 1$, *be a complete bipartite signed graph whose negative edges induce a path on* $2r+1$ *vertices with both pendent vertices of the path* P_{2r+1} *in* V_q. *Then the characteristic polynomial of* $\left(K_{p,q}, P_{2r+1}^{-}\right)$ *is given as*

$$\phi\left(\left(K_{p,q}, P_{2r+1}^{-}\right), x\right) = x^{p+q-2r-2}\phi\left(Z', x^2\right),$$

where

$$Z' = \begin{pmatrix} q & q-4 & q-8 & q-8 & \cdots & q-8 & c(q-4) \\ q-4 & q & q-4 & q-8 & \cdots & q-8 & c(q-4) \\ q-8 & q-4 & q & q-4 & \ddots & \vdots & \vdots \\ q-8 & q-8 & q-4 & q & \ddots & q-8 & c(q-4) \\ \vdots & \vdots & \ddots & \ddots & \ddots & q-4 & c(q-4) \\ q-8 & q-8 & \cdots & q-8 & q-4 & q & c(q-4) \\ c(q-4) & c(q-4) & \cdots & c(q-4) & c(q-4) & c(q-4) & q(p-r) \end{pmatrix}$$

is a positive semidefnite matrix of order $r+1$ and $c = \sqrt{p-r}$.

The next example gives the spectrum of a complete bipartite signed graph whose negative edges induce a path on 5 vertices.

Example 4.7. Let $\left(K_{p,q}, P_5^-\right)$, $p \leq q$, be a complete bipartite signed graph whose negative edges induce a path on 5 vertices with both pendent vertices of the path P_5 in V_q. By Theorem 4.6 (Part (iii)), the characteristic polynomial of $\left(K_{p,q}, P_5^-\right)$ is given by

$$\phi\left(\left(K_{p,q}, P_5^-\right), x\right) = x^{p+q-6}\phi\left(\begin{pmatrix} q & q-4 & c(q-4) \\ q-4 & q & c(q-4) \\ c(q-4) & c(q-4) & q(p-2) \end{pmatrix}, x^2\right),$$

where $c = \sqrt{p-2}$. To determine the spectrum of $(K_{p,q}, P_5^-)$, it is enough to consider the matrix

$$Z' = \begin{pmatrix} q & q-4 & c(q-4) \\ q-4 & q & c(q-4) \\ c(q-4) & c(q-4) & q(p-2) \end{pmatrix}.$$

Clearly, 4 is an eigenvalue of the matriz Z' corresponding to an eigenvector $(1, -1, 0)^\top$. To compute the other two eigenvalues of Z', we use the fact that the sum and product of the eigenvalues of Z' are equal to the trace and determinant, respectively. Then, we obtain the eigenvalues as

$$\frac{pq - 4 \pm \sqrt{p^2q^2 - 56pq + 128p + 96q - 240}}{2}.$$

Thus, the spectrum of $(K_{p,q}, P_5^-)$ is given as

$$\operatorname{Spec}((K_{p,q}, P_5^-)) = \begin{pmatrix} \mu_1 & \mu_2 & 2 & 0 & -2 & -\mu_2 & -\mu_1 \\ 1 & 1 & 1 & p+q-6 & 1 & 1 & 1 \end{pmatrix},$$

where

$$\mu_1, \mu_2 = \sqrt{\frac{pq - 4 \pm \sqrt{p^2q^2 - 56pq + 128p + 96q - 240}}{2}}.$$

5 Eigenvalues of $\left(K_{p,q}, H_{r,n}^-\right)$

The complete bipartite signed graph Γ whose negative edges induce a 1-regular graph of different orders has been studied in [2]. In this section, we consider the complete bipartite signed graph $\left(K_{p,q}, H^-_{r,n}\right)$, $p \leq q$, whose negative edges induce an r-regular subgraph H (not necessarily connected) of order n. We find a relation between the eigenvalues of this complete bipartite signed graph and the non-negative eigenvalues of H. The other eigenvalues of $\left(K_{p,q}, H^-_{r,n}\right)$ are also determined. We start with the following lemma.

Lemma 5.1. *Let $\left(K_{k,k}, H^-_{r,2k}\right)$ be a complete bipartite signed graph whose negative edges induce an r-regular subgraph H of order $2k$. If the eigenvalues of H are $\mu_1 = r \geq \mu_2 \geq \cdots \geq \mu_{2k} = -r$, then $-2\mu_i$ is an eigenvalue of $\left(K_{k,k}, H^-_{r,2k}\right)$ for $i = 2, \ldots, 2k-1$. Moreover, the other two eigenvalues of $\left(K_{k,k}, H^-_{r,2k}\right)$ are $k-2r$ and $-k+2r$.*

Proof. Let $A(H,-) = -A(H)$ be the adjacency matrix of $(H,-)$. Therefore, with a suitable labelling of the vertices of $\left(K_{k,k}, H^-_{r,2k}\right)$, we observe that

$$A\left(K_{k,k}, H^-_{r,2k}\right) = \begin{pmatrix} O_{k\times k} & A_{k\times k} \\ A_{k\times k} & O_{k\times k} \end{pmatrix} = A(K_{k,k}) - 2A(H), \tag{5.1}$$

where the $(k-2r)$-regular symmetric matrix $A_{k\times k}$ is the spectral block of the adjacency matrix of the signed graph $\left(K_{k,k}, H^-_{r,2k}\right)$. As the matrices $A(K_{k,k})$ and $A(H)$ commute, therefore they are simultaneously diagonalizable. Let $\{x_1, x_2, \ldots, x_{2k}\}$ be an orthogonal basis of $\mathbb{R}^{2k}$ consisting of the eigenvectors of $A(H)$ and $A(K_{k,k})$ with $x_1 = J_{2k\times 1} = (1, \ldots, 1)^T \in \mathbb{R}^{2k}$. Then, we have

$$(A(K_{k,k}) - 2A(H))x_1 = (k-2r)x_1.$$

Thus, $(k-2r)$ is an eigenvalue of $A(K_{k,k}) - 2A(H)$. To find the other eigenvalues of $A(K_{k,k}) - 2A(H)$, we use the facts that $\operatorname{Spec}(A(K_{k,k}) - 2A(H)) \subseteq \operatorname{Spec}(A(K_{k,k})) + \operatorname{Spec}(-2A(H))$ and the spectrum of $\left(K_{k,k}, H^-_{r,2k}\right)$ is symmetric with respect to origin. Thus,

$$(A(K_{k,k}) - 2A(H))x_i = -2\mu_i x_i, \;\; i = 2, 3, \ldots, 2k-1$$

and

$$(A(K_{k,k}) - 2A(H))x_{2k} = (-k+2r)x_{2k}.$$

This proves the result. □

The eigenvalues of a complete bipartite signed graph whose negative edges induce a regular graph H is completely determined by the non-negative eigenvalues of H and can be seen in the following result.

Theorem 5.2. *Let $\left(K_{p,q}, H^-_{r,2k}\right)$, $p \leq q$, be a complete bipartite signed graph whose negative edges induce an r-regular subgraph H of order $2k$. Then the following statements hold:*

(i) $\eta(\left(K_{p,q}, H^-_{r,2k}\right)) \geq p+q-2k-2$.

(ii) *If the first k largest non-negative eigenvalues of H are $\mu_1 = r \geq \mu_2 \geq \cdots \geq \mu_k \geq 0$, then*

  (a) $\pm\, 2\mu_i$ *is an eigenvalue of $\left(K_{p,q}, H^-_{r,2k}\right)$, for $i = 2, \ldots, k$ when $p+q \geq 2k+2$.*

  (b) $\pm\, 2\mu_i$ *is an eigenvalue of $\left(K_{p,q}, H^-_{r,2k}\right)$, for $i = 2, \ldots, k-1$ when $p+q < 2k+2$.*

Moreover, the other four eigenvalues of $\left(K_{p,q}, H^{-}_{r,2k}\right)$ *are*

$$\pm\sqrt{\frac{pq+(k-2r)^2-k^2\pm\theta}{2}},$$

where

$$\theta=\sqrt{(pq+(k-2r)^2-k^2)^2-4((k-2r)^2+k(q-k)-\tfrac{k(q-2r)^2}{q})(q(p-k))}.$$

Proof. Consider the matrix which is given in Equation (3.3)

$$Z_1=\begin{pmatrix} X_{r\times s}X^{\top}_{s\times r}+(q-s)J_{r\times r} & \sqrt{p-r}(Y+(q-s)J_{r\times 1}) \\ \sqrt{p-r}(Y+(q-s)J_{r\times 1})^{\top} & q(p-r)\end{pmatrix}, \tag{5.2}$$

where Y is the column vector of the row sums of the spectral block $X_{r\times s}$ of the adjacency matrix of the induced signed subgraph $\left(K_{k,k}, H^{-}_{r,2k}\right)$ of $\left(K_{p,q}, H^{-}_{r,2k}\right)$ which contains all the negative edges. By Equation (5.1), it is easy to see that $r=s=k$, $X_{r\times s}X^{\top}_{s\times r}=A^2_{k\times k}$ and $Y=(k-2r)J_{k\times 1}$. Now, the matrix Z_1 takes the form

$$Z_1=\begin{pmatrix} A^2_{k\times k}+(q-k)J_{k\times k} & \sqrt{p-k}(q-2r)J_{k\times 1} \\ \sqrt{p-k}(q-2r)J^{\top}_{1\times k} & q(p-k)\end{pmatrix}.$$

The matrix $A^2_{k\times k}$ is $(k-2r)^2$-regular and hence commutes with $(q-k)J_{k\times k}$. Thus, it is easy to see that $(k-2r)^2-k(q-k)$ is an eigenvalue of $A^2_{k\times k}+(q-k)J_{k\times k}$ corresponding to an eigenvector $J_{k\times 1}$. Also, by Equation (5.1), Corollary 2.4 and Lemma 5.1, we have

$$(A^2_{k\times k}+(q-k)J_{k\times k})x_i=4\mu_i^2x_i, i=2,\dots,k,$$

where $\{x_1,x_2,\dots,x_k\}$ is an orthogonal basis of $\mathbb{R}^k$ with $x_1=J_{k\times 1}$ and μ_i is non-negative eigenvalue of H. Define $y_i=[x_i\ \ 0]^{\top}\in\mathbb{R}^{k+1}$, $i=2,\dots,k$. Then

$$Z_1y_i=4\mu_i^2y_i,\ i=2,\dots,k.$$

Therefore, $4\mu_i^2$, $i=2,\dots,k$ is an eigenvalue of Z_1. Let α_1 and α_2 be the other two eigenvalues of Z_1. We have

$$\alpha_1+\alpha_2+\sum_{i=2}^{k}4\mu_i^2=tr(Z_1)=k(q-2r)+q(p-k)$$

and

$$(k-2r)^2+\sum_{i=2}^{k}4\mu_i^2=tr(A^2_{k\times k})=k(k-2r).$$

This yields that

$$\alpha_1+\alpha_2=pq+(k-2r)^2-k^2. \tag{5.3}$$

By the Schur complement formula, the determinant of a 2×2 block matrix Z_1 is given by

$$\begin{vmatrix} A^2_{k\times k}+(q-k)J_{k\times k} & \sqrt{p-k}(q-2r)J_{k\times 1} \\ \sqrt{p-k}(q-2r)J^{\top}_{1\times k} & q(p-k)\end{vmatrix}=|q(p-k)|\,|M|,$$

where $M = A^2_{k\times k} + (q-k)J_{k\times k} - \frac{(q-2r)^2}{q}J_{k\times k}$. Now, clearly the eigenvalues of the matrix $A^2_{k\times k} + (q-k)J_{k\times k} - \frac{(q-2r)^2}{q}J_{k\times k}$ are $(k-2r)^2 + k(q-k) - \frac{k(q-2r)^2}{q}$ and $4\mu_i^2$, $i = 2, \dots, k$, where μ_i is the non-negative eigenvalue of H. Thus, we have

$$\alpha_1\alpha_2 = ((k-2r)^2 + k(q-k) - \frac{k(q-2r)^2}{q})(q(p-k)). \tag{5.4}$$

Equations (5.3) and (5.4) imply that

$$\alpha_1, \alpha_2 = \frac{pq + (k-2r)^2 - k^2 \pm \theta}{2},$$

where $\theta = \sqrt{(pq + (k-2r)^2 - k^2)^2 - 4((k-2r)^2 + k(q-k) - \frac{k(q-2r)^2}{q})(q(p-k))}$. Hence, by Equation (3.3), we have $\eta(\left(K_{p,q}, H^-_{r,2k}\right)) \geq p+q-2k-2$ and with the fact that $\pm\alpha$ is an eigenvalue of $\left(K_{p,q}, H^-_{r,2k}\right)$ whenever α^2 is an eigenvalue of Z_1, the proof follows. □

The eigenvalues of a complete bipartite signed graph whose positive edges induce a regular graph H is completely determined by the non-negative eigenvalues of H as can be seen in the following corollary.

Corollary 5.3. *Let $\left(K_{p,q}, H^+_{r,2k}\right)$, $p \leq q$, be a complete bipartite signed graph whose positive edges induce an r-regular subgraph H of order $2k$. Then the following statements hold:*

(i) $\eta(\left(K_{p,q}, H^+_{r,2k}\right)) \geq p+q-2k-2$.

(ii) *If the first k largest non-negative eigenvalues of H are $\mu_1 = r \geq \mu_2 \geq \dots \geq \mu_k \geq 0$, then*

   (a) $\pm\, 2\mu_i$ *is an eigenvalue of $\left(K_{p,q}, H^+_{r,2k}\right)$, for $i = 2, \dots, k$ when $p+q \geq 2k+2$.*

   (b) $\pm\, 2\mu_i$ *is an eigenvalue of $\left(K_{p,q}, H^+_{r,2k}\right)$, for $i = 2, \dots, k-1$ when $p+q < 2k+2$.*

   Moreover, the other four eigenvalues of $\left(K_{p,q}, H^+_{r,2k}\right)$ are

$$\pm\sqrt{\frac{pq + (k-2r)^2 - k^2 \pm \theta}{2}},$$

   where

$$\theta = \sqrt{(pq + (k-2r)^2 - k^2)^2 - 4((k-2r)^2 + k(q-k) - \frac{k(q-2r)^2}{q})(q(p-k))}.$$

Finally, the necessary and sufficient condition for a complete bipartite signed graph whose negative edges induce a regular graph to be nonsingular is given below.

Corollary 5.4. *Let $\left(K_{p,q}, H^-_{r,2k}\right)$ (resp.$\left(K_{p,q}, H^+_{r,2k}\right)$) be a complete bipartite signed graph whose negative edges (resp. positive edges) induce an r-regular subgraph H of order $2k$. Then the signed graph $\left(K_{p,q}, H_{r,2k}\right)$ is nonsingular if and only if the graph H is nonsingular and $p = q = k \neq 2r$ or $p = q = k+1$.*

Conclusion and future research work

In this paper, we obtained a lower bound for the nullity of a complete bipartite signed graph and proved that

$$\eta((K_{p,q},\sigma)) \geq p+q-2k-2, \tag{5.5}$$

where $k = \min(r,s)$ and $(K_{p,q},\sigma)[U_r \cup V_s]$, $r \leq p$ and $s \leq q$, is an induced signed subgraph on minimum $r+s$ vertices, which contains all the negative edges of the signed graph $(K_{p,q},\sigma)$. We showed that this lower bound is best possible for a complete bipartite signed graph as shown in Figure 1. Therefore, the following becomes interesting.

Problem 1. To characterize all complete bipartite signed graphs for which the equality holds in inequality (5.5).

Furthermore, we determine, (1) the spectrum of a complete bipartite signed graph whose negative edges induce either disjoint complete bipartite subgraphs or a path, (2) the spectrum of a complete bipartite signed graph whose negative edges (positive edges) induce a regular subgraph, along with a relation between the eigenvalues of this complete bipartite signed graph and the non-negative eigenvalues of the regular subgraph. Thus, the following becomes interesting.

Problem 2. To determine the spectrum of a complete bipartite signed graph whose negative edges induce either co-regular graph, threshold graph, tree or k-cyclic graph.

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