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The fullerene graphs with a perfect star packing*

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Abstract

Fullerene graph G is a connected plane cubic graph with only pentagonal and hexagonal faces, which is the molecular graph of carbon fullerene. A spanning subgraph of G is called a perfect star packing in G if its each component is isomorphic to $K_{1,3}$. For an independent set $D \subseteq V(G)$, if each vertex in $V(G) \setminus D$ has exactly one neighbor in D, then D is called an efficient dominating set of G. In this paper we show that the number of vertices of a fullerene graph admitting a perfect star packing must be divisible by 8. This answers an open problem asked by Došlić et al. and also shows that a fullerene graph with an efficient dominating set has $8n$ vertices. In addition, we find some counterexamples for the necessity of Theorem 14 of paper of Došlić et al. from 2020 and list some subgraphs that preclude the existence of a perfect star packing of type $P0$.

Keywords: Fullerene graph, perfect star packing, efficient dominating set.

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1 Introduction

A chemical graph is a simple finite graph in which vertices denote the atoms and edges denote the chemical bonds in underlying chemical structure. Perfect matchings of a chemical graph correspond to Kekulé structures of the molecule, which feature in the calculation of molecular energies associated with benzenoid hydrocarbon molecules [20]. Alternating sextet faces (sextet patterns) also play a meaningful role in the prediction of molecular stability, in particular, but not only, in benzenoid compounds. Although for fullerenes, the above two structures do not play the same role as in benzenoid compounds, they have received considerable attention in recent years, see [1, 4, 8, 13, 17, 21, 32, 33] etc..

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A perfect matching in a graph G may be viewed as a collection of subgraphs of G, each of which is isomorphic to K_2, whose vertex sets partition the vertex set of G. This is naturally generalized by replacing K_2 by an arbitrary graph H. For a given graph H, an H*-packing* of G is the set of some vertex disjoint subgraphs, each of which is isomorphic to H. From the optimization point of view, the maximum H-packing problem is to find the maximum number of vertex disjoint copies of H in G called the *packing number*. An H-packing in G is called *perfect* if it covers all the vertices of G. If H is isomorphic to K_2, the maximum (perfect) H-packing problem becomes the familiar maximum (perfect) matching problem. If H is the cycle C_6 of length 6, for a fullerene or a hexagonal system G, the packing number is related to the Clar number (the maximum number of mutually disjoint sextet patterns) of G. If H is the star graph $K_{1,3}$, it is the maximum star packing problem. If a $K_{1,3}$-packing covers all the vertices of G, we call it being a *perfect star packing*. For a given family $\mathcal{F}$ of graphs, an H-packing concept can also be generalized to an $\mathcal{F}$-packing (we refer the reader to [29] for the definition).

Packing in graphs is an effective tool as it has lots of applications in applied sciences. H-packing, is of practical interest in the areas of scheduling [5], wireless sensor tracking [6], wiring-board design, code optimization [23] and many others. Packing problems were already studied for carbon nanotubes [2]. Packing lines in a hypercube had been studied in [15]. H-packing was determined for honeycomb [29] and hexagonal network [28]. For representing chemical compounds or to problems of pattern recognition and image processing, P_3-packing has some applications in chemistry [30]. Packing stars in fullerene graph have been investigated in [14] by Doslić et al.. For any integer $n \geq 5$, they found a fullerene graph of order $8n$ which has a perfect star packing. So they raised an open problem "Is there a fullerene on $8n + 4$ vertices with a perfect star packing?".

In the following section we introduce necessary preliminaries and characterize the classical fullerenes which have a perfect star packing. Section 3 gives a negative answer to the open problem asked by Doslić et al. [14]. This implies that a fullerene graph with an efficient dominating set must has $8n$ vertices. In Section 4, we generalize the Proposition 1 in reference [14] and give three counterexamples for the necessity of Theorem 14 in the same paper. We also list some subgraphs that preclude the existence of a perfect star packing of type $P0$.

2 Characterization of fullerenes with a perfect star packing

A *fullerene* graph (simply fullerene) is a cubic 3-connected plane graph with only pentagonal and hexagonal faces. By the Euler formula, each fullerene graph has exactly 12 pentagons. Such graphs are suitable models for carbon fullerene molecules: carbon atoms are represented by vertices, whereas edges represent chemical bonds between two atoms (see [16, 26]). For all even $n \geq 24$ and $n = 20$, Grünbaum and Motzkin [19] showed that there exists a fullerene graph with n vertices. Using a similar approach, Klein and Liu [24] proved that a fullerene graph with isolated pentagons of order n exists for $n = 60$ and for each even $n \geq 70$. We refer the reader to the reference [16] for more details on fullerene graphs.

A cycle of a fullerene graph G is a *facial cycle* if it is the boundary of a face in G, otherwise, it is a *non-facial* cycle. Clearly, each pentagon and hexagon in G is a facial cycle since G is 3-connected and any 3-edge-cut is trivial [31]. In paper [14], the authors obtained the following basic conclusions.

Proposition 2.1 ([14]). *Let S be a perfect star packing of fullerene graph G. Then each pentagon of G can contain at most one center of a star in S.*

Lemma 2.2 ([14]). *Let S be a perfect star packing of fullerene graph G. Then a vertex shared by two pentagons of G cannot be the center of a star in S.*

Recall that a vertex set X of a graph G is said to be *independent* if any two vertices in X are not adjacent in G. A cycle $C = v_1v_2 \cdots v_kv_1$ in G is called *induced* if v_i has only two adjacent vertices v_{i+1} and v_{i-1} around the k vertices $v_1, v_2, \cdots, v_k$ (note that $i + 1 := 1$ if $i = k$, and $i - 1 := k$ if $i = 1$). Otherwise, there exists some i and $j \notin \{i-1, i+1\}$ such that v_i and v_j are adjacent in G, the edge v_iv_j is a *chord* of C and C is not induced. A subgraph R of a graph G is *spanning* if R covers all the vertices of G. For a vertex v of a graph G, we call vertex u being a *neighbor* of v in G if u is adjacent to v in G.

Theorem 2.3. *Let G be a fullerene graph. Then G has a perfect star packing if and only if G has an independent vertex set S^* such that each component of $G - S^*$ is an induced cycle in G.*

Proof. If G has a perfect star packing S, then S is a spanning subgraph of G and any component in S is isomorphic to a star graph $K_{1,3}$. Let S^* be the set of all 3-degree vertices in S. Clearly, S^* is an independent vertex set in G and any vertex in $G - S^*$ has degree 2. So each component of $G - S^*$ is an induced cycle in G.

Let S^* be an independent vertex set of G such that each component of $G - S^*$ is an induced cycle in G. Clearly, each vertex in S^* and its three neighbors induce a star graph $K_{1,3}$. We collect all these star graphs and denote this set by $\mathcal{H}$. For any vertex x on a cycle C in $G - S^*$, x has exactly one neighbor in S^* since G is 3-regular and induced cycle C is a component of $G - S^*$. So $\mathcal{H}$ is a spanning subgraph of G and each component of $\mathcal{H}$ is a star graph $K_{1,3}$, that is, $\mathcal{H}$ is a perfect star packing of G. □

We note that star graph $K_{1,3}$ has exactly one *center* (the vertex of degree 3) and three leaves. For a perfect star packing S of fullerene graph G, each 1-degree vertex in S is a *leaf*. In the following, we denote by $C(S)$ the set of all the centers of stars in S.

Remark 2.4. Let S be a perfect star packing of fullerene graph G. Then

(1) $C(S)$ is an independent vertex set in G.

(2) Any leaf in S has exactly one neighbor belonging to $C(S)$ and has exactly two neighbors being leaves in S.

(3) Each cycle in $G - C(S)$ does not have a chord.

Proposition 2.5. *Each hexagon can contain at most two centers of a perfect star packing of fullerene graph G. If a hexagon h contains two such centers, then they are antipodal points on the hexagon h.*

Proof. Let h be a hexagon in G. We denote the six vertices of h by $v_1, v_2, \ldots, v_6$ in the clockwise direction. If vertex v_1 is the center of a star H in a perfect star packing S of G, then v_2 and v_6 are two leaves in H. Hence both v_3 and v_5 are leaves in S by Remark 2.4(2). Clearly, v_4 could be the center of a star in S. Hence h has at most two centers of S and if h contains two such centers, then they are antipodal points on h. □

3 The order of fullerenes with a perfect star packing

To show the main conclusion, we need to prepare as follows.

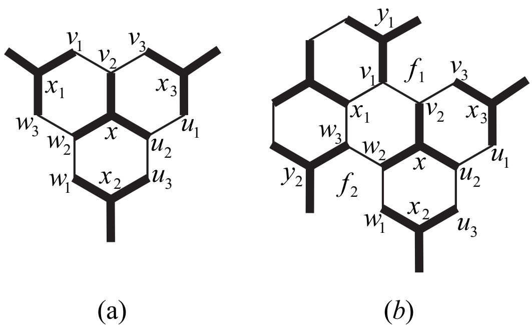

Figure 1: (a) Type 1; (b) Type 2; (c) Type 3.

Lemma 3.1. *Let S be a perfect star packing of fullerene graph G. Then for any vertex $x \in C(S)$, all the vertices on the three faces sharing x are covered by S as Type* 1*, Type* 2 *or Type* 3 *(see* Figure 1*, S are depicted in bold lines).*

Proof. By the Lemma 2.2, at most one of the three faces sharing x is a pentagon since $x \in C(S)$. There are two cases as follows.

Case 1: The three faces sharing x are all hexagons.

Clearly, x has three antipodal points on the three hexagons sharing x, denoted by x_1, x_2 and x_3 respectively as depicted in Figure 1(a). By Remark 2.4(2), the two neighbors v_1 and v_3 of v_2 are leaves in S. Similarly, u_1, u_3, w_1 and w_3 are also leaves in S. We claim that at least two of x_1, x_2 and x_3 are centers of stars in S. If x_1 is not the center of a star in S, then x_1 is a leaf in S. So the third neighbor of v_1, say y_1, is the center of a star in S (see Figure 1(b)). Similarly, the third neighbor of w_3, say y_2, is also the center of a star in S. Since the three vertices v_1, v_2 and v_3 are leaves in S and $y_1 \in C(S)$, the face f_1 has only one center of S by Propositions 2.5 and 2.1. Hence the two neighbors of v_3 on f_1 are leaves. By Remark 2.4(2), x_3 is the center of a star in S, that is, $x_3 \in C(S)$. Similarly, w_1 is a leaf in S and the two neighbors of w_1 on f_2 are all leaves in S. Hence $x_2 \in C(S)$. So at least two of x_1, x_2 and x_3 belong to $C(S)$. If exactly two of x_1, x_2 and x_3 belong to $C(S)$, without loss of generality, we suppose that $x_2, x_3 \in C(S)$, then all the vertices on the three faces sharing x are covered by S as Type 2. If all the three vertices x_1, x_2 and x_3 belong to $C(S)$ (see Figure 1(a)), then all the vertices on the three faces sharing x are covered by S as Type 1.

Case 2: Exactly one of the three faces sharing x is a pentagon.

By Proposition 2.1, w_1 and u_3 are leaves in S (see Figure 1(c)). Hence $x_4, x_3 \in C(S)$ and f is a hexagon by Remark 2.4(2) and Proposition 2.5. By Remark 2.4(2), the neighbor w_3 of w_2 is a leaf in S since the neighbor x of w_2 belongs to $C(S)$. Hence the other vertices on f_1 except for x_4 are all leaves in S by Propositions 2.1 and 2.5. This follows that the neighbor x_1 of w_3 is the center of a star in S by Remark 2.4(2). Similarly, we can show

$x_2 \in C(S)$. Hence all the vertices on the three faces sharing x are covered by S as Type 3 (see Figure 1(c)). □

Corollary 3.2. *Let S be a perfect star packing of fullerene graph G. If a pentagon P of G has a vertex $x \in C(S)$, then $G - C(S)$ has a non-facial cycle C of G such that the path $P - x$ is a subgraph of C.*

Proof. By Proposition 2.2, x is shared by this pentagon P and two hexagons. So all the vertices on the three faces sharing x are covered by S as Type 3 (see Figure 1(c)). Clearly, the path $P - x$ is a subgraph of a cycle C in $G - C(S)$ and C is a non-facial cycle of G. □

We note that 3-connected graphs have only one embedding up to equivalence [12]. If we embed a fullerene graph G in the plane, then any non-facial cycle C of G as a Jordan curve separates the plane into two regions, denoted by R_1^* and R_2^*, each of which has the entire C as its frontier. We denote the subgraph of G induced by the vertices lying in the interior of R_i^* by G_i, $i = 1, 2$. Here we note that $\{V(G_1), V(G_2), V(C)\}$ is a partition of all the vertices of G. We say that C *divide* the graph G into two *sides* G_1 and G_2.

Theorem 3.3. *Let S be a perfect star packing of fullerene graph G and C be a cycle in $G - C(S)$. Then $C(S)$ does not have a vertex which has three neighbors on C.*

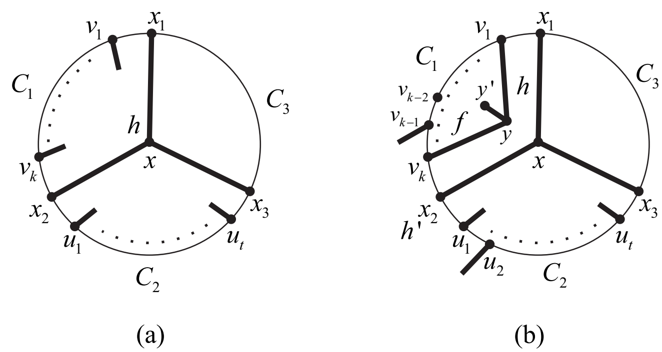

Figure 2: $x \in C(S)$ has three neighbors on C.

Proof. If C is a facial cycle of G, then C is a pentagon or a hexagon. The conclusion clearly holds. Now, let C be a non-facial cycle of G. Then C divides G into two sides, denoted by H_1 and H_2 respectively. We note that all vertices on C are leaves in S since C is a cycle in $G - C(S)$. On the contrary, we suppose that there is a vertex $x \in C(S)$ which has three neighbors on C, denoted by x_1, x_2 and x_3 respectively. Without loss of generality, we suppose that $x \in V(H_1)$ (see Figure 2(a)). The three vertices separate the circle C into three sections, denoted by C_1, C_2 and C_3 respectively, each of which is a path with x_i and x_{i+1} as two terminal ends, $i = 1, 2, 3$ (if $i = 3$, then $i + 1 := 1$). From Lemma 3.1 we know that at most one of $x_1C_1x_2x$, $x_2C_2x_3x$ and $x_3C_3x_1x$ is a facial cycle of G since C is a cycle in $G - C(S)$. Next, we suppose that $x_1C_1x_2x$ and $x_2C_2x_3x$ are non-facial cycles of G. Let $C_1 = x_1v_1v_2 \cdots v_kx_2$, $C_2 = x_2u_1u_2 \cdots u_tx_3$. So $k \geq 5$ and

$t \geq 5$ since any non-facial cycle of G has length at least 8. By Remark 2.4(3), C does not have a chord. So $v_1 v_k \notin E(G)$ and $u_1 u_t \notin E(G)$. This implies that h is a hexagon face of G, and x_1, x, x_2 and v_1, v_k are five vertices on h. We denote the sixth vertex of h by y. Clearly, $y \in V(H_1)$ by the planarity of G (see Figure 2(b)). Similarly, both u_1 and u_t have a common neighbor in H_1.

Since S is a perfect star packing of G and the two neighbors x_1 and v_2 of v_1 are leaves in S, y is the center of a star in S. If the third neighbor of y is on C, then it is on C_1, denoted it by v_r. The three neighbors of y separate the circle C into three sections, two of which are subgraphs of C_1, denoted by C_1^1 and C_1^2 respectively. As the above discussion, we know that one of $v_1 C_1^1 v_r y$ and $v_r C_1^2 v_k y$ is a non-facial cycle of G. By the recursive process and the finiteness of the order of G, we can suppose that the third neighbor of y is not on C, and denoted it by y'.

See Figure 2(b), the five vertices $v_{k-1}, v_k, x_2, u_1, u_2$ belong to a common facial cycle h' of G. Since C does not have a chord by Remark 2.4(3), v_{k-1} and u_2 are not adjacent in G. So h' is a hexagon. By the planarity of G, v_{k-1} and u_2 have a common neighbor in H_2. so v_{k-2}, v_{k-1}, v_k, y and y' are on a face of G, say f. If f is a pentagon, then v_{k-2} is adjacent to y'. So all the three neighbors of v_{k-2} are leaves in S. This implies a contradiction since v_{k-2} is also a leaf in S. If f is a hexagon, then v_{k-2} and y' have a common neighbor, denoted by z. Clearly, z is v_{k-3} or not. For $z = v_{k-3}$, the three neighbors of v_{k-3} are all leaves in S, a contradiction. For $z \neq v_{k-3}$, by Remark 2.4(2), z is a leaf in S since y' has a neighbor $y \in C(S)$. So the three neighbors of v_{k-2} are all leaves in S, a contradiction. All these contradictions imply that $C(S)$ does not have a vertex which has three neighbors on C. □

Let S be a perfect star packing of fullerene graph G and C be a cycle in $G - C(S)$ which is a non-facial cycle of G. C divides G into two sides, denoted by H_1 and H_2 respectively. Set C^i be the set of all the vertices on C each of which has a neighbor in H_i, $i = 1, 2$. Clearly, $\{C^1, C^2\}$ is a partition of $V(C)$. $G[C^i]$ is a vertex induced subgraph of G which has vertex set C^i and any two vertices of C^i are adjacent if and only if they are adjacent in G. See Figure 4, $G[C^1]$ is depicted as red and $G[C^2]$ is depicted as blue. In the following, we use these symbols no longer explaining.

Lemma 3.4. *For $i = 1, 2$, if a vertex x on C has a neighbor in H_i, then the component of the induced subgraph $G[C^i]$ which contains x is a path with 2 or 3 vertices.*

Proof. We suppose that x on C has a neighhbor in H_1. For the convenience of the following description, set $C := x v_1 v_2 \cdots v_k x$. Since C is a cycle in $G - C(S)$ which is a non-facial cycle of G, the length of C is at least 8. So $k \geq 7$. There are three cases for the two neighbors v_1 and v_k of x on C.

Case 1: Both v_1 and v_k have neighbors in H_2.

In this case, the three vertices v_1, x and v_k lie on the same face f of G (see Figure 3(a)). Since all the vertices on C are leaves in S, the other neighbor of v_1 (resp. v_k) which is not on C is the center of a star in S. So f has two vertices in $C(S)$ which are the centers of two stars in S covered v_1 and v_k, respectively. So f is a hexagon by Proposition 2.1. But the case cannot hold by Propositions 2.5.

Case 2: Both v_1 and v_k have neighbors in H_1.

In this case, the five vertices $v_2, v_1, x, v_k, v_{k-1}$ belong to a facial cycle h of G (see Figure 3(b)). We claim that both v_2 and v_{k-1} have neighbors in H_2. Otherwise, at least

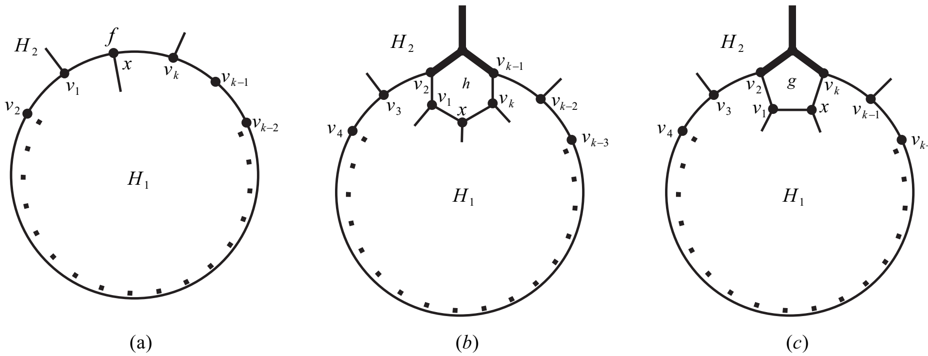

Figure 3: (a) v_1 and v_k have neighbors in H_2; (b) v_1 and v_k have neighbors in H_1; (c) v_1 has a neighbor in H_1, v_k has one in H_2.

one of v_2 and v_{k-1} has a neighbor in H_1. If v_2 has a neighbor in H_1 and v_{k-1} has a neighbor in H_2, then the six vertices $v_3, v_2, v_1, x, v_k, v_{k-1}$ lie on a face h of G. So h is a hexagon and C has a chord v_3v_{k-1}, a contradiction. For v_2 having a neighbor in H_2 and v_{k-1} having a neighbor in H_1, we can also obtain a chord of C, a contradiction. If both v_2 and v_{k-1} have neighbors in H_1, then the seven vertices $v_3, v_2, v_1, x, v_k, v_{k-1}, v_{k-2}$ belong to a common face h of G. This implies that G has a facial cycle of length at least 7, a contradiction. So both v_2 and v_{k-1} have neighbors in H_2, and $v_2, v_1, x, v_k, v_{k-1}$ lie on a hexagon h of G (see Figure 3(b)). Since C does not have a chord, the path v_1xv_k is a connected component of the induced subgraph $G[C^1]$.

Case 3: v_1 has a neighbor in H_1 and v_k has a neighbor in H_2, or v_1 has a neighbor in H_2 and v_k has a neighbor in H_1.

By symmetry, it is sufficient to consider that v_1 has a neighbor in H_1 and v_k has a neighbor in H_2. If v_2 has a neighbor in H_1, then v_3 must have a neighbor in H_2, otherwise, C has a chord or G has a facial cycle of length at least seven, a contradiction. As the proof of Case 2, v_3, v_2, v_1, x, v_k lie on a hexagonal facial cycle. So the path v_2v_1x is a connected component of the induced subgraph $G[C^1]$. Now, we suppose that v_2 has a neighbor in H_2. Then the four vertices v_k, x, v_1, v_2 lie on the same face g of G. Since v_k, x, v_1, v_2 are all leaves in S, g is a pentagon and v_2, v_k have a common neighbor in H_2 which is the center of a star in S (see Figure 3(c)). So the path xv_1 is a connected component of the induced subgraph $G[C^1]$.

In summary, the component of the induced subgraph $G[C^1]$ which contains x is a path with 2 or 3 vertices since C does not have a chord. □

In addition, we have the following Lemma.

Lemma 3.5. *Each component of $G[C^i]$ is a path with 2 or 3 vertices, $i = 1, 2$.*

Proof. For any vertex x on C, x must have exactly one neighbor in H_1 or H_2 since G is 3-regular and C does not have a chord. Without loss of generality, we suppose that x has exactly one neighbor in H_1. By Lemma 3.4, the component of the induced subgraph $G[C^1]$ which contains x is a path with 2 or 3 vertices. We note that the choice of x is arbitrary. So the conclusion holds. □

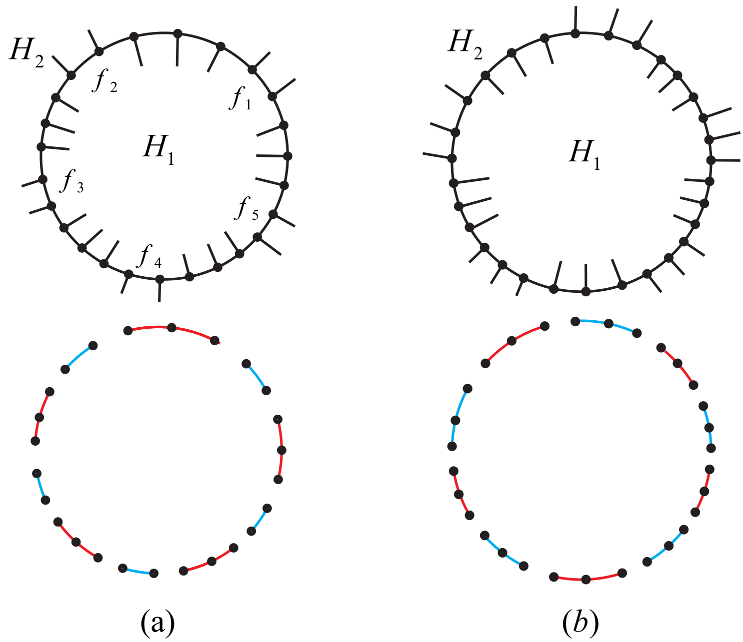

Figure 4: (a) A cycle C of length 25; (b) A cycle C of length 30. ($G[C^1]$ is red and $G[C^2]$ is blue.)

Proposition 3.6. *Let $C = v_0v_1 \cdots v_{k-1}$ be a non-facial cycle in $G - C(S)$ (In the following, the subscript is modulo k).*

(i) *If both v_i and v_{i+1} have neighbors in H_1 (resp. H_2) and v_{i-1} and v_{i+2} have neighbors in H_2 (resp. H_1), then the four vertices v_{i-1}, v_i, v_{i+1} and v_{i+2} lie on a pentagon of G.*

(ii) *If v_i, v_{i+1}, v_{i+2} have neighbors in H_1 (resp. H_2) and v_{i-1} and v_{i+3} have neighbors in H_2 (resp. H_1), then the five vertices $v_{i-1}, v_i, v_{i+1}, v_{i+2}$ and v_{i+3} lie on a hexagon of G.*

(iii) *For $j = 1, 2$, if both v_i and v_{i+1} have neighbors in H_j (we denote the two edges incident to v_i and v_{i+1} not lie in C by e_i and e_{i+1}, respectively), then the facial cycle containing both e_i and e_{i+1} is a hexagon, and two antipodal points on this hexagon are centers of two stars in the perfect star packing S.*

Proof. Cases (i) and (ii) can be easily obtained from the proof of the Cases 2 and 3 of Lemma 3.4 (see Figure 3). Since all the vertices on C are leaves in the perfect star packing S, the other end of e_i (resp. e_{i+1}) which is not on C, denoted by u_i (resp. u_{i+1}), is the center of a star in S. We know that any facial cycle of G is a pentagon or a hexagon. So u_i and u_{i+1} are distinct. By Lemmas 2.1 and 2.5, the facial cycle containing both e_i and e_{i+1} is a hexagon, and u_i and u_{i+1} are antipodal points on this hexagon. □

For example, in Figure 4, except for $f_i, i \in \{1, 2, 3, 4, 5\}$ the other faces sharing edges with C are all hexagons. Moreover, how the vertices on C being covered by S is determined.

We recall that the union of two graphs G_1 and G_2 is denoted by $G_1 \cup G_2$, which has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Let n_3 be the number of the components of $G[C^1] \cup G[C^2]$ each of which is isomorphic to a path with 3 vertices. Similarly, n_2 is the number of the components of $G[C^1] \cup G[C^2]$ each of which is isomorphic to a path with 2 vertices. For example, $n_3 = n_2 = 5$ in Figure 4(a) and $n_3 = 10$, $n_2 = 0$ in Figure 4(b).

Observation 1. $n_2 + n_3$ is even.

Proposition 3.7. *Let S be a perfect star packing of fullerene graph G and C a cycle in $G - C(S)$ which is a non-facial cycle of G. Then the length of C is $3n_3 + 2n_2$, and the length of C has the the same parity with n_2 and n_3.*

Proof. Clearly, the length of C is $3n_3 + 2n_2$ by Lemma 3.5. So n_3 is odd if and only if the length of C is odd. Since $n_2 + n_3$ is even by Observation 1, the parity of n_2 and n_3 are same. Then we are done. $\square$

Theorem 3.8. *Let S be a perfect star packing of fullerene graph G. Then $G - C(S)$ has even number of odd cycles.*

Proof. If $G - C(S)$ does not have a non-facial cycle of G, then any pentagon of G does not have a vertex in $C(S)$ by Corollary 3.2. So all the vertices on pentagons are leaves in S. It implies that $G - C(S)$ has exactly twelve odd cycles, each of which is a pentagon. Next, we suppose that $G - C(S)$ has a non-facial cycle of G, denoted by C.

Claim 1: If C is an even cycle, then G has even number of pentagons which share edges with C. If C is an odd cycle, then G has odd number of pentagons which share edges with C.

By Proposition 3.6, the number of pentagons which share edges with C is equal to n_2. By Proposition 3.7, n_2 and the length of C have the same parity. So the Claim holds.

Claim 2: Any pentagon of G shares edges with at most one non-facial cycle in $G - C(S)$.

Let P be a pentagon of G. By Proposition 2.1, P has at most one vertex which is the center of a star in S. If P does not have a vertex in $C(S)$, then P is a cycle in $G - C(S)$. By Theorem 2.3, each component of $G - C(S)$ is an induced cycle of G. So P does not share edges with any non-facial cycle in $G - C(S)$. If P has a vertex $x \in C(S)$, then by Corollary 3.2 $P - x$ is a subgraph of a non-facial cycle in $G - C(S)$. So P shares edges with exactly one non-facial cycle in $G - C(S)$.

Now, we consider the following two cases for the non-facial cycles in $G - C(S)$.

Case 1: $G - C(S)$ does not have a non-facial cycle of odd length.

Then any non-facial cycle C in $G - C(S)$ is of even length. By the above Claims, there are even number of pentagons in G such that they share edges with C. Since G has exactly twelve pentagons, there are even number of pentagons in G each of which does not share edges with non-facial cycles in $G - C(S)$. These pentagons must be cycles in $G - C(S)$ by Corollary 3.2. Hence $G - C(S)$ has even number of odd cycles.

Case 2: $G - C(S)$ has some non-facial cycle of odd length.

Suppose that $G - C(S)$ has exactly k non-facial cycles of odd length. We denote the number of pentagons in G each of which does not share edges with non-facial cycles in

$G - C(S)$ by p. These p pentagons must be cycles in $G - C(S)$ by Corollary 3.2. So $G - C(S)$ has $p + k$ odd length cycles. Next, we show that p and k have the same parity. If p is odd, then G has odd number of pentagons each of which share edges with exactly one non-facial cycle in $G - C(S)$ since G has exactly 12 pentagons. By the above Claims, for each even length non-facial cycle in $G - C(S)$, G has even number of pentagons which share edges with the cycle, and for each odd length non-facial cycle in $G - C(S)$, G has odd number of pentagons which share edges with the cycle. So $G - C(S)$ has odd number of non-facial cycles of odd length. This means that k is odd. For p being even, we can similarly show that k is even. So k and p have the same parity and $p + k$ is even. □

Clearly, for a fullerene graph G with a perfect star packing, its order must be divisible by 4. So the order of G is $8k$ or $8k + 4$ for some positive integer k. Now, we can obtain the following main theorem which illustrates that the order of G can not be $8k + 4$.

Theorem 3.9. *If fullerene graph G has a perfect star packing, then the order of G is divisible by* 8.

Proof. We suppose that S is a perfect star packing of G and $\mathcal{C}_o$ and $\mathcal{C}_e$ are the collections of all the odd cycles and even cycles in $G - C(S)$, respectively. Then we have the following equation.

$$\begin{aligned} |V(G)| &= |C(S)| + \sum_{C \in \mathcal{C}_o} |C| + \sum_{C \in \mathcal{C}_e} |C| \\ &= \frac{|V(G)|}{4} + \sum_{C \in \mathcal{C}_o} |C| + even. \end{aligned} \tag{3.1}$$

By Theorem 3.8, $\mathcal{C}_o$ has even number of elements. Combining the above equation, we know that $\frac{|V(G)|}{4} \times 3$ is even. Hence $\frac{|V(G)|}{4}$ is even, that is, the order of G is divisible by 8. □

This theorem is equivalent to the following corollary.

Corollary 3.10. *A fullerene graph with order $8n + 4$ does not have a perfect star packing.*

We recall that a *dominating set* of a graph G is a set D of vertices such that each vertex in $V(G) - D$ is adjacent to a vertex in D. Moreover, if each vertex in $V(G) - D$ is adjacent to exactly one vertex in D and D is an independent vertex set, then D is called *efficient*. The problem of determining the existence of efficient dominating sets in some families of graphs was first investigated by Biggs [7] and Kratochvil [25]. Later Livingston and Stout [27] studied the existence and construction of efficient dominating sets in families of graphs arising from the interconnection networks of parallel computers. It is algorithmically hard to find an efficient dominating set [3]. For more results and some historical background regarding efficient dominating set, we refer the reader to [9, 10, 11, 22] etc..

From the definitions of the efficient dominating set and the perfect star packing of a fullerene graph, the following proposition is a natural result.

Proposition 3.11 ([14])**.** *A fullerene graph G with n vertices has a perfect star packing if and only if G has an efficient dominating set of cardinality $\frac{n}{4}$.*

Combining Theorem 3.9 and Proposition 3.11, we get the following theorem.

Theorem 3.12. *The order of a fullerene graph with an efficient dominating set is $8n$.*

4 Some other conclusions

Došlić et al. gave the following necessary condition in terms of graph spectra.

Proposition 4.1 ([14]). *If a fullerene graph G has a perfect star packing, then -1 must be an eigenvalue of the adjacency matrix of G.*

The proof of this Theorem can be translate to a simple r-regular graph. Here for completeness, we prove as follows. For the definition of eigenvalues of the adjacency matrix of a graph, we refer the reader to [18].

Theorem 4.2. *If a simple r-regular graph G has a perfect $K_{1,r}$-packing S, then -1 must be an eigenvalue of the adjacency matrix of G.*

Proof. Let $C(S)$ be the set of centers of stars $K_{1,r}$ in S. We define the characteristic vector $\overrightarrow{c} \in \mathbb{R}^{|V(G)|}$ of $C(S)$ as follows: $c_i = 1$ if $i \in C(S)$, otherwise $c_i = 0$. Since G is a r-regular graph, we have $A\overrightarrow{u} = r\overrightarrow{u}$, where A is the adjacency matrix of G and $\overrightarrow{u}$ is the all one vectors. Let $\overrightarrow{w} = \overrightarrow{u} - (r+1)\overrightarrow{c}$. As $A\overrightarrow{c} = \overrightarrow{u} - \overrightarrow{c}$, we have

$$A\overrightarrow{w} = A\overrightarrow{u} - (r+1)A\overrightarrow{c} = r\overrightarrow{u} - (r+1)\overrightarrow{u} + (r+1)\overrightarrow{c} = (r+1)\overrightarrow{c} - \overrightarrow{u} = -\overrightarrow{w} \quad (4.1)$$

This means that -1 is an eigenvalue of A. □

For a perfect star packing S of fullerene graph G, if for each center $x \in C(S)$, all the three faces of G sharing x are hexagons, then we call S being *type* $P0$. For such perfect star packing, the following corollary holds.

Corollary 4.3. *If a fullerene graph G has a perfect star packing S of type $P0$, then $G - C(S)$ does not have a non-facial cycle of odd length.*

Proof. By the contrary, we suppose that $G - C(S)$ has a non-facial cycle C of odd length. By the Claim 1 of Theorem 3.8, G has a pentagon P which share edges with C. This implies that P contains the center y of a star in S. So one of the three faces of G sharing y is not a hexagon. This contradicts that S is of type $P0$. So $G - C(S)$ does not have a non-facial cycle of odd length. □

In the above Corollary, we note that $G - C(S)$ may have non-facial cycles of even lengths (see Figure 5, the blue cycle in C_{120}).

Now, we point out the flaw of the Theorem 14 in [14].

Theorem 4.4 ([14]). *A fullerene graph on $8n$ vertices has a perfect star packing of type $P0$ if and only if it arises from some other fullerene via the chamfer transformation.*

Readers can consult reference [14] to see the chamfer transformation. Here for completeness, we introduce it as follows. Let F be a fullerene graph. In each face g of F, we draw a polygon with the same number of sides as g. For each vertex $v \in V(F)$, we connect v with three new vertices each of which is inside exactly one face of F incident with v (see Figure 6, the vertices of original fullerene C_{20} are black, the new vertices are blue, each black vertex are connected to three blue vertices). We notice that each new vertex must be adjacent to exactly one vertex of F in this process, and the edges do not intersect inside. Finally, we remove all the edges of F. The resulting graph is called arising from F via the *chamfer transformation*. For example, (see Figure 6) the graph $C_{80}(I_h)$ arises from C_{20}

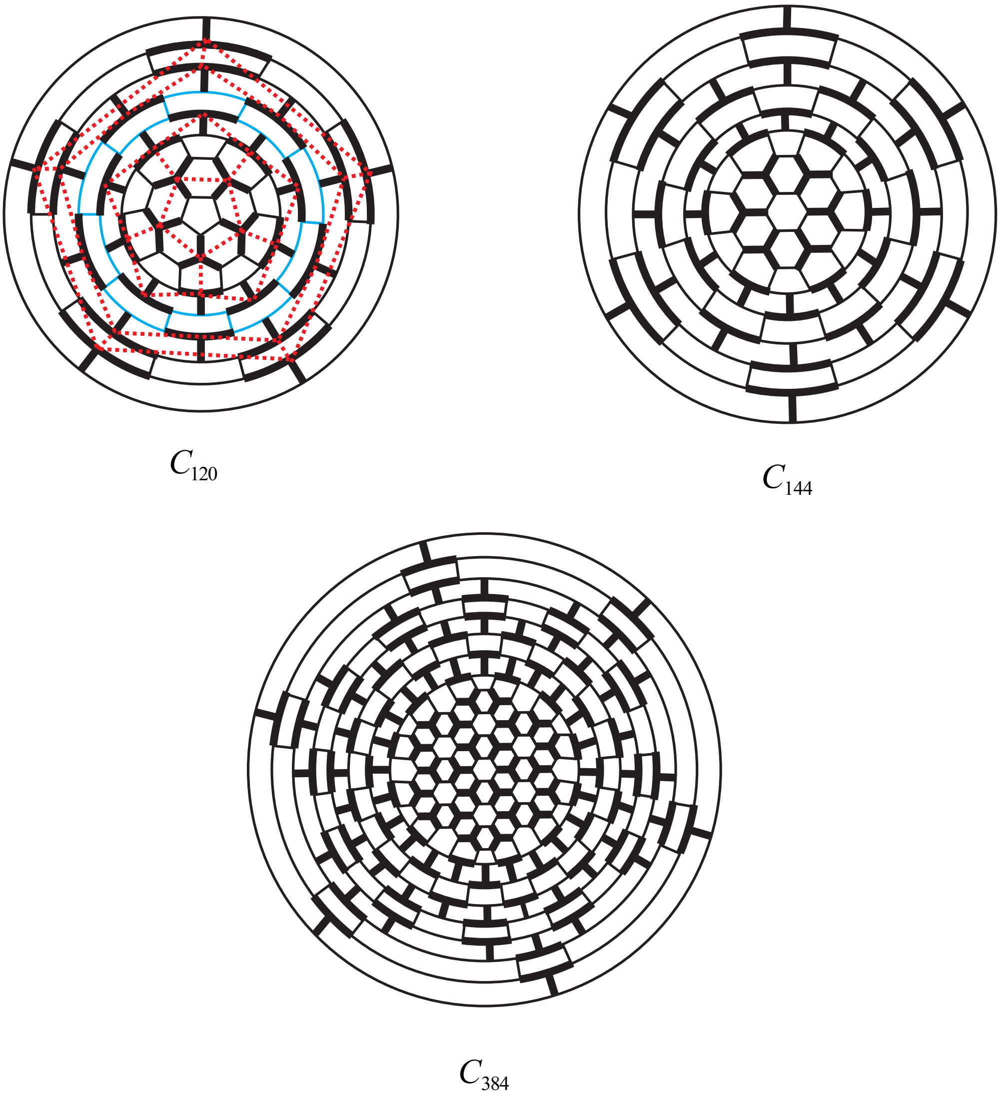

Figure 5: Each of $C_{120}, C_{144}, C_{384}$ has a unique perfect star packing of type $P0$ which is depicted in bold edges.

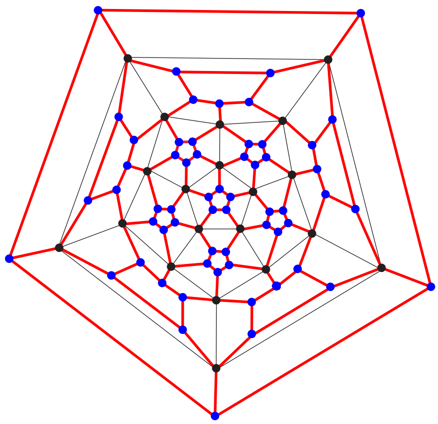

Figure 6: C_{20} is drawn in black line, $C_{80}(I_h)$ is drawn in red line.

via the chamfer transformation, and all the black vertices are the centers of stars in a perfect star packing of type $P0$ of $C_{80}(I_h)$.

For a perfect star packing S of type $P0$ in fullerene graph G, we construct a new graph with respect to S, and denoted it by G^S. $V(G^S) := C(S)$ and any two vertices in $V(G^S)$ are adjacent if and only if they belong to the same hexagon of G. In the proof of the necessity of the Theorem 4.4, there exist the following problem. G^S is planar, but does not have to be 3-regular, 3-connected and have only pentagonal and hexagonal faces. For example, it is easy to check that the fullerene graph C_{120} (resp. C_{144}, C_{384}) has a unique perfect star packing S_1 (resp. S_2, S_3) of type $P0$ (as depicted in bold edges in Figure 5). $C_{120}^{S_1}$, $C_{144}^{S_2}$ and $C_{384}^{S_3}$ are planar and not connected (the red dashed line in Figure 5 is the $C_{120}^{S_1}$, and here we omit the $C_{144}^{S_2}$ and $C_{384}^{S_3}$). In fact, we have Lemma 4.5.

I would like to thank Tomislav Došlić for conversations and email exchanges related to the contents of this paragraph.

Lemma 4.5. *The three fullerene graphs C_{120}, C_{144} and C_{384} as depicted in* Figure 5 *cannot arise from some other fullerene via the chamfer transformation.*

Proof. On the contrary, we suppose that C_{120} can arise from some fullerene F via the chamfer transformation. Then C_{120} has a perfect star packing S of type $P0$ which corresponds to the chamfer transformation of F, that is, all the vertices of F are the centers of stars in S. This means that $C_{120}^S = F$.

We can check that C_{120} has a unique perfect star packing of type $P0$, denoted by S_1 (as depicted in bold edges in Figure 5). So $S_1 = S$. However, $C_{120}^{S_1}$ is not connected (as depicted by red dotted lines in Figure 5). So $S_1 \neq S$, a contradiction.

For the other two fullerenes C_{144} and C_{384}, we can also check that each of them has a unique perfect star packing of type $P0$ (as depicted in bold edges in Figure 5). As the above proof, they also cannot arise from any fullerene graphs via the chamfer transformations. □

From Lemma 4.5 we know that the necessity of Theorem 4.4 does not hold, however, its sufficiency is right. So it can be corrected as follows.

Theorem 4.6. *A fullerene graph that arises from some other fullerene via the chamfer transformation must have a perfect star packing of type $P0$.*

If fullerene graph G has two pentagons sharing an edge xy, then x (resp. y) can not be center of a star in a perfect star packing of G by Lemma 2.2. Since all the three neighbors of x belong to pentagons of G, G does not have a perfect star packing of type $P0$. Hence if a fullerene graph has a perfect star packing of type $P0$, then all its pentagons are isolated. Next we list some other forbidden subgraphs for guaranteeing a fullerene graph to own a perfect star packing of type $P0$.

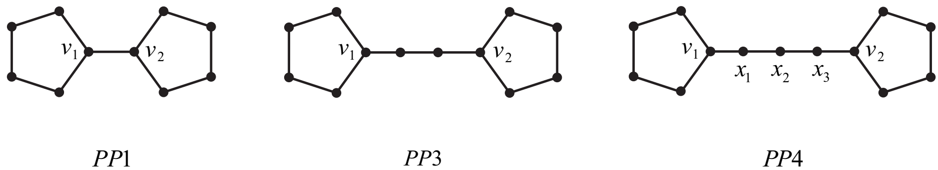

Figure 7: Three forbidden configurations.

Proposition 4.7. *If a fullerene graph G contains a subgraph $PP1, PP3$ or $PP4$ (see* Figure 7*), then it cannot have a perfect star packing of type $P0$.*

Proof. By the contrary, we suppose that G has a perfect star packing of type $P0$, denoted by S. Clearly, the vertices v_1 and v_2 (see Figure 7) are leaves in S. If $PP4$ is a subgraph of G, then x_1 is the center of a star in S since all vertices on a pentagon are leaves in S. So x_2 is a leaf in S. By Remark 2.4(2), the neighbor x_3 of x_2 is also a leaf in S. This implies that all the three neighbors of v_2 are leaves in S, a contradiction. For subgraphs $PP1$ and $PP3$, we can similarly show that v_1 or v_2 have all its three neighbors being leaves in S, a contradiction. □

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