ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 223–253
https://doi.org/10.26493/1855-3974.1695.144
(Also available at http://amc-journal.eu)

The orientable genus of the join of a cycle and a complete graph

Dengju Ma *

School of Sciences, Nantong University, 226019, Nantong, China

Han Ren †

Department of Mathematics, East China Normal University, 200062, Shanghai, China

Received 8 May 2018, accepted 22 April 2019, published online 2 October 2019

Abstract

Let m and n be two integers. In the paper we show that the orientable genus of the join of a cycle C_m and a complete graph K_n is $\lceil \frac{(m-2)(n-2)}{4} \rceil$ if $n = 4$ and $m \geq 12$, or $n \geq 5$ and $m \geq 6n - 13$.

Keywords: Surface, orientable genus of a graph, join of two graphs.

Math. Subj. Class.: 05C10

1 Introduction

Let G and H be two disjoint graphs. The *join* of G with H, denoted by $G + H$, is the graph obtained from the union of G and H by adding edges joining every vertex of G to every vertex of H. A cycle with m vertices is denoted by C_m, and a complete graph with n vertices denoted by K_n.

Our investigation of the orientable genus of $C_m + K_n$ is inspired by the problem of the critical graphs on surfaces. A graph G is *k-critical* if $\chi(G) = k$ but $\chi(G') < k$ for every proper subgraph of G, where $\chi(H)$ denotes the chromatic number of a graph H. If G_1 is k-critical and G_2 is l-critical, it is known that $G_1 + G_2$ is $(k + l)$-critical. Since an odd cycle is 3-critical and K_n is n-critical, the join of an odd cycle and K_n is $(n + 3)$-critical. Also, there are only finite many k-critical graphs on a surface if $k \geq 7$ ([4, 6, 7, 13]). So it is an interesting problem to explore the orientable genus of the join of an odd cycle (or a cycle) and K_n.

*Corresponding author. Supported by NNSFC under the granted number 11171114.

†Supported by NNSFC under the granted number 11171114, Science and Technology Commission of Shanghai Municipality under grant No. 13dz2260400.

E-mail addresses: ma-dj@163.com (Dengju Ma), hren@math.ecnu.edu.cn (Han Ren)

Let us look back the history of studying the orientable genus of the join of two graphs. Let $\bar{K}_t$ be the compliment graph of K_t. The complete bipartite graph $K_{m,n}$ and K_n $(n \geq 2)$ can be viewed as $\bar{K}_m + \bar{K}_n$ and $K_1 + K_{n-1}$, respectively. It is cheerful that the orietable genera of K_n and $K_{m,n}$ have been determined ([10, 11]). Upon the orientable genus of $\bar{K}_m + K_n$ there are some results. Craft [3] verified that $\bar{K}_m + K_n$ has the same orientable genus as that of $K_{m,n}$, when n is even and $m \geq 2n - 4$. Ellingham and Stephens [5] determined the orientable genus of $\bar{K}_m + K_n$ if n is even and $m \geq n$, or $n = 2^p + 2$ for $p \geq 3$ and $m \geq n - 1$, or $n = 2^p + 1$ for $p \geq 3$ and $m \geq n + 1$. Korzhik [8] contributed many results on the orientable genus of $\bar{K}_m + K_n$ with $m \leq n - 2$.

Let $m \geq 3$ and $n \geq 1$ be two integers. If $n = 1$, then $C_m + K_n$ is a planar graph. If $n = 2$, then $C_m + K_n$ has a minor isomorphic to K_5. So the orientable genus of $C_m + K_2$ is at least one. Since $C_m + K_2$ can be embedded on the torus, the orientable genus of $C_m + K_2$ is one. If $n = 3$, then K_n is exactly the cycle C_3. Craft [2] has proved that the orientable genus of $C_m + C_3$ is $\lceil \frac{m-2}{4} \rceil$. What is the orientable genus of $C_m + K_n$ if $n \geq 4$? In the paper we shall show that the orientable genus of $C_m + K_n$ is $\lceil \frac{(m-2)(n-2)}{4} \rceil$ if $n = 4$ and $m \geq 12$, or $n \geq 5$ and $m \geq 6n - 13$.

Since $K_{m,n}$ is a spanning subgraph of $C_m + K_n$, a lower bound of the oreintable genus of $C_m + K_n$ is that of $K_{m,n}$, which is $\lceil \frac{(m-2)(n-2)}{4} \rceil$. The key to determine the orientable genus of $C_m + K_n$ is the construction of an embedding of $C_m + K_n$ on the orientable surface of genus $\lceil \frac{(m-2)(n-2)}{4} \rceil$. We mainly use two methods of adding tubes to construct an embedding of $C_m + K_n$. Our general strategy of constructing an embedding is as follows. First, we construct an embedding of a spanning subgraph of $C_m + K_n$ which contains C_m, a spanning subgraph of K_n, and some edges between C_m and K_n on some orientable surface. Second, we apply the first method of adding tubes described in Section 2 to attach all the rest edges in K_n and some edges between C_m and K_n. Third, we apply the second method of adding tubes described in Section 2 to attach all the rest edges between C_m and K_n.

The remainder of the section is contributed for some terms. The other undefined terms can be found in [1, 9], or [14].

A *surface* is a compact connected 2-dimensional manifold without boundary. The orientable surface S_g $(g \geq 0)$ can be obtained from a sphere with g handles attached, where g is called the *genus* of S_g. A graph G is able to embed in a surface S if it can be drawn in the surface such that any edge does not pass through any vertex and any two edges do not cross each other. The *orientable genus* of a connected graph G, denoted by $\gamma(G)$, is the smallest nonnegative integer g such that G can be embedded in the orientable surface S_g.

An embedding Π of a connected graph in a surface S is called 2*-cell embedding* if any connected component of $S - \Pi$, called a face, is homeomorphic to an open disc. In a 2-cell embedding of a connected graph G, the boundary of a face in Π is a closed walk of G, which is called the *facial walk*. If a facial walk is a cycle, then it is called a *facial cycle*. Let v be a vertex of a graph G embedded on a surface. A local rotation π_v at the vertex v is a cyclic permutation of the edges incident with v. Suppose that v is incident with edges $vu_1, vu_2, \ldots, vu_n$ in this order. Then π_v can be written by $u_1, u_2, \ldots, u_n$. Furthermore, if $i_1, i_2, \ldots, i_k$ are k continuous numbers in $\{1, 2, \ldots, n\}$, where $2 \leq k \leq n$, then we call $u_{i_1}, u_{i_2}, \ldots, u_{i_k}$ a segment of the local rotation at v.

A graph H is a *supergraph* of G if G is a subgraph of H. If a cycle with n (≥ 3) vertices $v_1, v_2, \ldots, v_n$ in this order, then it is written by $v_1 v_2 \ldots v_n v_1$ and it is always oriented by this order.

2 Two methods of constructing embeddings

Let D_1 and D_2 be two facial cycles of a 2-cell embedding on a surface S such that the orientation of D_1 is the reverse of that of D_2. By adding a tube T to the surface S between D_1 and D_2, we mean that we cut two holes Δ_1 and Δ_2 in S such that Δ_i is in the interior of D_i and orient the boundary of Δ_i as that of D_i, then the tube T welds Δ_1 with Δ_2 in such a way that the rim of one of the ends of T coincides with the boundary of Δ_1 and the rim of the other end of T coincides with the boundary of Δ_2.

Lemma 2.1. *Suppose that G is a graph which has a vertex subset*

$$\{w_0, z_1, z_2, \ldots, z_t\} \cup \{x_i \mid i = 1, 2, \ldots, 2t\} \cup \{y_j \mid j = 1, 2, \ldots, 4t\},$$

where $z_1, z_2, \ldots, z_t$ need not be different, and suppose that G contains no edges in the set

$$\begin{aligned} E' = &\{w_0x_i \mid i = 1, 2, \ldots, 2t\} \cup \{x_iy_j \mid i = 1, 2, \ldots, 2t; j = 1, 2, \ldots, 4t\} \\ &\cup (\{x_ix_{i+1}, \ldots, x_ix_{2t} \mid i = 1, 2, \ldots, 2t-1\} \setminus \{x_{2i-1}x_{2i} \mid i = 1, 2, \ldots, t\}). \end{aligned}$$

Suppose that Π is a 2-cell embedding of G on the orientable surface S_g with the following properties:

- *(i) For $i = 1, 2, \ldots, t$, $R_{0,i} = w_0y_{4i-3}y_{4i-2}w_0$ and $R'_{0,i} = w_0y_{4i-1}y_{4i}w_0$ are facial cycles of Π.*
- *(ii) For $i = 1, 2, \ldots, t$, $Q_{0,i} = z_ix_{2i-1}x_{2i}z_i$ is a facial cycle of Π such that $Q_{0,i}$ has not any common vertex with each of $R_{0,1}, \ldots, R_{0,t}, R'_{0,1}, \ldots, R'_{0,t}$.*

Then there is a supergraph H of G satisfying the following conditions:

- *(i) E' is an edge subset of $E(H)$.*
- *(ii) H has an embedding on the orientable surface of genus $g + 2t^2$ such that it has a set of t facial 3-cycles $\{Q_{t,i} \mid Q_{t,i} = y_{l_i}x_{2i-1}x_{2i}y_{l_i}, i = 1, 2, \ldots, t\}$, where y_{l_i} is some vertex in $\{y_{4i-3}, y_{4i-2}, y_{4i-1}, y_{4i} \mid i = 1, 2, \ldots, t\}$.*

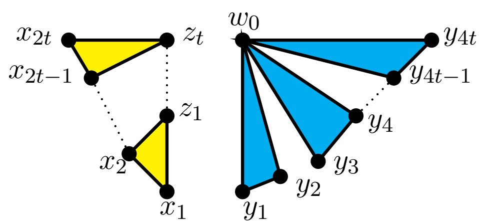

Figure 1: A local structure in Π.

Remark 2.2.

(1) A local structure of Π is shown in Figure 1.

(2) An application of Lemma 2.1 to the construction of an embedding of $C_m + K_n$ is as follows. After an embedding of a spanning subgraph of $C_m + K_n$ on some orientable surface has been constructed, all the rest edges of K_n and some edges between C_m and K_n can be attached by applying Lemma 2.1.

Proof. We shall construct an embedding on the surface of genus $g + 2t^2$ from the embedding Π by applying the operation of adding tubes t times. Every time $2t$ tubes are added to the present surface.

For $i = 1, 2, \ldots, t$, the tube $T_{0,i}$ is added between $Q_{0,i}$ and $R_{0,i}$. Next, the five edges $w_0x_{2i}, x_{2i-1}y_{4i-3}, x_{2i-1}y_{4i-2}, x_{2i}y_{4i-3}$ and $x_{2i}y_{4i-2}$ are drawn on $T_{0,i}$ in the way shown in (1) of Figure 2. For $i = 1, 2, \ldots, t$, let $Q'_{0,i} = y_{4i-2}x_{2i-1}x_{2i}y_{4i-2}$.

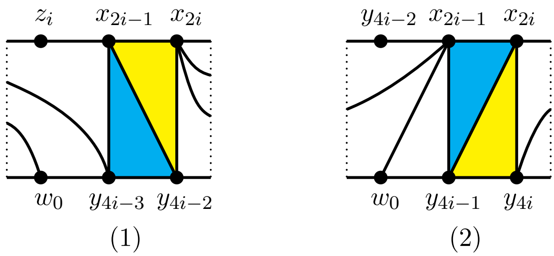

Figure 2: Two drawings of five edges on a tube.

For $i = 1, 2, \ldots, t$, the tube $T'_{0,i}$ is added between $Q'_{0,i}$ and $R'_{0,i}$. Next, the five edges w_0x_{2i-1}, $x_{2i-1}y_{4i-1}$, $x_{2i-1}y_{4i}$, $x_{2i}y_{4i-1}$ and $x_{2i}y_{4i}$ are drawn on $T'_{0,i}$ in the way shown in (2) of Figure 2.

Need to say that the rectangle represents a tube and that the two dot curves are identified with each other in Figure 2. In the rest of the paper we always use a rectangle to represent a tube and the two dot curves in the rectangle are always identified with each other.

For the convenience of argument, the way of drawing edges shown in (i) of Figure 2 is called the *drawing of Type-i* for $i = 1, 2$. To help the readers to understand how those $2t$ tubes are added and how five edges are drawn on each tube, we give an example that $t = 5$ which is shown in Figure 3. The diagrams in Figure 3 are partitioned into four columns from left to right. The three rectangles in the first column respectively represent $T_{0,1}, T_{0,2}$ and $T_{0,3}$ from top to bottom, and the two rectangles in the third column respectively represent $T_{0,4}$ and $T_{0,5}$ from top to bottom. Similarly, the three rectangles in the second column respectively represent $T'_{0,1}, T'_{0,2}$ and $T'_{0,3}$, and the two rectangles in the fourth column respectively represent $T'_{0,4}$ and $T'_{0,5}$.

After those $2t$ tubes have been added, there are three sets of facial 3-cycles which are

$$\begin{aligned}
\mathcal{X}_1 &= \{Q_{1,i} \mid Q_{1,i} = y_{4i-1}x_{2i-1}x_{2i}y_{4i-1}, i = 1, 2, \ldots, t\},\\
\mathcal{Y}_1 &= \{R_{1,i} \mid R_{1,i} = x_{2i-1}y_{4i-3}y_{4i-2}x_{2i-1}, i = 1, 2, \ldots, t\}, \text{ and}\\
\mathcal{Y}'_1 &= \{R'_{1,i} \mid R'_{1,i} = x_{2i}y_{4i-1}y_{4i}x_{2i}, i = 1, 2, \ldots, t\}.
\end{aligned}$$

For the convenience of argument, we now define t permutations. For $k = 0, 1, \ldots, t-1$, we define the permutation τ_k on the set $\{1, 2, \ldots, t\}$ as follows. For $i = 1, 2, \ldots, t$,

$$\tau_k(i) \equiv i + (-1)^{k+1}k \pmod{t},$$

where $0 \le i + (-1)^{k+1}k \le t - 1$.

Obviously, τ_0 is the identity mapping on $\{1, 2, \ldots, t\}$. For $0 \le k \le t - 1$, we define

$$\tau'_k(i) \equiv \begin{cases} \tau_k(i) \pmod{t}, & \text{if } k = 0,\\ \tau_0\tau_1 \cdots \tau_k(i) \pmod{t}, & \text{if } 1 \le k \le t-1, \end{cases}$$

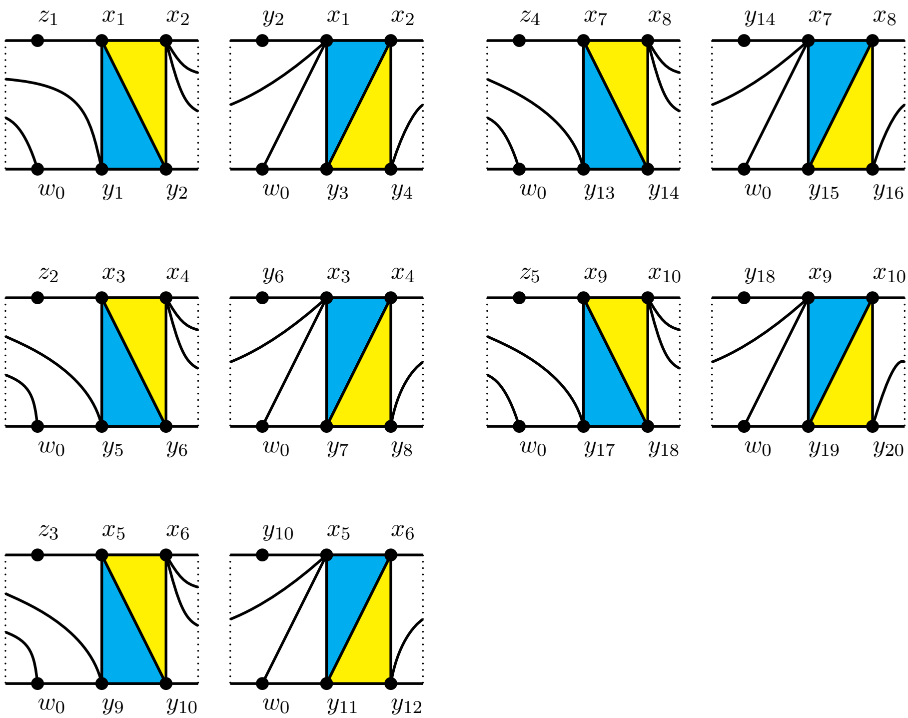

Figure 3: The first operation of adding $2t$ tubes when $t = 5$.

where $0 \le \tau'_k(i) \le t-1$ and $\tau_0\tau_1\cdots\tau_k$ is the product of $\tau_0, \tau_1, \dots, \tau_k$ in this order. For example, $\tau_0\tau_1(1) = \tau_1(\tau_0(1)) = 2$.

Thus, $Q_{1,i}$, $R_{1,i}$ and $R'_{1,i}$ can be alternately expressed as follows:

$$\begin{aligned} Q_{1,i} &= y_{4\tau'_0(i)-1}x_{2i-1}x_{2i}y_{4\tau'_0(i)-1}, \\ R_{1,i} &= x_{2i-1}y_{4\tau'_0(i)-3}y_{4\tau'_0(i)-2}x_{2i-1}, \text{ and} \\ R'_{1,i} &= x_{2i}y_{4\tau'_0(i)-1}y_{4\tau'_0(i)}x_{2i}. \end{aligned}$$

We continue to add tubes, and consider two cases.

Case 1: $t \equiv 1 \pmod 2$. In this case we firstly add t tubes $T_{1,1}, \dots, T_{1,t}$ to the present surface such that $T_{1,i}$ is between $Q_{1,i}$ and $R_{1,\tau_1(i)}$. Note that

$$\begin{aligned} R_{1,\tau_1(i)} &= x_{2\tau_1(i)-1}y_{4\tau_0\tau_1(i)-3}y_{4\tau_0\tau_1(i)-2}x_{2\tau_1(i)-1}, \text{ i.e.,} \\ R_{1,\tau_1(i)} &= x_{2\tau_1(i)-1}y_{4\tau'_1(i)-3}y_{4\tau'_1(i)-2}x_{2\tau_1(i)-1}. \end{aligned}$$

For $i = 1, 2, \dots, t$, the five edges $x_{2i-1}y_{4\tau'_1(i)-3}$, $x_{2i-1}y_{4\tau'_1(i)-2}$, $x_{2i}y_{4\tau'_1(i)-3}$, $x_{2i}y_{4\tau'_1(i)-2}$ and $x_{2i}x_{2\tau_1(i)-1}$ are drawn on $T_{1,i}$ in the way of the drawing of Type-1. Thus, there is a set $\mathcal{X}'_1$ of t facial 3-cycles, where

$$\mathcal{X}'_1 = \{Q'_{1,i} \mid Q'_{1,i} = y_{4\tau'_1(i)-2}x_{2i-1}x_{2i}y_{4\tau'_1(i)-2}, i = 1, 2, \dots, t\}.$$

Next, the t tubes $T'_{1,1}, \dots, T'_{1,t}$ are added to the present surface such that $T'_{1,i}$ is between $Q'_{1,i}$ and $R'_{1,\tau_1(i)}$. Then the five edges $x_{2i-1}y_{4\tau'_1(i)-1}$, $x_{2i-1}y_{4\tau'_1(i)}$, $x_{2i}y_{4\tau'_1(i)-1}$, $x_{2i}y_{4\tau'_1(i)}$

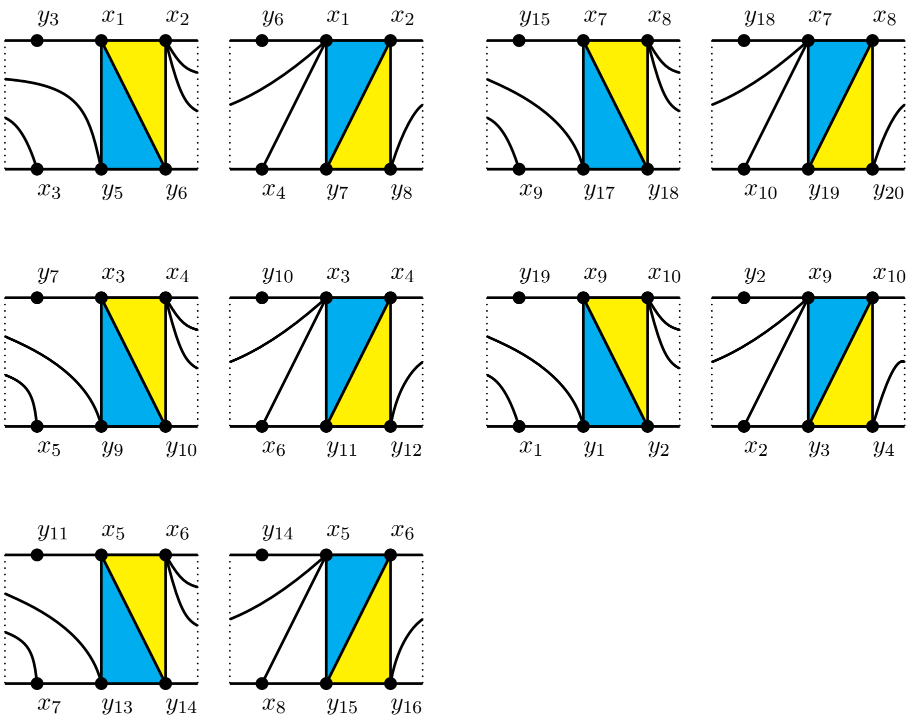

Figure 4: The second operation of adding $2t$ tubes when $t = 5$.

and $x_{2i}x_{2\tau_1(i)}$ are drawn on $T'_{1,i}$ in the way of the drawing of Type-2. For example, if $t = 5$, the above operation of adding $2t$ tubes is shown in Figure 4. The order of diagrams in Figure 4 is as that in Figure 3.

After those $2t$ tubes have been added, there are three sets $\mathcal{X}_2$, $\mathcal{Y}_2$, and $\mathcal{Y}'_2$ of facial 3-cycles which are

$$\begin{aligned}
\mathcal{X}_2 &= \{Q_{2,i} \mid Q_{2,i} = y_{4\tau'_1(i)-1}x_{2i-1}x_{2i}y_{4\tau'_1(i)-1}, i = 1, 2, \ldots, t\},\\
\mathcal{Y}_2 &= \{R_{2,i} \mid R_{2,i} = x_{2i-1}y_{4\tau'_1(i)-3}y_{4\tau'_1(i)-2}x_{2i-1}, i = 1, 2, \ldots, t\}, \text{ and}\\
\mathcal{Y}'_2 &= \{R'_{2,i} \mid R'_{2,i} = x_{2i}y_{4\tau'_1(i)-1}y_{4\tau'_1(i)}x_{2i}, i = 1, 2, \ldots, t\}.
\end{aligned}$$

In general, if the s-th operation ($s \geq 1$) of adding $2t$ tubes has been applied, then there are three sets of facial 3-cycles, i.e.,

$$\mathcal{X}_s = \{Q_{s,i} \mid i = 1, 2, \ldots, t\}, \qquad \mathcal{Y}_s = \{R_{s,i} \mid i = 1, 2, \ldots, t\}, \text{ and}$$
$$\mathcal{Y}'_s = \{R'_{s,i} \mid i = 1, 2, \ldots, t\}.$$

Next, we apply the $(s+1)$-th of adding $2t$ tubes $T_{s,1}, \ldots, T_{s,t}, T'_{s,1}, \ldots, T'_{s,t}$ to the present surface satisfying the following conditions.

(1) If $1 \leq s \leq \frac{t-1}{2}$, then the tube $T_{s,i}$ is added between $Q_{s,i}$ and $R_{s,\tau_s(i)}$, where $i = 1, 2, \ldots, t$. In this case $R_{s,\tau_s(i)} = x_{2\tau_s(i)-1}y_{4\tau'_s(i)-3}y_{4\tau'_s(i)-2}x_{2\tau_s(i)-1}$. Next,

the five edges

$$x_{2i-1}y_{4\tau_s'(i)-3}, \quad x_{2i-1}y_{4\tau_s'(i)-2}, \quad x_{2i}y_{4\tau_s'(i)-3}, \quad x_{2i}y_{4\tau_s'(i)-2}, \quad \text{and} \quad x_{2i}x_{2\tau_s(i)-1}$$

are drawn on $T_{s,i}$ in the way of the drawing of Type-1. After those t tubes have been added, there is a set $\mathcal{X}_s'$ of t facial 3-cycles, where

$$\mathcal{X}_s' = \{Q_{s,i}' \mid Q_{s,i}' = y_{4\tau_s'(i)-2}x_{2i-1}x_{2i}y_{4\tau_s'(i)-2},\ i = 1, 2, \ldots, t\}.$$

For $i = 1, 2, \ldots, t$, the tube $T_{s,i}'$ is added between $Q_{s,i}'$ and $R_{s,\tau_s(i)}'$. Note that $R_{s,\tau_s(i)}' = x_{2\tau_s(i)}y_{4\tau_s'(i)-1}y_{4\tau_s'(i)}x_{2\tau_s(i)}$. Next, the five edges

$$x_{2i-1}y_{4\tau_s'(i)-1}, \quad x_{2i-1}y_{4\tau_s'(i)}, \quad x_{2i}y_{4\tau_s'(i)-1}, \quad x_{2i}y_{4\tau_s'(i)}, \quad \text{and} \quad x_{2i-1}x_{2\tau_s(i)}$$

are drawn on $T_{s,i}'$ in the way of the drawing of Type-2.

After the $(s+1)$-th operation of adding $2t$ tubes has been applied, there are three sets $\mathcal{X}_{s+1}$, $\mathcal{Y}_{s+1}$, and $\mathcal{Y}_{s+1}'$ of facial 3-cycles which are

$$\begin{aligned}
\mathcal{X}_{s+1} &= \{Q_{s+1,i} \mid Q_{s+1,i} = y_{4\tau_s'(i)-1}x_{2i-1}x_{2i}y_{4\tau_s'(i)-1},\ i = 1, 2, \ldots, t\},\\
\mathcal{Y}_{s+1} &= \{R_{s+1,i} \mid R_{s+1,i} = x_{2i-1}y_{4\tau_s'(i)-3}y_{4\tau_s'(i)-2}x_{2i-1},\ i = 1, 2, \ldots, t\}, \text{ and}\\
\mathcal{Y}_{s+1}' &= \{R_{s+1,i}' \mid R_{s+1,i}' = x_{2i}y_{4\tau_s'(i)-1}y_{4\tau_s'(i)}x_{2i},\ i = 1, 2, \ldots, t\}.
\end{aligned}$$

(2) If $\frac{t+1}{2} \le s \le t-1$, suppose that k and k' are the maximum even and odd numbers which are not more than $\frac{t-1}{2}$, respectively. There are two cases to consider.

If $s = \frac{t+1}{2}, \frac{t+1}{2} + 2, \ldots, \frac{t+1}{2} + k$, then the tube $T_{s,i}$ is added between $Q_{s,i}$ and $R_{s,\tau_s(i)}'$. Next, the five edges

$$x_{2i-1}y_{4\tau_s'(i)-1}, \quad x_{2i-1}y_{4\tau_s'(i)}, \quad x_{2i}y_{4\tau_s'(i)-1}, \quad x_{2i}y_{4\tau_s'(i)}, \quad \text{and} \quad x_{2i}x_{2\tau_s(i)}$$

are drawn on $T_{s,i}$ in the way of the drawing of Type-1. After those t tubes have been added, there is a set $\mathcal{X}_s'$ of t facial 3-cycles, where

$$\mathcal{X}_s' = \{Q_{s,i}' \mid Q_{s,i}' = y_{4\tau_s'(i)}x_{2i-1}x_{2i}y_{4\tau_s'(i)},\ i = 1, 2, \ldots, t\}.$$

For $i = 1, 2, \ldots, t$, the tube $T_{s,i}'$ is added between $Q_{s,i}'$ and $R_{s,\tau_s(i)}$. Then the five edges

$$x_{2i-1}y_{4\tau_s'(i)-3}, \quad x_{2i-1}y_{4\tau_s'(i)-2}, \quad x_{2i}y_{4\tau_s'(i)-3}, \quad x_{2i}y_{4\tau_s'(i)-2}, \quad \text{and} \quad x_{2i-1}x_{2\tau_s(i)-1}$$

are drawn on $T_{s,i}'$ in the way of the drawing of Type-2. In this case there are three sets $\mathcal{X}_{s+1}$, $\mathcal{Y}_{s+1}$, and $\mathcal{Y}_{s+1}'$ of facial 3-cycles which are

$$\begin{aligned}
\mathcal{X}_{s+1} &= \{Q_{s+1,i} \mid Q_{s+1,i} = y_{4\tau_s'(i)-3}x_{2i-1}x_{2i}y_{4\tau_s'(i)-3},\ i = 1, 2, \ldots, t\},\\
\mathcal{Y}_{s+1} &= \{R_{s+1,i} \mid R_{s+1,i} = x_{2i}y_{4\tau_s'(i)-3}y_{4\tau_s'(i)-2}x_{2i},\ i = 1, 2, \ldots, t\}, \text{ and}\\
\mathcal{Y}_{s+1}' &= \{R_{s+1,i}' \mid R_{s+1,i} = x_{2i-1}y_{4\tau_s'(i)-1}y_{4\tau_s'(i)}x_{2i-1},\ i = 1, 2, \ldots, t\}.
\end{aligned}$$

If $s = \frac{t+1}{2} + 1, \frac{t+1}{2} + 3, \ldots, \frac{t+1}{2} + k'$, then the tube $T_{s,i}$ is added between $Q_{s,i}$ and $R_{s,\tau_s(i)}$. Next, the five edges

$$x_{2i-1}y_{4\tau_s'(i)-3}, \quad x_{2i-1}y_{4\tau_s'(i)-2}, \quad x_{2i}y_{4\tau_s'(i)-3}, \quad x_{2i}y_{4\tau_s'(i)-2}, \quad \text{and} \quad x_{2i}x_{2\tau_s(i)}$$

are drawn on $T_{s,i}$ in the way of the drawing of Type-1. After those t tubes have been added, there is a set $\mathcal{X}_s'$ of t facial 3-cycles, where $\mathcal{X}_s'$ is the same as in (1). For $i = 1, 2, \ldots, t$, the tube $T_{s,i}'$ is added between $Q_{s,i}'$ and $R_{s,\tau_s(i)}'$. Then the five edges

$$x_{2i-1}y_{4\tau_s'(i)-1}, \quad x_{2i-1}y_{4\tau_s'(i)}, \quad x_{2i}y_{4\tau_s'(i)-1}, \quad x_{2i}y_{4\tau_s'(i)}, \quad \text{and} \quad x_{2i-1}x_{2\tau_s(i)-1}$$

are drawn on $T_{s,i}'$ in the way of the drawing of Type-2. In this case there are three sets $\mathcal{X}_{s+1}$, $\mathcal{Y}_{s+1}$, and $\mathcal{Y}_{s+1}'$ of facial 3-cycles which are the same as in (1), respectively.

Need to say that x_{2i} and x_{2i-1} are connected with $x_{2\tau_s(i)}$ and $x_{2\tau_s(i)-1}$ in (2), respectively. However, x_{2i} and x_{2i-1} are connected with $x_{2\tau_s(i)-1}$ and $x_{2\tau_s(i)}$ in (1), respectively.

The above operation of adding $2t$ tubes is not stopped until the t-th operation of adding $2t$ tubes has been applied. Let Π' be the obtained embedding. Then Π' has a set $\mathcal{X}_t$ of t facial 3-cycles, where

$$\begin{aligned}\mathcal{X}_t = \{Q_{t,i} \mid Q_{t,i} &= y_{4\tau_t'(i)-3}x_{2i-1}x_{2i}y_{4\tau_t'(i)-3}, \text{ if } t = \tfrac{t+1}{2} + k, \text{ or} \\ Q_{t,i} &= y_{4\tau_t'(i)-1}x_{2i-1}x_{2i}y_{4\tau_t'(i)-1}, \text{ if } t = \tfrac{t+1}{2} + k'\}.\end{aligned}$$

Since there are $2t \times t$ $(= 2t^2)$ tubes being used all together, Π' is an embedding on the orientable surface of genus $g + 2t^2$.

Let H be the graph corresponding to Π'. We need to show that H satisfies the demands of the theorem. Before the proof, we give an example that $t = 5$ to illustrate how all 50 tubes are added and how all desired edges are attached. The former two operations of adding 10 tubes are shown in Figure 3 and Figure 4, respectively. The latter three operations of adding 10 tubes are shown in Figure 5. Need to say that the five rectangles in the first column upon (3) respectively represent $T_{2,1}, \ldots, T_{2,5}$, and the five rectangles in the second column upon (3) respectively represent $T_{2,1}', \ldots, T_{2,5}'$ in Figure 5. Similarly, the first column upon (4) respectively represent $T_{3,1}, \ldots, T_{3,5}$, and the second column upon (4) respectively represent $T_{3,1}', \ldots, T_{3,5}'$ in Figure 5. The order in (5) in Figure 5 is the same as that in Figure 3.

We now show that H satisfies all demands of the theorem.

Claim 2.3. *w_0 is connected with each of $x_1, x_2, \ldots, x_{2t}$.*

According to the first operation of adding $2t$ tubes, Claim 2.3 is obvious.

Claim 2.4. *For $i = 1, 2, \ldots, 2t$ and $j = 1, 2, \ldots, 4t$, x_i is connected with y_j in H.*

For $i = 1, 2, \ldots, 2t$, each of x_{2i-1} and x_{2i} is connected with $y_{4\tau_s'(i)-3}$, $y_{4\tau_s'(i)-2}$, $y_{4\tau_s'(i)-1}$, and $y_{4\tau_s'(i)}$ after the $(s+1)$-th operation of adding $2t$-tubes has been applied, where $1 \le s \le t-1$. Considering that any two of $y_{4\tau_s'(i)-3}$, $y_{4\tau_s'(i)-2}$, $y_{4\tau_s'(i)-1}$, and $y_{4\tau_s'(i)}$ are distinct, it is sufficient to show the following proposition.

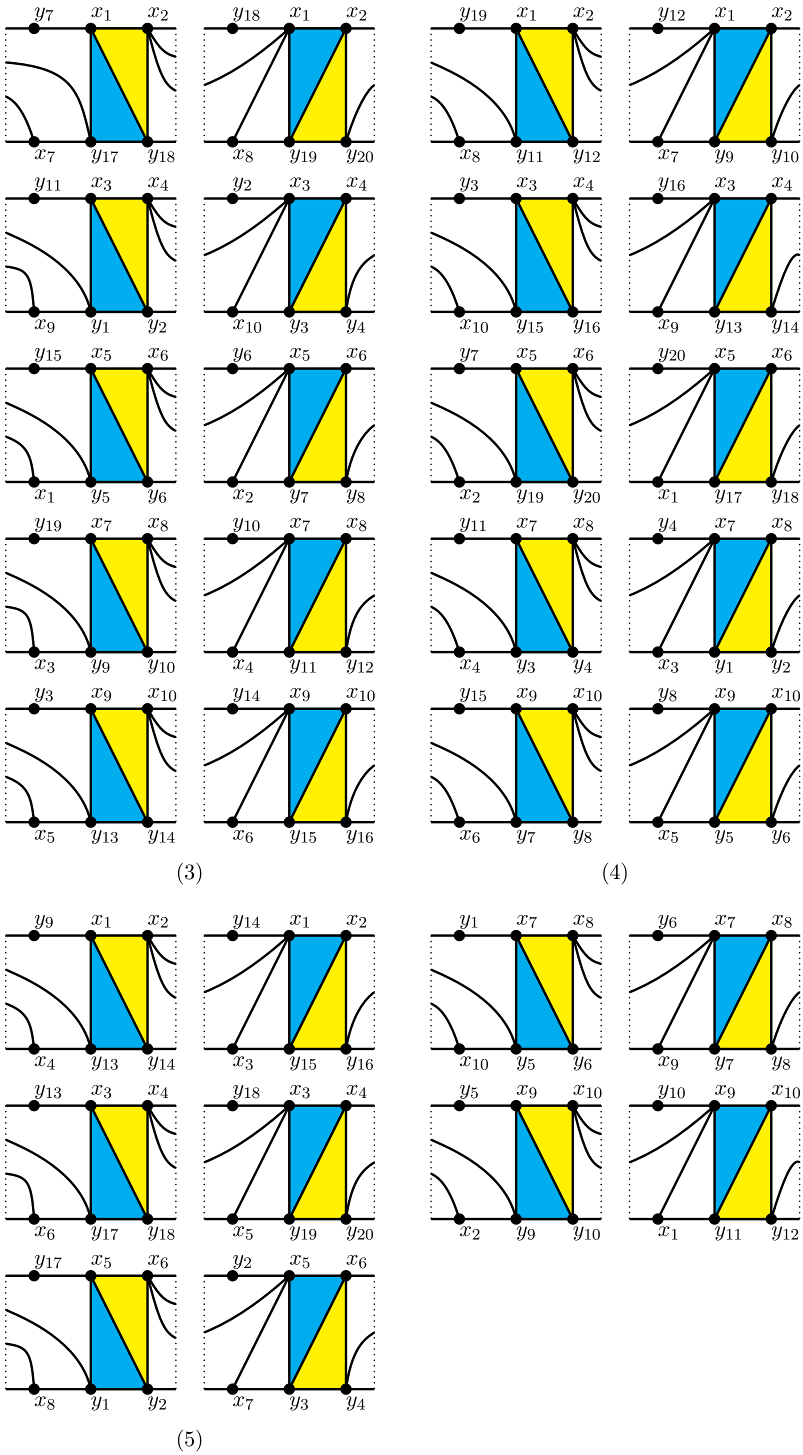

Figure 5: The latter three operations of adding $2t$ tubes when $t = 5$.

Proposition 2.5. *For $i = 1, 2, \ldots, t$, $\tau'_{s_1}(i) \neq \tau'_{s_2}(i)$ if $1 \leq s_1, s_2 \leq t-1$ and $s_1 \neq s_2$.*

Assume for the sake of contradiction that there are two distinct number s_1 and s_2 such that $\tau'_{s_1}(i) = \tau'_{s_2}(i)$ for some i. Without loss of generality, suppose that $s_1 > s_2$. Since $\tau'_s(i) \equiv \tau_0\tau_1 \cdots \tau_s(i) \pmod t$ and $\tau_j(i) \equiv i + (-1)^{j+1}j \pmod t$, we have that

$$\tau'_{s_1}(i) \equiv i + \sum\nolimits_{k=0}^{s_1}(-1)^{k+1}k \equiv \tau'_{s_2}(i) \equiv i + \sum\nolimits_{l=0}^{s_2}(-1)^{l+1}l (\mathrm{mod\, t}).$$

Hence

$$\sum\nolimits_{k=0}^{s_1}(-1)^{k+1}k \equiv \sum\nolimits_{l=0}^{s_2}(-1)^{l+1}l \pmod t.$$

Thus,

$$\sum\nolimits_{k=s_2+1}^{s_1}(-1)^{k+1}k \equiv 0 \pmod t.$$

Since $1 \leq s_1 \leq t-1$, we have that

$$\sum\nolimits_{k=s_2+1}^{s_1}(-1)^{k+1}k \not\equiv 0 \pmod t.$$

Then there is a contradiction. Thus, the proposition is verified.

Claim 2.6. *H contains the edge set*

$$\{x_ix_{i+1}, \ldots, x_ix_{2t} \mid i = 1, 2, \ldots, 2t-1\} \setminus \{x_{2i-1}x_{2i} \mid i = 1, 2, \ldots, t\}.$$

In fact, there are $2t$ edges being added such that each has the form x_kx_j $(k \neq j)$ except for the form $x_{2i-1}x_{2i}$ after the $(s+1)$-th operation of adding $2t$ tubes has been applied, where $1 \leq s \leq t-1$. So there are $2t(t-1)$ edges of the form x_ix_j being added after the t-th operation of adding tubes has been applied. We now show that any two edges in those $2t(t-1)$ edges are different. We need the following proposition.

Proposition 2.7. *Suppose that s_1 and s_2 are two distinct integers such that $1 \leq s_1, s_2 \leq t-1$. If $s_1 + s_2 \equiv 0 \pmod t$, then $\tau_{s_1}(i) = \tau_{s_2}(i)$.*

In fact,

$$\tau_{s_1}(i) \equiv i + (-1)^{s_1+1}s_1 \equiv i + (-1)^{t-s_2+1}(t-s_2) \equiv i + (-1)^{t-s_2}s_2 \pmod t.$$

Since $t \equiv 1 \pmod 2$, $(-1)^{t-s_2} = (-1)^{s_2+1}$. So $\tau_{s_1}(i) \equiv i + (-1)^{s_2+1}s_2 \pmod t$. In other words, $\tau_{s_1}(i) = \tau_{s_2}(i)$.

According to the rule of the $(s+1)$-th operation of adding $2t$ tubes, x_{2i} and x_{2i-1} are respectively connected with $x_{2\tau_s(i)-1}$ and $x_{2\tau_s(i)}$ if $1 \leq s \leq \frac{t-1}{2}$, and x_{2i} and x_{2i-1} are respectively connected with $x_{2\tau_s(i)}$ and $x_{2\tau_s(i)-1}$ if $\frac{t+1}{2} \leq s \leq t-1$. By Proposition 2.7, the pair of vertices connected with the pair of x_{2i-1} and x_{2i} in the s_2-th operation of adding $2t$ tubes is the same as the pair connected with the pair of x_{2i-1} and x_{2i} in the s_1-th operation of adding $2t$ tubes if $s_1 + s_2 \equiv 0 \pmod t$ and $1 \leq s_1, s_2 \leq t-1$. But the methods of two connections are different.

We now view the pair of x_{2i-1} and x_{2i} as a vertex u_i, where $i \in \{1, 2, \ldots, t\}$. In order to show Claim 2.6, it is sufficient to show that u_p is connected with u_q, where $p, q \in \{1, 2, \ldots, t\}$ and $p \neq q$. For the purpose, it is sufficient to show that there exists some k such that $\tau_k(p) = q$ or $\tau_k(q) = p$. By Proposition 2.7, it is sufficient to show that for any two distinct number $i, j \in \{1, 2, \ldots, t\}$, there exists some $k \in \{1, 2, \ldots, \frac{t-1}{2}\}$ such that $\tau_k(i) \equiv j \pmod t$ or $\tau_k(j) \equiv i \pmod t$.

Without loss of generality, suppose that $j > i$. If $j - i \equiv 1 \pmod 2$, there are two cases to consider. If $j - i \leq \frac{t-1}{2}$, let $k = j - i$. Then

$$\tau_k(i) \equiv i + (-1)^{k+1}k \equiv i + (j-i) \equiv j \pmod t.$$

So $\tau_k(i) = j$. If $j - i > \frac{t+1}{2}$, let $k = t - (j - i)$. Then

$$\tau_k(i) \equiv i + (-1)^{k+1}k \equiv i - t + j - i \equiv j \pmod t.$$

So $\tau_k(i) = j$. If $j - i \equiv 0 \pmod 2$, there are two cases to consider. If $j - i \leq \frac{t-1}{2}$, let $k = j - i$. Then

$$\tau_k(j) \equiv j + (-1)^{k+1}k \equiv j - (j-i) \equiv i \pmod t.$$

Thus, $\tau_k(j) = i$. If $j - i > \frac{t+1}{2}$, let $k = t - (j - i)$. Then

$$\tau_k(j) \equiv j + (-1)^{k+1}k \equiv j + t - j + i \equiv i \pmod t.$$

So $\tau_k(j) = i$.

Therefore, u_p is connected with u_q, where $p \neq q$. Thus, Claim 2.6 has been proved.

Case 2: $t \equiv 0 \pmod 2$. We proceed the similar argument to that in Case 1. Let $\mathcal{X}_s$, $\mathcal{Y}_s$, and $\mathcal{Y}'_s$ be the sets of facial 3-cycles defined in Case 1. When the $(s+1)$-th operation of adding $2t$ tubes $T_{s,1}, \ldots, T_{s,t}, T'_{s,1}, \ldots, T'_{s,t}$ will be applied, it satisfies the following conditions.

(1) If $1 \leq s \leq \frac{t}{2} - 1$, then the ways of adding $2t$ tubes and drawing the five edges are similar to that in (1) of Case 1.

(2) If $s = \frac{t}{2}$, we consider two cases. If $1 \leq i \leq \frac{t}{2}$, then the tube $T_{\frac{t}{2},i}$ is added between $Q_{\frac{t}{2},i}$ and $R_{\frac{t}{2},\tau_{\frac{t}{2}}(i)}$, and the five edges

$$x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)-3}, \qquad x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)-2}, \qquad x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-3},$$
$$x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-2}, \quad \text{and} \quad x_{2i-1}x_{2\tau_{\frac{t}{2}}(i)-1}$$

are drawn on $T_{\frac{t}{2},i}$ in the way of the drawing of Type-1.

If $\frac{t}{2} + 1 \leq i \leq t$, then the tube $T_{\frac{t}{2},i}$ is added between $Q_{\frac{t}{2},i}$ and $R'_{\frac{t}{2},\tau_{\frac{t}{2}}(i)}$, and the five edges

$$x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)-1}, \qquad x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)}, \qquad x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-1},$$
$$x_{2i}y_{4\tau'_{\frac{t}{2}}(i)}, \quad \text{and} \quad x_{2i}x_{2\tau_{\frac{t}{2}}(i)}$$

are drawn on $T_{\frac{t}{2},i}$ in the way of the drawing of Type-1.

After those t tubes have been added, there is a set $\mathcal{X}'_{\frac{t}{2}}$ of t facial 3-cycles, where

$$\begin{aligned}\mathcal{X}'_{\frac{t}{2}} = \{Q'_{\frac{t}{2},i} \mid Q'_{\frac{t}{2},i} &= y_{4\tau'_{\frac{t}{2}}(i)-2}x_{2i-1}x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-2}, \text{ if } i = 1, 2, \ldots, \tfrac{t}{2}, \text{ or} \\ Q'_{\frac{t}{2},i} &= y_{4\tau'_{\frac{t}{2}}(i)}x_{2i-1}x_{2i}y_{4\tau'_{\frac{t}{2}}(i)}, \text{ if } i = \tfrac{t}{2}+1, \tfrac{t}{2}+2, \ldots, t-1\}.\end{aligned}$$

Next, if $1 \leq i \leq \frac{t}{2}$, then the tube $T'_{\frac{t}{2},i}$ is added between $Q'_{\frac{t}{2},i}$ and $R'_{\frac{t}{2},\tau_{\frac{t}{2}}(i)}$, and the five edges

$$x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)-1}, \qquad x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)}, \qquad x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-1},$$
$$x_{2i}y_{4\tau'_{\frac{t}{2}}(i)}, \quad \text{and} \quad x_{2i-1}x_{2\tau_{\frac{t}{2}}(i)}$$

are drawn on $T'_{\frac{t}{2},i}$ in the way of the drawing of Type-2. If $\frac{t}{2}+1 \leq i \leq t$, then the tube $T'_{\frac{t}{2},i}$ is added between $Q'_{\frac{t}{2},i}$ and $R_{\frac{t}{2},\tau_{\frac{t}{2}}(i)}$, and the five edges

$$x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)-3}, \qquad x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)-2}, \qquad x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-3},$$
$$x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-2}, \quad \text{and} \quad x_{2i-1}x_{2\tau_{\frac{t}{2}}(i)-1}$$

are drawn on $T'_{\frac{t}{2},i}$ in the way of the drawing of Type-2. There are three sets $\mathcal{X}_{\frac{t}{2}+1}$, $\mathcal{Y}_{\frac{t}{2}+1}$, and $\mathcal{Y}'_{\frac{t}{2}+1}$ of facial 3-cycles, where

$$\begin{aligned}
\mathcal{X}_{\frac{t}{2}+1} = \{Q_{\frac{t}{2}+1,i} \mid Q_{\frac{t}{2}+1,i} &= y_{4\tau'_{\frac{t}{2}}(i)-1}x_{2i-1}x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-1}, \text{ if } i = 1, \dots, \tfrac{t}{2}, \text{ or} \\
Q_{\frac{t}{2}+1,i} &= y_{4\tau'_{\frac{t}{2}}(i)-3}x_{2i-1}x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-3}, \text{ if } i = \tfrac{t}{2}+1, \dots, t\}, \\
\mathcal{Y}_{\frac{t}{2}+1} = \{R_{\frac{t}{2}+1,i} \mid R_{\frac{t}{2}+1,i} &= x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)-3}y_{4\tau'_{\frac{t}{2}}(i)-2}x_{2i-1}, \text{ if } i = 1, \dots, \tfrac{t}{2}, \text{ or} \\
R_{\frac{t}{2}+1,i} &= x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)-1}y_{4\tau'_{\frac{t}{2}}(i)}x_{2i-1} \text{ if } i = \tfrac{t}{2}+1, \dots, t\}, \\
\mathcal{Y}'_{\frac{t}{2}+1} = \{R'_{\frac{t}{2}+1,i} \mid R'_{\frac{t}{2}+1,i} &= x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-1}y_{4\tau'_{\frac{t}{2}}(i)}x_{2i}, \text{ if } i = 1, \dots, \tfrac{t}{2}, \text{ or} \\
R'_{\frac{t}{2}+1,i} &= x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-3}y_{4\tau'_{\frac{t}{2}}(i)-2}x_{2i} \text{ if } i = \tfrac{t}{2}+1, \dots, t\}.
\end{aligned}$$

(3) If $\frac{t}{2}+1 \leq s \leq t-1$, then the tube $T_{s,i}$ is added between $Q_{s,i}$ and $R'_{s,\tau_s(i)}$. Since $R'_{s,\tau_s(i)}$ has two forms, we say that

- $R'_{s,\tau_s(i)}$ is of Class 1 if $R'_{s,\tau_s(i)}$ has the form $x_{2i}y_{4\tau'_s(i)-1}y_{4\tau'_s(i)}x_{2i}$, and
- $R'_{s,\tau_s(i)}$ is of Class 2 if $R'_{s,\tau_s(i)}$ has the form $x_{2i}y_{4\tau'_s(i)-3}y_{4\tau'_s(i)-2}x_{2i}$.

Similarly, we say that

- $R_{s,\tau_s(i)}$ is of Class 1 if $R_{s,\tau_s(i)}$ has the form $x_{2i-1}y_{4\tau'_s(i)-1}y_{4\tau'_s(i)}x_{2i-1}$, and
- $R_{s,\tau_s(i)}$ is of Class 2 if $R_{s,\tau_s(i)}$ has the form $x_{2i-1}y_{4\tau'_s(i)-3}y_{4\tau'_s(i)-2}x_{2i-1}$.

If $R'_{s,\tau_s(i)}$ is of Class 1, then the five edges

$$x_{2i-1}y_{4\tau'_s(i)-1}, \qquad x_{2i-1}y_{4\tau'_s(i)}, \qquad x_{2i}y_{4\tau'_s(i)-1},$$
$$x_{2i}y_{4\tau'_s(i)}, \quad \text{and} \quad x_{2i}x_{2\tau_s(i)}$$

are drawn on $T_{s,i}$ in the way of the drawing of Type-1. If $R'_{s,\tau_s(i)}$ is of Class 2, then the five edges

$$x_{2i-1}y_{4\tau'_s(i)-3}, \qquad x_{2i-1}y_{4\tau'_s(i)-2}, \qquad x_{2i}y_{4\tau'_s(i)-3},$$
$$x_{2i}y_{4\tau'_s(i)-2}, \quad \text{and} \quad x_{2i}x_{2\tau_s(i)}$$

are drawn on $T_{s,i}$ in the way of the drawing of Type-1. Then there is a set $\mathcal{X}'_s$ of t facial cycles, where

$$\mathcal{X}'_s = \{Q'_{s,i} \mid Q_{s,i} = y_{4\tau'_s(i)-2}x_{2i-1}x_{2i}y_{4\tau'_s(i)-2}, \text{ if } R'_{s,\tau_s(i)} \text{ is of Class 1, or } Q'_{s,i} = y_{4\tau'_s(i)}x_{2i-1}x_{2i}y_{4\tau'_s(i)}, \text{ if } R'_{s,\tau_s(i)} \text{ is of Class 2}\}.$$

Next, the tube $T'_{s,i}$ is added between $Q'_{s,i}$ and $R_{s,\tau_s(i)}$. If $R_{s,\tau_s(i)}$ is of Class 1, then the five edges

$$x_{2i-1}y_{4\tau'_s(i)-1}, \quad x_{2i-1}y_{4\tau'_s(i)}, \quad x_{2i}y_{4\tau'_s(i)-1}, \quad x_{2i}y_{4\tau'_s(i)}, \quad \text{and} \quad x_{2i}x_{2\tau_s(i)}$$

are drawn on $T'_{s,i}$ in the way of the drawing of Type-2. If $R_{s,\tau_s(i)}$ is of Class 2, then the five edges

$$x_{2i-1}y_{4\tau'_s(i)-3}, \quad x_{2i-1}y_{4\tau'_s(i)-2}, \quad x_{2i}y_{4\tau'_s(i)-3}, \quad x_{2i}y_{4\tau'_s(i)-2}, \quad \text{and} \quad x_{2i}x_{2\tau_s(i)}$$

are drawn on $T_{s,i}$ in the way of the drawing of Type-2. Then there are three sets $\mathcal{X}_{s+1}$, $\mathcal{Y}_{s+1}$ and $\mathcal{Y}'_{s+1}$ of t facial cycles, where

$$\begin{aligned}
\mathcal{X}_{s+1} = \{Q_{s+1,i} \mid\, & Q_{s+1,i} = y_{4\tau'_s(i)-2}x_{2i-1}x_{2i}y_{4\tau'_s(i)-2}, \text{ if } R'_{s,\tau_s(i)} \text{ is of Class 1,}\\
& \text{or } Q_{s+1,i} = y_{4\tau'_s(i)}x_{2i-1}x_{2i}y_{4\tau'_s(i)}, \text{ if } R'_{s,\tau_s(i)} \text{ is of Class 2}\},\\
\mathcal{Y}_{s+1} = \{R_{s+1,i} \mid\, & R_{s+1,i} = x_{2i-1}y_{4\tau'_s(i)-3}y_{4\tau'_s(i)-2}x_{2i-1}, \text{ if } R_{s,\tau_s(i)} \text{ is of Class 1,}\\
& \text{or } R_{s+1,i} = x_{2i-1}y_{4\tau'_s(i)-1}y_{4\tau'_s(i)}x_{2i-1}, \text{ if } R_{s,\tau_s(i)} \text{ is of Class 2}\},\\
\mathcal{Y}'_{s+1} = \{R'_{s+1,i} \mid\, & R'_{s+1,i} = x_{2i}y_{4\tau'_s(i)-3}y_{4\tau'_s(i)-2}x_{2i}, \text{ if } R'_{s,\tau_s(i)} \text{ is of Class 1,}\\
& \text{or } R'_{s+1,i} = x_{2i}y_{4\tau'_s(i)-1}y_{4\tau'_s(i)}x_{2i}, \text{ if } R'_{s,\tau_s(i)} \text{ is of Class 2}\}.
\end{aligned}$$

The above operation of adding $2t$ tubes is not stopped until the t-th operation of adding $2t$ tubes has been applied. Let Π' be the obtained embedding and let H the graph corresponding to Π'. Clearly, Π' is an embedding on the orientable surface of genus $g + 2t^2$, and Π' has a set $\mathcal{X}_t$ of t facial 3-cycles in which each has the form $Q_{t,i} = y_{l_i}x_{2i-1}x_{2i}y_{l_i}$, where $y_{l_i} \in \{y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j} \mid j = 1, 2, \ldots, t\}$.

In order to help readers to understand the procedure of adding tubes in this case, we give an example that $t = 4$ which is shown in Figure 6. For $i = 1, 2, 3, 4$, the four rectangles in the first column of (i) respectively represent $T_{i,1}, \ldots, T_{i,4}$ from top to bottom, and the four rectangles the second column of (i) respectively represent $T'_{i,1}, \ldots, T'_{i,4}$ from top to bottom.

We need to show that H satisfies the demands of the theorem. Obviously, w_0 is connected with each of $x_1, x_2, \ldots, x_{2t}$ in H. By the similar argument as in Case 1, one can show that for $i = 1, 2, \ldots, 2t$ and $j = 1, 2, \ldots, 4t$, x_i is connected with y_j in H.

Claim 2.8. *H contains the edge set*

$$\{x_ix_{i+1}, \ldots, x_ix_{2t} \mid i = 1, 2, \ldots, 2t-1\} \setminus \{x_{2i-1}x_{2i} \mid i = 1, 2, \ldots, t\}.$$

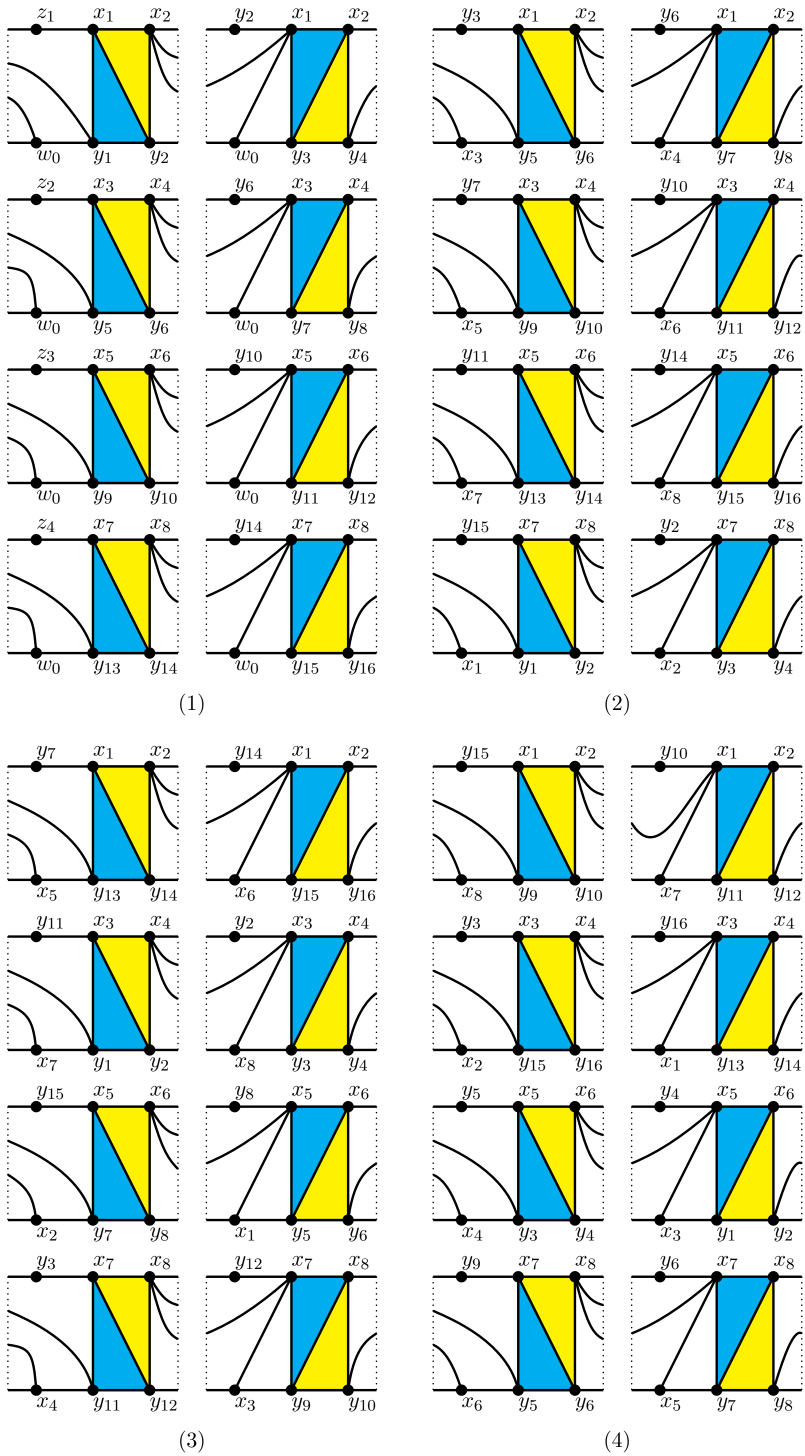

Figure 6: The operations of adding $2t$ tubes when $t = 4$.

We proceed the similar argument to that in Claim 2.6. Obviously, there are $2t(t-1)$ edges of the form x_kx_j ($k \neq j$) except for the form $x_{2i-1}x_{2i}$ after the t-th operation of adding $2t$ tubes has been applied. According to the rule of the $(s+1)$-th operation of adding $2t$ tubes, x_{2i} and x_{2i-1} are connected with $x_{2\tau_s(i)-1}$ and $x_{2\tau_s(i)}$, respectively, if $1 \leq s \leq \frac{t}{2}-1$ or $s = \frac{t}{2}$ and $i = 1, 2, \ldots, \frac{t}{2}$, and x_{2i} and x_{2i-1} are connected with $x_{2\tau_s(i)}$ and $x_{2\tau_s(i)-1}$, respectively, if $\frac{t}{2}+1 \leq s \leq t-1$ or $s = \frac{t}{2}$ and $i = \frac{t}{2}+1, \frac{t}{2}+2, \ldots, t$. We now consider the relation between $\tau_{s_1}(i)$ and $\tau_{s_2}(i)$, where $1 \leq s_1, s_2 \leq t-1$ and $s_1 + s_2 \equiv 0 \pmod t$. We have the following proposition.

Proposition 2.9. *Suppose that s_1 and s_2 are two integers such that $1 \leq s_1, s_2 \leq t-1$. If $s_1+s_2 \equiv 0 \pmod t$, then $\tau_{s_1}(t-i) = t - \tau_{s_2}(i)$ or $\tau_{s_2}(i) = t - \tau_{s_1}(t-i)$.*

In fact,

$$\begin{aligned}\tau_{s_1}(t-i) &\equiv t-i+(-1)^{s_1+1}s_1 \equiv t-i+(-1)^{t-s_2+1}(t-s_2)\\ &\equiv t-i+(-1)^{t-s_2}s_2 \pmod t.\end{aligned}$$

Since $t \equiv 0 \pmod 2$, $(-1)^{t-s_2} = (-1)^{s_2}$. So

$$\tau_{s_1}(t-i) \equiv t-i+(-1)^{s_2}s_2 \equiv t-(i+(-1)^{s_2+1}s_2) \equiv t-\tau_{s_2}(i) \pmod t.$$

In other words, $\tau_{s_1}(t-i) = t-\tau_{s_2}(i)$, or $\tau_{s_2}(i) = t-\tau_{s_1}(t-i)$.

Thus, the pair of vertices of the form $x_{2\tau_{s_2}(i)-1}$ and $x_{2\tau_{s_2}(i)}$ connected with the pair of x_{2i-1} and x_{2i} in the (s_2+1)-th operation of adding $2t$ tubes is the same as the pair of vertices of the form $x_{2(t-\tau_{s_1}(t-i))-1}$ and $x_{2(t-\tau_{s_1}(t-i))}$ connected with the pair of x_{2i-1} and x_{2i} in the (s_1+1)-th operation of adding $2t$ tubes if $0 \leq s_1, s_2 \leq t-1$ and $s_1+s_2 \equiv 0 \pmod t$. But the methods of two connections are different. We now view the pair of x_{2i-1} and x_{2i} as a vertex u_i, where $i \in \{1, 2, \ldots, t\}$. In order to show Claim 2.8, it is sufficient to show that u_p is connected with u_q, where $p, q \in \{1, 2, \ldots, t\}$ and $p \neq q$. For the purpose, it is sufficient to show that there exists some k such that $\tau_k(p) = q$ or $\tau_k(q) = p$. By Proposition 2.9, it is sufficient to show that for any two distinct numbers $i, j \in \{1, 2, \ldots, \frac{t}{2}\}$, there exists some $k \in \{1, 2, \ldots, t\}$ such that $\tau_k(i) = j$ or $\tau_k(j) = i$.

Without loss of generality, suppose that $j > i$. If $j - i \equiv 1 \pmod 2$, let $k = j - i$. Then

$$\tau_k(i) \equiv i+(-1)^{k+1}k \equiv i+(j-i) \equiv j \pmod t.$$

So $\tau_k(i) = j$. If $j-i \equiv 0 \pmod 2$, let $k = j-i$. Then

$$\tau_k(j) \equiv j+(-1)^{k+1}k \equiv j-(j-i) \equiv i \pmod t.$$

So $\tau_k(j) = i$. Hence u_p is connected with u_q for $p \neq q$. Thus, Claim 2.8 has been proved.

Therefore, the obtained embedding is as required. □

In the proof of Lemma 2.1, we apply the operation of adding $2t$ tubes t times starting from $\mathcal{X}_0$, $\mathcal{Y}_0$ and $\mathcal{Y}'_0$ to construct an embedding of H, where $\mathcal{X}_0 = \{Q_{0,i} \mid i = 1, 2, \ldots, t\}$, $\mathcal{Y}_0 = \{R_{0,i} \mid i = 1, 2, \ldots, t\}$, $\mathcal{Y}'_0 = \{R'_{0,i} \mid i = 1, 2, \ldots, t\}$. We call the above procedure *the operation of adding $2t^2$ tubes starting from $\mathcal{X}_0$, $\mathcal{Y}_0$ and $\mathcal{Y}'_0$*. Lemma 2.10 below is an analogue of Lemma 2.1. The vertex w_0 in Lemma 2.1 is replaced with two vertices w'_0, w''_0 in Lemma 2.10, and the others are not changed. The proof is similar to that in the proof of Lemma 2.1, which is omitted here.

Lemma 2.10. *Suppose that G is a graph which has a vertex subset*

$$\{w_0', w_0'', z_1, z_2, \dots, z_t\} \cup \{x_i \mid i = 1, 2, \dots, 2t\} \cup \{y_j \mid j = 1, 2, \dots, 4t\},$$

where $z_1, z_2, \dots, z_t$ need not be different, and suppose that G contains no edges in the set

$$\begin{aligned} E' = \{&w_0' x_{2i-1}, w_0'' x_{2i} \mid i = 1, 2, \dots, t\} \cup \{x_i y_j \mid i = 1, 2, \dots, 2t; j = 1, 2, \dots, 4t\} \\ &\cup (\{x_i x_{i+1}, \dots, x_i x_{2t} \mid i = 1, 2, \dots, 2t-1\} \setminus \{x_{2i-1} x_{2i} \mid i = 1, 2, \dots, t\}). \end{aligned}$$

Suppose that Π is a 2-cell embedding of G on the orientable surface S_g with the following properties:

(i) For $i = 1, 2, \dots, t$, $R_{0,i} = w_0' y_{4i-3} y_{4i-2} w_0$ and $R_{0,i}' = w_0'' y_{4i-1} y_{4i} w_0$ are facial cycles of Π.

(ii) For $i = 1, 2, \dots, t$, $Q_{0,i} = z_i x_{2i-1} x_{2i} z_i$ is a facial cycle of Π such that $Q_{0,i}$ has not any common vertex with each of $R_{0,1}, \dots, R_{0,t}, R_{0,1}', \dots, R_{0,t}'$.

Then there is a supergraph H of G satisfying the following conditions:

(i) E' is an edge subset of $E(H)$.

(ii) H has an embedding on the orientable surface of genus $g + 2t^2$ such that it has a set of t facial 3-cycles $\{Q_{t,i} \mid Q_{t,i} = y_{l_i} x_{2i-1} x_{2i} y_{l_i}, i = 1, 2, \dots, t\}$, where y_{l_i} is some vertex in $\{y_{4i-3}, y_{4i-2}, y_{4i-1}, y_{4i} \mid i = 1, 2, \dots, t\}$.

We now introduce another method of constructing an embedding, which is used in the proof of Lemma 2.11.

Lemma 2.11. *Let k and l be two positive integers. Suppose that G has a vertex subset*

$$\{w, z\} \cup \{x_i, y_j \mid i = 1, 2, \dots, 2l,\ j = 1, 2, \dots, 2k\},$$

and suppose that G contains no edges in

$$E' = \{x_i y_j \mid i = 1, 2, \dots, 2l,\ j = 1, 2, \dots, 2k\}.$$

If G has a 2-cell embedding Π on the orientable surface S_g such that $F_i = w x_{2i-1} x_{2i} w$ and $F_j' = z y_{2j-1} y_{2j} z$ are facial cycles in Π for $i = 1, 2, \dots, l$ and $j = 1, 2, \dots, k$, then there is a supergraph H of G with the following properties:

(i) E' is an edge subset of H.

(ii) H has an embedding on the orientable surface of genus $g + kl$ such that it has a set of l facial 3-cycles in which each has the form $y_{h_i} x_{2i-1} x_{2i} y_{h_i}$, where $y_{h_i} \in \{y_1, y_2, \dots, y_{2k}\}$.

Proof. We construct an embedding from Π as follows.

(1) Let $D_{1,1} = F_1$. Then the tube $T_{1,1}$ is added between $D_{1,1}$ and F_1'. Next, the four edges $x_1 y_1$, $x_1 y_2$, $x_2 y_1$ and $x_2 y_2$ are drawn on $T_{1,1}$ in the way shown in Figure 7. Let $D_{1,2} = y_1 x_1 x_2 y_1$, and let $Q_{1,1} = x_2 y_1 y_2 x_2$. The tube $T_{1,2}$ is now added between $D_{1,2}$ and F_2', and the four edges $x_1 y_3$, $x_1 y_4$, $x_2 y_3$ and $x_2 y_4$ are drawn on it in the similar way as in Figure 7. Let $D_{1,3} = y_3 x_1 x_2 y_3$ and $Q_{1,2} = x_2 y_3 y_4 x_2$. Then $D_{1,3}$ and F_3' are dealt with as $D_{1,2}$ and F_2', and so on. The procedure is not stopped until F_k' has been dealt with. Thus, we obtain k facial cycles $Q_{1,1}, \dots, Q_{1,k}$, where $Q_{1,i} = x_2 y_{2i-1} y_{2i} x_2$. Moreover, both x_1 and x_2 are connected with each of $y_1, y_2, \dots, y_{2k}$.

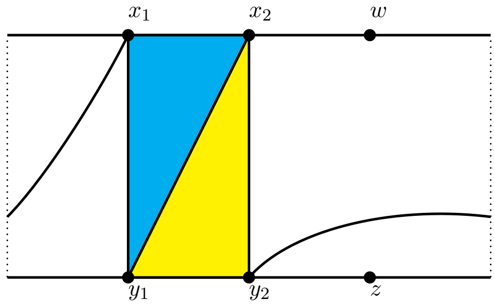

Figure 7: The drawing of the four edges in $T_{1,1}$.

(2) Let $\mathcal{Q}_1 = \{Q_{1,1}, Q_{1,2}, \ldots, Q_{1,k}\}$. Then the tube $T_{2,1}$ is added between F_2 and $Q_{1,1}$, and the four edges x_3y_1, x_3y_2, x_4y_1 and x_4y_2 are drawn on it in the similar way as in Figure 7, and so on. The procedure is stopped till $Q_{1,k}$ has been dealt with. Then we obtain a set of facial walks $\mathcal{Q}_2 = \{Q_{2,1}, Q_{2,2}, \ldots, Q_{2,k}\}$ such that $Q_{2,i} = x_4y_{2i-1}y_{2i}x_4$. Moreover, both x_3 and x_4 are connected with each of $y_1, y_2, \ldots, y_{2k}$.

(3) $\mathcal{Q}_2$ and F_3 are dealt with in the similar way to that of $\mathcal{Q}_1$ and F_2, and so on. The procedure is stopped till F_l has been dealt with. Then x_i is connected with each of $y_1, y_2, \ldots, y_{2k}$ for $i = 1, 2, \ldots, 2l$, and there is a set of l facial 3-cycles in which each has the form $y_{h_i}x_{2i-1}x_{2i}y_{h_i}$. Moreover, there are kl tubes to be added to the primitive surface all together. So the obtained embedding Π' is one on the orientable surface of genus $g + kl$. Let H be the graph corresponding to Π'. It is easy to find that E' is an edge set of H. $\square$

Let $\mathcal{F}_1 = \{F_1, F_2, \ldots, F_l\}$, and let $\mathcal{F}_2 = \{F'_1, F'_2, \ldots, F'_k\}$. We call the procedure of constructing an embedding in the proof of Lemma 2.11 *the operation of adding tubes with respect to $\mathcal{F}_1$ and $\mathcal{F}_2$.*

3 An upper bound for $\gamma(C_m + K_n)$ if m is odd

From now on we always suppose that $m \geq 3$ and $n \geq 4$, that $C_m = u_1u_2 \ldots u_mu_1$, and that the vertex set of K_n is $\{v_1, v_2, \ldots, v_n\}$. If no confusion occur, a face and its boundary in an embedding are not distinguished in the rest of the paper.

Lemma 3.1. *Suppose that $m \equiv 1 \pmod 2$ and $n \equiv 0 \pmod 4$. If $m \geq 4n - 5$, then*

$$\gamma(C_m + K_n) \leq \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

Proof. We shall construct an embedding of $C_m + K_n$ on the oreintable surface of genus $\lceil \frac{(m-2)(n-2)}{4} \rceil$ in the following steps.

(1) In the step we shall construct an embedding on a sphere in which each of v_1 and v_2 is connected with each of $u_1, u_2, \ldots, u_m$, and each of u_1 and u_2 is connected with each of $v_1, v_2, \ldots, v_n$.

First, C_m is placed in the equator of the sphere, and both v_1 and v_2 are situated at the northern pole and the southern pole, respectively. Second, each of v_1 and v_2 joins to each of $u_1, u_2, \dots, u_m$, and the path $P = v_3 v_4 \dots v_n$ is placed in the interior of the face $v_1 u_1 u_2 v_1$ such that v_3 is near to v_1. Third, v_3 joins to v_1, and each of u_1 and u_2 joins to each of $v_3, v_4, \dots, v_n$. Thus, we obtain an embedding Π_1 on the sphere, which is shown in Figure 8.

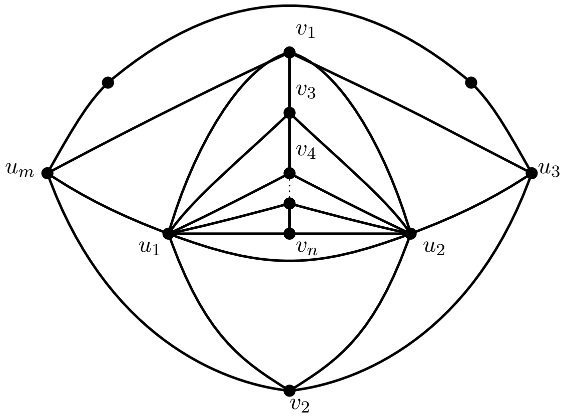

Figure 8: The embedding Π_1.

(2) In the step we shall add $\frac{n}{4}$ tubes to the sphere such that u_3 is connected with each of $v_3, v_4, \dots, v_n$, and v_1 joins to v_2.

The tube T_1 is now added between the facial cycles $u_2 v_3 v_4 u_2$ and $v_2 u_2 u_3 v_2$. Next, the edge $u_2 v_3$ is redrawn such that it is on T_1 and a segment of local rotation at u_2 in clockwise is that v_4, v_1, u_3, v_3. Then there is a facial walk $W_1 = u_3 v_2 u_2 v_3 v_1 u_2 v_4 v_3 u_2 u_3$. Let $Z_1 = u_3 v_2 u_2 v_3 v_1 u_2 v_4 v_3$. Then $W_1 = Z_1 u_2 u_3$.

The tube T_2 is added between the facial cycle $u_2 v_8 v_7 u_2$ and W_1. Then the two edges $u_2 v_7$ and $u_2 v_6$ are redrawn on T_2 such that a segment of local rotation at u_2 in clockwise is that u_3, v_7, v_6, v_3. Thus, there is a facial walk $W_2 = Z_1 u_2 v_6 v_5 u_2 v_8 v_7 u_2 u_3$. Let $Z_2 = u_2 v_6 v_5 u_2 v_8 v_7$. Thus, $W_2 = Z_1 Z_2 u_2 u_3$.

For $i = 3, 4, \dots, \frac{n}{4}$, the tube T_i is added between the facial cycle $u_2 v_{4i} v_{4i-1} u_2$ and W_{i-1}. Next, both edges $u_2 v_{4i-1}$ and $u_2 v_{4i-2}$ are redrawn on T_i such that a segment of local rotation at u_2 in clockwise is that u_3, v_{4i-1}, v_{4i-2} and v_{4i-5}. Then there is a facial walk $W_i = Z_1 Z_2 \dots Z_{i-1} u_2 v_{4i-2} v_{4i-3} u_2 v_{4i} v_{4i-1} u_2 u_3$. Let $Z_i = u_2 v_{4i-2} v_{4i-3} u_2 v_{4i} v_{4i-1}$. Thus, $W_i = Z_1 Z_2 \dots Z_i u_2 u_3$.

After the tube $T_{\frac{n}{4}}$ has been added, there is a facial walk $W_{\frac{n}{4}} = Z_1 Z_2 \dots Z_{\frac{n}{4}-1} u_2 u_3$. For $i = 2, 3, \dots, \frac{n}{4}$, each of v_{4i-3}, v_{4i-2}, v_{4i-1} and v_{4i} appears in Z_i once, but it does not appear in Z_j if $i \neq j$. Also, v_4 appears in Z_1 once, but it does not appear in Z_j if $j \neq 1$. In the interior of the face $W_{\frac{n}{4}}$, u_3 joins to each of $v_4, v_5, \dots, v_n$, and v_1 joins to v_2. For example, if $n = 8$, W_2 and all added edges in the interior of W_2 are shown in Figure 9. Let Π_2 be the embedding obtained from Π_1 by the above operation of adding tubes. Then Π_2 is an embedding on the surface of genus $\frac{n}{4}$.

(3) In the step we shall add $2(\frac{n}{2}-1)^2$ tubes to the present surface satisfying the following conditions:

(i) v_1 is connected with each of $v_3, v_4, \dots, v_n$,

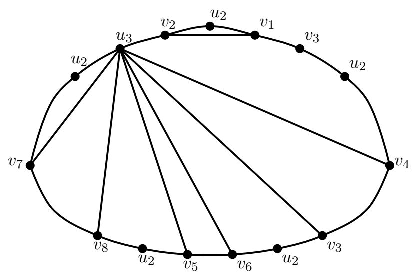

Figure 9: W_2 and all edges added in the interior of W_2.

(ii) for $i = 3, 4, \ldots, n$ and $j = 4, 5, \ldots, 2n-1$, v_i is connected with u_j, and

(iii) all edges in the set

$$\{v_iv_{i+1}, \ldots, v_iv_n \mid i = 3, \ldots, n-1\} \setminus \{v_{2i+1}v_{2i+2} \mid i = 1, \ldots, \tfrac{n-2}{2}\}$$

are added.

For the above purpose, let

$$\mathcal{X}_0 = \{Q_{0,i} \mid Q_{0,i} = u_1v_{2i+1}v_{2i+2}u_1,\ i = 1, 2, \ldots, \tfrac{n}{2} - 1\},$$
$$\mathcal{Y}_0 = \{R_{0,i} \mid R_{0,i} = v_1u_{4i}u_{4i+1}v_1,\ i = 1, 2, \ldots, \tfrac{n}{2} - 1\}, \text{ and}$$
$$\mathcal{Y}'_0 = \{R'_{0,i} \mid R'_{0,i} = v_1u_{4i+2}u_{4i+3}v_1,\ i = 1, 2, \ldots, \tfrac{n}{2} - 1\}.$$

Then we apply the operation of adding $2(\frac{n}{2} - 1)^2$ tubes starting from $\mathcal{X}_0$, $\mathcal{Y}_0$, and $\mathcal{Y}'_0$. By Lemma 2.1, an embedding Π_3 is obtained which satisfies all the requirements and contains a set $\mathcal{A}_0 = \{A_{0,1}, A_{0,2}, \ldots, A_{0,\frac{n}{2}-1}\}$ of facial 3-cycles such that $A_{0,i}$ has the form $u_{k_i}v_{2i+1}v_{2i}u_{k_i}$, where $u_{k_i} \in \{u_j \mid j = 4, 5, \ldots, 2n-1\}$.

(4) In the step we shall add $2(\frac{n}{2} - 1)^2$ tubes to present surface satisfying the following conditions:

(i) v_2 is connected with $v_3, v_4, \ldots, v_n$,

(ii) for $i = 3, 4, \ldots, n$ and $j = 2n, 2n+1, \ldots, 4n-5$, v_i is connected with u_j.

For the above purpose, let

$$\mathcal{B}_0 = \{B_{0,i} \mid B_{0,i} = v_2u_{2n+4i-4}u_{2n+4i-3}v_2,\ i = 1, 2, \ldots, \tfrac{n}{2} - 1\}, \text{ and}$$
$$\mathcal{B}'_0 = \{B'_{0,i} \mid B'_{0,i} = v_2u_{2n+4i-2}u_{2n+4i-1}v_2,\ i = 1, 2, \ldots, \tfrac{n}{2} - 1\}.$$

We now apply the operation of adding $2(\frac{n}{2} - 1)^2$ tubes starting from $\mathcal{A}_0$, $\mathcal{B}_0$, and $\mathcal{B}'_0$. By Lemma 2.1, an embedding Π_4 is obtained which satisfies all the requirements and contains a set $\mathcal{F} = \{F_1, F_2, \ldots, F_{\frac{n}{2}-1}\}$ of facial 3-cycles such that F_i has the form $u_{l_i}v_{2i+1}v_{2i+2}u_{l_i}$, where $u_{l_i} \in \{u_j \mid j = 2n, 2n+1, \ldots, 4n-5\}$. At last, all edges of the form v_iv_j added in the above operations are deleted, since these edges have been existed. Note that the deletion of these edges does not affect each cycle in $\mathcal{F}$.

(5) If $m = 4n-5$, then there is nothing to do. If $m > 4n-5$, then we shall add tubes to the present surface such that v_i is connected with each of $u_{4n-4}, \dots, u_m$ for $i = 3, 4, \dots, n$.

Let
$$\mathcal{D} = \{D_i \mid D_i = v_1 u_{4n+2i-6} u_{4n+2i-5} v_1,\ i = 1, 2, \dots, \tfrac{m-4n+5}{2}\}.$$
We now use the operation of adding tubes respect to $\mathcal{F}$ and $\mathcal{D}$. By Lemma 2.11, there are $\frac{(n-2)(m-4n+5)}{4}$ tubes being used, and v_i is connected with u_j, where $i \in \{3, 4, \dots, n\}$ and $j \in \{4n-4, 4n-3, \dots, m\}$. Let Π_5 be the obtained embedding. Then it is an embedding of $C_m + K_n$ on the surface of genus
$$\frac{n}{4} + \frac{(n-2)^2}{2} + \frac{(n-2)^2}{2} + \frac{(n-2)(m-4n+5)}{4}.$$
By simple counting, we have that
$$\frac{n}{4} + \frac{(n-2)^2}{2} + \frac{(n-2)^2}{2} + \frac{(n-2)(m-4n+5)}{4} = \frac{n}{4} + \frac{(n-2)(m-3)}{4}.$$
Since $n \equiv 0 \pmod 4$,
$$\left\lceil \frac{(m-2)(n-2)}{4} \right\rceil = \left\lceil \frac{n-2}{4} \right\rceil + \frac{(n-2)(m-3)}{4} = \frac{n}{4} + \frac{(n-2)(m-3)}{4}.$$
So
$$\frac{n}{4} + \frac{(n-2)^2}{2} + \frac{(n-2)^2}{2} + \frac{(n-2)(m-4n+5)}{4} = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$
Hence, $\gamma(C_m + K_n) \le \lceil \frac{(m-2)(n-2)}{4} \rceil$. □

Lemma 3.2. *Suppose that* $m \equiv 1 \pmod 2$ *and* $n \equiv 2 \pmod 4$. *If* $m \ge 4n-3$, *then*
$$\gamma(C_m + K_n) \le \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

Proof. We construct an embedding of $C_m + K_n$ in the similar way to that in the proof of Lemma 3.1.

(1) First, place C_m, v_1, and v_2 on a sphere and add edges as (1) in the proof of Lemma 3.1. Let $F_1 = v_1 u_1 u_2 v_1$, $F_2 = v_1 u_2 u_3 v_1$, and $F_3 = v_1 u_4 u_5 v_1$. The path $P = v_7 v_8 \dots v_n$ is now placed in the interior of F_1, and each of u_1 and u_2 joins to each of $v_7, v_8, \dots, v_n$. Next, both v_3 and v_5 are placed in the interior of F_2, and they join to each of u_2 and u_3, respectively. Similarly, both v_4 and v_6 are placed in the interior of F_3, and they join to each of u_4 and u_5, respectively. Let Π_1 be the obtained embedding on the sphere, which is shown in Figure 10.

The edge $u_3 u_4$ is now deleted from Π_1. Then the face $v_1 u_3 u_4 v_1$ and the face $v_2 u_3 u_4 v_2$ are merged into a face $F_4 = v_1 u_3 v_2 u_4 v_1$. Next, the edge $v_1 v_2$ is drawn in the interior of F_4. Let $F_5 = u_2 v_3 u_3 v_5 u_2$ and $F_6 = u_4 v_4 u_5 v_6 u_4$. The tube T_1 is added between F_5 and F_6. Then the five edges are drawn on T_1 in the way shown in (1) in Figure 11. Let $F_7 = u_2 v_3 u_4 v_6 u_2$ and $F_8 = u_3 v_4 u_5 v_5 u_3$. Next, the tube T_2 is

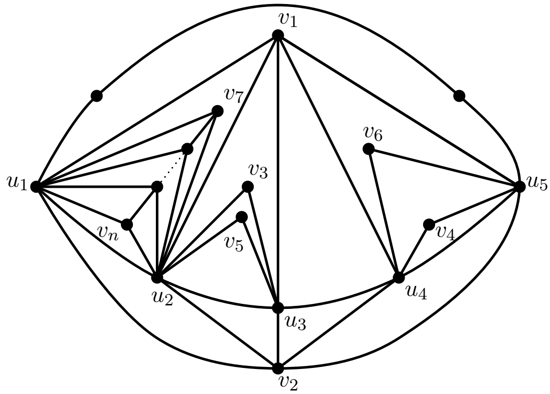

Figure 10: The embedding Π_1.

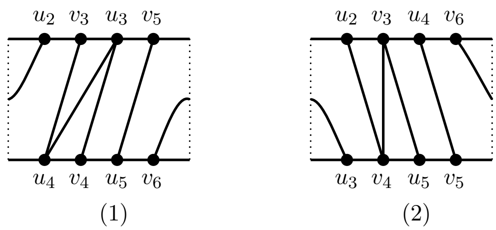

Figure 11: The drawing of edges on T_1 or T_2.

added between F_7 and F_8. Then the five edges are drawn on T_2 in the way shown in (2) in Figure 11.

We observe that the local rotation at u_2 in clockwise is that $u_1, v_n, \ldots, v_1, v_3, v_4, v_6, v_5, u_3, v_2$. Let $F_9 = u_2v_6u_3v_4u_2$, which is a facial cycle (refer to (2) in Figure 11). Let $F_{10} = u_1v_nu_2u_1$ (refer to Figure 10) if $n > 6$, or $F_{10} = u_1v_1u_2u_1$ if $n = 6$. The tube T_3 is now added between F_9 and F_{10}. Then the edges u_2v_5 and u_2v_4 are redrawn on T_3 such that a segment of the local rotation at u_2 is that $u_1, v_6, v_4, v_n, v_3, v_5$. Thus, there is a facial walk $W_1' = u_1u_2v_4v_3u_2v_5u_5v_6u_2v_nu_1$. Next, u_1 joins to each of v_3, v_4, v_5, v_6, and v_5 joins to v_6. Then there are two facial cycles $Q_{0,1} = u_1v_4v_3u_1$ and $Q_{0,2} = u_1v_5v_6u_1$.

(2) If $n = 6$, there is nothing to do. If $n > 6$, then we shall add $\frac{3(n-2)}{4}$ tubes to the present surface such that u_i is connected with each of $v_3, v_4, \ldots, v_n$ for $i = 3, 4, 5$.

Let $F_{11} = v_1u_3v_3u_2v_1$ (refer to Figure 10). For $i = 1, 2, \ldots, \frac{n-6}{4}$, let $F_i' = u_2v_{4i+4}v_{4i+5}u_2$. The tube T_1' is added between F_1' and F_{11}. Then two edges u_2v_{4i+4} and u_2v_{4i+5} are redrawn on T_1'. There is a facial walk $W_1 = u_2v_3u_3v_1u_2v_9v_{10}u_2 v_7v_8u_2$. For $i = 2, \ldots, \frac{n-6}{4}$, the tube T_i' is added between F_i' and W_{i-1}, where W_{i-1} is a facial walk which contains $v_7, \ldots, v_{4i+2}$ after T_{i-1}' has added. Next, both u_2v_{4i+4} and u_2v_{4i+5} are redrawn on T_i' and a segment in the local rotation at u_2 in clockwise is that $u_{4(i-1)+5}$, u_{4i+4}, u_{4i+5}, and u_3. After the tube $T'_{\frac{n-6}{4}}$ has been added, there is a facial walk $W_{\frac{n-6}{4}}$ which contains u_3, $v_7, v_8, \ldots, v_n$. Moreover, each of $v_7, v_8, \ldots, v_n$ appears in $W_{\frac{n-6}{4}}$ once. Next, u_3 joins to each of $v_7, v_8, \ldots, v_n$. There are $\frac{n-6}{2}$ facial 3-cycles $D_1, D_2, \ldots, D_{\frac{n-6}{2}}$, where $D_i = u_3v_{2i+5}v_{2i+6}u_3$.

Let $F_{12} = u_4v_4u_5u_4$ (refer to Figure 10). Let $\mathcal{F} = \{F_{12}\}$, and let $\mathcal{D} = \{D_1, D_2, \ldots, D_{\frac{n-6}{2}}\}$. Using the operation of adding tubes with respect to $\mathcal{D}$ and $\mathcal{F}$, each of u_4 and u_5 is connected with each of $v_7, v_8, \ldots, v_n$. By Lemma 2.11, there are $\frac{n-6}{2}$ tubes being used. Also, there are $\frac{n-6}{2}$ facial cycles $Q_{0,3}, \ldots, Q_{0,\frac{n-2}{2}}$ in which $Q_{0,i}$ has the form $u_{l_i}v_{2i+1}v_{2i+2}u_{l_i}$, where $u_{l_i} \in \{u_4, u_5\}$. Let Π_2 be the embedding obtained from Π_1 by the above procedures. Then Π_2 is an embedding on the surface of genus $3 + \frac{n-6}{4} + \frac{n-6}{2}$ $(= \frac{3(n-2)}{4})$. Moreover, u_i is connected with each of $v_1, v_2, \ldots, v_n$ for $i = 1, 2, \ldots, 5$.

(3) For $i = 1, 2, \ldots, \frac{n-6}{2}$, let $R_{0,i} = v_1u_{4i+2}u_{4i+3}v_1$, and let $R'_{0,i} = v_1u_{4i+4}u_{4i+5}v_1$. Let $\mathcal{X}_0 = \{Q_{0,i+2} \mid i = 1, 2, \ldots, \frac{n-6}{2}\}$, $\mathcal{Y}_0 = \{R_{0,i} \mid i = 1, 2, \ldots, \frac{n-6}{2}\}$, and $\mathcal{Y}'_0 = \{R'_{0,i} \mid i = 1, 2, \ldots, \frac{n-6}{2}\}$. Next procedures are similar to that in (4) and (5) in the proof of Lemma 3.1. Note that $\frac{(m-5)(n-2)}{4}$ tubes are added to the present surface such that v_i is connected with u_j for $i = 3, 4, \ldots, n$ and $j = 6, 7, \ldots, m$. Thus, an embedding Π_3 of $C_m + K_n$ on the surface of genus $\frac{3(n-2)}{4} + \frac{(m-5)(n-2)}{4}$ is obtained. Since $n \equiv 2 \pmod 4$, $\lceil \frac{(m-2)(n-2)}{4} \rceil = \frac{3(n-2)}{4} + \frac{(m-5)(n-2)}{4}$. Thus, Π_3 is the desired embedding. Since the operation of adding $n-2$ tubes is used twice, m is at least $5 + 4(n-2)$ $(= 4n - 3)$. □

Lemma 3.3. *Suppose that* $m \equiv 1 \pmod 2$ *and* $n \equiv 1 \pmod 2$. *If* $m \geq 6n - 13$, *then*

$$\gamma(C_m + K_n) \leq \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil .$$

Proof. We consider two cases.

Case 1: $m \equiv 1 \pmod 4$. In this case we construct an embedding of $C_m + K_n$ in the following steps.

(1) The path $P_m = u_1u_2 \ldots u_m$ is placed in the equator of a sphere. The edge v_1v_2 is situated in the northern pole and the vertex v_3 placed at the southern pole. Next, each of v_1 and v_3 joins to each of $u_1, u_2, \ldots, u_{\frac{m+1}{2}}$, and each of v_1 and v_2 joins to each of $u_{\frac{m+3}{2}}, u_{\frac{m+5}{2}}, \ldots, u_m$. Also, v_1 joins to v_3, and v_2 joins to $u_{\frac{m+1}{2}}$. Thus, an embedding Π_1 on the sphere is obtained. For example, the embedding Π_1 is shown in Figure 12 if $m = 17$.

(2) In this step we shall construct an embedding on the surface of genus $\frac{m-1}{4}$ such that v_2 is connected with $u_1, u_2, \ldots, u_{\frac{m-1}{2}}$, v_3 connected with $u_{\frac{m+3}{2}}, u_{\frac{m+5}{2}}, \ldots, u_m$, and u_1 connected with u_m.

For $i = 1, 2, \ldots, \frac{m-1}{4}$, let $F_i = v_3u_{2i-1}u_{2i}v_3$ and $F'_i = v_2u_{m+1-2i}u_{m+2-2i}v_2$. The tube T_1 is added between F_1 and F'_1, and the five edges are drawn on T_1 in the way shown in (1) in Figure 13. The tube T_2 is added between F_2 and F'_2, and the five edges are drawn on T_1 in the way shown in (2) of Figure 13.

For $i = 3, 4, \ldots, \frac{m-1}{4}$, the tube T_i is added between F_i and F'_i. Then the four edges v_3u_{m+2-2i},v_3u_{m+1-2i}, v_2u_{2i-1}, and v_2u_{2i} are drawn on T_i in the way shown in (2) of Figure 13, but v_2v_3 is not added. Thus, v_3 is connected with each of

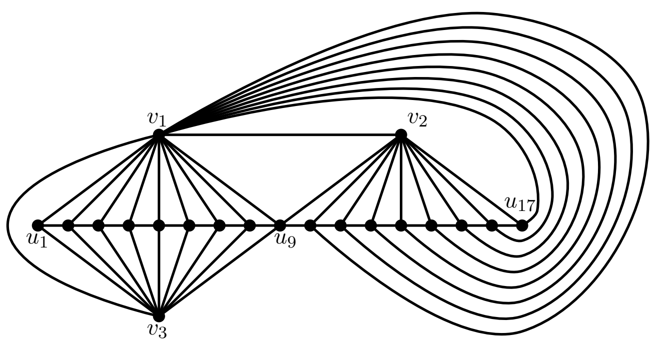

Figure 12: The embedding Π_1.

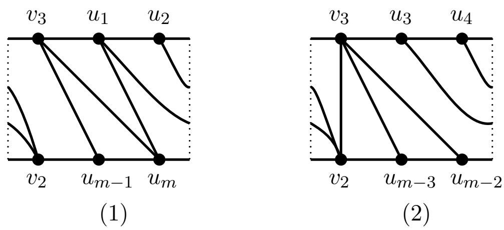

Figure 13: The drawing of edges on T_1 or T_2.

$u_{\frac{m+3}{2}}, u_{\frac{m+5}{2}}, \ldots, u_m, v_2$. Next, v_2 connected with each of $u_1, u_2, \ldots, u_{\frac{m-1}{2}}$. Let Π_2 be the obtained embedding. Note that there are two sets $\mathcal{Z}_0$ and $\mathcal{Z}_0'$ in Π_2, where

$$\mathcal{Z}_0 = \{Z_{0,i} \mid Z_{0,i} = v_2u_{2i-1}u_{2i}v_2,\ i = 1, 2, \ldots, \tfrac{m-1}{4}\} \text{ and}$$
$$\mathcal{Z}_0' = \{Z_{0,i}' \mid Z_{0,i}' = v_3u_{m+1-2i}u_{m+2-2i}v_3,\ i = 1, 2, \ldots, \tfrac{m-1}{4}\}.$$

(3) In this step $\lceil\frac{n-2}{4}\rceil$ tubes will be added to the present surface such that v_i is connected with $u_{\frac{m+1}{2}}, u_{\frac{m+3}{2}}, u_{\frac{m+5}{2}}$ for $i = 4, 5, \ldots, n$.

The path $P = v_4v_5\ldots v_n$ is now placed in the interior of $Z'_{0,\frac{m-1}{4}}$ such that v_4 is near to v_3. Then each of $u_{\frac{m+3}{2}}$ and $u_{\frac{m+5}{2}}$ joins to each of $v_4, v_5, \ldots, v_n$. For $i = 1, 2, \ldots, \lceil\frac{n-1}{4}\rceil$, let $D_i = u_{\frac{m+3}{2}}v_{4i}v_{4i+1}u_{\frac{m+3}{2}}$.

If $n \equiv 1 \pmod 4$, then $\lceil\frac{n-4}{4}\rceil = \frac{n-1}{4}$. The tube T_1' is now added between $D' = v_2u_{\frac{m+1}{2}}u_{\frac{m+3}{2}}v_2$ and D_1. Next, the edge $u_{\frac{m+3}{2}}v_4$ is redrawn on T_1'. Then we obtain a facial walk W_1 which contains $u_{\frac{m+1}{2}}$ and v_4. For $i = 2, 3, \ldots, \frac{n-1}{4}$, the tube T_i' is added between D_i and W_{i-1}, where W_{i-1} is a facial walk which contains $u_{\frac{m+1}{2}}$ and $u_{\frac{m+3}{2}}$ obtained by adding the tube T_{i-1}. Then two edges $u_{\frac{m+3}{2}}v_{4i-1}$ and $u_{\frac{m+3}{2}}v_{4i}$ are redrawn on T_i'. After the tube $T'_{\frac{n-1}{4}}$ has been added, there is a facial walk $W_{\frac{n-1}{4}}$ which contains $u_{\frac{m+1}{2}}, v_4, \ldots, v_n$. Next, $u_{\frac{m+1}{2}}$ joins to v_i if v_i appears once in $W_{\frac{n-1}{4}}$ or a copy of v_i if it appears more than once in $W_{\frac{n-1}{4}}$.

If $n \equiv 3 \pmod 4$, then $\lceil\frac{n-4}{4}\rceil = \frac{n-3}{4}$. We add $\frac{n-3}{4}$ tubes in the similar way to that in the above paragraph. The difference is that two edge $u_{\frac{m+3}{2}}v_{4i+1}$ and $u_{\frac{m+3}{2}}v_{4i+2}$

are redrawn on T_i' for $i = 1, 2, \dots, \frac{n-3}{4}$.

Let Π_3 be the embedding obtained from Π_2 by the above operation of adding tubes. Clearly, $u_{\frac{m+1}{2}}$, $u_{\frac{m+3}{2}}$, and $u_{\frac{m+5}{2}}$ are connected with each of $v_1, v_2, \dots, v_n$.

(4) In the step we proceed the similar argument as in (3) and (4) of the proof of Lemma 3.1. Let

$$\mathcal{X}_0 = \{Q_{0,i} \mid Q_{0,i} = u_{\frac{m+5}{2}} v_{2i+2} v_{2i+3} u_{\frac{m+5}{2}},\ i = 1, 2, \dots, \tfrac{n-3}{2}\},$$
$$\mathcal{Y}_0 = \{Z_{0,i} \mid i = 1, 2, \dots, \tfrac{n-3}{2}\}, \text{ and}$$
$$\mathcal{Y}_0' = \{Z_{0,i}' \mid i = 1, 2, \dots, \tfrac{n-3}{2}\}.$$

Then we apply the operation of adding $2(\frac{n-3}{2})^2$ tubes starting from $\mathcal{X}_0$, $\mathcal{Y}_0$, and $\mathcal{Y}_0'$. By Lemma 2.10, we have the following results:

(i) v_2 is connected with each of $v_4, v_6, \dots, v_{n-1}$, and v_3 connected with each of $v_5, v_7, \dots, v_n$.

(ii) For $i = 4, 5, \dots, n$ and $j = 1, 2, \dots, \frac{n-3}{2}$, v_i is connected with u_{2j-1}, u_{2j}, u_{m+1-2j}, u_{m+2-2j}.

(iii) There is a set

$$\{v_i v_{i+1}, \dots, v_i v_n \mid i = 1, 2, \dots, n-1\} \setminus \{v_4 v_5, v_6 v_7, \dots, v_{n-1} v_n\}.$$

(iv) There is a set

$$\mathcal{A}_0 = \{A_{0,1}, A_{0,2}, \dots, A_{0,\frac{n-3}{2}}\}$$

of facial cycles such that $A_{0,i}$ has the form $u_{l_i} v_{2i+1} v_{2i} u_{l_i}$, where $u_{l_i} \in \{u_1, \dots, u_{n-3}\} \cup \{u_{m-n+4}, \dots, u_m\}$.

Unfortunately, v_2 is not connected with each of $v_5, v_7, \dots, v_n$ and v_3 is not connected with each of $v_4, v_6, \dots, v_{n-1}$. In order to attach the edges $v_2v_5, \dots, v_2v_n$, $v_3v_4, \dots, v_3v_{n-1}$, we apply the operation of adding $2(\frac{n-3}{2})^2$ tubes again. Let

$$\mathcal{B}_0 = \{B_{0,i} \mid B_{0,i} = v_3 u_{m-n+4-2i} u_{m-n+5-2i} v_3,\ i = 1, 2, \dots, \tfrac{n-3}{2}\} \text{ and}$$
$$\mathcal{B}_0' = \{B_{0,i}' \mid B_{0,i}' = v_2 u_{n-4+2i} u_{n-3+2i} v_2,\ i = 1, 2, \dots, \tfrac{n-3}{2}\}.$$

We now apply the operation of adding $2(\frac{n-3}{2})^2$ tubes starting from $\mathcal{A}_0$, $\mathcal{B}_0$ and $\mathcal{B}_0'$. By Lemma 2.10, we have the following results:

(i) v_2 is connected with each of $v_5, v_7, \dots, v_n$, and v_3 connected with each of $v_4, v_6, \dots, v_{n-1}$.

(ii) For $i = 4, 5, \dots, n$ and $j = 1, 2, \dots, \frac{n-3}{2}$, v_i is connected with u_{n-4+2j}, $u_{n-3+2j}, u_{m-n+4-2j}, u_{m-n+5-2j}$.

(iii) There is a set

$$\mathcal{L}_0 = \{L_{0,1}, L_{0,2}, \dots, L_{0,\frac{n-3}{2}}\}$$

of $\frac{n-3}{2}$ facial cycles such that $L_{0,i}$ has the form $u_{h_i} v_{2i+1} v_{2i} u_{h_i}$, where $u_{h_i} \in \{u_{n-4+2j}, u_{n-3+2j}, u_{m-n+6-2j}, u_{m-n+5-2j} \mid j = 1, \dots, \frac{n-3}{2}\}$.

Need to say that all edges of the form $v_k v_l$ added in the above operations are deleted, since they have been existed.

For $i = 1, 2, \ldots, \frac{n-3}{2}$, let $F_{0,i} = v_1 u_{2n-7+2i} u_{2n-6+2i} v_1$ and $F'_{0,i} = v_1 u_{m-2n+7-2i} u_{m-2n+8-2i} v_1$. Let $\mathcal{F}_0 = \{F_{0,i} \mid i = 1, 2, \ldots, \frac{n-3}{2}\}$, and let $\mathcal{F}'_0 = \{F'_{0,i} \mid i = 1, 2, \ldots, \frac{n-3}{2}\}$. We apply the operation of adding $2(\frac{n-3}{2})^2$ tubes starting from $\mathcal{L}_0$, $\mathcal{F}_0$, and $\mathcal{F}'_0$. By Lemma 2.1, v_1 is connected with each of $v_4, v_5, \ldots, v_n$, and there is a set $\mathcal{N}_0 = \{N_{0,1}, N_{0,2}, \ldots, N_{0,\frac{n-3}{2}}\}$ of $\frac{n-3}{2}$ facial cycles such that $N_{0,i}$ has the form $u_{k_i} v_{2i+1} v_{2i} u_{k_i}$, where $u_{k_i} \in \{u_{2n-7+2j}, u_{2n-6+2j} u_{m-2n+7-2j}, u_{m-2n+8-2j} \mid j = 1, \ldots, \frac{n-3}{2}\}$. Next, all added edges of the form $v_i v_j$ $(i, j \neq 1)$ are deleted, since they have been existed.

(5) In this step we proceed the similar argument to (5) in the proof of Lemma 3.1. For $i = 1, \ldots, \frac{1}{2}(\frac{m-1}{2} - 3n + 9)$, let $M_i = v_1 u_{3n-10+2i} u_{3n-9+2i} v_1$, and $M'_i = v_1 u_{m-3n+10-2i} u_{m-3n+11+2i} v_1$. Clearly, $M'_{\frac{1}{2}(\frac{m-1}{2}-3n+9)}$ is exactly the cycle $v_1 u_{\frac{m+3}{2}} u_{\frac{m+5}{2}} v_1$. Since $u_{\frac{m+3}{2}}$ and $u_{\frac{m+5}{2}}$ are connected with each of $v_1, \ldots, v_n$, $M'_{\frac{1}{2}(\frac{m-1}{2}-3n+9)}$ should be neglected. Let

$$\mathcal{M} = \{M_i, M'_i \mid i = 1, \ldots, \tfrac{1}{2}(\tfrac{m-1}{2} - 3n + 9)\} \setminus \{M'_{\frac{1}{2}(\frac{m-1}{2}-3n+9)}\}.$$

Next, we apply the operation of adding tubes with respect to $\mathcal{M}$ and $\mathcal{N}_0$. There are $\frac{[m-6(n-3)-3](n-3)}{4}$ tubes being added to the present surface. Since $m \equiv 1 \pmod 2$ and $n \equiv 1 \pmod 4$, we have that

$$\left\lceil \frac{(m-2)(n-2)}{4} \right\rceil = \frac{(m-3)(n-3)}{4} + \frac{m-1}{4} + \left\lceil \frac{n-4}{4} \right\rceil$$

and

$$\frac{[m-6(n-3)-3](n-3)}{4} + \frac{m-1}{4} + \left\lceil \frac{n-4}{4} \right\rceil + 6\left(\frac{n-3}{2}\right)^2 = \frac{(m-3)(n-3)}{4} + \frac{m-1}{4} + \left\lceil \frac{n-4}{4} \right\rceil.$$

Hence an embedding of $C_m + K_n$ on the surface of genus $\lceil \frac{(m-2)(n-2)}{4} \rceil$ is obtained.

Need to say that the operations of adding $2(\frac{n-3}{2})^2$ tubes are used three times, m is at least $6(n-3)$ $(= 6n - 18)$. If $u_{\frac{m+1}{2}}, u_{\frac{m+3}{2}}$, $u_{\frac{m+5}{2}}$ and $M_{\frac{1}{2}(\frac{m-1}{2}-3n+9)}$ are considered, m is at least $6n - 18 + 5$ $(= 6n - 13)$.

Case 2: $m \equiv 3 \pmod 4$. In this case we shall construct an embedding of $C_m + K_n$ in the similar way to that in Case 1.

(1) P_m, v_1, v_2, and v_3 are placed in a sphere as in Case 1. Next, each of v_1 and v_3 is connected with each of $u_1, u_2, \ldots, u_{\frac{m+1}{2}}$, and each of v_1 and v_2 is connected with each of $u_{\frac{m+3}{2}}, u_{\frac{m+5}{2}}, \ldots, u_m$. Also, v_2 is connected with $u_{\frac{m+1}{2}}$, and v_3 is connected with $u_{\frac{m+3}{2}}$. Then we obtain an embedding Π_1 on the sphere. For example, Π_1 is shown in Figure 14 if $m = 15$.

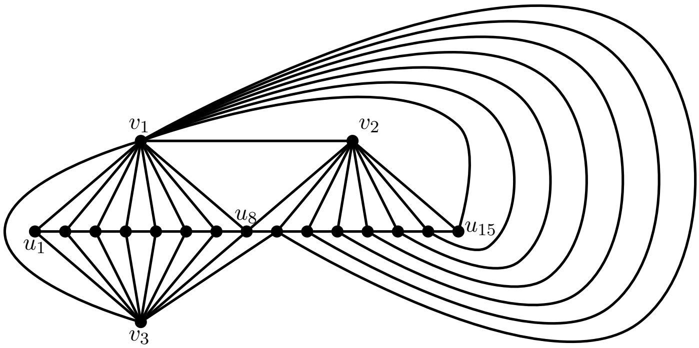

Figure 14: The embedding Π_1.

(2) As in (2) in Case 1, $\frac{m-3}{4}$ tubes are added to the sphere satisfying the following conditions:

(i) u_1 is connected with u_m,

(ii) v_2 is connected with each of $u_1, u_2, \ldots, u_{\frac{m-3}{2}}$,

(iii) v_3 is connected with each of $u_{\frac{m+5}{2}}, u_{\frac{m+7}{2}}, \ldots, u_m$.

Let Π_2 be the obtained embedding. Then it is an embedding on the surface of the genus $\frac{m-3}{4}$.

(3) The path $P = v_4v_5 \ldots v_n$ is now placed in the interior of $v_2u_{\frac{m+1}{2}}u_{\frac{m+3}{2}}v_2$. Then each of $u_{\frac{m+1}{2}}$ and $u_{\frac{m+3}{2}}$ joins to each of $v_4, v_5, \ldots, v_n$. For $j = 1, 2, \ldots, \lceil\frac{n-2}{4}\rceil$, let $D_j = u_{\frac{m+1}{2}}v_{4i}v_{4i+1}u_{\frac{m+1}{2}}$. If $n \equiv 1 \pmod 4$, then $\frac{n-1}{4}$ $(= \lceil\frac{n-2}{4}\rceil)$ tubes $T_1', T_2', \ldots, T_{\frac{n-1}{4}}'$ are added to the present surface one by one such that $u_{\frac{m+1}{2}}v_5$ is redrawn on T_1', and $u_{\frac{m+1}{2}}v_{4i}$ and $u_{\frac{m+1}{2}}v_{4i+1}$ are redrawn on T_i' for $i = 2, 3, \ldots, \frac{n-1}{4}$. If $n \equiv 3 \pmod 4$, then $\frac{n+1}{4}$ $(= \lceil\frac{n-2}{4}\rceil)$ tubes $T_1', T_2', \ldots, T_{\frac{n+1}{4}}'$ are added to the present surface one by one such that $u_{\frac{m+1}{2}}v_4$ is drawn on T_1', and $u_{\frac{m+1}{2}}v_{4i+3}$ and $u_{\frac{m+1}{2}}v_{4i}$ are redrawn on T_i' for $i = 2, 3, \ldots, \frac{n+1}{4}$. As in Case 1, there is a facial walk $W_{\lceil\frac{n-2}{4}\rceil}$ which contains $u_{\frac{m-1}{2}}, v_4, \ldots, v_n$ and v_2. Next, $u_{\frac{m-1}{2}}$ joins to v_j if it appears once in $W_{\lceil\frac{n-2}{4}\rceil}$ or a copy of v_j if it appears more than once in $W_{\lceil\frac{n-2}{4}\rceil}$, where v_j is a vertex in $v_4, v_5, \ldots, v_n$ and v_2. Let Π_3 be the obtained embedding. Then it is an embedding on the surface of the genus $\frac{m-3}{4} + \lceil\frac{n-2}{4}\rceil$.

(4) In this step we proceed the similar argument as in (4) and (5) in Case 1. There are $\frac{(m-3)(n-3)}{4}$ tubes being added to the present surface. The detail is omitted here. Let Π_4 be the obtained embedding. Then it is an embedding of $C_m + K_n$ on the surface of genus $\frac{m-3}{4} + \lceil\frac{n-2}{4}\rceil + \frac{(m-3)(n-3)}{4}$. Need to say that for the purpose that each of v_1, v_2 and v_3 is connected with $v_4, \ldots, v_n$, we need add at least $6(\frac{n-3}{2})^2$ tubes. Since each of $u_{\frac{m-1}{2}}$, $u_{\frac{m+1}{2}}$ and $u_{\frac{m+3}{2}}$ has been connected with each of $v_4, \ldots, v_n$, m is at least $3 + 6(n-3)$ $(= 6n - 15)$.

Since $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 2$, we have that $\lceil \frac{(m-2)(n-2)}{4} \rceil = \frac{m-3}{4} + \lceil \frac{n-2}{4} \rceil + \frac{(m-3)(n-3)}{4}$. So Π_4 is an embedding of $C_m + K_n$ on the surface of genus $\lceil \frac{(m-2)(n-2)}{4} \rceil$. □

4 An upper bound for $\gamma(C_m + K_n)$ if m is even

In the section we shall study the orientable genus of $C_m + K_n$ if m is even.

Lemma 4.1. *Suppose that* $m \equiv 0 \pmod 2$. *If* $m \geq 8$, *then*

$$\gamma(C_m + K_4) \leq \left\lceil \frac{m-2}{2} \right\rceil.$$

Proof. We firstly construct an embedding on a sphere. C_m, v_1, and v_2 are placed in the sphere as in the proof of Lemma 3.1, and each of v_1 and v_2 joins to $u_1, u_2, \ldots, u_n$. Let $F_1 = v_1u_1u_2v_1$ and $F_2 = v_2u_3u_4v_2$. Next, the vertex v_3 is placed in the interior of F_1 and is connected with to u_1, u_2, and v_1, and the vertex v_4 is placed in the interior of F_2 and is connected with u_3, u_4, and v_2. At last, the tube T_1 is added between the facial cycle $v_3u_1u_2v_3$ and the facial cycle $v_4u_3u_4v_4$. Then six edges are drawn on T_1 in the way shown in (1) of Figure 15.

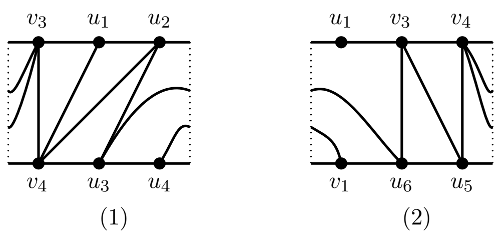

Figure 15: Two drawings of edges on T_1 or T_2.

Note that there are two edges connecting u_2 and u_3. Let $F_3 = v_1u_2u_3v_1$ and $F_4 = v_2u_2u_3v_2$. We now delete the edge u_2u_3 which is a common edge of F_3 and F_4. Then F_3 and F_4 are merged into a facial cycle $F_5 = v_1u_2v_2u_3v_1$. Next, the edge v_1v_2 is drawn in the interior of F_5.

Let $F_6 = u_1v_3v_4u_1$ (refer to (1) of Figure 15), and let $F_7 = v_1u_5u_6v_1$. The tube T_2 is now added between F_6 and F_7. Then the five edges are drawn on T_2 in the way shown in (2) in Figure 15. Let $F_8 = u_5v_3v_4u_5$ (refer to (2) of Figure 15), and let $F_9 = v_2u_8u_7v_2$. Then the tube T_3 is added between F_8 and F_9. Next, the five edges v_3u_8, v_3u_7, v_4u_7, v_4u_8 and v_4v_2 are drawn on T_3 in the similar way to that in (2) in Figure 15. Thus, v_i is connected with v_j if $i \neq j$. If $m = 8$, there is nothing to do. If $m > 8$, let $\mathcal{F} = \{F' \mid F' = u_7v_3v_4u_7\}$, and let $\mathcal{Q} = \{Q_i \mid Q_i = v_1u_{7+2i}u_{8+2i}v_1, i = 1, 2, \ldots, \frac{m-8}{2}\}$. We apply the operation of adding $\frac{m-8}{2}$ tubes with respect to $\mathcal{F}$ and $\mathcal{Q}$ to realize an embedding of $C_m + K_4$. Thus, there are $\frac{m-8}{2} + 3$ $(= \frac{m-2}{2})$ tubes being used. Hence, $\gamma(C_m + K_4) \leq \lceil \frac{m-2}{2} \rceil$. □

Lemma 4.2. *Suppose that* $m \equiv 0 \pmod 2$ *and* $n \equiv 0 \pmod 2$. *If* $n \geq 6$ *and* $m \geq 4n - 4$, *then*

$$\gamma(C_m + K_n) \leq \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

Proof. We construct an embedding of $C_m + K_n$ in the following steps.

(1) The cycle C_m and vertices v_1,v_2 are placed in a sphere as in the proof of Lemma 3.1. Next, each of v_1 and v_2 joins to $u_1, u_2, \dots, u_m$. Let $F_1 = v_1u_1u_2v_1$ and $F_2 = v_1u_3u_4v_1$. The two vertices v_4 and v_6 are placed in the interior of F_1, and each of u_1 and u_2 joins to each of v_4 and v_6 such that there are two facial 4-cycles $F'_1 = u_1v_4u_2v_6u_1$ and $F'_2 = v_1u_1v_6u_2v_1$. The two vertices v_3 and v_5 are placed in the interior of F_2, and each of u_3 and u_4 joins to each of v_3 and v_5 such that there are two facial 4-cycle $F'_3 = u_3v_3u_4v_5u_3$ and $D'_1 = u_3u_4v_5u_3$. The path $P = v_7v_8 \dots v_n$ is placed in the interior of F'_2 such that v_7 is near to v_6. Next, each of u_1 and u_2 joins to each of $v_7, v_8, \dots, v_n$. The obtained embedding is denoted by Π_1.

(2) In the step each of u_1, u_2, u_3 and u_4 will be connected with each of $v_3, v_4, \dots, v_n$, and v_1 is connected with v_2. For the above purpose, the tube T_1 is firstly added between F'_1 and F'_3, and the five edges u_1v_5, u_2v_3, u_3v_4, u_4v_6 and u_2u_3 are drawn on T_1 in the way shown in (1) of Figure 16. Thus, there are two edges connecting u_2 and u_3. The edge u_2u_3 which is the common edge of facial cycles $v_1u_2u_3v_1$ and $v_2u_2u_3v_2$ is deleted. Then there is a facial cycle $F_3 = v_1u_2v_2u_3v_1$. Next, v_1 joins to v_2 in the interior of F_3. The tube T_2 is now added between the facial cycles $u_1v_4u_3v_5u_1$ and $u_2v_3u_4v_6u_2$ (refer to (1) in Figure 16), and the six edges u_1v_3, u_2v_5, u_3v_6, u_4v_4, v_3v_4 and v_5v_6 are drawn on T_2 in the way shown in (2) of Figure 16.

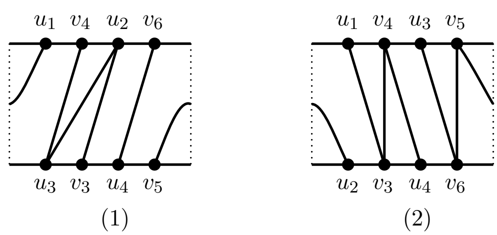

Figure 16: Two drawings of edges on T_1 or T_2.

For $i = 1, 2, \dots, \frac{n-6}{2}$, let $D_i = u_2v_{2i+5}v_{2i+6}u_2$. Let $\mathcal{D} = \{D_i \mid i = 1, 2, \dots, \frac{n-6}{2}\}$ and $\mathcal{D}' = \{D'_1\}$. We apply the operation of adding tubes with respect to $\mathcal{D}$ and $\mathcal{D}'$ such that both u_3 and u_4 are connected with each of $v_7, v_8, \dots, v_n$. By Lemma 2.11, there are $\frac{n-6}{2}$ tubes being used. Let Π_2 be the obtained embedding.

(3) We proceed a similar argument to that in (3) in the proof of Lemma 3.2. We shall add $\frac{(m-4)(n-2)}{4}$ tubes to the present surface to realize an embedding Π_3 of $C_m + K_n$. The detail is omitted here. For the purpose that each of v_1 and v_2 joins to each of $v_3, \dots, v_n$, $2(\frac{n-2}{2})^2$ tubes will be used by Lemma 2.1. So m is at least $4 + 4 \times \frac{n-2}{2}$ $(= 4n - 4)$.

Obviously, Π_3 is an embedding of $C_m + K_n$ on the surface of genus $2 + \frac{n-6}{2} + \frac{(m-4)(n-2)}{4}$. Since $m \equiv 0 \pmod 2$ and $n \equiv 0 \pmod 2$, we have that

$$\left\lceil \frac{(m-2)(n-2)}{4} \right\rceil = 2 + \frac{n-6}{2} + \frac{(m-4)(n-2)}{4}.$$

So $\gamma(C_m + K_n) \le \lceil \frac{(m-2)(n-2)}{4} \rceil$. □

Lemma 4.3. *Suppose that* $m \equiv 0 \pmod 2$ *and* $n \equiv 1 \pmod 2$. *If* $m \geq 6n - 14$ *and* $n \geq 5$, *then*

$$\gamma(C_m + K_n) \leq \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

Proof. We proceed a similar argument to that in the proof of Lemma 3.3.

(1) Let $P_m = u_1u_2\ldots u_m$. Then P_m, v_1, v_2, and v_3 are placed in a sphere as in (1) in the proof of Lemma 3.3. If $m \equiv 0 \pmod 4$, then each of v_1 and v_3 joins to each of $u_1, u_2, \ldots, u_{\frac{m}{2}}$ such that v_1u_i and v_3u_i are in the upper side and lower side of P_m, respectively. Next, each of v_2 and v_1 joins to each of $u_{\frac{m+2}{2}}, u_{\frac{m+4}{2}}, \ldots, u_m$ such that v_2u_i and v_1u_i are in the upper side and lower side of P_m, respectively. Also, v_1 joins to v_3. If $m \equiv 2 \pmod 4$, then each of v_1 and v_3 joins to each of $u_1, u_2, \ldots, u_{\frac{m}{2}}$ such that v_1u_i and v_3u_i are in the upper side and lower side of P_m, respectively. Next, each of v_2 and v_1 joins to each of $u_{\frac{m+2}{2}}, u_{\frac{m+4}{2}}, \ldots, u_m$ such that v_2u_i and v_1u_i are in the upper side and lower side of P_m, respectively. Also, v_1 joins to v_3, v_2 joins to $u_{\frac{m}{2}}$, and v_3 joins to $u_{\frac{m+2}{2}}$. Let Π_1 be the obtained embedding on the sphere.

(2) As in (2) in the proof of Lemma 3.3, there are $\frac{m}{4}$ tubes being added to the sphere if $m \equiv 0 \pmod 4$, or there are $\frac{m-2}{4}$ tubes being added to the sphere if $m \equiv 2 \pmod 4$, such that each of v_2 and v_3 is connected with all rest vertices in $u_1, u_2, \ldots, u_m$. Also, u_1 is connected with u_m, and v_2 is connected with v_3. Need to say that $\lceil \frac{m-2}{4} \rceil = \frac{m}{4}$ if $m \equiv 0 \pmod 4$, or $\lceil \frac{m-2}{4} \rceil = \frac{m-2}{4}$ if $m \equiv 2 \pmod 4$. Thus, there are $\lceil \frac{m-2}{4} \rceil$ tubes being used in the above procedure.

(3) Let $P' = v_4v_5\ldots v_n$. If $m \equiv 0 \pmod 4$, then P' is placed in the facial cycle $v_1u_1u_2v_1$, and each of u_1 and u_2 is connected with $v_4, v_5, \ldots, v_n$. If $m \equiv 2 \pmod 4$, then P' is placed in the facial cycle $v_1u_{\frac{m}{2}}u_{\frac{m}{2}+1}v_1$, and each of $u_{\frac{m}{2}}$ and $u_{\frac{m}{2}+1}$ is connected with $v_4, v_5, \ldots, v_n$.

Let

$$\mathcal{X}_0 = \{Q_{0,i} \mid Q_{0,i} = u_2v_{2i+2}v_{2i+3}u_2,\ i = 1, 2, \ldots, \tfrac{n-3}{2}\} \text{ if } m \equiv 0 \pmod 4, \text{ or}$$
$$\mathcal{X}_0 = \{Q_{0,i} \mid Q_{0,i} = u_{\frac{m}{2}}v_{2i+2}v_{2i+3}u_{\frac{m}{2}},\ i = 1, 2, \ldots, \tfrac{n-3}{2}\} \text{ if } m \equiv 2 \pmod 4.$$

Let

$$\mathcal{Y}_0 = \{R_{0,i} \mid R_{0,i} = v_2u_{2i+1}u_{2i}v_2,\ i = 1, 2, \ldots, \tfrac{n-3}{2}\}, \text{ and}$$
$$\mathcal{Y}'_0 = \{R'_{0,i} \mid R'_{0,i} = v_3u_{m+1-2i}u_{m+2-2i}v_3,\ i = 1, 2, \ldots, \tfrac{n-3}{2}\}.$$

We apply the operation of adding $2(\frac{n-3}{2})^2$ tubes starting from $\mathcal{X}_0$, $\mathcal{Y}_0$ and $\mathcal{Y}'_0$. Next procedures are similar to that in (4) in the proof of Lemma 3.3. Eventually, we obtain an embedding of $C_m + K_n$ by adding $\frac{(m-2)(n-3)}{4}$ tubes. Note that for the purpose that each of v_1, v_2 and v_3 is connected with each of $v_4, v_5, \ldots, v_n$, we need to add at least $3 \times 2 \times \frac{n-3}{2}$ tubes by Lemma 2.10. Thus, $m \geq 6(n-3) + 2 + 2 = 6n - 14$ if $m \equiv 0 \pmod 4$, or $m \geq 6(n-3) + 2 = 6n - 16$ if $m \equiv 2 \pmod 4$.

Since $m \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$, $\lceil \frac{(m-2)(n-2)}{4} \rceil = \frac{(m-2)(n-3)}{4} + \lceil \frac{m-2}{4} \rceil$. Since $\frac{m}{4} = \lceil \frac{m-2}{4} \rceil$ if $m \equiv 0 \pmod 4$, or $\frac{m-2}{4} = \lceil \frac{m-2}{4} \rceil$ if $m \equiv 2 \pmod 4$, the obtained embedding is an embedding of $C_m + K_n$ on the surface of genus $\lceil \frac{(m-2)(n-2)}{4} \rceil$. □

5 Conclusions

Lemma 5.1 ([10]). *If $m \geq 2$ and $n \geq 2$, then*

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

Considering that $K_{m,n}$ is a subgraph of $C_m + K_n$, Theorem 5.2 follows from Lemmas 3.1, 3.2, and 3.3, Lemmas 4.1, 4.2, and 4.3, and Lemma 5.1.

Theorem 5.2. *Suppose that m and n are two integers. Then*

$$\gamma(C_m + K_n) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$$

if $n \geq 4$ and m, n satisfy one of the following conditions:

(1) $m \equiv 1 \pmod 2$, $n \equiv 0 \pmod 2$, *and* $m \geq 4n - 5$,

(2) $m \equiv 1 \pmod 2$, $n \equiv 1 \pmod 2$, *and* $m \geq 6n - 13$,

(3) $m \equiv 0 \pmod 2$, $n \equiv 0 \pmod 2$, *and* $m \geq 4n - 4$,

(4) $m \equiv 0 \pmod 2$, $n \equiv 1 \pmod 2$, *and* $m \geq 6n - 14$.

Obviously, the maximal value in $4n-5, 4n-4, 6n-13$ and $6n-14$ is 12 if $n = 4$, or $6n - 13$ if $n \geq 5$. The result below follows from Lemma 5.1 and Theorem 5.2 directly.

Corollary 5.3. *Suppose that m and n are two integers. Let G_1 be a spanning subgraph of C_m, and let G_2 be a spanning subgraph of K_n. If $n = 4$ and $m \geq 12$, or $n \geq 5$ and $m \geq 6n - 13$, then*

$$\gamma(G_1 + G_2) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

Since $K_{r,s,t}$ ($r \geq s \geq t \geq 3$) is a spanning subgraph of $C_r + K_{s+t}$, we have the following result by Theorem 5.2.

Corollary 5.4. *If $r \geq s \geq t \geq 3$ and $r \geq 6(s+t) - 13$, then*

$$\gamma(K_{r,s,t}) = \left\lceil \frac{(r-2)(s+t-2)}{4} \right\rceil.$$

Therefore, Stahl and White's conjecture ([12]) on the orientable genus of the complete tripartite graph $K_{r,s,t}$ holds if $r \geq s \geq t \geq 3$ and $r \geq 6(s+t) - 13$.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory*, volume 244 of *Graduate Texts in Mathematics*, Springer, New York, 2008, doi:10.1007/978-1-84628-970-5.

[2] D. L. Craft, *Surgical Techniques for Constructing Minimal Orientable Imbeddings of Joins and Compositions of Graphs*, Ph.D. thesis, Western Michigan University, 1991, `https://search.proquest.com/docview/303919820`.

[3] D. L. Craft, On the genus of joins and compositions of graphs, *Discrete Math.* **178** (1998), 25–50, doi:10.1016/s0012-365x(97)81815-8.

[4] G. A. Dirac, The coloring of maps, *J. London Math. Soc.* **28** (1953), 476–480, doi:10.1112/jlms/s1-28.4.476.

[5] M. N. Ellingham and D. C. Stephens, The orientable genus of some joins of complete graphs with large edgeless graphs, *Discrete Math.* **309** (2009), 1190–1198, doi:10.1016/j.disc.2007.12.098.

[6] T. Gallai, Kritische Graphen I, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **8** (1963), 165–192.

[7] T. Gallai, Kritische Graphen II, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **8** (1963), 373–395.

[8] V. P. Korzhik, Triangular embeddings of $K_n - K_m$ with unboundedly large m, *Discrete Math.* **190** (1998), 149–162, doi:10.1016/s0012-365x(98)00040-5.

[9] B. Mohar and C. Thomassen, *Graphs on Surfaces*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, Maryland, 2001.

[10] G. Ringel, Das Geschlecht des vollständigen paaren Graphen, *Abh. Math. Sem. Univ. Hamburg* **28** (1965), 139–150, doi:10.1007/bf02993245.

[11] G. Ringel, *Map Color Theorem*, volume 209 of *Die Grundlehren der mathematischen Wissenschaften*, Springer-Verlag, Heidelberg, 1974, doi:10.1007/978-3-642-65759-7.

[12] S. Stahl and A. T. White, Genus embeddings for some complete tripartite graphs, *Discrete Math.* **14** (1976), 279–296, doi:10.1016/0012-365x(76)90042-x.

[13] C. Thomassen, Color-critical graphs on a fixed surface, *J. Comb. Theory Ser. B* **70** (1997), 67–100, doi:10.1006/jctb.1996.1722.

[14] A. T. White, *Graphs, Groups and Surfaces*, volume 8 of *North-Holland Mathematics Studies*, North-Holland, Amsterdam, 1973, doi:10.1016/s0304-0208(08)x7063-x.