ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 11 (2016) 157–173

On factorisations of complete graphs into circulant graphs and the Oberwolfach problem

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Received 21 November 2014, accepted 5 August 2015, published online 3 October 2015

Abstract

Various results on factorisations of complete graphs into circulant graphs and on 2-factorisations of these circulant graphs are proved. As a consequence, a number of new results on the Oberwolfach Problem are obtained. For example, a complete solution to the Oberwolfach Problem is given for every 2-regular graph of order $2p$ where $p \equiv 5 \pmod 8$ is prime.

Keywords: Oberwolfach problem, graph factorisations, graph decompositions, 2-factorisations.

Math. Subj. Class.: 05C70, 05C51, 05B30

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1 Introduction

The Oberwolfach problem was posed by Ringel in the 1960s and is first mentioned in [16]. It concerns graph factorisations. A *factor* of a graph is a spanning subgraph and a *factorisation* is a decomposition into edge-disjoint factors. A factor that is regular of degree k is called a k-*factor*. If each factor of a factorisation is a k-factor, then the factorisation is called a k-*factorisation*, and if each factor is isomorphic to a given graph F, then we say it is a *factorisation into* F.

Let F be an arbitrary 2-regular graph and let n be the order of F. If n is odd, then the *Oberwolfach Problem* $\mathrm{OP}(F)$ asks for a 2-factorisation of K_n into F, and if n is even, then $\mathrm{OP}(F)$ asks for a 2-factorisation of $K_n - I$ into F, where $K_n - I$ denotes the graph obtained from K_n by removing the edges of a 1-factor.

The Oberwolfach Problem has been solved completely when F consists of isomorphic components [1, 3, 18], when F has exactly two components [29], when F is bipartite [5, 17] and in numerous special cases. See [7] for a survey of results up to 2006. It is known that there is no solution to $\mathrm{OP}(F)$ for $F \in \{C_3 \cup C_3, C_4 \cup C_5, C_3 \cup C_3 \cup C_5, C_3 \cup C_3 \cup C_3 \cup C_3\}$, but a solution exists for every other 2-regular graph of order at most 40 [13].

In [8], it was shown that the Oberwolfach Problem has a solution for every 2-regular graph of order $2p$ where p is any of the infinitely many primes congruent to $5 \pmod{24}$, and for every 2-regular graph whose order is in an infinite family of primes congruent to $1 \pmod{16}$. In this paper we extend these results as follows. We show that $\mathrm{OP}(F)$ has a solution for every 2-regular graph of order $2p$ where p is any prime congruent to $5 \pmod 8$ (see Theorem 4.2), and we obtain solutions to $\mathrm{OP}(F)$ for broad classes of 2-regular graphs in many other cases (see Theorems 4.3 and 4.4). We also obtain results on the generalisation of the Oberwolfach Problem to factorisations of complete multigraphs into isomorphic 2-factors (see Theorem 5.4). Our results are obtained by constructing various factorisations of complete graphs into circulant graphs in Section 2, and then showing in Section 3 that these circulant graphs can themselves be factored into isomorphic 2-regular graphs in a wide variety of cases.

2 Factorising complete graphs into circulant graphs

Let $G = (G, \cdot)$ be a finite group with identity e and let S be a subset of G such that $e \notin S$ and $s \in S$ implies $s^{-1} \in S$. The *Cayley graph on* G *with connection set* S, denoted $\mathrm{Cay}(G\,;S)$, has the elements of G as its vertices and g is adjacent to $g \cdot s$ for each $s \in S$ and each $g \in G$. A Cayley graph on a cyclic group is called a *circulant graph*. We use the following standard notation. The ring of integers modulo n is denoted by $\mathbb{Z}_n$, the multiplicative group of units modulo n is denoted by $\mathbb{Z}_n^*$ and, when b divides $|\mathbb{Z}_n^*|$, the subgroup $\{x^b : x \in \mathbb{Z}_n^*\}$ of index b in $\mathbb{Z}_n^*$ is denoted by $(\mathbb{Z}_n^*)^b$.

In this section we consider factorisations of K_n for n odd (in Section 2.1) and of $K_n - I$ for n even (in Section 2.2) into circulant graphs. A 2-regular graph is a circulant if and only if its components are all isomorphic. Thus, for each 2-regular circulant graph F, there exists a factorisation of K_n (if F has odd order) or of $K_n - I$ (if F has even order) into F; except that there is no such factorisation when $F \in \{C_3 \cup C_3, C_3 \cup C_3 \cup C_3 \cup C_3\}$. Considerably less is known for factorisations into circulant graphs of degree greater than 2. Some factorisations into $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1, 2\})$ and $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1, 2, 3, 4\})$ are given in [4] and [8] respectively, and some further results, including results on self-complementary and almost self-complementary circulant graphs, appear in [2, 14, 15, 26].

2.1 Factorising complete graphs of odd order

In this subsection we will construct factorisations of complete graphs of odd order into isomorphic circulant graphs by finding certain partitions of cyclic groups. Problems concerning such partitions have been well studied, for example see [28], and existing results overlap with some of the results in this subsection. In particular, Theorem 2.3 below is a consequence of Lemma 3.1 of [24].

Lemma 2.1. *Let s be an integer, let $p \equiv 1 \pmod{2s}$ be prime, and let $S = \pm\{d_1, d_2, \dots, d_s\} \subseteq \mathbb{Z}_p^*$. Further, suppose a and b are integers such that $2abs = p-1$, let $G = (\mathbb{Z}_p^*)^b$, and let $H = (\mathbb{Z}_p^*)^{bs}$. If $d_1, d_2, \dots, d_s$ represent the s distinct cosets of G/H, then there exists a $2s$-factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; S)$.*

Proof. For each $x \in \mathbb{Z}_p$ let $xS = \{xy : y \in S\}$. Since p is prime, $\mathrm{Cay}(\mathbb{Z}_p\,; xS) \cong \mathrm{Cay}(\mathbb{Z}_p\,; S)$ for any $x \in \mathbb{Z}_p \setminus \{0\}$. If there is a partition of $\mathbb{Z}_p^*$ into sets $x_1S, x_2S, \dots, x_{ab}S$ where $x_i \in \mathbb{Z}_p \setminus \{0\}$ for $i = 1, 2, \dots, ab$, then $\{\mathrm{Cay}(\mathbb{Z}_p\,; x_iS) : i = 1, 2, \dots, ab\}$ is the required $2s$-factorisation of K_p. We now present such a partition.

Let ω be a generator of $\mathbb{Z}_p^*$. Thus, $H = \omega^0, \omega^{bs}, \omega^{2bs}, \dots, \omega^{(2a-1)bs}$, and $\omega^{abs} = -1 \in H$. Let $A = \omega^0, \omega^{bs}, \omega^{2bs}, \dots, \omega^{(a-1)bs}$, so that $H = A \cup -A$ (A is a set of representatives for the cosets in H of the order 2 subgroup of H). Since $d_1, d_2, \dots, d_s$ represent distinct cosets of G/H, it is easy to see that $\{xS : x \in A\}$ is a partition of G. Thus, if B is a set of representatives for the cosets of $\mathbb{Z}_p^*/G$, then $\{xyS : x \in A, y \in B\}$ is the required partition of $\mathbb{Z}_p^*$. □

Note that upon putting $s = 1$ in Lemma 2.1 we obtain the Hamilton decomposition

$$\{\mathrm{Cay}(\mathbb{Z}_p\,; \{\pm 1\}), \mathrm{Cay}(\mathbb{Z}_p\,; \{\pm 2\}), \dots, \mathrm{Cay}(\mathbb{Z}_p\,; \{\pm \tfrac{p-1}{2}\})\}$$

of K_p. We will be mostly interested in applications of Lemma 2.1 where the connection set S is $\pm\{1,2\}$, $\pm\{1,2,3\}$, $\pm\{1,3,4\}$ or $\pm\{1,2,3,4\}$. The factorisations given by Lemma 2.1 have the property that each factor is invariant under the action of $\mathbb{Z}_p$. It is worth mentioning that for $S \in \{\pm\{1,2\}, \pm\{1,2,3\}, \pm\{1,3,4\}, \pm\{1,2,3,4\}\}$, the construction given in Lemma 2.1 yields every $2s$-factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; S)$ with this property. This follows from the results in [9] and [22], together with Turner's result [30] that for p prime $\mathrm{Cay}(\mathbb{Z}_p\,; S) \cong \mathrm{Cay}(\mathbb{Z}_p\,; S')$ if and only if there exists an $\alpha \in \mathbb{Z}_p^*$ such that $S' = \alpha S$.

Theorem 2.2. *If $p \equiv 1 \pmod 4$ is prime and 4 divides the order of k in $\mathbb{Z}_p^*$, then there is a factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1, k\})$.*

Proof. Apply Lemma 2.1 with $S = \pm\{1, k\}$ taking G to be the subgroup of $\mathbb{Z}_p^*$ generated by k, and H to be the index 2 subgroup of G. □

Theorem 2.3. *If $p \equiv 1 \pmod 6$ is prime such that $2, 3 \notin (\mathbb{Z}_p^*)^3$ and $6 \in (\mathbb{Z}_p^*)^3$, then there is a factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1,2,3\})$.*

Proof. It follows from $2, 3 \notin (\mathbb{Z}_p^*)^3$ and $6 \in (\mathbb{Z}_p^*)^3$ that 1, 2 and 3 represent the three cosets of $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^3$. Thus, we obtain the required factorisation by applying Lemma 2.1 with $b = 1$. □

Theorem 2.4. *If $p \equiv 1 \pmod 6$ is prime such that $2, 3, 6 \notin (\mathbb{Z}_p^*)^3$, then there is a factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1,3,4\})$.*

Proof. It follows from $2, 3, 6 \notin (\mathbb{Z}_p^*)^3$ that 1, 3 and 4 represent the three cosets of $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^3$. Thus, we obtain the required factorisation by applying Lemma 2.1 with $b = 1$. □

The primes less than 1000 to which Theorem 2.3 applies are

$$7, 37, 139, 163, 181, 241, 313, 337, 349, 379, 409, 421, 541, 571, 607, 631, 751, 859, 877, 937,$$

and the primes less than 1000 to which Theorem 2.4 applies are

$$13, 19, 79, 97, 199, 211, 331, 373, 463, 487, 673, 709, 769, 823, 829, 883, 907.$$

In the next theorem we show that there are infinitely many primes to which Theorem 2.3 applies, and also infinitely many primes to which Theorem 2.4 applies.

Theorem 2.5. *There are infinitely many values of p such that p is prime, $p \equiv 1 \pmod 6$, $2, 3 \notin (\mathbb{Z}_p^*)^3$ and $6 \in (\mathbb{Z}_p^*)^3$, and there are infinitely many values of p such that p is prime, $p \equiv 1 \pmod 6$ and $2, 3, 6 \notin (\mathbb{Z}_p^*)^3$.*

Proof. Assume $p \equiv 1 \pmod 6$. Let $\mathbb{F}_p$ be the field with p elements. We use standard definitions and results from algebraic number theory, as found in [20]. The result essentially follows from the Chebotarev Density Theorem.

Let ω be a primitive cube root of unity, $\lambda = \sqrt[3]{2}$ be a cube root of 2 and $\rho = \sqrt[3]{3}$ a cube root of 3. Consider the following tower of fields:

$$M = \mathbb{Q}(\omega, \lambda, \rho) \supseteq L = \mathbb{Q}(\omega, \lambda) \supseteq K = \mathbb{Q}(\omega) \supseteq \mathbb{Q}.$$

Let $\mathbb{O}_K$, $\mathbb{O}_L$ denote the rings of integers of K and L respectively. We may ignore the finitely many ramified primes. Thus let p be a prime number, sufficiently large that it is unramified in M, let $\mathfrak{p}$ be a prime in K extending p and $\mathfrak{P}$ a prime in L extending $\mathfrak{p}$. Let $\mathbb{K} = \mathbb{O}_K/\mathfrak{p}$ and $\mathbb{L} = \mathbb{O}_L/\mathfrak{P}$ be the residue fields. We view $\mathbb{K}$ as embedded in $\mathbb{L}$ via the map $x + \mathfrak{p} \mapsto x + \mathfrak{P}$. As $p \equiv 1 \pmod 6$, p splits in K and $\mathbb{K} = \mathbb{O}_K/\mathfrak{p} \simeq \mathbb{F}_p$.

Since M and L are splitting fields, M/K and L/K are Galois extensions. The Galois group of M/K is $\mathsf{Gal}(M/K) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ generated by the maps $\alpha\colon \lambda \mapsto \lambda\omega$ and $\beta\colon \rho \mapsto \rho\omega$. The *Frobenius map* of $\mathbb{L}/\mathbb{K}$ is the map $x \mapsto x^{|\mathbb{L}|}$. The *Frobenius element* $\sigma_{\mathfrak{p}}^L$ is the element of $\mathsf{Gal}(L/K)$ inducing the Frobenius map on $\mathbb{L}/\mathbb{K}$. (A priori $\sigma_{\mathfrak{p}}^L$ could also depend on the choice of $\mathfrak{P}$ extending $\mathfrak{p}$, but this is not the case since Gal(L/K) is abelian; see [20, III.2.1].) Define $\sigma_{\mathfrak{p}}^M \in \mathsf{Gal}(M/K)$ analogously. Then $\sigma_{\mathfrak{p}}^L$ is the restriction of $\sigma_{\mathfrak{p}}^M$ to L by [20, III.2.3].

By definition of $\mathbb{L}$, for all sufficiently large $p \equiv 1 \pmod 6$, $2 \in (\mathbb{Z}_p^*)^3$ if and only if $\mathbb{L} = \mathbb{K}$. But $\mathbb{L} = \mathbb{K}$ if and only if $\sigma_{\mathfrak{p}}^L$ is the identity map, and it follows that $2 \in (\mathbb{Z}_p^*)^3$ if and only if $\sigma_{\mathfrak{p}}^M \in \langle\beta\rangle$. Similarly, $3 \in (\mathbb{Z}_p^*)^3$ if and only if $\sigma_{\mathfrak{p}}^M \in \langle\alpha\rangle$ and $6 \in (\mathbb{Z}_p^*)^3$ if and only if $\sigma_{\mathfrak{p}}^M \in \langle\alpha\beta\rangle$. In summary:

$$\begin{array}{lcl} 2, 3 \notin (\mathbb{Z}_p^*)^3,\ 6 \in (\mathbb{Z}_p^*)^3 & \Longleftrightarrow & \sigma_{\mathfrak{p}}^M \in \{\alpha\beta, \alpha^2\beta^2\}. \\ 2, 3, 6 \notin (\mathbb{Z}_p^*)^3 & \Longleftrightarrow & \sigma_{\mathfrak{p}}^M \in \{\alpha^2\beta, \alpha\beta^2\}. \end{array}$$

The Chebotarev Density Theorem [20, V.10.4] implies that for each $\theta \in \mathsf{Gal}(M/K)$, the set of primes $\mathfrak{p}$ of K (unramified in M) for which $\sigma_{\mathfrak{p}}^M = \theta$ is infinite. Thus each of the two conditions for $\sigma_{\mathfrak{p}}^M$ displayed above holds infinitely often. □

It is possible to describe the primes in Theorem 2.5 more explicitly. Given $p \equiv 1 \pmod 6$, factoring the ideal $p\mathbb{O}_K$ and taking norms, one shows there exist unique $c, d \in$

$\mathbb{Z}$ with $d > 0$, $\gcd(c,d) = 1$, $c \equiv 2 \pmod 3$ and $4p = (2c - 3d)^2 + 27d^2$. Let $t(p) = (c \pmod 6,\ d \pmod 6)$. There are 9 possible values for $t(p)$: $(2,1)$, $(2,3)$, $(2,5)$, $(5,0)$, $(5,1)$, $(5,2)$, $(5,3)$, $(5,4)$ and $(5,5)$. The Chebotarev density theorem implies that each of the 9 possible $t(p)$ values occurs "equally often" (that is, for a subset of the primes $p \equiv 1 \pmod 6$ of relative density $1/9$). Using cubic reciprocity [19, Ch. 9] one calculates that $2, 3 \notin (\mathbb{Z}_p^*)^3$ and $6 \in (\mathbb{Z}_p^*)^3$ if and only if $t(p) = (2,1)$ or $(5,5)$, while $2,3,6 \notin (\mathbb{Z}_p^*)^3$ if and only if $t(p) = (2,5)$ or $(5,1)$. Each case occurs for $2/9$ of the primes that are $1 \pmod 6$.

The above applications of Lemma 2.1 have all been with $b = 1$. We note however that the conditions of Lemma 2.1 are never satisfied when $S = \pm\{1,2,3,4\}$ and $b = 1$. This is because 2 is a quadratic residue when $p \equiv 1 \pmod 8$, which means that both 1 and 4 are in H. The factorisations of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1,2,3,4\})$ in [8] were obtained by applying Lemma 2.1 with $b = 2$ so that G and H have index 2 and 8, respectively, in $\mathbb{Z}_p^*$. Another example where Lemma 2.1 can be applied with $b \neq 1$ is when $p = 919$, $S = \pm\{1,2,3\}$, $a = 51$ and $b = 3$. This yields a factorisation of K_{919} into $\mathrm{Cay}(\mathbb{Z}_{919}\,; \pm\{1,2,3\})$. Such a factorisation cannot be obtained by applying Lemma 2.1 with $b = 1$ because 1, 2 and 3 are all cubes in $\mathbb{Z}_{919}^*$.

The following lemma can be used to obtain factorisations of K_p, for certain values of p, in which some of the factors are isomorphic to $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1,2,3\})$ and the others are isomorphic to $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1,2,3,4\})$.

Lemma 2.6. *Let p be prime, let H be the subgroup of $\mathbb{Z}_p^*$ generated by $\{-1,6\}$, and let d be the order of $2H$ in $\mathbb{Z}_p^*/H$. If there exist nonnegative integers α and β such that $d = 3\alpha + 4\beta$, then there is a factorisation of K_p into $\frac{\alpha(p-1)}{2d}$ copies of $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1,2,3\})$ and $\frac{\beta(p-1)}{2d}$ copies of $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1,2,3,4\})$.*

Proof. It is sufficient to partition $\mathbb{Z}_p^*$ into $\frac{\alpha(p-1)}{2d}$ 6-tuples of the form $\pm\{x,2x,3x\}$ and $\frac{\beta(p-1)}{2d}$ 8-tuples of the form $\pm\{x,2x,3x,4x\}$. Since $d = 3\alpha + 4\beta$, there is a partition

$$\begin{array}{l}\{\{2^{r_i-1}H, 2^{r_i}H, 2^{r_i+1}H\} : i = 1,\dots,\alpha\} \cup \\ \{\{2^{r_i-1}H, 2^{r_i}H, 2^{r_i+1}H, 2^{r_i+2}H\} : i = \alpha+1,\dots,\alpha+\beta\}\end{array}$$

of $\{H, 2H, \dots, 2^{d-1}H\}$. But $6 \in H$ implies $2^{r_i-1}H = 3 \cdot 2^{r_i}H$ for $i = 1,2,\dots,\alpha+\beta$. Thus, we can rewrite our partition of $\{H, 2H, \dots, 2^{d-1}H\}$ as

$$\{\{H_i, 2H_i, 3H_i\} : i = 1,\dots,\alpha\} \cup \{\{H_i, 2H_i, 3H_i, 4H_i\} : i = \alpha+1,\dots,\alpha+\beta\},$$

where $H_i = 2^{r_i}H$ for $i = 1,\dots,\alpha+\beta$.

Since $-1 \in H$, for $i = 1,\dots,\alpha$, $H_i \cup 2H_i \cup 3H_i$ can be partitioned into $\frac{|H|}{2}$ 6-tuples of the form $\pm\{x,2x,3x\}$, and for $i = \alpha+1,\dots,\alpha+\beta$, $H_i \cup 2H_i \cup 3H_i \cup 4H_i$ can be partitioned into $\frac{|H|}{2}$ 8-tuples of the form $\pm\{x,2x,3x,4x\}$. If $\mathcal{R}$ is the set of all $\alpha\frac{|H|}{2}$ of these 6-tuples and $\mathcal{S}$ is the set of all $\beta\frac{|H|}{2}$ of these 8-tuples, then $\mathcal{R} \cup \mathcal{S}$ is a partition of the subgroup $G = H \cup 2H \cup \cdots \cup 2^{d-1}H$ of $\mathbb{Z}_p^*$. Thus, if $g_1, g_2, \dots, g_t$ ($t = \frac{p-1}{d|H|}$) represent the cosets of $\mathbb{Z}_p^*/G$, then

$$\{g_iR : R \in \mathcal{R}, i = 1,\dots,t\} \cup \{g_iS : S \in \mathcal{S}, i = 1,\dots,t\}$$

is a partition of $\mathbb{Z}_p^*$ into $t\alpha\frac{|H|}{2} = \frac{\alpha(p-1)}{2d}$ 6-tuples of the form $\pm\{x, 2x, 3x\}$ and $t\beta\frac{|H|}{2} = \frac{\beta(p-1)}{2d}$ 8-tuples of the form $\pm\{x, 2x, 3x, 4x\}$. This is the required partition of $\mathbb{Z}_p^*$. □

Notice that any 6-factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1, 2, 3\})$ given by Lemma 2.1 can also be obtained via Lemma 2.6. For if $1, 2, 3$ represent the three distinct cosets of G/H (where $G = (\mathbb{Z}_p^*)^b$ and $H = (\mathbb{Z}_p^*)^{3b}$, and $p - 1 = 6ab$), then it follows that $\{-1, 6\} \subseteq H$ and $2H$ has order 3 in G/H. This means that if H' is the subgroup of $\mathbb{Z}_p^*$ generated by $\{-1, 6\}$, then $H' \leq H$ and 3 divides the order d of $2H'$ in $\mathbb{Z}_p^*/H'$. Thus, we can obtain our 6-factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1, 2, 3\})$ by applying Lemma 2.6 with $\alpha = \frac{d}{3}$ and $\beta = 0$. Similarly, any 8-factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1, 2, 3, 4\})$ given by Lemma 2.1 can be obtained by applying Lemma 2.6 with $\alpha = 0$ and $\beta = \frac{d}{4}$.

However, Lemma 2.6 gives us additional factorisations such as the following. When $p = 101$ we have $H = \pm\{1, 6, 14, 17, 36\}$, and $2H$ has order $d = 10$ in $\mathbb{Z}_p^*/H$. Taking $\alpha = 2$ and $\beta = 1$, we obtain a factorisation of K_{101} into 10 copies of $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1, 2, 3\})$ and 5 copies of $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1, 2, 3, 4\})$. Of course, 101 is neither $1 \pmod 6$ nor $1 \pmod 8$, so there is neither a 6-factorisation nor an 8-factorisation of K_{101}.

2.2 Factorising complete graphs of even order

In this section we construct factorisations of $K_{2p} - I$ where the factors are all isomorphic to $\mathrm{Cay}(\mathbb{Z}_{2p}\,; \pm\{1, 2\})$ or all isomorphic to $\mathrm{Cay}(\mathbb{Z}_{2p}\,; \pm\{1, 2, 3, 4\})$. We do this by considering $K_{2p} - I$ as a Cayley graph on a dihedral group and partitioning its connection set to generate the factors. The dihedral group D_{2p} of order $2p$ has elements $r_0, r_1, r_2, \ldots, r_{p-1}, s_0, s_1, s_2, \ldots, s_{p-1}$ and satisfies

$$r_i \cdot r_j = r_{i+j}, \quad r_i \cdot s_j = s_{i+j}, \quad s_i \cdot r_j = s_{i-j}, \quad s_i \cdot s_j = r_{i-j}$$

where arithmetic of subscripts is carried out modulo p.

Lemma 2.7. *If $p \geq 3$ is prime, then*

$$\mathrm{Cay}(D_{2p}\,; \{r_{\pm i}, s_j, s_{i+j}\}) \cong \mathrm{Cay}(\mathbb{Z}_{2p}\,; \pm\{1, 2\})$$

for all $i \in \mathbb{Z}_p \setminus \{0\}$ and all $j \in \mathbb{Z}_p$.

Proof. An isomorphism is given by

$$\begin{array}{cccccccccccc} r_0 & r_i & r_{2i} & r_{3i} & \cdots & r_{(p-1)i} & s_j & s_{i+j} & s_{2i+j} & s_{3i+j} & \cdots & s_{(p-1)i+j} \\ \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ 0 & 2 & 4 & 6 & \cdots & 2p-2 & 2p-1 & 1 & 3 & 5 & \cdots & 2p-3 \end{array}$$

□

Lemma 2.8. *If $p \geq 5$ is prime, then*

$$\mathrm{Cay}(D_{2p}\,; \{r_{\pm i}, r_{\pm 2i}, s_j, s_{i+j}, s_{2i+j}, s_{3i+j}\}) \cong \mathrm{Cay}(\mathbb{Z}_{2p}\,; \pm\{1, 2, 3, 4\})$$

for all $i \in \mathbb{Z}_p \setminus \{0\}$ and all $j \in \mathbb{Z}_p$.

Proof. An isomorphism is given by

$$\begin{array}{cccccccccccc} r_0 & r_i & r_{2i} & r_{3i} & \dots & r_{(p-1)i} & s_j & s_{i+j} & s_{2i+j} & s_{3i+j} & \dots & s_{(p-1)i+j} \\ \downarrow & \downarrow & \downarrow & \downarrow & \dots & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \dots & \downarrow \\ 0 & 2 & 4 & 6 & \dots & 2p-2 & 2p-3 & 2p-1 & 1 & 3 & \dots & 2p-5 \end{array}$$

□

Theorem 2.9. *For each odd prime p, there is a factorisation of* $K_{2p} - I$ *into* $\mathrm{Cay}(\mathbb{Z}_{2p}\,;\pm\{1, 2\})$.

Proof. The required factorisation is $\mathcal{F} = \{X_i : i \in \mathbb{Z}_p \setminus \{0\}\}$ where

$$X_i = \mathrm{Cay}(D_{2p}\,; \{r_{\pm 2i}, s_i, s_{-i}\})$$

for $i \in \mathbb{Z}_p \setminus \{0\}$. Note that $X_i = X_{-i}$ so $|\mathcal{F}| = \frac{p-1}{2}$ as required. Lemma 2.7 guarantees that $X_i \cong \mathrm{Cay}(\mathbb{Z}_{2p}\,;\pm\{1, 2\})$ for each $i \in \mathbb{Z}_p \setminus \{0\}$. Also, r_0 is the identity of D_{2p} and each element of $D_{2p} \setminus \{r_0, s_0\}$ occurs in exactly one X_i. Thus, $\mathcal{F}$ is a factorisation of $\mathrm{Cay}(D_{2p}\,; D_{2p} \setminus \{r_0, s_0\}) \cong K_{2p} - I$ where the 1-factor I is $\mathrm{Cay}(D_{2p}\,; \{s_0\})$. □

Following work of Davenport [10, Theorem 5] and Weil, a special case of a result due to Moroz [23] yields the following. If $p \equiv 1 \pmod 4$ is prime and $p > 8 \times 10^6$, then there exists an integer x such that x, $x + 1$, $x + 2$, $x + 3$ represent all four distinct cosets of $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^4$. A computer search using PARI/GP [25] verifies in a few minutes that such an x also exists for all $p < 8 \times 10^6$ with $p \equiv 1 \pmod 4$, with the exceptions $p = 13$ and $p = 17$. Thus, we have the following result.

Lemma 2.10. *If* $p \equiv 1 \pmod 4$ *is prime with* $p \notin \{13, 17\}$*, then there exists an* $x \in \mathbb{Z}_p^*$ *such that* x, $x + 1$, $x + 2$ *and* $x + 3$ *represent all four distinct cosets of* $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^4$.

Theorem 2.11. *If* $p \equiv 5 \pmod 8$ *is prime, then there is a factorisation of* $K_{2p} - I$ *into* $\mathrm{Cay}(\mathbb{Z}_{2p}\,;\pm\{1, 2, 3, 4\})$*; except that there is no factorisation of* $K_{26} - I$ *into* $\mathrm{Cay}(\mathbb{Z}_{2p}\,;\pm\{1, 2, 3, 4\})$.

Proof. We first observe that there is no factorisation of $K_{26} - I$ into graph $\mathrm{Cay}(\mathbb{Z}_{2p}\,;\pm\{1, 2, 3, 4\})$. If such a factorisation exists, then we can assume without loss of generality that the vertex set is $\mathbb{Z}_{26}$ and that $\mathrm{Cay}(\mathbb{Z}_{26}\,;\pm\{1, 2, 3, 4\})$ is a factor. But no edge of $\mathrm{Cay}(\mathbb{Z}_{26}\,;\pm\{7\})$ (for example) occurs in a complete subgraph of order 5 in $\mathrm{Cay}(\mathbb{Z}_{26}\,;\pm\{5, 6, 7, 8, 9, 10, 11, 12, 13\})$. Since $\mathrm{Cay}(\mathbb{Z}_{26}\,;\pm\{1, 2, 3, 4\})$ contains a complete subgraph of order 5, it follows that there is no factorisation of $K_{26} - I$ into graph $\mathrm{Cay}(\mathbb{Z}_{2p}\,;\pm\{1, 2, 3, 4\})$.

Let $p \equiv 5 \pmod 8$ be prime with $p \neq 13$. By Lemma 2.10, there exists an $x \in \mathbb{Z}_p^*$ such that x, $x + 1$, $x + 2$ and $x + 3$ represent all four distinct cosets of $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^4$. By Lemma 2.8,

$$\mathrm{Cay}(D_{2p}\,; \{r_{\pm 1}, r_{\pm 2}, s_x, s_{x+1}, s_{x+2}, s_{x+3}\}) \cong \mathrm{Cay}(\mathbb{Z}_{2p}\,;\pm\{1, 2, 3, 4\}).$$

Now let $H = (\mathbb{Z}_p^*)^4$ act on the subscripts of the connection set $\{r_{\pm 1}, r_{\pm 2}, s_x, s_{x+1}, s_{x+2}, s_{x+3}\}$ and consider the collection $S_1, S_2, \dots, S_{\frac{p-1}{4}}$ of subsets of D_{2p} thus formed.

We show that $\{\mathrm{Cay}(D_{2p}\,; S_i) : i = 1, 2, \ldots, \frac{p-1}{4}\}$ is a factorisation of $K_{2p} - I$ into $\mathrm{Cay}(\mathbb{Z}_{2p}\,; \pm\{1, 2, 3, 4\})$. If $h \in H$, then

$$\mathrm{Cay}(D_{2p}\,; \{r_{\pm h}, r_{\pm 2h}, s_{hx}, s_{h(x+1)}, s_{h(x+2)}, s_{h(x+3)}\}) \cong \mathrm{Cay}(\mathbb{Z}_{2p}\,; \pm\{1, 2, 3, 4\})$$

by Lemma 2.8 (indeed this is true for any $h \in \mathbb{Z}_p^*$) so it remains only to verify that we have a decomposition of $K_{2p} - I$. To do this we observe that $S_1, S_2, \ldots, S_{\frac{p-1}{4}}$ partitions $D_{2p} \setminus \{r_0, s_0\}$ (r_0 is the identity in D_{2p} and $\mathrm{Cay}(D_{2p}\,; \{s_0\})$ is a 1-factor in K_{2p}). We have $Hx \cup H(x+1) \cup H(x+2) \cup H(x+3) = \mathbb{Z}_p \setminus \{0\}$. Also, since $p \equiv 5 \pmod 8$ we have $-1 \in (\mathbb{Z}_p^*)^2$, $-1 \notin (\mathbb{Z}_p^*)^4$ and $2 \notin (\mathbb{Z}_p^*)^2$ (by the law of quadratic reciprocity). Thus, $\{\pm h : h \in H\} \cup \{\pm 2h : h \in H\} = \mathbb{Z}_p \setminus \{0\}$. So $S_1, S_2, \ldots, S_{\frac{p-1}{4}}$ does indeed partition $D_{2p} \setminus \{r_0, s_0\}$ and we have the required decomposition. □

3 2-factorisations of circulant graphs

In this section we present various results on 2-factorisations of circulant graphs, beginning with a couple of known results. Lemma 3.1 was proved independently in [4] and [27], and is a special case of a result in [6]. Lemma 3.2 was proved in [8].

Lemma 3.1. ([4, 27]) *If $n \geq 5$ and F is any 2-regular graph of order n, then there is a 2-factorisation of $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1, 2\})$ into a copy of F and a Hamilton cycle.*

Lemma 3.2. ([8]) *If $n \geq 9$ and F is a 2-regular graph of order n, then there is a 2-factorisation of $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1, 2, 3, 4\})$ into F with the definite exceptions of $F = C_4 \cup C_5$ and $F = C_3 \cup C_3 \cup C_3 \cup C_3 \cup C_3$, and the following possible exceptions.*

(1) $F = C_3 \cup C_3 \cup \cdots \cup C_3$ *when* $n \equiv 3, 6 \pmod 9$, $n \geq 21$.

(2) $F = C_4 \cup C_4 \cup \cdots \cup C_4$ *when* $n \equiv 4 \pmod 8$, $n \geq 20$.

(3) $F = C_3 \cup C_3 \cup \cdots \cup C_3 \cup C_4$ *when* $n \equiv 1 \pmod 3$, $n \geq 19$.

(4) $F = C_3 \cup C_4 \cup C_4 \cup \cdots \cup C_4$ *when* $n \equiv 7 \pmod 8$, $n \geq 23$.

We now obtain results on 2-factorisations of $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1, 2, 3\})$, but first we need some definitions and notation. For each $m \geq 1$, the graph with vertex set $\{0, 1, \ldots, m+2\}$ and edge set $\{\{i, i+1\}, \{i+1, i+3\}, \{i, i+3\} : i = 0, 1, \ldots, m-1\}$ is denoted by $J_m^{1,2,3}$. If F is a 2-regular graph of order m, and there exists a decomposition $\{H_1, H_2, H_3\}$ of $J_m^{1,2,3}$ into F such that

(1) $V(H_1) = \{0, 1, \ldots, m+2\} \setminus \{m, m+1, m+2\}$,

(2) $V(H_2) = \{0, 1, \ldots, m+2\} \setminus \{0, 2, m+1\}$, and

(3) $V(H_3) = \{0, 1, \ldots, m+2\} \setminus \{0, 1, m+2\}$,

then we shall write $J_m^{1,2,3} \mapsto F$. Notice that for $i = 1, 2, 3$, the subgraph H_i of $J_m^{1,2,3}$ contains exactly one vertex from each of $\{0, m\}$, $\{1, m+1\}$ and $\{2, m+2\}$.

Lemma 3.3. *If $n \geq 7$ and F is a 2-regular graph of order n such that there exists a decomposition $J_n^{1,2,3} \mapsto F$, then there exists a 2-factorisation of $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1, 2, 3\})$ into F.*

Proof. For each $i \in \{0, 1, 2\}$, identify vertex i of $J_n^{1,2,3}$ with vertex $n+i$. The resulting graph is $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1, 2, 3\})$ and the 2-regular graphs in the decomposition $J_n^{1,2,3} \mapsto F$ become the required 2-factors. □

Lemma 3.4. *If F and F' are vertex-disjoint 2-regular graphs and there exist decompositions $J_{|V(F)|}^{1,2,3} \mapsto F$ and $J_{|V(F')|}^{1,2,3} \mapsto F'$, then there exists a decomposition $J_{|V(F)|+|V(F')|}^{1,2,3} \mapsto F \cup F'$.*

Proof. Let r and s be the respective orders of F and F', let $\{H_1, H_2, H_3\}$ be a decomposition $J_r^{1,2,3} \mapsto F$ and let $\{H_1', H_2', H_3'\}$ be a decomposition $J_s^{1,2,3} \mapsto F'$. Apply the translation $x \mapsto x + r$ to the decomposition $\{H_1', H_2', H_3'\}$ to obtain a decomposition $\{H_1'', H_2'', H_3''\}$ of a copy of $J_s^{1,2,3}$ having vertex set $r, r+1, \ldots, r+s+2$ (H_i'' being the translation of H_i' for $i \in \{1, 2, 3\}$). It is clear that $\mathcal{D} = \{H_1 \cup H_1'', H_2 \cup H_2'', H_3 \cup H_3''\}$ is a decomposition $J_{r+s}^{1,2,3} \mapsto F \cup F'$. Properties (1)-(3) in the definition of $J_r^{1,2,3} \mapsto F$ ensure that H_i and H_i'' are vertex-disjoint for $i \in \{1, 2, 3\}$, and that

(1) $V(H_1 \cup H_1'') = \{0, 1, \ldots, r+s+2\} \setminus \{r+s, r+s+1, r+s+2\}$,

(2) $V(H_2 \cup H_2'') = \{0, 1, \ldots, r+s+2\} \setminus \{0, 2, r+s+1\}$, and

(3) $V(H_3 \cup H_3'') = \{0, 1, \ldots, r+s+2\} \setminus \{0, 1, r+s+2\}$.

□

Lemma 3.5. *For each $m \geq 4$, $J_m^{1,2,3} \mapsto C_m$.*

Proof. For $m \in \{4, 5, 6\}$, H_1, H_2, H_3 are as defined in the following table.

m	H_1	H_2	H_3
4	$(0,1,2,3)$	$(1,3,6,4)$	$(2,4,3,5)$
5	$(0,1,2,4,3)$	$(1,3,5,7,4)$	$(2,3,6,4,5)$
6	$(0,1,2,5,4,3)$	$(1,3,5,8,6,4)$	$(2,4,7,5,6,3)$

For $m \geq 7$ and odd

- H_1 contains the edges $\{0, 1\}$, $\{1, 2\}$, $\{0, 3\}$, $\{m-2, m-1\}$ and $\{i, i+2\}$ for $i \in \{2, 3, \ldots, m-3\}$,
- H_2 contains the edges $\{1, 3\}$, $\{m-2, m\}$, $\{m, m+2\}$, $\{m-1, m+2\}$, $\{i, i+1\}$ for $i \in \{4, 6, \ldots, m-3\}$ and $\{i, i+3\}$ for $i \in \{1, 3, \ldots, m-4\}$, and
- H_3 contains the edges $\{2, 3\}$, $\{m-2, m+1\}$, $\{m-1, m\}$, $\{m-1, m+1\}$, $\{i, i+1\}$ for $i \in \{3, 5, \ldots, m-4\}$ and $\{i, i+3\}$ for $i \in \{2, 4, \ldots, m-3\}$.

For $m \geq 8$ and even

- H_1 contains the edges $\{0, 1\}$, $\{1, 2\}$, $\{3, 4\}$, $\{0, 3\}$, $\{2, 5\}$, $\{m-2, m-1\}$ and $\{i, i+2\}$ for $i \in \{4, 5, \ldots, m-3\}$,
- H_2 contains the edges $\{1, 3\}$, $\{1, 4\}$, $\{3, 5\}$, $\{m-2, m\}$, $\{m, m+2\}$, $\{m-1, m+2\}$, $\{i, i+1\}$ for $i \in \{5, 7, \ldots, m-3\}$ and $\{i, i+3\}$ for $i \in \{4, 6, \ldots, m-4\}$, and
- H_3 contains the edges $\{2, 4\}$, $\{m-2, m+1\}$, $\{m-1, m\}$, $\{m-1, m+1\}$, $\{i, i+1\}$ for $i \in \{2, 4, \ldots, m-4\}$ and $\{i, i+3\}$ for $i \in \{3, 5, \ldots, m-3\}$.

□

Lemma 3.6. *For $m = 8$ and for each $m \geq 10$, $J_m^{1,2,3} \mapsto C_3 \cup C_{m-3}$.*

Proof. For $m \in \{8, 10, 11\}$, H_1, H_2, H_3 are as defined in the following table.

m	
8	$H_1 = (4,6,7) \cup (0,1,2,5,3)$ $H_2 = (7,8,10) \cup (1,3,6,5,4)$ $H_3 = (2,3,4) \cup (5,7,9,6,8)$
10	$H_1 = (7,8,9) \cup (0,1,2,4,5,6,3)$ $H_2 = (1,3,4) \cup (5,7,6,9,12,10,8)$ $H_3 = (2,3,5) \cup (4,6,8,11,9,10,7)$
11	$H_1 = (8,9,10) \cup (0,1,2,4,5,7,6,3)$ $H_2 = (1,3,4) \cup (5,6,9,11,13,10,7,8)$ $H_3 = (2,3,5) \cup (4,6,8,11,10,12,9,7)$

For $m \geq 12$ and even

- H_1 consists of the 3-cycle $(m-3, m-2, m-1)$ and the $(m-3)$-cycle with edges $\{0,1\}$, $\{0,3\}$, $\{1,2\}$, $\{2,4\}$, $\{m-5, m-4\}$, $\{i, i+1\}$ for $i \in \{4, 6, \ldots, m-6\}$ and $\{i, i+3\}$ for $i \in \{3, 5, \ldots, m-7\}$,
- H_2 consists of the 3-cycle $(1,3,4)$ and the $(m-3)$-cycle with edges $\{5,7\}$, $\{m-5, m-2\}$, $\{m-4, m-3\}$, $\{m-2, m\}$, $\{m, m+2\}$, $\{m-1, m+2\}$, $\{i, i+1\}$ for $i \in \{5, 7, \ldots, m-7\}$ and $\{i, i+3\}$ for $i \in \{6, 8, \ldots, m-4\}$, and
- H_3 consists of the 3-cycle $(2,3,5)$ and the $(m-3)$-cycle with edges $\{4,6\}$, $\{4,7\}$, $\{m-2, m+1\}$, $\{m-3, m\}$, $\{m-1, m\}$, $\{m-1, m+1\}$ and $\{i, i+2\}$ for $i \in \{6, 7, \ldots, m-4\}$.

For $m \geq 13$ and odd

- H_1 consists of the 3-cycle $(m-3, m-2, m-1)$ and the $(m-3)$-cycle with edges $\{0,1\}$, $\{0,3\}$, $\{1,2\}$, $\{2,4\}$, $\{3,6\}$, $\{4,5\}$, $\{5,7\}$, $\{m-5, m-4\}$, $\{i, i+1\}$ for $i \in \{7, 9, \ldots, m-6\}$ and $\{i, i+3\}$ for $i \in \{6, 8, \ldots, m-7\}$,
- H_2 consists of the 3-cycle $(1,3,4)$ and the $(m-3)$-cycle with edges $\{5,6\}$, $\{m-5, m-2\}$, $\{m-4, m-3\}$, $\{m-2, m\}$, $\{m, m+2\}$, $\{m-1, m+2\}$, $\{i, i+1\}$ for $i \in \{6, 8, \ldots, m-7\}$ and $\{i, i+3\}$ for $i \in \{5, 7, \ldots, m-4\}$, and
- H_3 consists of the 3-cycle $(2,3,5)$ and the $(m-3)$-cycle with edges $\{4,6\}$, $\{4,7\}$, $\{m-2, m+1\}$, $\{m-3, m\}$, $\{m-1, m\}$, $\{m-1, m+1\}$ and $\{i, i+2\}$ for $i \in \{6, 7, \ldots, m-4\}$.

□

Lemma 3.7. *Let $n \geq 7$ and let F be a 2-regular graph of order n. If $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F)$ where $\nu_m(F)$ denotes the number of m-cycles in F, then there exists a 2-factorisation of* $\mathrm{Cay}(\mathbb{Z}_n; \pm\{1,2,3\})$ *into F.*

Proof. If $n \geq 7$ and F is a 2-regular graph of order n such that $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F)$, then F can be written as a vertex-disjoint union of 2-regular graphs $G_1, G_2, \ldots, G_t$ where each G_i is isomorphic to either

- C_m with $m \geq 4$, or

- $C_3 \cup C_{m-3}$ with $m = 8$ or $m \geq 10$.

By Lemmas 3.5 and 3.6 we have a decomposition $J^{1,2,3}_{|V(G_i)|} \mapsto G_i$ for $i = 1, 2, \ldots, t$. Applying Lemma 3.4 we obtain a decomposition $J^{1,2,3}_n \mapsto F$, and from this we obtain the required 2-factorisation of $\mathrm{Cay}(\mathbb{Z}_n; \pm\{1, 2, 3\})$ into F by applying Lemma 3.3. □

We can obtain an analogue of Lemma 3.7 for $\mathrm{Cay}(\mathbb{Z}_n; \pm\{1, 3, 4\})$ by using similar methods, but we will require F to have girth at least 6. The graph with vertex set $\{0, 1, \ldots, m+3\}$ and edge set $\{\{i, i+1\}, \{i+1, i+4\}, \{i, i+4\} : i = 0, 1, \ldots, m-1\}$ is denoted by $J^{1,3,4}_m$. We write $J^{1,3,4}_m \mapsto F$ when there exists a decomposition $\{H_1, H_2, H_3\}$ of $J^{1,3,4}_m$ into a 2-regular graph F such that

(1) $V(H_1) = \{0, 1, \ldots, m+3\} \setminus \{m, m+1, m+2, m+3\}$,

(2) $V(H_2) = \{0, 1, \ldots, m+3\} \setminus \{0, 3, m+1, m+2\}$, and

(3) $V(H_3) = \{0, 1, \ldots, m+3\} \setminus \{0, 1, 2, m+3\}$.

Notice that for $i = 1, 2, 3$, the subgraph H_i of $J^{1,3,4}_m$ contains exactly one vertex from each of $\{0, m\}$, $\{1, m+1\}$, $\{2, m+2\}$ and $\{3, m+3\}$. It is clear that the proofs of Lemmas 3.3 and 3.4 can be easily modified to give the following two results.

Lemma 3.8. *If $n \geq 9$ and F is a 2-regular graph of order n such that there exists a decomposition $J^{1,3,4}_n \mapsto F$, then there exists a 2-factorisation of $\mathrm{Cay}(\mathbb{Z}_n; \pm\{1, 3, 4\})$ into F.*

Lemma 3.9. *If F and F' are vertex-disjoint 2-regular graphs and there exist decompositions $J^{1,3,4}_{|V(F)|} \mapsto F$ and $J^{1,3,4}_{|V(F')|} \mapsto F'$, then there exists a decomposition $J^{1,3,4}_{|V(F)|+|V(F')|} \mapsto F \cup F'$.*

Lemmas 3.8 and 3.9 allow us to obtain 2-factorisations of $\mathrm{Cay}(\mathbb{Z}_n; \pm\{1, 3, 4\})$ via the same method we used in the case of $\mathrm{Cay}(\mathbb{Z}_n; \pm\{1, 2, 3\})$, providing we can find appropriate decompositions of $J^{1,3,4}_m$. We now do this.

Lemma 3.10. *For $m = 6$, $m = 7$ and each $m \geq 9$, $J^{1,3,4}_m \mapsto C_m$.*

Proof. For $m \in \{6, 7, 9, 10\}$, H_1, H_2, H_3 are as defined in the following table.

m	H_1	H_2	H_3
6	$(0, 1, 5, 2, 3, 4)$	$(1, 2, 6, 9, 5, 4)$	$(3, 6, 5, 8, 4, 7)$
7	$(0, 1, 2, 3, 6, 5, 4)$	$(1, 4, 7, 10, 6, 2, 5)$	$(3, 4, 8, 5, 9, 6, 7)$
9	$(0, 1, 2, 3, 7, 6, 5, 8, 4)$	$(1, 4, 7, 8, 12, 9, 6, 2, 5)$	$(3, 4, 5, 9, 8, 11, 7, 10, 6)$
10	$(0, 1, 2, 3, 6, 9, 5, 8, 7, 4)$	$(1, 4, 8, 9, 13, 10, 7, 6, 2, 5)$	$(3, 4, 5, 6, 10, 9, 12, 8, 11, 7)$

For $m \geq 11$ and odd

- H_1 contains the edges $\{0, 1\}$, $\{0, 4\}$, $\{1, 2\}$, $\{2, 3\}$, $\{3, 7\}$, $\{5, 6\}$, $\{m-3, m-2\}$, $\{m-5, m-1\}$, $\{m-4, m-1\}$ and $\{i, i+4\}$ for $i \in \{4, 5, \ldots, m-6\}$,
- H_2 contains the edges $\{1, 4\}$, $\{1, 5\}$, $\{2, 5\}$, $\{2, 6\}$, $\{4, 7\}$, $\{m, m+3\}$, $\{m-1, m+3\}$, $\{m-2, m-1\}$, $\{m-3, m\}$, $\{i, i+1\}$ for $i \in \{7, 9, \ldots, m-4\}$ and $\{i, i+3\}$ for $i \in \{6, 8, \ldots, m-5\}$, and

- H_3 contains the edges $\{3,4\}$, $\{3,6\}$, $\{4,5\}$, $\{m-1,m\}$, $\{m-2,m+1\}$, $\{m-1,m+2\}$, $\{m-4,m\}$, $\{m-3,m+1\}$, $\{m-2,m+2\}$, $\{i,i+1\}$ for $i \in \{6,8,\dots,m-5\}$ and $\{i,i+3\}$ for $i \in \{5,7,\dots,m-6\}$.

For $m \geq 12$ and even

- H_1 contains the edges $\{0,1\}$, $\{0,4\}$, $\{1,2\}$, $\{2,3\}$, $\{3,6\}$, $\{4,7\}$, $\{5,6\}$, $\{5,9\}$, $\{m-5,m-2\}$, $\{m-4,m-3\}$, $\{m-4,m-1\}$, $\{m-2,m-1\}$, $\{i,i+1\}$ for $i \in \{7,9,\dots,m-7\}$ and $\{i,i+3\}$ for $i \in \{8,10,\dots,m-6\}$,
- H_2 contains the edges $\{1,4\}$, $\{1,5\}$, $\{2,5\}$, $\{2,6\}$, $\{4,8\}$, $\{m-6,m-2\}$, $\{m-5,m-4\}$, $\{m-5,m-1\}$, $\{m-3,m-2\}$, $\{m-3,m\}$, $\{m-1,m+3\}$, $\{m,m+3\}$, $\{i,i+1\}$ for $i \in \{6,8,\dots,m-8\}$ and $\{i,i+3\}$ for $i \in \{7,9,\dots,m-7\}$, and
- H_3 contains the edges $\{3,4\}$, $\{3,7\}$, $\{4,5\}$, $\{5,8\}$, $\{6,9\}$, $\{m-6,m-5\}$, $\{m-4,m\}$, $\{m-3,m+1\}$, $\{m-2,m+1\}$, $\{m-2,m+2\}$, $\{m-1,m\}$, $\{m-1,m+2\}$ and $\{i,i+4\}$ for $i \in \{6,7,\dots,m-7\}$.

□

Lemma 3.11. *For each $m \geq 14$, $J_m^{1,3,4} \mapsto C_8 \cup C_{m-8}$.*

Proof. For $m \in \{14,15,16,17\}$, H_1, H_2, H_3 are as defined in the following table.

m	
14	$H_1 = (0,1,2,3,7,8,5,4) \cup (6,9,13,12,11,10)$ $H_2 = (8,11,14,17,13,10,9,12) \cup (1,4,7,6,2,5)$ $H_3 = (7,10,14,13,16,12,15,11) \cup (3,4,8,9,5,6)$
15	$H_1 = (0,1,2,3,6,5,8,4) \cup (7,10,14,13,9,12,11)$ $H_2 = (1,4,7,8,9,6,2,5) \cup (10,11,14,18,15,12,13)$ $H_3 = (8,11,15,14,17,13,16,12) \cup (3,4,5,9,10,6,7)$
16	$H_1 = (0,1,5,6,2,3,7,4) \cup (8,9,10,11,15,14,13,12)$ $H_2 = (1,2,5,9,6,7,8,4) \cup (10,13,16,19,15,12,11,14)$ $H_3 = (3,4,5,8,11,7,10,6) \cup (9,12,16,15,18,14,17,13)$
17	$H_1 = (0,1,2,3,7,6,5,4) \cup (8,9,13,16,12,15,14,10,11)$ $H_2 = (1,4,8,12,9,6,2,5) \cup (7,10,13,14,17,20,16,15,11)$ $H_3 = (3,4,7,8,5,9,10,6) \cup (11,12,13,17,16,19,15,18,14)$

For $m \geq 18$ and even

- H_1 consists of the 8-cycle $(0,1,5,6,2,3,7,4)$ and the $(m-8)$-cycle with edges $\{8,9\}$, $\{9,10\}$, $\{10,11\}$, $\{8,12\}$, $\{m-5,m-1\}$, $\{m-4,m-3\}$, $\{m-3,m-2\}$, $\{m-2,m-1\}$ $\{i,i+1\}$ for $i \in \{12,14,\dots,m-6\}$ and $\{i,i+3\}$ for $i \in \{11,13,\dots,m-7\}$,
- H_2 consists of the 8-cycle $(1,2,5,9,6,7,8,4)$ and the $(m-8)$-cycle with edges $\{10,13\}$, $\{11,12\}$, $\{m-6,m-2\}$, $\{m-5,m-2\}$, $\{m-4,m-1\}$, $\{m-3,m\}$, $\{m-1,m+3\}$, $\{m,m+3\}$ and $\{i,i+4\}$ for $i \in \{10,11,\dots,m-7\}$, and
- H_3 consists of the 8-cycle $(3,4,5,8,11,7,10,6)$ and the $(m-8)$-cycle with edges $\{9,12\}$, $\{9,13\}$, $\{m-4,m\}$, $\{m-3,m+1\}$, $\{m-2,m+1\}$, $\{m-2,m+2\}$, $\{m-1,m\}$, $\{m-1,m+2\}$, $\{i,i+1\}$ for $i \in \{13,15,\dots,m-5\}$ and $\{i,i+3\}$ for $i \in \{12,14,\dots,m-6\}$.

For $m \geq 19$ and odd

- H_1 consists of the 8-cycle $(0, 1, 2, 3, 7, 6, 5, 4)$ and the $(m-8)$-cycle with edges $\{8, 9\}$, $\{8, 11\}$, $\{9, 13\}$, $\{10, 11\}$, $\{10, 14\}$, $\{12, 15\}$, $\{12, 16\}$, $\{m-4, m-1\}$, $\{m-3, m-2\}$ and $\{i, i+4\}$ for $i \in \{13, 14, \ldots, m-5\}$,
- H_2 consists of the 8-cycle $(1, 4, 8, 12, 9, 6, 2, 5)$ and the $(m-8)$-cycle with edges $\{7, 10\}$, $\{7, 11\}$, $\{10, 13\}$, $\{11, 15\}$, $\{m-4, m-3\}$, $\{m-3, m\}$, $\{m-2, m-1\}$, $\{m-1, m+3\}$, $\{m, m+3\}$, $\{i, i+1\}$ for $i \in \{13, 15, \ldots, m-6\}$ and $\{i, i+3\}$ for $i \in \{14, 16, \ldots, m-5\}$, and
- H_3 consists of the 8-cycle $(3, 4, 7, 8, 5, 9, 10, 6)$ and the $(m-8)$-cycle with edges $\{11, 12\}$, $\{11, 14\}$, $\{12, 13\}$, $\{m-4, m\}$, $\{m-3, m+1\}$, $\{m-2, m+1\}$, $\{m-2, m+2\}$, $\{m-1, m\}$, $\{m-1, m+2\}$, $\{i, i+1\}$ for $i \in \{14, 16, \ldots, m-5\}$ and $\{i, i+3\}$ for $i \in \{13, 15, \ldots, m-6\}$.

□

Lemma 3.12. $J_{24}^{1,3,4} \mapsto C_8 \cup C_8 \cup C_8$.

Proof. Take

$$H_1 = (0, 1, 2, 3, 6, 5, 8, 4) \cup (7, 10, 9, 12, 13, 14, 15, 11) \cup (16, 17, 18, 19, 23, 22, 21, 20),$$
$$H_2 = (1, 4, 7, 8, 9, 6, 2, 5) \cup (10, 11, 12, 15, 16, 13, 17, 14) \cup (18, 21, 24, 27, 23, 20, 19, 22), \text{ and}$$
$$H_3 = (3, 4, 5, 9, 13, 10, 6, 7) \cup (8, 11, 14, 18, 15, 19, 16, 12) \cup (17, 20, 24, 23, 26, 22, 25, 21).$$

□

The following result is an analogue of Lemma 3.7 for 2-factorisations of $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1, 3, 4\})$.

Lemma 3.13. *If $n \geq 9$ and F is a 2-regular graph of order n with girth at least 6, then there exists a 2-factorisation of $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1, 3, 4\})$ into F.*

Proof. If $n \geq 9$ and F is a 2-regular graph of order n with girth at least 6, then F can be written as a vertex-disjoint union of 2-regular graphs $G_1, G_2, \ldots, G_t$ where each G_i is isomorphic to either

- C_m with $m = 6, 7$ or $m \geq 9$,
- $C_8 \cup C_{m-8}$ with $m \geq 14$, or
- $C_8 \cup C_8 \cup C_8$.

By Lemmas 3.10, 3.11 and 3.12 we have a decomposition $J_{|V(G_i)|}^{1,3,4} \mapsto G_i$ for $i = 1, 2, \ldots, t$. Applying Lemma 3.9 we obtain a decomposition $J_n^{1,3,4} \mapsto F$, and from this we obtain the required 2-factorisation of $\mathrm{Cay}(\mathbb{Z}_n;\, \pm\{1, 3, 4\})$ into F by applying Lemma 3.8. □

4 2-factorisations and the Oberwolfach Problem

In this section we use results from the preceding sections to obtain results on the Oberwolfach Problem (and an additional result on 2-factorisations of $K_n - I$ into a number of specified 2-factors and Hamilton cycles). We will also use the following corollary of Lemma 3.2 which was proved in [8].

Lemma 4.1. ([8]) *If there exists a factorisation of K_n or of $K_n - I$ into* $\mathrm{Cay}(\mathbb{Z}_n\,;\pm\{1,2,3,4\})$*, then* $\mathrm{OP}(F)$ *has a solution for each 2-regular graph F of order n, with the exception that there is no solution to* $\mathrm{OP}(C_4 \cup C_5)$*.*

Theorem 4.2. *If $p \equiv 5 \pmod 8$ is prime, then* $\mathrm{OP}(F)$ *has a solution for every 2-regular graph F of order $2p$.*

Proof. The case $p = 13$ is covered in [13]. For $p \neq 13$, Theorem 2.11 gives us a factorisation of $K_{2p} - I$ into $\mathrm{Cay}(\mathbb{Z}_{2p}\,;\pm\{1,2,3,4\})$ and the result then follows by Lemma 4.1. □

Theorem 4.3. *Let $\mathcal{P}$ be the set of primes given by $p \in \mathcal{P}$ if and only if $p \geq 7$ and neither 4 nor 32 is in the subgroup of $\mathbb{Z}_p^*$ generated by $\{-1,6\}$. Then $\mathcal{P}$ is infinite and if $p \in \mathcal{P}$, then* $\mathrm{OP}(F)$ *has a solution for every 2-regular graph F of order p satisfying $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F)$ where $\nu_m(F)$ denotes the number of m-cycles in F.*

Proof. Let p be prime such that $p \equiv 1 \pmod 6$, $2,3 \notin (\mathbb{Z}_p^*)^3$ and $6 \in (\mathbb{Z}_p^*)^3$. Theorem 2.5 says that there are infinitely many such p. We shall show that $p \in \mathcal{P}$, which shows that $\mathcal{P}$ is also infinite. We have $-1 \in (\mathbb{Z}_p^*)^3$, and this together with the fact that $6 \in (\mathbb{Z}_p^*)^3$ implies that the subgroup of $\mathbb{Z}_p^*$ generated by $\{-1,6\}$ is a subgroup of $(\mathbb{Z}_p^*)^3$. Since it follows from $2 \notin (\mathbb{Z}_p^*)^3$ that $4, 32 \notin (\mathbb{Z}_p^*)^3$, neither 4 nor 32 is in the subgroup of $\mathbb{Z}_p^*$ generated by $\{-1,6\}$. That is, $p \in \mathcal{P}$.

Now let p be an arbitrary element of $\mathcal{P}$ and let G be the subgroup of $\mathbb{Z}_p^*$ generated by $\{-1,6\}$. The condition that neither 4 nor 32 is in G implies that the order d of $2G$ in $\mathbb{Z}_p^*/G$ is neither 1, 2 nor 5, and so there exist non-negative integers α and β such that $d = 3\alpha + 4\beta$. Thus, by Lemma 2.6 there is a factorisation of K_p in which each factor is either $\mathrm{Cay}(\mathbb{Z}_p\,;\pm\{1,2,3\})$ or $\mathrm{Cay}(\mathbb{Z}_p\,;\pm\{1,2,3,4\})$.

Let F be a 2-regular graph of order p satisfying $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F)$. Lemma 3.7 gives us a 2-factorisation of $\mathrm{Cay}(\mathbb{Z}_p\,;\pm\{1,2,3\})$ into F, and Lemma 3.2 gives us a 2-factorisation of $\mathrm{Cay}(\mathbb{Z}_p\,;\pm\{1,2,3,4\})$ (the facts that p is prime and that $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F)$ imply that F is not amongst the possible exceptions listed in Lemma 3.2). The result follows. □

Theorem 4.4. *Let $\mathcal{P}$ be the set of primes such that $p \in \mathcal{P}$ if and only if $p \equiv 1 \pmod 6$ and $2,3,6 \notin (\mathbb{Z}_p^*)^3$. Then $\mathcal{P}$ is infinite and if $p \in \mathcal{P}$, then* $\mathrm{OP}(F)$ *has a solution for every 2-regular graph F of order p with girth at least 6.*

Proof. By Theorem 2.5, $\mathcal{P}$ is infinite. If $p \in \mathcal{P}$, then Theorem 2.4 gives us a factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,;\pm\{1,3,4\})$, and the result then follows by applying Lemma 3.13 to each factor ($7 \notin \mathcal{P}$ so Lemma 3.13 can indeed be applied). □

For each odd prime p, the following theorem states there is a 2-factorisation of $K_{2p} - I$ into $\frac{p-1}{2}$ prescribed 2-factors and $\frac{p-1}{2}$ Hamilton cycles.

Theorem 4.5. *If p is an odd prime and $G_1, G_2, \ldots, G_{\frac{p-1}{2}}$ are 2-regular graphs of order $2p$, then there is a 2-factorisation $\{F_1, F_2, \ldots, F_{p-1}\}$ of $K_{2p} - I$ such that $F_i \cong G_i$ for $i = 1, 2, \ldots, \frac{p-1}{2}$ and F_i is a Hamilton cycle for $i = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p-1$.*

Proof. By Theorem 2.9 there is a factorisation of $K_{2p} - I$ into $\mathrm{Cay}(\mathbb{Z}_p\,;\pm\{1,2\})$. By Lemma 3.1, each copy of $\mathrm{Cay}(\mathbb{Z}_p\,;\pm\{1,2\})$ can be factored into any specified 2-regular graph of order $2p$ and a Hamilton cycle. The result follows. □

5 Isomorphic 2-factorisations of complete multigraphs

The complete multigraph of order n and multiplicity s is denoted by sK_n. It has s distinct edges joining each pair of distinct vertices.

Lemma 5.1. *If p is an odd prime and $S = \pm\{d_1, d_2, \dots, d_s\} \subseteq \mathbb{Z}_p^*$, then there exists a $2s$-factorisation of sK_p into $\mathrm{Cay}(\mathbb{Z}_p\,; S)$.*

Proof. The required factorisation is given by $\{\mathrm{Cay}(\mathbb{Z}_p\,; \omega^i S) : i = 0, 1, \dots, \frac{p-3}{2}\}$ where ω is primitive in $\mathbb{Z}_p$ and $\omega^i S = \{\omega^i s : s \in S\}$. □

Theorem 5.2. *If p is an odd prime and F is any 2-regular graph of order p satisfying $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F)$, where $\nu_m(F)$ denotes the number of m-cycles in F, then there exists a 2-factorisation of $3K_p$ into F.*

Proof. The cases $p = 3$ and $p = 5$ are trivial so assume $p \geq 7$. By Lemma 5.1 there exists a 6-factorisation of $3K_p$ into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1, 2, 3\})$, and by Lemma 3.7 each such 6-factor has a 2-factorisation into F. □

Theorem 5.3. *If p is an odd prime and F is any 2-regular graph of order p, then there exists a 2-factorisation of $4K_p$ into F.*

Proof. The cases $p = 3$ and $p = 5$ are trivial. Since solutions to $\mathrm{OP}(C_7)$ and $\mathrm{OP}(C_3 \cup C_4)$ exist, the case $p = 7$ can be dealt with by taking four copies of these 2-factorisations of K_7. So we may assume $p \geq 11$. By Lemma 5.1 there exists an 8-factorisation of $4K_p$ into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1, 2, 3, 4\})$, and by Lemma 3.2 each such 8-factor has a 2-factorisation into F; except in the case where F is one of the listed exceptions or possible exceptions in Lemma 3.2. These are easily dealt with as follows. Since p is prime the only relevant exceptions are $F = C_3 \cup C_3 \cup \dots \cup C_3 \cup C_4$ where the number of copies of C_3 is at least 5, and $F = C_3 \cup C_4 \cup C_4 \cup \dots \cup C_4$ where the number of copies of C_4 is odd and at least 5. However, it is known that for each such F, there is a 2-factorisation of K_p into F; the former case is covered in [11], and the latter case is covered in [21]. Thus, by taking four copies of these 2-factorisations of K_p, we obtain the required 2-factorisations of $4K_p$. □

Theorem 5.4. *Let p be an odd prime and let F be a 2-regular graph of order p. If $\lambda \equiv 0 \pmod 4$, then there exists a 2-factorisation of λK_p into F. Moreover, if F satisfies $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F)$, where $\nu_m(F)$ denotes the number of m-cycles in F, then the result also holds for $\lambda = 3$ and for all $\lambda \geq 6$.*

Proof. For the given values of λ, it is trivial to factorise λK_p such that each factor is either $3K_p$ or $4K_p$, and with each factor being $4K_p$ when $\lambda \equiv 0 \pmod 4$. Thus, the result follows by Theorems 5.2 and 5.3. □

Acknowledgement. The authors gratefully acknowledge the support of the Australian Research Council via grants DE120100040, DP0770400, DP120100790, DP120103067, DP150100506, DP150100530 and DP130102987.

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