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Bled'11 – Part 2

This is the second of two special issues of *Ars Mathematica Contemporanea* dedicated to the 7th Slovenian Graph Theory Conference (Bled'11). The conference was held (by tradition) at Lake Bled, Slovenia, and took place from June 19 to June 25, 2011. In total, 9 keynote lectures and 213 contributed talks (within 16 Minisymposia and a general session) were given.

The first special issue (published as the first issue of Volume 6 of *Ars Mathematica Contemporanea*) contained 15 research papers. The current issue contains another 18 papers, on some more of the high quality research presented at the conference, and accepted for publication after a thorough refereeing process. These papers, together with the 15 from the first special issue, present the readers of *Ars Mathematica Contemporanea* with a selection of 33 fascinating and valuable contributions, covering a wide range of aspects of graph theory.

A small number of submissions for these special issues have not yet completed the peer review process. Those that succeed will be published in forthcoming regular issues of *Ars Mathematica Contemporanea*.

In a sense, this special issue represents a formal conclusion to the Bled'11 conference. We are already looking forward to the next Bled conference, to be held in June 2015.

Klavdija Kutnar and Primož Šparl

Guest Editors

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Compression ratio of Wiener index in 2-d rectangular and polygonal lattices

Jelena Sedlar *
Faculty of civil engeneering, architecture and geodesy
Matice hrvatske 15, HR-21000 Split, Croatia

Damir Vukičević
Faculty of Science, University of Split, Nikole Tesle 12, HR-21000 Split, Croatia

Franco Cataldo , Ottorino Ori
Actinium Chemical Research, Via Casilina 1626/A, 00133 Rome, Italy

Ante Graovac
Faculty of Science, University of Split, Nikole Tesle 12, HR-21000 Split, Croatia
IMC, University of Dubrovnik, Branitelja Dubrovnika 29, HR-20000 Dubrovnik, Croatia

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Abstract

In this paper, we establish leading coefficient of Wiener index for open and closed 2-dimensional rectangular lattices, for various open and closed polygonal lattices, and for open and closed multidimensional cubes. These results enable us to establish compression ratio of Wiener index when number of rows and columns in the lattice tends to infinity.

Keywords: Graph theory, 2D rectangular and polygonal lattices, Wiener index, Compression ratio.

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*corresponding author
E-mail addresses: jsedlar@gradst.hr (Jelena Sedlar), vukicevi@pmfst.hr (Damir Vukičević), franco.cataldo@fastwebnet.it (Franco Cataldo), ottorino.ori@alice.it (Ottorino Ori), graovac@irb.hr (Ante Graovac)

1 Introduction

Topological indices are very important in chemistry, since they can be used for modeling and prediction of many chemical properties. One of the most famous and the most researched indices is Wiener index. Topological indices are invariants defined on graphs representing various chemical compounds. For example, one such compound is graphene which is represented with hexagonal lattice graph. Also, nanotubes and nanotori received much attention recently, and they are represented with a graph which is rectangle shaped lattice with opposite sides identified.

Recent theoretical investigations point out that the minimization of distance-based graph invariants, namely the Wiener index W [7] and the topological efficiency index ρ recently introduced [3], provides the fast determination of the subsets of isomers with relative structural stability of a given chemical structure. This method has been applied to important classes of carbon hexagonal systems like fullerenes, graphene with nanocones and graphene. This elegant computational topological approach quickly sieves the most stable C_{66} cages among 4478 distinct isomers as reported in [8]. Moreover, the same method gives the correct numbers of NMR resonance peaks and relative intensities. The Wiener index has been computed for monodimensional infinite lattices to describe conductibilty features of conjugated polymers [1]. Present article reports about a relevant property - the compression factor [3] - of the topological invariants computed on infinite lattices. For different topological indices, there is quite much recent interest [4-6] in the ratio of the value of the index on open and closed lattices (i.e. nanotubes and nanotori). In this article, we investigate compression ratio of 2-dimensional and multidimensional rectangular lattices, and various 2-dimensional polygonal lattices.

The present paper is organized as follows. In the second section named 'Preliminaries', we introduce some basic notions and notation that will be used throughout the paper. In the third section, we establish the leading coefficient in Wiener index for open and closed 2-dimensional rectangular lattices, which leads us to compression ratio in asymptotic case. In the fourth section, we use the results from the second section to derive the same kind of result for hexagonal and similar lattices. In the fifth section, we establish the limit of compression ratio for d-dimensional rectangular cube. Finally, in the last section named 'Conclusion', we summarize the main results of this paper.

2 Preliminaries

In this paper, we consider only simple connected graphs. We will use the following notation: G for graph, $V(G)$ or just V for its set of vertices, $E(G)$ or just E for its set of edges. With N we will denote number of vertices in a graph. For two vertices $u, v \in V$, we define *distance* $d(u, v)$ of u and v as the length of shortest path connecting u and v. Given the notion of distance, Wiener index of a graph G is defined [9] as

$$W(G) = \sum_{u,v \in V} d(u, v).$$

Here, the pair of vertices u, v is unordered. In some literature, the summation goes over all ordered pairs (u, v) of vertices and then the sum needs to be multiplied by a half. Now, let us introduce some special kinds of graphs that will be of interest to us in this paper, namely open and closed lattice graphs. First, let $R_{n,k}$ be rectangular lattice containing $2n$ rows and $2kn$ columns of squares. Lattices $R_{3,1}$ and $R_{3,2}$ are shown in Figure 1.

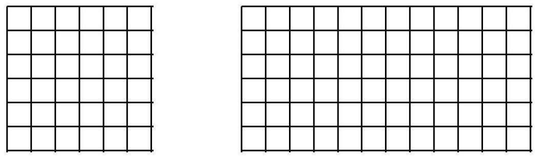

Figure 1: Lattices $R_{3,1}$ and $R_{3,2}$.

Let us denote vertices of $R_{n,k}$ with integer coordinates (i,j) for $i = 0,\ldots,2nk$ and $j = 0,\ldots,2n$ as if the lattice was placed in first quadrant of Cartesian coordinate system. Therefore

$$V(R_{n,k}) = \{v_{i,j} : i = 0,\ldots,2nk \text{ and } j = 0,\ldots,2n\}.$$

Open lattice is a graph ${}^{O}R_{n,k}$ obtained from $R_{n,k}$ by deleting vertices $v_{0,j}$ and $v_{i,0}$. Closed lattice is a graph ${}^{C}R_{n,k}$ obtained from $R_{n,k}$ by identifying vertices $v_{0,j}$ and $v_{2nk,j}$, and also vertices $v_{i,0}$ and $v_{i,2n}$. Therefore, open and closed lattice graphs have the same number of vertices which is

$$\left|V({}^{O}R_{n,k})\right| = \left|V({}^{C}R_{n,k})\right| = 4n^2k.$$

Further, for a fixed integer k let us consider polygons P_k with $4k+2$ vertices. Let $L_{n,k}$ be rectangle shaped lattice consisting of $2n$ rows of polygons P_k, with rows containing n and $n-1$ polygons alternatively, such that two neighboring polygons from one row share exactly two vertices, while two neighboring polygons from different rows share exactly $k+1$ vertices. For $k = 1$, lattice $L_{n,k}$ is actually hexagonal lattice with $2n$ rows and n columns of hexagons. Note that lattice $L_{n,k}$ can be considered as subgraph of rectangular lattice $R_{n,k}$ assigning to polygons the size of $2k$ square cells. Therefore, we can use the same vertex notation in $L_{n,k}$ as in $R_{n,k}$. Lattices $L_{3,1}$ and $L_{3,2}$ for $k = 1$ and $k = 2$ are shown in Figure 2. Again, open and closed lattices ${}^{O}L_{n,k}$ and ${}^{C}L_{n,k}$ can be obtained as for rectangular lattice. Finally, a d-dimensional rectangular cube R_n^d is defined in a following manner: set of vertices $V(R_n^d)$ is defined with

$$V(R_n^d) = \{(x_1,\ldots,x_d) : x_i \in \{0,1,\ldots,n\}\},$$

and two vertices $(x_1,\ldots,x_d)$ and $(y_1,\ldots,y_d)$ are connected with an edge if and only if there is a coordinate j such that $x_i = y_i$ for every $i \in \{1,\ldots,d\}\setminus\{j\}$, and for j-th coordinate holds $|y_j - x_j| = 1$. Open lattice ${}^{O}R_n^d$ is a graph obtained from R_n^d by deleting all vertices that have at least one zero coordinate. Closed lattice ${}^{C}R_n^d$ is a graph obtained from R_n^d by identifying every pair of vertices $(x_1,\ldots,x_d)$ and $(y_1,\ldots,y_d)$ such that $x_i = y_i$ for every $i \in \{1,\ldots,d\}\setminus\{j\}$ and for j-th coordinate holds $x_j = 0$ and $y_j = n$. Obviously, the number of vertices in ${}^{O}R_n^d$ and ${}^{C}R_n^d$ is the same and it equals n^d.

Now, if ${}^{O}G$ is an open lattice graph and ${}^{C}G$ its closed version, compression ratio of G is defined [3] as

$$cr(G) = \frac{W({}^{C}G)}{W({}^{O}G)}.$$

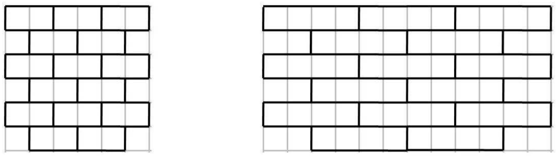

Figure 2: Lattices $L_{3,1}$ and $L_{3,2}$.

Obviously, Wiener index depends on the size of the lattice i.e. on n and k, and therefore compression ratio depends on them too. Our goal is to establish the limit of compression ratio for $R_{n,k}$, $L_{n,k}$ and R_n^d when n tends to infinity for a fixed k.

3 Compression ratio of $R_{n,k}$

For $k = 1$ (i.e. for square shaped rectangular lattices), the result $cr(R_{n,k}) = \frac{3}{4}$ was already obtained in reference [4]. The result $\frac{3}{4}$ for bidimensional square is the same as the result for monodimensional lattices (polymer chains) obtained in [3]. We will here derive the same result for $k > 1$, i.e. for some rectangle shaped lattices. Let k be fixed integer. We have following theorems.

Theorem 3.1. *For the lattice graph* ${}^O R_{n,k}$*, Wiener index* $W({}^O R_{n,k})$ *is a polynomial in* n *of degree* 5 *whose leading coefficient equals*

$$\frac{16}{3}k^2\left(1+k\right).$$

Proof. For vertices $v_{i,j}$ and $v_{p,q}$ of ${}^O R_{n,k}$ holds

$$d\left(v_{i,j}, v_{p,q}\right) = |p-i| + |q-j|\,.$$

Obviously,

$$W({}^O R_{n,k}) = \sum_{i=1}^{2kn}\sum_{j=1}^{2n}\sum_{p=i}^{2kn}\sum_{q=1}^{2n} d(v_{i,j}, v_{p,q}) - \sum_{i=1}^{2kn}\sum_{j=1}^{2n}\sum_{q=j}^{2n} d(v_{i,j}, v_{i,q}).$$

The second (subtracted) sum does not influence leading term in n, therefore we can neglect it. To avoid absolute value, we can rewrite the first sum as

$$W({}^O R_{n,k}) \approx 2\sum_{i=1}^{2kn}\sum_{j=1}^{2n}\sum_{p=i}^{2kn}\sum_{q=j}^{2n} d(v_{i,j}, v_{p,q}) - \sum_{i=1}^{2kn}\sum_{j=1}^{2n}\sum_{p=i}^{2kn} d(v_{i,j}, v_{p,j}).$$

Again, the second (subtracted) sum does not influence leading term in n, therefore we can neglect it. Now we have

$$W({}^O R_{n,k}) \approx 2\sum_{i=1}^{2kn}\sum_{j=1}^{2n}\sum_{p=i}^{2kn}\sum_{q=j}^{2n} (p-i+q-j)$$

and the result then follows by easy calculation.

□

Theorem 3.2. *For the lattice graph ${}^{C}R_{n,k}$, Wiener index $W({}^{C}R_{n,k})$ is a polynomial in n of degree 5 whose leading coefficient equals*

$$4k^2\,(1+k)\,.$$

Proof. Obviously, all vertices in ${}^{C}R_{n,k}$ have the same sums of distances to all other vertices. Therefore, to obtain $W({}^{C}R_{n,k})$ it is enough to calculate distances from one vertex ($v_{1,1}$ is easiest for calculation) to all other vertices. Since ${}^{C}R_{n,k}$ is a torus, to do that we will calculate the sum of distances from $v_{1,1}$ to $v_{p,q}$ where $1 \leq p \leq kn$ and $1 \leq q \leq n$ and multiply it by 4. Now, the obtained number should be multiplied by number $2n \cdot 2kn$ of vertices in the lattice, and then divided by 2 since each distance was counted twice. Therefore we have

$$W({}^{C}R_{n,k}) \approx \frac{1}{2}\cdot 2n\cdot 2kn\cdot 4\cdot\sum_{p=1}^{kn}\sum_{q=1}^{n}(p-1+q-1)\,,$$

and the result now follows by direct calculation.

□

Corollary 3.3. *Holds*

$$\lim_{n\to\infty} cr(R_{n,k}) = \frac{3}{4}.$$

4 Compression ratio of $L_{n,k}$

This section is devoted to the exact determination of the compression factors for the 2-dimensional polygonal lattice $L_{n,k}$. We start from a numerical example devoted to the case of the graphene lattice. Figure 3 shows the rectangular $L_{3,1}$ portion of this hexagonal system.

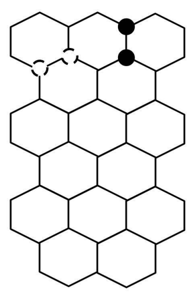

Figure 3: View of $L_{3,1}$ hexagonal lattice with bold vertices common to neighboring polygons along one row, whereas the dotted ones are shared by two neighboring polygons from two different rows.

The numerical determination of invariants $W({}^{O}G)$ and $W({}^{C}G)$ for this infinite graph is based on the results summarized in Table 1 where, for an increasing number of vertices N, values of both descriptors are listed. The exact polynomial forms are given in Table 1 for ${}^{O}W$ and ${}^{C}W$ with leading terms ${}^{O}W \approx \frac{2N^{\frac{5}{2}}}{5}$ and ${}^{C}W \approx \frac{7N^{\frac{5}{2}}}{24}$ respectively, producing

N	${}^{O}\mathbf{W} = \frac{1}{15}\left(6N^{\frac{5}{2}} - 5N^{\frac{3}{2}} - N^{\frac{1}{2}}\right)$	${}^{C}\mathbf{W} = \frac{1}{24}\left(7N^{\frac{5}{2}} - 4N^{\frac{3}{2}}\right)$
36	3 038	2 232
100	39 666	29 000
196	214 214	156 408
324	753 882	550 152
484	2 057 902	1 501 368
676	4 746 690	3 462 472
900	9 710 998	7 083 000

Table 1: Exact polynomial forms for the Wiener index of the open (${}^{O}W$) and closed (${}^{C}W$) rectangular graphene lattices $L_{n,1}$ with N vertices.

the value $cr(L_{n,1}) = \frac{35}{48}$ for the compression factor of rectangular graphene. More details about the numerical determination of various topological descriptors of the graphene rectangular lattices are given in [2].

Now, we want to establish the compression ratio of $L_{n,k}$. We will use the fact that $L_{n,k}$ can be considered as the subgraph of $R_{n,k}$. Therefore, distances between vertices in $L_{n,k}$ for some pairs of vertices are equal as in $R_{n,k}$, while for some other pairs of vertices are greater than in $R_{n,k}$. We will establish for which pairs of vertices the distance is greater in $L_{n,k}$ than in $R_{n,k}$, and how much greater. We will not establish the exact value of Wiener index for ${}^{O}L_{n,k}$ or ${}^{C}L_{n,k}$ as that would be tedious for all the possible cases, but we will neglect some quantities which do not influence the leading coefficient.

Theorem 4.1. *For the lattice graph ${}^{O}L_{n,k}$, Wiener index $W({}^{O}L_{n,k})$ is a polynomial in n of degree 5 whose leading coefficient equals*

$$\frac{16}{15}k^2\left(7k+5\right).$$

Proof. Since $L_{n,k}$ is a subgraph of $R_{n,k}$ that means distances in $L_{n,k}$ are equal or greater than in $R_{n,k}$. Therefore

$$W({}^{O}L_{n,k}) = W({}^{O}R_{n,k}) + \Delta.$$

If we establish leading coefficient in Δ, then by combining that result with Theorem 3.1 we obtain desired result. Therefore, we are interesting in establishing Δ, i.e. for which pairs of vertices the distance in $L_{n,k}$ is greater than in $R_{n,k}$ and how much greater. For a vertex $v_{i,j}$ lattice $L_{n,k}$ can be divided into four areas as illustrated with Figure 4. Vertices in areas left and right to $v_{i,j}$ have the same distance to $v_{i,j}$ in $L_{n,k}$ as in $R_{n,k}$, while the vertices in areas up and down to $v_{i,j}$ have the greater distance to $v_{i,j}$ in $L_{n,k}$ than in $R_{n,k}$. For easier calculation, we will approximate zig-zag lines that divide $L_{n,k}$ into areas with lines

$$q = q_1(p) = j - \frac{1}{k}(p-i),$$
$$q = q_2(p) = j + \frac{1}{k}(p-i).$$

as also illustrated in Figure 4, and we denote upper and lower areas with $L_{n,k}^{A}$ and left and right areas with $L_{n,k}^{B}$. With such an approximation, we make an error in some vertices near

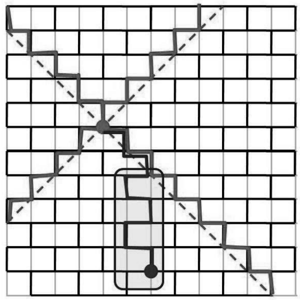

Figure 4: Division of lattice $L_{n,k}$ into areas for a vertex $v_{i,j}$.

the lines, but since number of such vertices is linear in n, that error does not influence the leading term of Δ and we can neglect it. Now that we established the pairs of vertices for which the distance is greater, we want to establish how much greater. Obviously, the trouble is if we have to go vertically, since some vertical edges are missing now. Therefore, we go vertically as little as we can (the shortest path is illustrated in Figure 4), and for each vertical step the path is k edges longer in $L_{n,k}$ than in $R_{n,k}$. Now, to calculate all these exactly, we have to divide into cases, regarding the position of $v_{i,j}$ (since then lines $q_1(p)$ and $q_2(p)$ intersect boundaries in different sides, which influences calculation). The division into cases is illustrated with Figure 5. There are four areas, but upper and lower

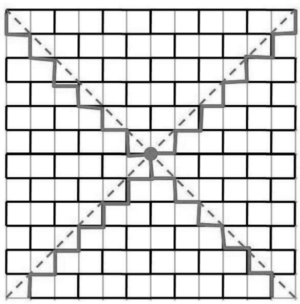

Figure 5: Division into cases with respect to the position of $v_{i,j}$ in the lattice.

are equal up to symmetry, and also left and right. Again, we will approximate division with lines

$$j = j_1(i) = n - \frac{1}{k}(i - kn),$$
$$j = j_2(i) = n + \frac{1}{k}(i - kn)$$

and denote upper and lower areas with $L_{n,k}^{(1)}$ and left and right areas with $L_{n,k}^{(2)}$. This approximation again produces an error in calculating $W(L_{n,k})$ but not great enough to influence the leading term, therefore we can again neglect it. Before we proceed note that lines we introduced can also be expressed as $i = i_1(j)$, $i = i_2(j)$, $p = p_1(q)$ and $p = p_2(q)$. Now we distinguish two cases.

CASE I: Let $v_{i,j} \in L^{(1)}_{n,k}$. For an arbitrary vertex $v_{p,q}$, we are interested in establishing the difference in $d(v_{i,j}, v_{p,q})$ between $L_{n,k}$ and $R_{n,k}$. The difference is greater than zero only if $v_{p,q} \in L^A_{n,k}$. To calculate the difference Δ_1 that occurs for pairs of vertices in this case, we have to divide $L^A_{n,k}$ into 4 subareas A_1, A_2, A_3, A_4 as illustrated in Figure 6. Now

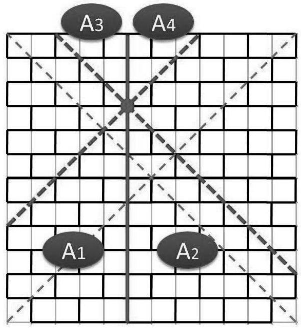

Figure 6: The division of the area $L^A_{n,k}$ into subareas in the case $v_{i,j} \in L^{(1)}_{n,k}$.

we have

$$\Delta(A_1) = 2\sum_{j=n}^{2n}\sum_{i=i_1(j)}^{i_2(j)}\sum_{p=1}^{i}\sum_{q=1}^{q_2(p)} (q_2(p) - q)\cdot k,$$

$$\Delta(A_2) = 2\sum_{j=n}^{2n}\sum_{i=i_1(j)}^{i_2(j)}\sum_{p=i}^{2kn}\sum_{q=1}^{q_1(p)} (q_1(p) - q)\cdot k,$$

$$\Delta(A_3) = 2\sum_{j=n}^{2n}\sum_{i=i_1(j)}^{i_2(j)}\sum_{p=p_1(2n)}^{i}\sum_{q=q_1(p)}^{2n} (q - q_1(p))\cdot k,$$

$$\Delta(A_4) = 2\sum_{j=n}^{2n}\sum_{i=i_1(j)}^{i_2(j)}\sum_{p=i}^{p_2(2n)}\sum_{q=q_2(p)}^{2n} (q - q_2(p))\cdot k.$$

Therefore $\Delta_1 = \Delta(A_1)+\Delta(A_2)+\Delta(A_3)+\Delta(A_4)$ and by direct calculation we establish that Δ_1 is a polynomial in n of degree 5 with leading coefficient being $\frac{14}{5}k^3$.

CASE II: Let $v_{i,j} \in L^{(2)}_{n,k}$. Again, the difference $d(v_{i,j}, v_{p,q})$ is greater than zero only if $v_{p,q} \in L^A_{n,k}$. Again, we divide $L^A_{n,k}$ into four areas $A_1, \ldots, A_4$ as shown in Figure 7. Now, we calculate

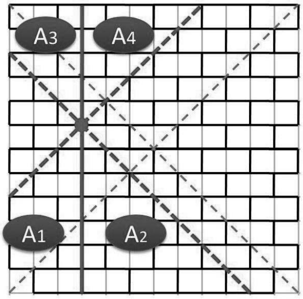

Figure 7: The division of the area $L_{n,k}^{A}$ into subareas in the case $v_{i,j} \in L_{n,k}^{(2)}$.

$$\Delta(A_1) = 2\sum_{i=1}^{kn}\sum_{j=j_2(i)}^{j_1(i)}\sum_{p=1}^{i}\sum_{q=1}^{q_2(p)}(q_2(p)-q)\cdot k,$$

$$\Delta(A_2) = 2\sum_{i=1}^{kn}\sum_{j=j_2(i)}^{j_1(i)}\sum_{p=i}^{p_1(1)}\sum_{q=1}^{q_1(p)}(q_1(p)-q)\cdot k,$$

$$\Delta(A_3) = 2\sum_{i=1}^{kn}\sum_{j=j_2(i)}^{j_1(i)}\sum_{p=1}^{i}\sum_{q=q_1(p)}^{2n}(q-q_1(p))\cdot k,$$

$$\Delta(A_4) = 2\sum_{i=1}^{kn}\sum_{j=j_2(i)}^{j_1(i)}\sum_{p=i}^{p_2(2n)}\sum_{q=q_2(p)}^{2n}(q-q_2(p))\cdot k.$$

Now $\Delta_2 = \Delta(A_1)+\Delta(A_2)+\Delta(A_3)+\Delta(A_4)$ and by direct calculation we establish that Δ_2 is a polynomial in n of degree 5 with leading coefficient being $\frac{22}{15}k^3$.
Therefore, we conclude that Δ is a polynomial in n of degree 5 with leading coefficient being equal to

$$\frac{1}{2}\left(\frac{14}{5}k^3+\frac{22}{15}k^3\right)=\frac{32}{15}k^3.$$

Now the result follows from this and Theorem 3.1.

□

Theorem 4.2. *For the lattice graph* ${}^{C}L_{n,k}$*, Wiener index* $W({}^{C}L_{n,k})$ *is a polynomial in* n *of degree* 5 *whose leading coefficient equals*

$$\frac{4}{3}k^2\left(4k+3\right).$$

Proof. Let us introduce the same vertex notation in ${}^{C}L_{n,k}$ as in ${}^{C}R_{n,k}$ (we can do that as ${}^{C}L_{n,k}$ is a subgraph of ${}^{C}R_{n,k}$). Since, ${}^{C}L_{n,k}$ is a subgraph of ${}^{C}R_{n,k}$ we have

$$W\left({}^{C}L_{n,k}\right)=W\left({}^{C}R_{n,k}\right)+\Delta.$$

Therefore, if we establish Δ, the result will follow from Theorem 3.2. Since all vertices in ${}^{C}L_{n,k}$ are equivalent in the sense that they have the same distances to all other vertices, it is enough to calculate difference in distances for one vertex, and then multiply it by number of vertices, and divide by two since each difference is thus calculated twice. It is easiest if we calculate for $v_{1,1}$. Since ${}^{C}L_{n,k}$ is a torus, we will calculate the difference in distance from $v_{1,1}$ to $v_{p,q}$ where $1 \leq p \leq kn$ and $1 \leq q \leq n$ and multiply it by 4. In that area difference in distances is greater than 0 only if $1 \leq p \leq kn$ and $q_1(p) \leq q \leq n$ where

$$q_1(p) = \frac{1}{k} \cdot p.$$

Therefore,

$$\Delta \approx \frac{1}{2} \cdot 2kn \cdot 2n \cdot 4 \cdot \sum_{p=1}^{kn} \sum_{q=q_1(p)}^{n} (q - q_1(p)).$$

By direct calculation, we obtain that Δ is a polynomial in n of degree 5 with leading coefficient being

$$\frac{4}{3}k^3.$$

Now the result follows from Theorem 3.2.

□

Corollary 4.3. *Holds*

$$\lim_{n\to\infty} cr(L_{n,k}) = \frac{5\,(4k+3)}{4\,(7k+5)}.$$

Now, we can derive from this result compression ratio for some specific lattices. For example, lattice $L_{n,1}$ is hexagonal lattice with $2n$ rows and n columns of hexagons. Therefore, from Corollary 4.3 follows that compression factor for such lattice equals

$$\lim_{n\to\infty} cr(L_{n,1}) = \frac{5\,(4+3)}{4\,(7+5)} = \frac{35}{48} = 0.729\,17$$

confirming the result numerically derived in [2]. For $k = 2$, lattice $L_{n,k}$ is a lattice consisting of 10-gons, with $2n$ rows and n columns of rectangular unit cells with size $2k = 4$ squares as in Figure 2. From Corollary 4.3 follows that compression factor for such lattice equals

$$\lim_{n\to\infty} cr(L_{n,2}) = \frac{5\,(4\cdot 2+3)}{4\,(7\cdot 2+5)} = \frac{55}{76} = 0.723\,68.$$

Corollary 4.4. *Holds*

$$\lim_{n\to\infty} cr(L_{n,k}) = \lim_{n\to\infty} cr(R_{n,k}) - \frac{k}{4(7k+5)}.$$

Invariant $\lim_{n\to\infty} cr(L_{n,k})$ *reaches its maximum value* $\lim_{n\to\infty} cr(L_{n,1}) = \frac{35}{48}$, *whereas for large* k *the limit* $\lim_{n\to\infty} cr(L_{n,k})$ *constantly decreases toward its lower limit* $\frac{5}{7}$.

5 Compression ratio of R_n^d

In this section, we will establish the limit of compression ratio of d-dimensional rectangular cube R_n^d when n tends to infinity, and we will show that it does not depend on dimension d.

Theorem 5.1. *Holds*

$$\lim_{n\to\infty} cr\left(R_n^d\right) = \frac{3}{4}.$$

Proof. Let us denote $[n] = \{1, 2, \ldots, n\}$. Note that

$$\begin{aligned}
\lim_{n\to\infty} \frac{W\left({}^O R_n^d\right)}{n^{2d+1}} &= \lim_{n\to\infty} \frac{1}{n^{2d+1}} \left(\sum_{(x_1,\ldots x_d)\in[n]^d} \sum_{(y_1,\ldots y_d)\in[n]^d} \sum_{1\le i\le d} |y_i - x_i| \right) = \\
&= \lim_{n\to\infty} \frac{1}{n^{2d+1}} \left(d \cdot n^{2d-2} \sum_{1\le i\le j\le n} (j-i) \right) = \\
&= \lim_{n\to\infty} \frac{1}{n^{3}} \left(d \cdot \sum_{1\le i\le j\le n} (j-i) \right) = \\
&= \frac{d}{6.}
\end{aligned}$$

On the other hand

$$\begin{aligned}
\lim_{n\to\infty} \frac{W\left({}^C R_n^d\right)}{n^{2d+1}} &= \lim_{n\to\infty} \frac{1}{n^{2d+1}} \left(\frac{1}{2} \cdot n^d \cdot 2^d \cdot \sum_{(x_1,\ldots,x_d)\in[n/2]^d} \sum_{1\le i\le d} (x_i - 1) \right) = \\
&= \lim_{n\to\infty} \frac{1}{n^{d+1}} \left(2^{d-1} \cdot d \cdot \left(\frac{n}{2}\right)^{d-1} \sum_{1\le i\le \frac{n}{2}} (i-1) \right) = \\
&= \lim_{n\to\infty} \frac{1}{n^{2}} \left(d \cdot \sum_{1\le i\le \frac{n}{2}} (i-1) \right) = \\
&= \frac{d}{8}.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} cr(R_n^d) = \lim_{n\to\infty} \frac{W\left({}^C R_n^d\right)}{W\left(R_n^d\right)} = \lim_{n\to\infty} \frac{\frac{W\left({}^C R_n^d\right)}{n^{2d+1}}}{\frac{W(R_n^d)}{n^{2d+1}}} = \frac{\frac{d}{8}}{\frac{d}{6}} = \frac{3}{4}.$$

□

6 Conclusion

In this paper, we studied compression ratio for open and closed 2-dimensional rectangular lattices $R_{n,k}$ and 2-dimensional polygonal lattices $L_{n,k}$. For that purpose, we established that leading coefficient in $W({}^O R_{n,k})$ equals $\frac{16}{3}k^2\,(1+k)$ (Theorem 3.1), while in $W(^C R_{n,k})$ equals $4k^2(1+k)$ (Theorem 3.2), which yields $\lim_{n\to\infty} cr(R_{n,k}) = \frac{3}{4}$ (Corollary 3.3). Also, we established that leading coefficient in $W(^O L_{n,k})$ equals $\frac{16}{15}k^2$ $(7k+5)$ (Theorem 4.1), while leading coefficient in $W(^C L_{n,k})$ equals $\frac{4}{3}k^2(4k+3)$ (Theorem 4.2), which yields $\lim_{n\to\infty} cr(L_{n,k}) = \frac{5(4k+3)}{4(7k+5)}$ (Corollary 4.3). Lattice $L_{n,1}$ is hexagonal lattice with $2n$ rows and n columns of hexagons, for which therefore holds $\lim_{n\to\infty} cr(L_{n,1}) = \frac{35}{48}$. Lattice $L_{n,2}$ is 10-gonal lattice with $2n$ rows and n columns, for which therefore $\lim_{n\to\infty} cr(L_{n,2}) = \frac{5(8+3)}{4(14+5)} = \frac{55}{76}$. Finally, we established that for d-dimensional rectangular cube holds $\lim_{n\to\infty} cr\left(R_n^d\right) = \frac{3}{4}$ (Theorem 5.1).

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On the packing chromatic number of square and hexagonal lattice*

Danilo Korže
FERI, University of Maribor, Smetanova 17, SI-2000 Maribor, Slovenia

Aleksander Vesel †
Faculty of Natural Sciences and Mathematics, University of Maribor
Koroška cesta 160, SI-2000 Maribor, Slovenia

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Abstract

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that the vertex set $V(G)$ can be partitioned into disjoint classes $X_1, \ldots, X_k$, with the condition that vertices in X_i have pairwise distance greater than i. We show that the packing chromatic number for the hexagonal lattice $\mathcal{H}$ is 7. We also investigate the packing chromatic number for infinite subgraphs of the square lattice $\mathbb{Z}^2$ with up to 13 rows. In particular, we establish the packing chromatic number for $P_6 \square \mathbb{Z}$ and provide new upper bounds on these numbers for the other subgraphs of interest. Finally, we explore the packing chromatic number for some infinite subgraphs of $\mathbb{Z}^2 \square P_2$. The results are partially obtained by a computer search.

Keywords: Packing chromatic number, hexagonal lattice, square lattice, computer search.

Math. Subj. Class.: 05C70, 05C85

1 Introduction and preliminaries

The packing coloring was introduced by Goddard et al. in [6] under the name broadcast coloring. The concept comes from the regulations concerning the assignment of broadcast frequencies to radio stations. In particular, two radio stations which are assigned the same frequency must be placed sufficiently far apart so that neither broadcast interferes with the reception of the other. Moreover, the geographical distance between two radio stations

* Supported by the Ministry of Science of Slovenia under the grant 0101-P-297.
† Corresponding author.
E-mail addresses: korze@uni-mb.si (Danilo Korže), vesel@uni-mb.si (Aleksander Vesel)

which are assigned the same frequency is directly related to the power of their broadcast signals. These frequency restrictions have inspired the graphical coloring problem defined below.

A k-coloring of a graph G is a function f from $V(G)$ onto a set $C = \{1, 2, \dots, k\}$ (with no additional constraints). The elements of C are called *colors*. Let X_i denote the set of vertices with the image (color) i. Note that $X_1, ..., X_k$ is partition of the vertex set of G into disjoint (color) classes.

Let $X_1, ..., X_k$ be a partition of the vertex set of G with respect to the following constraints: each color class X_i is a set of vertices with the property that any distinct pair $u, v \in X_i$ satisfies $d_G(u, v) > i$. Here $d_G(u, v)$ denotes the usual shortest path distance between u and v. Then X_i is said to be an *i-packing*, while such a partition is called a *packing k-coloring*. The smallest integer k for which there exists a packing k-coloring of G is called the *packing chromatic number of* G and it is denoted by $\chi_\rho(G)$.

Let $G = (V, E)$ be a graph. A *walk* is a sequence of vertices $v_1, v_2, \dots, v_k$ and edges $v_i v_{i+1}$, $1 \le i \le k - 1$. A *path* on n vertices is a walk on n distinct vertices and denoted P_n. A walk is *closed* if $v_1 = v_n$. A closed walk in which all vertices (except the first and the last) are different, is a *cycle*. The cycle on n vertices is denoted C_n. For $u, v \in V(G)$, $d_G(u, v)$ or $d(u, v)$ denotes the length of the shortest walk (i.e., the number of edges on the shortest walk) in G from u to v. These definitions extend naturally to directed graphs.

A set $S \subseteq V(G)$ is *independent* if $xy \notin E(G)$ for any pair of vertices $x, y \in S$. Cardinality of a largest independent set S of G is the *independence* number $\alpha(G)$ of G.

This paper studies the packing chromatic number of hexagonal lattice and of some infinite subgraphs of square lattice. Section 2 contains the search for the lower bound on the packing chromatic number in hexagonal lattice. The bound is obtained by a computer program using the dynamic approach for computing graph invariants, as described in the first part of the section. Section 3 discusses the packing chromatic number for some infinite subgraphs of the square lattice. We establish the packing chromatic number for $P_6 \square \mathbb{Z}$ and provide upper bounds on these numbers for $P_n \square \mathbb{Z}$, where $7 \le n \le 13$. We conclude the paper with the packing chromatic number for $C_4 \square \mathbb{Z}$ as well as with some partial results on upper bounds for some infinite subgraphs of $\mathbb{Z}^2 \square P_2$ provided in Section 4.

The results in our paper were partially obtained by computers, mainly in Windows environment, but some also using Linux Ubuntu operating system. The machines used for computations were also diverse: Intel i7 930 based personal computer, Intel Q9400 based machine and a computer cluster (with up to 24 processor cores). All computations were carried out during six months, starting in the middle of 2010. The development environment and class libraries Lazarus (version of Pascal language) were used to write all necessary programs.

2 Hexagonal lattice

The hexagonal lattice $\mathcal{H}$ plays a crucial role in many network applications, particularly in frequency assignments, e.g. see [5]. It was proved by Brešar et al. [1] that the packing chromatic number of the infinite hexagonal lattice lies between 6 and 8. The result was improved by Fiala et al. [3], where the packing 7-coloring of the hexagonal lattice is presented.

We show in this section that actual lower bound on the packing chromatic number of the infinite hexagonal lattice is 7 and therefore $\chi_\rho(\mathcal{H}) = 7$.

We now present the algorithms, that have been used to provide the main result. We first describe the concept needed to describe our computer checking. The idea is introduced in [9] in a more general framework, but for our purposes the following description will be sufficient.

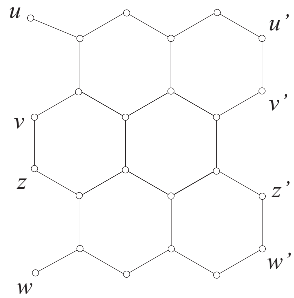

Figure 1: Graph H_1.

Observe first the graph H_1 depicted in Fig. 1. We construct H_i for $i > 1$ as follows. Take the graph which is composed of an isomorphic copy of H_{i-1} and of an isomorphic copy of H_1. Then add additional four edges that connect vertices u', v', w' , and z' of the last added copy of H_1 in H_{i-1} with the vertices u, v, w, and z of the new copy of H_1. As an example see Fig. 2 where H_2 is depicted.

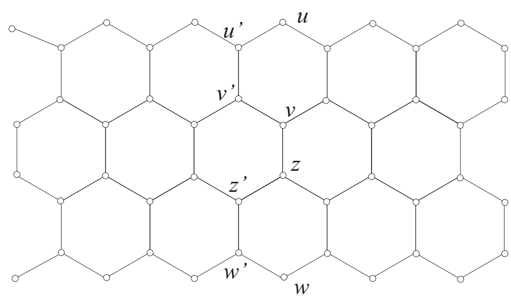

Figure 2: Graph H_2.

Obviously, H_i is a subgraph of $\mathcal{H}$ for $i \geq 1$.

We next define a directed graph D_k as follows.

The vertices of D_k are all packing k-colorings of H_1. Let u and v be two distinct vertices of D_k. Then $\widetilde{uv}$ denotes a k-coloring of H_2 such that u and v induce the respective packing k-coloring of the first and the second copy of H_1. Note that $\widetilde{uv}$ need not to be a packing k-coloring of H_2. uv is an arc in D_k if and only if $\widetilde{uv}$ is a packing k-coloring of H_2.

Lemma 2.1. *Let $k \leq 6$. Then H_i admits a packing k-coloring if and only if D_k possesses a walk $P = v_1, v_2, \ldots, v_i$ with v_j corresponding to the j-th copy of H_1.*

Proof. Suppose first that D_k possesses a walk $P = v_1, v_2, \ldots, v_i$. If $i = 2$, then P is an arc from v_1 to v_2 in D_k and the claim is obvious. Let then $i > 2$. Suppose the claim holds for $P' = v_1, v_2, \ldots, v_{i-1}$, i.e. H_{i-1} admits a packing k-coloring. Since P has an arc from v_{i-1} to v_i, the corresponding colorings induce a packing k-coloring in a copy of H_2 that corresponds to v_{i-1} and v_i. In order to see that the assertion holds, note that the distance between a vertex of a copy of H_1 that corresponds to v_i and a vertex of a copy of H_1 that corresponds to v_{i-2} is at least 7.

Suppose now that H_i admits a packing k-coloring. If $i = 2$, then by definition of D_k a packing k-coloring of H_2 induce an arc in D_k. Let then $i > 2$. Note that H_i is composed of an isomorphic copy of H_{i-1}, say X, and of an isomorphic copy of H_1, say Y. Y is connected in H_i to an isomorphic copy of H_1, say Z. Suppose the claim holds for H_{i-1} and let $P' = v_1, v_2, \ldots, v_{i-1}$ denote a walk in D_k that corresponds to X. Since H_i admits a packing k-coloring, Y induces a packing k-coloring of H_1, say v_i. Y and Z together induce a packing k-coloring of H_2 and therefore $v_{i-1}v_i$ forms an arc in D_k. Then $P = v_1, v_2, \ldots, v_{i-1}, v_i$ is a walk in D_k and the proof is complete. □

Lemma 2.2. *Let $k \leq 6$. Then $\mathcal{H}$ admits a packing k-coloring only if D_k contains a closed directed walk.*

Proof. Let $\mathcal{H}$ for a given k admit a packing k-coloring denoted f. Suppose that D_k is acyclic. Since H_1 is finite, there is obviously only a finite number of vertices (packing k-colorings of H_1) in D_k, say n_k. Let then $d < n_k$ denotes the length of a longest directed path in D_k. Take now a subgraph of $\mathcal{H}$ isomorphic to H_{d+2}. A restriction of f to H_{d+2} is obviously a packing k-coloring of H_{d+2}. From Lemma 2.1 it follows that H_{d+2} admits a packing k-coloring if and only if D_k possesses a walk $P = v_1, v_2, \ldots, v_{d+2}$ with v_j corresponding to a packing k-coloring of the j-th copy of H_1. But since D_k is acyclic, the length of the longest walk in D_k is at most d and we obtain a contradiction.

□

Theorem 2.3. $\chi_\rho(\mathcal{H}) = 7$.

Proof. Since it is proved in [1] that the packing chromatic number of the infinite hexagonal lattice is at least 6 and since in [3] a coloring of the hexagonal lattice using 7 colors is presented, we have to show that $\mathcal{H}$ does not admit a packing 6-coloring.

We first constructed the graph D_6 by using a computer program. The graph consists of 26660 vertices with a maximum output degree of 37 (see also the concluding remark). By the depth first search algorithm we next established that D_6 is an acyclic graph. From Lemma 2.2 then it follows that the hexagonal lattice cannot admit a packing 6-coloring. This assertion completes the proof. □

An alternative approach to prove Theorem 2.3 is to use a naive brute force search for a large enough subgraph of $\mathcal{H}$. The approach used in the proof Theorem 2.3, however, is potentially much more interesting and utile in order to search for the packing chromatic number in other families of graphs since it uses the packing k-colorings of a relatively small graph.

3 Square lattice

Cartesian product of graphs provide a setting which has been widely used in designing large scale computer systems and interconnection networks. The *Cartesian product* of graphs G

and H is the graph $G\Box H$ with vertex set $V(G) \times V(H)$ and $(x_1, x_2)(y_1, y_2) \in E(G\Box H)$ whenever $x_1y_1 \in E(G)$ and $x_2 = y_2$, or $x_2y_2 \in E(H)$ and $x_1 = y_1$. The Cartesian product is commutative and associative, having the trivial graph as a unit, cf. [8]. The subgraph of $G\Box H$ induced by $u \times V(H)$ is isomorphic to H and it is called an H-fiber.

It will be convenient to view the square lattice as the Cartesian product of two infinite paths, i.e $\mathbb{Z}\Box\mathbb{Z}$.

Goddard et al. [6] determined the packing chromatic number for infinite subgraphs of the square lattice $\mathbb{Z}^2$ with up to 5 rows. In the same paper the question of determining the packing chromatic number of the infinite square lattice was posed. The best upper bound 17 was given by Holub and Soukal [7], while the best lower bound 12 was determined by Ekstein et al. [2].

We have considered infinite subgraphs of $\mathbb{Z}\Box\mathbb{Z}$ with up to 13 rows. The main results are summarized in the following proposition.

Proposition 3.1.
(i) $\chi_\rho(P_6\Box\mathbb{Z}) = 10$,
(ii) $\chi_\rho(P_7\Box\mathbb{Z}) \leq 11$,
(iii) $\chi_\rho(P_8\Box\mathbb{Z}) \leq 12$,
(iv) $\chi_\rho(P_9\Box\mathbb{Z}) \leq 13$,
(v) $\chi_\rho(P_{10}\Box\mathbb{Z}) \leq 14$,
(vi) $\chi_\rho(P_{11}\Box\mathbb{Z}) \leq 14$,
(vii) $\chi_\rho(P_{12}\Box\mathbb{Z}) \leq 15$.
(viii) $\chi_\rho(P_{13}\Box\mathbb{Z}) \leq 15$.

Proof. Note first that if f is a packing k-coloring of $P_n\Box C_\ell$, $k < \ell$, then we can construct from f a packing k-coloring of $P_n\Box P_m$ for every m. One can use f to color every P_n-fibre $(u_j \times P_n)$ of $P_n\Box P_m$ in the same way as the P_n-fibre $(v_{j \bmod \ell} \times P_n)$ of $P_n\Box C_\ell$

1	2	1	3	1	2	1	3	1	2	1	3	1	10
3	1	4	1	8	1	5	1	9	1	4	1	2	1
1	6	1	2	1	3	1	2	1	3	1	7	1	5
2	1	3	1	7	1	4	1	6	1	2	1	3	1
1	9	1	5	1	2	1	3	1	5	1	8	1	4
3	1	2	1	3	1	10	1	2	1	3	1	2	1

Figure 3: A packing 10-coloring of $P_6\Box C_{14}$

1	2	1	3	1	2	1	3	1	2	1	3	1	4	1	5
3	1	6	1	4	1	7	1	5	1	6	1	2	1	7	1
1	8	1	2	1	3	1	2	1	3	1	9	1	3	1	2
4	1	3	1	5	1	10	1	4	1	2	1	5	1	11	1
1	2	1	9	1	2	1	3	1	8	1	3	1	2	1	3
5	1	7	1	3	1	6	1	2	1	7	1	4	1	6	1
1	3	1	2	1	4	1	5	1	3	1	2	1	3	1	2

Figure 4: A packing 11-coloring of $P_7\Box C_{16}$

1	2	1	3	1	2	1	3	1	2	1	3	1	4
3	1	5	1	4	1	10	1	11	1	5	1	2	1
1	8	1	2	1	3	1	2	1	3	1	6	1	9
2	1	3	1	6	1	5	1	4	1	7	1	3	1
1	4	1	7	1	2	1	3	1	2	1	12	1	5
3	1	2	1	3	1	9	1	8	1	3	1	2	1
1	11	1	5	1	4	1	2	1	5	1	4	1	10
2	1	3	1	2	1	3	1	6	1	2	1	3	1

Figure 5: A packing 12-coloring of $P_8 \square C_{14}$

1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3
4	1	5	1	6	1	4	1	5	1	7	1	4	1	5	1	6	1	7	1
1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2
12	1	8	1	7	1	10	1	11	1	6	1	9	1	13	1	4	1	5	1
1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3
6	1	4	1	9	1	5	1	4	1	8	1	5	1	7	1	10	1	11	1
1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2
7	1	5	1	13	1	6	1	7	1	12	1	4	1	6	1	5	1	4	1
1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3

Figure 6: A packing 13-coloring of $P_9 \square C_{20}$

In order to obtained the upper bounds, we therefore first tried to find a packing k-coloring of $P_n \square C_m$ for every n of the interest with k (and m) being as small as possible.

The obtained colorings for $n \in \{6, 7, 8, 9, 11, 13\}$ are depicted in Figs. 3 - 8, while a packing 14-coloring of $P_{10} \square C_{16}$ and a packing 15-coloring of $P_{12} \square C_{16}$ can be obtained from the first 10 rows of the packing 14-coloring of $P_{11} \square C_{16}$ depicted in Fig. 7 and the first 12 rows of the packing 15-coloring of $P_{13} \square C_{16}$ depicted in Fig. 8, respectively.

In order to provide the lower bound for $\chi_\rho(P_6 \square \mathbb{Z})$ we applied the backtracking search, e.g. see [10], adapted to packing colorings. Since the procedure did not find a packing 9-coloring in $\chi_\rho(P_6 \square P_{12})$, the assertion follows. $\square$

Results in Proposition 3.1 provide general upper bounds for infinite families of Cartesian products of two paths. For some graphs of these families however, better bounds or even the exact numbers can be computed. The results are depicted on the web page presented in the concluding remark. We again applied the backtracking search, which it is guaranteed to find a solution, if one exits, but it is relatively time consuming and therefore not usable for larger graphs.

The colorings depicted in Figs. 3 - 8 have something in common: every second vertex in a row (column) is colored by the color 1. We therefore conjecture, that if a packing k-coloring of $P_n \square P_m$ exists, one can always find a packing k-coloring such that the class X_1 is distributed as described above. This conjecture is formally stated below.

Conjecture 3.2. *Let $n \geq 4$ and let $\chi_\rho(P_m \square P_n) = k$. Then exists a packing k-coloring of $P_m \square P_n$ with $|X_1| = \alpha(P_m \square P_n) = \lceil \frac{nm}{2} \rceil$.*

1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3
4	1	5	1	8	1	4	1	6	1	7	1	5	1	12	1
1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2
7	1	9	1	13	1	5	1	10	1	11	1	4	1	6	1
1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3
5	1	4	1	6	1	7	1	4	1	5	1	8	1	14	1
1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2
10	1	11	1	5	1	12	1	9	1	6	1	7	1	4	1
1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3
6	1	7	1	4	1	8	1	5	1	4	1	13	1	5	1
1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2

Figure 7: A packing 14-coloring of $P_{11} \square C_{16}$

1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3
4	1	5	1	8	1	14	1	9	1	6	1	7	1	12	1
1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2
6	1	7	1	4	1	15	1	5	1	4	1	10	1	5	1
1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3
11	1	9	1	5	1	6	1	7	1	8	1	13	1	4	1
1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2
5	1	4	1	10	1	12	1	4	1	5	1	6	1	7	1
1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3
8	1	6	1	7	1	5	1	11	1	9	1	4	1	14	1
1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2
4	1	5	1	13	1	4	1	6	1	7	1	5	1	15	1
1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3

Figure 8: A packing 15-coloring of $P_{13} \square C_{16}$

If the conjecture holds, the vertices of the class X_1 can be fixed and therefore the backtracking is capable to provide results for much larger graph. In order to provide lower bounds for graphs of moderate size we therefore applied backtracking with no additional constraints, while for larger graphs the vertices of the class X_1 were fixed. The results of these computations are summarized in Table 1. The results in the table are of two types: exact values and upper bounds. Some of the upper bounds are exact values of χ_ρ if the Conjecture 3.2 holds. If a value k in the table is exact, that means that a packing k-coloring for the graph of interest is found and that the backtracking procedure confirmed that a packing $(k-1)$-coloring does not exist. An upper bound k means that a packing k-coloring for the graph of interest exists, but we could not prove that a packing $(k-1)$-coloring does not exist. On the other hand, if an upper bound k is marked with asterisk, the backtracking proved that a packing $(k-1)$-coloring with the vertices of the class X_1 fixed as stated in the conjecture does not exist.

$m\backslash n$	6	7	8	9	10	11	12	13	14−15	16−24	25−27	28−41	> 41
6	8	9	9	9	9	9	10	10	10	10	10	10	10
7		9	9	9	10	10	10	10	10	≤11*	≤11*	≤11*	≤11*
8			9	10	10	10	≤11*	≤11*	≤11*	≤11*	≤11*	≤12	≤12
9				10	≤11*	≤11*	≤11*	≤11*	≤11*	≤12*	≤12*	≤12*	≤13
10					≤11*	≤11*	≤11*	≤12*	≤12*	≤12*	≤14	≤14	≤14
11						≤11*	≤12*	≤12*	≤14	≤14	≤14	≤14	≤14
12							≤12*	≤12*	≤15	≤15	≤15	≤15	≤15
13								≤13	≤15	≤15	≤15	≤15	≤15

Table 1: Packing chromatic numbers and bounds for $P_m \square P_n$.

4 Subgraphs of $\mathbb{Z}\square\mathbb{Z}\square P_2$

It is known that $\chi_\rho(\mathbb{Z}^3) = \infty$ [4]. Moreover even the packing chromatic number of $\mathbb{Z}\square\mathbb{Z}\square P_2$ is unbounded [3]. On the other hand, it was proved that $\chi_\rho(G\square\mathbb{Z}) < \infty$ for any finite graph G [3].

Hence, it is worthy to study the packing chromatic number of some infinite subgraphs of $\mathbb{Z}\square\mathbb{Z}\square P_2$. In particular we considered $C_4\square\mathbb{Z}$, $C_6\square\mathbb{Z}$, $C_8\square\mathbb{Z}$, $C_{10}\square\mathbb{Z}$, $C_{12}\square\mathbb{Z}$, and $P_2\square P_3\square\mathbb{Z}$. We were able to obtain exact results for the packing chromatic number of $C_4\square\mathbb{Z}$, while for the other families some partial results and bounds were found.

Proposition 4.1.
(i) $\chi_\rho(C_4\square\mathbb{Z}) = 9$,
(ii) $\chi_\rho(C_6\square\mathbb{Z}) \leq 13$,
(iii) $\chi_\rho(C_8\square\mathbb{Z}) \leq 15$,
(iv) $\chi_\rho(C_{10}\square\mathbb{Z}) \leq 22$,
(v) $\chi_\rho(C_{12}\square\mathbb{Z}) \leq 17$,
(vi) $\chi_\rho(P_2\square P_3\square\mathbb{Z}) \leq 18$.

Proof. The upper bounds follow from the packing 9-coloring of $C_4\square C_{16}$ and from the packing 15-coloring of $C_8\square C_{24}$ depicted in Fig 9 and Fig 10, respectively. The packing 13-coloring of $C_6\square C_{48}$, the packing 22-coloring of $C_{10}\square C_{48}$, the packing 17-coloring of $C_{12}\square C_{48}$ and the packing 18-coloring of $P_2\square P_3\square C_{48}$ can be obtained from the authors or from the web page presented in the concluding remark.

1	4	1	6	1	5	1	8	1	4	1	6	1	5	1	9
2	1	3	1	2	1	3	1	2	1	3	1	2	1	3	1
1	5	1	7	1	4	1	9	1	5	1	7	1	4	1	8
3	1	2	1	3	1	2	1	3	1	2	1	3	1	2	1

Figure 9: A packing 9-coloring of $C_4\square C_{16}$

The lower bound for $C_4\square P_{10}$ is obtained by using the backtracking procedure which confirms that 8-coloring of $C_4\square P_{10}$ does not exist. □

Note that a coloring of G which provides the upper bound in Proposition 4.1 has the vertices of the class X_1 distributed such that the cardinality of X_1 equals the independence number of G. We therefore generalize Conjecture 3.2 as follows.

Let X_m denote P_m or C_m.

1	13	1	8	1	4	1	5	1	9	1	4	1	5	1	8	1	4	1	5	1	9	1	4
3	1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2	1
1	6	1	7	1	12	1	14	1	6	1	7	1	13	1	15	1	6	1	7	1	10	1	5
2	1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3	1
1	4	1	9	1	5	1	4	1	8	1	5	1	4	1	9	1	5	1	4	1	8	1	11
3	1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2	1
1	5	1	15	1	6	1	7	1	10	1	11	1	6	1	7	1	12	1	14	1	6	1	7
2	1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3	1

Figure 10: A packing 15-coloring of $C_8 \Box C_{24}$

Conjecture 4.2. *Let $n \geq 4$ and let $\chi_\rho(X_m \Box P_\ell \Box P_n) = k$. Then there exists a packing k-coloring of $X_m \Box P_\ell \Box P_n$ such that $|X_1| = \alpha(X_m \Box P_\ell \Box P_n)$.*

Analogous as in Section 3 we therefore applied backtracking with no additional constraints for graphs of moderate size, while for larger graphs the vertices of the class X_1 were fixed. The packing colorings of the graphs of interest can be obtained from the authors or from the web page presented in the concluding remark. The results of these computations are summarized in Table 2, where an upper bound k marked with asterisk means that the backtracking proved that a packing $(k-1)$-coloring with the vertices of the class X_1 fixed as stated in the conjecture does not exist.

$m \backslash n$	2	3	4	5	6	7	8	9	10	11	12−15	16−18	19−34	> 34
4	5	5	7	7	7	7	8	8	9	9	9	9	9	9
6	5	8	8	8	10	10	11	11	11	12	≤ 12*	≤ 12*	≤ 13	≤ 13
8	7	7	9	9	10	10	11	≤12*	≤12*	≤ 13*	≤ 13*	≤ 14	≤ 14	≤ 15
$P_2 \Box P_3$	5	8	8	10	10	11	≤12*	≤12*	≤14	≤ 15	≤ 18	≤ 18	≤ 18	≤ 18

Table 2: Packing chromatic numbers for $C_m \Box P_n$ and $P_2 \Box P_3 \Box P_n$ (below).

Concluding remark

All obtained packing colorings as well as the graph D_6 can be obtained from the authors or directly from the web page `http://matematika-racunalnistvo.fnm.uni-mb.si/personal/vesel/constructions.aspx`.

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Product irregularity strength of certain graphs

Marcin Anholcer *

Poznań University of Economics, Al. Niepodległości 10, 61-875 Poznań, Poland

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Abstract

Consider a simple graph G with no isolated edges and at most one isolated vertex. A labeling $w : E(G) \to \{1, 2, \dots, m\}$ is called *product - irregular*, if all product degrees $pd_G(v) = \prod_{e \ni v} w(e)$ are distinct. The goal is to obtain a product - irregular labeling that minimizes the maximal label. This minimal value is called *the product irregularity strength* and denoted $ps(G)$. We give the exact values of $ps(G)$ for several families of graphs, as complete bipartite graphs $K_{m,n}$, where $2 \le m \le n \le \binom{m+2}{2}$, some families of forests, including complete d-ary trees, and other graphs with $\delta(G) = 1$.

Keywords: Product-irregular labeling, product irregularity strength, tree.

Math. Subj. Class.: 05C05, 05C15, 05C78

1 Introduction

Assume we are given simple undirected graph $G = (V(G), E(G))$ with neither loops nor isolated edges and with at most one isolated vertex. Let us define integer labelling $w : E(G) \to \{1, 2, \dots, s\}$. For every vertex $v \in V(G)$ we define the *product degree* as

$$pd_G(v) = \begin{cases} \prod_{e \ni v} w(e), & d_G(v) > 0, \\ 0, & d_G(v) = 0 \end{cases} \tag{1.1}$$

(where $d_G(v)$ denotes the degree of vertex v in G).
We call w *product-irregular* if for every pair of vertices $u, v \in V(G)$, $u \neq v$

$$pd_G(u) \neq pd_G(v). \tag{1.2}$$

The *product irregularity strength* $ps(G)$ of G is the smallest value of s that allows some product-irregular labelling.

*http://kbo.ue.poznan.pl/anholcer
E-mail address: m.anholcer@ue.poznan.pl (Marcin Anholcer)

The concept was introduced by M. Anholcer in [4]. As we can see, it is the multiplicative version of the well known *irregularity strength* introduced by Chartrand et al. in [5] and studied by numerous authors (the best result for general graphs can be found in Kalkowski, Karoński and Pfender [6], while e.g. trees and forests have been studied e.g. by Aigner and Triesch [1], and Amar and Togni [3]). On the other hand, the problem of founding the product irregularity strength of graph is connected with the product antimagic labellings investigated e.g. by Pikhurko [7]. Indeed, the result from the last publication implies that for sufficiently large graphs

$$ps(G) \leq |E(G)|\,.$$

Let n_d denote the number of vertices of degree d, where $\delta(G) \leq d \leq \Delta(G)$. In [4] M. Anholcer showed that

$$ps(G) \geq \max_{\delta(G)\leq d\leq\Delta(G)} \left\{ \left\lceil \frac{d}{e} n_d^{1/d} - d + 1 \right\rceil \right\}.$$

This reduces to

$$ps(G) \geq \left\lceil \frac{r}{e} n^{1/r} - r + 1 \right\rceil.$$

in the case of r-regular graph. Note that these bounds are not tight. Also the bounds on $ps(C_n)$ were given, where C_n is cycle on n vertices. It was proved that if $n > 17$, then

$$ps(C_n) \geq \left\lceil \left(\frac{n}{1 - \ln 2} \right)^{1/2} \right\rceil,$$

while for every $\varepsilon > 0$ there exists n_0 such that for $n > n_0$

$$ps(C_n) \leq \lceil (1+\varepsilon)\sqrt{2n} \ln n \rceil.$$

Similarly the upper bounds on the irregularity strength of grids and toroidal grids were proved:

$$ps(T_{n_1\times n_2\times\cdots\times n_k}) \leq \lceil (1+\varepsilon)\sqrt{2}(\sum_{j=1}^{k} \sqrt{n_j}) \ln(\sum_{j=1}^{k} n_j) \rceil,$$

$$ps(G_{n_1\times n_2\times\cdots\times n_k}) \leq \lceil (1+\varepsilon)\sqrt{2}(\sum_{j=1}^{k} \sqrt{n_j}) \ln(\sum_{j=1}^{k} n_j) \rceil.$$

In [9] Skowronek-Kaziów showed, that

Proposition 1.1. *For every* $n \geq 3$

$$ps(K_n) = 3.$$

Let us recall that n_d denotes the number of vertices of degree d in G, where $\delta(G) \leq d \leq \Delta(G)$. In this paper we are going to give the exact value of $ps(G)$ for complete bipartite graphs $K_{m,n}$, where $2 \leq m \leq n \leq \binom{m+2}{2}$, and some families of graphs with $\delta(G) = 1$. The main results are as follows.

Proposition 1.2. *Let m and n be two integers such that $2 \leq m \leq n$. Then*

$$ps(K_{m,n}) = 3$$

if and only if $n \leq \binom{m+2}{2}$. Otherwise $ps(K_{m,n}) \geq 4$.

Theorem 1.3. *Let $D \geq 3$ be arbitrary integer. For almost all forests F such that*

(i) $\Delta(F) = D$, $n_2 = 0$ and $n_0 \leq 1$,

(ii) if we remove all the pendant edges, then in the resulting forest F', $n_2 = 0$,

the product irregularity strength equals to

$$ps(F) = n_1.$$

The proofs of the above results are given in two following sections.

2 Complete bipartite graphs

Proof of Proposition 1.2

Let $K_{m,n} = (U, V, E)$, where $U = \{u_1, \ldots, u_m\}$, $V = \{v_1, \ldots, v_n\}$ and $E = \{\{u_i, v_j\}, 1 \leq i \leq m, 1 \leq j \leq n\}$. If we used only labels 1 and 2, we would be able to obtain at most $n + 1$ distinct products, $1, 2, \ldots, 2^n$ while we have $n + m \geq n + 2$ vertices. Thus $ps(K_{n,n}) \geq 3$. On the other hand, assume that we are using only the labels 1, 2 and 3. The number of possible multisets of m elements is equal to $\binom{m+2}{2}$, and it is the maximal number distinct products for the vertices in V. Thus it is impossible to find product-irregular labeling of $K_{m,n}$ if $|V| = n > \binom{m+2}{2}$. Now we are going to prove that labels 1, 2 and 3 are enough if $m \leq n \leq \binom{m+2}{2}$.

Let us consider the set of all $\binom{m+2}{2}$ multisets of m elements equal to either 1, 2 or 3. Let us denote the elements of j^{th} multiset, where $1 \leq j \leq \binom{m+2}{2}$, with a_j^i, where $1 \leq i \leq m$. Assume they are arranged in non decreasing order, i.e. in such a way that $a_j^i \leq a_j^{i+1}$ for $1 \leq j \leq \binom{m+2}{2}$ and $1 \leq i \leq m-1$. Now we arrange the obtained sequences in decreasing lexicographic order, i.e. in such a way that for every $1 \leq j_1 < j_2 \leq \binom{m+2}{2}$ there exists i_0, $1 \leq i_0 \leq m$ such that $a_{j_1}^i = a_{j_2}^i$ if $1 \leq i < i_0$ and $a_{j_1}^{i_0} > a_{j_2}^{i_0}$. Now if $m < n$, then we put

$$w(\{u_i, v_j\}) = a_j^i, 1 \leq i \leq m, 1 \leq j \leq n.$$

Observe that the weighted degrees in V are distinct, as the respective multisets are. It is also straightforward to see that the degrees in U are distinct, as the numbers of factors equal to 3 are. Moreover the number of factors different than 1 in the weighted degrees in V are equal at most m, while in U they equal at least $m + 1$. Thus finally the obtained labeling is product-irregular.

If $m = n$, then we label any $K_{n-1,n}$ subgraph of $K_{n,n}$ as above and then put 1 on all the edges incident to the remaining vertex. As in the case $m = n - 1$ none of the vertices obtains the weighted degree 1, the resulting labeling is product-irregular. □

3 Graphs with $\delta(G) = 1$

Proof of Theorem 1.3

Let us consider a forest F. We distinguish two kinds of non pendant vertices. The *external vertex* is such a vertex that at least one of its neighbours is pendant vertex. The *internal vertex* has no pendant vertices in the neighbourhood.

The product degree of every pendant vertex is equal to the label of the only edge incident to it. Thus $ps(F) \geq n_1$. So in order to prove the theorem we have to show that there exists a product-irregular labelling of F with n_1 labels.

The proof consists of two parts. First using the Probabilistic Method (more precisely the Linearity of Expectation, see e.g. [2], pp.13-21) we will prove the existence of partial labeling that distinguishes the product degrees of internal vertices. Then, by labeling the pendant edges we will extend the product-irregular labeling on whole forest F.

Let us choose the label for every non-pendant edge uniformly at random from the set of all odd primes p, $n_1^{1/2} < p \leq n_1$. The number of such primes $\pi_{1/2}$ equals

$$\pi_{1/2} = \pi(n_1) - \pi(n_1^{1/2}) > \frac{n_1 - 2.51012{n_1}^{1/2}}{\ln n_1}$$

provided $n_1 \geq 17$ (see e.g. [8]).

Let us enumerate in any way all the m_1 pairs of non-adjacent internal vertices and m_2 pairs of adjacent internal vertices. As for every forest we have

$$n_1 \geq 2 + \sum_{i=3}^{D} (i-2)n_i > \sum_{i=3}^{D} n_i,$$

it follows that the total number of internal vertices n_{int} satisfies the inequality

$$n_{int} < \frac{n_1}{2}.$$

Thus

$$m_2 \leq n_{int} - 1 < \frac{n_1}{2}$$

and

$$m_1 \leq \binom{n_{int}}{2} < \frac{n_1^2}{4}.$$

For every i, $1 \leq i \leq m_1 + m_2$, let v_i and u_i be the vertices forming pair i and let X_i be random variable such that

$$X_i = \begin{cases} 1, & pd(u_i) = pd(v_i), \\ 0, & otherwise. \end{cases}$$

We have

$$E(X_i) = Pr(pd(u_i) = pd(v_i)) \leq \begin{cases} D!{\pi_{1/2}}^{-2}, & u_i \sim v_i, \\ D!{\pi_{1/2}}^{-3}, & otherwise. \end{cases}$$

Let X be random variable counting the"bad" pairs. Of course,

$$E(X) = \sum_{i=1}^{m_1+m_2} E(X_i) \leq m_1 D!{\pi_{1/2}}^{-3} + m_2 D!{\pi_{1/2}}^{-2} < 1, \tag{3.1}$$

if only n_1 is large enough.

This implies that for almost every forest satisfying given conditions, there exists a labeling $w^\star$ using only the primes lower or equal to n_1 that distinguishes all the internal vertices. Moreover, for every internal vertex v we have

$$pd(v) > n_1$$

and

$$pd(v) \bmod 2 = 1.$$

Let us choose any such labeling. Now we are going to extend $w^\star$ on the pendant edges in order to obtain complete labeling w. We will do it in two steps. Let n_{ext} be the number of external vertices. In the first step, for each such vertex we leave two pendant edges incident to it not labeled. If there are other pendant edges (i.e. for at least one external vertex v, $d(v) > 3$), then we put on them distinct labels from the set $\{1, , \ldots, n_1 - 2n_{ext}\}$ (in any order). Let $pd^\star(v)$ be the product of all edges incident with v that have been labeled. In the second step we order the external vertices with non-decreasing value of $pd^\star(v)$. Then we label two edges incident with i^{th} external vertex ($1 \le i \le n_{ext}$) using labels $n_1 - 2n_{ext} + i$ and $n_1 - i + 1$. Observe that the products of the pairs of labels increase with i.

After the second step, the product degrees of external vertices satisfy the conditions:

$$pd(v_i) < pd(v_j) \text{ if } i < j,$$

$$pd(v_i) > n_1$$

and

$$pd(v_i) \bmod 2 = 0.$$

For pendant vertices we have in turn:

$$pd(v_i) \le n_1$$

and

$$pd(v_i) \ne pd(v_j) \text{ if } i \ne j.$$

As there can be at most one vertex v with $pd(v) = 0$, this finishes the proof. □

Corollaries

From the above one can deduce the following two corollaries.

Corollary 3.1. *Let $d \ge 2$ be arbitrary integer. Then for almost all the complete d-ary trees*

$$ps(T) = n_1.$$

Proof. We proceed as in the proof of Theorem 1.3. Even if $d = 2$, there is only one vertex of degree 2 (the root) and its product degree is the product of two (not necessarily distinct) primes $p_1, p_2 > n_1^{1/2}$. It distinguishes this vertex from all other internal and pendant vertices. And even if $n_1 \in \{p_1, p_2\}$, it is impossible to obtain the external vertex with same product degree, as the triple $1, p_1, p_2$ cannot appear (pendant edges should be labeled 1 and n_1 this time, what would imply $n_1 \bmod 2 = 0$, contradiction). Thus all the product degrees are distinct. □

One of the most important facts used in the proof of Theorem 1.3 is that $n_1 > n/2$. However, as it may be easily checked, the inequality analogous to (3.1) will be satisfied even for smaller number of pendant vertices.

Corollary 3.2. *Let $D \geq d \geq 3$ be arbitrary integers. For almost all graphs G such that*

(i) $\delta(G) = 1$, $\Delta(G) = D$,

(ii) if we remove all the pendant edges, then for the resulting graph G', $\delta(G') = d \geq 4$,

(iii) $n_1 >> n^{2/(d-1)} \ln n$ *(i.e.* $n = o(\frac{n_1^{(d-1)/2}}{\ln n_1})$*),*

(iv) none of the external vertices is adjacent to exactly one pendant vertex,

the product irregularity strength equals to

$$ps(G) = n_1.$$

Proof. We proceed as in the proof of Theorem 1.3. The difference is that this time we do not distinguish pairs of adjacent and non-adjacent vertices. The inequality (3.1) takes the form:

$$E(X) < \binom{n}{2} D! \pi_{1/2}{}^{-d+1} < 1,$$

and the deterministic part of the proof remains unchanged. □

Two simple observations

Finally let us add two simple observations on some special families of trees.

Proposition 3.3. *Let $K_{1,n}$ be star with n pendant vertices, $n \geq 2$. Then*

$$ps(K_{1,n}) = \max\{3, n\}.$$

Proof. In the case $n = 2$, two labels 1 and 2 are not enough (either we use two equal labels and obtain same product degrees of pendant vertices, or we label the edges with 1 and 2 and obtain two product degrees 2). On the other hand, using labels 2 and 3 we produce product-irregular labeling. If $n \geq 3$, we need at least n labels to distinguish the product degrees of pendant vertices and this is enough, as the product degree of central vertex equals $n! > n$. □

Centipede Q_n is the graph with $V(Q_n) = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ and $E(Q_n) = \{\{u_i, v_i\}, 1 \leq i \leq n\} \cup \{\{v_i, v_{i+1}\}, 1 \leq i \leq n-1\}$.

Proposition 3.4. *Let Q_n be a centipede, $n \geq 2$. Then*

$$ps(Q_n) = \max\{3, n\}.$$

Proof. If $n = 2$, two labels are not enough as it would be possible to obtain at most three distinct products 1, 2 and 4 and we have four vertices. If $n \geq 3$, we need at least n labels to distinguish all the pendant vertices. So $ps(Q_n) \geq \max\{3, n\}$. The product-irregular labelings realizing this bound are given as follows:

(i) If $n = 2$, put $w(\{u_i, v_i\}) = i, i = 1,2$ and $w(\{v_1, v_2\}) = 3$. Then $pd(u_1) = 1$, $pd(u_2) = 2$, $pd(v_1) = 3$ and $pd(v_2) = 6$.

(ii) If $n = 3$, put $w(\{u_2, v_2\}) = 1$, $w(\{u_1, v_1\}) = w(\{v_1, v_2\}) = 2$, $w(\{u_3, v_3\}) = w(\{v_2, v_3\}) = 3$. Then $pd(u_1) = 2$, $pd(u_2) = 1$, $pd(u_3) = 3$, $pd(v_1) = 4$, $pd(v_2) = 6$ and $pd(v_3) = 9$.

(iii) If $n \geq 4$, put $w(\{u_1, v_1\}) = n-1$, $w(\{u_n, v_n\}) = n-2$, $w(\{u_{n-1}, v_{n-1}\}) = n$, $w(\{u_i, v_i\}) = i-1, 2 \leq i \leq n-2$ and $w(\{v_i, v_{i+1}\}) = n, 1 \leq i \leq n-1$. Then product degrees of pendant vertices are distinct numbers from the set $\{1, 2, \ldots, n\}$ and the product degrees of external vertices - distinct numbers from the set $\{n(n-2), n(n-1)\} \cup \{in^2, 1 \leq i \leq n-3\} \cup \{n^3\}$.

□

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Polytopes associated to dihedral groups

Barbara Baumeister *
Universität Bielefeld, Germany

Christian Haase †
Goethe-Universität Frankfurt, Germany

Benjamin Nill ‡
Case Western Reserve University, Cleveland, OH, USA

Andreas Paffenholz §
Technische Universität Darmstadt, Germany

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Abstract

In this note we investigate the convex hull of those $n \times n$ permutation matrices that correspond to symmetries of a regular n-gon. We give the complete facet description. As an application, we show that this yields a Gorenstein polytope, and we determine the Ehrhart h^*-vector.

Keywords: Permutation polytopes, dihedral groups, lattice polytopes.

Math. Subj. Class.: 20B35, 52B12; 05E10, 52B05, 52B20

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E-mail addresses: b.baumeister@math.uni-bielefeld.de (Barbara Baumeister), haase@math.uni-frankfurt.de (Christian Haase), benjamin.nill@case.edu (Benjamin Nill), paffenholz@mathematik.tu-darmstadt.de (Andreas Paffenholz)

1 Introduction

To any finite group G of real $n \times n$ permutation matrices we can associate the *permutation polytope* $P(G)$ given by the convex hull of these matrices in the vector space $\mathbb{R}^{n \times n}$. A well-known example of such a polytope is the Birkhoff polytope B_n, which is defined as the convex hull of all $n \times n$ permutation matrices [9, 8]. This polytope appears in various contexts in mathematics from optimization to statistics to enumerative combinatorics. (See, e.g., [24, 20, 21, 2, 1].) It is also a famous example of a Gorenstein polytope (see Section 5). Gorenstein polytopes turn up in connection to mirror symmetry in theoretical physics.

Guralnick and Perkinson [15] studied polytopes associated to general subgroups G of the symmetric group and proved results about their dimension, and about the diameter of their vertex-edge graph. A systematic exposition of general permutation polytopes is given in [5]. There, we studied which groups lead to affinely equivalent polytopes, we considered products of groups and polytopes, classified low-dimensional cases, and we formulated several open conjectures.

In order to get an intuition about what one can expect in general, it is instructive to consider some special classes of permutation groups. A seemingly very difficult case is when G equals the group of even permutation matrices. Just to exhibit exponentially many facets is already a daunting task, for this see [17]. Even for cyclic G we showed in [6] that these polytopes have a surprisingly complex and not yet fully understood facet structure.

In [12] Collins and Perkinson studied polytopes given by Frobenius groups. A special case is the dihedral group D_n for n odd, which was considered in more detail by Steinkamp [22]. Since D_n is the automorphism group of a regular n-gon, the cases where n is even and odd are quite different.

The most recent paper on permutation polytopes [11] focused on determining the volumes of permutation polytopes associated to cyclic groups, dihedral groups, and Frobenius groups. In order to compute the volume of $P(D_n)$, the authors find a Gale dual combinatorial description, which they use to provide an explicit formula for the Ehrhart polynomial of $P(D_n)$.

The dihedral group D_n is the automorphism group $\mathrm{Aut}(C_n)$ of a cycle C_n, and any permutation matrix $M(\sigma)$ of an element $\sigma \in D_n$ commutes with the adjacency matrix A of C_n. So any point in $P(D_n)$ commutes with A, and

$$P(D_n) \subseteq \{M \in \mathbb{R}^{n \times n} \mid M \text{ is doubly stochastic and } MA = AM\}.$$

Here, a matrix is doubly stochastic if all entries are non-negative and each row and column sum is 1. Tinhofer [24] asks, more generally, for a classification of those undirected graphs G where the two sets above are equal, i.e. where the commutation condition $MA = AM$ already suffices to characterize the elements of $P(\mathrm{Aut}(G))$ among all doubly stochastic matrices. The Birkhoff-von Neumann theorem is the special case where A is the unit matrix. Tinhofer shows that this also holds for the adjacency matrices of cycles and trees [24, Theorems 2&3].

In this note, we independently investigate $P(D_n)$ in a more direct and elementary way. We give a complete list of its facet inequalities (Theorem 3.3, Theorem 4.1). As an application, we observe that these lattice polytopes are Gorenstein polytopes, and we get a nice description of the generating function of their Ehrhart polynomials (Theorem 5.3, Corollary 5.4).

Acknowledgments: Many results are based upon experiments and computations using the package `polymake` [14] by Gawrilow and Joswig. We would like to thank the referees for carefully reading and improving the text.

2 Notation and preliminary results

Let S_n be the permutation group on $n \geq 3$ elements. Every permutation $\sigma \in S_n$ can be represented by an $n \times n$ matrix M_σ with entries $\delta_{i,(j)\sigma}$. So the entries are in $\{0, 1\}$ and there is exactly one 1 in each row and column. Notice that we apply matrices and permutations from the right. We can view such a matrix as a vector in $\mathbb{R}^{n^2}$. For a subgroup G of S_n we define the polytope

$$P_G := \operatorname{conv}(M_\sigma \mid \sigma \in G) .$$

This is a $0/1$-polytope, so all matrices are in fact vertices of the polytope.

We denote by D_n the subgroup of S_n corresponding to the symmetry group of the regular n-gon, the *dihedral group of order* $2n$. This group is generated by two elements. If n is odd, then these may taken to be the rotation ρ of the n-gon by $360/n$ degrees, and the reflection τ along a line through one vertex and the midpoint of the opposite edge. If n is even, then the second generator τ is instead the reflection along a line through two opposite vertices. Thus ρ is the permutation $(1, 2, \dots, n)$ and τ the reflection $(2, n)(3, n - 1) \cdots ((n + 1)/2, (n + 3)/2)$ if n is odd and $(2, n)(3, n - 1) \cdots (n/2, (n/2) + 2)$ if n is even.

The associated permutation polytope is the convex hull of the corresponding matrices,

$$\mathrm{DP}_n := \operatorname{conv}(M_\sigma \mid \sigma \in D_n) .$$

The dihedral group D_n has $2n$ elements

$$\rho^0, \rho^1, \rho^2, \dots, \rho^{n-1}, \tau, \tau\rho, \tau\rho^2, \tau\rho^3, \dots, \tau\rho^{n-1}.$$

We label the vertices of DP_n by $v_0, \dots, v_{n-1}, w_0, \dots, w_{n-1}$ in this order. Let us give a more convenient way to write these matrices.

Let I be the n-dimensional identity matrix and R be the $n \times n$ matrix that has 0's everywhere except at the n entries (i, j), where $0 \leq i, j \leq n - 1$ and $j \equiv i + 1 \mod n$:

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} .$$

Reading the matrices M_σ row by row, we can identify M_σ with a (row) vector in $\mathbb{R}^{n^2}$. For instance, the 2×2 identity matrix would be identified with $(1\ 0\ 0\ 1)$. Under this identification the vertices of DP_n are (in the order given above) the rows of the $2n \times n^2$ matrix

$$\begin{bmatrix} R^0 & R^1 & R^2 & \cdots & R^{n-1} \\ R^0 & R^{-1} & R^{-2} & \cdots & R^{-(n-1)} \end{bmatrix} .$$

Permuting the coordinates (corresponding to a linear automorphism of $\mathbb{R}^{n^2}$) we may write the vertices in the form

$$V = \begin{bmatrix} I & I & I & \cdots & I \\ I & R^{-2} & R^{-4} & \cdots & R^{-2(n-1)} \end{bmatrix}. \tag{2.1}$$

Clearly, the first $2n$ coordinates of the vertices linearly determine the remaining coordinates. So we can project onto $\mathbb{R}^{2n}$ without changing the combinatorics of the polytope. Hence, we observe that the dimension of DP_n is at most $2n$.

3 The situation for odd n

In this section we completely describe DP_n for n odd. As it will turn out, it is useful to introduce a new polytope that will serve as a basic building block for both situations of even n and odd n.

Definition 3.1. Let Q_n be the polytope defined as the convex hull of the rows of the $2n \times n^2$ matrix

$$W := \begin{bmatrix} I & I & I & \dots & I \\ I & R^1 & R^2 & \dots & R^{n-1} \end{bmatrix}. \tag{3.1}$$

While Q_n differs from DP_n for even n, for odd n the R^{2k} for $0 \le k \le n-1$ are a permutation of the R^k for $0 \le k \le n-1$. So we deduce from (2.1) that, for n odd, Q_n is up to a permutation of coordinates just the polytope DP_n.

Proposition 3.2. *For odd n, the polytopes DP_n and Q_n are affinely isomorphic.* □

The following theorem examines the structure of Q_n for arbitrary n. For n odd, this result is a special case of Theorem 4.4 in [12].

Let us fix some convenient notation. We denote by Δ_r the r-dimensional simplex. We also use for any two integers s, k, the term $[s]_k \in \{0, \dots, k-1\}$ to denote the remainder of s upon division by k. The *free sum* of two polytopes P and P' of dimensions d and d' is the polytope

$$P \oplus P' := \mathrm{conv}(\{(p,0) \in \mathbb{R}^{d+d'} \mid p \in P\} \cup \{(0,p') \in \mathbb{R}^{d+d'} \mid p' \in P'\}).$$

Theorem 3.3 (Collins&Perkinson [12])**.** *Let n be odd or even. The polytope Q_n has dimension $2n-2$ and is a free sum of two copies of Δ_{n-1}. Taking coordinates $x_0, \dots, x_{n^2-1}$ for $\mathbb{R}^{n\times n}$, its affine hull is given by the equations*

$$1 = \sum_{i=ln}^{(l+1)n-1} x_i \tag{aff}$$

$$0 = x_{kn+[j]_n} - x_{(k+1)n+[j]_n} - x_{(k+1)n+[j+1]_n} + x_{(k+2)n+[j+1]_n} \tag{$A_{j,k}$}$$

for $0 \le l \le n-1$, $0 \le j \le n-2$, $0 \le k \le n-3$.

An irredundant system of inequalities defining the polytope inside its affine hull is given by the inequalities

$$x_i \ge 0$$

for $0 \le i \le n^2 - 1$.

Proof. All the given equations are satisfied by the vertices of Q_n. There are n equations of type (aff) and $n^2 - 3n + 2$ equations of type $(A_{j,k})$. They are easily seen to be linearly independent, so the dimension of Q_n is at most $2n - 2$. On the other hand, deleting any row of W leaves us with a linearly (and hence affinely) independent set of row vectors. (Observe that deleting a row leaves us with a column that contains exactly one 1.) Hence, $\dim(Q_n) = 2n - 2$ and the given equations define the affine hull of Q_n in $\mathbb{R}^{n^2}$.

Further, we see that every $2n - 1$ of the $2n$ rows of W span the affine hull of Q_n. So any facet of Q_n has $2n - 2$ vertices. Since the inequalities $x_j \geq 0$ are 0 on exactly $2n - 2$ of the rows, they all define facets.

In order to prove that Q_n is a free sum of simplices we observe that the first n and the last n vertices define $(n-1)$-dimensional simplices sitting in transversal subspaces (intersecting in the matrix corresponding to the row vector $(1/n, \dots, 1/n)$). Therefore, the combinatorial dual of Q_n corresponds to the product of Δ_{n-1} with itself. In particular, Q_n has precisely n^2 facets, so the facet description given above is complete. □

4 The situation for even n

Recall that the *join* $P \star Q$ of two polytopes P and Q is the convex hull of $P \cup Q$ after embedding P and Q in skew affine subspaces. The dimension of $P \star Q$ equals $\dim(P) + \dim(Q) + 1$. For instance, the join of two intervals is a tetrahedron.

Theorem 4.1. *Let n be even. The polytope DP_n is a join of two copies of $Q_{n/2}$. In particular, its dimension is $2n - 3$.*

Combined with Theorem 3.3, this result gives a complete description of the facet inequalities and the affine hull equations of DP_n for n even.

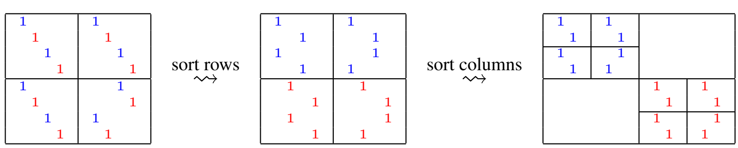
Proof. Permuting the coordinates, we can transform V (see (2.1)) into

$$\begin{bmatrix} I & I & I & \cdots & I & & I & I & I & \cdots & I \\ R^0 & R^2 & R^4 & \cdots & R^{n-2} & & R^0 & R^2 & R^4 & \cdots & R^{n-2} \end{bmatrix}.$$

Clearly, projecting onto the first $\frac{n^2}{2}$ coordinates yields an affine isomorphism of DP_n onto the convex hull of the rows of the $2n \times \frac{n^2}{2}$ matrix

$$\begin{bmatrix} I & I & I & \dots & I \\ R^0 & R^2 & R^4 & \dots & R^{n-2} \end{bmatrix}.$$

In the representation given by this matrix let us partition the set of $2n$ vertices (labelled from 0 to $2n - 1$) into two sets: consisting of the n rows with even index and the n rows with odd index.



Then we permute the $\frac{n^2}{2}$ coordinates in such a way that in the first set of rows (corresponding to even vertices) all nonzero entries are in the first half (i.e. in the first $\frac{n^2}{4}$ columns).

Then all nonzero entries in the second set of rows (corresponding to the odd vertices) will be in the second half (i.e. in the last $\frac{n^2}{4}$ columns). By a permutation of the coordinates within the first half we get that the rows of even vertices yield precisely the vertex set of $Q_{n/2} \times \{0\}$ (for $0 \in \mathbb{R}^{\frac{n^2}{4}}$). In the same way, the coordinates in the second half can be permuted so that the rows of odd vertices equal the vertices of $\{0\} \times Q_{n/2}$ (for $0 \in \mathbb{R}^{\frac{n^2}{4}}$). Since 0 is not in the affine hull of $Q_{n/2}$, we deduce that DP_n is a join of two copies of $Q_{n/2}$. Hence, its dimension equals $2\dim(Q_{n/2}) + 1 = 2(n-2) + 1 = 2n - 3$ by Theorem 3.3. □

5 Lattice properties

DP_n and Q_n are *lattice polytopes*, i.e. their vertices lie in the lattice $\mathbb{Z}^{n^2}$ of integral vectors. It is readily checked that all above affine isomorphisms respect lattice points. In this section, we will show that these lattice polytopes have especially nice properties which allow us to completely describe their Ehrhart h^*-vectors.

A d-dimensional lattice polytope P containing 0 in its interior is *reflexive*, if its polar (or dual) polytope

$$P^* := \{x \in \mathbb{R}^d \mid \langle x, v\rangle \geq -1 \;\forall\, v \in P\}$$

is again a lattice polytope (in the dual lattice). This notion was introduced by Batyrev in [3]. A generalization of this is the class of Gorenstein polytopes. A lattice polytope is a *Gorenstein polytope of codegree* k, if there is a positive integer k and an interior lattice point m in kP such that $kP - m$ is a reflexive polytope. Such polytopes play an important role in the classification of Calabi-Yau manifolds for string theory. See [4] for basic properties. The next proposition tells us that the polytopes Q_n belong to this class. The *normalized volume of* $\mathbb{R}^n$ is the volume form which assigns to the standard simplex the volume 1.

Proposition 5.1. *Let n be odd or even. The polytope Q_n is Gorenstein of codegree n and normalized volume n.*

Proof. By Theorem 3.3, the point $\frac{1}{n}(1, 1, \ldots, 1)$ is an interior point of Q_n with equal integral distance $1/n$ to all facets, and $m := (1, 1, \ldots, 1)$ is the unique interior lattice point in nQ_n. Hence $nQ_n - m$ is a reflexive polytope.

By Theorem 3.3, all facets of Q_n are simplices of facet width 1, hence they are all unimodular. As we have seen, multiplying by n gives (up to translation) a reflexive polytope with the unique interior lattice point $m = (1, 1, \ldots, 1)$. The normalized volume of nQ_n is the sum of the volumes of n^2 pyramids over facets with apex m. But in nQ_n each facet has normalized volume n^{2n-3}, and the apex has lattice distance 1 from the facet, so each pyramid has normalized volume n^{2n-3}. There are n^2 of these pyramids, so the normalized volume of nQ_n equals n^{2n-1}. Dividing by n^{2n-2} to get from nQ_n back to Q_n gives the normalized volume n of Q_n. □

A polytope P is *compressed* if every so-called pulling triangulation is regular and unimodular. Equivalently, P is compressed if for any supporting inequality $a^t x \leq b$ with a primitive integral normal a, i.e. with a normal vector whose entries are integers and which is not an integral multiple of some other integer vector, the polytope is contained in the set $\{x \mid b - 1 \leq a^t x \leq b\}$. For a more detailed explanation of these terms we refer to [13].

This property has strong implications on the associated toric ideal, see e.g. [23]. The next proposition follows immediately from Theorem 1.1 of [19] and Theorem 3.3.

Proposition 5.2. *Let n be odd or even. The polytope Q_n is compressed.* □

The *Ehrhart polynomial* $L_P(k) := |kP \cap \mathbb{Z}^d|$ of a d-dimensional lattice polytope counts the number of integral points in integral dilates of P. It is well known that the generating function of L_P is given by

$$\sum_{m\geq 0} L_P(m)t^m \;=\; \frac{h^*(t)}{(1-t)^{d+1}}$$

for some polynomial h^* of degree at most d with integral non-negative coefficients, see [7]. Hence, determining the Ehrhart polynomial is equivalent to finding the h^*-vector (also called the δ-vector) of coefficients of $h^*(t)$. As is well-known, P is Gorenstein if and only if the h^*-vector is symmetric. The following theorem shows that in our case this vector has a particularly nice form.

Theorem 5.3. *Let n be odd or even. The h^*-vector of Q_n satisfies $h_i^* = 1$ for $0 \leq i \leq n-1$ and $h_i^* = 0$ otherwise.*

Proof. Since the codegree of Q_n is n and its dimension is $2n-2$ by Theorem 3.3, the maximal non-zero entry of the h^*-vector has to be h_{n-1}^*, see [7]. By a theorem of Bruns and Römer [10] we know that the h^*-vector of a Gorenstein polytope that has a regular unimodular triangulation is symmetric and unimodal. In particular, $h_i^* \geq 1$ for $i = 0, \ldots, n-1$. Since by Proposition 5.1 the sum of the entries of the h^*-vector equals n, the statement follows. □

In particular, if n is odd, the previous result describes the h^*-vector of DP_n. Finally, let us deal with the even case.

Corollary 5.4. *Let n be even. The h^*-vector of DP_n equals*

$$(1, 2, 3, \ldots, \frac{n}{2}-1, \frac{n}{2}, \frac{n}{2}-1, \ldots, 2, 1).$$

In particular, the polytope DP_n is Gorenstein of codegree n and normalized volume $n^2/4$.

Proof. By the proof of Theorem 4.1, DP_n is given up to coordinate permutation as the convex hull of the rows of

$$\begin{bmatrix} \tilde{W} & 0 \\ 0 & \tilde{W} \end{bmatrix},$$

where $\tilde{W}$ is the $n \times (\frac{n}{2})^2$ matrix whose rows are the vertices of $Q_{\frac{n}{2}}$ as given in (3.1). The integral linear functional which sums the first $\frac{n}{2}$ coordinates evaluates to 1 on the first $\frac{n}{2}$ rows, and to 0 on the second half. Hence, the two copies of $Q_{\frac{n}{2}}$ (say, $P_1 \times \{0\}$ and $\{0\} \times P_2$) have lattice distance 1 in the lattice $\mathbb{Z}^{\frac{n^2}{2}} \cap \operatorname{aff} \mathrm{DP}_n$. In other words, there is an affine isomorphism respecting lattice points which maps DP_n onto the convex hull of $P_1 \times \{0\} \times \{1\}$ and $\{0\} \times P_2 \times \{0\}$ in $\mathbb{R}^{\frac{n^2}{2}+1}$. Therefore, the statement follows from the well-known fact [7, Example 3.32] that in this case the h^*-polynomial equals the product of the h^*-polynomials of P_1 and P_2. □

6 Substructures

In [5] the authors discussed which subgroups of a permutation group yield faces of $P(G)$. An obvious class of such subgroups are stabilizers:

Take a partition $[n] := \{1, \dots, n\} = \bigsqcup I_i$. Then the polytope of the stabilizer of the subsets I_i

$$\mathrm{stab}(G; (I_i)_i) := \{\sigma \in G \mid \sigma(I_i) = I_i \text{ for all } i\} \leq G$$

is a face of $P(G)$. The authors conjecture that there are no other examples.

Conjecture 5.8 [5] Let $G \leq S_n$. Suppose $H \leq G$ is a subgroup such that $P(H) \preceq P(G)$ is a face. Then $H = \mathrm{stab}(G; (I_i)_i)$ for a partition $[n] = \bigsqcup I_i$.

We have verified the conjecture for $G = S_n$ as well as for cyclic subgroups $G \leq S_n$, see Proposition 5.9 of [5]. Meanwhile Jessica Nowack and Daniel Heinrich studied this question for the dihedral groups in their Diploma theses.

Proposition 6.1. *(Heinrich, Nowack [16, 18]) Conjecture 5.8 holds for $G = D_n \leq S_n$ for every n.*

Sketch of the proof. For n odd Heinrich first shows that, if H is the subgroup of all rotations of G, then P_H is not a face of P_G. The remaining subgroups are precisely the stabilizers of their orbits, see Theorem 7.1.1 of [16].

For n even the main work is to show that the subgroup of all rotations, the subgroup of the squares of the rotations and finally the subgroup generated by the squares of the rotations and by the reflections through two edges are precisely those subgroups H of G for which P_H is not a face of P_G. Nowack shows that the remaining subgroups are precisely the stabilizers of their orbits, see Section 4.2 of [18]. □

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A class of semisymmetric graphs

Li Wang

School of Mathematical Sciences, Capital Normal University, Beijing, 100048, P R China
and
School of Mathematics and Information Sciences, Henan Polytechnic University, Jiaozuo, 454000, P R China

Shaofei Du *

School of Mathematical Sciences, Capital Normal University, Beijing, 100048, P R China

Xuewen Li

Department of Mathematics and Information Sciences, Tangshan Normal University, Tangshan, 063000, P R China

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Abstract

A simple undirected graph is said to be *semisymmetric* if it is regular and edge-transitive but not vertex-transitive. Every semisymmetric graph is a bipartite graph with two parts of equal size. Let p be a prime. In this paper, a class of semisymmetric graphs of order $2p^3$ are determined. This work is a partial result for our long term goal to classify all semisymmetric graphs of order $2p^3$.

Keywords: Permutation group, primitive group, vertex-transitive graph, semisymmetric graph.

Math. Subj. Class.: 05C10, 05C25, 20B25

1 Introduction

All graphs considered in this paper are finite, undirected, connected and simple. For a graph X, we use $V(X)$, $E(X)$ and $A := \mathrm{Aut}(X)$ to denote its vertex set, edge set and the full automorphism group, respectively. The graph is said to be vertex-transitive and edge-transitive, if A acts transitively on $V(X)$ and $E(X)$, respectively. If X is bipartite with bipartition $V(X) = W(X) \cup U(X)$, we let A^+ be the subgroup of A preserving both

*Corresponding author.

E-mail addresses: wanglimath@hpu.edu.cn (Li Wang), dushf@mail.cnu.edu.cn (Shaofei Du), lixuewen@sina.com (Xuewen Li)

$W(X)$ and $U(X)$. Since X is connected, we have that either $|A : A^+| = 2$ or $A = A^+$, depending on whether or not there exists an automorphism which interchanges the two parts. For $G \leq A^+$, the graph X is said to be *G-semitransitive* if G acts transitively on both $W(X)$ and $U(X)$, and *semitransitive* if X is A^+-semitransitive.

We call a graph *semisymmetric* if it is regular and edge-transitive but not vertex-transitive. It is easy to see that every semisymmetric graph is a semitransitive bipartite graph with two parts of equal size.

The first person who studied semisymmetric graphs was Folkman. In 1967 he constructed several infinite families of such graphs and proposed eight open problems see [13]. Afterwards, Bouwer, Titov, Klin, I.V. Ivanov, A.A. Ivanov and others did much work on semisymmetric graphs see [2, 4, 16, 17, 18, 30]. They gave new constructions of such graphs and nearly solved all of Folkman's open problems. In particular, Iofinova and Ivanov [16] in 1985 classified biprimitive semisymmetric cubic graphs using group-theoretical methods. This was the first classification theorem for such graphs. More recently, following some deep results in group theory which depend on the classification of finite simple groups and some methods from graph coverings, some new results on semisymmetric graphs have appeared. For instance, in [11] Du and Xu classified semisymmetric graphs of order $2pq$ for two different primes p and q. For more papers on semisymmetric graphs see [5, 7, 8, 9, 10, 11, 12, 19, 21, 22, 23, 24, 25, 26, 27, 28, 33].

In [13], Folkman proved that there are no semisymmetric graphs of order $2p$ and $2p^2$ where p is a prime. Then we are interesting in determining semisymmetric graphs of order $2p^3$, where p is prime. Since the smallest semisymmetric graphs have the order 20 [13], we let $p \geq 3$. It is proved in [25] that the Gray graph of order 54 is the only cubic semisymmetric graph of order $2p^3$. To classify all semisymmetric graphs of order $2p^3$ is still one of attractive and difficult problems. These graphs X are naturally divided into two subclasses:

(1) Aut(X) acts unfaithfully on at least one part;

(2) Aut(X) acts faithfully on both parts.

Now we are going to concentrate on Subclass (1). To state our main theorem, we first introduce two concepts.

Let Y be a connected semitransitive and edge-transitive graph with bipartition $V(Y) = W(Y) \cup U(Y)$, where $W(Y) = Z_p^3$ and $U(Y) = Z_p^2$ for an odd prime p. For distinguishing the vertices of $W(Y)$ and $U(Y)$ convenience, the vertices of $W(Y)$ and $U(Y)$ are denoted by $(i, j, k, 0)$ and $(y, z, 1)$, respectively, where $i, j, k, y, z \in Z_p$. Now we define a bipartite graph X with bipartition $W(X) \cup U(X)$, where

$$W(X) = W(Y),\ U(X) = Z_p \times U(Y) = \{(x, y, z, 1) \mid x, y, z \in Z_p\},$$

such that two vertices $(i, j, k, 0) \in W(X)$ and $(x, y, z, 1) \in U(X)$ are adjacent if $\{(i, j, k, 0), (y, z, 1)\} \in E(Y)$. From now on, we shall say that the graph X is the *graph expanded from* Y and that the graph Y is the *graph contracted from* X. Clearly X is edge-transitive and regular. Furthermore, since for any $(y, z, 1) \in U(Y)$, the p vertices $\{(x, y, z, 1) \mid x \in Z_p\}$ in $U(X)$ have the same neighborhood, X is semisymmetric, provided there exist no two vertices in $W(X)$ which have the same neighborhood. Clearly, Aut(X) acts unfaithfully on $W(X)$ and Aut$(X)/S_p^p \cong$ Aut(Y).

Note that the semisymmetric graphs where two vertices have the same neighbourhood have been studied in several papers see [11, 20, 29, 33], with different definitions, for instance, *X is a derived graph from Y*, *X is a unworthy graph*, *X is contracted to from Y* and so on.

Let Y be a connected graph and $\mathcal{B}$ an imprimitive system of $\mathrm{Aut}(Y)$. Define a graph Z with the vertex set $\mathcal{B}$ such that two blocks are adjacent in Z if there exists at least one edge in Y between two blocks. This graph Z is called the *block graph of Y*. Moreover, if $\mathcal{B}$ is the set of orbits of some nontrival normal subgroup N of $\mathrm{Aut}(Y)$, then we call Z the *block graph induced by N*.

The following proposition gives a characterization for Subclass (1) given in [31]:

Proposition 1.1. *Suppose X is a semisymmetric graph of order $2p^3$, where p is an odd prime, such that $\mathrm{Aut}(X)$ acts unfaithfully on at least one part. Then $\mathrm{Aut}(X)$ must act unfaithfully on one part and faithfully on the other part, and X is the graph expanded from the graph Y with bipartition $V(Y) = W(Y) \cup U(Y)$, where $W(Y) = Z_p^3$ and $U(Y) = Z_p^2$. Moreover, we have that either*

(1) $p = 3$, $\mathrm{Aut}(Y) \cong S_3 \wr S_3$ *which acts primitively on $W(Y)$; or*

(2) $\mathrm{Aut}(Y)$ *has blocks of length p^2 on $W(Y)$ and of length p on $U(Y)$. Let $\overline{Y}$ be the block graph of Y. Then either*

(2.1) *the block graph $\overline{Y}$ is of valency at least 3, and* $\mathrm{Aut}(Y)$ *is solvable and contains a normal regular subgroup on $W(Y)$; or*

(2.2) *the block graph $\overline{Y}$ is of valency 2, where* $\mathrm{Aut}(Y)$ *may be solvable or insolvable.*

Following Proposition 1.1, in this paper we shall determine the graphs in Case (2.2), while Cases (1) and (2.1) will be determined in our another paper. Before giving the main theorem of this paper, we first define six families of graphs Y.

Definition 1.2. We shall define six families of bipartite graphs X with bipartition $V(X) = W(X) \cup U(X)$, where

$$W(X) = \{(i, j, k, 0) \mid i, j, k \in Z_p\}, \quad U(X) = \{(x, y, z, 1) \mid x, y, z \in Z_p\},$$

and edge set

$$\begin{aligned} E(X) \;=\; & \{\{(i,j,k,0),(x,i+b,k+\tfrac{p-1}{2},1)\} \mid i,j,k,x \in Z_p, b \in \Sigma\} \cup \\ & \{\{(i,j,k,0),(x,j+sb,k+\tfrac{p+1}{2},1)\} \mid i,j,k,x \in Z_p, b \in \Sigma\}, \end{aligned}$$

where $s = \theta^{\frac{p-1}{2r}}$, $Z_p^* = \langle \theta \rangle$ for the family of graphs $X_2(p, r)$, $s = 1$ for other five families of graphs $X_i(p, r)$, and Σ is given by

(1) **Graphs** $X_1(p, r)$**:** Let $p \geq 3$ and let Σ be a subgroup of Z_p^* of order r, where $(p, r) \neq (7, 3), (11, 5)$. Moreover, the valency of $X_1(p, r)$ is $2pr$ and the smallest examples are $X_1(3, 1)$ and $X_1(3, 2)$.

(2) **Graphs** $X_2(p, r)$**:** Let $p \geq 5$ and let Σ be a subgroup of Z_p^* of order $r \geq 2$, where $(p, r) \neq (7, 3), (11, 5)$ and $2r \mid (p - 1)$. Moreover, the valency of $X_2(p, r)$ is $2pr$ and the smallest example is $X_2(5, 2)$.

(3) **Graphs** $X_3(11, 5)$: Let $p = 11$ and let $\Sigma = \{0, 2, 3, 4, 8\} \subset Z_{11}$. Moreover, the valency of $X_3(11, 5)$ is 110.

(4) **Graphs** $X_4(11,6)$**:** Let $p = 11$ and $\Sigma = \{1,5,6,7,9,10\} \subset Z_{11}$. Moreover, the valency of $X_4(11,6)$ is 132.

(5) **Graphs** $X_5(p,r)$**:** Choose a point $\langle v\rangle$ and a hyperplane $\mathcal{L}$ in the project space $\mathrm{PG}(n-1,q)$, where $\frac{q^n-1}{q-1} = p \geq 7$, and let $G = \langle t\rangle$ be a Singer subgroup of $\mathrm{PGL}(n,q)$. Let $\Sigma = \{l \in Z_p \mid \langle v\rangle \in \mathcal{L}^{t^l}\}$, where $r = |\Sigma| = \frac{q^{n-1}-1}{q-1}$. Moreover, the valency of $X_5(p,r)$ is $2p\frac{q^{n-1}-1}{q-1}$ and the smallest example is $X_5(7,3)$.

(6) **Graphs** $X_6(p,r)$: Adopting the same notation as in (5), set $\Sigma = \{l \in Z_p \mid \langle v\rangle \notin \mathcal{L}^{t^l}\}$, where $r = q^{n-1}$. Moreover, the valency of $X_6(p,r)$ is $2pq^{n-1}$ and the smallest example is $X_6(7,4)$.

Remark 1.3. For $1 \leq i \leq 6$, let $X_i(p,r)$ be as in Definition 1.2. Then

(1) For any given $y,z \in Z_p$, the p vertices $\{(x,y,z,1) \mid x \in Z_p\}$ have the same neighborhood. Let $Y_i(p,r)$ be the contracted graph from $X_i(p,r)$, obtained by contracting each such p vertices into one vertex while preserving the adjacent relation, that is,

$$W(Y_i(p,r)) = W(X_i(p,r)),\ U(Y_i(p,r)) = \{(y,z,1) \mid y,z \in Z_p\}.$$

Then we shall see from the proof of Theorem 1.4 that $\mathrm{Aut}(Y_i(p,r)) = K \rtimes D_{2p}$, where the subgroup K is the following

(i) $Y_1(p,r)$ and $Y_2(p,r)$: $K = S_p^p$ if $r \in \{1,p-1\}$; $K = (Z_p \rtimes Z_r)^p$ if $r \notin \{1,p-1\}$;

(ii) $Y_3(11,5)$ and $Y_4(11,6)$: $K = (\mathrm{PSL}(2,11))^p$;

(iii) $Y_5(p,r)$ and $Y_6(p,r)$: $K = (\mathrm{P\Gamma L}(n,q))^p$.

(2) For any $k \in Z_p$, let

$$W_k(Y) = \{(i,j,k,0) \in W(Y) \mid i,j \in Z_p\}, \quad U_z(Y) = \{(y,z,1) \mid y \in Z_p\}.$$

Then we shall see from the proof of Theorem 1.4 that $\{W_k(Y) \mid k \in Z_p\}$ and $\{U_z(Y) \mid z \in Z_p\}$ are orbits of the group K on $W(Y)$ and $U(Y)$, respectively. Let $\overline{Y}$ be the block graph induced by K. Then $\overline{Y}$ is a cycle of length $2p$.

Now we give the main theorem of this paper.

Theorem 1.4. *For an odd prime p, suppose that X is a semisymmetric graph of order $2p^3$ expanded from a graph Y such that* $\mathrm{Aut}(Y)$ *has the blocks of length p^2 on $W(Y)$ and of length p on $U(Y)$ while the block graph $\overline{Y}$ is a cycle of length $2p$. Then X is isomorphic to one of graphs $X_i(p,r)$ where $1 \leq i \leq 6$, defined in Definition 1.2.*

After this introductory section, some preliminary results will be given in Section 2, and the main theorem will be proved in Sections 3. For group-theoretic concepts and notation not defined here the reader is refereed to [6, 15].

2 Preliminaries

First we introduce some notation. By H char G, we mean that H is a characteristic subgroup of G. Given a group G and a subgroup H of G, by $\mathrm{Cos}(G,H)$ we denote the set of right cosets of H in G. The action of G on $\mathrm{Cos}(G,H)$ is always assumed to be the right multiplication action. For two subgroups $N \lhd G$ and $H \le G$, by $N \rtimes H$ we denote the semi-direct product of N by H, where N is normal. For a group G, by Exp (G) we denote the least common multiple of orders of all the elements of G. By $H \wr K$, we denote the wreath product of H and K.

A group-theoretic construction of semitransitive and semisymmetric graphs were given in [11]. Here we quote one definition and two results.

Definition 2.1. Let G be a group, let L and R be subgroups of G and let D be a union of double cosets of R and L in G, namely, $D = \bigcup_i Rd_iL$. Define a bipartite graph $X = (G, L, R; D)$ with bipartition $V(X) = \mathrm{Cos}(G,L) \cup \mathrm{Cos}(G,R)$ and edge set $E(X) = \{(Lg, Rdg) \bigm| g \in G, d \in D\}$. This graph is called the bi-coset graph of G with respect to L, R and D.

Proposition 2.2. *[11] The graph $X = \mathbf{B}(G, L, R; D)$ is a well-defined bipartite graph. Under the right multiplication action of G on $V(X)$, the graph X is G-semitransitive. The kernel of the action of G on $V(X)$ is $\mathrm{Core}_G(L) \cap \mathrm{Core}_G(R)$, the intersection of the cores of the subgroups L and R in G. Furthermore, we have*

- (i) *X is G-edge-transitive if and only if $D = RdL$ for some $d \in G$;*
- (ii) *the degree of any vertex in $\mathrm{Cos}(G,L)$ (resp. $\mathrm{Cos}(G,R)$) is equal to the number of right cosets of R (resp. L) in D (resp. D^{-1}), so X is regular if and only if $|L| = |R|$;*
- (iii) *X is connected if and only if G is generated by elements of $D^{-1}D$;*
- (vi) *$X \cong \mathbf{B}(G, L^a, R^b; D')$ where $D' = \bigcup_i R^b\ (b^{-1}d_ia)L^a$, for any $a, b \in G$;*
- (v) *$X \cong \mathbf{B}(\hat{G}, L^\sigma, R^\sigma; D^\sigma)$ where σ is an isomorphism from G to $\hat{G}$ (it does not appear in [11] but it is easy to prove.)*

Proposition 2.3. *[11] Suppose Y is a G-semitransitive graph with bipartition $V(Y) = U(Y) \cup W(Y)$. Take $u \in U(Y)$ and $w \in W(Y)$. Set $D = \{g \in G \bigm| w^g \in Y_1(u)\}$. Then D is a union of double cosets of G_w and G_u in G, and $Y \cong \mathbf{B}(G, G_u, G_w; D)$.*

Proposition 2.4. *[32, 11.6, 11.7] Every permutation group of prime degree p is either insolvable and 2-transitive, or isomorphic to $Z_p \rtimes Z_s$ for some s dividing $p-1$.*

Proposition 2.5. *[14] The insolvable permutation groups of prime degree p are given as follows, where T denotes be the socle of the group and H denotes a point stabilizer of T:*

- (i) *$T = A_p$ and $H = A_{p-1}$;*
- (ii) *$T = \mathrm{PSL}(n,q)$ and H is the stabilizer of a projective point or a hyperplane in $\mathrm{PG}(n-1,q)$, and $|T : H| = (q^n-1)/(q-1) = p$;*
- (iii) *$T = \mathrm{PSL}(2,11)$ and $H = A_5$, and T has two conjugacy classes of subgroups isomorphic to A_5;*
- (iv) *$T = M_{11}$ and $H = M_{10}$;*

(v) $T = M_{23}$ *and* $H = M_{22}$.

Lemma 2.6. *[31] Let G be an imprimitive transitive group of degree p^2 with $p \geq 3$ and $p^3 \mid |G|$. Suppose that G has an imprimitive system $\mathcal{B}$ with p-blocks and the kernel K. Let P be a Sylow p-subgroup of G and $N = P \cap K$. Then*

(1) Exp $(P) \leq p^2$, $|Z(P)| = p$ *and* $P = N\langle t\rangle$, *where* $t^p \in Z(P)$;

(2) *K is solvable, N* char *K and so $N \lhd G$, provided either $p = 3$; or $p \geq 5$ and $|N| \leq p^{p-1}$.*

3 Proof of the main theorem

To prove Theorem 1.4, we assume that p is an odd prime and that X is a semisymmetric graph of order $2p^3$ expanded from the graph Y, where $\mathrm{Aut}(Y)$ acts edge transitively on Y and has blocks of length p^2 on $W(Y)$ and of length p on $U(Y)$, and the block graph $\overline{Y}$ is a cycle C_{2p} of length $2p$.

Let $F = \mathrm{Aut}(Y)$ and let

$$\mathcal{B} = \{B_0, B_1, \cdots, B_{p-1}\} \qquad \text{and} \qquad \mathcal{B}' = \{B'_0, B'_1, \cdots, B'_{p-1}\}$$

be blocks system of F on $U(Y)$ and $W(Y)$, respectively. Label

$$E(\overline{Y}) = \{(B_0, B'_{\frac{p+1}{2}}), (B'_{\frac{p+1}{2}}, B_1), \cdots, (B_{\frac{p-1}{2}}, B'_0), (B'_0, B_{\frac{p+1}{2}}), \cdots, (B_{p-1}, B'_{\frac{p-1}{2}}), (B'_{\frac{p-1}{2}}, B_0)\},$$

so that $\overline{Y} \cong C_{2p}$. Set

$$\sigma = (0, 1, \cdots, p-1) \quad \text{and} \quad \tau = (0)(1, -1) \cdots (\frac{p-1}{2}, \frac{p+1}{2}) \in S_p.$$

Then $\mathrm{Aut}(\overline{Y}) \cong \langle \sigma, \tau\rangle \cong D_{2p}$, by defining $(B_i)^\gamma = B_{i^\gamma}$ and $(B'_j)^\gamma = B'_{j^\gamma}$ for any $\gamma \in \langle \sigma, \tau\rangle$.

Label the vertices in B_i by a_{ji} for $j \in Z_p$. By considering the imprimitive action of F on $U(Y)$, we know that

$$F \leq S_p \wr \langle \sigma, \tau\rangle = S_p^p \rtimes \langle \sigma, \tau\rangle,$$

where, for any

$$e = (e^{(0)}, e^{(1)}, \cdots, e^{(p-1)}) \in S_p^p \quad \text{and} \quad \gamma \in \langle \sigma, \tau\rangle,$$

we have

$$a_{ji}{}^{(e;\gamma)} = a_{j^{e^{(i)}} i^\gamma}.$$

In particular, by identifying $(1, \gamma)$ with γ so that $a_{ji}^\gamma = a_{ji^\gamma}$, we have that $\langle \sigma, \tau\rangle$ can be viewed as a subgroup of F.

From now on, for any $t \in T \leq S_p$ and $i \in Z_p$, we set

$$t_i = (\overbrace{1, 1, \cdots, t}^{i+1}, 1, \cdots, 1) \quad \text{and} \quad T_i = \langle t_i \mid t \in T\rangle,$$

where T_i acts transitively on B_i and fixes B_j pointwise for all $j \neq i$. Moreover, we have

$$t_i^{(e;\gamma)} = t_{i^\gamma}^{e^{(i)}},\ T_0^{\sigma^i} = T_{0^{\sigma^i}} = T_i,\ B_0^{\sigma^i} = B_i.$$

Since K^{B_i} is a transitive group of degree p, following Propositions 2.4 and 2.5 we need to consider the following four cases separately in four subsections:

(i) $p \geq 5$ and K^{B_i} is insolvable;

(ii) $p \geq 5$ and $K^{B_i} \cong Z_p \rtimes Z_r$ for $r \neq 1$;

(iii) $p \geq 5$ and $K^{B_i} \cong Z_p$.

(iv) $p = 3$.

3.1 K^{B_i} is insolvable for $p \geq 5$

Lemma 3.1. *Suppose that $p \geq 5$ and K^{B_i} is insolvable. Then Y is isomorphic to one of the following graphs:*

(i) $Y_1(p, r)$, *and* $\mathrm{Aut}(Y) = S_p \wr D_{2p}$, *where* $r = 1$ *or* $p - 1$;

(ii) $Y_3(11, 5)$ *and* $Y_4(11, 6)$, *and* $\mathrm{Aut}(Y) = \mathrm{PSL}(2, 11) \wr D_{22}$;

(iii) $Y_5(p, \frac{q^{n-1}-1}{q-1})$ *and* $Y_6(p, q^{n-1})$, *and* $\mathrm{Aut}(Y) = \mathrm{P\Gamma L}(n, q) \wr D_{2p}$.

Proof. Suppose that $p \geq 5$ and K^{B_i} is insolvable. Then by Lemma 2.6 we know that $K = T_0 \times T_1 \times \cdots \times T_{p-1}$, where T is an insolvable group of degree p and T_i is defined as before. In particular, a Sylow p-subgroup of F is of order p^{p+1}, and so we may assume that F contains σ defined as above.

Let $u \in B_0$ and take an element $g_0 \in F_u \setminus K$. Since g_0 fixes B_0 setwise and exchanges $B'_{\frac{p-1}{2}}$ and $B'_{\frac{p+1}{2}}$, there exists a $d = (d^{(0)}, d^{(1)}, \cdots, d^{(p-1)}) \in S_p^p$ such that $g_0 = d\tau$, where τ is defined as before. Since $F/K \cong D_{2p}$, by considering the order of F we get $F = KR$ where $R = \langle \sigma, d\tau \rangle$.

Let $H_0 = (T_0)_u$. Then

$$K_u = H_0 \times T_1 \times \cdots \times T_{p-1} \quad \text{and} \quad F_u = K_u \rtimes \langle d\tau \rangle.$$

By $K_u^{d\tau} = K_u$, we know that $d^{(0)} \in N_{(S_p)_0}(H_0)$ and $d^{(i)} \in N_{(S_p)_i}(T_i)$ for $i \neq 0$.

Now $d\tau$ fixes the block B'_0 setwise and exchanges $B_{\frac{p-1}{2}}$ and $B_{\frac{p+1}{2}}$. Take $w \in B'_0$. Since $T_{\frac{p-1}{2}} \times T_{\frac{p+1}{2}}$ fixes u and acts transitively on B'_0, there exists a $k \in T_{\frac{p-1}{2}} \times T_{\frac{p+1}{2}} \leq K_u$ such that $kd\tau$ fixes both u and w, where without loss of generality, we denote kd by d again so that $d\tau$ fixes both u and w. Then

$$K_w = T_0 \times \cdots \times T_{\frac{p-3}{2}} \times L_{\frac{p-1}{2}} \times N_{\frac{p+1}{2}} \times T_{\frac{p+3}{2}} \times \cdots \times T_{p-1} \quad \text{and} \quad F_w = K_w \langle d\tau \rangle.$$

By $K_w^{d\tau} = K_w$, we know that $L = N$ and $d^{(i)} \in N_{(S_p)_i}(L_i)$ for $i \in \{\frac{p-1}{2}, \frac{p+1}{2}\}$.

Now the corresponding groups H and L are two maximal subgroups of T of index p. Following Proposition 2.5 we need to consider three cases separately.

(1) H and L are conjugate in T.

Without loss of generality, let $H = L$. For any almost simple group T in S_p, its point stabilizers have two orbits in each block B_i with the respective length 1 and $p - 1$. We may therefore let $T = S_p$ so that $H = S_{p-1}$ and $F = S_p^p R = S_p^p \langle \sigma, d\tau \rangle = S_p^p \langle \sigma, \tau \rangle$. Thus, we may set $d = 1$. For later use, we set $t = (0, 1, \cdots, p-1)$, $\Sigma_1 = \{0\}$ and $\Sigma_2 = Z_p^*$.

(2) $\mathrm{soc}(T) = \mathrm{PSL}(2, 11)$, and H and L are not conjugate in T.

In this case $T = \text{PSL}(2, 11)$, and T has two nonequivalent representations on the set of right cosets of A_5 of cardinality 11. Now $F = T^p\langle\sigma, d\tau\rangle$. Since $d^{(i)} \in N_{S_{11}}(T_i) = T_i$, we have $d \in T^p$ and so $F = T^p\langle\sigma, \tau\rangle$. Therefore, we set $d = 1$.

Moreover, T may be considered as the automorphism group of a (11, 5, 2)-design $\mathcal{D}$. Let $V = Z_{11}$ be the point set and let $t = (0, 1, \cdots, 10)$ be an element of order 11 in T. Then $M = \{0, 2, 3, 4, 8\} \subset V$ is a block (see [1, p.55]) of $\mathcal{D}$. Without loss of generality, we choose L and H to be the stabilizes of the block M and point 0, respectively. Again, for later use, set $\Sigma_3 = M$ and $\Sigma_4 = Z_{11} \setminus M$.

(3) $\text{soc}(T) = \text{PSL}(n, q)$, and H and L are not conjugate in T.

In this case, $\text{PSL}(n, q) \leq T \leq N_{S_p}(\text{PSL}(n, q))$=$\text{P}\Gamma\text{L}(n, q)$. With the same reason as (1), we let $T = N_{S_p}(\text{PSL}(n, q))$ and $d = 1$. Let S_1 and S_2 be the set of points and hyperplanes of $\text{PG}(n, q-1)$, respectively, where $|S_1| = |S_2| = \frac{q^n-1}{q-1} = p$. Without loss of generality, we choose L and H to be the stabilizers of a given point $\langle v\rangle$ and a hyperplane $\mathcal{L}$, respectively. Let $G = \langle t\rangle \cong Z_p$ be a singer subgroup of $\text{PGL}(n, q)$. Let

$$\Sigma_5 = \{l \in Z_p \mid \langle v\rangle \in \mathcal{L}^{t^l}\},\ \Sigma_6 = Z_p \setminus \Sigma_1 = \{l \in Z_p \mid \langle v\rangle \notin \mathcal{L}^{t^l}\},$$

where $|\Sigma_5| = \frac{q^{n-1}-1}{q-1}$ and $|\Sigma_6| = q^{n-1}$.

Now for the above three cases (1)-(3), we have

$$\text{Cos}(F, F_w) = \{F_w t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}} \sigma^k \mid i, j, k \in Z_p\},\ \text{Cos}(F, F_u) = \{F_u t_0^y \sigma^z \mid y, z \in Z_p\}.$$

Clearly, F_w has two orbits on $B_{\frac{p-1}{2}} \cup B_{\frac{p+1}{2}}$, that is,

$$D_l = \{F_u t_0^b \sigma^{\frac{p-1}{2}}, F_u t_0^b \sigma^{\frac{p+1}{2}} \mid b \in \Sigma_l\},$$

where $l = 1, 3, 4, 5, 6$. For any point $F_w t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}} \sigma^k$ in $W(Y)$, since

$$\begin{aligned} F_u t_0^b \sigma^{\frac{p-1}{2}} t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}} \sigma^k &= F_u t_0^b (t^{\sigma^{\frac{p+1}{2}}}_{\frac{p-1}{2}})^i (t^{\sigma^{\frac{p+1}{2}}}_{\frac{p+1}{2}})^j \sigma^{\frac{p-1}{2}+k} = F_u t_0^b t_0^i t_1^j \sigma^{\frac{p-1}{2}+k} \\ &= F_u t_0^{i+b} \sigma^{k+\frac{p-1}{2}}, \end{aligned}$$

and similarly,

$$F_u t_0^b \sigma^{\frac{p+1}{2}} t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}} \sigma^k = F_u t_0^{j+b} \sigma^{k+\frac{p+1}{2}},$$

its neighbor is

$$D_l t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}} \sigma^k = \{F_u t_0^{i+b} \sigma^{k+\frac{p-1}{2}}, F_u t_0^{j+b} \sigma^{k+\frac{p+1}{2}} \mid b \in \Sigma_l\}.$$

By labeling $F_w t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}} \sigma^k$ by $(i, j, k, 0)$ and $F_u t_0^y \sigma^z$ by $(y, z, 1)$, we get the respective edge set of two graphs $Y(l)$

$$\begin{aligned} E_l \quad = \quad & \{((i, j, k, 0), (y, z, 1)) \mid y = i + b, z = k + \tfrac{p-1}{2}; \text{and} \\ & y = j + b, z = k + \tfrac{p+1}{2}, i, j, k, y, z \in Z_p, b \in \Sigma_l\}. \end{aligned}$$

In cases (2) and (3), we get the graphs $Y_3(11,5), Y_4(11,6)$ with the automorphism group $\mathrm{PSL}(2,11) \wr D_{2p}$, and $Y_5(p,r), Y_6(p,r)$ with the automorphism group $\mathrm{P\Gamma L}(n,q) \wr D_{2p}$.

For case (1), the graph with the edge set E_2 is exactly $Y_1(p,p-1)$ for $p \geq 5$. As for the graph with the edge set E_1, let ϕ be a map on $W(Y) \cup U(Y)$ which fixes $W(Y)$ pointwise and sends $(y,z,1)$ to $(y+1,z,1)$. Then ϕ is an isomorphism between the present graph and $Y_1(p,1)$. From the proof we know that both $Y_1(p,1)$ and $Y_1(p,p-1)$ have the automorphism group $S_p \wr D_{2p}$. □

3.2 $K^{B_i} \cong Z_p \rtimes Z_r$ for $p \geq 5$ and $r \neq 1$

Lemma 3.2. *Suppose $K^{B_i} \cong Z_p \rtimes Z_r$ for $p \geq 5$ and $r \neq 1$. Then $Y \cong Y_1(p,r)$ or $Y_2(p,r)$ where $p \geq 5$, $r \neq 1, p-1$ and $(p,r) \neq (7,3),(11,5)$, where* $\mathrm{Aut}(Y) = (Z_p \rtimes Z_r) \wr D_{2p}$.

Proof. Step 1: Determination of the structure of F.

Proof. Suppose $K^{B_i} \cong Z_p \rtimes Z_r$ for $r \neq 1$. Let $S = \langle t \rangle \rtimes \langle c \rangle \cong Z_p \rtimes Z_{p-1} \leq S_p$. Then we may set $T = \langle t \rangle \rtimes \langle h \rangle$, where $h = c^{\frac{p-1}{r}}$. Let P be a Sylow p-subgroup of F and take $d_0\sigma \in P$ where

$$d_0 \in \langle t_0 \rangle \times \langle t_1 \rangle \times \cdots \times \langle t_{p-1} \rangle \cong Z_p^p.$$

Then $K \leq T^p$ and $F = K\langle d_0\sigma, d\tau \rangle$ for some $d \in S_p^p$. Moreover,

$$F \leq \hat{F} = T^p\langle d_0\sigma, d\tau \rangle = T^p\langle \sigma, d\tau \rangle \quad \text{and} \quad \langle \sigma, d\tau \rangle / (T^p \cap \langle \sigma, d\tau \rangle) \cong D_{2p}.$$

Let $w \in B_0'$ and $(B_0', B_{\frac{p-1}{2}}), (B_0', B_{\frac{p+1}{2}}) \in E(\overline{Y})$. Let $(w,u_1) \in E(Y)$ for $u_1 \in B_{\frac{p-1}{2}}$. Then $E = (w,u_1)^F \leq (w,u_1)^{\hat{F}}$. Since the orbits of F_w and $\hat{F}_w$ on the block $B_{\frac{p-1}{2}}$ in $U(Y)$ are completely the same, we have $|(w,u_1)^{\hat{F}}| = 2rp^3 = |E|$, which implies $E = (w,u_1)^{\hat{F}}$. Therefore, we may just consider the case $F = \hat{F} = T^p\langle \sigma, d\tau \rangle$.

As in the last Lemma, we choose two vertices $u \in B_0$ and $w \in B_0'$ which are fixed by $d\tau$. Without loss of generality, let $H = \langle h \rangle$ so that

$$F_u = (H_0 \times T_1 \times \cdots \times T_{p-1})\langle d\tau \rangle,\ F_w = (T_0 \times T_1 \times \cdots \times H_{\frac{p-1}{2}} \times H_{\frac{p+1}{2}} \times \cdots \times T_{p-1})\langle d\tau \rangle.$$

We then need to determine the element d.

Let $d = (d^{(0)}, d^{(1)}, \cdots, d^{(p-1)}) \in S_p^p$. Since $d\tau$ normalizes K, K_u and K_w, it follows that $d^{(i)} \in N_{S_p}(H_i) = \langle c \rangle$ for $i \in \{0, \frac{p\pm1}{2}\}$, and $d^{(i)} \in N_{S_p}(T_i) = S = \langle t \rangle \langle c \rangle$ for $i \notin \{0, \frac{p\pm1}{2}\}$. Suppose that $i \in \{0, \frac{p\pm1}{2}\}$ and write $d^{(i)} = t^m c^n$. Since $T_i \leq F_u$ and F_w we may re-choose $d^{(i)} = c^n$. Therefore, for any $i \in Z_p$, we get

$$d^{(i)} \in \langle c \rangle. \tag{1}$$

Since $(d\tau)^2 \in K$, we have

$$d\tau d\tau = ((d^{(0)})^2, d^{(1)}d^{(p-1)}, \cdots, d^{(p-1)}d^{(1)}) \in K,$$

and by taking into account (1) we get

$$(d^{(0)})^2, d^{(1)}d^{(p-1)}, \cdots, d^{(\frac{p-1}{2})}d^{(\frac{p+1}{2})} \in H. \tag{2}$$

Since K_w fixes only one point $u^{\sigma^{\frac{p-1}{2}}}$ in $B_{\frac{p-1}{2}}$ and $u^{\sigma^{\frac{p+1}{2}}}$ in $B_{\frac{p+1}{2}}$ and since $d\tau$ normalizes K_w and exchanges $B_{\frac{p-1}{2}}$ and $B_{\frac{p+1}{2}}$, it follows that $d\tau$ must exchange these two points. Therefore,

$$\begin{aligned} F_u\sigma^{\frac{p-1}{2}}(d\tau) &= F_u(d^{(\frac{p-1}{2})}, d^{(\frac{p+1}{2})}, \cdots, d^{(p-1)}, d^{(0)}, d^{(1)}, \cdots, d^{(\frac{p-3}{2})})\tau\sigma^{\frac{p+1}{2}} \\ &= F_u d(d^{(\frac{p-1}{2})}, d^{(\frac{p-3}{2})}, \cdots, d^{(1)}, d^{(0)}, d^{(p-1)}, \cdots, d^{(\frac{p+1}{2})})\sigma^{\frac{p+1}{2}} \\ &= F_u(d^{(0)}d^{(\frac{p-1}{2})}, d^{(1)}d^{(\frac{p-3}{2})}, \cdots, d^{(\frac{p-1}{2})}d^{(0)}, d^{(\frac{p+1}{2})}d^{(p-1)}, \\ &\quad \cdots, d^{(p-1)}d^{(\frac{p+1}{2})})\sigma^{\frac{p+1}{2}} \\ &= F_u\sigma^{\frac{p+1}{2}}. \end{aligned}$$

Hence,

$$d^{(0)}d^{(\frac{p-1}{2})}, d^{(1)}d^{(\frac{p-3}{2})}, \cdots, d^{(\frac{p-1}{2})}d^{(0)}, d^{(\frac{p+1}{2})}d^{(p-1)}, \cdots, d^{(p-1)}d^{(\frac{p+1}{2})} \in H. \quad (3)$$

From (2) and (3) we get

$$d^{(0)}, d^{(1)} \cdots, d^{(p-1)} \in H, \text{ or } d^{(0)}, d^{(1)} \cdots, d^{(p-1)} \in \langle c^{\frac{p-1}{2r}} \rangle \setminus H \text{ if } 2r \mid (p-1). \quad (4)$$

Therefore, if $2r \nmid (p-1)$ then we set $d = 1$; if $2r \mid (p-1)$, we set $d = (c'^m, c'^m, \cdots c'^m)$ where $c' = c^{\frac{p-1}{2r}}$ and $m = 0, 1$. To unify these two cases, in the first case we still write $d = (c'^m, c'^m, \cdots c'^m)$ for $m = 0$.

Suppose that $2r \mid (p-1)$. Let $F_1 = K \rtimes \langle \sigma, \tau \rangle$ and $F_2 = K \rtimes \langle \sigma, d\tau \rangle$, where $d = (c', \cdots, c')$ with $c' = c^{\frac{p-1}{2r}}$, noting that $c' \notin T$. we may then state the following fact

Fact: $F_1 \ncong F_2$

Proof: Assume the contrary. Suppose that γ is an isomorphism from F_1 to F_2. Since $\langle (t, t, \cdots, t) \rangle$ is characteristic in F_1 and F_2, we get

$$\gamma((t, t \cdots t)) = (t^k, t^k, t^k \cdots, t^k)$$

for some $k \in F_p^*$. Assume that $\gamma(\tau) = ed\tau$, where $e = (e^{(0)}, e^{(1)} \cdots, e^{(p-1)}) \in K$. Since

$$\tau^{-1}(t, t, \cdots, t)\tau = (t, t, \cdots, t).$$

we have

$$\gamma(\tau^{-1})\gamma((t, t, \cdots, t))\gamma(\tau) = \gamma(t, t, \cdots, t),$$

that is

$$(ed\tau)^{-1}(t^k, t^k, \cdots, t^k)(ed\tau) = (t^k, t^k, \cdots, t^k),$$

which implies

$$(t^k)^{e^{(0)}c'} = t^k.$$

Therefore, $e^{(0)}c' \in \langle t \rangle$ and so $c' \in T$, a contradiction.

Step 2: Determination of the bicoset graphs.

Set $D(l) = F_u t^l \sigma^{\frac{p-1}{2}} F_w$ and by $Z = Z(p, r, d, l)$ we denote the corresponding bicoset graph. We consider two cases separately.

(1) $l = 0$.

Since $F_u\sigma^{\frac{p-1}{2}}K_w = F_u\sigma^{\frac{p-1}{2}}$ and $F_u\sigma^{\frac{p-1}{2}}K_w\langle d\tau\rangle = F_u\sigma^{\frac{p+1}{2}}$, we have

$$D(0) = F_u\sigma^{\frac{p-1}{2}}F_w = \{F_u\sigma^{\frac{p-1}{2}}, F_u\sigma^{\frac{p+1}{2}}\}.$$

For any point $F_w t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}}\sigma^k$ in $W(Z)$, since

$$F_u\sigma^{\frac{p-1}{2}} t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}}\sigma^k = F_u(t^{\sigma^{\frac{p+1}{2}}}_{\frac{p-1}{2}})^i(t^{\sigma^{\frac{p+1}{2}}}_{\frac{p+1}{2}})^j\sigma^{k+\frac{p-1}{2}} = F_u t^i_0 t^j_1\sigma^{k+\frac{p-1}{2}} = F_u t^i_0\sigma^{k+\frac{p-1}{2}},$$

and similarly

$$F_u\sigma^{\frac{p+1}{2}} t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}}\sigma^k = F_u t^j_0\sigma^{k+\frac{p+1}{2}},$$

its neighbor is $N = \{F_u t^i_0\sigma^{k+\frac{p-1}{2}}, F_u t^j_0\sigma^{k+\frac{p+1}{2}}\}$. In this case, $d(w) = 2$. Let

$$\rho : F_w t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}}\sigma^k \to F_w t^{i+1}_{\frac{p-1}{2}} t^{j+1}_{\frac{p+1}{2}}\sigma^k, \quad F_u t^y_0\sigma^z \to F_u t^y_0\sigma^z$$

be the mapping of $V(Z(p,r,d,0))$ to $V(Y_1(p,1))$. Then one may check that ρ is an isomorphism from $Z(p,r,d,0)$ to $Y_1(p,1)$. Therefore, $\mathrm{Aut}(Z(p,r,d,0)) \cong S_p \wr D_{2p}$, contrary to our hypothesis $K^{B_i} \cong Z_p \rtimes Z_r$.

(2) $l \neq 0$.

In $S_p \wr D_{2p}$, there exists some $c^{l'}$ such that the inner automorphism $I(c^{l'})$ fixes F_u and F_w and maps $D(1)$ to $D(l)$. Therefore, up to graph isomorphism, we only consider $Z(p,r,d,1)$.

Since

$$F_u t_0\sigma^{\frac{p-1}{2}}K_w = F_u t_0\sigma^{\frac{p-1}{2}}(T_0 \times T_1 \times \cdots \times H_{\frac{p-1}{2}} \times H_{\frac{p+1}{2}} \times \cdots T_{p-1}) = F_u(t^H)_0\sigma^{\frac{p-1}{2}},$$

$$\begin{aligned} F_u t_0\sigma^{\frac{p-1}{2}}K_w d\tau &= F_u(t^H)_0\sigma^{\frac{p-1}{2}}d\tau = F_u(t^H)_0(c'^m, c'^m, \cdots, c'^m)\tau\sigma^{\frac{p+1}{2}} \\ &= F_u(t^{Hc'^m})_0\sigma^{\frac{p+1}{2}}, \end{aligned}$$

it follows that

$$D(1) = \{F_u(t^{h'})_0\sigma^{\frac{p-1}{2}}, F_u(t^{h'c'^m})_0\sigma^{\frac{p+1}{2}} \mid h' \in H\}.$$

For any $i, j, k \in Z_p$, since

$$F_u(t^{h'})_0\sigma^{\frac{p-1}{2}} t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}}\sigma^k = F_u(t^{h'})_0 t^i_0\sigma^{k+\frac{p-1}{2}} = F_u(t^{h'+i})_0\sigma^{k+\frac{p-1}{2}}$$

and

$$F_u(t^{h'c'^m})_0\sigma^{\frac{p+1}{2}} t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}}\sigma^k = F_u(t^{h'c'^m})_0 t^j_0\sigma^{k+\frac{p+1}{2}} = F_u(t^{h'c'^m+j})_0\sigma^{k+\frac{p+1}{2}},$$

the neighbor of $F_w t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}}\sigma^k$ is

$$\{F_u(t^{h'+i})_0\sigma^{k+\frac{p-1}{2}}, F_u(t^{h'c'^m+j})_0\sigma^{k+\frac{p+1}{2}} \mid h' \in H\}.$$

Suppose $d(w) = 2(p-1)$. Then $K^{B_i} \cong Z_p \rtimes Z_{p-1}$ and $F = K\langle\sigma,\tau\rangle$. In this case, for any $F_w t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}} \sigma^k$ in $W(Y)$, its neighbor is

$$N = \{F_u(t^{h'+i})_0\sigma^{k+\frac{p-1}{2}}, F_u(t^{h'+j})_0\sigma^{k+\frac{p+1}{2}} \mid h' \in H\}$$

where $H \cong Z_{p-1}$. Clearly, the corresponding graph is isomorphic to $Y_1(p,p-1)$, with $K^{B_i} \cong S_p$, a contradiction. Therefore, $2 < d(w) < 2(p-1)$.

Let $t^{h'} = t^b$ and $b \in \Sigma$ where Σ is a subgroup of Z_p^* of order r. Define a mapping ϕ from $V(Z)$ to $V(Y_2(p,r))$, for $r \neq 1, p-1$, by

$$F_w t^i_{p-1} t^j_1 \sigma^k \to F_w t^i_{p-1} t^j_1 \sigma^k, \quad \text{and} \quad F_u t^y_0 \sigma^z \to F_u t^y_0 \sigma^z.$$

Then ϕ is clearly an isomorphism between the two graphs.

Step 3: Determination of isomorphic classes and automorphism groups.

Let $\tilde{A} = \text{Aut}(Z)$ and K_Z be the kernel of $\tilde{A}$ on $\overline{Z}$, where $\overline{Z}$ is a cycle of length $2p$. Clearly, $\tilde{A}/K \cong D_{2p}$.

If $(p,r) = (7,3)$ and $(11,5)$, then Z is $Y_2(7,3)$ with $K^{B_i} \cong P\Gamma L(3,2)$ and $Y_1(11,5)$ with $\text{PSL}(2,11)$, respectively, contradicting our condition.

Suppose that $(p,r) \neq (7,3),(11,5)$. Since $r \mid (p-1)$ and $r \neq 1, p-1$, K^{B_i} can not be insolvable and hence an affine group. Therefore, $K \leq K_Z = (Z_p \rtimes Z_r)^p$ and then $K_Z = K$. Therefore, $\tilde{A} = F$.

In the case of $2r \mid (p-1)$, let $F_1 = K \rtimes \langle\sigma,\tau\rangle$ and $F_2 = K \rtimes \langle\sigma, d\tau\rangle$ be defined as in Step 1. Let Z_1 and Z_2 be the corresponding graphs. Suppose that ρ is an isomorphism from Z_1 to Z_2. Then $F_2 \leq \langle\rho^{-1}F_1\rho, F_2\rangle \leq \text{Aut}(Z_2) = F_2$. Therefore, $F_2 = \rho^{-1}F_1\rho \cong F_1$, a contradiction. Therefore, the two graphs are not isomorphic.

□

3.3 $K^{B_i} \cong Z_p$ for $p \geq 5$

Lemma 3.3. *The case $K^B \cong Z_p$ cannot occur.*

Proof. Suppose that $K^B \cong Z_p$. Then $|E(Y)| = 2p^3$. As above, let $w \in B'_0$ and $(B'_0, B_{\frac{p-1}{2}}), (B'_0, B_{\frac{p+1}{2}}) \in E(\overline{Y})$. Let $(w,u_1) \in E(Y)$ for $u_1 \in B_{\frac{p-1}{2}}$. Then $E = (w,u_1)^F$. We may consider the group $\hat{F} = S_p^p \rtimes \langle\sigma,\tau\rangle \geq F$. From the proof of Lemma 3.1, we may construct two representations of $\hat{F}$ with respective degree p^3 and p^2 such that both K_w and $(S_p^p)_w$ fix u_1. Then $(w,u_1)^F \subset (w,u_1)^{\hat{F}}$. Since $|(w,u_1)^F| = 2p^3 = |(w,u_1)^{\hat{F}}|$, we have $(w,u_1)^{\hat{F}} = (w,u_1)^F = E(Y)$ and so $\text{Aut}(Y) \cong \hat{F}$, contrary to our hypothesis $K^{B_i} \cong Z_p$. Therefore, this case cannot occur. □

3.4 $p = 3$

Lemma 3.4. *If $p = 3$, then $Y \cong Y_1(3,r)$ for $r = 1, 2$.*

Proof. In this case, take $F = S_3 \wr D_6$ and $H = L = Z_2$. Checking the proof of Lemma 3.1(1), one may find that the arguments in there still hold for $p = 3$. Therefore, $Y \cong Y_1(3,r)$ for $r = 1, 2$. □

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Cayley graphs on nilpotent groups with cyclic commutator subgroup are hamiltonian

Ebrahim Ghaderpour , Dave Witte Morris

Department of Mathematics and Computer Science, University of Lethbridge Lethbridge, Alberta, T1K 3M4, Canada

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Abstract

We show that if G is any nilpotent, finite group, and the commutator subgroup of G is cyclic, then every connected Cayley graph on G has a hamiltonian cycle.

Keywords: Cayley graph, hamiltonian cycle, nilpotent group, commutator subgroup.

Math. Subj. Class.: 05C25, 05C45

1 Introduction

It has been conjectured that every connected Cayley graph has a hamiltonian cycle. See [4, 10, 11, 14, 15, 17] for references to some of the numerous results on this problem that have been proved in the past forty years, including the following theorem that is the culmination of papers by Marušič [12], Durnberger [5, 6], and Keating-Witte [9]:

Theorem 1.1 (D. Marušič, E. Durnberger, K. Keating, and D. Witte, 1985)**.** Let G be a nontrivial, finite group. If the commutator subgroup $[G, G]$ of G is cyclic of prime-power order, then every connected Cayley graph on G has a hamiltonian cycle.

It is natural to try to prove a generalization that only assumes the commutator subgroup is cyclic, without making any restriction on its order, but that seems to be an extremely difficult problem: at present, it is not even known whether all connected Cayley graphs on dihedral groups have hamiltonian cycles. (See [1, 2] and [15, Cor. 5.2] for the main results that have been proved for dihedral groups.) In this paper, we replace the assumption on the order of $[G, G]$ with the rather strong assumption that G is nilpotent:

Theorem 1.2. Let G be a nontrivial, finite group. If G is nilpotent, and the commutator subgroup of G is cyclic, then every connected Cayley graph on G has a hamiltonian cycle.

E-mail addresses: Ebrahim.Ghaderpoor@uleth.ca (Ebrahim Ghaderpour), Dave.Morris@uleth.ca, http://people.uleth.ca/∼dave.morris/ (Dave Witte Morris)

The proof of this theorem is based on a variant of the method of D. Marušič [12] that established theorem 1.1 (cf. [9, Lem. 3.1]).

Remark 1.3. Here are some previous results on the hamiltonicity of the Cayley graph $\mathrm{Cay}(G;S)$ when G is nilpotent:

1. Assume G is nilpotent, the commutator subgroup of G is cyclic, and S has only two elements. Then a hamiltonian cycle in $\mathrm{Cay}(G;S)$ was found in [9, §6] (see proposition 3.16). The present paper generalizes this by eliminating the restriction on the cardinality of the generating set S.
2. For Cayley graphs on nilpotent groups (without any assumption on the commutator subgroup) it was recently shown that if the valence is at most 4, then there is a hamiltonian *path* (see [13]).
3. Every nilpotent group is a direct product of p-groups. For p-groups, it is known that every Cayley graph has a hamiltonian cycle ([16], see theorem 3.13). Unfortunately, we do not know how to extend this to direct products.
4. Every abelian group is nilpotent. It is well known (and easy to prove) that Cayley graphs on abelian groups always have hamiltonian cycles. In fact, there is usually a hamiltonian path from any vertex to any other vertex (see [3]).

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2 Assumptions and notation

We begin with some standard notation:

Notation 2.1. Let G be a group, and let S be a subset of G.

- $\mathrm{Cay}(G;S)$ denotes the *Cayley graph* of G with respect to S. Its vertices are the elements of G, and there is an edge joining g to gs for every $g \in G$ and $s \in S$.
- $G' = [G, G]$ denotes the commutator subgroup of G.
- $S^r = \{\, s^r \mid s \in S \,\}$ for any $r \in \mathbb{Z}$. (Similarly, $G^r = \{\, g^r \mid g \in G \,\}$.)
- $S^{\pm 1} = S \cup S^{-1}$.
- $\#S$ is the cardinality of S.

Note that if S happens to be a cyclic subgroup of G, then S^r is a subgroup of S.

We now fix notation designed specifically for our proof of theorem 1.2:

Notation 2.2.

- G is a nilpotent, finite group,
- N is a cyclic, normal subgroup of G that contains G',
- $g \mapsto \overline{g}$ is the natural homomorphism from G to $G/N = \overline{G}$,
- $S = \{\sigma_1, \sigma_2, \ldots, \sigma_\ell\}$ is a subset of G, such that

  ◦ $\overline{S}$ is a minimal generating set for $\overline{G}$, and
  ◦ $\ell = \#S = \#\overline{S} \geq 2$,

  The minimality of $\overline{S}$ implies that $e \notin S$, and that if $s \in S$ and $|s| \geq 3$, then $s^{-1} \notin S$.

- $S_k = \{\, \sigma_i \mid i \leq k \,\}$ for $1 \leq k \leq \ell$,
- $G_k = \langle S_k \rangle N$, and
- $m_k = |\overline{G_k} : \overline{G_{k-1}}|$.

Definition 2.3.

- If $(s_i)_{i=1}^n$ is a sequence of elements of $S^{\pm 1}$ and $\overline{g} \in \overline{G}$, we use $\overline{g}(s_i)_{i=1}^n$ to denote the walk in $\mathrm{Cay}(\overline{G}; \overline{S})$ that visits (in order) the vertices

$$\overline{g}, \overline{gs_1}, \overline{gs_1s_2}, \ldots, , \overline{gs_1s_2 \cdots s_n}.$$

- If $C = \overline{g}(s_i)_{i=1}^n$ is any oriented cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$, its *voltage* is $\prod_{i=1}^n s_i$. This is an element of N, and it may be denoted ΠC.
- For $S_0 \subset S$, we say the walk $\overline{g}(s_i)_{i=1}^n$ *covers* $S_0^{\pm 1}$ if it contains an oriented edge labeled s and a (different) oriented edge labeled s^{-1}, for every $s \in S_0$. (That is, there exist i, j with $i \neq j$, such that $s_i = s$ and $s_j = s^{-1}$. When $|s| = 2$, this means $s_i = s_j = s$.)
- $\mathcal{V}_k$ is the set of voltages of oriented hamiltonian cycles in the graph $\mathrm{Cay}(\overline{G_k}; \overline{S_k})$ that cover $S_k^{\pm 1}$.

Notation 2.4. For $k \in \mathbb{Z}^+$, we use $(s_1, \ldots, s_n)^k$ to denote the concatenation of k copies of the sequence $(s_1, \ldots, s_n)$. Abusing notation, we often write s^k and s^{-k} for

$$(s)^k = (s, s, \ldots, s) \text{ and } (s^{-1})^k = (s^{-1}, s^{-1}, \ldots, s^{-1}),$$

respectively. Furthermore, we often write $\big((s_1, \ldots, s_m), (t_1, \ldots, t_n)\big)$ to denote the concatenation $(s_1, \ldots, s_m, t_1, \ldots, t_n)$. For example, we have

$$\big((a^3, b)^2, c^{-2}\big)^2 = (a, a, a, b, a, a, a, b, c^{-1}, c^{-1}, a, a, a, b, a, a, a, b, c^{-1}, c^{-1}).$$

The following well-known, elementary observation is the foundation of our proof:

Lemma 2.5 (“Factor Group Lemma” [17, §2.2])**.** Suppose

- N is a cyclic, normal subgroup of G,
- $C = \overline{g}(s_i)_{i=1}^n$ is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$, and
- the voltage ΠC generates N.

Then $(s_1, \ldots, s_n)^{|N|}$ is a hamiltonian cycle in $\mathrm{Cay}(G; S)$.

With this in mind, we let $N = G'$, and we would like to find a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$ whose voltage generates N. In almost all cases, we will do this by induction on

$\ell = \#S$, after substantially strengthening the induction hypothesis. Namely, we consider the following assertion (α_k^ϵ) for $2 \leq k \leq \ell$ and $\epsilon \in \{1, 2\}$:

$$\begin{array}{c}\text{there exists } h_k \in N\text{, such that, for every } x \in N,\\ \big(\mathcal{V}_k \cap h_k (G_k')^\epsilon\big) x \text{ contains a generator of}\\ \text{a subgroup of } N \text{ that contains } (G_k')^\epsilon.\end{array} \tag{α_k^ϵ}$$

For $\epsilon = 2$, we also consider the following slightly stronger condition, which we call α_k^{2+}:

$$\alpha_k^2 \text{ holds, and } \langle h_k, (G_k')^2 \rangle \text{ contains } G_k'. \tag{α_k^{2+}}$$

Lemma 2.6. Let $N = G'$. If either α_ℓ^1 or α_ℓ^{2+} holds, then there is a hamiltonian cycle in $\operatorname{Cay}(\overline{G}; \overline{S})$ whose voltage generates N.

Proof. Note that $G_\ell' = G' = N$. Since $\mathcal{V}_\ell$ consists of voltages of hamiltonian cycles in $\operatorname{Cay}(\overline{G}; \overline{S})$, it suffices to find an element of $\mathcal{V}_\ell$ that generates G_ℓ'.

If we assume α_ℓ^1, then the desired conclusion is immediate, by taking $x = e$ in that assertion.

Similarly, if we assume α_ℓ^{2+}, then taking $x = e$ in α_ℓ^2 tells us that some element γ of $\mathcal{V}_\ell \cap h_\ell (G_\ell')^2$ generates a subgroup of N that contains $(G_\ell')^2$. Then, since $\gamma \in (G_\ell')^2 h_\ell$, and $\langle h_\ell, (G_\ell')^2 \rangle$ contains G_ℓ', we have

$$N \supset \langle \gamma \rangle = \langle \gamma, (G_\ell')^2 \rangle = \langle h_\ell, (G_\ell')^2 \rangle \supset G_\ell' = N. \qquad \square$$

Remark 2.7.

1. If $|G_k'|$ is odd, then $(G_k')^2 = G_k'$, so we have $\alpha_k^1 \Leftrightarrow \alpha_k^2 \Leftrightarrow \alpha_k^{2+}$ in this case. Thus, the parameter ϵ is only of interest when $|G'|$ is even.
2. It is not difficult to see that $\alpha_k^1 \Rightarrow \alpha_k^{2+}$, but we do not need this fact.

Our proof of α_ℓ^1 or α_ℓ^{2+} is by induction on k. Here is the outline:

I. We prove a base case of the induction: α_2^2 is usually true (see proposition 4.1).

II. We prove an induction step: under certain conditions, $\alpha_k^1 \Rightarrow \alpha_{k+1}^1$ and $\alpha_k^{2+} \Rightarrow \alpha_{k+1}^{2+}$ (see proposition 5.4).

III. We prove α_ℓ^1 or α_ℓ^{2+} is usually true, by bridging the gap between α_2^2 and either α_3^1 or α_3^{2+}, and then applying the induction step (see corollary 6.1 and proposition 6.2).

3 Preliminaries

3A Remarks on voltage

Remark 3.1. By definition, it is clear that all translates of an oriented cycle C in $\operatorname{Cay}(\overline{G}; \overline{S})$ have the same voltage. That is,

$$\Pi\big(\overline{g}(s_i)_{i=1}^n\big) = \Pi\big((s_i)_{i=1}^n\big).$$

Remark 3.2. If $|N|$ is square-free (which is usually the case in this paper), then N is contained in the center of G (because N is the direct product of normal subgroups of prime order, and it is well known that those are all in the center [8, Thm. 4.3.4]). In this situation, the voltage of a cycle in $\mathrm{Cay}(\overline{G};\overline{S})$ is independent of the starting point that is chosen for its representation. That is, if $(t_i)_{i=1}^n$ is a cyclic rotation of $(s_i)_{i=1}^n$, so there is some $r \in \{0, 1, 2, \ldots, n\}$ with $t_i = s_{i+r}$ for all i (where subscripts are read modulo n), then

$$\Pi(t_i)_{i=1}^n = s_{r+1}s_{r+2}\cdots s_n\, s_1s_2\cdots s_r = (s_1\cdots s_r)^{-1}\big(\Pi(s_i)_{i=1}^n\big)s_1\cdots s_r = \Pi(s_i)_{i=1}^n,$$

because $\Pi(s_i)_{i=1}^n \in N \subset Z(G)$.

The following observation is useful for calculating voltages:

Lemma 3.3. If $a, b \in G$, $G' \subset Z(G)$, and $p_1, q_1, \ldots, p_r, q_r \in \mathbb{Z}$, then

$$a^{p_1}b^{q_1}a^{p_2}b^{q_2}\cdots a^{p_r}b^{q_r} = a^{p_1+\cdots+p_r}b^{q_1+\cdots+q_r}[a, b]^{-\Sigma},$$

where $\Sigma = \sum_{i>j} p_iq_j$.

Proof. The desired conclusion is easily proved by induction on r, using the fact that, since $G' \subset Z(G)$, we have $[a^p, b^q] = [a, b]^{pq}$ for $p, q \in \mathbb{Z}$ [7, Lem. 2.2(i)]. □

3B Facts from group theory

Lemma 3.4. If $|G'_k|$ is square-free, then $|G'_k/G'_{k-1}|$ is a divisor of both $|\overline{G_{k-1}}|$ and $|\overline{G_k}/\overline{G_{k-1}}|$.

Proof. We may assume $k = \ell$, so $G = G_k$. Let p be a prime factor of $|G'/G'_{k-1}|$, let P be the Sylow p-subgroup of G, and let $\varphi\colon G \to P$ be the natural projection. Since $|G'|$ is square-free, it suffices to show that $|\overline{G_{k-1}}|$ and $|\overline{G_k}/\overline{G_{k-1}}|$ are divisible by p.

We may assume $|G'| = p$ and $G'_{k-1} = \{e\}$, by modding out the unique subgroup of index p in G'. Therefore $\varphi(G_{k-1})$ is abelian, so it is a proper subgroup of P. Since $G' = P' \subset \Phi(P)$, this implies $\varphi(G_{k-1})G'$ is a proper subgroup of P, so its index is divisible by p. Hence $|\overline{G}/\overline{G_{k-1}}|$ is divisible by p.

There must be some $t \in S_{k-1}$, such that $[\sigma_k, t]$ is nontrivial. Hence $\varphi(t) \notin Z(G) \supset G'$, so p is a divisor of $|\overline{\varphi(t)}|$, which is a divisor of $|\overline{G_{k-1}}|$. □

The following fact is well known and elementary, but we do not know of a reference in the literature. It relies on our assumption that G' is cyclic.

Lemma 3.5. We have $\langle\, [s, t] \mid s, t \in S\,\rangle = G'$ if $N \subset Z(G)$.

Proof. Let $H = \langle\, [s, t] \mid s, t \in S\,\rangle$. Then H is a normal subgroup of G, because every subgroup of a cyclic, normal subgroup is normal. In G/H, every element of S commutes with all of the other elements of S (and with all of N), so G/H is abelian. Hence $G' \subset H$. □

3C Elementary facts about cyclic groups of square-free order

Lemma 3.6. Assume $|N|$ is square-free, and H and K are two subgroups of N. Then:

1. There is a unique subgroup $K^\perp$ of N, such that $N = K \times K^\perp$.
2. $K^\perp$ is a normal subgroup of G.
3. $K \subseteq H$ iff $\underline{H} = \underline{N}$ in $\underline{G} = G/K^\perp$.

Proof. (1) Since N is cyclic, it has a unique subgroup of any order dividing $|N|$; let $K^\perp$ be the subgroup of order $|N/K|$. Since $|N|$ is square-free, we have $\gcd\big(|K|, |K^\perp|\big) = 1$, so $N = K \times K^\perp$.

(2) It is well known that every subgroup of a cyclic, normal subgroup is normal (because no other subgroup of N has the same order).

(3) We prove only the nontrivial direction. Since $\underline{H} = \underline{N}$, we know that $|K| = |\underline{N}|$ is a divisor of $|H|$. So $|H|$ has a subgroup whose order is $|K|$. Since K is the only subgroup of N with this order, we must have $K \subseteq H$. □

Remark 3.7. When we want to show that some subgroup H of N contains some other subgroup K, Lemma 3.6 often allows us to assume $K = N$ (by modding out $K^\perp$), which means we wish to prove $H = N$.

Lemma 3.8. Suppose

- γ is a generator of N,
- $x \in N$, and
- $a \geq \max\big(|N|, 5\big)$.

Then, for some i with $1 \leq i \leq \lfloor (a-1)/2 \rfloor$, we have $N^2 \subseteq \langle \gamma^{-2i} x \rangle$.

Proof. Write $x = \gamma^h$, where $1 \leq h \leq |N|$, choose $r \in \{1, 2\}$ such that $h - r$ is even, and let

$$i = \begin{cases} r & \text{if } h \in \{1, 2\}, \\ (h-r)/2 & \text{if } h > 2. \end{cases}$$

Then $h - 2i \in \{\pm r\} \subset \{\pm 1, \pm 2\}$, so $N^2 \subseteq \langle \gamma^{h-2i} \rangle = \langle \gamma^{-2i} x \rangle$. □

Lemma 3.9. If

- N is a cyclic group of square-free order,
- $m \geq |N|$,
- $k \geq 2$,
- $T = \{\gamma_1, \ldots, \gamma_k\}$ generates N,
- $h \in N$, and
- $\mathrm{Cay}(N; T)$ is not bipartite,

then we may choose a sequence $(j_i)_{i=1}^{m-1}$ of elements of $\{1, 2, \ldots, k\}$, and $\gamma_i^* \in \{\gamma_{j_i}^{\pm 1}\}$ for each i, such that $\gamma_{i+1}^* = \gamma_i^*$ whenever $j_{i+1} = j_i$, and

$$\langle h \gamma_1^* \gamma_2^* \cdots \gamma_{m-1}^* \rangle = N. \tag{3.10}$$

Proof. Let us assume $|N| > 3$. (The smaller cases are very easy to address individually.)

We begin by finding $\gamma_1^*, \gamma_2^*, \ldots, \gamma_{m-1}^* \in T^{\pm 1}$, such that $\langle h \gamma_1^* \gamma_2^* \cdots \gamma_{m-1}^* \rangle$ contains N^2 (or N, if appropriate), but without worrying about the requirement that $\gamma_{i+1}^* = \gamma_i^*$ whenever $j_{i+1} = j_i$.

Let γ be a generator of N, and assume $h^{-1}\gamma \neq e$ (by replacing γ with its inverse, if necessary). Since $\mathrm{Cay}(N; T)$ is not bipartite, there is a walk $(\gamma_i^*)_{i=1}^r$ from e to $h^{-1}\gamma$, such that $r \equiv m - 1 \pmod 2$.

We now show the walk can be chosen to satisfy the additional constraint that $r < |N|$ (so $r \leq m-1$). We know that $\mathrm{Cay}(N;T)$ has a hamiltonian cycle C (since N is abelian). Since $\mathrm{Cay}(N;T)$ is not bipartite, C must have a chord L of even length. We may assume one endpoint of L is e, since $\mathrm{Cay}(N;T)$ is vertex transitive. Let z be the other endpoint of L. Being a hamiltonian cycle, C can be written as the union of two edge-disjoint paths from e to $h^{-1}\gamma$. Let P be the one of these paths that contains a subpath of even length from e to z, and let $\widehat{P}$ be obtained from P by replacing this subpath with the edge L. Then P and $\widehat{P}$ are two paths from e to $h^{-1}\gamma$. Both have length less than $|N|$, and their lengths are of opposite parity.

Now

$$h\gamma_1^*\gamma_2^*\cdots\gamma_r^*(\gamma_1\gamma_1^{-1})^{(m-1-r)/2} = \gamma \text{ generates } N,$$

as desired.

To complete the proof of the lemma, we modify the above sequence $\gamma_1^*, \gamma_2^*, \ldots, \gamma_{m-1}^*$ to satisfy the condition that $\gamma_{i+1}^* = \gamma_i^*$ whenever $j_{i+1} = j_i$. First of all, since N is commutative, we may collect like terms, and thereby write

$$\gamma_1^*\gamma_2^*\cdots\gamma_{m-1}^* = \gamma_1^{m_1}\gamma_2^{m_2}\cdots\gamma_k^{m_k}\gamma_1^{-n_1}\gamma_2^{-n_2}\cdots\gamma_k^{-n_k}$$

where $m_1+\cdots+m_k+n_1+\cdots+n_k = m-1$. Notice that if m_k and n_1 are both nonzero, then no occurrence of γ_i is immediately followed by γ_i^{-1}; so we have $\gamma_{i+1}^* = \gamma_i^*$ whenever $j_{i+1} = j_i$, as desired. Therefore, by permuting $\gamma_1, \ldots, \gamma_k$, we may assume $m_i = n_i = 0$ for all $i > 1$. Also, we may assume m_1 and n_1 are both nonzero, for otherwise we have $\gamma_i^* = \gamma_j^*$ for all i and j. Then, since $\gamma_1\gamma_1^{-1} = \gamma_2\gamma_2^{-1}$, we have

$$\gamma_1^*\gamma_2^*\cdots\gamma_{m-1}^* = \gamma_1^{m_1}\gamma_1^{-n_1} = \gamma_1^{m_1-1}\gamma_2\gamma_1^{-(n_1-1)}\gamma_2^{-1}.$$

We can assume $n_1 \geq m_1$ (by replacing γ_1 with its inverse, if necessary). Then

$$n_1 \geq \left\lceil \frac{m-1}{2} \right\rceil \geq \left\lceil \frac{|N|-1}{2} \right\rceil \geq 2,$$

so this new representation of the same product satisfies the condition that γ_i is never immediately followed by γ_i^{-1}. This completes the proof. □

Corollary 3.11. Assume $|N|$ is square-free and $k \geq 2$ (and $\epsilon \in \{1,2\}$, as usual). For convenience, let $m = m_{k+1}$ and $a = \sigma_{k+1}$. If $h \in N$, then there exists a sequence $(s_i)_{i=1}^{m-1}$ of elements of S_k, and $s_i^* \in \{s_i^{\pm 1}\}$ for each i, such that $s_{i+1}^* = s_i^*$ whenever $s_{i+1} = s_i$, and

$$\left\langle h \prod_{i=1}^{m-1} [a, s_i^*],\ (G_k')^\epsilon \right\rangle \text{ contains } \begin{cases} G_{k+1}' & \text{if there exist } s, s' \in S_k, \text{ such that } \\ & |[a,s]| \text{ is even and } |[a,s']| \text{ is odd,} \\ (G_{k+1}')^\epsilon & \text{if } |G_{k+1}'/G_k'| \text{ is odd,} \\ (G_{k+1}')^2 & \text{otherwise.} \end{cases} \tag{3.12}$$

Proof. For each $s \in S_k$, let $\gamma_s = [a,s]$. Also, let

$$T = \{\, \gamma_s \mid s \in S_k \,\} \subseteq G_{k+1}' \subseteq N.$$

Let

$$\widehat{\epsilon} = \begin{cases} 1 & \text{if there exist } t, t' \in T \text{, such that } |t| \text{ is even and } |t'| \text{ is odd,} \\ \epsilon & \text{if } |G'_{k+1}/G'_k| \text{ is odd,} \\ 2 & \text{otherwise.} \end{cases}$$

Lemma 3.6 allows us to assume $(G'_{k+1})^{\widehat{\epsilon}} = N$, by modding out $\big((G'_{k+1})^{\widehat{\epsilon}}\big)^{\perp}$. We can also assume $(G'_k)^{\epsilon}$ is trivial, by modding it out.

We claim that $\langle T \rangle = N$. We have

$$\begin{aligned} \big\langle T, G'_k \big\rangle &\supseteq \big\langle \{\, [\sigma_{k+1}, s] \mid s \in S_k \,\}, \{\, [s,t] \mid s,t \in S_k \,\} \big\rangle \\ &= \big\langle \{\, [s,t] \mid s,t \in S_{k+1} \,\} \big\rangle \\ &= G'_{k+1} && \text{(see Lemma 3.5)} \\ &= N. \end{aligned}$$

Since $(G'_k)^{\epsilon}$ is trivial, this implies $|N : \langle T \rangle| \le \epsilon$. Thus, if the claim is not true, must have $|N : \langle T \rangle| = \epsilon = 2$. In particular, $|N|$ is even. Since $(G'_{k+1})^{\widehat{\epsilon}} = N$, we conclude that $\widehat{\epsilon} = 1$, so the definition of $\widehat{\epsilon}$ (together with the fact that $\epsilon = 2$) implies that T contains an element of even order. So $|\langle T \rangle|$ is even, which contradicts the fact that $|N : \langle T \rangle| = 2$ is even (and $|N|$ is square-free). This completes the proof of the claim.

We also claim that $\mathrm{Cay}(N; T)$ is not bipartite. We may assume $|N|$ is even (for otherwise the claim is obvious). Since $(G'_{k+1})^{\widehat{\epsilon}} = N$, this implies $\widehat{\epsilon} = 1$. However, the claim is also obviously true if there exist $t, t' \in T$, such that $|t|$ is even and $|t'|$ is odd. Hence, we may assume $|G'_{k+1}/G'_k|$ is odd, and $\epsilon = \widehat{\epsilon} = 1$. Since $(G'_k)^{\epsilon}$ is trivial, this implies $|G'_k|$ is odd, which contradicts the fact that $|N|$ is even. This completes the proof of the claim.

The desired conclusion is now immediate from Lemma 3.9 (since $\gamma_s = [a, s]$ and $(\gamma_s)^{-1} = [a, s^{-1}]$). □

3D Results from [9] and [16]

The following result from [16] allows us to assume G is not a 3-group. (Since we always assume that G' is cyclic, a short proof of the special case we need can be found in [15, Thm. 6.1].)

Theorem 3.13 (Witte [16])**.** If $|G|$ is a power of some prime p, then every connected Cayley graph on G has a hamiltonian cycle.

The following simple observation usually allows us to assume $|N|$ is square-free.

Lemma 3.14 ([9, Lem. 3.2])**.** Let $\underline{G} = G/\Phi(N)$, where $\Phi(N)$ is the Frattini subgroup of N [8, §10.4]. Then:

1. $|\underline{N}|$ is square-free, and
2. if there is a hamiltonian cycle in $\mathrm{Cay}(\underline{G}/\underline{N}; S)$ whose voltage generates $\underline{N}$, then there is a hamiltonian cycle in $\mathrm{Cay}(G/N; S)$ whose voltage generates N.

Lemma 3.15 (Keating-Witte [9, Case 6.1])**.** If $|\overline{G_2}|$ is even, then $\mathrm{Cay}\big(\overline{G_2}; \overline{S_2}\big)$ has a hamiltonian cycle whose voltage is a generator of G'_2.

Proof. For the reader's convenience, we provide a proof. We may assume $|\overline{\sigma_1}|$ is even (by interchanging σ_1 and σ_2 if necessary). For convenience, let $n = |\overline{\sigma_1}|$ and $m = m_2$. Then

$$\big(\sigma_2^{m-1}, (\sigma_1, \sigma_2^{-(m-2)}, \sigma_1, \sigma_2^{m-2})^{(n-2)/2}, \sigma_1, \sigma_2^{-(m-1)}, \sigma_1^{-(n-1)}\big)$$

is a hamiltonian cycle in $\mathrm{Cay}(\overline{G_2}; \overline{S_2})$.

Lemma 3.14 allows us to assume $|G_2'|$ is square-free, which implies G_2' is in the center of G_2 (see remark 3.2). Also, from Lemma 3.4, we know that $|[\sigma_1, \sigma_2]|$ is a divisor of both m and n. Therefore

$$\begin{aligned}(\sigma_1\sigma_2^{-(m-2)}\sigma_1\sigma_2^{m-2})^{(n-2)/2} &= \big(\sigma_1^2\,[\sigma_1, \sigma_2^{m-2}]\big)^{(n-2)/2}\\ &= \sigma_1^{n-2}\,[\sigma_1, \sigma_2]^{(m-2)(n-2)/2} = \sigma_1^{n-2}\,[\sigma_1, \sigma_2]^2\end{aligned}$$

and

$$[\sigma_2^{-(m-1)}, \sigma_1^{-(n-1)}] = [\sigma_2, \sigma_1]^{(m-1)(n-1)} = [\sigma_2, \sigma_1] = [\sigma_1, \sigma_2]^{-1},$$

so the voltage of this cycle is

$$\begin{aligned}&\sigma_2^{m-1}\big(\sigma_1\sigma_2^{-(m-2)}\sigma_1\sigma_2^{m-2}\big)^{(n-2)/2}\sigma_1\sigma_2^{-(m-1)}\sigma_1^{-(n-1)}\\ &\quad= \sigma_2^{m-1}\big(\sigma_1^{n-2}\,[\sigma_1, \sigma_2]^2\big)\sigma_1\sigma_2^{-(m-1)}\sigma_1^{-(n-1)}\\ &\quad= \sigma_2^{m-1}\sigma_1^{n-1}\sigma_2^{-(m-1)}\sigma_1^{-(n-1)}\,[\sigma_1, \sigma_2]^2\\ &\quad= [\sigma_2^{-(m-1)}, \sigma_1^{-(n-1)}]\,[\sigma_1, \sigma_2]^2\\ &\quad= [\sigma_1, \sigma_2],\end{aligned}$$

which generates G_2' (see Lemma 3.5). □

The following result allows us to assume $\ell \geq 3$.

Proposition 3.16 (Keating-Witte [9, §6])**.** If $\ell = 2$ and $N = G'$, then $\mathrm{Cay}\big(G; S\big)$ has a hamiltonian cycle.

Proof. For the reader's convenience, we provide a proof (using the main result of Section 4 below). We may assume $|G/G'|$ is odd, for otherwise a hamiltonian cycle is obtained by combining Lemma 3.15 with the Factor Group Lemma (2.5). We may also assume that $|G|$ is not a power of 3, for otherwise theorem 3.13 applies. This implies it is not the case that $|\overline{s}| = 3$ for every $s \in S$.

If $|G'|$ is square-free, then proposition 4.1 tells us that α_2^2 is true. Also, since $|G/G'|$ is odd, we know $|G'|$ is odd (cf. Lemma 3.4), so α_2^2 implies that α_2^1 is true (see remark 2.7(1)). Therefore, the Factor Group Lemma (2.5) provides a hamiltonian cycle in $\mathrm{Cay}(G; S)$ (see Lemma 2.6). Then Lemma 3.14 tells us there is a hamiltonian cycle even without the assumption that $|G'|$ is square-free. □

4 Base case of the inductive construction

Recall that the condition α_k^ϵ is defined in Section 2.

Proposition 4.1 (cf. [9, Case 6.2])**.** Assume $|N|$ is square-free (and $\ell \geq 2$). Then α_2^2 is true unless $|G_2'| = m_2 = |\overline{\sigma_1}| = |\overline{\sigma_2}| = 3$.

Proof. For convenience, let

$$a = \sigma_1, \ b = \sigma_2, \ \text{and} \ m = m_2,$$

and define r by

$$\overline{b}^m = \overline{a}^r \ \text{and} \ 0 < r \leq |\overline{a}|.$$

We may assume:

- $\ell = 2$, so $S = S_2 = \{a, b\}$ and $G = G_2$.
- $(G')^2$ is nontrivial. (Otherwise, the condition about generating $(G')^2$ is automatically true, so it suffices to show $\mathcal{V}_2 \neq \emptyset$, which is easy.)
- Either $|\overline{a}|$ is even, or m is odd (by interchanging σ_1 and σ_2 if necessary).
- $|\overline{a}| \neq 3$ (by interchanging σ_1 and σ_2 if necessary: if $|\overline{\sigma_1}| = |\overline{\sigma_2}| = 3$, then $m = 3$ and, from Lemma 3.4, we also have $|G'| = 3$, which means we are in a case in which the statement of the proposition does not make any claim).
- $r \geq |\overline{a}|/2$ (by replacing a with its inverse if necessary).

Note that $|G'|$ is a divisor of both $|\overline{a}|$ and m (see Lemma 3.4). Since $(G')^2$ is nontrivial, this implies that $|\overline{a}|$ and m both have at least one odd prime divisor.

Case 1. *Assume* $m = 3$. Since $|G'|$ is a divisor of m, we must have $|G'| = 3$, so $|\overline{a}|$ must be divisible by 3. Then, since $|\overline{a}| \neq 3$, we must have $|\overline{a}| \geq 6$. Furthermore, by applying Lemma 3.4 with a and b interchanged, we see that $|\overline{G}/\langle \overline{b} \rangle|$ is also divisible by $|G'| = 3$, which means that r is divisible by 3.

We claim that it suffices to find two elements $\gamma_1, \gamma_2 \in \mathcal{V}_2$, such that $\gamma_1 \neq \gamma_2$ and $\gamma_1 \in \gamma_2 G'$. To see this, note that, for any $x \in N$, there is some $i \in \{1, 2\}$, such that $\langle \gamma_i x \rangle$ has nontrivial projection to G' (with respect to the unique direct-product decomposition $N = G' \times (G')^{\perp}$). Since $|G'|$ is prime, this implies that the projection is all of G', so Lemma 3.6 tells us that $\langle \gamma_i x \rangle$ contains G'. This establishes α_2^1, which is equivalent to α_2^2 (see remark 2.7(1)). This completes the proof of the claim.

Assume, for the moment, that $r = 3$. Then, since $r \geq |\overline{a}|/2$ and $|\overline{a}| \geq 6$, we must have $|\overline{a}| = 6$. Here are two hamiltonian cycles in $\mathrm{Cay}(\overline{G}; \overline{a}, \overline{b})$ that cover $S^{\pm 1}$:

$$(b^{-1}, a^{-2}, b^{-4}, a^{-2}, b^{-1}, a^3, b^2, a, b^{-2})$$

and

$$(b^{-1}, a^{-2}, b^{-1}, a, b^{-1}, a^{-1}, b^{-2}, a^{-1}, b^{-1}, a^2, b^2, a, b^{-2})$$

(see Figure 1). By using Lemma 3.3, we see that their voltages are

$$b^{-1}a^{-2}b^{-4}a^{-2}b^{-1}a^3b^2ab^{-2} = b^{-6}[a, b]^{-(-10)} = b^{-6}[a, b]$$

and

$$b^{-1}a^{-2}b^{-1}ab^{-1}a^{-1}b^{-2}a^{-1}b^{-1}a^2b^2ab^{-2} = b^{-6}[a, b]^{-(-8)} = b^{-6}[a, b]^2,$$

respectively. So we may let $\gamma_1 = b^{-6}[a, b]$ and $\gamma_2 = b^{-6}[a, b]^2$.

We may now assume $r \geq 6$ (since r is divisible by 3). Let

$$I = \begin{cases} \{0, 1\} & \text{if } r \neq |\overline{a}|, \\ \{1, 2\} & \text{if } r = |\overline{a}|. \end{cases}$$

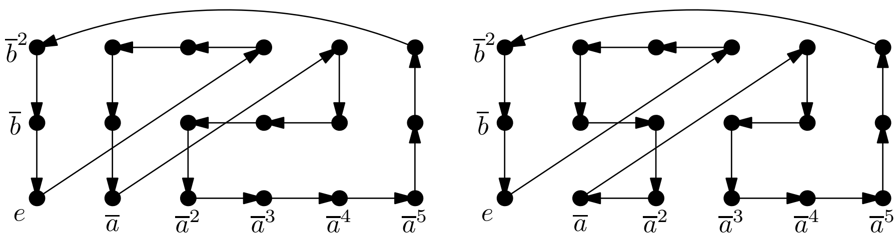

Figure 1: Two hamiltonian cycles in $\mathrm{Cay}\big(\overline{G}; \{\overline{a}, \overline{b}\}\big)$ when $m = r = 3$.

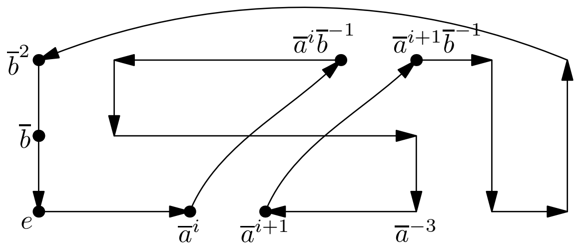

Figure 2: A hamiltonian cycle in $\mathrm{Cay}\big(\overline{G}; \{\overline{a}, \overline{b}\}\big)$ when $m = 3$ and $r \geq 6$.

Then, for $i \in I$, we have $0 \leq i \leq |\overline{a}| - 4$, and $4 \leq r - i \leq |\overline{a}| - 1$, so the walk

$$C_i = \big(a^i, b^{-1}, a^{-(|\overline{a}|-r+i-1)}, b^{-1}, a^{|\overline{a}|-4}, b^{-1}, a^{-(|\overline{a}|-i-4)}, b^{-1}, a^{r-i-3}, b^{-2}, a, b^2, a, b^{-2}\big)$$

is as pictured in Figure 2. It is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{a}, \overline{b})$ that covers $S^{\pm 1}$. Furthermore, since

$$a^i b^{-1} a^{-i} = b^{-1}(b a^i b^{-1} a^{-i}) = b^{-1}[b^{-1}, a^{-i}] = b^{-1}[b, a]^i = b^{-1}[a, b]^{-i},$$

its voltage is of the form $h_2\,[a, b]^{-2i}$, where h_2 is independent of i. Thus, we may let

$$\{\gamma_1, \gamma_2\} = \{\, h_2\,[a, b]^{-2i} \mid i \in I \,\}.$$

Case 2. *Assume* $m \neq 3$. (Cf. [9, Case 4.3].) Since m and $|\overline{a}|$ both have at least one odd prime divisor, we must have $m \geq 5$ and $|\overline{a}| \geq 5$. Let

$$X = \begin{cases} \big(b^{-(m-2)}, a, b^{m-3}, a^{|\overline{a}|-3}, b^{-1}, (a^{-(|\overline{a}|-4)}, b^{-1}, a^{|\overline{a}|-4}, b^{-1})^{(m-3)/2}\big) & \text{if } |\overline{a}| \text{ is odd,} \\ \big(b^{-1}, (b^{-(m-3)}, a, b^{m-3}, a)^{(|\overline{a}|/2)-1}, b^{-(m-2)}\big) & \text{if } |\overline{a}| \text{ is even.} \end{cases}$$

For each i with $1 \leq i \leq \lfloor (|\overline{a}| - 1)/2 \rfloor$, we have $1 \leq i \leq \min\big(r - 1, |\overline{a}| - 3\big)$ (since $r \geq |\overline{a}|/2$ and $|\overline{a}| \geq 5$), so we may let

$$C_i = \big(a^i, b^{-1}, a^{-(|\overline{a}|+i-r-1)}, X, a^{-(|\overline{a}|-i-2)}, b^{-1}, a^{r-i-1}, b^{-(m-1)}\big)$$

(see Figures 3 and 4). Then C_i is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{a}, \overline{b})$.

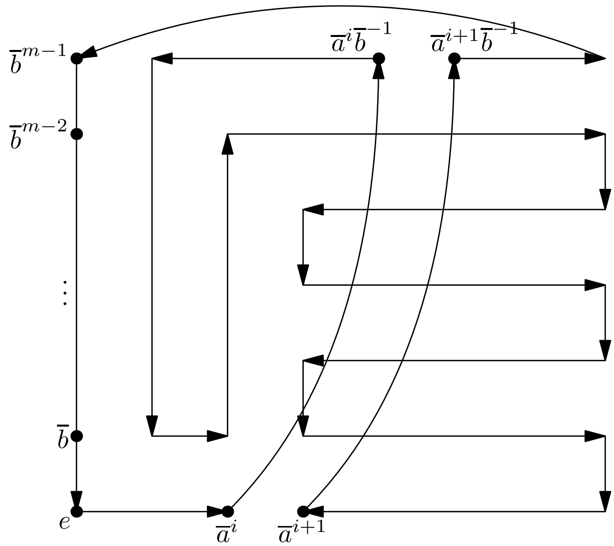

Figure 3: A hamiltonian cycle C_i in $\mathrm{Cay}\big(\overline{G}; \{\overline{a}, \overline{b}\}\big)$ when $m = |\overline{G}/\langle\overline{a}\rangle|$ is odd.

Note that both possibilities for X contain oriented edges labelled a, b, and b^{-1}. Furthermore, since $|\overline{a}| - i - 2 \geq 1$, we see that C_i also contains at least one oriented edge labelled a^{-1}. Therefore C_i covers $\{a, b, a^{-1}, b^{-1}\} = S^{\pm 1}$.

As in Case 1, the voltage ΠC_i of C_i is of the form $h_2\,[a, b]^{-2i}$, where h_2 is independent of i. Since $|\overline{a}| \geq |G'|$ (see Lemma 3.4) and $\langle [a, b] \rangle = G'$ (see Lemma 3.5), Lemma 3.8 (combined with Lemma 3.6) tells us that for any $x \in N$, we may choose i so that $\langle (\Pi C_i)x \rangle$ contains $(G')^2$. □

5 The main induction step

The induction step of our proof uses the following well-known gluing technique that is illustrated in Figure 5.

Definition 5.1. Let

- C_1 and C_2 be two vertex-disjoint oriented cycles in $\mathrm{Cay}(\overline{G}; \overline{S})$,
- $g \in G$, and
- $a, s \in S$.

If

- C_1 contains the oriented edge $\overline{g}(s)$, and
- C_2 contains the oriented edge $\overline{gsa}(s^{-1})$,

then we use $C_1 \#_s^a C_2$ to denote the oriented cycle obtained from $C_1 \cup C_2$ as in Figure 5, by

- removing the oriented edges $\overline{g}(s)$ and $\overline{gsa}(s^{-1})$, and

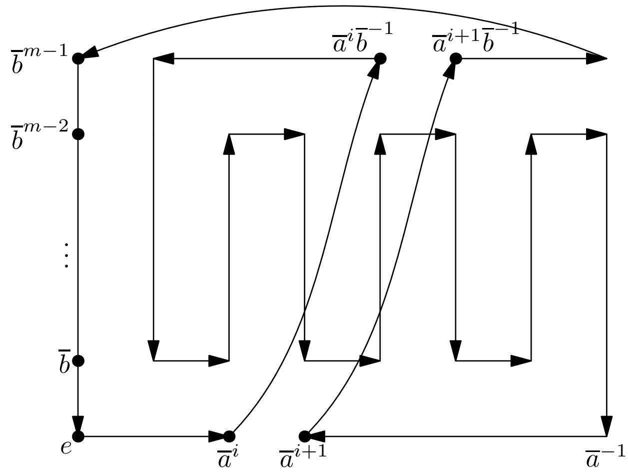

Figure 4: A hamiltonian cycle C_i in $\mathrm{Cay}\big(\overline{G}; \{\overline{a}, \overline{b}\}\big)$ when $|\overline{a}|$ is even.

- inserting the oriented edges $\overline{g}(a)$ and $\overline{gsa}(a^{-1})$.

This is called the *connected sum* of C_1 and C_2.

Lemma 5.2. If C_1, C_2, g, s, and a are as in definition 5.1, and $N \subset Z(G)$, then

$$\Pi(C_1 \#_s^a C_2) = (\Pi C_1)(\Pi C_2)[a, s].$$

Proof. Write $C_1 = \overline{gs}(s_i)_{i=1}^m$ and $C_2 = \overline{ga}(t_j)_{j=1}^n$. Then

$$C_1 \#_s^a C_2 = \overline{gsa}\big(a^{-1}, (s_i)_{i=1}^{m-1}, a, (t_j)_{j=1}^{n-1}\big),$$

so

$$\begin{aligned}
\Pi(C_1 \#_s^a C_2) &= a^{-1} \left(\prod_{i=1}^{m-1} s_i\right) a \left(\prod_{j=1}^{n-1} t_j\right) \\
&= a^{-1} \left(\prod_{i=1}^{m} s_i\right) s_m^{-1}\, a \left(\prod_{j=1}^{n} t_j\right) t_n^{-1} \\
&= a^{-1}\ (\Pi C_1)\, s^{-1}\, a\ (\Pi C_2)\, s \\
&= (\Pi C_1)\, (\Pi C_2)\ a^{-1} s^{-1} a s \qquad\qquad (\Pi C_i \in N \subset Z(G)) \\
&= (\Pi C_1)\, (\Pi C_2)\ [a, s].
\end{aligned}$$

□

Corollary 5.3. Assume

- $2 \leq k < \ell$, and (to eliminate some subscripts) $m = m_{k+1}$ and $a = \sigma_{k+1}$,
- $\pi_1, \pi_2, \ldots, \pi_m$ are elements of $\mathcal{V}_k$,

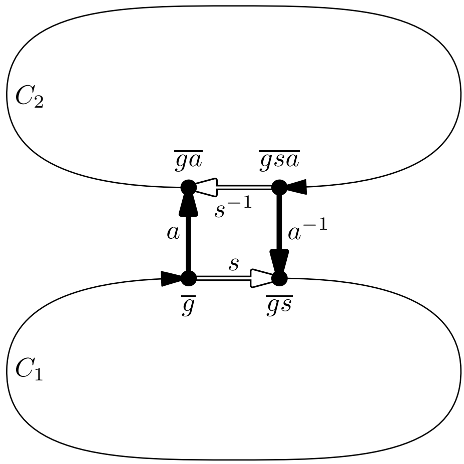

Figure 5: C_1 and C_2 are merged into a single cycle by replacing the two white edges labelled s and s^{-1} with the two black edges labelled a and a^{-1}.

- $s_1, s_2, \ldots, s_{m-1}$ are elements of S_k, and, for each i, a choice $s_i^* \in \{s_i^{\pm 1}\}$ has been made in such a way that if $s_{i+1} = s_i$, then $s_{i+1}^* = s_i^*$, and
- $N \subset Z(G)$.

Then there is a hamiltonian cycle in $\mathrm{Cay}(\overline{G_{k+1}}; \overline{S_{k+1}})$ that covers $S_{k+1}^{\pm 1}$, and whose voltage is

$$\left(\prod_{i=1}^{m} \pi_i\right)\left(\prod_{i=1}^{m-1} [a, s_i^*]\right).$$

Proof. For each i, let C_i be an oriented hamiltonian cycle in $\mathrm{Cay}(\overline{G_k}; \overline{S_k})$ that covers $S_k^{\pm 1}$, and has voltage π_i. We inductively construct sequences $(g_i)_{i=1}^m$ and $(x_i)_{i=1}^m$ of elements of G_k, as follows.

Let $g_1 = e$. Since C_1 covers $S_k^{\pm 1}$, we know there is some $x_1 \in G_k$, such that ag_1C_1 contains the oriented edge $\overline{ax_1}(s_1^*)$.

Now, suppose $g_1, x_1, g_2, x_2, \ldots, g_i, x_i \in G_k$ are given, such that the connected sum

$$ag_1C_1 \#_{s_1^*}^a a^2 g_2 C_2 \#_{s_2^*}^a \cdots \#_{s_{i-1}^*}^a a^i g_i C_i$$

exists, and contains the oriented edge $\overline{a^i x_i}(s_i^*)$. Since C_{i+1} covers $S_k^{\pm 1}$, we know that C_{i+1} contains an oriented edge labelled $(s_i^*)^{-1}$, and a different oriented edge that is labelled s_{i+1}^*. Therefore, there exist $g_{i+1}, x_{i+1} \in G_k$, such that

$$a^{i+1} g_{i+1} C_{i+1} \text{ contains the oriented edges } \overline{a^{i+1} x_i s_i^*}\big((s_i^*)^{-1}\big) \text{ and } \overline{a^{i+1} x_{i+1}}(s_{i+1}^*).$$

The first of these edges is removed when we form the connected sum

$$\big(ag_1C_1 \#_{s_1^*}^a a^2 g_2 C_2 \#_{s_2^*}^a \cdots \#_{s_{i-1}^*}^a a^i g_i C_i\big) \;\#_{s_i^*}^a\; a^{i+1} g_{i+1} C_{i+1},$$

but the second edge remains, and will be used to form the next connected sum (unless $i+1=m$).

Since each C_i is a hamiltonian cycle in $\mathrm{Cay}(\overline{G_k};\overline{S_k})$, the resulting connected sum

$$ag_1C_1 \#^a_{s_1^*} a^2g_2C_2 \#^a_{s_2^*} \cdots \#^a_{s_{m-1}^*} a^m g_m C_m$$

passes through all of the vertices in $\overline{aG_k} \cup \overline{a^2G_k} \cup \cdots \cup \overline{a^mG_k}$. That is, it passes through every element of $\overline{G_{k+1}}$, so it is a hamiltonian cycle in $\mathrm{Cay}(\overline{G_{k+1}};\overline{S_{k+1}})$. Its voltage is calculated by repeated application of Lemma 5.2.

To complete the proof, we verify that the hamiltonian cycle covers $S_{k+1}^{\pm 1}$. Since each C_i covers $S_k^{\pm 1}$, the disjoint union

$$ag_1C_1 \cup a^2g_2C_2 \cup \cdots \cup a^m g_m C_m$$

contains (at least) m disjoint pairs of edges labelled s and s^{-1}, for each $s \in S_k$. Each invocation of the connected sum removes only one such pair, and the operation is performed only $m-1$ times, so at least one of the m pairs must remain, for each $s \in S_k$. Therefore, the hamiltonian cycle covers $S_k^{\pm 1}$. Also, the cycle certainly covers $a^{\pm 1}$, since each invocation of the connected sum inserts a pair of edges labelled a and a^{-1}. Hence, the hamiltonian cycle covers $S_k^{\pm 1} \cup \{a^{\pm 1}\} = S_{k+1}^{\pm 1}$. □

We can now prove the main result of this section. (Recall that the condition α_k^ϵ is defined in Section 2.)

Proposition 5.4. Assume $|N|$ is square-free and $|G'_{k+1}/G'_k|$ is odd. Then

1. $\alpha_k^1 \Rightarrow \alpha_{k+1}^1$, and
2. $\alpha_k^{2+} \Rightarrow \alpha_{k+1}^{2+}$ if $|[s,t]|$ is even for all $s,t \in S_{k+1}$ with $s \neq t$.

Proof. For convenience, let $m = m_{k+1}$ and $a = \overline{\sigma_{k+1}}$. Let $h_k \in N$ be as in (α_k^ϵ), and choose an oriented hamiltonian cycle C in $\mathrm{Cay}(\overline{G_k};\overline{S_k})$ that covers $S_k^{\pm 1}$, and has its voltage in $h_k(G'_k)^\epsilon$. There is no harm in assuming that the voltage is precisely h_k. Let

$$h_{k+1} = (h_k)^m [a,\sigma_1]^{m-1}.$$

Given any $x \in N$, corollary 3.11 provides a sequence $(s_i)_{i=1}^{m-1}$ of elements of S_k, and $s_i^* \in \{s_i^{\pm 1}\}$ for each i, such that $s_{i+1}^* = s_i^*$ whenever $s_{i+1} = s_i$, and

$$\left\langle x\,(h_k)^m \prod_{i=1}^{m-1} [a, s_i^*],\ (G'_k)^\epsilon \right\rangle \text{ contains } (G'_{k+1})^\epsilon. \tag{5.5}$$

From (α_k^ϵ) we know there exists $\pi \in \mathcal{V}_k \cap h_k\,(G'_k)^\epsilon$, such that, if we let

$$\gamma = \pi\,(h_k)^{m-1} \prod_{i=1}^{m-1} [a, s_i^*],$$

then $\langle x\gamma\rangle$ contains $(G'_k)^\epsilon$. Since $\pi \equiv h_k \pmod{(G'_k)^\epsilon}$, combining this with (5.5) shows that $\langle x\gamma\rangle$ contains $(G'_{k+1})^\epsilon$. Also, since we are assuming $|[a,s_i^*]|$ is even if $\epsilon = 2$, we have $[a,s_i^*] \equiv [a,\sigma_1] \pmod{(G'_{k+1})^\epsilon}$ for all i, so

$$\gamma \in (h_k)^m [a,\sigma_1]^{m-1} (G'_{k+1})^\epsilon = h_{k+1}(G'_{k+1})^\epsilon.$$

Furthermore, corollary 5.3 tells us that there is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$ whose voltage is γ, and this hamiltonian cycle covers $S^{\pm 1}$. This establishes α_{k+1}^{ϵ}.

Now, if $\epsilon = 2$, then our assumptions imply that $|h_k|$ and $|[a, \sigma_1]|$ are both even. Since m and $m-1$ are of opposite parity, this implies that $|h_{k+1}|$ is even, so $\langle h_k, (G'_{k+1})^2 \rangle$ contains G'_{k+1}. This establishes α_{k+1}^{2+}. □

6 Combining the base case with the induction step

Recall that the condition α_k^{ϵ} is defined in Section 2.

Corollary 6.1. Assume $|N|$ is square-free and $\ell \geq 2$. If $|G'|$ is odd, then α_ℓ^1 is true unless $|G'| = |\overline{s}| = 3$ for all $s \in S$.

Proof. Assume it is not the case that $|G'| = |\overline{s}| = 3$ for all $s \in S$. Then we may assume (by permuting the elements of S) that either $|G'_2| \neq 3$ or $|\sigma_1| \neq 3$. Therefore proposition 4.1 tells us that α_2^2 is true. Also, since $|G'|$ is odd, we have $\alpha_2^2 \Leftrightarrow \alpha_2^1$ (see remark 2.7(1)), so α_2^1 is true. Then repeated application of proposition 5.4(1) establishes α_ℓ^1. □

Proposition 6.2. Assume $|N|$ is square-free and $\ell \geq 3$. If $|G'|$ is even, then:

1. α_ℓ^1 is true if there exist $s, t \in S$, such that $|[s,t]|$ is odd and $s \neq t$.
2. α_ℓ^{2+} is true if $|[s,t]|$ is even for all $s, t \in S$ with $s \neq t$.

Proof. Since $|G'|$ is even, we may assume (by permuting the elements of S) that $|[\sigma_3, \sigma_1]|$ is even. It suffices to prove α_3^1 or α_3^{2+} (as appropriate), for then repeated application of proposition 5.4 establishes the desired conclusion. Thus, we may assume $\ell = 3$, so $G_3 = G$. Let $m = m_3$ and $a = \sigma_3 = \sigma_\ell$.

By permuting the elements of S, we may assume that either:

odd case: $|[\sigma_3, \sigma_2]|$ is odd, or

even case: $|[s,t]|$ is even for all $s, t \in S$ with $s \neq t$.

Furthermore, in the even case, we may assume that either:

even subcase: $\langle \overline{\sigma_1}, \overline{\sigma_2} \rangle$ has even index in $\overline{G}$, or

odd subcase: $\langle \overline{s}, \overline{t} \rangle$ has odd index in $\overline{G}$, for all $s, t \in S$, such that $s \neq t$.

Since $|[\sigma_3, \sigma_1]|$ is even, we know $|\overline{\sigma_1}|$ is even (see Lemma 3.4). In particular, we have $|\overline{\sigma_1}| \neq 3$, so proposition 4.1 tells us that α_2^2 is true.

We now use a slight modification of the proof of proposition 5.4. Let $h_2 \in N$ be as in (α_k^{ϵ}) (with $k = \epsilon = 2$), and choose an oriented hamiltonian cycle C in $\mathrm{Cay}(\overline{G_2}; \overline{S_2})$ that covers $S_2^{\pm 1}$, and has its voltage in $h_2(G'_2)^2$. There is no harm in assuming that the voltage is precisely h_2.

Since $|\overline{\sigma_1}|$ is even, we know $|\overline{G_2}|$ is even. Therefore, Lemma 3.15 provides a hamiltonian cycle C' in $\mathrm{Cay}(\overline{G_2}; \overline{S_2})$, such that $\Pi C'$ is a generator of G'_2. Let

$$h' = \begin{cases} \Pi C' & \text{in the odd subcase of the even case,} \\ h_2 & \text{in all other cases.} \end{cases}$$

Let $h_3 = (h_2)^{m-1} h' [a, \sigma_1]^{m-1}$.

Given any $x \in N$, corollary 3.11 provides a sequence $(s_i)_{i=1}^{m-1}$ of elements of S_2, and $s_i^* \in \{s_i^{\pm 1}\}$ for each i, such that $s_{i+1}^* = s_i^*$ whenever $s_{i+1} = s_i$, and

$$\left\langle x(h_2)^{m-1} h' \prod_{i=1}^{m-1} [a, s_i^*],\ (G_2')^2 \right\rangle \text{ contains } (G')^2. \tag{6.3}$$

Furthermore, in the odd case, the choices can be made so that (6.3) holds with G' in the place of $(G')^2$.

From α_2^2, we know there exists $\pi \in \mathcal{V}_2 \cap h_2\,(G_2')^2$, such that, if we let

$$\gamma = \pi\,(h_2)^{m-2} h' \prod_{i=1}^{m-1} [a, s_i^*],$$

then

$$\langle x\gamma \rangle \text{ contains } (G')^2. \tag{6.4}$$

It is clear from the definitions that $\gamma \in h_3 G_3'$. Furthermore, we have $\gamma \in h_3 (G_3')^2$ in the even case.

corollary 5.3 tells us that there is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$ whose voltage is γ, and this hamiltonian cycle covers $S^{\pm 1}$. We now consider various cases individually.

Case 1. *The odd case.* Recall that, in this case, (6.3) holds with G' in the place of $(G')^2$. Since $\pi \equiv h_2 \pmod{(G_2')^2}$, combining this with (6.4) shows that $\langle x\gamma \rangle$ contains all of G'. This establishes α_3^1.

Case 2. *The even subcase of the even case.* In this subcase, we know m is even, $h' = h_2$, and $|[a, \sigma_1]|$ is even. Since $h_3 = (h_2)^m [a, \sigma_1]^{m-1}$, we see that $|h_3|$ is even, so $\langle h_3, (G_3')^2 \rangle$ contains G_3'. This establishes α_3^{2+}.

Case 3. *The odd subcase of the even case.* In this subcase, we know $m - 1$ is even, and $h' = \Pi C'$ has even order. Therefore $|h_3|$ is even, so $\langle h_3, (G_3')^2 \rangle$ contains G_3'. This establishes α_3^{2+}. □

We can now establish our main theorem:

Proof of theorem 1.2. We may assume:

- $\ell \geq 3$, for otherwise proposition 3.16 applies.
- $|G|$ is not a power of 3, for otherwise theorem 3.13 applies.

Let S be a minimal generating set of G, and let $N = G'$. Note that $\overline{S}$ is a minimal generating set of $\overline{G}$ (because G' is contained in the Frattini subgroup $\Phi(G)$ [8, Thm. 10.4.3]).

We claim there is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$ whose voltage generates G'. While proving this, there is no harm in assuming that $|G'|$ is square-free (see Lemma 3.14). Also note that, since $|G|$ is not a power of 3, we cannot have $|G'| = |\overline{s}| = 3$ for all $s \in S$. Then, by applying either corollary 6.1 or proposition 6.2 (depending on the parity of $|G'|$), we obtain either α_ℓ^1 or α_ℓ^{2+}. Each of these yields the desired hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$ (see Lemma 2.6).

Now that the claim has been verified, the Factor Group Lemma (2.5) provides a hamiltonian cycle in $\mathrm{Cay}(G; S)$. □

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Poly-antimatroid polyhedra

Yulia Kempner
Holon Institute of Technology, Israel

Vadim E. Levit
Ariel University, Israel

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Abstract

The notion of "antimatroid with repetition" was conceived by Bjorner, Lovasz and Shor in 1991 as an extension of the notion of antimatroid in the framework of non-simple languages. Further they were investigated by the name of "poly-antimatroids" (Nakamura, 2005, Kempner & Levit, 2007), where the set system approach was used. If the underlying set of a poly-antimatroid consists of n elements, then the poly-antimatroid may be represented as a subset of the integer lattice $\mathbb{Z}^n$. We concentrate on geometrical properties of two-dimensional ($n = 2$) poly-antimatroids - poly-antimatroid polygons, and prove that these polygons are parallelogram polyominoes. We also show that each two-dimensional poly-antimatroid is a poset poly-antimatroid, i.e., it is closed under intersection.

The convex dimension $cdim(S)$ of a poly-antimatroid S is the minimum number of maximal chains needed to realize S. While the convex dimension of an n-dimensional poly-antimatroid may be arbitrarily large, we prove that the convex dimension of an n-dimensional poset poly-antimatroid is equal to n.

Keywords: Antimatroid, polyhedron, convex dimension, lattice animal, polyomino.

Math. Subj. Class.: 05B35

1 Preliminaries

An antimatroid is an accessible set system closed under union [3]. An algorithmic characterization of antimatroids based on the language definition was introduced in [5]. Another algorithmic characterization of antimatroids that depicted them as set systems was developed in [14]. Dilworth (1940) was the first to study antimatroids, using another axiomatization based on lattice theory, and they have been frequently rediscovered in other contexts. The most updated survey on the subject may be found in [19].

E-mail addresses: yuliak@hit.ac.il (Yulia Kempner), levitv@ariel.ac.il (Vadim E. Levit)

Antimatroids can be viewed as a special case of either greedoids or semimodular lattices, and as a generalization of partial orders and distributive lattices. While classical examples of antimatroids connect them with posets, chordal graphs, convex geometries, etc., game theory gives a framework in which antimatroids are interpreted as permission structures for coalitions [1]. There are also rich connections between antimatroids and cluster analysis [16]. In mathematical psychology, antimatroids are used to describe feasible states of knowledge of a human learner [9].

A poly-antimatroid [21] is a generalization of the notion of the antimatroid to multisets. If the underlying set of a poly-antimatroid consists of n elements, then the n-dimensional poly-antimatroid may be represented as a subset of the integer lattice $\mathbb{Z}^n$.

In this paper we investigate the correspondence between poly-antimatroids and polyominoes. In the digital plane $\mathbb{Z}^2$, *a polyomino* [12] is a finite connected union of unit squares without cut points. If we replace each unit square of a polyomino by a vertex at its center, we obtain an equivalent object named *a lattice animal* [13]. Further, we use the name polyomino for the two equivalent objects.

A polyomino is called column-convex (row-convex) if all its columns (rows) are connected. In other words, each column/row has no holes. A convex polyomino is both row-convex and column-convex. The parallelogram polyominoes [6], sometimes known as staircase polygons [4, 13, 22], are a particular case of this family. In the staircase polygon each element may be reached from the source (lowest left) point by a path made only of north and east unit steps, and similarly this element may be reached from the target (highest right) point by a path made only of south and west unit steps. Hence staircase polygons are defined by a pair of monotone north-east paths that have common ending points.

We prove that a representation of a two-dimensional poly-antimatroid on the plane is a staircase polygon.

Let E be a finite set. A *set system* over E is a pair $(E, \mathcal{F})$, where $\mathcal{F}$ is a family of sets over E, called *feasible* sets. We will use $X \cup x$ for $X \cup \{x\}$, and $X - x$ for $X - \{x\}$.

Definition 1.1. [18]A finite non-empty set system $(E, \mathcal{F})$ is an *antimatroid* if

$(A1)$ for each non-empty $X \in \mathcal{F}$, there exists $x \in X$ such that $X - x \in \mathcal{F}$

$(A2)$ for all $X, Y \in \mathcal{F}$, and $X \nsubseteq Y$, there exists $x \in X - Y$ such that $Y \cup x \in \mathcal{F}$.

Any set system satisfying $(A1)$ is called *accessible.*

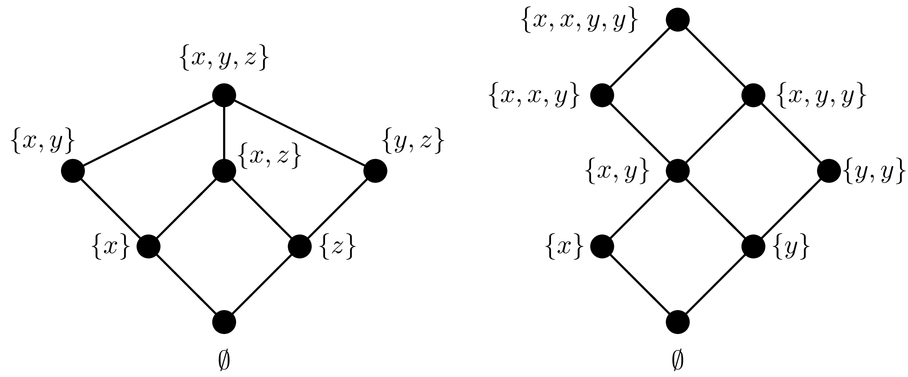
In addition, we use the following characterization of antimatroids.

Proposition 1.2. *[18] For an accessible set system $(E, \mathcal{F})$ the following statements are equivalent:*

(i) $(E, \mathcal{F})$ *is an antimatroid*

(ii) $\mathcal{F}$ *is closed under union ($X, Y \in \mathcal{F} \Rightarrow X \cup Y \in \mathcal{F}$)*

An "antimatroid with repetition" was invented by Bjorner, Lovasz and Shor [2]. Further it was investigated by the name of "poly-antimatroid" as a generalization of the notion of the antimatroid for multisets. A *multiset* A over E is a function $f_A : E \to N$, where $f_A(e)$ is a number of repetitions of an element e in A. *A poly-antimatroid* is a finite non-empty multiset system (E, S) that satisfies the antimatroid properties $(A1)$ and $(A2)$. So antimatroids may be considered as a particular case of poly-antimatroids. Examples of an antimatroid $(\{x, y, z\}, \mathcal{F})$ and a poly-antimatroid $(\{x, y\}, S)$ are illustrated in Figure 1.



(a) (b)

Figure 1: (a) Antimatroid. (b) Poly-antimatroid.

Definition 1.3. A multiset system (E, S) satisfies the *chain property* if for all $X, Y \in \mathcal{F}$, and $X \subset Y$, there exists a chain $X = X_0 \subset X_1 \subset ... \subset X_k = Y$ such that $X_i = X_{i-1} \cup x_i$ and $X_i \in S$ for $0 \leq i \leq k$.

It is easy to see that the chain property follows from $(A2)$, but they are not equivalent. It is clear that each poly-antimatroid satisfies the chain property.

Antimatroids have already been investigated within the framework of lattice theory by Dilworth [7]. The feasible sets of an antimatroid ordered by inclusion form a lattice, with lattice operations: $X \vee Y = X \cup Y$, and $X \wedge Y$ is the maximal feasible subset of set $X \cap Y$ called a *basis*. Since an antimatroid is closed under union, it has only one basis.

A finite lattice L is called *join-distributive* [3] if for any $x \in L$ the interval $[x, y]$ is Boolean, where y is the join of all elements covering x. Such lattices have appeared under several different names, e.g. locally free lattices [18] and upper locally distributive lattices (ULD) [10, 20]. Distributive lattices are exactly those that are both upper and lower locally distributive.

Theorem 1.4. *[3, 18] A finite lattice is join-distributive if and only if it is isomorphic to the lattice of feasible sets of some antimatroid.*

It easy to see that feasible sets of a poly-antimatroid ordered by inclusion form a join-distributive lattice as well.

Consider a particular case of antimatroids called *poset antimatroids* [18]. A poset antimatroid has as its feasible sets the lower sets of a poset (partially ordered set). Poset antimatroids can be characterized as the unique antimatroids, which are closed under intersection [18]. We extend this definition to poly-antimatroids.

Definition 1.5. A poly-antimatroid is called a poset poly-antimatroid if it is closed under intersection.

Evidently, feasible sets of a poset poly-antimatroid ordered by inclusion form a distributive lattice.

2 Two-dimensional poly-antimatroids and polyominoes

In this section we consider a geometric characterization of two-dimensional poly-antimatroids.

Let $E = \{x, y\}$. In this case each point $A = (x_A, y_A)$ in the digital plane $\mathbb{Z}^2$ may be considered as a multiset A over E, where x_A is a number of repetitions of an element x, and y_A is a number of repetitions of an element y in multiset A. Thus a set of points in the digital plane $\mathbb{Z}^2$ that satisfies the properties of an antimatroid is a representation of a two-dimensional poly-antimatroid.

Definition 2.1. A set of points S in the digital plane $\mathbb{Z}^2$ is a poly-antimatroid polygon if
$(A1)$ for every point $(x_A, y_A) \in S$, such that $(x_A, y_A) \neq (0, 0)$,
either $(x_A - 1, y_A) \in S$ or $(x_A, y_A - 1) \in S$
$(A2)$ for all $A \not\subseteq B \in S$,
if $x_A \geq x_B$ and $y_A \geq y_B$ then either $(x_B + 1, y_B) \in S$ or $(x_B, y_B + 1) \in S$
if $x_A \leq x_B$ and $y_A \geq y_B$ then $(x_B, y_B + 1) \in S$
if $x_A \geq x_B$ and $y_A \leq y_B$ then $(x_B + 1, y_B) \in S$

Notice that accessibility implies $\emptyset \in S$.
For example, see a poly-antimatroid polygon in Figure 2.

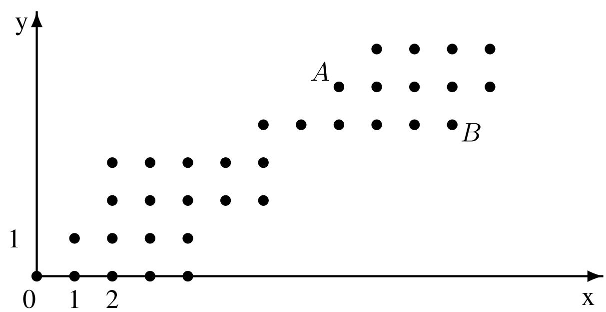


Figure 2: A poly-antimatroid polygon.

We use the following notation [17]. If $A = (x, y)$ is a point in a digital plane, the *4-neighborhood* $N_4(x, y)$ is the set of points

$$N_4(x, y) = \{(x - 1, y), (x, y - 1), (x + 1, y), (x, y + 1)\}$$

and *8-neighborhood* $N_8(x, y)$ is the set of points

$$N_8(x, y) = \{(x - 1, y), (x, y - 1), (x + 1, y), (x, y + 1), (x - 1, y - 1),$$

$$(x - 1, y + 1), (x + 1, y - 1), (x + 1, y + 1)\}.$$

Let m be any of the numbers 4 or 8. A sequence $A_0, A_1, ..., A_n$ is called an N_m*-path* if $A_i \in N_m(A_{i-1})$ for each $i = 1, 2, ...n$. Any two points $A, B \in S$ are said to be N_m-connected in S if there exists an N_m-path $A = A_0, A_1, ..., A_n = B$ from A to B such that $A_i \in S$ for each $i = 1, 2, ...n$. A digital set S is an N_m*-connected* set if any two points

P,Q from S are N_m-connected in S. An N_m*-connected component* of a set S is a maximal subset of S, which is N_m-connected.

An N_m-path $A = A_0, A_1, ..., A_n = B$ from A to B is called *a monotone increasing* N_m-path if $A_i \subset A_{i+1}$ for all $0 \le i < n$, i.e.,

$$(x_{A_i} < x_{A_{i+1}}) \wedge (y_{A_i} \le y_{A_{i+1}}) \text{ or } (x_{A_i} \le x_{A_{i+1}}) \wedge (y_{A_i} < y_{A_{i+1}}).$$

The chain property and the fact that the family of feasible sets of a poly-antimatroid is closed under union mean that for each two points A, B: if $B \subset A$, then there is a monotone increasing N_4-path from B to A, and if A is incomparable with B, then there is a monotone increasing N_4-path from both A and B to $A \cup B = (\max(x_A, x_B), \max(y_A, y_B))$. In particular, for each $A \in S$ there is a monotone decreasing N_4-path from A to 0. So, we can conclude that a poly-antimatroid polygon is an N_4-connected component in the digital plane $\mathbb{Z}^2$.

Definition 2.2. A point set $S \subseteq \mathbb{Z}^2$ is defined to be orthogonally convex if, for every line L that is parallel to the x-axis ($y = y^*$) or to the y-axis ($x = x^*$), the intersection of S with L is empty, a point, or a single interval ($[(x_1, y^*), (x_2, y^*)] = \{(x_1, y^*), (x_1 + 1, y^*), ..., (x_2, y^*)\}$).

It follows immediately from the chain property that every poly-antimatroid polygon S is both connected and orthogonally convex.

In what follows we prove that poly-antimatroid polygons are closed not only under union, but under intersection as well.

Lemma 2.3. *A poly-antimatroid polygon is closed under intersection, i.e., if two points* $A = (x_A, y_A)$ *and* $B = (x_B, y_B)$ *belong to a poly-antimatroid polygon S, then the point* $A \cap B = (min(x_A, x_B), min(y_A, y_B)) \in S$.

Proof. The claim of the lemma is evident for two comparable points. Consider two incomparable points A and B, and assume without loss of generality that $x_A < x_B$ and $y_A > y_B$. Then there is a monotone decreasing N_4-path from A to 0, and so there is a point $C = (x_C, y_B) \in S$ on this path with $x_C \le x_A$. Hence, the point $A \cap B$ belongs to S, since it is located on the monotone increasing N_4-path from C to B. □

Thus, every poly-antimatroid polygon is a distributive lattice polyhedron [11], since $x, y \in S \Rightarrow min(x, y), max(x, y) \in S$.

Consider the following rectangles:

$$R = \{(x, y) \in \mathbb{Z}^2 : x_{\min} \le x \le x_{\max} \wedge y_{\min} \le y \le y_{\max}\} \text{ and } |R| > 1$$

Definition 2.4. The sequence of n rectangles $C_1, C_2, ..., C_n$ is called regular if

(a) $x^0_{\min} = y^0_{\min} = 0$

(b) $x^i_{\min} \le x^{i+1}_{\min} \wedge y^i_{\min} \le y^{i+1}_{\min}$
and for each $1 \le i \le n-1$ at least one of the inequality is strong

(c) $x^{i+1}_{\min} \le x^i_{\max} \wedge y^{i+1}_{\min} \le y^i_{\max}$ for each $1 \le i \le n-1$

(d) $x^i_{\max} \le x^{i+1}_{\max} \wedge y^i_{\max} \le y^{i+1}_{\max}$
and for each $1 \le i \le n-1$ at least one of the inequality is strong

Lemma 2.3 implies that every poly-antimatroid polygon is a union of rectangles built on each pair of incomparable points. The following is even more explicit.

Lemma 2.5. *Every poly-antimatroid polygon is a regular sequence of rectangles.*

Proof. Consider the set of rectangles built on pairs of incomparable points and leave only maximal rectangles, i.e., rectangles that are not covered completely by other rectangles. These rectangles forms a regular sequence. Indeed, the property (a) and (c) follows from the definition of a poly-antimatroid polygon (A1).

Suppose there are two maximal rectangles R_1 and R_2 with two incomparable minimal points $(x^1_{\min}, y^1_{\min})$ and $(x^2_{\min}, y^2_{\min})$. Then, since poly-antimatroid polygons are closed under union and under intersection, these rectangles are covered by two rectangles. The minimal and maximal points of the first rectangle are

$$(\min(x^1_{\min}, x^2_{\min}), \min(y^1_{\min}, y^2_{\min})), (\min(x^1_{\max}, x^2_{\max}), \min(y^1_{\max}, y^2_{\max})).$$

The minimal and maximal points of the second rectangle are

$$(\max(x^1_{\min}, x^2_{\min}), \max(y^1_{\min}, y^2_{\min})), (\max(x^1_{\max}, x^2_{\max}), \max(y^1_{\max}, y^2_{\max}))$$

respectively. There are the cases that these two rectangles are identical. See Figure 3. Thus the rectangles R_1 and R_2 are not maximal and so all minimal points are comparable. The same is true for maximal points as well. □

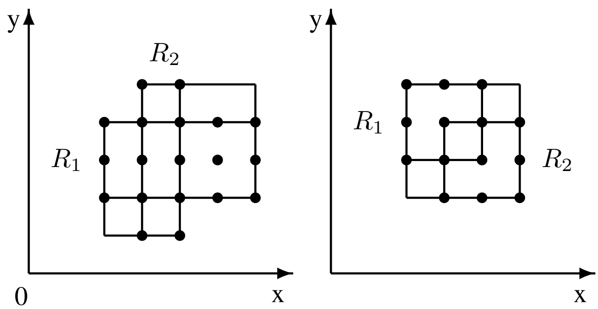


Figure 3: Two examples for proof of Lemma 2.5.

The following theorem shows that a poly-antimatroid polygon is a parallelogram polyomino.

Theorem 2.6. *A set of points S in the digital plane $\mathbb{Z}^2$ is a poly-antimatroid polygon if and only if it is an orthogonally convex N_4-connected set that is bounded by two monotone increasing N_4-paths between $(0, 0)$ and the maximum point of the set (x_{max}, y_{max}).*

To prove the "if" part of Theorem 2.6 it remains to give a definition of the boundary.

A point A in set S is called an *interior point* in S if $N_8(A) \in S$. A point in S which is not an interior point is called a *boundary point*. All boundary points of S constitute the boundary of S. We can see a poly-antimatroid polygon with its boundary in Figure 4.

Since poly-antimatroid polygons are closed under union and under intersection, there are six types of boundary points that we divide into two sets – lower and upper boundary:

$$\mathcal{B}_{lower} = \{(x, y) \in S : (x + 1, y) \notin S \vee (x, y - 1) \notin S \vee (x + 1, y - 1) \notin S\}$$

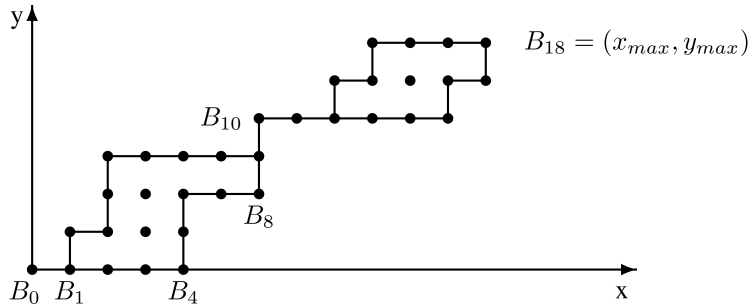


Figure 4: A boundary of a poly-antimatroid polygon.

$$\mathcal{B}_{upper} = \{(x, y) \in S : (x - 1, y) \notin S \vee (x, y + 1) \notin S \vee (x - 1, y + 1) \notin S\}$$

It is possible that $\mathcal{B}_{lower} \cap \mathcal{B}_{upper} \neq \emptyset$. For example, the point B_{10} in Figure 4 belongs to both the lower and upper boundaries.

From Lemma 2.5 it follows immediately that lower and upper boundaries are boundaries of regular sequence of rectangles and so a poly-antimatroid polygon is an orthogonally convex N_4-connected set bounded by two monotone increasing N_4-paths.

The following lemma is the "only-if" part of Theorem 2.6.

Lemma 2.7. *An orthogonally convex N_4-connected set S that is bounded by two monotone increasing N_4-paths between $(0, 0)$ and (x_{max}, y_{max}) is a poly-antimatroid polygon.*

Proof. By Definition 2.1 we have to check the two properties (A1) and (A2):

(A1) Let $A = (x, y) \in S$. If A is an interior point in S then $(x - 1, y) \in S$ and $(x, y-1) \in S$. If A is a boundary point, then the previous point on the boundary ($(x, y-1)$ or $(x - 1, y)$) belongs to S.

(A2) Let $A \nsubseteq B \in S$. Consider two cases:

(i) $x_a \geq x_b$ and $y_a \geq y_b$. If B is an interior point in S then $(x_b + 1, y_b) \in S$ and $(x_b, y_b+1) \in S$. If B is a boundary point, then the next point on the boundary ((x_b, y_b+1) or $(x_b + 1, y_b)$) belongs to S.

(ii)$x_a \leq x_b$ and $y_a \geq y_b$. We have to prove that $(x_b, y_b+1) \in S$. Suppose the opposite. Then the point B is an upper boundary point. Since $y_a \geq y_b$ there exists an upper boundary point (x_a, y) with $y \geq y_b$ that contradicts the monotonicity of the boundary. □

Corollary 2.8. *Any poly-antimatroid polygon S may be represented as the* ***union*** *of its boundary:*

$$S = \mathcal{B}_{lower} \vee \mathcal{B}_{upper} = \{X \cup Y : X \in \mathcal{B}_{lower}, Y \in \mathcal{B}_{upper}\}$$

3 Convex dimension

Definition 3.1. Convex dimension [18] $cdim(S)$ of any antimatroid S is the minimum number of maximal chains

$$\emptyset = X_0 \subset X_1 \subset ... \subset X_k = X_{\max} \text{ with } X_i = X_{i-1} \cup x_i$$

whose union gives the antimatroid S.

The set of maximal chains sufficient to realize a poly-antimatroid is called *a convex realizer*.

The result of Corollary 2.8 shows that the convex dimension of a two-dimensional poly-antimatroid equals two. Note, that the convex dimension of an arbitrary three-dimensional poly-antimatroid may be arbitrarily large [8]. Let S be a set of points:

$$S = \{(x, y, z) : (0 \leq x, y \leq N) \wedge (0 \leq z \leq 1) \wedge (z = 1 \Rightarrow x + y \geq N)\}.$$

It is easy to check that S is a three-dimensional poly-antimatroid. Consider $N + 1$ points $(x, y, 1)$ with $x + y = N$. Since each of these points cannot be represented as a union of any points from S with smaller coordinates, the convex dimension of S is at least $N + 1$.

In the sequel we prove that the convex dimension of an n-dimensional poset poly-antimatroids is at most n.

An *endpoint* of a feasible set X is an element $e \in X$ such that $X - e$ is a feasible set too. A feasible set that has only one endpoint is called a *path* of the antimatroid. It is easy to see that a path is an union-irreducible element of the lattice and each feasible set is the union of its path subsets. The family of ideals in the set of paths partially ordered by set inclusion forms *path poset* antimatroid.

Theorem 3.2. *[18] The convex dimension of antimatroid is equal to the width of its path poset.*

It is easy to check that the theorem holds for poly-antimatroids as well.

Denote by $dim(S)$ the *order dimension* of the lattice of feasible multisets of a poly-antimatroid. The order (or the Dushnik-Miller) dimension of a poset is the smallest number of total orders the intersection of which gives the partial order.

Corollary 3.3. *[18]*

$$dim(S) \leq cdim(S)$$

Since any n-dimensional poly-antimatroid may be represented as a subset of the integer lattice $\mathbb{Z}^n$, its order dimension is at most n [23].

For poset poly-antimatroids their convex dimension should equal to order dimension, since feasible sets of a poset poly-antimatroid form a distributive lattice and order dimension of distributive lattice is equal to the width of its path poset (ideals of the union-irreducible elements)[23]. Eventually, we obtain the following.

Proposition 3.4. *The convex dimension of the n-dimensional poset poly-antimatroids is at most n.*

4 Conclusions

It turned out that a two-dimensional case of a poly-antimatroid known in this paper as a poly-antimatroid polygon is equivalent to special cases of polyominoes, lattice animals, and staircase polygons. It seems to be a challenging problem to generalize the path structure of poly-antimatroid polygons to upper dimensions.

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The $L_2(11)$-subalgebra of the Monster algebra

Sophie Decelle *

Department of Mathematics, Imperial College, 180 Queen's Gate, London, SW7 2AZ, UK

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Abstract

We study a subalgebra V of the Monster algebra, $V_{\mathbb{M}}$, generated by three Majorana axes $a_x,\ a_y$ and a_z indexed by the $2A$-involutions $x,\ y$ and z of $\mathbb{M}$, the Monster simple group. We use the notation $V = \langle\langle a_x, a_y, a_z\rangle\rangle$. We assume that xy is another $2A$-involution and that each of $xz,\ yz$ and xyz has order 5. Thus a subgroup G of $\mathbb{M}$ generated by $\{x,\ y,\ z\}$ is a non-trivial quotient of the group $G^{(5,5,5)} = \langle x,\ y,\ z \mid x^2,\ y^2,\ (xy)^2,\ z^2,\ (xz)^5,\ (yz)^5,\ (xyz)^5\rangle$. It is known that $G^{(5,5,5)}$ is isomorphic to the projective special linear group $L_2(11)$ which is simple, so that G is isomorphic to $L_2(11)$. It was proved by S. Norton that (up to conjugacy) G is the unique $2A$-generated $L_2(11)$-subgroup of $\mathbb{M}$ and that $K = C_{\mathbb{M}}(G)$ is isomorphic to the Mathieu group M_{12}. For any pair $\{t,\ s\}$ of $2A$-involutions, the pair of Majorana axes $\{a_t,\ a_s\}$ generates the dihedral subalgebra $\langle\langle a_t, a_s\rangle\rangle$ of $V_{\mathbb{M}}$, whose structure has been described in [16]. In particular, the subalgebra $\langle\langle a_t, a_s\rangle\rangle$ contains the Majorana axis a_{tst} by the conjugacy property of dihedral subalgebras. Hence from the structure of its dihedral subalgebras, V coincides with the subalgebra of $V_{\mathbb{M}}$ generated by the set of Majorana axes $\{a_t \mid t \in T\}$, indexed by the 55 elements of the unique conjugacy class T of involutions of $G \cong L_2(11)$. We prove that V is 101-dimensional, linearly spanned by the set $\{\ a_t \cdot a_s \mid s,\ t \in T\ \}$, and with $C_{V_{\mathbb{M}}}(K) = V \oplus \iota_{\mathbb{M}}$, where $\iota_{\mathbb{M}}$ is the identity of $V_{\mathbb{M}}$. Lastly we present a recent result of Á. Seress proving that V is equal to the algebra of the unique Majorana representation of $L_2(11)$.

Keywords: Majorana representation, Monster group, Conway-Griess-Norton algebra.

Math. Subj. Class.: 20C99, 20F05, 20C34, 20D05

1 Main result

We let $\big(V_{\mathbb{M}},\ \cdot\ ,(\ ,\ )\big)$ be the Monster algebra, a commutative non-associative algebra of dimension $196,884$ over $\mathbb{R}$, as described in [2]. As an $\mathbb{R}\mathbb{M}$-module, $V_{\mathbb{M}} = V'_{\mathbb{M}} \oplus \mathbb{1}_{\mathbb{M}}$, where $V'_{\mathbb{M}}$ is the minimal faithful irreducible $\mathbb{R}\mathbb{M}$-module of dimension $196,883$ and $\mathbb{1}_{\mathbb{M}}$

*This article was written under the supervision of Prof. Alexander A. Ivanov.
E-mail address: sophie.decelle@imperial.ac.uk (Sophie Decelle)

is the trivial $\mathbb{RM}$-module which is the $\mathbb{R}$-span of the identity $\iota_{\mathbb{M}}$ of the algebra $V_{\mathbb{M}}$. The automorphism group of $\big(V_{\mathbb{M}},\ \cdot\ ,(\ ,\ )\big)$ is $\mathbb{M}$ the Monster simple group ([2], [7]). By $2A$ we denote the conjugacy class of involutions in $\mathbb{M}$ with the largest centraliser as in the Atlas [3]. For each $2A$ involution t of $\mathbb{M}$, the centraliser $C_{\mathbb{M}}(t) \cong 2.BM$ stabilises a 2-subspace W of $V_{\mathbb{M}}$ which has two non-trivial idempotents a_t and $\iota_{\mathbb{M}} - a_t$. In [2], J. Conway constructed an $\mathbb{M}$-invariant bijection ψ sending each $2A$ involution t to the non-trivial idempotent a_t of W with eigenvalue 1 and multiplicity 1 . We denote by $a_t := \psi(t)$ the image of t. In [8] A. A. Ivanov axiomatises some of the properties of the idempotents a_t into the definition of a Majorana axis.

A Majorana axis a of a real commutative non-associative algebra $\big(V,\ \cdot\ ,(\ ,\ )\big)$, where $\cdot$ associates with $(\ ,\ )$ in the sense that $(u \cdot v,\ w) = (u,\ v \cdot w)$ for all $u,\ v,\ w \in V$, is an idempotent of length 1, whose adjoint operator ad_a is semi-simple on V with spectrum $\{1, 0, \frac{1}{2^2}, \frac{1}{2^5}\}$. The eigenspaces of ad_a are denoted by $V_\mu^{(a)}$, with μ an eigenvalue, and satisfy the following conditions. The 1-eigenvectors of ad_a are exactly the scalar multiples of a. There exists a linear transformation $\tau(a)$ of V, called a Majorana involution, negating the $\frac{1}{2^5}$-eigenvectors, fixing the other eigenvectors and preserving both the algebra and inner products. Lastly there exists a linear transformation $\sigma(a)$ of $V_+^{(a)} = V_1^{(a)} \oplus V_0^{(a)} \oplus V_{\frac{1}{2^2}}^{(a)}$ negating the $\frac{1}{2^2}$-eigenvectors, fixing the 0- and 1-eigenvectors, and preserving both products on $V_+^{(a)}$. From [8], this definition is equivalent to the 'Fusion Rules'. For two eigenvectors $u \in V_\lambda^{(a)}$ and $v \in V_\mu^{(a)}$ of a fixed Majorana axis a, the Fusion Rules specify in which part of the spectrum of ad_a the product $u \cdot v$ lies.

Sp	1	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
1	1	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
0	0	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
$\frac{1}{2^2}$	$\frac{1}{2^2}$	$\frac{1}{2^2}$	$1, 0$	$\frac{1}{2^5}$
$\frac{1}{2^5}$	$\frac{1}{2^5}$	$\frac{1}{2^5}$	$\frac{1}{2^5}$	$1, 0, \frac{1}{2^2}$

Table 1: Fusion rules

Definition 1.1. We denote by $\langle\langle A \rangle\rangle$ the subalgebra of $V_{\mathbb{M}}$ generated by a set A of Majorana axes.

The classification of subalgebras $\langle\langle a_t, a_s \rangle\rangle$ of $V_{\mathbb{M}}$, where $\{a_s,\ a_t\}$ is a pair of Majorana axes, was started in [2] and completed in [16]. We call them dihedral subalgebras as the corresponding pair of $2A$-involutions $\{t,\ s\}$ generates a dihedral subgroup of $\mathbb{M}$. We say the dihedral subalgebra has type C if the product of involutions ts belongs to the conjugacy class C of $\mathbb{M}$.

Some subalgebras of $V_{\mathbb{M}}$ generated by triples of Majorana axes are described by A. A. Ivanov et al in [11], [12], [13], [10], and [9].

In this paper, we investigate a subalgebra $V = \langle\langle a_x, a_y, a_z\rangle\rangle$ of $V_{\mathbb{M}}$ such that the dihedral subalgebra $\langle\langle a_x, a_y\rangle\rangle$ has type $2A$ and each of the dihedral subalgebras $\langle\langle a_x, a_z\rangle\rangle$, $\langle\langle a_y, a_z\rangle\rangle$, and $\langle\langle a_{xy}, a_z\rangle\rangle$ has type $5A$. The vector a_{xy} is the Majorana axis $\psi(xy)$ (since a dihedral subalgebra $\langle\langle a_s, a_t\rangle\rangle$ of type $2A$ contains the axis a_{st}).

Keeping in mind the bijection ψ we might ask whether there exists a subgroup of $\mathbb{M}$ generated by a triple of $2A$ involutions $\{x,\ y,\ z\}$ satisfying the relations:

$$x^2 = y^2 = z^2 = (xy)^2 = (xz)^5 = (yz)^5 = (xyz)^5 = 1.$$

A group affording the presentation

$$\langle x,\ y,\ z \mid x^2,\ y^2,\ (xy)^2,\ z^2,\ (xz)^5,\ (yz)^5,\ (xyz)^5\rangle$$

defines the Coxeter group $G^{(5,5,5)}$ and from [4] it is isomorphic to the projective special linear group $L_2(11)$. From classical results on $L_n(p^k)$, [5], $L_2(11)$ is a simple group of order $660 = 2^2 \cdot 3 \cdot 5 \cdot 11$ and it has a single conjugacy class of involutions which we denote by T, and whose size is 55.

Proposition 1.2. *There exists a monomorphism $\iota : L_2(11) \hookrightarrow \mathbb{M}$ such that $\iota(T) \subseteq 2A$ and ι is unique up to conjugacy in $\mathbb{M}$.*

Proof. In Table 5 of [17] S. Norton gives the list of simple subgroups of $\mathbb{M}$ having their elements of order 5 in the $\mathbb{M}$-conjugacy class $5A$. For $\iota(T)$ it is a requirement since if a product of two $2A$ involutions has order 5 it belongs to the conjugacy class $5A$ of $\mathbb{M}$ [2]. By Norton's list there is only one conjugacy class of groups isomorphic to $L_2(11)$ containing $5A$ elements and their involutions belong to class $2A$. □

Throughout the paper ι denotes the monomorphism as in Proposition 1.2, $G \cong L_2(11)$ denotes the image of ι, and T denotes the conjugacy class of involutions in G.

By the conjugacy property of dihedral subalgebras[1], the axis a_{tst} is contained in $\langle\langle a_t, a_s\rangle\rangle$. Hence from the dihedral subalgebras of V, we can restate our aim to be the study of the subalgebra V of $V_{\mathbb{M}}$ generated by the set of 55 Majorana axes $\{a_t \mid t \in T\}$. We determine the dimension of V and find a spanning set for V. In the next section we prove the following theorem.

Theorem. *Let V be the subalgebra of $V_{\mathbb{M}}$ generated by the set of 55 Majorana axes $\{a_t \mid t \in T\}$, where T is the class of involutions of the unique $2A$-generated $L_2(11)$-subgroup G of $\mathbb{M}$. Then*

(1) $dim(V) = 101$,

(2) *V is linearly spanned by the set $\{a_t \cdot a_s \mid t,\ s\ \in T\}$.*

(3) *If $K = C_{\mathbb{M}}(G)$ then $C_{V_{\mathbb{M}}}(K) = V \oplus \iota_{\mathbb{M}}$.*

[1]Let t and s be two $2A$ involutions, then tst is an involution conjugate to s. Hence tst is a $2A$ involution with corresponding Majorana axis $a_{tst} := \psi(tst)$.

In the last section we give some evidence towards the uniqueness of the map $\psi : t \to a_t$, where $t \in T$, within the class of Majorana representations of $L_2(11)$ satisfying conditions $(2A)$ and $(3A)$ (the terminology is explained in the last section)[2]. Lastly we state a recent result of Á. Seress proving that V is equal to the algebra of the unique Majorana representation of $L_2(11)$.

2 Some properties of $L_2(11)$

We present some of the standard properties of $G \cong L_2(11)$ used when calculating inner product values for V.

The group G is the automorphism group of the $(11, 5, 2)$-biplane, which we denote $\mathcal{B}$ (see [19]).
$\mathcal{B}$ is a 2-symmetric design with 11 points, $\{p_1, \ldots, p_{11}\}$, and 11 lines, $\{l_1, \ldots, l_{11}\}$, such that each line contains 5 points, each point lies on 5 lines, two lines intersect in exactly 2 points, and two points share exactly 2 lines. We call the incidence relation $p_i \in l_j$ a flag, which we denote $\alpha_{i,j}$, and the relation $p_i \notin l_j$ an anti-flag, which we denote by $w_{i,j}$.

From [14], the lines of $\mathcal{B}$ can be obtained by finding a difference set l_1 of size 5, with elements from $\mathbb{Z}_{11}$, such that every integer modulo 11 appears exactly twice as a difference $i - j$ mod 11 for i and j in l_1. We have that $l_1 = \{1, 3, 4, 5, 9\}$, which is the set of non-zero perfect squares in $\mathbb{Z}_{11}$, and all other lines l_k can be defined by $l_k = \big\{1 + k,\ 3 + k,\ 4 + k,\ 5 + k,\ 9 + k\big\}$, where $k \in \mathbb{Z}_{11}^*$ and addition is modulo 11.
The incidence matrix $\mathcal{N}$ of $\mathcal{B}$ is given below with the rows indexed by the points of $\mathcal{B}$, the columns indexed by the lines, and each flag is represented by a $'1'$ and each anti-flag by a $'0'$.

1	0	1	1	1	0	0	0	1	0	0
0	1	0	1	1	1	0	0	0	1	0
0	0	1	0	1	1	1	0	0	0	1
1	0	0	1	0	1	1	1	0	0	0
0	1	0	0	1	0	1	1	1	0	0
0	0	1	0	0	1	0	1	1	1	0
0	0	0	1	0	0	1	0	1	1	1
1	0	0	0	1	0	0	1	0	1	1
1	1	0	0	0	1	0	0	1	0	1
1	1	1	0	0	0	1	0	0	1	0
0	1	1	1	0	0	0	1	0	0	1

We can represent G as a permutation group on 11 letters, so that $G \subset Sym(11)$, by letting G act on the indices of the points or lines such that the incidence structure of $\mathcal{B}$ is preserved.

The stabiliser $G(\alpha_{i,j})$ of a flag $\alpha_{i,j}$ is isomorphic to A_4, the stabiliser $G(w_{k,l})$ of an anti-flag $w_{k,l}$ is isomorphic to D_{10}, and the stabiliser of a line (or a point) is isomorphic to A_5. We can associate to a flag $\alpha_{i,j}$ a unique subgroup $S(\alpha_{i,j}) \cong C_2 \times C_2$ and to an anti-flag $w_{k,l}$ a unique subgroup $S(w_{k,l}) \cong C_5$ such that $N_G\big(S(\alpha_{i,j})\big) = G(\alpha_{i,j})$ and $N_G\big(S(w_{k,l})\big) = G(w_{k,l})$. It is easy to see that each involution t stabilises 3 flags and to deduce that $C_G(t) \cong$

[2]When the first draft of this article was written, the author has learned that Ákos Seress has written a *GAP* program, [6], capable of checking this uniqueness conjecture.

D_{12}. Similarly for $\langle h\rangle$ a subgroup of order 3 we can deduce $N_G(\langle h\rangle) \cong D_{12}$. There are only one class of involutions and one class of elements of order 3 in G, so we can let d be the G-invariant bijection between subgroups of order 2 and 3 sending each involution t to the unique subgroup of order 3 commuting with t. Furthermore, by [14], G contains one class of subgroups isomorphic to the Frobenius group of order 55, which we denote F_{55}. These are the four conjugacy classes of maximal subgroups of G; two non-conjugate classes of subgroups isomorphic to A_5 each of size 11 and each stabilising a point or a line, one class of subgroups isomorphic to D_{12}, and one class of subgroups isomorphic to the Frobenius group F_{55}.

3 The algebra *V*

We start this section by finding an upper bound for $dim(V)$ based on the work of S. Norton ([15], [16], and [17]). We then calculate the Gram matrix of a particular subset of V which provides a lower bound for $dim(V)$.

3.1 S. Norton's observations

The upper bound on $dim(V)$ stems from the following inclusion.

Lemma 3.1. $V \subseteq C_{V_{\mathbb{M}}}\big(C_{\mathbb{M}}(G)\big)$

Proof. By the definition of a Majorana axis, a_t is fixed by $C_{\mathbb{M}}(t) \cong 2.BM$. Therefore $C_{\mathbb{M}}(G) = C_{\mathbb{M}}\big(\langle x, y, z\rangle\big) = \bigcap_{t=x,y,z} C_{\mathbb{M}}(t)$ fixes $V = \langle\langle a_x, a_y, a_z\rangle\rangle$ by $\mathbb{M}$-invariance of the algebra $V_{\mathbb{M}}$. □

We denote by K the group $C_{\mathbb{M}}(G)$. The dimension of the fixed space of K in $V_{\mathbb{M}}$ can be obtained by calculating the fusion of the character table of K in that of $\mathbb{M}$ (since the character of $V'_{\mathbb{M}}$ is known [3]). It is equal to the inner product of characters $\langle \chi_{V_{\mathbb{M}}} \downarrow_K, \mathbb{1}_K\rangle_{\mathbb{R}K}$, where $\mathbb{1}_K$ is the trivial character of K, and $\chi_{V_{\mathbb{M}}} \downarrow_K$ is the character of $V_{\mathbb{M}}$ restricted to K. We thus need to determine the isomorphism type of K and the inclusions of the conjugacy classes of G and K into those of $\mathbb{M}$.

We call an A_5-subgroup H of $\mathbb{M}$ an A_5 of type $(2A, 3A, 5A)$ if the elements of order 2, 3 and 5 of H are in the $\mathbb{M}$-conjugacy classes $2A$, $3A$ and $5A$ respectively. Clearly all A_5-subgroups of G are of type $(2A, 3A, 5A)$.

Proposition 3.2. (i) *For K as above, $K \cong M_{12}$.*

(ii) *All A_5-subgroups H as above are conjugate in $\mathbb{M}$ and $C_{\mathbb{M}}(H) \cong A_{12}$.*

(iii) *The conjugacy classes of G fuse into those $\mathbb{M}$ as follows:*

Class in G	$1a$	$2a = T$	$3a$	$5a$	$5b$	$6a$	$11a$	$11b$
Class in $\mathbb{M}$	$1A$	$2A$	$3A$	$5A$	$5A$	$6A$	$11A$	$11A$

Proof. The result from part (i) can be read from the entry 31 of Table 3 of [15]. Part (ii) is proved in Lemma 4 of [15]. To prove (iii) we carry on from the proof of Proposition 1.1. From Table 5 of [17] we deduce the inclusion $3a \subset 3A$. In the character table of $\mathbb{M}$, given in [3], the information on p-powers[3] of elements $g \in 6A$ gives $g^2 \in 3A$ and $g^3 \in 2A$, and $6A$ is the unique conjugacy class of elements of order 6 with those p-powers, hence to avoid a contradiction we must have $6a \subset 6A$. Since $\mathbb{M}$ has a unique class $11A$ of elements of order 11 the classes $11a$ and $11b$ are subsets of $11A$. □

Proposition 3.3. *For the algebra $C_{V_{\mathbb{M}}}(K)$ we have $dim(C_{V_{\mathbb{M}}}(K)) = 102$.*

Within the proof of Proposition 3.3 we determine the fusion of the conjugacy classes of K into those of $\mathbb{M}$. We follow the Atlas's notation, [3], by writing the conjugacy of elements of order N in $\mathbb{M}$: $NA,\ NB, \ldots (etc)$ in increasing order of the size of the class. Similarly for K we use the notation $NA_K,\ NB_K, \cdots (etc)$. The character tables used are those of the Atlas [3].

Proof. By part (i) of the previous proposition we have the inclusion of groups $H \subset G$, where $H \cong A_5$ is of type $(2A, 3A, 5A)$, which implies $K \subset C_{\mathbb{M}}(H) \cong A_{12}$. In A_{12}, the elements with cycle decompositions $2^2 1^8$, $2^4 1^4$, and 2^6 have 8, 4, and no fixed points respectively in the natural action of A_{12} on 12 points, and so by Lemma 6 in [15] they are mapped to the $\mathbb{M}$-conjugacy classes $2A,\ 2B$ and $2A$ respectively. There is a doubly transitive action of M_{12} on 12 points with character $\chi_1 + \chi_{11a}$ where χ_1 is the trivial character of M_{12} and χ_{11a} is the first irreducible character of degree 11 (as in the Atlas, [3]). This character takes the value 0 for the elements in the class $2A_{M_{12}}$, and the value 4 for the elements in the class $2B_{M_{12}}$, hence $2A_{M_{12}} \subset 2A$ and $2B_{M_{12}} \subset 2B$.

The structure class constants[4] for any pair of $2A$ involutions in $\mathbb{M}$ give the number of elements in each conjugacy class of $\mathbb{M}$ expressible as a product of two $2A$ involutions. The constants can be calculated directly from the character table. For $\mathbb{M}$ the product of two $2A$ involutions lies in either of the $\mathbb{M}$ classes : $1A,\ 2A,\ 2B,\ 3A,\ 3C,\ 4A,\ 4B,\ 5A$ or $6A$ (see [2] or [15]). Similarly for $K \cong M_{12}$ we obtain that the product of two $2A_K$ involutions lies in either of the K classes $1A_K,\ 2A_K,\ 2B_K,\ 3B_K,\ 4A_K,\ 4B_K,\ 5A_K$ or $6A_K$. To avoid a contradiction on the monomorphism ι we have $5A_K \subset 5A$, and $6A_K \subset 6A$ and $3A_K$ is a subset of either $3A$ or $3C$. The class $6A_K$ has p-powers $3B_K$, $2A_K$ in K, and the class $6A$ has p-powers $3A$, $2A$ in $\mathbb{M}$. Hence $3B_K \subset 3A$. From lemma 6 of [15] no elements of order 3 in $C_{\mathbb{M}}(H) \cong A_{12}$ belongs to class $3C$ of $\mathbb{M}$, hence $3A_K$ belongs to either $3A$ or $3B$. If $3A_K \subset 3A$ then $6B_K \subset 6C$ and if $3B_K \subset 3B$ then $6B_K$ is in either $6B$ or $6E$ according to the relevant p-powers in K and $\mathbb{M}$. We determine the fusion in $\mathbb{M}$ of $3A_K$ and $6B_K$ at the end of the proof. The classes $4A_K,\ 4B_K$ contain products of $2A$ involutions and their squares lie in class $2B_K \subset 2B$ hence $4A_K,\ 4B_K \subset 4A$ as $4A$ is the unique class of elements of order 4 squaring to $2B$. The classes $8A_K,\ 8B_K$ have their squares in classes $4A_K,\ 4B_K$ respectively, and in $\mathbb{M}$ the unique conjugacy class of elements of order 8 squaring to $4A$ is $8B$. Hence $8A_K,\ 8B_K \subset 8A$. The class $10A_K$ in M_{12} has p-powers $5A_K \subset 5A$ and $2A_K \subset 2A$ and in $\mathbb{M}$ the class $10A$ is the unique class of elements of

[3]For a finite group L, the p-power line in the character table of L records for each conjugacy class C of L, and for each prime p dividing the order of the elements of C, to which conjugacy class the p^{th}-power of the elements of C belongs to.

[4]For a finite group L, the structure class constants give the number of solutions $s_{1,2,3}$ to equations in the group of the type $x_1.x_2 = x_3$, where each x_i belongs to a conjugacy class C_i of L. From the table of complex characters of L:

$$s_{1,2,3} = \frac{|L|}{|C_L(x_1)| \cdot |C_L(x_2)|} \sum_{\chi \in \mathrm{Irr(L)}} \frac{\chi(x_1)\chi(x_2)\overline{\chi(x_3)}}{\chi(\mathbb{1})}$$

order 10 with such p-powers so that $10A_K \subset 10A$. There is a unique class of elements of order 11 in $\mathbb{M}$ so $11A_K,\ 11B_K \subset 11A$. If $3A_K \subset 3A$ then $6B_K \subset 6C$ and the completed fusion of conjugacy classes of K in those of $\mathbb{M}$ gives a value of $\langle \chi_{V_{\mathbb{M}}} \downarrow_K, \mathbb{1}_K \rangle_{\mathbb{R}K}$ which is not integral, a contradiction. Hence $3A_K \subset 3B$ and $6B_K$ is in $6B$ or $6E$. To determine which, we look at the fusion of the conjugacy classes of $A := C_{\mathbb{M}}(H) \cong A_{12}$ in $\mathbb{M}$. Apart from the conjugacy classes $6G_A,\ 9A_A,\ 9B_A$ and $9C_A$, the fusion of the classes of A in $\mathbb{M}$ is straightforward using the information on p-powers and the fusion of the classes of K in $\mathbb{M}$ already obtained. From a calculation of S. Shpectorov in [12] we know that $\langle \chi_{V_{\mathbb{M}}} \downarrow_A, \mathbb{1}_A \rangle_{\mathbb{R}A} = 26$. This can only happen if $6G_A \subset 6B$ and $9A_A,\ 9B_A,\ 9C_A \subset 9A$. In particular elements of order 6 in A_{12} cannot be subsets of $6E$, hence neither can the elements of order 6 in K. Hence $6B_K \subset 6B$. We have obtained the fusion of K in $\mathbb{M}$

Class in K	$1A_K$	$2A_K$	$2B_K$	$3A_K$	$3B_K$	$4A_K$	$4B_K$
p-powers	A	A	A	A	A	B	B
Class in $\mathbb{M}$	$1A$	$2A$	$2B$	$3B$	$3A$	$4A$	$4A$

	$5A_K$	$6A_K$	$6B_K$	$8A_K$	$8B_K$	$10A_K$	$11A_K$
	A	BA	AB	A	B	AA	A
	$5A$	$6B$	$6A$	$8B$	$8B$	$10A$	$11A$

and we can now compute the inner product of real characters of K

$$\langle \chi_{V_{\mathbb{M}}} \downarrow_K, \mathbb{1}_K \rangle_{\mathbb{R}K} = 102.$$

□

The following useful observation was made by S. Norton (in a more general context).

Lemma 3.4. *The identity $\iota_{\mathbb{M}}$ of $V_{\mathbb{M}}$ cannot be contained in V.*

Proof. The groups G and K centralise each other in $\mathbb{M}$ and $G \times K$ is a subgroup of $\mathbb{M}$. The unique conjugacy class of involutions T of G is in class $2A$ of $\mathbb{M}$, and we have proved that the classes $2A_K$ and $2B_K$ are in $2A$ and $2B$ respectively.

Claim : there exist elements $s \in T$ and $t \in 2A_K$ such that the element ts of $G \times K$ is in class $2B$ of $\mathbb{M}$.

From Proposition 3.2, part (iii), the element s can be taken from a A_5-subgroup X of G of type $(2A, 3A, 5A)$. From the proof of Proposition 3.2, the elements of $2A_K$ act fixed-point freely on 12 points, and the centraliser in $\mathbb{M}$ of an A_5-subgroup of type $(2A, 3A, 5A)$ is isomorphic to A_{12}. By Table 4, line 7 of [17], there exists a subgroup $Y \subseteq A_{12},\ Y \cong A_5$, that acts transitively on 12 points. Let $t \in Y$. Then, by Table 3, line 8 of [17], the involutions in the diagonal subgroups of $X \times Y$ are in $2B$. In particular $ts \in 2B$ and the claim is proved.
Since $ts \in 2B$, the axes a_t and a_s generate a dihedral algebra of type $2B$ and $a_t \cdot a_s = 0$ (see [2] or [11]). For all $z \in T$ there is an element $g \in G$ such that $z = s^g$, so by invariance of the algebra product $(a_t \cdot a_s)^g = 0 = a_t \cdot a_z$ since G normalises K. Now, V is generated by the 55 Majorana axes a_z for $z \in T$ and the 0-eigenspace of a_t is closed under the algebra product, so if the identiy $\iota_{\mathbb{M}}$ were in V we would get the contradiction $a_t \cdot \iota_{\mathbb{M}} = 0$. □

The identity of a commutative algebra being unique and therefore stable under the automorphism group we have $\iota_{\mathbb{M}} \in C_{V_{\mathbb{M}}}(K)$. And since $\iota_{\mathbb{M}}$ is not in V we obtain the main result of this subsection.

Proposition 3.5. *For the algebra V we have $dim(V) \leq 101$.* □

3.2 Inner product values for *V*

In this subsection we calculate all inner products on a well-chosen subset of V and compute the rank of the corresponding Gram matrix to bound below the dimension of V. We do so using the information on some subalgebras of V which have already been classified.

3.2.1 Dihedral subalgebras of *V*

The algebra V contains the dihedral subalgebras of types $2A$, $3A$, $5A$, and $6A$ (obtained by calculating the relevant structure class constants in the character table of G). From [16], for each type of dihedral algebra, we know the dimension of the algebra, and a basis for which all algebra and inner products are known. We follow the exposition given in [8] which is now accepted as standard in the Majorana Theory and where a different scaling to [16] is used. Table 2 is taken from [8], which notation we explain below.

Each dihedral subalgebra corresponds to a dihedral subgroup D of $\mathbb{M}$ generated by two $2A$ involutions t and s, whose product we denote by $\rho := ts$. We denote by $a_0,\ a_1$ and a_i the Majorana axes $a_t,\ a_s$ and $a_{t\rho^i}$ in Table 2.
In the subalgebra of type $2A$, we have $a_\rho = \psi(\rho)$ which is also a Majorana axis, and in the types $3A$ and $5A$ the vectors u_ρ and w_ρ are introduced to close the algebra product. They correspond to elements of order 3 or 5 in D respectively. From [2], the 1-dimensional subspace linearly spanned by the vector u_ρ or w_ρ is invariant under the normaliser $N_{\mathbb{M}}(\langle\rho\rangle)$ which is isomorphic to $3.F_{24}$ or $(D_{10} \times F_5).2$ respectively. Also, in the type $3A$ the vector itself is stable under $N_{\mathbb{M}}(\langle\rho\rangle)$, so that $u_\rho = u_{\rho^{-1}}$, and in the type $5A$ the vector is stabilised up to negation $w_\rho = -w_{\rho^2} = -w_{\rho^3} = w_{\rho^4}$.

Any element of order 3 or 5 in G can be expressed as a product of two involutions, and any two involutions correspond to a dihedral subalgebra of V. Hence to study the algebra V we can consider the span of the vectors corresponding to the cyclic subgroups of order $2,\ 3$ and 5 of G.
We let $G^{(i)}$ be a set of non-trivial representatives of each cyclic subgroup of order i for $i = 2, 3, 5$, of size 55, 55 and 66 respectively, where for $i = 5$ the representatives are taken from the same conjugacy class of G. We use the notation $A := \{a_t \mid t \in G^{(2)}\}$, $U := \{u_h \mid h \in G^{(3)}\}$, and $W := \{w_f \mid f \in G^{(5)}\}$, and we let $S := A \cup U \cup W$.

3.2.2 A_5 subalgebras of *V*

The algebra V also contains 22 A_5-subalgebras of type $(2A, 3A, 5A)$ as G contains two conjugacy classes of A_5-subgroups of type $(2A, 3A, 5A)$, of size 11 each. The structure of a subalgebra V_H generated by the Majorana axes indexed by the involutions of an A_5-subgroup H of type $(2A, 3A, 5A)$ follows from [12]. We reformulate it so as to present V_H as a subalgebra of $V_{\mathbb{M}}$ generated by a triple of Majorana axes.

Type	Basis	Products and angles
2A	a_0, a_1, a_ρ	$a_0 \cdot a_1 = \frac{1}{2^3}(a_0 + a_1 - a_\rho),\ a_0 \cdot a_\rho = \frac{1}{2^3}(a_0 + a_\rho - a_1)$ $(a_0, a_1) = (a_0, a_\rho) = (a_1, a_\rho) = \frac{1}{2^3}$
3A	$a_{-1}, a_0, a_1,$ u_ρ	$a_0 \cdot a_1 = \frac{1}{2^5}(2a_0 + 2a_1 + a_{-1}) - \frac{3^3 \cdot 5}{2^{11}} u_\rho$ $a_0 \cdot u_\rho = \frac{1}{3^2}(2a_0 - a_1 - a_{-1}) + \frac{5}{2^5} u_\rho$ $u_\rho \cdot u_\rho = u_\rho$ $(a_0, a_1) = \frac{13}{2^8}, (a_0, u_\rho) = \frac{1}{2^2}, (u_\rho, u_\rho) = \frac{2^3}{5}$
5A	$a_{-2}, a_{-1}, a_0,$ a_1, a_2, w_ρ	$a_0 \cdot a_1 = \frac{1}{2^7}(3a_0 + 3a_1 - a_2 - a_{-1} - a_{-2}) + w_\rho$ $a_0 \cdot a_2 = \frac{1}{2^7}(3a_0 + 3a_2 - a_1 - a_{-1} - a_{-2}) - w_\rho$ $a_0 \cdot w_\rho = \frac{7}{2^{12}}(a_1 + a_{-1} - a_2 - a_{-2}) + \frac{7}{2^5} w_\rho$ $w_\rho \cdot w_\rho = \frac{5^2 \cdot 7}{2^{19}}(a_{-2} + a_{-1} + a_0 + a_1 + a_2)$ $(a_0, a_1) = \frac{3}{2^7},\ (a_0, w_\rho) = 0, (w_\rho, w_\rho) = \frac{5^3 \cdot 7}{2^{19}}$
6A	$a_{-2}, a_{-1}, a_0,$ a_1, a_2, a_3 a_{ρ^3}, u_{ρ^2}	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-2} - a_{-1} - a_2 - a_3 + a_{\rho^3}) + \frac{3^2 \cdot 5}{2^{11}} u_{\rho^2}$ $a_0 \cdot a_2 = \frac{1}{2^5}(2a_0 + 2a_2 + a_{-2}) - \frac{3^3 \cdot 5}{2^{11}} u_{\rho^2}$ $a_0 \cdot u_{\rho^2} = \frac{1}{3^2}(2a_0 - a_2 - a_{-2}) + \frac{5}{2^5} u_{\rho^2}$ $a_0 \cdot a_3 = \frac{1}{2^3}(a_0 + a_3 - a_{\rho^3}), a_{\rho^3} \cdot u_{\rho^2} = 0, (a_{\rho^3}, u_{\rho^2}) = 0$ $(a_0, a_1) = \frac{5}{2^8}, (a_0, a_2) = \frac{13}{2^8}, (a_0, a_3) = \frac{1}{2^3}$

Table 2: Dihedral subalgebras

Proposition 3.6. *Let $V_H = \langle\langle a_x, a_y, a_z\rangle\rangle$ be a subalgebra of $V_{\mathbb{M}}$ where the dihedral subalgebra $\langle\langle a_x, a_y\rangle\rangle$ has type $2A$ and where the dihedral subalgebras $\langle\langle a_x, a_z\rangle\rangle$, $\langle\langle a_y, a_z\rangle\rangle$, and $\langle\langle a_{xy}, a_z\rangle\rangle$ have types $5A$, $5A$ and $3A$ respectively. Then V_H has dimension 26 and it is linearly spanned by the products of all pairs of Majorana axes indexed by the involutions of H.* □

For explicit formulas for the algebra product in V_H or a list of all inner product values, we refer the reader to [12]. In the rest of the paper we will simply refer to an A_5-subgroup H to mean an A_5-subgroup of type $(2A, 3A, 5A)$.

For an A_5-subgroup H, we denote by $H^{(2)},\ H^{(3)}$ and $H^{(5)}$ the sets of non-trivial conjugate representatives of cyclic subgroups of order $2,\ 3$ and 5 and in the corresponding algebra V_H we denote by $A_H,\ U_H$ and W_H the sets of vectors $\{a_t \mid t \in H^{(2)}\},\ \{u_h \mid h \in H^{(3)}\}$, and $\{w_f \mid f \in H^{(5)}\}$. Let w_H be the sum of all vectors in W_H. By [12], the set $S_H := A_H \cup U_H \cup W_H$ is a spanning set of size 31 for V_H, and V_H is 26-dimensional with a basis $A_H \cup U_H \cup \{w_H\}$. The five independent linear relations on S_H, which can be found in [12] or [16], are called the Norton Relations.

Proposition 3.7. ***The Norton Relations***
In the algebra V_H corresponding to an A_5-subgroup H of type $(2A, 3A, 5A)$, all vectors $w_f \in W_H$ satisfy:

$$w_f = \frac{1}{6} w_H + \frac{1}{2^7}\left(\sum_{t\in H_5^{(2)}(f)} a_t - \sum_{t\in H_3^{(2)}(f)} a_t\right) + \frac{3^2.5}{2^{12}}\left(\sum_{\substack{h\in H^{(3)},\\ o([h,f])=3}} u_h - \sum_{\substack{h\in H^{(3)},\\ o([h,f])=5}} u_h\right)$$

where

$$H_5^{(2)}(f) := \{t \in H^{(2)} \mid o(tf) = 5\},$$
$$H_3^{(2)}(f) := \{t \in H^{(2)} \mid o(tf) = 3\}.$$

□

We denote by $\mathcal{H}_1 = \{H_1, \ldots,\ H_{11}\}$ and $\mathcal{H}_2 = \{H'_1, \ldots,\ H'_{11}\}$ the two classes of A_5-subgroups in G. One class corresponds to the rows of $\mathcal{N}$ and the other to the columns, so the intersection between A_5's taken from different classes can be read directly from the entries of $\mathcal{N}$.

For a given A_5-subgroup H_i in G let W^i be the sum of all vectors in W_{H_i}. For a vector $w_f \in W_{H_i}$ we rewrite the Norton relation for w_f as

$$w_f = \frac{1}{6} W^i + \lambda\, A_i(f) + \mu\, U_i(f) \tag{3.1}$$

where the meaning of $\lambda,\ \mu,\ A_i(f)$ and $U_i(f)$ is clear from Proposition 3.7.

Corollary 3.8. *Let w be the sum of all vectors w_f in $W \subseteq V$. Then $S' := A \cup U \cup \{w\}$ is a spanning set of size 111 for S.*

Proof. Consider an A_5-subgroup $H_1 \in \mathcal{H}_1$. From $\mathcal{N}$, any subgroup $H_{i'} \in \mathcal{H}_2$ intersect H_1 in a D_{10} or an A_4. If $H_1 \cap H_{i'} \cong D_{10}$ there exists a representative f of $H_1^{(5)}$ in $H_1 \cap H_{i'}$ for which the Norton relations give

$$w_f = \begin{cases} \frac{1}{6}W^1 + \lambda\, A_1(f) + \mu\, U_1(f) \\ \\ \frac{1}{6}W^{i'} + \lambda\, A_{i'}(f) + \mu\, U_{i'}(f) \end{cases}$$

and so $W^{i'}$ is in $Sp(A \cup U \cup \{W^1\})$, the $\mathbb{R}$-linear span of $A \cup U \cup \{W^1\}$.
If $H_1 \cap H_{i'} \cong A_4$ then the situation can be visualized as the following submatrix of $\mathcal{N}$, where each anti-flag has been replaced by the unique element of $G^{(5)}$ stabilising it, and the rows and columns are indexed with the copy of A_5 stabilising the corresponding line or point of $\mathcal{B}$.

$$\begin{array}{c} \\ H_i' \\ H_j' \end{array} \begin{array}{c} \begin{array}{cc} H_1 & H_i \end{array} \\ \begin{pmatrix} 1 & w_g \\ w_f & w_k \end{pmatrix} \end{array}$$

From the Norton relations for $g \in H_i \cap H_{i'}$, $k \in H_i \cap H_{j'}$ and $f \in H_1 \cap H_{j'}$, we also get $W^{i'} \in Sp\big(A \cup U \cup \{W^1\}\big)$. From $\mathcal{N}$, for any subgroup $H_i \in \mathcal{H}_1$ there are 3 elements of $\mathcal{H}_2$ intersecting both H_1 and H_i in a D_{10}, with say $H_{l'}$ being one of them:

$$\begin{array}{cc} & \begin{array}{cc} H_1 & H_i \end{array} \\ H_{l'} & \begin{pmatrix} w_f & w_l \end{pmatrix} \end{array}$$

so the Norton relations for f and l give $W^i \in Sp\big(A \cup U \cup \{W^1\}\big)$. Hence there exists $v \in Sp(A \cup U)$ such that

$$22\, W^1 = \textstyle\sum_{H_i \in \mathcal{H}_1} W^i + \sum_{H_{i'} \in \mathcal{H}_2} W^{i'} + v,$$

and from $\mathcal{N}$ every element of $G^{(5)}$ is contained in exactly one element of $\mathcal{H}_1$ and one of $\mathcal{H}_2$ so that

$$\textstyle\sum_{H_i \in \mathcal{H}_1} W^i + \sum_{H_{i'} \in \mathcal{H}_2} W^{i'} = 2\, w,$$

where w is the sum of all vectors w_f in W, and hence

$$W^1 = \tfrac{1}{11} w + v' \quad \text{for some } v' \in Sp(A \cup U).$$

□

4 Inner product values

Definition 4.1. For each pair $(G^{(i)}, G^{(j)})$ with $i,\ j \in \{2, 3, 5\}$ we call the inner product values on $G^{(i)} \times G^{(j)}$ inner products of type **(i, j)**.

If we let $E^{(i)}$ be the equivalence class of elements of order i in G belonging to the same cyclic subgroup, then the orbits of G acting by conjugation on $E^{(i)} \times E^{(j)}$ form a subpartition of the distinct inner products values of type (i, j) (these orbits were calculated using [1]).

We will only explain the inner products values (u_k, v_l) for which the subgroup $\langle k, l\rangle$ is isomorphic to F_{55} or to the whole of G. They arise as the solutions of equations of intersecting subalgebras inside V, or equivalently as particular configurations of subgroups inside G, which can be read from the incidence matrix $\mathcal{N}$ or found using a code written in [1].

4.1 Inner products of type $(2,2)$

From the dihedral subalgebras of V we know all possible inner product values of any two Majorana axes in V, see [11].

Case	$o(ts)$	$\langle\langle a_t, a_s\rangle\rangle$	$\langle t, s\rangle$	(a_t, a_s)
1	1	$1A$	1	1
2	2	$2A$	D_4	$\frac{1}{2^3}$
3	3	$3A$	D_6	$\frac{13}{2^8}$
4	5	$5A$	D_{10}	$\frac{3}{2^7}$
5	6	$6A$	D_{12}	$\frac{5}{2^8}$

Table 3: Inner Products of type $(2,2)$, with $t, s \in G^{(2)}$

4.2 Inner products of type $(2,3)$

The value for case 5) of the inner product of type $(2,3)$ was computed using the following lemma.

Lemma 4.2. *For $t \in G^{(2)}$ and $h \in G^{(3)}$ such that $\langle t, h\rangle \cong L_2(11)$ we have*

$$(a_t, u_h) = \frac{1}{2.3.5}.$$

Proof. We fix an element $h \in G^{(3)}$ and we let $t \in G^{(2)}$ such that $\langle t, h\rangle = G$. Since $N_G(\langle h\rangle) \cong D_{12}$ then $\langle h\rangle$ is contained in exactly two distinct dihedral groups of order 6. Let S_h be one of the two sets of 3 involutions, $S_h := \{s, sh, sh^2\}$, such that $\langle S, h\rangle \cong D_6$, and up to permutation of the set S_h we have

$$\langle t,\ s\rangle \cong D_6,$$
$$\langle t,\ sh\rangle \cong D_{12},$$
$$\langle t,\ sh^2\rangle \cong D_{10}.$$

In the $3A$-dihedral subalgebra $\langle\langle a_s, u_h\rangle\rangle$ we have the equality

$$u_h = -\frac{2^{11}}{3^3.5}\Big[a_s \cdot a_{sh} - \frac{1}{2^5}(2a_s + 2a_{sh} - a_{sh^2})\Big],$$

so taking the inner product with a_t gives

$$(a_t\ ,\ u_h) = -\frac{2^{11}}{3^3.5}\Big[(a_t\ ,\ a_s \cdot a_{sh}) - \frac{1}{2^5}(a_t\ ,\ 2a_s + 2a_{sh} - a_{sh^2})\Big],$$

Case	$o(ht)$	$\langle t, h\rangle$	(a_t, u_h)
1	2	D_6	$\frac{1}{2^2}$
2	3	A_4	$\frac{1}{3^2}$
3	5	A_5	$\frac{1}{2.3^2}$
4	6	C_6	0
5	11	$L_2(11)$	$\frac{1}{2.3.5}$

Table 4: Inner Products of type $(2, 3)$, with $t \in G^{(2)}$ and $h \in G^{(3)}$

Case	$o(tf) = o(tf^{-1})$	$o([t, f])$	$\langle t, f\rangle$	(a_t, w_f)
1	2	5	D_{10}	0
2	3	5	A_5	$-\frac{7^2}{2^{14}}$
3	5	3	A_5	$\frac{7^2}{2^{14}}$
4	5	5	$L_2(11)$	$-\frac{1}{2^{14}}$
5	6	6	$L_2(11)$	$-\frac{3}{2^{12}}$
6	11	5	$L_2(11)$	$\frac{19}{2^{14}}$

Table 5: Inner Products of type $(2, 5)$, with $t \in G^{(2)}$ and $f \in G^{(5)}$

and by associativity of the algebra product with the inner product

$$(a_t,\ a_s \cdot a_{sh}) = (a_s,\ a_t \cdot a_{sh}).$$

Since $\langle t, sh\rangle \cong D_{12}$, the element $\rho = tsh$ has order 6 so the algebra product $a_t \cdot a_{sh}$ is contained in the dihedral algebra $\langle\langle a_t, a_{sh}\rangle\rangle$ of type $6A$, and so

$$(a_s,\ a_t \cdot a_{sh}) = \frac{1}{2^6}(a_s,\ a_t + a_{sh} - a_{t\rho^2} - a_{t\rho^3} - a_{t\rho^4} - a_{t\rho^5} + a_{\rho^3}) + \frac{3^2.5}{2^{11}}(a_s, u_{\rho^2}),$$

where $\langle s, \rho^2\rangle \cong A_4$, so the value of $(a_s\ ,\ u_{\rho^3})$ is known to be $\frac{1}{9}$ from [11]. Since all the required inner products are now known, one can compute $(a_t, u_h) = \frac{1}{2.3.5}$.

□

4.3 Inner products of type $(2, 5)$

The next lemma justifies the values found in cases $4)$, $5)$ and $6)$. We omit its proof which is similar to the proof of Lemma 4.2.

Lemma 4.3. *Let $t \in G^{(2)}$ and $f \in G^{(5)}$ such that $\langle t,\ f\rangle \cong L_2(11)$. Then exactly one of the following holds.*

(i) *There exists $s \in G^{(2)}$ commuting with t and inverting f, and*

$$(a_t, w_f) = -\frac{1}{2^{11}} + \frac{1}{2^6}p - \frac{1}{2^3}q, \ \text{where}$$

$$p = 7(a_t, a_{sf}) + (a_t, a_{sf^2}) \ \text{and} \ q = (a_{ts}, a_{sf}).$$

(ii) *There exists $s \in G^{(2)}$ inverting f and generating with t a dihedral group of order 6, and*

$$(a_t, w_f) = \frac{3^2}{2^{14}} + \frac{1}{2^7}p - \frac{3^3.5}{2^{11}}q, \ \text{where}$$

$$p = (a_t,\ 5a_{sf} + a_{sf^2} + a_{sf^3} + a_{sf^4}) \ \text{and} \ q = (u_{st}, a_{sf}).$$

(iii) *There exists $s \in G^{(2)}$ inverting f and generating with t a dihedral group of order 12, and*

$$(a_t, w_f) = -\frac{3}{2^{15}} + \frac{1}{2^6}p + \frac{1}{2^7}q - \frac{3^2.5}{2^{11}}r, \ \text{where}$$

$$p = (a_{sf},\ a_{\rho^3} - 2a_t - a_{t\rho^2} - a_{t\rho^3} - a_{t\rho^4} - a_{t\rho^5}),\ \ q = (a_t,\ a_{sf^2} + a_{sf^3} + a_{sf^4}) \ \text{and} \ r = (u_{\rho^2}, a_{sf}),$$

$$\text{for } \rho := ts \text{ of order } 6 \text{ in } \langle t,\ s\rangle \cong D_{12}.$$

□

4.4 Inner products of type $(3, 3)$

In the next lemma part (i) addresses cases $4)$ and $6)$ and part (ii) addresses case $5)$. The lemma assumes all products of type $(2, 3)$ are known.

Lemma 4.4. *Let $h,\ k \in G^{(3)}$ with $\langle h, k\rangle \cong L_2(11)$. Then exactly one of the following holds.*

(i) *There exists an involution t inverting both h and k, and*

$$(u_h, u_k) = \frac{2^4}{3^5.5}(5 - 2^3.3^2p + 2^6q), \ \text{where}$$

$$p = (u_h, a_{tk^2}) = (u_k, a_{th^2}) \ \text{and}$$
$$q = (a_{th}, a_{tk^2}) + (a_{th}, a_{tk}) + (a_{th^2}, a_{tk}) + (a_{th^2}, a_{tk^2}).$$

(ii) *There exists an involution t inverting h and generating with k an alternating group A_4, and*

$$(u_h, u_k) = \frac{2^5}{3^3.5}\left(\frac{1}{2^2.3} - p - 2q - r\right), \ \text{where}$$

$$p = (a_{th},\ 3u_{tk} - 4u_{tk} - 4u_{tk^2}),\ \ q = (a_{th^2}, u_k) \ \text{and} \ r = (a_{th}, u_k).$$

□

Case	$\{o(hk), o(hk^{-1})\}$	$\langle h, k\rangle$	(u_h, u_k)
1	$\{1,3\}$	C_3	$\frac{2^3}{5}$
2	$\{2,3\}$	A_4	$\frac{2^3.17}{3^4.5}$
3	$\{5,5\}$	A_5	$\frac{2^4}{3^4.5}$
4	$\{5,6\}$	$L_2(11)$	$\frac{2^3}{3.5^2}$
5	$\{5,11\}$	$L_2(11)$	$\frac{2^3.7}{3^3.5^2}$
6	$\{6,6\}$	$L_2(11)$	$\frac{2^5}{3^4.5}$

Table 6: Inner Products of type $(3,3)$, with $h, k \in G^{(3)}$

4.5 Inner products of type $(3, 5)$

Part (i) of the next lemma addresses case 3), and part (ii) addresses cases 5) and 6).

Lemma 4.5. *Let $h \in G^{(3)}$ and $f \in G^{(5)}$ such that $\langle h,\ f\rangle \cong L_2(11)$. Then exactly one of the following holds.*

(i) *There exists an involution t inverting f and h, and*

$$(u_h, w_f) = -\tfrac{2^{11}}{3^3.5}(p - \tfrac{1}{2^5}q), \text{ where}$$

$$p = \tfrac{7}{2^{12}}(a_{tf} + a_{tf^4} - a_{tf^2} - a_{tf^3},\ a_{th}) + \tfrac{7}{2^5}(w_f, a_{th}) \text{ and}$$
$$q = (2a_t + 2a_{th} + a_{th^2},\ w_f).$$

(ii) *There exists an involution t inverting f and generating with h a subgroup isomorphic to A_5, and*

$$(u_h, w_f) = \tfrac{7}{2^7}p + \tfrac{1}{2^7}q - \tfrac{1}{2^4}r, \text{ where}$$

$$p = (a_{sf} + a_{sf^4},\ u_h),\ \ q = (a_{sf^2} + a_{sf^3},\ u_h) \text{ and } r = (a_{sf},\ u_{sh} + u_{sh^2}).$$

□

The inner product value for case 4) of the inner product of type $(3, 5)$ can be found using the Norton relations inside some A_5-subalgebras of V, see equation (1). The proof of the following lemma uses similar arguments to the proof of Corollary 3.8. We use the notation of (1).

Lemma 4.6. *Let $f \in G^{(5)}$ and $h \in G^{(3)}$. If there exists an element $g \in G^{(5)}$ such that $A_1 := \langle f, g\rangle$ and $A_2 := \langle h, g\rangle$ are two non-conjugate A_5-subgroups of G, then*

$$(u_h, w_f) = \tfrac{1}{6}\big(u_h\ , W^2\big) + \tfrac{1}{2^7}\big(u_h\ , l_a\big) + \tfrac{3^2.5}{2^{12}}\big(u_h, l_u\big), \text{ where}$$
$$l_a = A_1(g) + A_1(f) - A_2(g) \text{ and } l_u = U_1(g) + U_1(f) - U_2(g).$$

□

Case	$o(hf)$	$o(hf^{-1})$	$\langle h,f\rangle$	(u_h,w_f)
1	2	5	A_5	$\frac{-5.7}{2^9.3^2}$
2	3	5	A_5	$\frac{5.7}{2^9.3^2}$
3	3	6	$L_2(11)$	$\frac{-67}{2^9.3^2.5}$
4	5	5	$L_2(11)$	$\frac{-1}{2^8.3^2.5}$
5	6	11	$L_2(11)$	$\frac{7}{2^6.3^2.5}$
6	11	11	$L_2(11)$	$\frac{-7}{2^7.3^2.5}$

Table 7: Inner Products of type $(3,5)$, with $h\in G^{(3)}$ and $f\in G^{(5)}$

4.6 Inner products of type $(5,5)$

In the next lemma, part (i) justifies the values of the inner product of type $(5,5)$ for the cases 2), 3) and 4), and part (ii) justifies case 6). The proof is similar to that of Corollary 3.8 and the notation is the same as the one used in the previous lemma.

Lemma 4.7. *Let $f,\ g\in G^{(5)}$ not contained in a common A_5-subgroup, with f and g belonging to the pairs $\{H_i,\ H_{i'}\}$ and $\{H_j,\ H_{j'}\}$ respectively, of distinct non-conjugate A_5-subgroups of G. Then exactly one of the following holds.*

(i) *$H_i\cap H_{j'}\cong D_{10}$, or $H_j\cap H_{i'}\cong D_{10}$, with k an element of order 5 in $H_i\cap H_{j'}$, say, then*

$$(w_f,w_g)=\frac{1}{6^2}(W^j,W^j)+\frac{1}{2^8.3}(l_a\,,W^j)+\frac{3.5}{2^{13}}(l_u,\ W^j)+\frac{1}{2^{14}}(l_a\,,A_j(g)) + \frac{3^2.5}{2^{19}}(l_a\,,\ U_j(g))+\frac{3^2.5}{2^{19}}(l_u\,,A_j(g))+\frac{3^4.5^2}{2^{24}}(l_u\,,U_j(g)),$$

where $l_a=A_j(k)-A_{i'}(k)+A_{i'}(f)$ and $l_u=U_j(k)-U_{i'}(k)+U_{i'}(f)$.

(iii) *$H_i\cap H_{j'}\cong H_j\cap H_{i'}\cong A_4$, and there exist two elements $k\neq l\in G^{(5)}$ such that k belongs to H_i and $H_{m'}$ and l belongs to H_j and $H_{m'}$, so that*

$$(w_f,w_g)=\frac{1}{6^2}(W^j,W^j)+\frac{1}{2^8.3}(l_a\,,W^j)+\frac{3.5}{2^{13}}(l_u\,,W^j)+\frac{1}{2^{14}}(l_a\,,A_j(g)) + \frac{3^2.5}{2^{19}}(l_a\,,U_j(g))+\frac{3^2.5}{2^{19}}(l_u\,,A_j(g))+\frac{3^4.5^2}{2^{24}}(l_u\,,U_j(g)),$$

where $l_a=A_j(l)-A_{m'}(l)-A_i(k)+A_{m'}(k)+A_i(f)$,
and $l_u=U_j(l)-U_{m'}(l)-U_i(k)+U_{m'}(k)+U_i(f)$.

□

Case	$\{o(fg), o(fg^{-1})\}$	$o([f,g])$	$\langle f,g\rangle$	(u_h, w_f)
1	$\{1,5\}$	1	C_5	$\frac{5^3.7}{2^{19}}$
2	$\{3,5\}$	5	A_5	$\frac{7.29}{2^{19}}$
3	$\{5,11\}$	11	F_{55}	$\frac{-11}{2^{19}}$
4	$\{3,6\}$	5	$L_2(11)$	$\frac{3.151}{2^{21}}$
5	$\{2,6\}$	5	$L_2(11)$	$\frac{157}{2^{20}}$
6	$\{5,11\}$	2	$L_2(11)$	$\frac{59}{2^{20}}$
7	$\{5,5\}$	3	$L_2(11)$	$\frac{-3.41}{2^{20}}$

Table 8: Inner Products of type $(5,5)$, with $f, g \in G^{(5)}$

Corollary 4.8. *The inner product values between the vector w, and the vectors $a_t \in A$, $u_h \in U$, $w_f \in W$ and w itself are as follows*

(i) $(a_t, w) = \frac{3^2}{2^{11}}$;

(ii) $(u_h, w) = -\frac{3^2}{2^7.5}$;

(iii) $(w_f, w) = \frac{3^3.5.17}{2^{18}}$;

(iv) $(w, w) = \frac{3^4.5.11.17}{2^{17}}$.

□

4.7 Dependence relations in the algebra

We let $V_{S'}$ be the $\mathbb{R}$-vector space having the subset $S' = A \cup U \cup \{w\}$ of V as a basis. We turn $V_{S'}$ into a G-module by the natural action of G on S', and we let π be the natural projection

$$\pi : V_{S'} \to V.$$

Using [1] we find the rank of the Gram matrix of the set S' and give a description of the kernel of π.

We recall the bijection d introduced at the beginning of section 2 between subgroups of order 2 and 3 in G:

$$d : G^{(2)} \to G^{(3)}$$
$$\langle t\rangle \to \langle h\rangle$$

since $\forall t \in G^{(2)}$ $\exists!$ $h \in G^{(3)}$ where $[t,\ h] = 1$. For a fixed involution $t \in G^{(2)}$ its normaliser $N_G(t) \cong D_{12}$ has the following orbits on $G^{(2)}$ (the action is conjugation):

$$O_1,\ O_3^1,\ O_3^2,\ O_6^1,\ O_6^2,\ O_6^3,\ O_6^4,\ O_{12}^1 \text{ and } O_{12}^2,$$

where the subscript indicates the size of the orbit. If we write $N_G(t) = \langle\rho\rangle \rtimes \langle s\rangle$ then $\rho^3 = t$, so $O_1 = \{\rho^3\}$, $O_3^1 = \{s,\ s\rho^2,\ s\rho^4\}$ and $O_3^2 = \{s\rho,\ s\rho^3,\ s\rho^5\}$ wlog. Further we can describe the orbits as follows:

$$\begin{aligned} O_3^1 \cup O_3^2 =& \{s \in G^{(2)} | \langle s,\ t\rangle \cong 2^2\} \\ O_6^1 \cup O_6^2 =& \{s \in G^{(2)} | \langle s,\ t\rangle \cong D_{12}\} \\ O_6^3 \cup O_6^4 =& \{s \in G^{(2)} | \langle s,\ t\rangle \cong D_6\} \\ O_{12}^1 \cup O_{12}^2 =& \{s \in G^{(2)} | \langle s,\ t\rangle \cong D_{10}\}. \end{aligned}$$

For $(t_1,\ t_2)$ in $O_3^1 \times O_6^2$ or $O_3^2 \times O_6^1$ the subgroup $\langle t_1,\ t_2\rangle$ in G is isomorphic to either 2^2 or D_{10}. For $(t_1,\ t_2)$ in $O_3^1 \times O_6^1$ or $O_3^2 \times O_6^2$ the subgroup $\langle t_1,\ t_2\rangle$ is isomorphic to either D_6 or D_{12}.

Proposition 4.9. (i) *The rank of the Gram matrix for the set* S' *is* 101.

(ii) *The kernel of* π *is* 10*-dimensional and consists of* 10 *linearly independent relations, between the vectors of* $A \cup U$*, taken from a set of* 55 G*-invariant relations* $R(t)$ *indexed by the involutions of* G*.*

For a fixed involution t in $G^{(2)}$, $R(t)$ *defines the following* $N_G(t)$*-invariant relation:*

$$R(t) := \sum_{r\in T_1} a_r - \sum_{s\in T_2} a_s + \frac{3^2.5}{2^5}\Big(\sum_{h\in d(T_1)} u_h - \sum_{k\in d(T_2)} u_k\Big) = 0,$$

where T_1 *and* T_2 *can taken to be* $O_3^1 \cup O_6^1$ *and* $O_3^2 \cup O_6^2$ *respectively (or vice versa).* □

From the rank of the Gram matrix of the set S' we obtain the following proposition.

Proposition 4.10. *For the algebra* V *we have* $dim(V) \geq 101$. □

The above, together with Proposition 3.5, proves that $dim(V) = 101$. Hence the set S' spans V, so that $\{a_t \cdot a_s \mid t,\ s \in T\}$ also spans V. From Lemma 3.4, the identity $\iota_{\mathbb{M}}$ of $V_{\mathbb{M}}$ is not in V. The space $C_{V_{\mathbb{M}}}(K)$ is 102-dimensional, containing $\iota_{\mathbb{M}}$ and having V as a subspace. Hence $C_{V_{\mathbb{M}}}(K)$ decomposes as $V \oplus \iota_{\mathbb{M}}$, and we have proved our main theorem.

5 A Majorana representation of $L_2(11)$

The dihedral and A_5-subalgebras of V can be characterised under the axioms of Majorana theory; they are equal to the algebra of the Majorana representations of the dihedral groups D_4 of type $2A$, D_6 of type $3A$, D_{10} of type $5A$, and D_{12} of type $6A$, and of the alternating group A_5 of type $(2A, 3A, 5A)$.

Majorana theory was introduced by A. A. Ivanov in [8] to axiomatise some of the properties of $V_{\mathbb{M}}$ and its Majorana axes. We refer the reader to [8] and [11] for a full description.

Definition 5.1. A **Majorana representation** of a finite group G is a tuple

$$\mathcal{R} = \big(G,\ T,\ X,\ (\ ,\ ),\ \cdot,\ \varphi,\ \psi\big),$$

where T is a union of conjugacy classes of involutions generating G, and X is a commutative non-associative $\mathbb{R}$-algebra endowed with an inner product $(\ ,\ )$ associating with its algebra product $\cdot$ in the sense that $(u\cdot v,\ w) = (u,\ v\cdot w)$ for all $u,\ v,\ w \in X$ and satisfying the Norton Inequality

$$(u\cdot u,\ v\cdot v) \geq (u\cdot v,\ u\cdot v), \text{ for all } u, v \in X.$$

The image of the homomorphism $\varphi : G \hookrightarrow GL(X)$ is an automorphism of $\big(X,\ \cdot\ ,\ (\ ,\ )\big)$, and the map ψ is an injection sending each involution t of T to a Majorana axis a_t of X, as defined in the second paragraph of section 1 (the properties of the spectrum of ad_a and the Fusion Rules are assumed to hold), such that φ and ψ commute in the sense that :

$$a_{g^{-1}tg} = (a_t)^{\varphi(g)} \text{ for every } g \in G.$$

We require that the algebra X be generated by the set of Majorana axes $\psi(T)$ and that it must satisfy conditions $(2A)$ and $(3A)$ below.

Conditions $(2A)$ and $(3A)$ ensure that when constructing X in the above definition we get the right number of $3A$ vectors u_h from the Majorana axes.

(2A) Let $t_0, t_1 \in T$ and $\rho := t_0t_1$ such that

(a) if $\rho \in T$ and the vectors a_{t_0}, a_{t_1} generate a dihedral subalgebra of type $2A$ then $a_\rho = \psi(\rho)$,

(b) if $\rho^i \in T$ for ρ of order 4 or 6 and the vectors a_{t_0} and a_{t_1} generate a subalgebra of type $4B$ or $6A$, then $\psi(\rho^i)$ coincides with the axis a_{ρ^i} ;

(3A) Let $t_0, t_1, t_2, t_3 \in T$ with $\langle t_0, t_1\rangle \cong \langle t_2, t_3\rangle \cong D_6$. We let $\rho_1 := t_0t_1$ and $\rho_2 := t_2t_3$ both of order 3.

If the following two conditions are satisfied:
(i) $\rho_1 = \rho_2$ or ρ_2^{-1}, and
(ii) the dihedral subalgebras generated by $\{a_{t_0}, a_{t_1}\}$ and $\{a_{t_2}, a_{t_3}\}$ have type $3A$,

then the corresponding $3A$-axial vectors u_{ρ_1} and u_{ρ_2} in the above subalgebras are equal in X.

We call $dim(X)$ the dimension of $\mathcal{R}$, and we say that $\mathcal{R}$ is based on an embedding of G into $\mathbb{M}$ if there exists a monomorphism $\iota : G \to \mathbb{M}$ with $\iota(T) \subset 2A$ and such that $\mathcal{R}$ is isomorphic to the subalgebra of $V_{\mathbb{M}}$ generated by the Majorana axes corresponding to $\iota(T)$.

Definition 5.2. The **shape** of a Majorana representation $\mathcal{R}$ of G specifies the types of dihedral subalgebras associated with all pairs of involutions on T.

Theorem 5.3. *A Majorana representation of* $G \cong L_2(11)$ *must have shape* $(2A, 3A, 5A, 6A)$.

Proof. Let $\mathcal{R}$ be a Majorana representation of G with associated algebra X. The group $L_2(11)$ has a single conjugacy class of involutions, $2a$, and a single class $3a$ of elements of order 3. From the structure class constants the product of any $2a$ involutions is in either of the $L_2(11)$ classes $1a,\ 2a,\ 3a,\ 5a,\ 5b$ or $6a$. Hence X contains dihedral subalgebras of type $5A$ and $6A$ since they are the only dihedral subalgebras associated with dihedral groups of order 10 and 12. By the inclusion of the dihedral subalgebras $3A \hookrightarrow 6A$ and $2A \hookrightarrow 6A$ the classes $3a$ and $2a$ are mapped to $3A$ and $2A$ under ψ. Hence X also contains dihedral subalgebras of type $3A$ and $2A$ and we have accounted for all possible dihedral subalgebras in X. □

Let $\mathcal{R}$ be a Majorana representation of G with associated algebra X. From the above theorem, $\mathcal{R}$ has the same shape as the subalgebra V of $V_{\mathbb{M}}$ and the same inner product values for the sets S' and S. Moreover the dihedral and A_5-subalgebras of X are equal to their Majorana representations from [11] and [12].

Proposition 5.4. (i) *The dihedral subalgebras of type* $2A$*,* $3A$*,* $5A$ *and* $6A$ *are equal to the unique Majorana representations of* D_4*,* D_6*,* D_{10}*, and* D_{12} *of shape* $2A$*,* $3A$*,* $5A$ *and* $6A$ *respectively.*

(ii) *The* A_5*-subalgebra of type* $(2A, 3A, 5A)$ *is equal to the unique Majorana representation of shape* $(2A, 3A, 5A)$ *of a group* A_5 *of type* $(2A, 3A, 5A)$*, which has dimension* 26. □

We would like to show that the shape of $\mathcal{R}$ uniquely determines the algebra product in X so that $X = V$. In particular it is necessary to find the closure of the algebra generated by S. This can be inspected computationally. We let $S^2 := \{u \cdot v \mid u,\ v \in S\}$ and $S^3 := \{(u \cdot v) \cdot w,\ \ u \cdot (v \cdot w) \mid u,\ v,\ \ \in S\}$ and for any positive integer n the set S^n is defined in a similar way. Already for the set of vectors $S \cup S^2$ the Majorana axioms yield a very large number of eigenvectors for each Majorana axis and the first computational step is to check whether or not the linear span of $S \cup S^2$ over $\mathbb{R}$ is contained in the closure of X. In fact during the reviewing stage of this paper, the author has learned that Á. Seress has proved in [18] that the system of linear equations in $S \cup S^2$, obtained from the eigenvectors of the axes $\{a_t \mid t \in T\}$, has a unique solution and that $dim_{\mathbb{R}}(X) = 101$. The result was obtained computationally with an algorithm written with [6].

Theorem 5.5. *The* $L_2(11)$*-subalgebra of the Monster algebra* $V_{\mathbb{M}}$ *is equal to* X*, the algebra corresponding to the unique Majorana representation of* $L_2(11)$*.* □

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On fat Hoffman graphs with smallest eigenvalue at least -3

Hye Jin Jang , Jack Koolen

Department of Mathematics, POSTECH, Pohang 790-784, South Korea

Akihiro Munemasa

Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan

Tetsuji Taniguchi

Matsue College of Technology, Nishiikuma-cho 14-4, Matsue, Shimane 690-8518, Japan

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Abstract

We investigate fat Hoffman graphs with smallest eigenvalue at least -3, using their special graphs. We show that the special graph $S(\mathfrak{H})$ of an indecomposable fat Hoffman graph $\mathfrak{H}$ is represented by the standard lattice or an irreducible root lattice. Moreover, we show that if the special graph admits an integral representation, that is, the lattice spanned by it is not an exceptional root lattice, then the special graph $S^-(\mathfrak{H})$ is isomorphic to one of the Dynkin graphs A_n, D_n, or extended Dynkin graphs $\tilde{A}_n$ or $\tilde{D}_n$.

Keywords: Hoffman graph, line graph, graph eigenvalue, special graph, root system.

Math. Subj. Class.: 05C50, 05C76

1 Introduction

Throughout this paper, a graph will mean an undirected graph without loops or multiple edges.

Hoffman graphs were introduced by Woo and Neumaier [5] to extend the results of Hoffman [3]. Hoffman proved what we would call Hoffman's limit theorem (Theorem 2.14) which asserts that, in the language of Hoffman graphs, the smallest eigenvalue of a fat Hoffman graph is a limit of the smallest eigenvalues of a sequence of ordinary graphs with increasing minimum degree. Woo and Neumaier [5] gave a complete list of fat indecomposable Hoffman graphs with smallest eigenvalue at least $-1-\sqrt{2}$. From their list, we

E-mail addresses: laluz@postech.ac.kr (Hye Jin Jang), koolen@postech.ac.kr (Jack Koolen)munemasa@math.is.tohoku.ac.jp (Akihiro Munemasa), tetsuzit@matsue-ct.ac.jp (Tetsuji Taniguchi)

find that only $-1, -2$ and $-1-\sqrt{2}$ appear as the smallest eigenvalues. This implies, in particular, that $-1, -2$ and $-1-\sqrt{2}$ are limit points of the smallest eigenvalues of a sequence of ordinary graphs with increasing minimum degree. It turns out that there are no others between -1 and $-1-\sqrt{2}$. More precisely, for a negative real number λ, consider the sequences

$$\eta_k^{(\lambda)} = \inf\{\lambda_{\min}(\Gamma) \mid \min\deg\Gamma \geq k,\ \lambda_{\min}(\Gamma) > \lambda\} \quad (k = 1, 2, \dots),$$
$$\theta_k^{(\lambda)} = \sup\{\lambda_{\min}(\Gamma) \mid \min\deg\Gamma \geq k,\ \lambda_{\min}(\Gamma) < \lambda\} \quad (k = 1, 2, \dots),$$

where Γ runs through graphs satisfying the conditions specified above, namely, Γ has minimum degree at least k and Γ has smallest eigenvalue greater than (or less than) λ. Then Hoffman [3] has shown that

$$\lim_{k\to\infty} \eta_k^{(-2)} = -1, \qquad \lim_{k\to\infty} \theta_k^{(-1)} = -2,$$
$$\lim_{k\to\infty} \eta_k^{(-1-\sqrt{2})} = -2, \qquad \lim_{k\to\infty} \theta_k^{(-2)} = -1-\sqrt{2}.$$

In other words, real numbers in $(-2,-1)$ and $(-1-\sqrt{2},-2)$ are ignorable if our concern is the smallest eigenvalues of graphs with large minimum degree. Woo and Neumaier [5] went on to prove that

$$\lim_{k\to\infty} \eta_k^{(\alpha)} = -1-\sqrt{2},$$

where $\alpha \approx -2.4812$ is a zero of the cubic polynomial x^3+2x^2-2x-2. Recently, Yu [6] has shown that analogous results for regular graphs also hold.

Woo and Neumaier [5, Open Problem 4] raised the problem of classifying fat Hoffman graphs with smallest eigenvalue at least -3. They also proposed a generalization of a concept of a line graph based on a family of isomorphism classes of Hoffman graphs. This enables one to define generalized line graphs in a very simple manner. As we shall see in Proposition 3.2, the knowledge of μ-saturated indecomposable fat Hoffman graphs gives a description of all fat Hoffman graphs with smallest eigenvalue at least μ. For $\mu = -3$, this in turn should give some information on the limit points of the smallest eigenvalues of a sequence of ordinary graphs with increasing minimum degree. Also, using the generalized concept of line graphs, we can expect to give a description of all graphs with smallest eigenvalue at least -3 and sufficiently large minimum degree. Thus our ultimate goal is to classify (-3)-saturated indecomposable fat Hoffman graphs, as proposed by Woo and Neumaier [5].

The purpose of this paper is to begin the first step of this classification, by determining their special graphs for such Hoffman graphs having an integral reduced representation. One of our main result is that the special graph $\mathcal{S}^-(\mathfrak{H})$ of such a Hoffman graph $\mathfrak{H}$ is isomorphic to one of the Dynkin graphs A_n, D_n, or extended Dynkin graphs $\tilde{\mathsf{A}}_n$ or $\tilde{\mathsf{D}}_n$. We also show that, even when the Hoffman graph $\mathfrak{H}$ does not admit an integral representation, its special graph $\mathcal{S}(\mathfrak{H})$ can be represented by one of the exceptional root lattices E_n ($n = 6, 7, 8$). This might mean that the rest of the work can be completed by a computer as in the classification of maximal exceptional graphs (see [1]). Indeed, if one attaches a fat neighbor to every slim vertex of an ordinary maximal exceptional graph, then we obtain a (-3)-indecomposable fat Hoffman graph. However, maximal graphs among (-3)-indecomposable fat Hoffman graphs represented in E_8 are usually much larger (see

Example 3.8 and the comment following it), so the problem is not as trivial as it looks. We plan to discuss in the subsequent papers the determination of these special graphs and the corresponding Hoffman graphs.

2 Hoffman graphs and their smallest eigenvalues

2.1 Basic theory of Hoffman graphs

In this subsection we discuss the basic theory of Hoffman graphs. Hoffman graphs were introduced by Woo and Neumaier [5], and most of this section is due to them. Since the sums of Hoffman graphs appear only implicitly in [5] and later formulated by Taniguchi [4], we will give proof of the results that use sums for the convenience of the reader.

Definition 2.1. A *Hoffman graph* $\mathfrak{H}$ is a pair (H, μ) of a graph $H = (V, E)$ and a labeling map $\mu : V \to \{f, s\}$, satisfying the following conditions:

(i) every vertex with label f is adjacent to at least one vertex with label s;

(ii) vertices with label f are pairwise non-adjacent.

We call a vertex with label s a *slim vertex*, and a vertex with label f a *fat vertex*. We denote by $V_s = V_s(\mathfrak{H})$ (resp. $V_f(\mathfrak{H})$) the set of slim (resp. fat) vertices of $\mathfrak{H}$. The subgraph of a Hoffman graph $\mathfrak{H}$ induced on $V_s(\mathfrak{H})$ is called the *slim subgraph* of $\mathfrak{H}$. If every slim vertex of a Hoffman graph $\mathfrak{H}$ has a fat neighbor, then we call $\mathfrak{H}$ *fat*.

For a vertex x of $\mathfrak{H}$ we define $N^f(x) = N^f_{\mathfrak{H}}(x)$ (resp. $N^s(x) = N^s_{\mathfrak{H}}(x)$) the set of fat (resp. slim) neighbors of x in $\mathfrak{H}$. The set of all neighbors of x is denoted by $N(x) = N_{\mathfrak{H}}(x)$, that is $N(x) = N^f(x) \cup N^s(x)$. In a similar fashion, for vertices x and y we define $N^f(x, y) = N^f_{\mathfrak{H}}(x, y)$ to be the set of common fat neighbors of x and y.

Definition 2.2. A Hoffman graph $\mathfrak{H}_1 = (H_1, \mu_1)$ is called an (induced) *Hoffman subgraph* of another Hoffman graph $\mathfrak{H} = (H, \mu)$, if H_1 is an (induced) subgraph of H and $\mu(x) = \mu_1(x)$ for all vertices x of $\mathfrak{H}_1$. Unless stated otherwise, by a subgraph we mean an induced Hoffman subgraph. For a subset S of $V_s(\mathfrak{H})$, we denote by $\langle\langle S\rangle\rangle_{\mathfrak{H}}$ the subgraph of $\mathfrak{H}$ induced on the set of vertices

$$S \cup \left(\bigcup_{x \in S} N^f_{\mathfrak{H}}(x) \right).$$

Definition 2.3. Two Hoffman graphs $\mathfrak{H} = (H, \mu)$ and $\mathfrak{H}' = (H', \mu')$ are called *isomorphic* if there exists an isomorphism from H to H' which preserves the labeling.

An ordinary graph H without labeling can be regarded as a Hoffman graph $\mathfrak{H} = (H, \mu)$ without any fat vertex, i.e., $\mu(x) = s$ for all vertices x. Such a graph is called a *slim graph*.

Definition 2.4. For a Hoffman graph $\mathfrak{H}$, let A be its adjacency matrix,

$$A = \begin{pmatrix} A_s & C \\ C^T & O \end{pmatrix}$$

in a labeling in which the fat vertices come last. *Eigenvalues* of $\mathfrak{H}$ are the eigenvalues of the real symmetric matrix $B(\mathfrak{H}) = A_s - CC^T$. Let $\lambda_{\min}(\mathfrak{H})$ denote the smallest eigenvalue of $\mathfrak{H}$.

Definition 2.5 ([5]). For a Hoffman graph $\mathfrak{H}$ and a positive integer n, a mapping $\phi : V(\mathfrak{H}) \to \mathbf{R}^n$ such that

$$(\phi(x), \phi(y)) = \begin{cases} m & \text{if } x = y \in V_s(\mathfrak{H}), \\ 1 & \text{if } x = y \in V_f(\mathfrak{H}), \\ 1 & \text{if } x \text{ and } y \text{ are adjacent in } \mathfrak{H}, \\ 0 & \text{otherwise,} \end{cases}$$

is called a *representation of norm* m. We denote by $\Lambda(\mathfrak{H}, m)$ the lattice generated by $\{\phi(x) \mid x \in V(\mathfrak{H})\}$. Note that the isomorphism class of $\Lambda(\mathfrak{H}, m)$ depends only on m, and is independent of ϕ, justifying the notation.

Definition 2.6. For a Hoffman graph $\mathfrak{H}$ and a positive integer n, a mapping $\psi : V_s(\mathfrak{H}) \to \mathbf{R}^n$ such that

$$(\psi(x), \psi(y)) = \begin{cases} m - |N^f_{\mathfrak{H}}(x)| & \text{if } x = y, \\ 1 - |N^f_{\mathfrak{H}}(x, y)| & \text{if } x \text{ and } y \text{ are adjacent,} \\ -|N^f_{\mathfrak{H}}(x, y)| & \text{otherwise.} \end{cases}$$

is called a *reduced representation of norm* m. We denote by $\Lambda^{\text{red}}(\mathfrak{H}, m)$ the lattice generated by $\{\psi(x) \mid x \in V_s(\mathfrak{H})\}$. Note that the isomorphism class of $\Lambda^{\text{red}}(\mathfrak{H}, m)$ depends only on m, and is independent of ψ, justifying the notation.

While it is clear that a representation of norm $m > 1$ is an injective mapping, a reduced representation of norm m may not be. See Section 4 for more details.

Lemma 2.7. *Let $\mathfrak{H}$ be a Hoffman graph having a representation of norm m. Then $\mathfrak{H}$ has a reduced representation of norm m, and $\Lambda(\mathfrak{H}, m)$ is isomorphic to $\Lambda^{\text{red}}(\mathfrak{H}, m) \oplus \mathbb{Z}^{|V_f(\mathfrak{H})|}$ as a lattice.*

Proof. Let $\phi : V(\mathfrak{H}) \to \mathbf{R}^n$ be a representation of norm m. Let U be the subspace of $\mathbf{R}^n$ generated by $\{\phi(x) \mid x \in V_f(\mathfrak{H})\}$. Let ρ, $\rho^\perp$ denote the orthogonal projections from $\mathbf{R}^n$ onto U, $U^\perp$, respectively. Then $\rho^\perp \circ \phi$ is a reduced representation of norm m, $\rho^\perp(\Lambda(\mathfrak{H}, m)) \cong \Lambda^{\text{red}}(\mathfrak{H}, m)$, and $\rho(\Lambda(\mathfrak{H}, m)) \cong \mathbb{Z}^{|V_f(\mathfrak{H})|}$. □

Theorem 2.8. *For a Hoffman graph $\mathfrak{H}$, the following conditions are equivalent:*

(i) *$\mathfrak{H}$ has a representation of norm m,*

(ii) *$\mathfrak{H}$ has a reduced representation of norm m,*

(iii) $\lambda_{\min}(\mathfrak{H}) \geq -m$.

Proof. From Lemma 2.7, we see that (i) implies (ii).

Let ψ be a reduced representation of $\mathfrak{H}$ of norm m. Then the matrix $B(\mathfrak{H}) + mI$ is the Gram matrix of the image of ψ, and hence positive semidefinite. This implies that $B(\mathfrak{H})$ has smallest eigenvalue at least $-m$ and hence $\lambda_{\min}(\mathfrak{H}) \geq -m$. This proves (ii) $\implies$ (iii).

The proof of equivalence of (i) and (iii) can be found in [5, Theorem 3.2]. □

From Theorem 2.8, we obtain the following lemma:

Lemma 2.9. *If* $\mathfrak{G}$ *is a subgraph of a Hoffman graph* $\mathfrak{H}$*, then* $\lambda_{\min}(\mathfrak{G}) \geq \lambda_{\min}(\mathfrak{H})$ *holds.*

Proof. Let $m := -\lambda_{\min}(\mathfrak{H})$. Then $\mathfrak{H}$ has a representation ϕ of norm m by Theorem 2.8. Restricting ϕ to the vertices of $\mathfrak{G}$ we obtain a representation of norm m of $\mathfrak{G}$, which implies $\lambda_{\min}(\mathfrak{G}) \geq -m$ by Theorem 2.8. □

In particular, if Γ is the slim subgraph of $\mathfrak{H}$, then $\lambda_{\min}(\Gamma) \geq \lambda_{\min}(\mathfrak{H})$.

Under a certain condition, equality holds in Lemma 2.9. To state this condition we need to introduce decompositions of Hoffman graphs. This was formulated first by the third author [4], although it was already implicit in [5].

Definition 2.10. Let $\mathfrak{H}$ be a Hoffman graph, and let $\mathfrak{H}^1$ and $\mathfrak{H}^2$ be two non-empty induced Hoffman subgraphs of $\mathfrak{H}$. The Hoffman graph $\mathfrak{H}$ is said to be the *sum* of $\mathfrak{H}^1$ and $\mathfrak{H}^2$, written as $\mathfrak{H} = \mathfrak{H}^1 \uplus \mathfrak{H}^2$, if the following conditions are satisfied:

(i) $V(\mathfrak{H}) = V(\mathfrak{H}^1) \cup V(\mathfrak{H}^2)$;

(ii) $\{V_s(\mathfrak{H}^1), V_s(\mathfrak{H}^2)\}$ is a partition of $V_s(\mathfrak{H})$;

(iii) if $x \in V_s(\mathfrak{H}^i)$, $y \in V_f(\mathfrak{H})$ and $x \sim y$, then $y \in V_f(\mathfrak{H}^i)$;

(iv) if $x \in V_s(\mathfrak{H}^1)$, $y \in V_s(\mathfrak{H}^2)$, then x and y have at most one common fat neighbor, and they have one if and only if they are adjacent.

If $\mathfrak{H} = \mathfrak{H}^1 \uplus \mathfrak{H}^2$ for some non-empty subgraphs $\mathfrak{H}^1$ and $\mathfrak{H}^2$, then we call $\mathfrak{H}$ *decomposable*. Otherwise $\mathfrak{H}$ is called *indecomposable*. Clearly, a disconnected Hoffman graph is decomposable.

It follows easily that the above-defined sum is associative and so that the sum

$$\mathfrak{H} = \biguplus_{i=1}^{n} \mathfrak{H}^i$$

is well-defined. The main reason for defining the sum of Hoffman graphs is the following lemma.

Lemma 2.11. *Let* $\mathfrak{H}$ *be a Hoffman graph and let* $\mathfrak{H}^1$ *and* $\mathfrak{H}^2$ *be two (non-empty) induced Hoffman subgraphs of* $\mathfrak{H}$ *satisfying* (i)–(iii) *of Definition 2.10. Let* ψ *be a reduced representation of* $\mathfrak{H}$ *of norm* m*. Then the following are equivalent.*

(i) $\mathfrak{H} = \mathfrak{H}^1 \uplus \mathfrak{H}^2$,

(ii) $(\psi(x), \psi(y)) = 0$ *for any* $x \in V_s(\mathfrak{H}^1)$ *and* $y \in V_s(\mathfrak{H}^2)$.

Proof. This follows easily from the definitions of $\mathfrak{H} = \mathfrak{H}^1 \uplus \mathfrak{H}^2$ and a reduced representation of norm m. □

Lemma 2.12. *If* $\mathfrak{H} = \mathfrak{H}^1 \uplus \mathfrak{H}^2$*, then*

$$\lambda_{\min}(\mathfrak{H}) = \min\{\lambda_{\min}(\mathfrak{H}^1), \lambda_{\min}(\mathfrak{H}^2)\}.$$

Proof. Let $m = -\min\{\lambda_{\min}(\mathfrak{H}^1), \lambda_{\min}(\mathfrak{H}^2)\}$. In view of Lemma 2.9 we only need to show that $\lambda_{\min}(\mathfrak{H}) \geq -m$. By Theorem 2.8, $\mathfrak{H}^i$ has a reduced representation $\psi^i : V(\mathfrak{H}^i) \to \mathbf{R}^{n_i}$ of norm m, for each $i = 1, 2$. Define $\psi : V(\mathfrak{H}) \to \mathbf{R}^{n_1} \oplus \mathbf{R}^{n_2}$ by $\psi(x) = \psi^1(x) \oplus 0$ if $x \in V(\mathfrak{H}^1)$, $\psi(x) = 0 \oplus \psi^2(x)$ otherwise. It is easy to check that ψ is a reduced representation of norm m, and the result then follows from Theorem 2.8. □

2.2 Hoffman's limit theorem

In this subsection, we state and prove Hoffman's limit theorem (Theorem 2.14) using the concept of Hoffman graphs.

Lemma 2.13. *Let $\mathfrak{G}$ be a Hoffman graph whose vertex set is partitioned as $V_1 \cup V_2 \cup V_3$ such that*

(i) $V_2 \cup V_3 \subset V_s(\mathfrak{G})$,

(ii) *there are no edges between V_1 and V_3,*

(iii) *every pair of vertices $x \in V_2$ and $y \in V_3$ are adjacent,*

(iv) *V_3 is a clique.*

Let $\mathfrak{H}$ be the Hoffman graph with the set of vertices $V_1 \cup V_2$ together with a fat vertex $f \notin V(\mathfrak{G})$ adjacent to all the vertices of V_2. If $\mathfrak{G}$ has a representation of norm m, then $\mathfrak{H}$ has a representation of norm

$$m + \frac{(m-1)|V_2|}{|V_3| + m - 1}.$$

Proof. Let $\phi : V(\mathfrak{G}) \to \mathbf{R}^d$ be a representation of norm m, and let

$$P = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

be the $|V(\mathfrak{G})| \times d$ matrix whose rows are the images of $V(\mathfrak{G}) = V_1 \cup V_2 \cup V_3$ under ϕ. Set

$$\begin{aligned} \boldsymbol{u} &= \frac{1}{\sqrt{|V_3|(|V_3| + m - 1)}} \sum_{x \in V_3} \phi(x), \\ \epsilon_1 &= 1 - \sqrt{\frac{|V_3|}{|V_3| + m - 1}}, \\ \epsilon_2 &= \sqrt{\frac{m-1}{|V_3| + m - 1}}. \end{aligned}$$

Let $\boldsymbol{j}$ denote the row vector of length $|V_2|$ all of whose entries are 1. Then

$$\boldsymbol{u}\boldsymbol{u}^T = 1, \tag{2.1}$$

$$P_1 \boldsymbol{u}^T = 0, \tag{2.2}$$

$$P_2 \boldsymbol{u}^T = (1 - \epsilon_1)\boldsymbol{j}^T, \tag{2.3}$$

$$\epsilon_2^2 = 2\epsilon_1 - \epsilon_1^2. \tag{2.4}$$

Fix an orientation of the complete digraph on V_2, and let B be the $|V_2| \times \binom{|V_2|}{2}$ matrix defined by

$$B_{\alpha,(\beta,\gamma)} = \delta_{\alpha\beta} - \delta_{\alpha\gamma} \quad (\alpha, \beta, \gamma \in V_2,\ \beta \neq \gamma).$$

Then

$$BB^T = |V_2| I - J. \tag{2.5}$$

We now construct the desired representation of $\mathfrak{H}$, as the row vectors of the matrix

$$D = \begin{pmatrix} P_1 & \epsilon_2\sqrt{|V_2|}I & 0 \\ P_2 + \epsilon_1 \boldsymbol{j}^T\boldsymbol{u} & 0 & \epsilon_2 B \\ \boldsymbol{u} & 0 & 0 \end{pmatrix}.$$

Then, using (2.1)–(2.5), we find

$$\begin{aligned} DD^T &= \begin{pmatrix} P_1 & \epsilon_2\sqrt{|V_2|}I & 0 \\ P_2 + \epsilon_1 \boldsymbol{j}^T\boldsymbol{u} & 0 & \epsilon_2 B \\ \boldsymbol{u} & 0 & 0 \end{pmatrix} \begin{pmatrix} P_1^T & P_2^T + \epsilon_1 \boldsymbol{u}^T\boldsymbol{j} & \boldsymbol{u}^T \\ \epsilon_2\sqrt{|V_2|}I & 0 & 0 \\ 0 & \epsilon_2 B^T & 0 \end{pmatrix} \\ &= \begin{pmatrix} P_1P_1^T + \epsilon_2^2|V_2|I & P_1P_2^T & 0 \\ P_2P_1^T & P_2P_2^T + (2\epsilon_1 - \epsilon_1^2)J + \epsilon_2^2(|V_2|I - J) & \boldsymbol{j}^T \\ 0 & \boldsymbol{j} & 1 \end{pmatrix} \\ &= \begin{pmatrix} P_1P_1^T + \epsilon_2^2|V_2|I & P_1P_2^T & 0 \\ P_2P_1^T & P_2P_2^T + \epsilon_2^2|V_2|I & \boldsymbol{j}^T \\ 0 & \boldsymbol{j} & 1 \end{pmatrix} \\ &= \begin{pmatrix} P_1P_1^T & P_1P_2^T & 0 \\ P_2P_1^T & P_2P_2^T & \boldsymbol{j}^T \\ 0 & \boldsymbol{j} & 1 \end{pmatrix} + \epsilon_2^2|V_2| \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Therefore, the row vectors of D define a representation of norm $m + \epsilon_2^2|V_2|$ of the Hoffman graph $\mathfrak{H}$. □

Theorem 2.14 (Hoffman)**.** *Let $\mathfrak{H}$ be a Hoffman graph, and let $f_1, \dots, f_k \in V_f(\mathfrak{H})$. Let $\mathfrak{G}^{n_1,\dots,n_k}$ be the Hoffman graph obtained from $\mathfrak{H}$ by replacing each f_i by a slim n_i-clique K^i, and joining all the neighbors of f_i with all the vertices of K^i by edges. Then*

$$\lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_k}) \geq \lambda_{\min}(\mathfrak{H}), \tag{2.6}$$

and

$$\lim_{n_1,\dots,n_k\to\infty} \lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_k}) = \lambda_{\min}(\mathfrak{H}). \tag{2.7}$$

Proof. We prove the assertions by induction on k. First suppose $k = 1$. Let $\mu_n = -\lambda_{\min}(\mathfrak{G}^n)$. Let $\mathfrak{H}^n$ denote the Hoffman graph obtained from $\mathfrak{H}$ by attaching a slim n-clique K to the fat vertex f_1, joining all the neighbors of f_1 and all the vertices of K by edges. Then $\mathfrak{H}^n$ contains both $\mathfrak{H}$ and $\mathfrak{G}^n$ as subgraphs, and $\mathfrak{H}^n = \mathfrak{H} \uplus \mathfrak{H}'$, where $\mathfrak{H}'$ is the subgraph induced on $K \cup \{f_1\}$. Since $\lambda_{\min}(\mathfrak{H}') = -1$, Lemma 2.12 implies

$$\lambda_{\min}(\mathfrak{H}) = \lambda_{\min}(\mathfrak{H}^n) \leq \lambda_{\min}(\mathfrak{G}^n) = -\mu_n.$$

Thus (2.6) holds for $k = 1$. Since n is arbitrary and $\{-\mu_n\}_{n=1}^{\infty}$ is decreasing, we see that $\lim_{n\to\infty} \mu_n$ exists and

$$\lambda_{\min}(\mathfrak{H}) \leq -\lim_{n\to\infty} \mu_n. \tag{2.8}$$

Since $\mathfrak{G}^n$ has a representation of norm μ_n, it follows from Lemma 2.13 that $\mathfrak{H}$ has a representation of norm

$$\mu_n + \frac{(\mu_n - 1)|N_{\mathfrak{H}}(f)|}{n + \mu_n - 1}.$$

By Theorem 2.8, we have

$$\lambda_{\min}(\mathfrak{H}) \geq -\mu_n - \frac{(\mu_n - 1)|N_{\mathfrak{H}}(f)|}{n + \mu_n - 1},$$

which implies

$$\lambda_{\min}(\mathfrak{H}) \geq - \lim_{n\to\infty} \mu_n. \tag{2.9}$$

Combining (2.9) with (2.8), we conclude that (2.7) holds for $k = 1$.

Next, suppose $k \geq 2$. Let $\mathfrak{G}^{n_1,\dots,n_{k-1}}$ be the Hoffman graph obtained from $\mathfrak{H}$ by replacing each f_i $(1 \leq i \leq k-1)$ by a slim n_i-clique K^i, and joining all the neighbors of f_i with all the vertices of K^i by edges. Then $\mathfrak{G}^{n_1,\dots,n_k}$ is obtained from $\mathfrak{G}^{n_1,\dots,n_{k-1}}$ by replacing f_k by a slim n_k-clique K^k, and joining all the neighbors of f_k with all the vertices of K^k by edges. Then it follows from the assertions for $k = 1$ that

$$\lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_k}) \geq \lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_{k-1}}), \tag{2.10}$$

and

$$\lim_{n_k\to\infty} \lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_k}) = \lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_{k-1}}).$$

This means that, for any $\epsilon > 0$, there exists N_1 such that

$$n_k \geq N_1 \implies 0 \leq \lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_k}) - \lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_{k-1}}) < \epsilon.$$

By induction, we have

$$\lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_{k-1}}) \geq \lambda_{\min}(\mathfrak{H}), \tag{2.11}$$

and

$$\lim_{n_1,\dots,n_{k-1}\to\infty} \lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_{k-1}}) = \lambda_{\min}(\mathfrak{H}). \tag{2.12}$$

Combining (2.10) with (2.11), we obtain (2.6), while (2.11) and (2.12) imply that there exists N_0 such that

$$n_1,\dots,n_{k-1} \geq N_0 \implies 0 \leq \lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_{k-1}}) - \lambda_{\min}(\mathfrak{H}) < \epsilon.$$

Setting $N = \max\{N_0, N_1\}$, we see that

$$n_1,\dots,n_k \geq N \implies 0 \leq \lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_k}) - \lambda_{\min}(\mathfrak{H}) < 2\epsilon.$$

This establishes (2.7). □

Corollary 2.15. *Let $\mathfrak{H}$ be a Hoffman graph. Let Γ^n be the slim graph obtained from $\mathfrak{H}$ by replacing every fat vertex f of $\mathfrak{H}$ by a slim n-clique $K(f)$, and joining all the neighbors of f with all the vertices of $K(f)$ by edges. Then*

$$\lambda_{\min}(\Gamma^n) \geq \lambda_{\min}(\mathfrak{H}),$$

and

$$\lim_{n\to\infty} \lambda_{\min}(\Gamma^n) = \lambda_{\min}(\mathfrak{H}).$$

In particular, for any $\epsilon > 0$, there exists a natural number n such that, every slim graph Δ containing Γ^n as an induced subgraph satisfies

$$\lambda_{\min}(\Delta) \leq \lambda_{\min}(\mathfrak{H}) + \epsilon.$$

Proof. Immediate from Theorem 2.14. □

3 Special graphs of Hoffman graphs

Definition 3.1. Let μ be a real number with $\mu \leq -1$ and let $\mathfrak{H}$ be a Hoffman graph with smallest eigenvalue at least μ. Then $\mathfrak{H}$ is called *μ-saturated* if no fat vertex can be attached to $\mathfrak{H}$ in such a way that the resulting graph has smallest eigenvalue at least μ.

Proposition 3.2. *Let μ be a real number, and let $\mathscr{H}$ be a family of indecomposable fat Hoffman graphs with smallest eigenvalue at least μ. The following statements are equivalent:*

(i) *every fat Hoffman graph with smallest eigenvalue at least μ is a subgraph of a graph $\mathfrak{H} = \biguplus_{i=1}^{n} \mathfrak{H}^i$ such that $\mathfrak{H}^i$ is a member of $\mathscr{H}$ for all $i = 1, \ldots, n$.*

(ii) *every μ-saturated indecomposable fat Hoffman graph is isomorphic to a subgraph of a member of $\mathscr{H}$.*

Proof. First suppose (i) holds, and let $\mathfrak{H}$ be a μ-saturated indecomposable fat Hoffman graph. Then $\mathfrak{H}$ is a fat Hoffman graph with smallest eigenvalue at least μ, hence $\mathfrak{H}$ is a subgraph of $\mathfrak{H}' = \biguplus_{i=1}^{n} \mathfrak{H}^i$, where $\mathfrak{H}^i$ is a member of $\mathscr{H}$ for $i = 1, \ldots, n$. Since $\mathfrak{H}$ is μ-saturated, it coincides with the subgraph $\langle\langle V_s(\mathfrak{H})\rangle\rangle_{\mathfrak{H}'}$ of $\mathfrak{H}'$. Since $\mathfrak{H}$ is indecomposable, this implies that $\mathfrak{H}$ is a subgraph of $\mathfrak{H}^i$ for some i.

Next suppose (ii) holds, and let $\mathfrak{H}$ be a fat Hoffman graph with smallest eigenvalue at least μ. Without loss of generality we may assume that $\mathfrak{H}$ is indecomposable and μ-saturated. Then $\mathfrak{H}$ is isomorphic to a subgraph of a member of $\mathscr{H}$, hence (i) holds. □

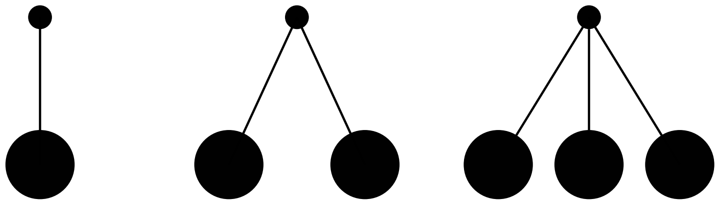
Definition 3.3. For a Hoffman graph $\mathfrak{H}$, we define the following three graphs $\mathcal{S}^-(\mathfrak{H})$, $\mathcal{S}^+(\mathfrak{H})$ and $\mathcal{S}(\mathfrak{H})$ as follows: For $\epsilon \in \{-, +\}$ define the *special ϵ-graph* $\mathcal{S}^\epsilon = (V_s(\mathfrak{H}), E^\epsilon)$ as follows: the set of edges E^ϵ consists of pairs $\{s_1, s_2\}$ of distinct slim vertices such that $\operatorname{sgn}(\psi(s_1), \psi(s_2)) = \epsilon$, where ψ is a reduced representation of $\mathfrak{H}$ of norm m. The graph $\mathcal{S}(\mathfrak{H}) := \mathcal{S}^+(\mathfrak{H}) \cup \mathcal{S}^-(\mathfrak{H}) = (V_s(\mathfrak{H}), E^- \cup E^+)$ is the *special graph* of $\mathfrak{H}$.

Note that the definition of the special graph $\mathcal{S}(\mathfrak{H})$ is independent of the choice of the norm m of a reduced representation ψ.

It is easy to determine whether a Hoffman graph $\mathfrak{H}$ is decomposable or not.

Lemma 3.4. *Let $\mathfrak{H}$ be a Hoffman graph. Let $V_s(\mathfrak{H}) = V_1 \cup V_2$ be a partition, and set $\mathfrak{H}^i = \langle\langle V_i\rangle\rangle_{\mathfrak{H}}$ for $i = 1, 2$. Then $\mathfrak{H} = \mathfrak{H}^1 \uplus \mathfrak{H}^2$ if and only if there are no edges connecting V_1 and V_2 in $\mathcal{S}(\mathfrak{H})$. In particular, $\mathfrak{H}$ is indecomposable if and only if $\mathcal{S}(\mathfrak{H})$ is connected.*

Proof. This is immediate from Definition 2.10(iv) and Definition 3.3. □



$\mathfrak{H}^{(1)}, \lambda_{\min} = -1$ $\mathfrak{H}^{(2)}, \lambda_{\min} = -2$ $\mathfrak{H}^{(3)}, \lambda_{\min} = -3$

Figure 1.

For an integer $t \geq 1$, let $\mathfrak{H}^{(t)}$ be the fat Hoffman graph with one slim vertex and t fat vertices.

Lemma 3.5. *Let t be a positive integer. If $\mathfrak{H}$ is a fat Hoffman graph with $\lambda_{\min}(\mathfrak{H}) \geq -t$ containing $\mathfrak{H}^{(t)}$ as a Hoffman subgraph, then $\mathfrak{H} = \mathfrak{H}^{(t)} \uplus \mathfrak{H}'$ for some subgraph $\mathfrak{H}'$ of $\mathfrak{H}$. In particular, if $\mathfrak{H}$ is indecomposable, then $\mathfrak{H} = \mathfrak{H}^{(t)}$.*

Proof. Let x be the unique slim vertex of $\mathfrak{H}^{(t)}$. Let ψ be a reduced representation of norm t of $\mathfrak{H}$. Then $\psi(x) = 0$, hence x is an isolated vertex in $\mathcal{S}(\mathfrak{H})$. Thus $\mathfrak{H} = \mathfrak{H}^{(t)} \uplus \langle\langle V_s(\mathfrak{H}) \setminus \{x\}\rangle\rangle_{\mathfrak{H}}$ by Lemma 3.4. □

Lemma 3.6. *Let $\mathfrak{H}$ be a fat Hoffman graph with smallest eigenvalue at least -3. Let ψ be a reduced representation of norm 3 of $\mathfrak{H}$. Then for any distinct slim vertices x, y of $\mathfrak{H}$, $(\psi(x), \psi(y)) \in \{1, 0, -1\}$.*

Proof. Since $\mathfrak{H}$ is fat, we have $(\psi(x), \psi(x)) \leq 2$ for all $x \in V_s(\mathfrak{H})$. Thus $|(\psi(x), \psi(y))| \leq 2$ for all $x, y \in V_s(\mathfrak{H})$ by Schwarz's inequality. Equality holds only if $\psi(x) = \pm\psi(y)$ and $(\psi(x), \psi(x)) = 2$. The latter condition implies $|N^f_{\mathfrak{H}}(x)| = 1$, hence $|N^f_{\mathfrak{H}}(x, y)| \leq 1$. Thus $(\psi(x), \psi(y)) \geq -1$, while $(\psi(x), \psi(y)) = 2$ cannot occur unless $x = y$, by Definition 2.6. Therefore, $|(\psi(x), \psi(y))| < 2$, and the result follows. □

Let $\mathfrak{H}$ be a fat Hoffman graph with smallest eigenvalue at least -3. Then by Lemma 3.6, the edge set of the special graph $\mathcal{S}^\epsilon(\mathfrak{H})$ is

$$\{\{x, y\} \mid x, y \in V_s(\mathfrak{H}),\ (\psi(x), \psi(y)) = \epsilon 1\},$$

for $\epsilon \in \{+, -\}$.

Theorem 3.7. *Let $\mathfrak{H}$ be a fat indecomposable Hoffman graph with smallest eigenvalue at least -3. Then every slim vertex has at most three fat neighbors. Moreover, the following statements hold:*

(i) *If some slim vertex has three fat neighbors, then $\mathfrak{H} \cong \mathfrak{H}^{(3)}$.*

(ii) *If no slim vertex has three fat neighbors and some slim vertex has exactly two fat neighbors, then $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3) \simeq \mathbb{Z}^n$ for some positive integer n.*

(iii) *If every slim vertex has a unique fat neighbor, then $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ is an irreducible root lattice.*

Proof. As the smallest eigenvalue is at least -3, every slim vertex has at most three fat neighbors.

If $|N^f_{\mathfrak{H}}(x)| = 3$ for some slim vertex x of $\mathfrak{H}$, then $\mathfrak{H}$ contains $\mathfrak{H}^{(3)}$ as a subgraph. Thus $\mathfrak{H} = \mathfrak{H}^{(3)}$ by Lemma 3.5, and (i) holds. Hence we may assume that $|N^f_{\mathfrak{H}}(x)| \le 2$ for all $x \in V_s(\mathfrak{H})$. Then for each $x \in V_s(\mathfrak{H})$ we have $\|\psi(x)\|^2 = 1$ (resp. 2) if and only if $|N^f_{\mathfrak{H}}(x)| = 2$ (resp. $|N^f_{\mathfrak{H}}(x)| = 1$). Suppose that $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ contains m linearly independent vectors of norm 1. We claim that $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ can be written as an orthogonal direct sum $\mathbb{Z}^m \oplus \Lambda'$, where Λ' is a lattice containing no vectors of norm 1. Indeed, if x is a slim vertex such that $\|\psi(x)\|^2 = 2$ and $\psi(x) \notin \mathbb{Z}^m$, then $\psi(x)$ is orthogonal to $\mathbb{Z}^m$. This implies $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3) = \mathbb{Z}^m \oplus \Lambda'$ and $\psi(V_s(\mathfrak{H})) \subset \mathbb{Z}^m \cup \Lambda'$.

If $m > 0$ and $\Lambda' \neq 0$, then the special graph $\mathcal{S}(\mathfrak{H})$ is disconnected. This contradicts the indecomposability of $\mathfrak{H}$ by Lemma 3.4. Therefore, we have either $m = 0$ or $\Lambda' = 0$. In the latter case, (ii) holds. In the former case, $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3) = \Lambda'$ is generated by vectors of norm 2, hence it is a root lattice. Again by the assumption and Lemma 3.4, (iii) holds. □

We shall see some examples for the case (ii) of Theorem 3.7 in the next section. As for the case (iii), $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ is an irreducible root lattice of type A_n, D_n or E_n. If $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ is not an irreducible root lattice of type E_n, then it can be imbedded into the standard lattice, hence the results of the next section applies. On the other hand, if $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ is an irreducible root lattice of type E_n, then it is contained in the irreducible root lattice of type E_8, and hence there are only finitely many possibilities. For example, Let Γ be any ordinary graph with smallest eigenvalue at least -2 (see [1] for a description of such graphs). Attaching a fat neighbor to each vertex of Γ gives a fat Hoffman graph with smallest eigenvalue at least -3. However, this Hoffman graph may not be maximal among fat Hoffman graphs with smallest eigenvalue at least -3. Therefore, we aim to classify fat Hoffman graphs with smallest eigenvalue at least -3 which are maximal in E_8. This may seem a computer enumeration problem, but it is harder than it looks.

Example 3.8. Let Π denote the root system of type E_8. Fix $\alpha \in \Pi$. Then there are elements $\beta_i \in \Pi$ $(i = 1, \dots, 28)$ such that

$$\{\beta \in \Pi \mid (\alpha, \beta) = 1\} = \bigcup_{i=1}^{28} \{\beta_i, \alpha - \beta_i\}.$$

Let V denote the set of 57 roots consisting of the above set and α. Then V is a reduced representation of a fat Hoffman graph $\mathfrak{H}$ with 29 fat vertices. The fat vertices of $\mathfrak{H}$ are f_i $(i = 0, 1, \dots, 28)$, f_0 is adjacent to α, and f_i is adjacent to $\beta_i, \alpha - \beta_i$ $(i = 1, \dots, 28)$. It turns out that $\mathfrak{H}$ is maximal among fat Hoffman graphs with smallest eigenvalue at least -3. Indeed, no fat vertex can be attached, since the root lattice of type E_8 is generated by $V \setminus \{\gamma\}$ for any $\gamma \in V$, and attaching another fat neighbor to γ would mean the existence of a vector of norm 1 in the dual lattice E_8^* of E_8. Since $\mathsf{E}_8^* = \mathsf{E}_8$, there are no vectors of norm 1 in E_8^*. This is a contradiction. If a slim vertex can be attached, then it can be represented by $\delta \in \Pi$ with $(\alpha, \delta) = 0$. Then there exists $i \in \{1, \dots, 28\}$ such that $(\beta_i, \delta) = \pm 1$. Exchanging β_i with $\alpha - \beta_i$ if necessary, we may assume $(\beta_i, \delta) = -1$. This implies that β_i and δ have a common fat neighbor. Since $(\beta_i, \alpha - \beta_i) = -1$, β_i and $\alpha - \beta_i$ have a common fat neighbor. Since β_i has a unique fat neighbor, δ and $\alpha - \beta_i$ have a common fat neighbor, contradicting $(\delta, \alpha - \beta_i) = 1$.

On the other hand, it is known that there is a slim graph Γ with 36 vertices represented by the root system of type E_8 (see [1]). Attaching a fat neighbor to each vertex of Γ gives a fat Hoffman graph $\mathfrak{H}'$ with smallest eigenvalue -3 such that $\Lambda^{\mathrm{red}}(\mathfrak{H}', 3)$ is isometric to the root lattice of type E_8. The graph $\mathfrak{H}'$ is not contained in $\mathfrak{H}$, so there seems a large number of maximal fat Hoffman graphs represented in the root lattice of type E_8.

4 Integrally represented Hoffman graphs

In this section, we consider a fat (-3)-saturated Hoffman graph $\mathfrak{H}$ such that $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ is a sublattice of the standard lattice $\mathbb{Z}^n$. Since any of the exceptional root lattices $\mathsf{E}_6, \mathsf{E}_7$ and E_8 cannot be embedded into the standard lattice (see [2]), this means that, in view of Theorem 3.7, $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ is isometric to $\mathbb{Z}^n$ or a root lattice of type A_n or D_n. Note that $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ cannot be isometric to the lattice A_1, since this would imply that $\mathfrak{H}$ has a unique slim vertex with a unique fat neighbor, contradicting (-3)-saturatedness. The following example gives a fat (-3)-saturated graph $\mathfrak{H}$ with $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3) \cong \mathbb{Z}$.

Example 4.1. Let $\mathfrak{H}$ be the Hoffman graph with vertex set $V_s(\mathfrak{H}) \cup V_f(\mathfrak{H})$, where $V_s(\mathfrak{H}) = \mathbb{Z}/4\mathbb{Z}$, $V_f(\mathfrak{H}) = \{f_i \mid i \in \mathbb{Z}/4\mathbb{Z}\}$, and with edge set

$$\{\{0,2\},\{1,3\}\} \cup \{\{i, f_j\} \mid i = j \text{ or } j+1\}.$$

Then $\mathfrak{H}$ is a fat indecomposable (-3)-saturated Hoffman graph such that $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ is isomorphic to the standard lattice $\mathbb{Z}$. Since $\mathcal{S}^-(\mathfrak{H})$ has edge set $\{\{i, i+1\} \mid i \in \mathbb{Z}/4\mathbb{Z}\}$, $\mathcal{S}^-(\mathfrak{H})$ is isomorphic to the Dynkin graph $\tilde{\mathsf{A}}_3$. Theorem 4.9 below implies that $\mathfrak{H}$ is maximal, in the sense that no fat indecomposable (-3)-saturated Hoffman graph contains $\mathfrak{H}$.

For the remainder of this section, we let $\mathfrak{H}$ be a fat indecomposable (-3)-saturated Hoffman graph such that $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ is isomorphic to a sublattice of the standard lattice $\mathbb{Z}^n$. Let ϕ be a representation of norm 3 of $\mathfrak{H}$. Then we may assume that ϕ is a mapping from $V(\mathfrak{H})$ to $\mathbb{Z}^n \oplus \mathbb{Z}^{V_f(\mathfrak{H})}$, where its composition with the projection $\mathbb{Z}^n \oplus \mathbb{Z}^{V_f(\mathfrak{H})} \to \mathbb{Z}^n$ gives a reduced representation $\psi : V_s(\mathfrak{H}) \to \mathbb{Z}^n$. It follows from the definition of a representation of norm 3 that

$$\phi(s) = \psi(s) + \sum_{f \in N^f_{\mathfrak{H}}(s)} \mathbf{e}_f,$$

$$\psi(s) = \sum_{j=1}^{n} \psi(s)_j \mathbf{e}_j,$$

$$\psi(s)_j \in \{0, \pm 1\},$$

and

$$|\{j \mid j \in \{1, \dots, n\},\ \psi(s)_j \in \{\pm 1\}\}| = 3 - |N^f_{\mathfrak{H}}(s)| \le 2. \tag{4.1}$$

Lemma 4.2. *If $i \in \{1, \dots, n\}$ and $\psi(s)_i \neq 0$ for some $s \in V_s(\mathfrak{H})$, then there exist $s_1, s_2 \in V_s(\mathfrak{H})$ such that $\psi(s_1)_i = -\psi(s_2)_i = 1$.*

Proof. By way of contradiction, we may assume without loss of generality that $i = n$, and $\psi(s)_n \in \{0, 1\}$ for all $s \in V_s(\mathfrak{H})$. Let $\mathfrak{G}$ be the Hoffman graph obtained from $\mathfrak{H}$ by attaching a new fat vertex g and join it to all the slim vertices s of $\mathfrak{H}$ satisfying $\psi(s)_n = 1$. Then the composition of $\psi : V_s(\mathfrak{H}) = V_s(\mathfrak{G}) \to \mathbb{Z}^n$ and the projection $\mathbb{Z}^n \to \mathbb{Z}^{n-1}$ gives a reduced representation of norm 3 of $\mathfrak{G}$. By Theorem 2.8, $\mathfrak{G}$ has smallest eigenvalue at least -3. This contradicts the assumption that $\mathfrak{H}$ is (-3)-saturated. $\square$

Proposition 4.3. *The graph $\mathcal{S}^-(\mathfrak{H})$ is connected.*

Proof. Before proving the proposition, we first show the following claim.

Claim 4.4. *Let s_1 and s_2 be two slim vertices such that $\phi(s_1)_i = 1$ and $\phi(s_2)_i = -1$ for some $i \in \{1, \ldots, n\}$. Then the distance between s_1 and s_2 is at most 2 in $\mathcal{S}^-(\mathfrak{H})$.*

By (4.1), we have $(\psi(s_1), \psi(s_2)) \in \{0, -1\}$. If $(\psi(s_1), \psi(s_2)) = -1$, then s_1 and s_2 are adjacent in $\mathcal{S}^-(\mathfrak{H})$ by the definition, hence the distance equals one. If $(\psi(s_1), \psi(s_2)) = 0$, then there exists a unique $j \in \{1, \ldots, n\}$ such that $\phi(s_1)_j = \phi(s_2)_j = \pm 1$. From Lemma 4.2, there exists a slim vertex s_3 such that $\phi(s_3)_j = -\phi(s_1)_j$. If $\phi(s_3)_i \neq 0$, then $(\psi(s_q), \psi(s_3)) \in \{\pm 2\}$ for some $q \in \{1, 2\}$, which is a contradiction. Hence $\phi(s_3)_i = 0$. This implies that $(\psi(s_i), \psi(s_3)) = -1$ for $i = 1, 2$, or equivalently, s_3 is a common neighbor of s_1 and s_2 in $\mathcal{S}^-(\mathfrak{H})$. This shows the claim.

Since $\mathfrak{H}$ is indecomposable, $\mathcal{S}(\mathfrak{H})$ is connected by Lemma 3.4. Thus, in order to show the proposition, we only need to show that slim vertices s_1 and s_2 with $(\psi(s_1), \psi(s_2)) = 1$ are connected by a path in $\mathcal{S}^-(\mathfrak{H})$. There exists $i \in \{1, \ldots, n\}$ such that $\phi(s_1)_i = \phi(s_2)_i = \pm 1$. From Lemma 4.2, there exists a slim vertex s_3 such that $\phi(s_3)_i = -\phi(s_1)_i$, and hence the distances between s_1 and s_3 and between s_3 and s_2 are at most 2 in $\mathcal{S}^-(\mathfrak{H})$ by Claim 4.4. Therefore, s_1 and s_2 are connected by a path of length at most 4 in $\mathcal{S}^-(\mathfrak{H})$. □

Lemma 4.5. *Let $\mathfrak{H}$ be a fat indecomposable (-3)-saturated Hoffman graph. Then the reduced representation of norm 3 of $\mathfrak{H}$ is injective unless $\mathfrak{H}$ is isomorphic to a subgraph of the graph given in Example 4.1.*

Proof. Suppose that the reduced representation ψ of norm 3 of $\mathfrak{H}$ is not injective. Then there are two distinct slim vertices x and y satisfing $\psi(x) = \psi(y)$. Then $(\psi(x), \psi(y)) = 0$ or 1.

If $(\psi(x), \psi(y)) = 0$, then $\psi(x) = \psi(y) = 0$, hence both x and y are isolated vertices, contradicting the assumption that $\mathfrak{H}$ is indecomposable.

Suppose $(\psi(x), \psi(y)) = 1$. Since $\mathcal{S}^-(\mathfrak{H})$ is connected by Proposition 4.3, there exists a slim vertex z such that $(\psi(x), \psi(z)) = -1$. Then we may assume

$$\begin{aligned}
\phi(x) &= \mathbf{e}_1 + \mathbf{e}_{f_1} + \mathbf{e}_{f_2}, \\
\phi(y) &= \mathbf{e}_1 + \mathbf{e}_{f_3} + \mathbf{e}_{f_4}, \\
\phi(z) &= -\mathbf{e}_1 + \mathbf{e}_{f_1} + \mathbf{e}_{f_3}.
\end{aligned}$$

If $\mathfrak{H}$ has another slim vertex w, then

$$\phi(w) = -\mathbf{e}_1 + \mathbf{e}_{f_2} + \mathbf{e}_{f_4},$$

and no other possibility occurs. Therefore, $\mathfrak{H}$ is isomorphic to either the graph given in Example 4.1, or its subgraph obtained by deleting one slim vertex. □

Lemma 4.6. *Suppose that $s \in V_s(\mathfrak{H})$ has exactly two fat neighbors in $\mathfrak{H}$. Then the following statements hold.*

(i) *for each $f \in N^f_{\mathfrak{H}}(s)$, $|N_{\mathcal{S}^-(\mathfrak{H})}(s) \cap N^s_{\mathfrak{H}}(f)| \leq 2$,*

(ii) *$|N_{\mathcal{S}^-(\mathfrak{H})}(s)| \leq 4$, and if equality holds, then $\mathcal{S}^-(\mathfrak{H})$ is isomorphic to the graph $\tilde{\mathsf{D}}_4$,*

(iii) *if* $|N_{\mathcal{S}^-(\mathfrak{H})}(s)| = 3$, *then two of the vertices in* $N_{\mathcal{S}^-(\mathfrak{H})}(s)$ *have* s *as their unique neighbor in* $\mathcal{S}^-(\mathfrak{H})$.

Proof. Let ψ be the reduced representation of norm 3 of $\mathfrak{H}$. Since s has exactly two fat neighbors, $(\psi(s), \psi(s)) = 1$. This means that we may assume without loss of generality $\psi(s) = \mathbf{e}_1$.

Let $f \in N^f_{\mathfrak{H}}(s)$. If t_1 and t_2 are distinct vertices of $N_{\mathcal{S}^-(\mathfrak{H})}(s) \cap N^s_{\mathfrak{H}}(f)$, then

$$\begin{aligned} 1 &\geq (\phi(t_1), \phi(t_2)) \\ &\geq (\psi(t_1) + \mathbf{e}_f, \psi(t_2) + \mathbf{e}_f) \\ &= (\psi(t_1), \psi(t_2)) + 1, \end{aligned}$$

Thus we have $(\psi(t_1), \psi(t_2)) \leq 0$. Since $t_1, t_2 \in N_{\mathcal{S}^-(\mathfrak{H})}(s)$, we have $(\psi(s), \psi(t_1)) = (\psi(s), \psi(t_2)) = -1$, and hence we may assume without loss of generality that

$$\psi(t_1) = -\mathbf{e}_1 + \mathbf{e}_2, \tag{4.2}$$

$$\psi(t_2) = -\mathbf{e}_1 - \mathbf{e}_2. \tag{4.3}$$

Now it is clear that there cannot be another $t_3 \in N_{\mathcal{S}^-(\mathfrak{H})}(s)$. Thus $|N_{\mathcal{S}^-(\mathfrak{H})}(s) \cap N^s_{\mathfrak{H}}(f)| \leq 2$. This proves (i).

As for (ii), let $N^f_{\mathfrak{H}}(s) = \{f, f'\}$. Then

$$|N_{\mathcal{S}^-(\mathfrak{H})}(s)| \leq |N_{\mathcal{S}^-(\mathfrak{H})}(s) \cap N^s_{\mathfrak{H}}(f)| + |N_{\mathcal{S}^-(\mathfrak{H})}(s) \cap N^s_{\mathfrak{H}}(f')| \leq 4$$

by (i). To prove (iii) and the second part of (ii), we may assume that $\{t_1, t_2\} = N_{\mathcal{S}^-(\mathfrak{H})}(s) \cap N_{\mathfrak{H}}(f)$. We claim that neither t_1 nor t_2 has a neighbor in $\mathcal{S}^-(\mathfrak{H})$ other than s. Suppose by contradiction, that $t_3 \neq s$ is a neighbor in $\mathcal{S}^-(\mathfrak{H})$ of t_1. By (4.2) (resp. (4.3)), f is the unique fat neighbor of t_1 (resp. t_2). In particular, f is also a neighbor of t_3. Observe

$$\begin{aligned} &1 \geq (\phi(s), \phi(t_3)) \geq (\psi(s), \psi(t_3)) + 1, \\ &1 \geq (\phi(t_i), \phi(t_3)) = (\psi(t_i), \psi(t_3)) + 1 \quad (i = 1, 2). \end{aligned}$$

Thus

$$\begin{aligned} (\mathbf{e}_1, \psi(t_3)) &\leq 0, \\ (-\mathbf{e}_1 \pm \mathbf{e}_2, \psi(t_3)) &\leq 0. \end{aligned}$$

These imply $(\mathbf{e}_1, \psi(t_3)) = (\mathbf{e}_2, \psi(t_3)) = 0$. But then $-1 = (\psi(t_1), \psi(t_3)) = (-\mathbf{e}_1 + \mathbf{e}_2, \psi(t_3)) = 0$. This is a contradiction, and we have proved our claim. Now (iii) is an immediate consequence of this claim.

Continuing the proof of the second part of (ii), if $|N_{\mathcal{S}^-(\mathfrak{H})}(s)| = 4$, then we may assume

$$\begin{aligned} \psi(t'_1) &= -\mathbf{e}_1 + \mathbf{e}_3, \\ \psi(t'_2) &= -\mathbf{e}_1 - \mathbf{e}_3, \end{aligned}$$

where $\{t'_1, t'_2\} = N_{\mathcal{S}^-(\mathfrak{H})}(s) \cap N^s_{\mathfrak{H}}(f')$. It follows that t_1, t_2, t'_1, t'_2 are pairwise non-adjacent in $\mathcal{S}^-(\mathfrak{H})$. By our claim, none of t_1, t_2, t'_1, t'_2 is adjacent to any vertex other than s in $\mathcal{S}^-(\mathfrak{H})$. Since $\mathcal{S}^-(\mathfrak{H})$ is connected by Proposition 4.3, $\mathcal{S}^-(\mathfrak{H})$ is isomorphic to the graph $\tilde{\mathrm{D}}_4$. □

Lemma 4.7. *Suppose that slim vertices* s, t^+, t^- *share a common fat neighbor and that they are represented as*

$$\psi(s) = \mathbf{e}_1 + \mathbf{e}_2,$$
$$\psi(t^\pm) = -\mathbf{e}_1 \pm \mathbf{e}_3.$$

If there exists a slim vertex t with

$$\psi(t) = -\mathbf{e}_2 + \mathbf{e}_j \quad \text{for some } j \notin \{1, 2, 3\},$$

then the vertices $t^\pm$ *have* s *as their unique neighbor in* $\mathcal{S}^-(\mathfrak{H})$.

Proof. Note that each of the vertices $s, t^\pm, t$ has a unique fat neighbor. Since $(\psi(s), \psi(t^\pm)) = (\psi(s), \psi(t)) = -1$, these vertices share a common fat neighbor f. Suppose that there exists a slim vertex t' adjacent to t^- in $\mathcal{S}^-(\mathfrak{H})$. This means $(\psi(t'), \psi(t^-)) = -1$. Since f is the unique fat neighbor of t^-, t' is adjacent to f, and hence $(\psi(t'), \psi(u)) \leq 0$ for $u \in \{t, t^+, s\}$. This is impossible. Similarly, there exists no slim vertex adjacent to t^+ in $\mathcal{S}^-(\mathfrak{H})$. □

Lemma 4.8. *Suppose that* $s \in V_s(\mathfrak{H})$ *has exactly one fat neighbor in* $\mathfrak{H}$. *Then the following statements hold:*

(i) $|N_{\mathcal{S}^-(\mathfrak{H})}(s)| \leq 4$, *and if equality holds, then* $\mathcal{S}^-(\mathfrak{H})$ *is isomorphic to the graph* $\tilde{\mathsf{D}}_4$,

(ii) *if* $|N_{\mathcal{S}^-(\mathfrak{H})}(s)| = 3$, *then two of the vertices in* $N_{\mathcal{S}^-(\mathfrak{H})}(s)$ *have* s *as their unique neighbor in* $\mathcal{S}^-(\mathfrak{H})$.

Proof. Let ψ be the reduced representation of norm 3 of $\mathfrak{H}$. Since s has exactly one fat neighbor, $(\psi(s), \psi(s)) = 2$. This means that we may assume without loss of generality $\psi(s) = \mathbf{e}_1 + \mathbf{e}_2$. Let f be the unique fat neighbor of s. If $t \in N_{\mathcal{S}^-(\mathfrak{H})}(s)$, then t is adjacent to f, hence

$$\psi(t) \in \{-\mathbf{e}_1, -\mathbf{e}_2\} \cup \{-\mathbf{e}_i \pm \mathbf{e}_j \mid 1 \leq i \leq 2 < j \leq n\}. \tag{4.4}$$

If $t, t' \in N_{\mathcal{S}^-(\mathfrak{H})}(s)$ are distinct, then

$$\begin{aligned} 1 &\geq (\phi(t), \phi(t')) \\ &\geq (\psi(t) + \mathbf{e}_f, \psi(t') + \mathbf{e}_f) \\ &= (\psi(t), \psi(t')) + 1, \end{aligned}$$

Thus we have $(\psi(t), \psi(t')) \leq 0$. If $|N_{\mathcal{S}^-(\mathfrak{H})}(s)| \geq 3$, then by (4.4), we may assume without loss of generality that there exists $t \in N_{\mathcal{S}^-(\mathfrak{H})}(s)$ with $\psi(t) = -\mathbf{e}_1 + \mathbf{e}_3$. Then $\psi(N_{\mathcal{S}^-(\mathfrak{H})}(s) \setminus \{t\})$ is contained in

$$\{-\mathbf{e}_2 - \mathbf{e}_3\}, \{-\mathbf{e}_2, -\mathbf{e}_1 - \mathbf{e}_3\}, \text{ or } \{-\mathbf{e}_1 - \mathbf{e}_3\} \cup \{-\mathbf{e}_2 \pm \mathbf{e}_j\}$$

for some j with $3 \leq j \leq n$. Thus $|N_{\mathcal{S}^-(\mathfrak{H})}(s)| \leq 4$, and equality holds only if

$$\psi(N_{\mathcal{S}^-(\mathfrak{H})}(s)) = \{-\mathbf{e}_1 \pm \mathbf{e}_3, -\mathbf{e}_2 \pm \mathbf{e}_j\}$$

for some j with $3 \leq j \leq n$. In this case, Lemma 4.7 implies that each of the vertices in $N_{\mathcal{S}^-(\mathfrak{H})}(s)$ has s as a unique neighbor. This means that $\mathcal{S}^-(\mathfrak{H})$ contains a subgraph isomorphic to the graph $\tilde{\mathsf{D}}_4$ as a connected component. Since $\mathcal{S}^-(\mathfrak{H})$ is connected by Proposition 4.3, we have the desired result.

To prove (ii), suppose $|N_{\mathcal{S}^-(\mathfrak{H})}(s)| = 3$. Then we may assume without loss of generality that $N_{\mathcal{S}^-(\mathfrak{H})}(s) = \{t, t^+, t^-\}$, where $\psi(t^\pm) = -\mathbf{e}_2 \pm \mathbf{e}_4$. Then by Lemma 4.7, the vertices $t^\pm$ have s as their unique neighbor in $\mathcal{S}^-(\mathfrak{H})$. □

Theorem 4.9. *Let $\mathfrak{H}$ be a fat indecomposable (-3)-saturated Hoffman graph such that $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ is isomorphic to a sublattice of the standard lattice $\mathbb{Z}^n$. Then $\mathcal{S}^-(\mathfrak{H})$ is a connected graph which is isomorphic to $\mathsf{A}_m, \mathsf{D}_m, \tilde{\mathsf{A}}_m$ or $\tilde{\mathsf{D}}_m$ for some positive integer m.*

Proof. From Proposition 4.3, $\mathcal{S}^-(\mathfrak{H})$ is connected. First we suppose that the maximum degree of $\mathcal{S}^-(\mathfrak{H})$ is at most 2. Then $\mathcal{S}^-(\mathfrak{H}) \cong \tilde{\mathsf{A}}_m$ or $\mathcal{S}^-(\mathfrak{H}) \cong \mathsf{A}_m$ for some positive integer m.

Next we suppose that the degree of some vertex s in $\mathcal{S}^-(\mathfrak{H})$ is at least 3. From Lemma 4.6(ii) and Lemma 4.8(i), $\deg_{\mathcal{S}^-(\mathfrak{H})}(s) \leq 4$, and $\mathcal{S}^-(\mathfrak{H}) \cong \tilde{\mathsf{D}}_4$ if $\deg_{\mathcal{S}^-(\mathfrak{H})}(s) = 4$. Thus, for the remainder of this proof, we suppose that the maximum degree of $\mathcal{S}^-(\mathfrak{H})$ is 3 and $\deg_{\mathcal{S}^-(\mathfrak{H})}(s) = 3$.

It follows from Lemma 3.5 that if $\mathfrak{H}$ has a subgraph isomorphic to $\mathfrak{H}^{(3)}$, then $\mathfrak{H} \cong \mathfrak{H}^{(3)}$, in which case $\mathcal{S}^-(\mathfrak{H})$ consists of a single vertex, and the assertion holds. Hence it remains to consider two cases: s is adjacent to exactly two fat vertices, and s is adjacent to exactly one fat vertex. In either cases, by Lemma 4.6(iii) or Lemma 4.8(ii), s has at most one neighbor t with degree greater than 1. Thus, the only way to extend this graph is by adding a slim neighbor adjacent to t. We can continue this process, but once we encounter a vertex of degree 3, then the process stops by Lemma 4.6(iii) or Lemma 4.8(ii). Thus, $\mathcal{S}^-(\mathfrak{H})$ is isomorphic to one of the graphs D_m or $\tilde{\mathsf{D}}_m$. □

Example 4.10. Let $n_1, \dots, n_k$ be positive integers satisfying $n_i \geq 2$ for $1 < i < k$. Set $m_j = \sum_{i=1}^{j} n_i$ and $\ell_j = m_j - j$ for $j = 0, 1, \dots, k$. Let $\mathfrak{H}$ be the Hoffman graph with $V_s(\mathfrak{H}) = \{v_i \mid i = 0, 1, \dots, m_k\}$, $V_f(\mathfrak{H}) = \{f_j \mid j = 0, 1, \dots, k+1\}$, and

$$\begin{aligned} E(\mathfrak{H}) =& \{\{v_i, v_{i'}\} \mid 1 \leq j \leq k,\ m_{j-1} < i+1 < i' \leq m_j\} \\ &\cup \{\{v_{m_j-1}, v_{m_j+1}\} \mid 1 < j < k\} \\ &\cup \{\{f_j, v_i\} \mid 1 \leq j < k,\ m_{j-1} \leq i \leq m_j\} \cup \{\{f_0, v_0\}, \{f_{k+1}, v_{m_k}\}\}. \end{aligned}$$

Then $\mathfrak{H}$ is a fat Hoffman graph with smallest eigenvalue at least -3, and $\mathcal{S}^-(\mathfrak{H})$ is isomorphic to the Dynkin graph A_{m_k+1}. Indeed, $\mathfrak{H}$ has a reduced representation ψ of norm 3 defined by

$$\psi(v_i) = \begin{cases} (-1)^j \mathbf{e}_{\ell_j} & \text{if } i = m_j, 0 \leq j \leq k, \\ (-1)^j (\mathbf{e}_{i-j} - \mathbf{e}_{i-j-1}) & \text{if } m_j < i < m_{j+1}, 0 \leq j < k. \end{cases}$$

Moreover, $\mathfrak{H}$ is (-3)-saturated. Indeed, suppose not, and let $\tilde{\mathfrak{H}}$ be a Hoffman graph obtained by attaching a new fat vertex f to $\mathfrak{H}$, and let $\tilde{\psi}$ be a reduced representation of norm 3 of $\tilde{\mathfrak{H}}$. If f is adjacent to v_{m_j} for some $j \in \{0, 1, \dots, k\}$, then v_{m_j} has three fat neighbors in $\tilde{\mathfrak{H}}$, hence $\tilde{\psi}(v_{m_j}) = 0$. This is absurd, since $(\tilde{\psi}(v_{m_j}), \tilde{\psi}(v_{m_j \pm 1})) = -1$. If f is adjacent to v_i with $m_{j-1} < i < m_j$, then $\|\tilde{\psi}(v_i)\| = 1$. Since $(\tilde{\psi}(v_{i-1}), \tilde{\psi}(v_i)) = (\tilde{\psi}(v_{i+1}), \tilde{\psi}(v_i)) =$

-1 while $(\tilde{\psi}(v_{i-1}), \tilde{\psi}(v_{i+1})) = 0$, we may assume $\tilde{\psi}(v_{i\pm1}) = \mathbf{e}_1 \pm \mathbf{e}_2$, $\tilde{\psi}(v_i) = -\mathbf{e}_1$. Then $i+1 < m_j$, and

$$\begin{aligned}
0 &= (\tilde{\psi}(v_{i+2}), \tilde{\psi}(v_{i-1})) \\
&= (\tilde{\psi}(v_{i+2}), \mathbf{e}_1 - \mathbf{e}_2) \\
&= (\tilde{\psi}(v_{i+2}), 2\mathbf{e}_1 - (\mathbf{e}_1 + \mathbf{e}_2)) \\
&= -2(\tilde{\psi}(v_{i+2}), \tilde{\psi}(v_i)) - (\tilde{\psi}(v_{i+2}), \tilde{\psi}(v_{i+1})) \\
&= 1,
\end{aligned}$$

which is absurd.

We note that the graph $\mathcal{S}^+(\mathfrak{H})$ has the following edges:

$$\{\{v_{m_j-1}, v_{m_j+1}\} \mid 1 < j < k\}.$$

Example 4.11. Let $\mathfrak{H}$ be the Hoffman graph constructed in Example 4.10 by setting $n_1 = 1$, $n_2 = 2$, and $n_3 = 1$. Let $\mathfrak{H}_0$ (resp. $\mathfrak{H}_1$) be the Hoffman graph obtained from $\mathfrak{H}$ by identifying the fat vertices f_4 and f_0 (resp. f_4 and f_1), and adding edges $\{v_0, v_2\}, \{v_2, v_4\}$. Then $\mathfrak{H}_0$ and $\mathfrak{H}_1$ are fat (-3)-saturated Hoffman graphs and $\mathcal{S}^-(\mathfrak{H}_i)$ is isomorphic to the Dynkin graph A_5 for $i = 0, 1$. We note that $\mathcal{S}^+(\mathfrak{H}_i)$ has two edges $\{v_0, v_2\}, \{v_2, v_4\}$.

Examples 4.10 and 4.11 indicate that $\mathfrak{H}$ is not determined by $\mathcal{S}^{\pm}(\mathfrak{H})$. We plan to discuss the classification of fat indecomposable (-3)-saturated Hoffman graphs with prescribed special graph in subsequent papers.

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Convex cycle bases

Marc Hellmuth
Center for Bioinformatics, Saarland University
Building E 2.1, D-66041 Saarbrücken, Germany

Josef Leydold *
Institute for Statistics and Mathematics, WU, Augasse 2-6, A-1090 Wien, Austria

Peter F. Stadler
Bioinformatics Group, Department of Computer Science; and Interdisciplinary Center of Bioinformatics, University of Leipzig, Härtelstraße 16-18, D-04107 Leipzig, Germany
Max-Planck-Institute for Mathematics in the Sciences
Inselstraße 22, D-04103 Leipzig, Germany
Fraunhofer Institut f. Zelltherapie und Immunologie
Perlickstraße 1, D-04103 Leipzig, Germany
Inst. f. Theoretical Chemistry, University of Vienna
Währingerstraße 17, A-1090 Wien, Austria
Center for non-coding RNA in Technology and Health, University of Copenhagen
Grønnegårdsvej 3, DK-1870 Frederiksberg, Denmark
Santa Fe Institute, 1399 Hyde Park Rd., Santa Fe, NM 87501, USA

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Abstract

Convex cycles play a role e.g. in the context of product graphs. We introduce convex cycle bases and describe a polynomial-time algorithm that recognizes whether a given graph has a convex cycle basis and provides an explicit construction in the positive case. Relations between convex cycles bases and other types of cycles bases are discussed. In particular we show that if G has a unique minimal cycle bases, this basis is convex. Furthermore, we characterize a class of graphs with convex cycles bases that includes partial cubes and hence median graphs.

Keywords: cycle basis, convex subgraph, isometric subgraph, Cartesian product, partial cubes

Math. Subj. Class.: 05C15, 05C10

* corresponding author
E-mail addresses: marc.hellmuth@bioinf.uni-sb.de (Marc Hellmuth), josef.leydold@wu.ac.at (Josef Leydold), studla@bioinf.uni-leipzig.de (Peter F. Stadler)

1 Introduction and basics

The cycle space $\mathcal{C}(G)$ of a simple, unweighted, undirected graph $G = (V, E)$ consists of all its Eulerian subgraphs (or generalized cycles), i.e., all the subgraphs of G for which every vertex has even degree. It is convenient in this context to interpret subgraphs of G as edge sets. The generalized cycles form a vector space over GF(2) with vector addition $X \oplus Y := (X \cup Y) \setminus (X \cap Y)$ and scalar multiplication $1 \cdot X = X$, $0 \cdot X = \emptyset$, for $X, Y \in \mathcal{C}(G)$. This vector space is generated by the elementary cycles of G, i.e., the connected subgraphs of G for which every vertex has degree 2. A basis $\mathcal{B}$ of the cycle space $\mathcal{C}$ is called a *cycle basis* of $G = (V, E)$ [9]. The dimension of the cycle space is the *cyclomatic number* $\mu(G)$ (or *first Betti number*). For a connected graph we have $\mu(G) = |E| - |V| + 1$. Notice that the cycle space of a graph is the direct sum of the cycle spaces of its 2-connected components.

Cycle bases of graphs have diverse applications in science and engineering. Examples include structural flexibility analysis [27], electrical networks [11], chemical structure storage and retrieval systems [15], scheduling problems [36], graph drawing [33], and biopolymer structures [34, 35]. Surveys and extensive references can be found in [19, 22, 28, 37].

A convexity space $(V, \mathfrak{C})$ [6] consists of a ground set V and a set $\mathfrak{C}$ of subsets of V satisfying

(C1) $\emptyset \in \mathfrak{C}$, $V \in \mathfrak{C}$, and

(C2) $K', K'' \in \mathfrak{C}$ implies $K' \cap K'' \in \mathfrak{C}$.

For a simple, undirected graph G with vertex set V, every set $\mathfrak{P}$ of paths on G defines a convexity space $(V, \mathfrak{C}(\mathfrak{P}))$ in the following way: A set of vertices K is $\mathfrak{P}$-convex, $K \in \mathfrak{C}(\mathfrak{P})$, if and only if, for every path $P \in \mathfrak{P}$ with both end vertices in K, all vertices of P are contained in K. This construction is discussed in detail in [14]. Several special types of paths $\mathfrak{P}$ have been studied in this context, most prominently the set of all paths [5], the set of all triangle paths [8], the set of all induced paths [13], and the set of all shortest paths [39].

We will be concerned here only with the latter definition of convexity, usually known as geodetic convexity, see Section 2 for a formal definition. Geodetically convex cycles play an important role in the theory of Cartesian graphs products and their isometric subgraphs. The absence of convex cycles longer than 4, for example, characterizes semi-median graphs [3]. Such long convex cycles furthermore play a role e.g. in Euler-type inequalities for partial cubes [31].

It appears natural, hence, to investigate whether there is a connection between the cycle space and the (geodetic) convexity space of a graph $G = (V, E)$. Note that the cycle space is defined on the edge set, while the convexity space is defined on the vertex set. Intuitively, this connection is made possible by the fact that induced elementary cycles in G are characterized by either their vertex sets or their edge sets.

Definition 1.1. A *convex cycle basis* of a graph G is a cycle basis that consists of convex elementary cycles.

We briefly consider a generalized definition of convex cycle bases relaxing the requirement for elementary basis cycles in the final section.

Cycle bases with special properties have been investigated in much detail in the literature. Examples include minimum cycle bases [2, 17, 19, 29, 44], (strictly) fundamental cycle bases [20, 32, 38], or (quasi) robust cycle bases [26, 40]. Here, we consider *convex* cycle bases. We show that convex cycle bases are not related to other types of cycle bases, we introduce a polynomial-time algorithm to compute a convex cycle basis for an arbitrary input graph, and we construct a class of graphs with convex cycles bases by means of Cartesian products that in particular includes partial cubes.

2 Geodetic convexity and characterization of convex cycles

For a graph G we denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. Similarly, we write $\mathcal{C}(G)$ for the cycle space of G. An edge that joins vertices x and y is denoted by the unordered pair $\{x, y\}$. The lengths $|P|$ and $|C|$ of a path P and a cycle C in G, respectively, is the number of their edges. For simplicity, we will refer to a path with end vertices u and v as uv-path. The distance $\mathrm{dist}_G(u, v)$ between two vertices u and v of G is the length of a shortest uv-path. It is well known that this distance forms a metric on V. The set of all shortest uv-paths will be denoted by $\mathbb{P}_G[u, v]$. The cardinality of this set, i.e., the number of shortest uv-paths, will be denoted by $S_{uv} = |\mathbb{P}_G[u, v]|$. A modification of Dijkstra's algorithm computing both the distance matrix of G and the matrix S is given in the appendix.

A subgraph H of G is *isometric* if $\mathrm{dist}_H(u, v) = \mathrm{dist}_G(u, v)$ holds for all $u, v \in V(H)$. We say that H is a *(geodetically) convex* subgraph of G if for all $u, v \in V(H)$, all shortest uv-paths $P \in \mathbb{P}_G[u, v]$ are contained in H. In the following, convex will always mean geodetically convex. The empty subgraph will be considered as convex. The intersection of convex subgraphs of G is again a convex subgraph of G [42].

Since H is an isometric subgraph of G if and only if H contains at least one $P \in \mathbb{P}_G[u, v]$ for every pair $u, v \in V(H)$, we see that convex implies isometric. Furthermore, if H is an isometric subgraph of G, it is in particular an induced subgraph of G. Finally, the connectedness of G implies that all its isometric subgraphs are connected.

Our first result characterizes elementary convex cycles.

Lemma 2.1. *Let C be an elementary cycle of G.*
If $|C|$ is odd, then C is convex if and only if for every edge $e = \{x, y\}$ in C there is a unique vertex z in C such that $\mathrm{dist}_G(x, z) = \mathrm{dist}_G(y, z) = (|C| - 1)/2$ and $S_{xz} = S_{yz} = 1$.
If $|C|$ is even, then C is convex if and only if for every edge $e = \{x, y\}$ in C there is a unique edge $h = \{u, v\}$ in C such that (i) $\mathrm{dist}_G(x, u) = \mathrm{dist}_G(y, v) = |C|/2 - 1$, (ii) $\mathrm{dist}_G(x, v) = \mathrm{dist}_G(y, u) = |C|/2$, (iii) $S_{xu} = S_{yv} = 1$, and (iv) $S_{xv} = S_{yu} = 2$.

Proof. Suppose C is convex. Consider two vertices p and q in C with $\mathrm{dist}_G(p, q) < |C|/2$. If C is convex, then the unique shortest path between p and q must run along C, so that $S_{pq} = 1$. Clearly, this condition characterizes convex cycles provided C is odd.

The situation is more complicated for even cycles. Let us first suppose that C is convex and fix an arbitrary edge $\{x, y\}$. In an even elementary cycle there is a unique edge $h = \{u, v\}$ satisfying (i) $\mathrm{dist}_C(x, u) = \mathrm{dist}_C(y, v) = |C|/2 - 1$, (ii) $\mathrm{dist}_C(x, v) = \mathrm{dist}_C(y, u) = |C|/2$. Isometry of C implies that properties (i) and (ii) are satisfied. The argument of the preceding paragraph shows that (iii) holds. For x, the only point in C at distance $|C|/2$ is v. Thus there are two paths P' and P'' in C of length $\mathrm{dist}_G(x, v) = |C|/2$. By the convexity of C, these paths are contained in C (so that $C = P' \cup P''$) and must be the only shortest paths connecting x and v; hence consequently $S_{xv} = 2$. An analogous argument shows that $S_{yu} = 2$.

In order to prove the converse, consider an even elementary cycle C satisfying (i) through (iv). Again we fix an arbitrary edge $\{x, y\}$ of C. Since C is even, there is a unique antipodal point v of x and a unique antipodal point u of y with $\mathrm{dist}_C(x, v) = \mathrm{dist}_C(y, u) = |C|/2$. We claim that $\{u, v\}$ is the required edge. If this were not the case, then there would be some other edge with both endpoints closer to x along C than v that satisfies condition (ii). This is impossible, however, since for such a vertex v' we would have $|C|/2 = \mathrm{dist}_G(x, v') \leq \mathrm{dist}_C(x, v') < \mathrm{dist}_C(x, v) = |C|/2$. We easily check that $\mathrm{dist}_C(x, u) = |C|/2 - 1$ and $\mathrm{dist}_C(y, v) = |C|/2 - 1$. By property (i), therefore, the paths from x to u and from y to v along C are shortest paths in G. Furthermore, the two paths from x to v along C via either u or y are also shortest paths in G by property (ii). Thus $\mathrm{dist}_C(x, q) = \mathrm{dist}_G(x, q)$ for all vertices q in C. Repeating this argument for all x in C shows that C is isometric. By property (iii), the shortest path from x to u is unique. Since all sub-paths of shortest paths are again shortest path, this is also true for all vertices q in C along the shortest path from x to u. The same is true for all q in C along the unique shortest path from v to y. By property (iv), finally, there are exactly two shortest paths from x to v. We have already seen that two of these run along either half of the cycle C. The same is true for the two paths connecting y with u. Thus all shortest path connecting a vertex q in C with either x or y are contained in C. Repeating the argument for all edges $\{x, y\}$ in C shows that C is convex. □

A direct consequence of Lemma 2.1 is that a cycle C in G can be efficiently tested for convexity provided both the distance matrix and the matrix S containing the number of shortest paths have been pre-computed: it suffices to verify, in constant time, the conditions of the lemma for each antipodal pair of edges or pair of edge and vertex, respectively. The test thus requires $\mathcal{O}(|C|)$ time provided that C is given as ordered list of its vertices.

As a simple corollary of Lemma 2.1 we have

Corollary 2.2. *Let C be an elementary convex cycle of G. Then, for every $e = \{x, y\} \in C$ there is a vertex z in C such that $C = P' \cup P'' \cup \{x, y\}$, $P' \in \mathbb{P}_G[x, z]$, and $P'' \in \mathbb{P}_G[y, z]$.*

A closely related, but much weaker, condition appears in the theory of minimal cycles bases [22]:

Definition 2.3. A cycle C is *edge-short* if it contains an edge $e = \{x, y\}$ and a vertex z such that $C = C_{xy,z} := \{x, y\} \cup P_{xz} \cup P_{yz}$ where P_{xz} and P_{yz} are shortest paths.

Corollary 2.4. *If C is an elementary convex cycle of G then it is edge-short.*

3 Convex cycle bases

Corollary 2.2 sets the stage for enumerating all elementary convex cycles in a graph. The following result establishes an upper bound and provides a polynomial time algorithm for this purpose.

Theorem 3.1. *Any graph $G = (V, E)$ contains at most $|E||V|$ elementary convex cycles. These can be constructed and listed in $\mathcal{O}(|E||V|^2)$ time.*

Proof. Every pair of an edge $e = \{x, y\}$ and a vertex z specifies at most one elementary convex cycle in the following way: If $\mathrm{dist}_G(x, z) = \mathrm{dist}_G(y, z)$ and $S_{xz} = S_{yz} = 1$ we set $C_{ez} := P_{xz} \cup P_{yz} \cup \{x, y\}$. If $\mathrm{dist}_G(x, z) = \mathrm{dist}_G(y, z) + 1$, $S_{xz} = 2$ and $S_{yz} = 1$, then we choose a neighbor u of z such that $\mathrm{dist}_G(x,u) = \mathrm{dist}_G(y,z)$, $S_{ux} = 1$ and $S_{uy} = 2$, and set $C_{ez} := P_{xu} \cup \{u,z\} \cup P_{yz} \cup \{x,y\}$. Note that the choice of u is unique if C is convex. The selection of these $|E|\,|V|$ candidates thus requires $\mathcal{O}(|E|\,|V|^2)$ time.

In order to efficiently retrieve each candidate cycle in $\mathcal{O}(|C|)$ time given $\{x,y\}$ and z we need the to know the predecessor π_{su} of u on the shortest path from s to u. Note that this information is needed only if $S_{su} = 1$. The modified Dijkstra algorithm in the Appendix computes this array without changing the asymptotic complexity of the shortest path algorithm. Since each candidate cycle can then be checked for convexity in $\mathcal{O}(|C|)$ time, the total effort to extract all elementary convex cycles is in $\mathcal{O}(|E||V|^2)$. □

This algorithm outlined in the proof of Theorem 3.1 can be regarded as a variant of Vismara's construction of prototypes of candidates for relevant cycles [44]. The fact that the number of elementary convex cycles in G is bounded by $|V|\,|E|$ immediately implies that a convex cycle basis can also be found in polynomial time:

Corollary 3.2. *For each graph $G = (V,E)$ it can be decided whether G has a convex cycle basis, and if so, a convex cycle bases can be constructed, in $\mathcal{O}(|E|^2\,|V|\,\mu(G)^2)$ time.*

Proof. Since the cycles of a graph form a matroid, the canonical greedy algorithm can be applied to find a maximum set of linearly independent elementary convex cycles, see e.g. [21]. G has a convex cycle basis if and only if this set has size $\mu(G) = |E| - |V| + 1$. For each of the at most $|V|\,|E|$ candidate cycles, this requires a test of linear independence with a partial basis that is not larger than $\mu(G) = |E| - |V| + 1$, i.e., $\mathcal{O}(|E|)$. Applying Gaussian elimination for this purpose, the total effort is bounded by $\mathcal{O}(|E|\,|V|^2) + \mathcal{O}(|E|^2\,|V|\mu(G)^2) = \mathcal{O}(|E|^2\,|V|\mu(G)^2)$ time. □

There are graphs that do not have a convex cycle basis. The complete bipartite graph $K_{2,3}$ is the simplest counter example (see Fig. 1). None of its three cycles (all have length 4) is convex.

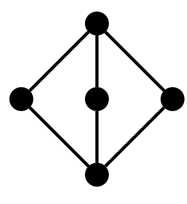

Figure 1: None of the cycles in $K_{2,3}$ is convex.

4 Relation of convex cycle bases to other types of cycle bases

Although we have an efficient algorithm to test whether a graph has a convex cycle basis, it will be interesting to characterize the class of graphs that admit convex cycle bases. However, we first investigate the relation between convex cycle bases and other types of cycle bases.

A procedure analogous to Corollary 3.2 was introduced in [22] for the purpose of retrieving minimal cycle bases from a candidate set of edge-short cycles. One would expect, therefore, that convex cycle bases and minimal cycles bases are closely related.

Convex cycle bases of a graph need not have the same length. Consider the graph that is obtained from the cube Q_3 where one edge is contracted. Then the four quadrangles and two triangles are convex and five of these form a convex cycle basis. Thus convex bases contain either exactly one or two triangles and thus may have different lengths.

The length $\ell(\mathcal{B})$ of a cycle basis $\mathcal{B}$ is the sum of the lengths of its generalized cycles: $\ell(\mathcal{B}) = \sum_{C\in\mathcal{B}} |C|$. A *minimum cycle basis* $\mathcal{M}$ is a cycle basis with minimum length. The generalized cycles in $\mathcal{M}$ are chord-less cycles (see [22]). Hence, we may consider elementary cycles instead of generalized cycles in the remaining part of this section. For the sake of completeness we note that a minimum cycle basis is a cycle basis in which the longest cycle has the minimum possible length [10].

The set $\mathcal{R}$ of *relevant cycles* of a graph is the union of its minimum cycle bases [41, 44]. In analogy to convex cycle bases one may want to consider *isometric cycle bases*, i.e., cycle bases consisting of isometric cycles.

Lemma 4.1. *All relevant cycles of a graph are isometric. Thus every minimal cycle basis is an isometric cycle basis.*

Proof. We start from Lemma 2 of [44]: If P is a subpath of a relevant cycle C such that $|P| \leq \frac{1}{2}|C|$, then P is a shortest path. It follows that every relevant cycle is isometric, and hence every minimal cycle basis of G consists of elementary isometric cycles. □

Theorem 4.2. *If G has a uniquely defined minimal cycle basis, then this minimal cycle basis is convex.*

Proof. Assume that G has a unique minimal cycle basis $\mathcal{B}$. By Lemma 4.1 the cycles of $\mathcal{B}$ are necessarily isometric. Now suppose that $C \in \mathcal{B}$ is not convex. Then there exist two vertices $u, v \in C$ and (at least) three edge disjoint uv-paths P, P' and P'' such that $|P| \geq |P'| = |P''|$ and $C = P \cup P'$. Hence there are two cycles $C_1 = P \cup P''$ and $C_2 = P' \cup P''$ with $|C| = |C_1| \geq |C_2|$. By construction C, C_1, and C_2 are linearly dependent and thus one of C_1 or C_2 cannot be represented as sum of cycles in $\mathcal{B} \setminus \{C\}$. Hence we get a new cycle basis $\mathcal{B}' = (\mathcal{B} \setminus \{C\}) \cup \{C'\}$ where C' is either C_1 or C_2. In either case we find $\ell(\mathcal{B}') \leq \ell(\mathcal{B})$ a contradiction to our assumption that $\mathcal{B}$ is the unique minimal cycle basis. □

As a consequence, we can conclude that Halin graphs that are not necklaces [43] and outerplanar graphs [35] have a convex cycle basis.

The converse of Theorem 4.2 is not true, however, as Figure 2 shows. This graph has a convex cycle basis but its minimal cycle basis is not uniquely defined. Even worse, none of its minimal cycle bases is convex.

A cycle basis $\mathcal{B} = \{C_1, \dots, C_{\mu(G)}\}$ of G is called *fundamental* [20, 46] if there is an ordering π such that for $2 \leq k \leq \mu(G)$:

$$C_{\pi(k)} \setminus \left(\bigcup_{j=1}^{k-1} C_{\pi(j)} \right) \neq \emptyset \; . \tag{4.1}$$

Fundamental cycle bases are obtained from ear decomposition, suggesting that there could be a relation between convex and fundamental cycles bases. Champetier's graph [4], however, has a cycle basis consisting entirely of triangles, which obviously is convex. On the other hand, this basis is not fundamental [1]. Conversely, fundamental cycle bases need not be convex, as shown, e.g., by the planar basis of $K_{2,3}$.

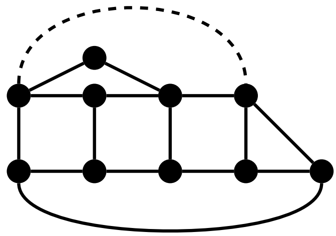

Figure 2: The cyclomatic number of the graph is 7. All minimal cycle bases consist of the two triangles, all quadrangles that do not contain the upper dashed edge and two of the three quadrangles that contain the upper dashed edge (which also includes the outer cycle). However, two of these three quadrangles that contain the upper dashed edge are not convex. Hence none of the minimal cycle bases is a convex cycle basis.
On the other hand there is a unique convex cycle basis that consists of all triangles, all quadrangles that do not contain the upper dashed edge, the outer quadrangle and the cycle of length 5 at the bottom.

5 Convexity in subgraphs and intersections

This section contains some auxiliary results which we will need for our investigation of isometric subgraphs in Section 6 below.

Lemma 5.1. *Let M be an isometric (convex) subgraph of G and $F \subseteq M$ be a subgraph of M. Then F is isometric (convex) in M if and only if it is isometric (convex) in G.*

Proof. If F is an isometric subgraph of G, then for each pair of vertices $u, v \in V(F)$, F contains a shortest uv-path. Since $F \subseteq M$, this path is also a shortest uv-path in M and hence F is isometric in M. If F is a convex subgraph of G, then it contains all shortest uv-paths which are also shortest paths in M and thus F is convex in M.

Now assume that F is not isometric in G. Then there exist two distinct vertices $u, v \in V(F) \subseteq V(M)$ such that there are shortest uv-paths P in G with $|P| < \mathrm{dist}_F(u, v)$. At least one of these paths must be contained in M since M is an isometric subgraph of G. Thus F cannot be an isometric subgraph of M, either. If F is not convex in G then there exist two distinct vertices $u, v \in V(F) \subseteq V(M)$ such that there is at least one shortest uv-path P which is not contained in F. Since M is convex, P must be contained in M and thus F cannot be convex in M, either. □

Lemma 5.2. *Let M be an isometric subgraph of G and F be a convex subgraph of G. Then $F \cap M$ is convex in M.*

Proof. For each pair of vertices $x, y \in V(F) \cap V(M)$, F contains all shortest xy-path in G. Since M is an isometric subgraph of G it must contain at least one of these and thus the proposition follows. □

Lemma 5.3. *Assume that G has a convex cycle basis. Let M be a convex subgraph of G that has a convex cycle basis $\mathcal{B}_M$. Then $\mathcal{B}_M$ can be extended to a convex cycle basis $\mathcal{B}_G$ of G.*

Proof. By Lemma 5.1 the cycles in $\mathcal{B}_M$ are convex subgraphs of G. By assumption there exists a convex cycle basis $\mathcal{B}'_G$ of G. By the *Austauschsatz* we can replace appropriate cycles in $\mathcal{B}'_G$ by the cycles in $\mathcal{B}_M$. Thus we obtain a convex cycle basis $\mathcal{B}_G$ of G which such that $\mathcal{B}_M \subseteq \mathcal{B}_G$ as claimed. □

Figure 3 shows that the converse of this lemma is not true in general: a convex subgraph of a graph that has a convex cycle basis need not necessarily have a convex cycle basis.

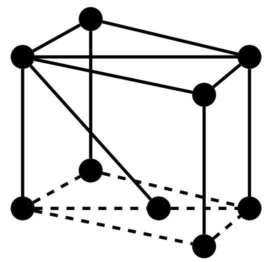

Figure 3: The cyclomatic number of this graph is $|E| - |V| + 1 = 16 - 9 + 1 = 8$. The three triangles and the five quadrangles that do not entirely consist of dashed edges form a convex cycle basis. The subgraph that consists of the dashed edges is convex but does not have a convex cycle basis (see Fig. 1).

6 Isometric subgraphs of Cartesian products

In this section, we will be concerned with the Cartesian product $G\Box H$ and its isometric and convex subgraphs. The Cartesian product has vertex set $V(G\Box H) = V(G) \times V(H)$; two vertices (x_G, x_H) and (y_G, y_H) are adjacent in $G\Box H$ if $\{x_G, y_G\} \in E(G)$ and $x_H = y_H$, or $\{x_H, y_H\} \in E(H)$ and $x_G = y_G$. For detailed information about product graphs we refer the interested reader to [18, 24].

For the Cartesian product $G\Box H$ the subgraph G^v induced by all vertices (x, v) with $x \in V(G)$ and a fixed vertex $v \in V(H)$ is called a *layer* of G (or G-layer) in $G\Box H$. The projection $\pi_G\colon G\Box H \to G$ is the usual weak homomorphism defined as $(x, y) \in V(G\Box H) \mapsto x \in V(G)$. Note that edges in G-layers are mapped into edges in G and edges in H-layers are mapped into vertices in G.

There is a close relationship between (geodetic) convexity and Cartesian products, see [7] for a general result. The fundamental result for this purpose is the distance lemma.

Proposition 6.1 (Distance Lemma, [23])**.** *Let $x = (x_G, x_H)$ and $y = (y_G, y_H)$ be arbitrary vertices of the Cartesian product $G\Box H$. Then*

$$\mathrm{dist}_{G\Box H}(x, y) = \mathrm{dist}_G(x_G, y_G) + \mathrm{dist}_H(x_H, y_H)\ .$$

Moreover, if P is a shortest xy-path in $G\Box H$, then $\pi_G(P)$ is a shortest $x_G y_G$-path in G.

It seems natural that convexity properties of products also hold for layers and projections.

Lemma 6.2 ([24])**.** *The layers G^v and H^w are convex subgraphs of the Cartesian product $G\Box H$. Moreover, if F is an isometric (convex) subgraph of $G\Box H$, then for all $v \in V(H)$*

and $w \in V(G)$ the following holds: $F \cap G^v$ and $F \cap H^w$ are isometric (convex) subgraphs of F, G^v and H^w, respectively.

Corollary 6.3. *Let M be an isometric subgraph of $G\Box H$. If (x_G, x_H) and (y_G, y_H) are two vertices in M with $x_G = y_G$, then there exists a shortest $x_H y_H$-path in $M \cap H^{x_G}$. Moreover, all shortest $(x_G, x_H)(y_G, y_H)$-paths in M are contained in H^{x_G}.*

Another consequence of the distance lemma is the following auxiliary result.

Lemma 6.4. *Let P be a shortest xy-path in $G\Box H$. Then $\pi_G(P)$ is a path with $|\pi_G(P)| = \operatorname{dist}_G(x_G, y_G) = \sum_{v\in H} |P \cap G^v|$, where the last term is the total number of edges of P in G-layers. The result holds analogously for $\pi_H(P)$.*

Proof. If (w, x_H) and (w, y_H) are two distinct points of P, then by Corollary 6.3 all shortest $x_H y_H$-paths are contained in layer H^w. Consequently, there cannot be two distinct edges e_1 and e_2 in G with $\pi_G(e_1) = \pi_G(e_2)$ that belong to P since otherwise P also must contain two shortest paths in different H-layers that connect corresponding vertices of these edges, that is, P would contain a cycle. Hence all vertices of $\pi_G(P)$ have degree 2 except its end vertices which have degree 1 (or 0 in the case where $\pi_G(P)$ is a single vertex). Thus $\pi_G(P)$ is path of length $|\pi_G(P)| = \operatorname{dist}_G(x_G, y_G) = \sum_{v\in H} |P \cap G^v|$, as claimed. □

Lemma 6.5. *For every isometric (convex) subgraph F of $G\Box H$, $\pi_G(F)$ is an isometric (convex) subgraph of G.*

Proof. Let $x = (x_G, x_H)$ and $y = (y_G, y_H)$ be two vertices in F. If F is isometric in $G\Box H$, then there exists a shortest xy-path P in F. By the distance lemma, $\pi_G(P)$ is a shortest $x_G y_G$-path in G and contained in $\pi_G(F)$. Thus $\pi_G(F)$ is an isometric subgraph of G. Now if $\pi_G(F)$ is not convex in G, then there exists a shortest $x_G y_G$-path P_G in G that is not contained in $\pi_G(F)$. Let P_H be a shortest $x_H y_H$-path in H. Then $P = P_G\Box\{x_h\} \cup P_H\Box\{y_G\}$ is a shortest xy-path as its length is $|P| = \operatorname{dist}_G(x_G, y_G) + \operatorname{dist}_H(x_H, y_H) = \operatorname{dist}_{G\Box H}(x, y)$. However, by construction P cannot be contained in F and hence F is not convex in $G\Box H$. Consequently, if F is convex in $G\Box H$, then $\pi_G(F)$ is convex in G, as claimed. □

On the other hand convexity and isometry properties of factors are also propagated to their Cartesian product. The following result is well-known and holds for more general notions of convexity.

Lemma 6.6 ([7])**.** *If F and M are convex subgraphs of G and H, respectively, then $F\Box M$ is a convex subgraph of $G\Box H$.*

The last lemma also holds for isometric subgraphs.

Lemma 6.7. *If F and M are isometric subgraphs of G and H, respectively, then $F\Box M$ is an isometric subgraph of $G\Box H$.*

Proof. Immediate corollary of the distance lemma. □

We now want to extend convex cycle bases of two graphs G and H to a cycle basis of their Cartesian product $G\Box H$. Let T_G and T_H denote spanning trees of G and H, respectively. Let

$$\mathcal{B}_\Box = \{e\Box f \colon e \in E(G),\ f \in T_H\} \cup \{e\Box f \colon e \in T_G,\ f \in E(H)\}\ . \tag{6.1}$$

Then for fixed vertices $v \in V(H)$ and $w \in V(G)$ and respective cycle basis $\mathcal{B}_G$ and $\mathcal{B}_H$

$$\{C^v : C \in \mathcal{B}_G\} \cup \{C^w : C \in \mathcal{B}_H\} \cup \mathcal{B}_\square \tag{6.2}$$

is a cycle basis of $G\square H$ [25].

Theorem 6.8. *Let G and H be two graphs that have convex cycle bases $\mathcal{B}_G$ and $\mathcal{B}_H$, respectively. Then their Cartesian product $G\square H$ has a convex cycle basis that can be constructed using Eq. (6.2).*

Proof. Notice that all quadrangles in $\mathcal{B}_\square$ are convex subgraphs in $G\square H$. By Lemma 5.1 C^v is a convex cycle in $G\square H$. Thus we get a convex cycle basis of $G\square H$ by means of basis (6.2) when both $\mathcal{B}_G$ and $\mathcal{B}_H$ are convex cycle basis. □

Remark 6.9. An analogous statement for the strong product (see [18]) is not true, as the strong product of an elementary cycle and an edge K_2 shows.

We have seen in Figure 3 that a convex subgraph of a graph that has a convex cycle basis does not necessarily have a convex cycle basis. However, a more restrictive property appears to propagate under the formation of Cartesian products: we consider the class of graphs for which every convex subgraph has a convex cycles basis.

Theorem 6.10. *Let G be a graph that has a convex cycle basis. Then every isometric subgraph M of $G\square K_2$ with $\pi_G(M) = G$ has a convex cycle basis.*

For the proof of this theorem we need some intermediate results.

Lemma 6.11. *Let C be an isometric elementary cycle in $G\square H$. Then one of the following holds:*

(1) $\pi_G(C) \cong K_1$, i.e., a single vertex, or

(2) $\pi_G(C) \cong K_2$, i.e., a single edge, or

(3) $\pi_G(C)$ is an isometric elementary cycle in G.

Proof. Notice that $\pi_G(C) = \bigcup_{v\in V(H)} \pi_G(C \cap G^v)$. Let $x = (x_G, x_H)$ and $y = (y_G, y_H)$ be two vertices in C with $x_G = y_G$ and $x_H \neq y_H$. Analogously to the proof of Lemma 6.4 no vertex v in $\pi_G(C)$ can have degree greater than 2. Now if $C \subseteq H^w$ for some $w \in V(G)$, then $\pi_G(C) = \{w\} \cong K_1$, i.e., case (1). If there is a vertex x where $\pi_G(x)$ has degree 1, then there exist two distinct vertices $u, v \in V(H)$ such that $\pi_G(C \cap G^u)$ and $\pi_G(C \cap G^v)$ have a common edge e. However, this only can happen if $\pi_G(C) = \{e\} \cong K_2$, i.e., for case (2). Otherwise, there would be two vertices y' and y'' in C so that $\pi_G(y') = \pi_G(y'')$ is adjacent to $\pi_G(x)$ with vertex degree larger than 1 in the projection, contradicting isometry of C. If we have neither case (1) nor case (2), then all vertices of $\pi_G(C)$ have degree 2 and hence $\pi_G(C)$ is an elementary cycle which is isometric in G by Lemma 6.5, i.e., case (3). □

Now let C be an elementary cycle in G and M be an isometric subgraph of $G\square K_2$. Let $\mathcal{Z}(C, M)$ denote the set of elementary cycles $C' \subseteq M$ that are convex in M and satisfy $\pi_G(C') = C$. We set $\mathcal{Z}(C, M) = \emptyset$ if no such cycle exists.

Lemma 6.12. *Let M be an isometric subgraph of $G\Box K_2$ and let $C \in G$ be a convex elementary cycle with $C \subseteq \pi_G(M)$. Then $\mathcal{Z}(C, M)$ is non-empty.*

Proof. First notice that $C\Box K_2$ is a convex subgraph of $G\Box K_2$ by Lemma 6.6. $M' = M \cap (C\Box K_2)$ is isometric in $C\Box K_2$ by Lemma 5.1 and convex in M by Lemma 5.2. Let M^1 and M^2 denote the respective intersections of M' with the two K_2-layers of $C\Box K_2$. If $M^1 \cong C$ (or $M^2 \cong C$) then M^1 (M^2) is a convex elementary cycle in $C\Box K_2$ by Lemma 6.2, and thus also in M' by Lemma 5.2. Otherwise, both M^1 and M^2 are paths of length $|M^i| \leq \frac{1}{2}|C|$ for $i = 1, 2$, since M' is isometric. As $\pi_C(M') = C$, $\pi_C(M^1) \cup \pi_C(M^2) = C$. Consequently, as M' is isometric, M' is an elementary cycle that is trivially convex in M'. In all cases $\mathcal{Z}(C, M')$ is non-empty. Since Lemma 5.1 implies that $\mathcal{Z}(C, M') \subseteq \mathcal{Z}(C, M)$, the proposition follows. □

Remark 6.13. The arguments in the proof of Lemma 6.12 together with the distance lemma also show that the elements of $\mathcal{Z}(C, M)$ form the set of all shortest cycles C' in M with the property $\pi_C(C') = C$.

Proof of Theorem 6.10. Let $\mathcal{B}_G$ be a convex cycles basis of G. Let $\mathcal{B}_\Box$ be as in (6.1) and define $\mathcal{B}_\mathcal{Z}$ be a set of cycles that contains exactly one cycle $C' \in \mathcal{Z}(C, M)$ for each $C \in \mathcal{B}_G$. By Lemma 6.12 all these sets $\mathcal{Z}(C, M)$ are non-empty. Clearly, the cycles in $\mathcal{B}_\Box \cup \mathcal{B}_\mathcal{Z}$ are linearly independent and thus form a cycle basis of $G\Box K_2$. Now let $\mathcal{B}_M$ be the set of all cycles in $\mathcal{B}_\Box \cup \mathcal{B}_\mathcal{Z}$ that are contained in M. By construction all cycles in $\mathcal{B}_M$ are convex subgraphs of M and $\mathcal{B}_\mathcal{Z} \subseteq \mathcal{B}_M$. Thus it remains to show that $|\mathcal{B}_M| = \mu(M)$. Let $\bar{m}_G$ and $\bar{m}_{K_2}$ denote the numbers of edges in $(G\Box K_2) \setminus M$ that lie in G-layers and K_2-layers, respectively. Let $\bar{n}$ be the number of vertices in $(G\Box K_2) \setminus M$. Since $\pi_G(M) = G$ and M is an isometric subgraph of $G\Box K_2$ we find that $\bar{m}_{K_2} = \bar{n}$. Thus $\mu(M) = (|E(G\Box K_2)| - \bar{m}_G - \bar{m}_{K_2}) - (|V(G\Box K_2)| - \bar{n}) + 1 = |E(G\Box K_2)| - |V(G\Box K_2)| + 1 - \bar{m}_G = \mu(G\Box K_2) - \bar{m}_G$. On the other hand, there are exactly $\bar{m}_G$ cycles in $\mathcal{B}_\Box$ that are not contained in M and hence $|\mathcal{B}_M| = \mu(G\Box K_2) - \bar{m}_G = \mu(M)$, i.e., $\mathcal{B}_M$ is a cycle basis of M. This finishes the proof of the theorem. □

We easily can generalize Theorem 6.10 to arbitrary isometric subgraphs of $G\Box K_2$.

Theorem 6.14. *Let G be a graph such that every isometric subgraph has a convex cycle basis. Then every isometric subgraph of $G\Box K_2$ also has a convex cycle basis.*

Proof. Let H be an isometric subgraph of $G\Box K_2$. By Lemma 6.5, $G' = \pi_G(H)$ is an isometric embedding into G and thus has a convex cycle basis by our assumptions. Hence by Theorem 6.10 every isometric subgraph M of $G'\Box K_2 \subseteq G\Box K_2$ has a convex cycle basis. □

Theorem 6.14 has quite strong implications. A d-dimensional *hypercube* is the d-fold product of K_2 by itself, $Q_d = \Box_{i=1}^d K_2$. *Partial cubes* are isometric subgraphs of Q_d and form a very rich graph class that contains hypercubes, trees, median graphs, tope graphs of oriented matroids, benzenoid graphs, tiled partial cubes, netlike partial cubes, and flip graphs of point sets that have no empty pentagons; see [30, 31] and references therein. As K_2 has a convex cycle basis (namely $\emptyset$) we immediately obtain the following results by a recursive application of Theorem 6.14.

Theorem 6.15. *Partial cubes have a convex cycle basis.*

Theorem 6.16. *Let G be a graph such that every isometric subgraph has a convex cycle basis and let Q be any partial cube. Then every isometric subgraph of $G\Box Q$ has a convex cycle basis.*

Proof. Let G be as claimed. Theorem 6.14 implies that every isometric subgraph of $G\Box K_2\Box\cdots\Box K_2 = G\Box Q_n$ has a convex cycle basis. Lemma 6.7 implies that $G\Box Q$ is an isometric subgraph of $G\Box Q_n$. Moreover, Lemma 5.1 implies that every isometric subgraph of $G\Box Q$ is an isometric subgraph of $G\Box Q_n$ and thus, has a convex cycles basis. $\Box$

Figure 4 shows that the class covered by Theorem 6.16 is much larger than the class of partial cubes. Recall that partial cubes are characterized by the so-called *Djoković-Winkler-Relation* Θ: Two edges $e = \{u, v\}$ and $f = \{x, y\}$ are in relation Θ, $(ef) \in \Theta$, if $\mathrm{dist}(u, x) + \mathrm{dist}(v, y) \neq \mathrm{dist}(u, y) + \mathrm{dist}(v, x)$. A graph is a partial cube if and only if it is bipartite and the relation Θ is an equivalence relation [47].

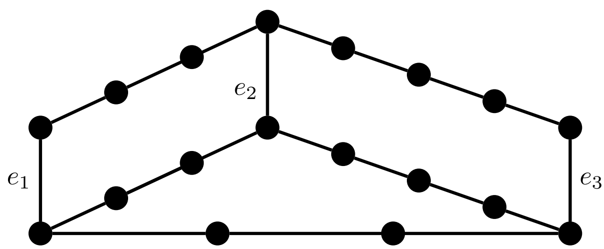

Figure 4: Observe that $(e_1e_2) \in \Theta$ and $(e_2e_3) \in \Theta$, but $(e_1e_3) \notin \Theta$. Thus Θ is not an equivalence relation. Therefore, this bipartite graph is not a partial cube. However, it has a convex cycle base consisting of the three planar faces.

It seems natural that Theorem 6.16 should remain true also for a more general type of Cartesian products. We state this as

Conjecture 6.17. *Let G_1 and G_2 be graphs such that each of their isometric subgraphs have convex cycle bases. Then every isometric subgraph of $G_1\Box G_2$ has a convex cycle basis.*

A further step towards a proof of this conjecture is given by the following special case:

Theorem 6.18. *Let G be a graph such that every isometric subgraph has a convex cycle basis and let C_n be an elementary cycle. Then every isometric subgraph of $G\Box C_n$ has a convex cycle basis.*

Notice that this theorem is an immediate corollary of Theorem 6.16 if n is even since cycles of even length are partial cubes [30, 45]. The proof of the general case is present after Lemma 6.19 below. For this purpose we first have to introduce a graph operation for the case when C is a cycle of odd length. So assume that $C = C_{2k-1}$ for some integer $k \geq 2$. First fix three vertices $u, v, w \in V(C)$ with $\{u, v\}, \{v, w\} \in E(C_{2k-1})$. Create a new cycle $C' \cong C_{2k}$ by splitting vertex v, that is, replace v by two vertices v' and v'' and the edges $\{u, v\}, \{v, w\}$ by three edges $\{u, v'\}, \{v', v''\}, \{v'', w\}$.

This splitting operation can be generalized to subgraphs F of $G\Box C$. In essence, we replace $F \cap G^v$ by $(F \cap G^v)\Box K_2$. In more detail, we introduce the graph operations Υ and its converse Υ^* as follows: For a fixed vertex $v \in C$, and any subgraph

$F \subseteq G\Box C$, we obtain the subgraph $\Upsilon(F) \subseteq G\Box C'$ by splitting all vertices $(x,v) \in F$ with $x \in G$ in the following way: Replace vertex (x,v) by (x,v') and (x,v''), and replace the edges $\{(x,u),(x,v)\}$, $\{(x,v),(x,w)\}$, and $\{(x,v),(y,v)\}$, when present, by the corresponding edges $\{(x,u),(x,v')\}$, $\{(x,v'),(x,v'')\}$, $\{(x,v''),(x,w)\}$, $\{(x,v'),(y,v')\}$ and $\{(x,v''),(y,v'')\}$. Conversely, for a subgraph $F' \subseteq B\Box C'$ we obtain the subgraph $\Upsilon^*(F') \subseteq G\Box C$ by contracting all edges $\{(x,v'),(x,v'')\} \in E(G\Box C')$ and remove possible double edges. This construction in particular has the property that $\Upsilon(G\Box C) = G\Box C'$ and $\Upsilon^*(G\Box C') = G\Box C$.

Lemma 6.19. *Let $C = C_{2k-1}$ be an elementary cycle of odd length $2k-1$. If P is a shortest xy-path in $G\Box C$, then $\Upsilon(P)$ contains a shortest $x'y'$-path P' in $G\Box C'$ where x' and y' are vertices in $\Upsilon(x)$ and $\Upsilon(y)$, resp.*

Proof. Let $x = (x_G, x_C)$ and $y = (y_G, y_C)$ be two vertices in $G\Box C$ and let $x' = (x'_G, x'_{C'})$ and $y' = (y'_G, y'_{C'})$ be two vertices in $\Upsilon(x)$ and $\Upsilon(y)$, resp. Let P' be a shortest $x'y'$-path in $\Upsilon(P)$. We have to show that P' is also a shortest $x'y'$-path in $G\Box C'$. Observe that Lemma 6.4 implies that $|\pi_G(P)| = |\pi_G(P')|$ and $|\pi_C(P)| = |\pi_{C'}(P')| - \delta(P')$ where $\delta(P') = 1$ if $\pi_{C'}(P')$ contains edge $\{v', v''\}$ and $\delta(P') = 0$ otherwise. Moreover, $\mathrm{dist}_C(x_C, y_C) \leq k-1$ and $\mathrm{dist}_{C'}(x'_{C'}, y'_{C'}) \leq k$. Now suppose that P' is not a shortest $x'y'$-path in $G\Box C'$. Then there exists a $x'y'$-path P'' that is strictly shorter than P', that is, $|\pi_{C'}(P'')| < |\pi_{C'}(P')| \leq k$. As P is a shortest xy-path we have $|\pi_C(\Upsilon^*(P''))| = |\pi_C(\Upsilon^*(P'))| = |\pi_C(P)| \leq k-1$. Again $|\pi_C(\Upsilon^*(P''))| = |\pi_{C'}(P'')| - \delta(P'')$. Consequently $\pi_{C'}(P'')$ must contain edge $\{v', v''\}$ while $\pi_{C'}(P')$ must not. Therefore $\pi_{C'}(P'') \cap \pi_{C'}(P') \cong C'$. However $|\pi_{C'}(P'')| + |\pi_{C'}(P')| < k + k = 2k = |C'|$, a contradiction. This completes the proof. □

Proof of Theorem 6.18. Let C be an odd cycle. Thus C' is even and hence a partial cube. Lemma 6.19 implies that $\Upsilon(M)$ is an isometric subgraph of $G\Box C'$ if M is an isometric subgraph in $G\Box C$. In this case, $\Upsilon(M)$ has a convex cycle basis $\mathcal{B}'$. Now consider a convex cycle $D' \in \mathcal{B}'$. Lemma 6.11 implies that $\Upsilon^*(D')$ is either an elementary cycle or $\Upsilon^*(D')$ is a single edge in layer G^v. The latter happens if and only if D' contains edges $\{(x,v'),(x,v'')\}$ and $\{(y,v'),(y,v'')\}$. In this case D' must be a convex quadrangle. There are $|E(M \cap G^v)|$ quadrangles of this type, and they form an linearly independent set $\mathcal{Q}$ of convex cycles. Thus we can assume, w.l.o.g., that they all are contained in $\mathcal{B}'$. Lemma 6.19 implies that $\Upsilon^*(D')$ is a convex subgraph of M. Thus let

$$\mathcal{B} := \left\{ \Upsilon^*(D') \middle| D' \in \mathcal{B}' \text{ and } \Upsilon^*(D') \text{ is an elementary cycle} \right\} .$$

The cycles in $\mathcal{B}$ are linearly independent: Consider any linear combination of the form $\sum_i \lambda_i \Upsilon^*(D'_i) = 0$. It follows that there is a corresponding linear combination $\sum_i \lambda_i D'_i = \sum_j \xi_j Q_j$, where $Q_j \in \mathcal{Q}$ is a quadrangle that is contracted to 0 by Υ^*. Since $\mathcal{B}'$ is linearly independent by assumption, all ξ_j and λ_i must be 0, however.

It remains to show that $|\mathcal{B}| = \mu(M)$. Observe that $\Upsilon(M)$ contains the subgraph induced by vertices (x,v') and (x,v'') if $(v,x) \in V(M)$ for some $x \in G$. Otherwise $\Upsilon(M)$ contains none of these two vertices. Thus we find for the cyclomatic number $\mu(M) = \mu(\Upsilon(M)) - |E(M \cap G^v)|$. On the other hand $|\mathcal{B}| = |\mathcal{B}'| - |E(M \cap G^v)| = \mu(\Upsilon(M)) - |E(M \cap G^v)| = \mu(M)$. This completes the proof. □

7 Convex Eulerian graphs that are not cycles

Convex cycles need not be elementary, even though they are necessarily connected when G is connected. Furthermore, the elementary cycles whose union forms convex Eulerian subgraph need not be convex themselves. An example is the $K_{2,4}$, which can be decomposed into two elementary but not convex squares. In fact, the sum of convex cycles typically is not convex:

Lemma 7.1. *Let C_1 and C_2 be two convex cycles in G. If $C_1 \oplus C_2$ is 2-connected, then $C_1 \oplus C_2$ is not convex.*

Proof. If $C_1 \oplus C_2$ is 2-connected, then it contains at least two distinct vertices $u, v \in V(C_1) \cap V(C_2)$. Since $C_1 \cap C_2$ is also convex, it contains the set of all shortest uv-path which cannot be empty as $u \neq v$. Consequently, $C_1 \oplus C_2 = (C_1 \cup C_2) \setminus (C_1 \cap C_2)$ cannot contain any of these shortest path and is thus not convex. □

If $C_1 \oplus C_2$ is convex for two convex cycles C_1 and C_2, then $C_1 \oplus C_2 = C_1 \cup C_2$ and connected (but not 2-connected). Thus $V(C_1) \cap V(C_2)$ consists of a single vertex. Notice, however, that even then $C_1 \oplus C_2$ need not be convex.

One may ask, therefore, whether the cycle space of a graph that does not have a convex cycle basis nevertheless may have a basis consisting of convex Eulerian subgraphs. The example in Figure 5 shows that this is indeed possible.

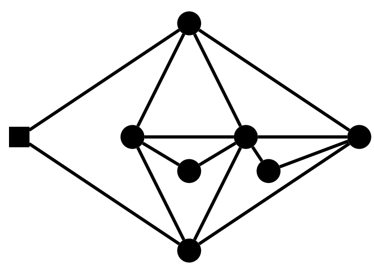

Figure 5: The 6 triangles and the whole graph are all convex cycles and form a cycle basis. However, there is no convex cycle basis according to Definition 1.1: none of the elementary cycle that pass through the square node is convex.

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Appendix A: Modified Dijkstra Algorithm

A shortest path algorithm that keeps track of the multiplicity of shortest paths and keeps some backtracing information is required as a pre-processing step in the computation of convex cycle bases. We use a modified version of Dijkstra's approach [12]. Let $\ell(x, y)$ denote the length of the edge $\{x, y\}$ in G, $d_{xy} = \mathrm{dist}_G(x, y)$ is the length and S_{xy} is the number of shortest paths in between x and y, π_{sx} is the predecessor of x along the *unique* shortest path from s to x, and $\pi_{sx} = \emptyset$ otherwise. Q denotes a priority queue sorted by d_{sx} for fixed s.

Input: $G = (V, E, \ell)$ /* an edge-weighted graph */
Output: Matrices $[S_{xy}]$, $[d_{xy}]$, and $[\pi_{xy}]$.

for all $s \in V$ **do**
  /* Modified Dijkstra algorithm */
  for all $v \in V$ **do**
    $d_{sv} = \infty$; $S_{sv} = 0$; $\pi_{sv} = \emptyset$
  $d_{ss} = 0$; $S_{ss} = 0$; $\pi_{ss} = s$
  $Q \leftarrow V$;
  while $(Q \neq \varnothing)$ **do**
    u := vertex with smallest d_{su}.
    if $(d_{su} = \infty)$ **then**
      break /* G not connected */
    remove u from Q
    for all neighbors $v \in Q \cap N(u)$ of u **do**
      $t := d_{su} + \ell(u, v)$
      if $(d_{sv} = t)$ **then**
        $S_{sv} := S_{sv} + 1$; $\pi_{s,v} = \emptyset$ /* more than one shortest path */
      if $(d_{sv} > t)$ **then**

17: $d_{sv} := t$; $S_{sv} := 1$; $\pi_{sv} = u$

The algorithm runs in $\mathcal{O}(|V|(|E| + |V|\log|V|))$ when the min-priority queue Q is implemented by means of a Fibonacci heap [16]. The modifications do not change the asymptotic complexity of the algorithm.

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From spanning forests to edge subsets

Martin Trinks *

Hochschule Mittweida, Faculty Mathematics / Sciences / Computer Science, Technikumplatz 17, 09648 Mittweida, Germany

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Abstract

We give some insight into Tutte's definition of internally and externally active edges for spanning forests. Namely we prove, that every edge subset can be constructed from the edges of exactly one spanning forest by deleting a unique subset of the internally active edges and adding a unique subset of the externally active edges.

Keywords: Spanning forests, edge subsets, Tutte polynomial.

Math. Subj. Class.: 05C30, 05C31

1 Introduction

The Tutte polynomial originally defined by a sum over spanning forests using (the number of) internally and externally active edges [12], can also be given as a sum over edge subsets [14, Equation (9.6.2)]. We show how both representations, as sum over spanning forests and as sum over edge subsets, are directly connected to each other.

Namely we prove, that every edge subset can be constructed from the edges of exactly one spanning forest by deleting a unique subset of the internally active edges and adding a unique subset of the externally active edges.

While seeking a generalization to matroids we observed that the statement is already given by Björner [4, Proposition 7.3.6]. It seems that this result is not well known in graph theory. Hence we state it explicitly in the special case of graphs and verify it graph-theoretically.

We apply this in some direct proofs for the equivalence of different representations of the Tutte polynomial, the chromatic polynomial, the reliability polynomial and the weighted graph polynomial.

Definition 1.1. A *graph* $G = (V, E)$ is an ordered pair of a set V, the vertex set, and a multiset E, the edge set, such that the elements of the edge set are one- and two-element subsets of the vertex set, $e \in \binom{V}{1} \cup \binom{V}{2}$ for all $e \in E$.

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E-mail address: trinks@hs-mittweida.de (Martin Trinks)

For a graph $G = (V, E)$, we denote the number of connected components of G by $k(G)$ and refer to G with the edge $e \in E$ deleted and with the edge $f \in \binom{V}{1} \cup \binom{V}{2}$ added by G_{-e} and G_{+f}, respectively.

Definition 1.2. Let $G = (V, E)$ be a graph and $A \subseteq E$ an edge subset of G. A graph $G\langle A\rangle = (V, A)$ is a *spanning subgraph* of G. A tree $T = (V, A)$ is a *spanning tree* of G. A forest $F = (V, A)$ is a *spanning forest* of G, if $k(G) = k(F)$. The *set of spanning trees* and the *set of spanning forests* of the graph G are denoted by $\mathcal{T}(G)$ and $\mathcal{F}(G)$, respectively.

While the term “spanning tree” is unambiguous, the term “spanning forest” is not, because not every spanning subgraph, which is a forest, is a “spanning forest”. A spanning forest is the union of spanning trees of each connected component.

In the following we consider graphs $G = (V, E)$ with a linear order $<$ on the edge set E. This linear order can be represented by a bijection $\beta\colon E \to \{1, \dots, |E|\}$ for all $e, f \in E$ with

$$e < f \Leftrightarrow \beta(e) < \beta(f). \tag{1.1}$$

Definition 1.3 (Section 3 in [12])**.** Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E and $F = (V, A) \in \mathcal{F}(G)$ a spanning forest of G. An edge $e \in A$ is *internally active* in F with respect to G and $<$, if there exists no edge $f \in E \setminus A$, such that $e < f$ and $F_{-e+f} \in \mathcal{F}(G)$. We denote the *set of internally active edges* and the *number of internally active edges* of F with respect to G and $<$ by $E_i(F, G, <)$ and $i(F, G, <)$, respectively.

An edge e in the spanning forest F is internally active, if it is the maximal edge of all edges in the cut crossed by itself (connecting the vertices in the connected components arising by deleting e from F). In other words, the edge e can not be replaced by a greater edge (not in the spanning forest), such that F remains a spanning forest.

Definition 1.4 (Section 3 in [12])**.** Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E and $F = (V, A) \in \mathcal{F}(G)$ a spanning forest of G. An edge $f \in E \setminus A$ is *externally active* in F with respect to G and $<$, if there exists no edge $e \in A$, such that $f < e$ and $F_{-e+f} \in \mathcal{F}(G)$. We denote the *set of externally active edges* and the *number of externally active edges* of F with respect to G and $<$ by $E_e(F, G, <)$ and $e(F, G, <)$, respectively.

An edge f not in the spanning forest is externally active, if it is the maximal edge of all edges in the cycle closed by itself. In other words, there is no greater edge (in the spanning forest), which can be replaced by f, such that F remains a spanning forest.

Definition 1.5 (Section 3 in [12])**.** Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E. The *Tutte polynomial* is defined as

$$T(G, x, y) = \sum_{F \in \mathcal{F}(G)} x^{i(F,G,<)} y^{e(F,G,<)}. \tag{1.2}$$

The primal usage of “a linear order on the edge set” seems to be by Whitney [18, Section 7]. Internally and externally active edges were probably first defined by Tutte [12, Section 3] to state the Tutte polynomial. This polynomial was originally introduced under the name “dichromate” for connected graphs [12, Equation (13)] and extended to disconnected graphs by the multiplicativity with respect to components [12, Equation (18)]. It was shown, that the value of the polynomial is independent of the linear order on the edge set [12, page 85-88]. For some background to the definition of internally and externally active edges and the Tutte polynomial we refer to [1, 9, 15]. For surveys on the Tutte polynomial and its applications we refer to [5, 7, 11].

2 Main theorem

The spanning forests and their internally and externally active edges can be used to generate all edge subsets. We use the disjoint union $\dot\cup$, the union of pairwise disjoint sets, in the statement of this main theorem below to indicate its bijectivity.

Theorem 2.1. *Let $G=(V,E)$ be a graph with a linear order $<$ on the edge set E. Then*

$$\dot\bigcup_{F=(V,A_f)\in\mathcal{F}(G)} \dot\bigcup_{\substack{A_i\subseteq E_i(F,G,<)\\ A_e\subseteq E_e(F,G,<)}} \{(A_f\setminus A_i)\cup A_e\} = \bigcup_{A\subseteq E}\{A\} = 2^E. \tag{2.1}$$

Proof. We prove that the function $m\colon \{(A_f,A_i,A_e)\mid F=(V,A_f)\in\mathcal{F}(G), A_i\subseteq E_i(F,G,<), A_e\subseteq E_e(F,G,<)\}\to 2^E$ with $m((A_f,A_i,A_e))=(A_f\setminus A_i)\cup A_e$ is a bijection.

First, we show that the function m is injective by an indirect proof. Assume it is not, that means there are two different triples $A^1=(A_f^1,A_i^1,A_e^1)$ and $A^2=(A_f^2,A_i^2,A_e^2)$, such that $m(A^1)=m(A^2)=A$.

If $A^1\neq A^2$, then $A_f^1\neq A_f^2$, because otherwise $A_i^1=A_f^1\setminus A=A_f^2\setminus A=A_i^2$ and $A_e^1=A\setminus A_f^1=A\setminus A_f^2=A_e^2$ and the triples would not be different.

As A_f^1 and A_f^2 are the edges of different spanning forests, there is an edge $g\in A_f^1\setminus A_f^2$. Furthermore, for any choice of g, there is an edge $h\in A_f^2\setminus A_f^1$, such that $(V,A_f^1)_{-g+h},(V,A_f^2)_{-h+g}\in\mathcal{F}(G)$. (There is at least one edge in the path connecting the incident vertices of g in (V,A_f^2), which is in the cut crossed by g in (V,A_f^1). These conditions ensure that we can "compare" the edges g and h, because g is in the cycle closed by adding h to A_f^1 and, equivalently, in the cut crossed by h in A_f^2, and vice versa.)

We distinguish whether g ($g\in A_f^1$ but $g\notin A_f^2$) and h ($h\notin A_f^1$ but $h\in A_f^2$) are in A or not:

- Case 1: $g\in A, h\in A$: We have a contradiction by
  - $g\in A\Rightarrow g\in A_e^2\Rightarrow h<g$,
  - $h\in A\Rightarrow h\in A_e^1\Rightarrow g<h$.
- Case 2: $g\in A, h\notin A$: We have a contradiction by
  - $g\in A\Rightarrow g\in A_e^2\Rightarrow h<g$,
  - $h\notin A\Rightarrow h\in A_i^2\Rightarrow g<h$.
- Case 3: $g\notin A, h\in A$: We have a contradiction by
  - $g\notin A\Rightarrow g\in A_i^1\Rightarrow h<g$,
  - $h\in A\Rightarrow h\in A_e^1\Rightarrow g<h$.

- Case 4: $g \notin A, h \notin A$: We have a contradiction by
  - $g \notin A \Rightarrow g \in A_i^1 \Rightarrow h < g$,
  - $h \notin A \Rightarrow h \in A_i^2 \Rightarrow g < h$.

Consequently there are no such triples A^1 and A^2, hence the function m is injective.

Second, we show that the function m is surjective by proving that for each edge set $A \subseteq E$ there is a spanning forest $F \in \mathcal{F}(G)$ and a triple (A_f, A_i, A_e) with $F = (V, A_f), A_i \subseteq E_i(F, G, <), A_e \subseteq E_e(F, G, <)$ such that $m((A_f, A_i, A_e)) = A$.

We arrange the edges of A and $E \setminus A$ in a sequence $e_1, \ldots, e_{|E|}$, such that the edges of A appear before the edges of $E \setminus A$, that the edges of A are increasing, and that the edges of $E \setminus A$ are decreasing, both with respect to $<$.

We start with the edgeless graph on the vertex set V and successively add the edges of E as they appear in the sequence to the graph, if the graph remains cycle-free. That means $G^0 = (V, \emptyset)$ and for $i \in \{1, \ldots, |E|\}$ we have

$$G^i = \begin{cases} G^{i-1}_{+e_i} & \text{if } G^{i-1}_{+e_i} \text{ is a forest,} \\ G^{i-1} & \text{if } G^{i-1}_{+e_i} \text{ is not a forest.} \end{cases}$$

Thus, $G^{|E|} = F = (V, A_f) \in \mathcal{F}(G)$ is a spanning forest of G.

An edge, which is in A but not in A_f, is not added to G^i, meaning that it would close a cycle consisting of earlier added and thus lesser edges of A, hence this edge is an externally active edge (maximal edge of the cycles closed by itself), $A \setminus A_f = A_e \subseteq E_e(F, G, <)$.

An edge, which is not in A but in A_f, is added to G^i, meaning that it is the first and thus greatest edges of $E \setminus A$ crossing the according cut and hence this edge is an internally active edge (maximal edge of the cut crossed by itself), $A_f \setminus A = A_i \subseteq E_i(F, G, <)$.

Consequently, for each edge subset $A \subseteq E$ there is a spanning forest $F \in \mathcal{F}(G)$ and an according triple (A_f, A_i, A_e) with $(A_f \setminus A_i) \cup A_e = A$, hence the function m is surjective. □

Corollary 2.2. *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E, $A \subseteq E$ an edge subset of G and $f(G, A)$ a function mapping in a commutative semigroup. Then*

$$\sum_{F=(V,A_f)\in\mathcal{F}(G)} \sum_{\substack{A=(A_f\setminus A_i)\cup A_e \\ A_i\subseteq E_i(F,G,<) \\ A_e\subseteq E_e(F,G,<)}} f(G, A) = \sum_{A\subseteq E} f(G, A). \tag{2.2}$$

Proof. The equation follows directly from Theorem 2.1. □

Corollary 2.3 (Theorem 3 in [7])**.** *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E. Then*

$$\sum_{F\in\mathcal{F}(G)} 2^{i(F,G,<)+e(F,G,<)} = 2^{|E|}. \tag{2.3}$$

Proof. The equation follows directly from Corollary 2.2 with $f(G, A) = 1$. □

To apply Theorem 2.1, the following lemma stating some kind of independence of the internally and externally active edges of a given spanning forest seems useful: Deleting an internally active edge splits a connected component, which can not be reconnected by adding externally active edges. Adding an externally active edge connects vertices already connected by a path, which can not be destroyed by deleting internally active edges.

Lemma 2.4. *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E and $F \in \mathcal{F}(G)$ a spanning forest of G. For all $e \in E_i(F, G, <)$ and $f \in E_e(F, G, <)$ it holds*

$$k(F) = k(F_{+f}) > k(F_{-e+f}) = k(F_{-e}) = k(F) - 1. \tag{2.4}$$

Proof. The first part, $k(F) = k(F_{+f}) > k(F_{-e}) = k(F) - 1$, follows directly from the definition of a spanning forest. The idea to prove the rest, $k(F_{-e+f}) = k(F_{-e})$, is already used in the case distinction in the proof of Theorem 2.1: The edge f can not reconnect the connected components arising from the deletion of e, because otherwise each of the two edges must be greater than the other. □

3 Applications of the main theorem

As an application of Theorem 2.1 we prove the equivalence of representations using sums over spanning forests/trees (spanning forest/tree representation) and sums over edge subsets (edge subset representation) for the Tutte polynomial, the chromatic polynomial, the reliability polynomial and (a derivation of) the weighted graph polynomial.

3.1 Edge subset representation of the Tutte polynomial

The edge subset representation of the Tutte polynomial was first given by Tutte stating the relation to the dichromatic polynomial [13, Equation (21)]. In this article, the dichromatic polynomial is defined by an edge subset representation and it is shown, that it satisfies recurrence relations [13, Equations (18) - (20)] analogous to the recurrence relations satisfied by the Tutte polynomial [12, Equations (18) - (20)].

Theorem 3.1 (Equation (9.6.2) in [14])**.** *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E. The Tutte polynomial has the edge subset representation*

$$T(G, x, y) = \sum_{A \subseteq E} (x - 1)^{k(G\langle A\rangle)-k(G)} (y - 1)^{|A|-|V|+k(G\langle A\rangle)}. \tag{3.1}$$

Proof. First, we expand the definition of the Tutte polynomial (Definition 1.5) using the binomial theorem:

$$\begin{aligned} T(G, x, y) &= \sum_{F \in \mathcal{F}(G)} x^{i(F,G,<)} y^{e(F,G,<)} \\ &= \sum_{F \in \mathcal{F}(G)} (x - 1 + 1)^{|E_i(F,G,<)|} (y - 1 + 1)^{|E_e(F,G,<)|} \\ &= \sum_{F \in \mathcal{F}(G)} \sum_{\substack{A_i \subseteq E_i(F,G,<) \\ A_e \subseteq E_e(F,G,<)}} (x - 1)^{|A_i|} (y - 1)^{|A_e|}. \end{aligned}$$

Second, we represent for each spanning forest F the number of internally and externally active edges in terms of the graph G and the spanning subgraph $G\langle A\rangle = (V, A)$ with

$A = (A_f \setminus A_i) \cup A_e$ using Lemma 2.4: If $G\langle A\rangle$ has more connected components than the graph G, each "additional" connected component results from deleting an internally active edge, i.e., $|A_i| = k(G\langle A\rangle) - k(G)$. If $G\langle A\rangle$ is not a forest, each "additional" edge (closing a cycle) results from adding an externally active edge, i.e., $|A_e| = |A| - |V| + k(G\langle A\rangle)$. Thus we have

$$\begin{aligned} T(G,x,y) &= \sum_{F=(V,A_f)\in\mathcal{F}(G)} \sum_{\substack{A=(A_f\setminus A_i)\cup A_e \\ A_i\subseteq E_i(F,G,<) \\ A_e\subseteq E_e(F,G,<)}} (x-1)^{|A_i|}(y-1)^{|A_e|} \\ &= \sum_{F=(V,A_f)\in\mathcal{F}(G)} \sum_{\substack{A=(A_f\setminus A_i)\cup A_e \\ A_i\subseteq E_i(F,G,<) \\ A_e\subseteq E_e(F,G,<)}} (x-1)^{k(G\langle A\rangle)-k(G)}(y-1)^{|A|-|V|+k(G\langle A\rangle)}. \end{aligned}$$

Finally, the statement follows by Corollary 2.2:

$$T(G,x,y) = \sum_{A\subseteq E} (x-1)^{k(G\langle A\rangle)-k(G)}(y-1)^{|A|-|V|+k(G\langle A\rangle)}. \qquad \square$$

3.2 Spanning forest representation of the chromatic polynomial

Definition 3.2 ([3])**.** Let $G = (V, E)$ be a graph. The *chromatic polynomial* $\chi(G, x)$ is the number of proper (vertex) colorings of G with at most x colors.

The spanning forest representation of the chromatic polynomial can be easily derived from its relation to the Tutte polynomial, which follows from the recurrence relations both polynomials satisfy. But the direct proof points out more clearly why internally and externally active edges make different contributions to the chromatic polynomial.

Theorem 3.3 (Theorem 14.1 in [2])**.** *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E. The chromatic polynomial has the spanning forest representation*

$$\chi(G,x) = (-1)^{|V|}(-x)^{k(G)} \sum_{\substack{F\in\mathcal{F}(G) \\ e(F,G,<)=0}} (1-x)^{i(F,G,<)}. \tag{3.2}$$

Proof. We start with the representation of the chromatic polynomial as sum over edge subsets [17, Section 2] and apply Corollary 2.2:

$$\begin{aligned} \chi(G,x) &= \sum_{A\subseteq E} x^{k(G\langle A\rangle)}(-1)^{|A|} \\ &= \sum_{F=(V,A_f)\in\mathcal{F}(G)} \sum_{\substack{A=(A_f\setminus A_i)\cup A_e \\ A_i\subseteq E_i(F,G,<) \\ A_e\subseteq E_e(F,G,<)}} x^{k(G\langle A\rangle)}(-1)^{|A|}. \end{aligned}$$

First, we analyze the contribution of the externally active edges $A_e \subseteq E_e(F, G, <)$ to the term $x^{k(G\langle A\rangle)}(-1)^{|A|}$: Each externally active edge $f \in E_e(F, G, <)$ contributes (independently) the factor -1 if $f \in A_e$ (the number of connected components is not influenced), and the factor 1 otherwise:

$$\begin{aligned}
\chi(G,x) &= \sum_{F=(V,A_f)\in\mathcal{F}(G)} \sum_{\substack{A'=A_f\setminus A_i\\ A_i\subseteq E_i(F,G,<)\\ A_e\subseteq E_e(F,G,<)}} x^{k(G\langle A'\cup A_e\rangle)}(-1)^{|A'\cup A_e|}\\
&= \sum_{F=(V,A_f)\in\mathcal{F}(G)} \sum_{\substack{A'=A_f\setminus A_i\\ A_i\subseteq E_i(F,G,<)\\ A_e\subseteq E_e(F,G,<)}} x^{k(G\langle A'\rangle)}(-1)^{|A'|}(-1)^{|A_e|}\\
&= \sum_{F=(V,A_f)\in\mathcal{F}(G)} \sum_{\substack{A'=A_f\setminus A_i\\ A_i\subseteq E_i(F,G,<)}} x^{k(G\langle A'\rangle)}(-1)^{|A'|} \sum_{A_e\subseteq E_e(F,G,<)} (-1)^{|A_e|}\\
&= \sum_{F=(V,A_f)\in\mathcal{F}(G)} \sum_{\substack{A'=A_f\setminus A_i\\ A_i\subseteq E_i(F,G,<)}} x^{k(G\langle A'\rangle)}(-1)^{|A'|}(1-1)^{e(F,G,<)}\\
&= \sum_{\substack{F=(V,A_f)\in\mathcal{F}(G)\\ e(F,G,<)=0}} \sum_{\substack{A'=A_f\setminus A_i\\ A_i\subseteq E_i(F,G,<)}} x^{k(G\langle A'\rangle)}(-1)^{|A'|}.
\end{aligned}$$

Second, we analyze the contribution of the internally active edges $A_i \subseteq E_i(F,G,<)$ to the term $x^{k(G\langle A\rangle)}(-1)^{|A|}$: Each internally active edge $e \in E_i(F,G,<)$ contributes (independently) the factor $-x$ if $e \in A_i$ (the number of connected components is increased by 1), and the factor 1 otherwise:

$$\begin{aligned}
\chi(G,x) &= \sum_{\substack{F=(V,A_f)\in\mathcal{F}(G)\\ e(F,G,<)=0}} \sum_{A_i\subseteq E_i(F,G,<)} x^{k(G\langle A_f\setminus A_i\rangle)}(-1)^{|A_f\setminus A_i|}\\
&= \sum_{\substack{F=(V,A_f)\in\mathcal{F}(G)\\ e(F,G,<)=0}} \sum_{A_i\subseteq E_i(F,G,<)} x^{k(G\langle A_f\rangle)}x^{|A_i|}(-1)^{|A_f|}(-1)^{|A_i|}\\
&= \sum_{\substack{F\in\mathcal{F}(G)\\ e(F,G,<)=0}} x^{k(G)}(-1)^{|V|-k(G)} \sum_{A_i\subseteq E_i(F,G,<)} (-x)^{|A_i|}\\
&= (-1)^{|V|}(-x)^{k(G)} \sum_{\substack{F\in\mathcal{F}(G)\\ e(F,G,<)=0}} (1-x)^{i(F,G,<)}. \qquad \square
\end{aligned}$$

The proof above also “includes” the Broken-cycle Theorem [18, Section 7], [6, Theorem 2.3.1]: The edge subsets not including any broken cycle are exactly the edge subsets resulting from spanning forests having no externally active edges by deleting a subset of internally active edges. Hence the Broken-cycle Theorem can be stated as

$$\chi(G,x) = \sum_{\substack{F=(V,A_f)\in\mathcal{F}(G)\\ e(F,G,<)=0}} \sum_{\substack{A'=A_f\setminus A_i\\ A_i\subseteq E_i(F,G,<)}} x^{k(G\langle A'\rangle)}(-1)^{|A'|} \tag{3.3}$$

$$= \sum_{\substack{F=(V,A_f)\in\mathcal{F}(G)\\ e(F,G,<)=0}} \sum_{\substack{A'=A_f\setminus A_i\\ A_i\subseteq E_i(F,G,<)}} x^{|V|-|A'|}(-1)^{|A'|}. \tag{3.4}$$

The connection between the spanning forest representation and the Broken-cycle Theorem is also given in [1].

3.3 Spanning tree representation of the reliability polynomial

The set of connected spanning subgraphs of a connected graph can be enumerated from the spanning trees by only adding externally active edges. We apply this insight to obtain a spanning tree representation of the reliability polynomial.

For a statement S, let $[S]$ equal 1, if S is true, and 0 otherwise [8].

Lemma 3.4 (Section 5, Item (19) in [16]). *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E. The generating function (in the indeterminant y) for the number of connected spanning subgraphs $S(G, y)$ has the spanning tree representation*

$$S(G, y) = \sum_{A \subseteq E} [k(G\langle A \rangle) = 1] y^{|A|} \tag{3.5}$$

$$= y^{|V|-1} \sum_{T \in \mathcal{T}(G)} (1 + y)^{e(T,G,<)}. \tag{3.6}$$

Proof. We start by applying Corollary 2.2:

$$\begin{aligned} S(G, y) &= \sum_{A \subseteq E} [k(G\langle A \rangle) = 1] y^{|A|} \\ &= \sum_{F=(V,A_f) \in \mathcal{F}(G)} \sum_{\substack{A=(A_f \setminus A_i) \cup A_e \\ A_i \subseteq E_i(F,G,<) \\ A_e \subseteq E_e(F,G,<)}} [k(G\langle A \rangle) = 1] y^{|A|}. \end{aligned}$$

The spanning subgraph $G\langle A \rangle$ is connected only if the graph G is connected, that means the spanning forests are spanning trees with $|V| - 1$ edges, and if no (internally active) edge is deleted from the spanning tree. It follows:

$$\begin{aligned} S(G, y) &= \sum_{T=(V,A_t) \in \mathcal{T}(G)} \sum_{\substack{A=A_t \cup A_e \\ A_e \subseteq E_e(T,G,<)}} y^{|A|} \\ &= \sum_{T=(V,A_t) \in \mathcal{T}(G)} \sum_{A_e \subseteq E_e(T,G,<)} y^{|A_t|} y^{|A_e|} \\ &= \sum_{T=(V,A_t) \in \mathcal{T}(G)} y^{|A_t|} \sum_{A_e \subseteq E_e(T,G,<)} y^{|A_e|} \\ &= y^{|V|-1} \sum_{T \in \mathcal{T}(G)} (1 + y)^{e(T,G,<)}. \end{aligned}$$

□

The probability, that all vertices of a graph are connected, if all edges of the graph are independently available with a probability p, is a polynomial in p, the reliability polynomial $R(G, p)$ [7, 16].

Definition 3.5. Let $G = (V, E)$ be a graph. The reliability polynomial is defined as

$$R(G, p) = \sum_{A \subseteq E} [k(G\langle A \rangle) = 1]\, p^{|A|} (1 - p)^{|E \setminus A|}. \tag{3.7}$$

Theorem 3.6 (Section 5, Item (15) in [16])**.** *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E. The reliability polynomial $R(G, p)$ has the spanning tree representation*

$$R(G,p) = (1-p)^{|E|-|V|+1} p^{|V|-1} \sum_{T \in \mathcal{T}(G)} \frac{1}{(1-p)^{e(T,G,<)}}. \tag{3.8}$$

Proof. We rewrite the definition of the reliability polynomial using $S(G, y)$:

$$\begin{aligned} R(G,p) &= \sum_{A \subseteq E} [k(G\langle A \rangle) = 1] p^{|A|} (1-p)^{|E \setminus A|} \\ &= \sum_{A \subseteq E} [k(G\langle A \rangle) = 1] \left(\frac{p}{1-p} \right)^{|A|} (1-p)^{|E|} \\ &= (1-p)^{|E|} S\left(G, \frac{p}{1-p}\right). \end{aligned}$$

From this the statement follows directly by Lemma 3.4. □

3.4 Spanning forest representation of a derivation of the weighted graph polynomial

For the graph polynomials above it was possible to derive a spanning forest/tree representation that depends only on the number of internally and externally active edges, independently of the corresponding edge sets. Obviously, this is not possible for every graph polynomial, also not for those having an edge subset representation.

The graph polynomial $U'(G, \bar{x}, y)$, a derivation of the weighted graph polynomial $U(G, \bar{x}, y)$ [10], is an example where only the contribution of the externally active edges can be summed up.

Definition 3.7. Let $G = (V, E)$ be a graph and $\bar{x} = (x_1, \ldots, x_{|V|})$. The graph polynomial $U'(G, \bar{x}, y)$ is defined as

$$U'(G, \bar{x}, y) = \sum_{A \subseteq E} \prod_{i=1}^{|V|} x_i^{k_i(G\langle A \rangle)} y^{|A|}, \tag{3.9}$$

where $k_i(G)$ denotes the number of connected components including exactly i vertices.

Theorem 3.8. *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E and $\bar{x} = (x_1, \ldots, x_{|V|})$. The (derivation of the) weighted graph polynomial $U'(G, \bar{x}, y)$ has the spanning forest representation*

$$U'(G, \bar{x}, y) = \sum_{F=(V,A_f) \in \mathcal{F}(G)} \sum_{\substack{A = A_f \setminus A_i \\ A_i \subseteq E_i(F,G,<)}} \prod_{i=1}^{|V|} x_i^{k_i(G\langle A \rangle)} y^{|A|} (1+y)^{e(F,G,<)}, \tag{3.10}$$

where $k_i(G)$ denotes the number of connected components including exactly i vertices.

Proof. We start by applying Corollary 2.2 and then sum up the contribution of the externally active edges (as in the proofs above):

$$\begin{aligned}
U'(G,\bar{x},y) &= \sum_{A\subseteq E}\prod_{i=1}^{|V|} x_i^{k_i(G\langle A\rangle)} y^{|A|}\\
&= \sum_{F=(V,A_f)\in\mathcal{F}(G)} \sum_{\substack{A=(A_f\setminus A_i)\cup A_e\\ A_i\subseteq E_i(F,G,<)\\ A_e\subseteq E_e(F,G,<)}} \prod_{i=1}^{|V|} x_i^{k_i(G\langle A\rangle)} y^{|A|}\\
&= \sum_{F=(V,A_f)\in\mathcal{F}(G)} \sum_{\substack{A=A_f\setminus A_i\\ A_i\subseteq E_i(F,G,<)}} \prod_{i=1}^{|V|} x_i^{k_i(G\langle A\rangle)} y^{|A|}(1+y)^{e(F,G,<)}. \qquad \square
\end{aligned}$$

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Revised and edge revised Szeged indices of graphs

Morteza Faghani , Ali Reza Ashrafi *

Department of Mathematics, Faculty of Mathematical Science, University of Kashan, Kashan 87317-51167, I. R. Iran

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Abstract

The revised Szeged index is a molecular structure descriptor equal to the sum of products $[n_u(e) + n_0(e)/2] \times [n_v(e) + n_0(e)/2]$ over all edges $e = uv$ of the molecular graph G, where $n_0(e)$ is the number of vertices equidistant from u and v, $n_u(e)$ is the number of vertices whose distance to vertex u is smaller than the distance to vertex v and $n_v(e)$ is defined analogously. In this paper, new formula for computing this molecular descriptor is presented by which it is possible to reprove most of results given in [M. Aouchiche and P. Hansen, On a conjecture about the Szeged index, *European J. Combin.* **31** (2010), 1662–1666]. We also present an edge version of this graph invariant. At the end of the paper an open question is presented.

Keywords: Szeged index, edge Szeged index, revised Szeged index, edge revised Szeged index.

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1 Introduction

We first describe some notations which will be kept throughout. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. If $e = uv \in E(G)$ then $d(u, v)$ stands for the distance between u and v in G. A *topological index* is a graph invariant applicable in chemistry. A topological index χ is called *distanced-based*, if χ is related to the distance function $d(-, -)$. The first use of a distance-based topological index occurred in the year 1947 in a seminal paper by an American chemist Harold Wiener [14].

Suppose G is a connected graph and $e = uv \in E(G)$. The quantities $n_0(e)$, $n_u(e)$ and $n_v(e)$ are defined to be the number of vertices equidistant from u and v, the number of vertices whose distance to vertex u is smaller than the distance to vertex v and the number

*Corresponding author.
E-mail addresses: m_faghani@pnu.ac.ir (Morteza Faghani), ashrafi@kashanu.ac.ir (Ali Reza Ashrafi)

of vertices closer to v than u, respectively. Similarly, the quantities $m_0(e)$, $m_u(e)$ and $m_v(e)$ are defined to be the number of edges equidistant from u and v, the number of edges whose distance to vertex u is smaller than the distance to vertex v and the number of edges closer to v than u, respectively. Here, for an edge $e = xy$ and vertex u, the distance between e and u is defined as $d_G(e, u) = Min\{d_G(x, u), d_G(y, u)\}$.

The *Szeged*, *edge Szeged*, *edge-vertex Szeged*, *vertex-edge Szeged*, *revised Szeged* and *edge revised Szeged* indices of G are defined as follows:

$$\begin{aligned}
Sz(G) &= \sum_{e=uv} [n_u(e) \times n_v(e)], \\
Sz_e(G) &= \sum_{e=uv} [m_u(e) \times m_v(e)], \\
Sz_{ev}(G) &= 1/2 \sum_{e=uv} [m_u(e) \times n_v(e) + m_v(e) \times n_u(e)], \\
Sz_{ve}(G) &= 1/2 \sum_{e=uv} [m_u(e) \times n_u(e) + m_v(e) \times n_v(e)], \\
Sz^*(G) &= \sum_{e=uv} [(n_u(e) + n_0(e)/2) \times (n_v(e) + n_0(e)/2)], \\
Sz_e^*(G) &= \sum_{e=uv} [(m_u(e) + m_0(e)/2) \times (m_v(e) + m_0(e)/2)].
\end{aligned}$$

It is worth mentioning here that the Szeged index was introduced by Ivan Gutman [4] and the name Szeged index was given in [5]. For the mathematical properties of this topological index we refer to [3, 9, 10]. The concept of edge Szeged index was introduced in [6] and mathematical properties of this graph invariant are studied in [2, 7, 8]. The revised Szeged index was introduced by Milan Randić [13] as a modification of the classical Wiener index. Nowadays the scientists prefer the name revised Szeged index for this distance-based topological index. The interested readers can consult [1, 11, 12, 15] for mathematical properties of this new topological index.

Throughout this section graph means finite simple connected graph. The notation is standard and can be taken from the standard books on graph theory.

2 Main results

In this section, we first present a new formula for computing revised Szeged index of graphs. Then apply this new formula to reprove all results given by Aouchiche and Hansen [1]. We also present an edge version of the revised Szeged index and extend the results given in the mentioned article to this new graph invariant. We begin by an example.

Example 2.1. Suppose $G_1 = K_n$, $G_2 = C_n$ and $G_3 = W_n$ denote the complete, cycle and wheel graphs of order n, and $G_4 = K_{m,n}$ is the complete bipartite graph with partitions of size m and n, respectively. Then,

- If $e = uv \in E(G_1)$ then $m_u = m_v = n - 2$ and $m_0 = \frac{n^2-5n+8}{2}$. Therefore, $Sz_e(G_1) = \frac{n(n-1)(n-2)^2}{2}$ and $Sz_e^*(G_1) = \frac{n^3(n-1)^3}{32}$.
- Suppose $e = uv$ is an arbitrary edge of G_2. If $n = 2k + 1$, then $m_u = m_v = k$ and so $m_0 = 1$. Therefore, $Sz_e(G_2) = (2k + 1)k^2 = \frac{n(n-1)^2}{4}$ and $Sz_e^*(G_2) = \frac{n^3}{4}$.

If $n = 2k$ then $m_u = m_v = k-1$ and so $m_0 = 2$. This implies that $Sz_e(G_2) = n(k-1)^2 = \frac{n(n-2)^2}{4}$ and $Sz_e^*(G_2) = \frac{n^3}{4}$.

- Consider the $n-$vertex wheel graph G_3, $n > 5$. If $e = uv$ is an edge of G_3 such that the vertex v is the center of G_3, then $m_u = 3$, $m_v = 2n-7$ and $m_0 = 3$. If both of u and v are not the center of G_3, then $m_u = 3$, $m_v = 3$ and $m_0 = 2n-8$. Therefore $Sz_e(G_3) = (n-1)(4n-5)$ and $Sz_e^*(G_3) = (n-1)(n^2+5n-73/4)$.
- Suppose $G_4 = K_{x,y}$, $x+y = n$, is the complete bipartite graph containing an arbitrary edge $e = uv$, where $deg(u) = x$ and $deg(v) = y$. Then we have $m_u = x-1, m_v = y-1, m_0 = xy-x-y+2$. This implies that $Sz_e(G_4) = xy(x-1)(y-1)$ and $Sz_e^*(G_4) = \frac{xy}{4}(x^2y^2 - x^2 - y^2 + 2xy)$.

Theorem 2.2. *Let G be an n-vertex and m-edge graph. Then*

$$Sz^*(G) = \frac{mn^2}{4} - \frac{1}{4}\sum_{e=uv}(n_u^2+n_v^2) + \frac{1}{2}Sz(G)$$

Proof. Since $n_u(e) + n_v(e) = n - n_0(e)$ we have:

$$\begin{aligned}
Sz^*(G) &= \sum_{e=uv}\left[(n_u+\frac{n_0}{2})(n_v+\frac{n_0}{2})\right] \\
&= \sum_{e=uv}\left[n_un_v + \frac{n_0}{2}(n_u+n_v) + \frac{1}{4}n_0^2\right] \\
&= \sum_{e=uv}\left[n_un_v + \frac{1}{2}((n-(n_u+n_v)))(n_u+n_v) + \frac{1}{4}(n-(n_u+n_v))^2\right] \\
&= \sum_{e=uv}\Big[n_un_v + \frac{n}{2}(n_u+n_v) - \frac{1}{2}(n_u+n_v)^2 \\
&\qquad + \frac{n^2}{4} - \frac{1}{2}n(n_u+n_v) + \frac{1}{4}(n_u+n_v)^2\Big] \\
&= \sum_{e=uv}\left[n_un_v + \frac{n^2}{4} - \frac{1}{4}(n_u^2+n_v^2+2(n_u)(n_v))\right] \\
&= \frac{1}{2}Sz(G) + \frac{mn^2}{4} - \frac{1}{4}\sum_{e=uv}[n_u^2+n_v^2].
\end{aligned}$$

proving the result. □

The next Corollary is already known result that stated and proven in [1].

Corollary 2.3. $Sz(G) \leq Sz^*(G) \leq \frac{mn^2}{4}$.

Proof. Since $n_u + n_v \leq n$, $(n_u+n_v)^2 \leq n^2$. So, $\sum_{e=uv}[n_u+n_v]^2 \leq mn^2$ and therefore $\frac{mn^2}{4} - \frac{1}{4}\sum_{e=uv}[n_u+n_v]^2 = \frac{mn^2}{4} - \frac{1}{4}\sum_{e=uv}[n_u^2+n_v^2] - \frac{1}{2}Sz(G) \geq 0$. Now, Theorem 2.2 implies that $Sz(G) \leq Sz^*(G)$, the left hand side of inequality. The right hand side is a direct consequence of Theorem 2.2 and the following inequality:

$$\frac{1}{2}Sz(G) - \frac{1}{4}\sum_{e=uv}[n_u^2+n_v^2] = -\frac{1}{4}\sum_{e=uv}[n_u-n_v]^2 \leq 0.$$

□

By a similar argument as Theorem 2.2, one can prove:

Theorem 2.4. *Let G be an n-vertex and m-edge graph. Then*

$$Sz_e^*(G) = \frac{m^3}{4} - \frac{1}{4}\sum_{e=uv}[m_u^2 + m_v^2] + \frac{1}{2}Sz_e(G).$$

Corollary 2.5. $Sz_e(G) \leq Sz_e^*(G) \leq \frac{m^3}{4}$.

Proof. The proof is similar to the proof of Corollary 2.3 and so omitted. □

Suppose G is a connected graph and u is a vertex of G. Define

$$D(u,G) = \sum_{x \in V(G)} [d_G(u,x)].$$

The graph G is called *distance-balanced* (or *transmission-regular* according to [1]) if for every $u, v \in V(G)$, $D(u,G) = D(v,G)$. Similarly for a vertex u and an edge $e = xy$ define $D_e(u,G) = \sum_{e \in E(G)}[d_G(e,u)$. A graph G is called *edge-distance-balanced* if for every vertices $u, v \in V(G)$, $D_e(u,G) = d_e(v,G)$.

Theorem 2.6. *Suppose u and v are vertices of a connected graph G. Then $m_u = m_v$ if and only if $D_e(u,G) = d_e(v,G)$.*

Proof. Let $e = uv$ be an arbitrary edge of G. We partition the edge set of G into three parts as follows:

- $M(u)$ is the set of all edges that are closer to u than v.
- $M(v)$ is the set of all edges that are closer to v than u.
- $M(o)$ is the set of all edges that are equidistant from u and v.

Suppose $m_u(e) = |M(u)|$, $m_v(e) = |M(v)|$ and $m_0(e) = |M(o)|$. Then we have :

$$\begin{aligned} D_e(u,G) &= \sum_{e \in E(G)} d_G(e,u) \\ &= \sum_{e \in M(u)} d_G(e,u) + \sum_{e \in M(v)} d_G(e,u) + \sum_{e \in M(0)} d_G(e,u) \\ &= \sum_{e \in M(u)} d_G(e,u) + \sum_{e \in M(v)} (1 + d_G(e,v)) + \sum_{e \in M(0)} d_G(e,u) \\ &= \sum_{e \in M(u)} d_G(e,u) + m_v(e) + \sum_{e \in M(v)} d_G(e,v) + \sum_{e \in M(0)} d_G(e,u). \end{aligned}$$

A similar argument shows that

$$D_e(v,G) = \sum_{e \in M(u)} d_G(e,u) + m_u(e) + \sum_{e \in M(v)} d_G(e,v) + \sum_{e \in M(0)} d_G(e,v).$$

But $D_e(u,G) - D_e(v,G) = m_v(e) - m_u(e)$ and so $m_u(e) = m_v(e)$ if and only if $D_e(u,G) = D_e(v,G)$. This complete our argument. □

Corollary 2.7. *If $Sz_e(G) = \frac{m^3}{4}$ then G is an edge-distance-balanced graph.*

Proof. If $Sz_e(G) = \frac{m^3}{4}$ then by Corollary 2.5, $Sz_e^*(G) = \frac{m^3}{4}$. Thus

$$\frac{1}{2}Sz_e(G) - \frac{1}{4}\sum_{uv\in E(G)}[m_u^2 + m_v^2] = -\frac{1}{4}\sum_{uv\in E(G)}[m_u - m_v]^2 = 0.$$

Therefore $m_u = m_v$. Now Theorem 2.6 implies that G is an edge-distanced-balanced graph. □

In the end of this paper, we compute an exact formula for the edge revised Szeged index of Cartesian product of graphs. To do this, we assume that G and H are connected graphs with vertex sets $V(G) = \{u_1, u_2, ..., u_r\}$ and $V(H) = \{v_1, v_2, \ldots, v_s\}$. We also assume that $|E(G)| = e_1$ and $|E(H)| = e_2$. Then by definition $V(G \times H) = V(G) \times V(H)$ and we have:

$$E(G\times H) = \{(u, v)(a, b) \mid [u = a, vb \in E(H)] \text{ or } [ua \in E(G), v = b]\}.$$

Clearly, $|E(G \times H)| = |V(G)||E(H)| + |V(H)||E(G)|$. To compute the edge revised Szeged index of $G \times H$ we partition the edge set of this graph into the following parts:

$$\begin{aligned} A_m &= \{(u_m, x)(u_m, y) \mid xy \in E(H)\}\ ;\ 1 \le m \le r,\\ B_t &= \{(a, v_t)(b, v_t) \mid ab \in E(G)\}\ ;\ 1 \le t \le s. \end{aligned}$$

Theorem 2.8. *(See [15, Lemmas 2 and 3]). With above notations we have:*
(a) If $e = (u_m, v_j)(u_m, v_q) \in A_m)$ then

$$m_{(u_m,v_j)}(e) = |V(G)|m_{v_j}(v_jv_q) + |E(G)|n_{v_j}(v_jv_q) = rm_{v_j}(H) + e_1n_{v_j}(H),$$
$$m_{(u_m,v_q)}(e) = |V(G)|m_{v_q}(v_jv_q) + |E(G)|n_{v_j}(v_jv_q) = rm_{v_q}(H) + e_1n_{v_q}(H).$$

(b) If $e = (u_i, v_t)(u_p, v_t) \in B_t$ then

$$m_{(u_i,v_t)}(e) = |V(H)|m_{u_i}(u_iu_p) + |E(H)|n_{u_i}(u_iu_p) = sm_{u_i}(G) + e_2n_{u_i}(G),$$
$$m_{(u_p,v_t)}(e) = |V(H)|m_{u_p}(u_iu_p) + |E(H)|n_{u_p}(u_iu_p) = sm_{u_p}(G) + e_2n_{u_p}(G).$$

Theorem 2.9. *With notation of Theorem 2.8, the edge revised Szeged index of Cartesian product of G and H can be computed as follows:*

$$\begin{aligned} Sz_e^*(G\times H) &= \frac{1}{2}r^3Sz_e(H) + r^2e_1Sz_{ev}(H) + \frac{1}{2}re_1^2Sz(H) + \frac{1}{4}re_2(re_2 + se_1)^2\\ &- r^2e_1Sz_{ve}(H) - \frac{1}{4}r^3\sum_{xy\in E(H)}[m_x^2(H) + m_y^2(H)]\\ &- \frac{1}{4}re_1^2\sum_{xy\in E(H)}[n_x^2(H) + n_y^2(H)] + \frac{1}{2}s^3Sz_e(G) + s^2e_2Sz_{ev}(G)\\ &+ \frac{1}{2}se_2^2Sz(G) + \frac{1}{4}se_1(se_1 + re_2)^2 - s^2e_2Sz_{ve}(G)\\ &- \frac{1}{4}s^3\sum_{ab\in E(G)}[m_a^2(G) + m_b^2(G)] - \frac{1}{4}se_2^2\sum_{ab\in E(G)}[n_a^2(G) + n_b^2(G)]. \end{aligned}$$

Proof. Let $e = (u_m, x)(u_m, y) \in A_m$. Then $m_0(e) = re_2 + se_1 - r(m_x(H) + m_y(H)) - e_1(n_x(H) + n_y(H))$. Set,

$$\begin{aligned} A &= \left[m_{(u_m,x)}(e) + \frac{m_0(e)}{2}\right] \times \left[m_{(u_m,y)}(e) + \frac{m_0(e)}{2}\right], \\ B &= \left[m_{(a,v_t)}(e) + \frac{m_0(e)}{2}\right] \times \left[m_{(b,v_t)}(e) + \frac{m_0(e)}{2}\right]. \end{aligned}$$

Then we have:

$$\begin{aligned} A &= \frac{1}{2}r^2 m_x(H)m_y(H) + \frac{1}{2}e_1 r(n_x(H)m_y(H) + n_y(H)m_x(H)) \\ &+ \frac{1}{2}e_1^2 n_x(H)n_y(H) + \frac{1}{4}(re_2 + se_1)^2 - \frac{1}{2}e_1 r(n_x(H)m_x(H) + n_y(H)m_y(H)) \\ &- \frac{1}{4}r^2(m_x^2(H) + m^2 y(H)) - \frac{1}{4}e_1^2(n_x^2(H) + n_y^2(H)). \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{(u_m,x)(u_m,y)\in A_m} & \left[m_{(u_m,x)}(e) + \frac{m_o(e)}{2}\right]\left[m_{(u_m,y)}(e) + \frac{m_0(e)}{2}\right] \\ &= 1/2r^2 Sz_e(H) + e_1 r Sz_{ev}(H) + \frac{1}{2}e_1^2 Sz(H) + \frac{1}{4}e_2(re_2 + se_1)^2 \\ &- e_1 r Sz_{ve}(H) - \frac{1}{4}r^2 \sum_{xy\in E(H)} [m_x^2(H) + m_y^2(H)] \qquad (2.1) \\ &- \frac{1}{4}e_1^2 \sum_{xy\in E(H)} [n_x^2(H) + n_y^2(H)]. \end{aligned}$$

Using a similar argument for the edge $e = (a, v_t)(b, v_t) \in B_t$, we have:

$$\begin{aligned} B &= \left[m_{(a,v_t)}(e) + \frac{m_0(e)}{2}\right]\left[m_{(b,v_t)}(e) + \frac{m_0(e)}{2}\right] \\ &= \frac{1}{2}s^2 m_{u_i}(G)m_{u_p}(G) + \frac{1}{2}e_2 s\left[n_a(G)m_b(G) + n_b(G)m_a(G)\right] \\ &+ \frac{1}{2}e_2^2 n_a(G)n_b(G) + \frac{1}{4}(se_1 + re_2)^2 - \frac{1}{2}e_2 s\left[n_a(G)m_a(G) + n_b(G)m_b(G)\right] \\ &- \frac{1}{4}s^2\left[m_a^2(G) + m_b^2(G)\right] - \frac{1}{4}e_2^2\left[n_a^2(G) + n_b^2(G)\right]. \end{aligned}$$

So,

$$\begin{aligned} \sum_{(a,v_t)(b,v_t)\in B_t} & \left[m_{(a,v_t)}(e) + \frac{m_0(e)}{2}\right]\left[m_{(b,v_t)}(e) + \frac{m_0(e)}{2}\right] \\ &= \frac{1}{2}s^2 Sz_e(G) + e_2 s Sz_{ev}(G) + \frac{1}{2}e_2^2 Sz(G) + \frac{1}{4}e_1(se_1 + re_2)^2 \\ &- e_2 s Sz_{ve}(G) - \frac{1}{4}s^2 \sum_{ab\in E(G)} \left[m_a^2(G) + m_b^2(G)\right] \qquad (2.2) \\ &- \frac{1}{4}e_2^2 \sum_{ab\in E(G)} \left[n_a^2(G) + n_b^2(G)\right]. \end{aligned}$$

Now multiplying Eq. (2.1) by r and Eq. (2.2) by s and summation of these values, the formula given in the theorem will be obtained. $\square$

3 Conclusions

Some of mathematicians recently focus on the revised Szeged index of graphs. In this paper a new formula for computing this topological index is presented by which it is possible to reprove some earlier results. We also investigate an edge version of this interesting topological index. We proved that the edge version of this graph invariant is more complicated than its vertex version. In the case of vertex version, it is easy to find an exact formula for the Cartesian product of graphs but in the edge version it is too difficult.

In Theorem 2.6 and Corollary 2.7, it is proved that $Sz_e(G) = \frac{m^3}{4}$ implies that G is an edge-balanced-distance graph. We end the paper with the following open question:

Question: Characterize graphs G such that $Sz_e(G) = m^3/4$.

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Petersen-colorings and some families of snarks

Jonas Hägglund
Department of Mathematics and Mathematical Statistics, Umeå University
SE-901 87 Umeå, Sweden

Eckhard Steffen
Paderborn Institute for Advanced Studies in Computer Science and Engineering,
Paderborn University, Zukunftsmeile 1, 33102 Paderborn, Germany

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Abstract

In this paper we study Petersen-colorings and strong Petersen-colorings on some well known families of snarks, e.g. Blanuša snarks, Goldberg snarks and flower snarks. In particular, it is shown that flower snarks have a Petersen-coloring but they do not have a strong Petersen-coloring. Furthermore it is proved that possible minimum counterexamples to Jaeger's Petersen-coloring conjecture do not contain a specific subdivision of $K_{3,3}$.

Keywords: Petersen colorings, strong Petersen colorings, snarks

Math. Subj. Class.: 05C15, 05C21, 05C70

1 Introduction

We study finite graphs G with vertex set $V(G)$ and edge set $E(G)$. If we distinguish an initial and a terminal end for every edge e, then we obtain a directed graph. For $S \subseteq V(G)$, the set of edges with initial end in S and terminal end in $V(G) - S$ is denoted by $\omega_G^+(S)$. We write $\omega_G^-(S) = \omega_G^+(V(G) - S)$ and $\omega_G(S) = \omega_G^+(S) \cup \omega_G^-(S)$. If S consists of a single vertex v we also write $\omega_G(v)$ instead of $\omega_G(\{v\})$. Subsets of $E(G)$ of the form $\omega_G(S)$ for $S \subseteq V(G)$ are called *cocycles* of G. If $R \subseteq E(G)$, then $G[R]$ denotes the graph with vertex set $V(G)$ and edge set R.

Given graphs G and H, we say that $f : E(G) \to E(H)$ is a H-coloring of G if it is a proper edge-coloring and for every $v \in V(G)$ there exists a $v' \in V(H)$ such that $f(\omega_G(v)) \subseteq \omega_H(v')$. That is, adjacent edges in G are mapped to adjacent edges in H. If H is the Petersen graph, we say that G has a *Petersen-coloring*.

E-mail addresses: jonas.hagglund@math.umu.se (Jonas Hägglund), eckhard.steffen@uni-paderborn.de (Eckhard Steffen)

Jaeger [6] studied nowhere-zero flow problems on graphs where the set of flow values are certain subsets of some Abelian group. He showed that a number of problems in graph theory such as the cycle double cover conjecture [9, 12] and Fulkerson's conjecture [2] (i.e. that every bridgeless cubic graph has six perfect matchings such that every edge is in precisely two of them) can be formulated in terms of such flows. He posed the following conjecture which would imply both previously mentioned conjectures, and many others, see [8].

Conjecture 1.1 (Petersen Coloring Conjecture [6])**.** *Every bridgeless cubic graph has a Petersen-coloring.*

In [5] an even more specific notion is introduced. Associate to G a directed graph dG with vertex set $V(dG) = V(G) \cup E(G)$, and to every edge $e = xy$ in G correspond two directed edges e_x and e_y with initial end e and terminal ends x and y, respectively. We say e_x is opposite to e_y and vice versa. Let G and G' be two graphs. A mapping ϕ from $E(dG)$ to $E(dG')$ is compatible, if for any two opposite edges e_1 and e_2 in dG, $\phi(e_1)$ and $\phi(e_2)$ are opposite edges in dG'.

For a cubic graph G the set of triples of edges of dG of the form $\omega_{dG}(v)$ is denoted by $T^+(dG)$, where v is a trivalent vertex in dG. $T^-(dG)$ is the set of triples of the form $\{e_1^-, e_2^-, e_3^-\}$ where $\{e_1, e_2, e_3\} \in T^+(dG)$ and e_i^- is opposite to e_i.

Let G and G' be two cubic graphs. A dG'-coloring of dG is a compatible mapping γ from $E(dG)$ to $E(dG')$ which maps every triple of $T^+(dG)$ to a triple of $T^+(dG') \cup T^-(dG')$. For the particular case when dG has a dG'-coloring and G' is the Petersen graph, we say that G is *strongly Petersen-colorable*.

Clearly, strongly Petersen-colorable graphs satisfy the Petersen-coloring conjecture and hence Fulkerson's and the cycle double cover conjecture as well. Jaeger [5] noticed that moreover these graphs also satisfy Tutte's 5-flow- and the orientable cycle double cover conjecture.

All these conjectures are trivially true for 3-edge-colorable cubic graphs. Hence we focus on bridgeless cubic graphs, which are not 3-edge-colorable; so called snarks. Snarks are of major interest in graph theory since they are potential counterexamples to many hard conjectures. Brinkmann et al. [1] generated all snarks with at most 36 vertices and they disproved a couple of conjectures concerning these graphs. The paper also gives an overview on conjectures which are related to snarks. In [11] it is shown that cubic graphs with high cyclic connectivity have a nowhere-zero 5-flow. This result can also be considered as a first approximation to a conjecture of Jaeger and Swart [7] who conjectured that every cyclically 7-edge connected cubic graph has a nowhere-zero 4-flow.

The paper is organized as follows. The next section delivers Jaeger's characterizations of Petersen-colorable and strongly Petersen-colorable graphs, [5]. We show that type 1 Blanuša snarks have a strong Petersen-coloring while flower snarks do not have such a coloring. We study the structure of a minimum counterexample to the Petersen-coloring conjecture and finally we show that the flower-, the Goldberg-, and all Blanuša snarks have a Petersen-coloring.

2 Normal 5-edge-colorings

Let G be a cubic graph and $\phi : E(G) \to \{1, 2, 3, 4, 5\}$ be a proper 5-edge-coloring. An edge $e = xy$ in G is *poor* if $|\phi(\omega(x)) \cup \phi(\omega(y))| = 3$ and it is *rich* if $|\phi(\omega(x)) \cup \phi(\omega(y))| =$

5. If every edge in G is either rich or poor, then ϕ is a *normal 5-edge-coloring*. Jaeger characterizes Petersen-colorable and strongly Petersen-colorable graphs in terms of normal 5-edge-colorings.

Theorem 2.1. *[5] A cubic graph is Petersen-colorable if and only if it has a normal 5-edge-coloring.*

Theorem 2.2. *[5] A cubic graph is strongly Petersen-colorable if and only if it has a normal 5-edge-coloring, and the set of poor edges forms a cocycle.*

If ϕ is a normal 5-edge-coloring of a graph G, such that the set of poor edges forms a cocycle, then we call ϕ a strong normal 5-edge-coloring. Jaeger [5] stated that cubic graphs with strong normal 5-edge-coloring do not contain a triangle (cf. Proposition 4.1).

3 Strong Petersen-colorings

3.1 Blanuša snarks

The generalized Blanuša snarks were introduced by Watkins in [13]. Let A be the graph formed by removing two adjacent vertices from the Petersen graph. The generalized Blanuša snarks of type 1 are formed by joining n copies of the graph A as depicted in Figure 1 and one copy of the graph P_2.

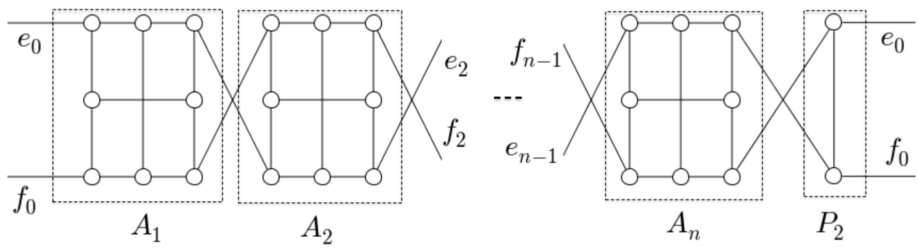

Figure 1: The generalized Blanuša snark of type 1.

Theorem 3.1. *Every generalized Blanuša snark of type 1 with an odd number of A-blocks is strongly Petersen-colorable.*

Proof. Let G_{2n-1} be a Blanuša snark of type 1 formed by blocks $A_1, \ldots, A_{2n+1}, P_2$ and let ϕ be the coloring of the even respectively odd blocks as shown in Figure 2. Then it is easy to see that ϕ is a normal edge-coloring where the set of poor edges is the set $\cup\{\omega(V(A_i))\}_2^{2n}$ and hence a cocycle. It now follows from Theorem 2.2 that G_{2n-1} is strongly Petersen-colorable. □

3.2 Flower snarks

In this section we will show that flower snarks do not have a strong Petersen-coloring.

Let G be a graph which has a normal 5-edge-coloring. We first study possible partitions of the edge set of C_6 (the cycle of length 6) into rich and poor edges. We denote the set of rich edges with R.

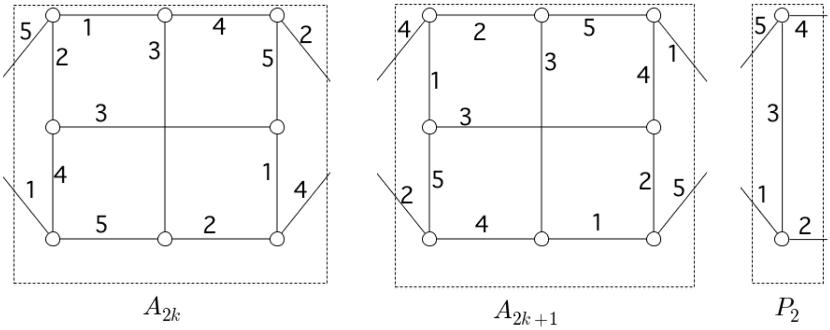

Figure 2: A normal edge-coloring ϕ of a generalized Blanuša snark of type 1 where the only poor edges are the diagonal edges between the blocks $A_1, \ldots, A_{2n+1}$.

Lemma 3.2. *Let G be a cubic graph that has a strong normal 5-edge-coloring. If C_6 is a subgraph of G, then the connected components of $C_6[E(C_6) \cap R]$ are either C_6 or two paths of length 2 or two isolated edges and two isolated vertices or six isolated vertices.*

Proof. Let G be a cubic graph that has a strong normal 5-edge-coloring ϕ, and that contains C_6 as a subgraph. Then the set of poor edges forms a cocycle by Theorem 2.2 and therefore, it partitions $V(G)$ into two sets S and S' such that the following two conditions are satisfied:

C1: If $e = vw$ is a poor edge, then $v \in S$ if and only if $w \in S'$.

C2: If $e = vw$ is a rich edge, then either $v, w \in S$ or $v, w \notin S$.

Taken into account these two conditions, it is easy to see that the following claim is true.

Claim 3.3. *The number of rich (poor) edges in C_6 is even.*

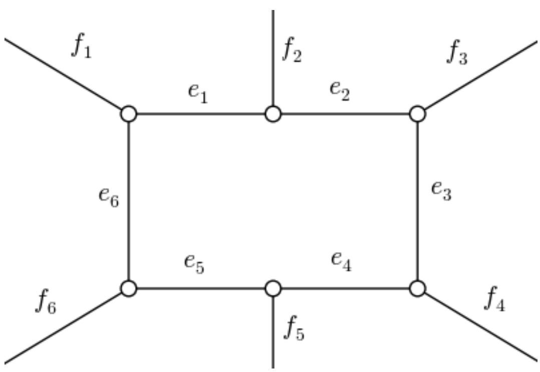

Figure 3: C_6

Let the edges of C_6 be labeled as indicated in Figure 3.

Claim 3.4. *The rich edges do not induce a path of length 4.*

Proof. Assume that e_1, e_2, e_3, e_4 are rich. W.l.o.g. we may assume that $\phi(e_1) = 1$, $\phi(f_1) = 2$, $\phi(e_6) = 3$, $\phi(f_2) = 4$, $\phi(e_2) = 5$. Then $\phi(f_3), \phi(e_3), \phi(f_4), \phi(e_4) \neq 5$. Hence $5 \in \{\phi(f_5), \phi(e_5)\}$. But on the other hand $\{\phi(e_5), \phi(f_5)\} = \{1, 3\}$ or $= \{2, 3\}$, a contradiction.

Claim 3.5. *The rich edges do not induce a path of length 3 and an isolated edge in* C_6.

Proof. Assume that e_1, e_2, e_3, e_5 are rich. W.l.o.g. we may assume that $\phi(e_1) = 1$, $\phi(f_1) = 2$, $\phi(e_6) = 3$, $\phi(f_2) = 4$, $\phi(e_2) = 5$. This implies, that $\{\phi(e_4), \phi(f_4)\} = \{1, 4\}$ and $\{\phi(e_4), \phi(f_5)\} = \{4, 5\}$; hence $\phi(e_4) = 4$. Thus $\phi(f_5) = 5 = \phi(f_4)$, a contradiction.

Claim 3.6. *The rich edges do not induce precisely one path of length 2 in* C_6.

Proof. Assume that e_1, e_2 are rich. W.l.o.g. we may assume that $\phi(e_1) = 1$, $\phi(f_1) = 2$, $\phi(e_6) = 3$, $\phi(f_2) = 4$, $\phi(e_2) = 5$. This implies that $3 \in \{\phi(f_5), \phi(e_4)\}$ and $5 \in \{\phi(e_4), \phi(f_4)\}$. On the other hand we have that $\{\phi(e_5), \phi(e_6), \phi(f_6)\} = \{1, 2, 3\}$ and hence $5 \notin \{\phi(e_4), \phi(f_5)\}$. But then $\phi(e_4) = 3$, $\phi(f_4) = 5$ and therefore $5 \in \{\phi(e_6), \phi(f_5)\}$, a contradiction. □

For the further study we will go a little bit more into the details of possible (strong) normal 5-edge-colorings.

Lemma 3.7. *Let G be a cubic graph that has a normal 5-edge-coloring ϕ. If C_6 is a subgraph of G and all its edges are rich, then $E(C_6)$ is partitioned into three color classes, say $\phi^{-1}(1)$, $\phi^{-1}(2)$, $\phi^{-1}(3)$, such that $e_i, e_{i+3} \in \phi^{-1}(i)$, for $i = 1, 2, 3$.*

Proof. Clearly, at least three colors appear at the edges of C_6 since for otherwise there are two edges of the same color with distance 1, contradicting the fact that all edges are rich.

If more than three colors appear at the edges of C_6, then there is a path of length 4, say e_1, e_2, e_3, e_4, whose edges are colored pairwise differently, say $\phi(e_i) = i$. W.l.o.g. we may assume that $\phi(f_2) = 4$ and $\phi(f_3) = 5$. Thus $\phi(f_4) = 1$, and since all edges are rich, it follows that $\{\phi(e_5), \phi(f_5)\} = \{2, 5\}$, $\{\phi(e_6), \phi(f_1)\} = \{3, 5\}$, and hence $\{\phi(e_6), \phi(f_6)\} = \{1, 3\}$ and $\{\phi(e_5), \phi(f_6)\} = \{2, 4\}$, a contradiction. It is easy to see that a coloring as stated in the claim exists. □

Lemma 3.8. *Let G be a cubic graph that contains C_6 as a subgraph and ϕ be a strong normal 5-edge-coloring. If precisely two edges of C_6 are rich, then they receive the same color.*

Proof. It follows from Lemma 3.2 that there are two non-isomorphic distributions of the rich edges.

1) The distance between the rich edges in C_6 is 2. Assume that e_1, e_4 are rich. W.l.o.g. we may assume that $\phi(e_1) = 1$, $\phi(f_1) = 2$, $\phi(e_6) = 3$, $\phi(f_2) = 4$, $\phi(e_2) = 5$. Assume to the contrary $\phi(e_4) \neq 1$.

Case 1: $\phi(e_5) = 1$. Then it follows that $\phi(f_3) = 1$ and $\phi(f_4) = 1$, contradicting the fact that e_4 is rich.

Case 2: $\phi(e_5) \neq 1$, i.e. $\phi(e_5) = 2$, and hence $\phi(f_6) = \phi(f_5) = 1$ and $\phi(e_4) = 3$. But $3 \notin \{\phi(e_2), \phi(f_3)\}$, a contradiction.

2) The distance between the rich edges in C_6 is 1. Assume that e_1, e_3 are rich. W.l.o.g. we may assume that $\phi(e_1) = 1$, $\phi(f_1) = 2$, $\phi(e_6) = 3$, $\phi(f_2) = 4$, $\phi(e_2) = 5$. Assume to the contrary $\phi(e_3) \neq 1$. Then $\phi(e_3) = 4$ and hence $4 \in \{\phi(e_5), \phi(f_5)\}$, and therefore

in any case $4 \in \{\phi(e_5), \phi(f_6), \phi(e_6)\}$. But on the other hand $\{\phi(e_5), \phi(f_6), \phi(e_6)\} = \{1, 2, 3\}$, a contradiction. □

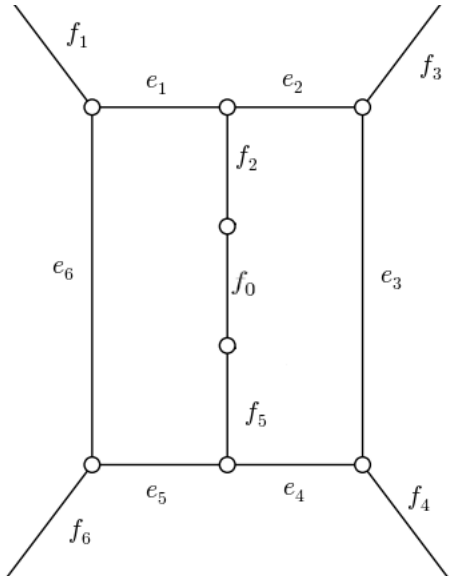

Figure 4: C_6^*

Let C_6^* be the graph of Figure 4 without the edges f_1, f_3, f_4, f_6. Our objective is to reduce the number of non-isomorphic partitions of the edge set of C_6^* into rich and poor edges to the five partitions shown in Figure 5.

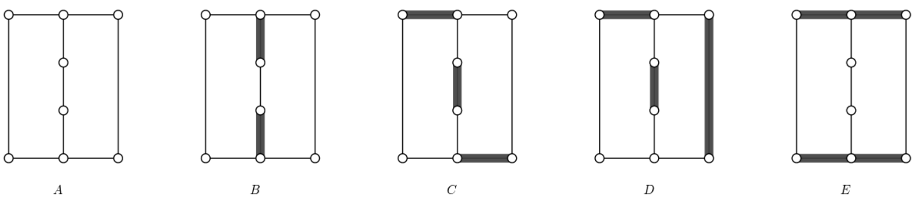

Figure 5: Five types of non-isormorphic partitions of $E(C_6^*)$ into rich and poor edges. (The rich edges are bold.)

Lemma 3.9. *Let G be a cubic graph that has a strong normal 5-edge-coloring. If C_6^* is a subgraph of G and E_p, E_r is a partition of the edges of $E(C_6^*)$ into poor and rich edges, then this partition is isomorphic to one of the types in Figure 5.*

Proof. The result follows by case checking along the number r of rich edges in C_6^*. Let the edges of C_6^* be labeled as in Figure 4. It contains three C_6 - with edge sets $\{e_1, e_2, \ldots, e_6\}$, $\{e_1, f_2, f_0, f_5, e_5, e_6\}$, and $\{e_2, e_3, e_4, f_5, f_0, f_2\}$ - which share pairwise a path of length 3.

$r = 0$: We obtain a partition of type A of Figure 5.

$r = 1$: Then there is a C_6 with an odd number of rich edges, contradicting Lemma 3.2.

$r = 2$: By Lemma 3.2 any of the three C_6 has either no rich edge or two rich edges, which induce two isolated edges. Now it is easy to see that type B of Figure 5 is the only solution (up to isomorphism).

$r = 3$: By Lemma 3.2 any of the three C_6 has two rich edges, which induce two isolated edges. It is easy to see that types C and D are the only possible solutions.

$r = 4$: The matching number of C_6^* is 4. If $r = 4$ and the four rich edges induce a matching, then there is a C_6 that contains an odd number of rich edges, a contradiction. Thus, by Lemma 3.2, we can assume that there is a C_6 such that the rich edges induce two paths of length 2. The only realizable partition is of type E of Figure 5 (up to isomorphism).

$5 \leq r \leq 8$: It is easy to see that Lemma 3.2 can not be satisfied for all three C_6 of C_6^*.

$r = 9$: In this case, we obtain a contradiction to Lemma 3.7. □

The following lemma easily follows from Lemma 3.8.

Lemma 3.10. *Let G be a cubic graph that has a strong normal 5-edge-coloring. If C_6^* is a subgraph of G and the edges of $E(C_6^*)$ are partitioned into poor and rich edges as shown in Figure 5 B, C or D, then the three rich edges receive the same color.*

The flower snarks are invented by Isaacs [4]. They are cyclically 6-edge connected and have girth 6, if $k \geq 3$. For $k \geq 1$, the flower snark J_{2k+1} has vertex set $V(J_{2k+1}) = \{a_i, b_i, c_i, d_i | i = 0, 1, \ldots, 2k\}$ and edge set $E(J_{2k+1}) = \{b_i a_i, b_i c_i, b_i d_i; a_i a_{i+1}; c_i d_{i+1}; d_i c_{i+1} | i = 0, 1 \ldots, 2k\}$ (indices are added modulo $2k + 1$).

Theorem 3.11. *For every $k \geq 1$, the flower snark J_{2k+1} is not strongly Petersen-colorable.*

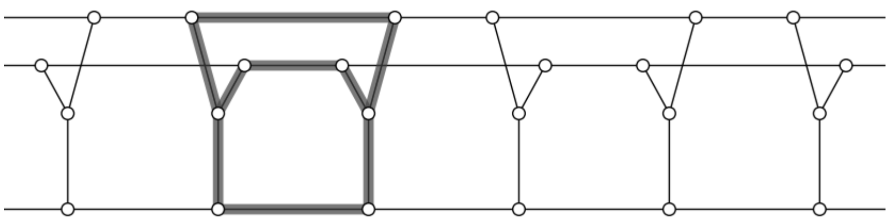

Figure 6: Substructure of J_{2k+1}

Proof. We show that the flower snarks do not have a strong normal 5-edge-coloring. Then the result follows with Theorem 2.2. Assume to the contrary that J_{2k+1} has a strong normal 5-edge-coloring ϕ. Let C_6^* be the graph as indicated in Figure 4. The flower snark J_{2k+1} can be considered as the union of $2k + 1$ copies $D_0, \ldots, D_{2k}$ of C_6^*, where D_i and D_{i+1} share precisely the subgraph which is induced by one vertex of degree 3 and its neighbors (indices are added modulo $2k + 1$); see Figure 6. By Lemma 3.9, the five partitions of the edges of C_6^* shown in Figure 5 are the only non-isomorphic types of possible partitions of the edges of C_6^* into rich and poor edges.

1) There is $i \in \{0 \ldots 2k\}$ such that D_i is of type E. Since D_i shares with D_{i+1} a vertex of degree 3 with its three incident edges, it follows that D_{i+1} is of type E as well.

Hence all D_i are of type E and therefore all edges of the inner cycle of length $2k+1$ are poor, contradicting our assumption, that J_{2k+1} has a strong normal 5-edge-coloring. Thus all D_i are not of type E.

2) There is $i \in \{0 \ldots 2k\}$ such that D_i is of type D. Then D_{i+1} can be of any other type different from E. We may assume that the edge b_ic_i is rich. Hence $c_{i-1}d_i$ and $a_{i-1}a_i$ are rich, too. All the other edges of D_i are poor. If D_{i+1} is of type C or D, then it follows, that two different rich edges, one of D_i and one of D_{i+1} are adjacent. By Lemma 3.10, they all have the same color, contradicting the fact that ϕ is a coloring. Thus D_{i+1} is of type B. On the other hand, D_{i-1} shares with D_i the vertex b_{i-1} of degree 3 which is incident to three poor edges. As above, it follows that D_{i-1} cannot be of type D; thus it is of type A. Since the number of the D_i is odd it follows that the types A, B, C and D cannot combined to get a coloring of J_{2k+1}. Thus all D_i are not of type D.

3) There is $i \in \{0 \ldots 2k\}$ such that D_i is of type A. Since D_i shares with D_{i+1} a vertex of degree 3 with its three incident edges, it follows that D_{i+1} is of type A as well. Not all D_j can be of type A since then J_{2k+1} has no rich edges and therefore it is 3-edge-colorable, a contradiction. Thus all D_i are of type B or C.

4) There is $i \in \{0 \ldots 2k\}$ such that D_i is of type B or C. It follows that D_{i+1} is of type B or C. It turns out, that in any case the two rich edges which are adjacent to the trivalent vertices b_i and b_{i+1} are of the form b_ic_i, $b_{i+1}d_{i+1}$ or b_id_i, $b_{i+1}c_{i+1}$. This implies that eventually two edges b_jc_j and b_jd_j are rich, contradicting the fact that every D_i is of type B or C.

Since the five types of Figure 5 are the only possible strong normal 5-edge-colorings of C_6^* and no combination of them yields a strong normal 5-edge-coloring of J_{2k+1}, it follows with Theorem 2.2 that J_{2k+1} has no strong Petersen-coloring. □

4 Structure of a possible minimum counterexample to the Petersen-coloring conjecture

Jaeger [6] showed that a possible minimum counterexample to the Petersen-coloring conjecture must be cyclically 4-edge connected snark.

If G contains a triangle, then let G^- be the graph obtained from G by contracting the triangle to a single vertex. Clearly, every normal 5-edge-coloring of G^- can be extended to one of G. On the hand, if a cubic graph G has a normal 5-edge-coloring then this coloring can be extended to any graph which is obtained from G by expanding a vertex to a triangle. The following proposition is a reformulation of Proposition 15 in [5].

Lemma 4.1. *Let ϕ be a normal 5-edge-coloring of a bridgeless cubic graph G. If there is an edge e which is contained in a triangle, then e is poor.*

Proof. Let $e_1 = v_1v_2$, $e_2 = v_2v_3$, $e_3 = v_3v_1$ be the edges of a triangle T in G and let f_i be the edge which is incident to v_i and not an edge of T. Assume that e_1 is rich, then $|\phi(\omega(v_1)) \cup \phi(\omega(v_2))| = 5$ and hence e_1, e_2, e_3, f_1, f_2 and f_3 have to receive pairwise different colors; contradicting the fact that ϕ is a 5-edge-coloring. □

Consider $K_{3,3}$ with partition sets $\{u, v, w\}$ and $\{v_1, v_2, v_3\}$. Let $K_{3,3}^*$ be the graph obtained from $K_{3,3}$ by subdividing the edges uv_i and wv_i by vertices u_i and w_i, respectively. Graph $K_{3,3}^*$ is shown in Figure 7.

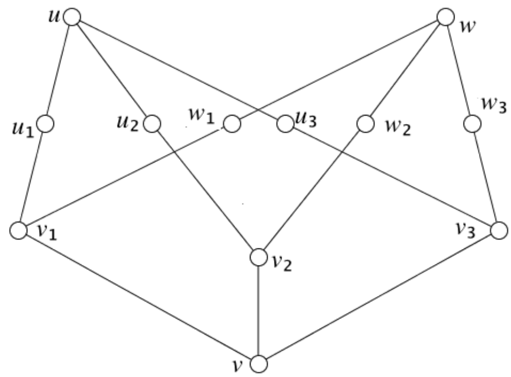

Figure 7: $K^*_{3,3}$

It is easy to see that the statements of this section are also true if we consider Fulkerson-colorings (i.e. a cover with six perfect matchings such that every edge is contained in precisely two of them) instead of Petersen-colorings.

Theorem 4.2. *If G is a minimum counterexample to the Petersen-coloring conjecture (or to the Fulkerson conjecture), then it does not contain $K^*_{3,3}$ as a subgraph.*

Proof. Let ϕ be a normal 5-edge-coloring of G, and assume that $K^*_{3,3}$ is a subgraph of G. Remove the vertices u and w and add edges u_iw_i, for $i = 1, 2, 3$, to obtain a cubic graph G'. Since G is cyclically 4-edge connected it follows that G' is bridgeless. Thus G' has a normal 5-edge-coloring ϕ' by induction hypothesis. Since u_i, v_i, w_i span a triangle in G' ($i = 1, 2, 3$), it follows by Lemma 4.1 that edge u_iw_i receives the same color as vv_i. Thus ϕ' is extendable to a normal 5-edge-coloring of G, a contradiction. The statement follows with Theorem 2.1. The proof for the Fulkerson conjecture is similar. □

This also yields a method to generated cubic graphs with normal 5-edge-colorings from smaller ones (with normal 5-edge-coloring). Let v be a vertex of a cubic graph with normal 5-edge-coloring ϕ, and let w_1, w_2, w_3 be the neighbors of v. Expand w_i to a triangles T_i with vertex set $\{w_{i,1}, w_{i,2}, w_{i,3}\}$ such that $v, w_{i,1}$ are incident, to obtain a graph G_1. Then ϕ can be extended to a normal 5-edge-coloring ϕ_1 on G_1. By Lemma 4.1 it follows that $\phi_1(vw_{i,1}) = \phi_1(w_{i,2}w_{i,3})$. Hence edges $w_{i,2}w_{i,3}$ can be removed and two vertices can be added so that we obtain a $K^*_{3,3}$ as a subgraph and a normal 5-edge-coloring of the new graph.

We will use this fact, to prove Conjecture 1.1 for flower snarks.

5 Petersen-colorings for some families of snarks

5.1 Flower snarks

If a cubic graph G contains a $K^*_{3,3}$ and we reduce it to a smaller graph G' as in the proof of Theorem 4.2, then G' contains three triangles. If we contract these three triangles to single vertices we obtain a new cubic graph G^* that has 8 vertices less than G. Let us say that G is $K^*_{3,3}$-reducible to G^*. Theorem 4.2 can be reformulated as follows:

Theorem 5.1. *Let G be a cubic graph that is $K^*_{3,3}$-reducible to a graph H. If H has a Petersen-coloring, then G has a Petersen-coloring.*

The following lemma is a simple consequence of Lemma 4.2 of [10].

Lemma 5.2. *For $k \geq 1$, the flower snark J_{2k+3} is $K^*_{3,3}$-reducible to J_{2k+1}.*

Since J_3 can be reduced to the Petersen graph by contracting the triangle to a single vertex, Theorem 5.1 and Lemma 5.2 imply the following theorem.

Theorem 5.3. *For all $k \geq 1$, the flower snark J_{2k+1} has a Petersen-coloring.*

5.2 Goldberg snarks

Let $k \geq 5$ be a odd integer. The Goldberg snark [3] G_k is formed from k copies $B_1, \ldots, B_k$ of the graph B in Figure 8 and the edges $\{a_i a_{i+1}, c_i b_{i+1}, e_i d_{i+1}\}$ for each $i \in \{1, 2, \ldots, k\}$ where indices are added modulo k.

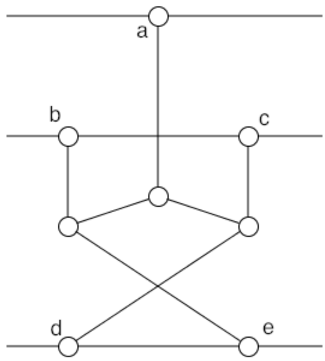

Figure 8: A block B in the Goldberg snark.

Theorem 5.4. *Every Goldberg snark G_k, where $k \geq 5$ is odd, has a Petersen-coloring.*

Proof. Let G_k be a Goldberg snark. Then G_k can be constructed from one 3-block (see Figure 10) and $\frac{k-3}{2}$ 2-blocks (see Figure 9). Using the normal 5-edge-colorings provided in Figure 9 and 10 it is easy to see that it will give a normal 5-edge-coloring of G_k. □

5.3 Blanuša snarks

Let G be a Blanuša snark of type 1 as defined in Section 3.1. If we color the blocks $A_1, \ldots, A_{r-1}$ as in figure 11 and A_r and C_1 as in figure 12 and 13, it is easy to see that we have a normal edge coloring of all such graphs.

The generalized Blanuša snarks of type 2 are formed by joining r copies of A and one copy of C_2 (see Figure 14). Once again it is straightforward to see that all such graphs has normal edges colorings by coloring $A_1, \ldots, A_{r-2}$ as in Figure 11, A_{r-1} as in Figure 13 and finally C_2 as in Figure 14.

From this we get the following theorem.

Theorem 5.5. *All generalized Blanuša snarks of type 1 and 2 have Petersen-colorings.*

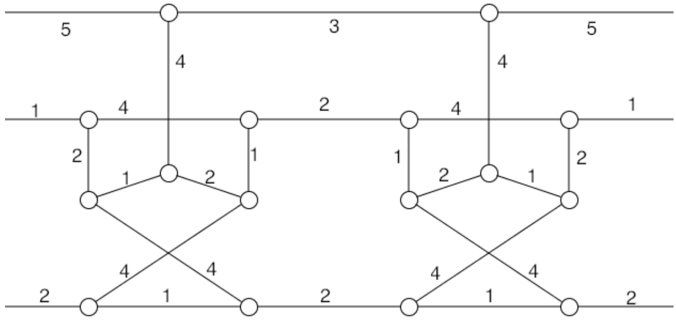

Figure 9: A 2-block in the Goldberg snark with a normal 5-edge-coloring.

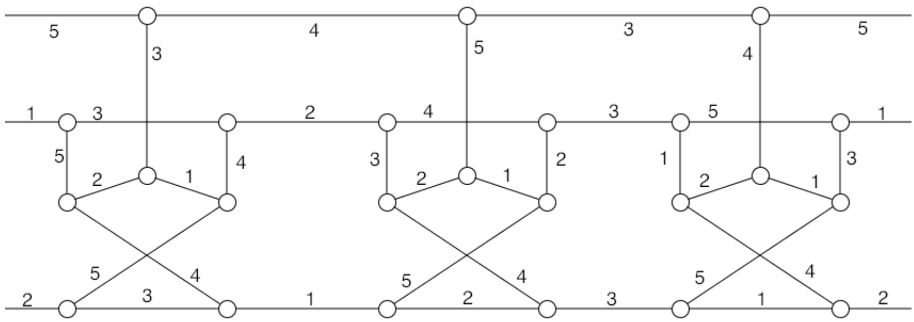

Figure 10: A 3-block in the Goldberg snark with a normal 5-edge-coloring.

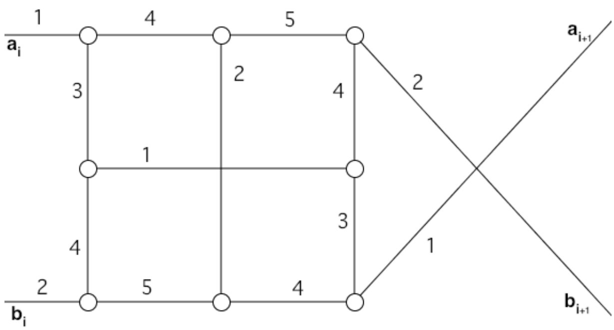

Figure 11: Block A_i in the generalized Blanuša snark.

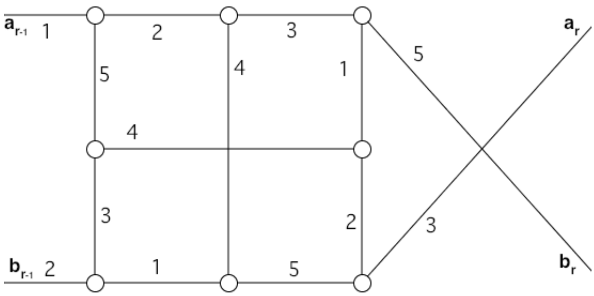

Figure 12: Block A_r in the generalized Blanuša snark.

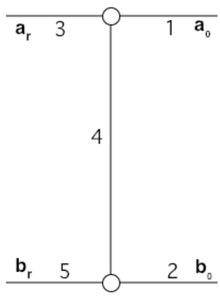

Figure 13: Block P_2 in the generalized Blanuša snark.

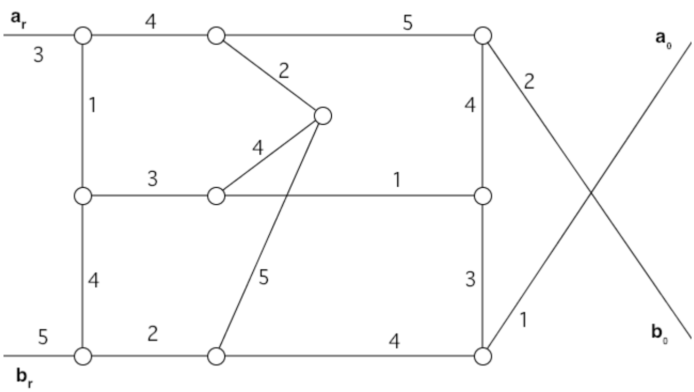

Figure 14: Block C_2 in the generalized Blanuša snark.

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Constructions for large spatial point-line (n_k) configurations

Gábor Gévay *

Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H-6720 Szeged, Hungary

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Abstract

Highly symmetric figures, such as regular polytopes, can serve as a scaffolding on which spatial (n_k) point-line configurations can be built. We give several constructions using this method in dimension 3 and 4. We also explore possible constructions of point-line configurations obtained as Cartesian products of smaller ones. Using suitable powers of well-chosen configurations, we obtain infinite series of (n_k) configurations for which both n and k are arbitrarily large. We also combine the method of polytopal scaffolding and the method of powers to construct further examples. Finally, we formulate an incidence statement concerning a (100_4) configuration in 3-space derived from the product of two complete pentalaterals; it is posed as a conjecture.

Keywords: Spatial configuration, Platonic solid, regular 4-polytope, product of configurations, incidence statement.

Math. Subj. Class.: 51A20, 51A45, 51E30, 52B15

1 Introduction

By a (p_q, n_k) *configuration* we mean a set consisting of p points and n lines such that k of the points lie on each line and q of the lines pass through each point [7, 16, 19, 21]. If, in particular, $p = n$, then $q = k$; in this case the notation (n_k) is used, and we speak of a *balanced configuration* [16]. We consider configurations embedded either in Euclidean or projective spaces.

In the last decades, there has been a revival of interest in point-line configurations; the developments and results are summarized in the quite recent research monograph by Branko Grünbaum [16]. This book deals predominantly with planar configurations. However, as the author notices in the Postscript, a "*...seemingly safe guess is that there will be interest in higher-dimensional analogues of the material described in this book*".

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E-mail address: gevay@math.u-szeged.hu (Gábor Gévay)

In fact, the theory was and has not been restricted to planar configurations. Research in higher dimensions go back to Cayley, Cremona, Veronese and others [4, 7, 10, 16]. One of the most well known spatial configurations is Reye's $(12_4, 16_3)$ configuration in projective 3-space [19, 25]. Its construction is based on the ordinary cube; the cube serves, so to say, as a "scaffolding": once the configuration has been built, the underlying cube is deleted. Moreover, the configuration inherits its symmetry (at least, in this case, in a combinatorial sense).

We apply the same building principle in several of our constructions. For an underlying polytope, we choose a regular polytope in dimension 3 or 4. In all but one of these cases, the geometric symmetry group of the configuration will be the same as that of the underlying polytope. We note that the term *geometric symmetry group*, or briefly, *symmetry group*, is meant in the usual sense; i.e., it denotes the group of isometries of the ambient (Euclidean) space that map the configuration onto itself. The combinatorial counterpart of this notion is the group of *automorphisms*, i.e., the group of incidence-preserving permutations of points and lines (both among themselves); Coxeter simply calls it the *group of the configuration* [7]. (The distinction between these two types of groups will play some role in Section 5.)

In Section 4 we apply the notion of a product, which can be considered as the Cartesian product of configurations. Using this tool, we construct two infinite series and a finite class of (n_k) configurations which are powers of smaller configurations. In this way very high n and k values can be attained (and so can the dimension of the space spanned by these configurations). We think this may be interesting for future research; for, as it is emphasized just recently, little is known in general on the existence of such large configurations [3].

We summarize our results in the following theorem.

Theorem 1.1. *There exist (n_k) configurations which form the following classes:*

1. *infinite series of type*

$$\big(18(t+1)_3\big), \quad \big(36(t+1)_3\big) \quad \textit{and} \quad \big(90(t+1)_3\big), \ \ (t = 1, 2, \dots);$$

   they have the symmetry group of a regular tetrahedron, cube and dodecahedron, respectively, and each spans the Euclidean space $\mathbb{E}^3$;

2. *infinite series of type*

$$\left(\left(\binom{2k+1}{2}^k\right)_{2k}\right), \ \ (k = 2, 3, \dots);$$

   they span the projective space $\mathbb{P}^{2k}$;

3. *infinite series of type*

$$\Big(\big((2k)^{2k-2}(2k+1)k\big)_{2k}\Big), \ \ (k = 2, 3, \dots);$$

   they span the projective space $\mathbb{P}^{2k}$;

4. *finite class of types*

$$(240_3), \quad (768_3) \quad \textit{and} \quad (28\,800_3),$$

with the symmetry group of a regular 4-simplex, a regular 4-cube, or a regular 120-cell, respectively; each spans the Euclidean space $\mathbb{E}^4$;

5. *finite class of types*

$$\begin{array}{ll} (14\,400_4) & \subset \mathbb{E}^6 \\ (5\,832\,200_6) & \subset \mathbb{E}^9 \\ (3\,317\,760\,000_8) & \subset \mathbb{E}^{12} \\ \left((2.43 \cdot 10^{10})_{10}\right) & \subset \mathbb{E}^{15} \\ \left((2^{18} \cdot 3^{12} \cdot 5^6)_{12}\right) & \subset \mathbb{E}^{18}, \end{array}$$

each full-dimensional in the given Euclidean space;

6. *sporadic examples of type*

$$(180_3), \quad (60_4), \quad (540_4) \quad \textit{and} \quad (780_4),$$

with the symmetry group of a regular dodecahedron, all of them spanning the Euclidean space $\mathbb{E}^3$.

In Sections 2–4 below, we construct all these configurations, and thus their existence is proved; the formula numbers of them at the location where they are actually constructed are as follows: 1: (3.8); 2: (4.1); 3: (4.2); 4: (3.14); 5: (4.5); 6: (3.10), (2.1), (2.2) and (2.3).

In each case, an essential part of the constructive proof is to exclude unintended incidences (i.e., incidences that do not belong to the given configuration, cf. [16], Section 2.6). We emphasize that this possibility has been checked in each case. However, in all but one case this part of the proof was omitted, to save space (the one exception is the case of type (2.2), where the problem has been indicated in Remark 2.2).

In stating our results, we avoid using the term "d-dimensional" for a configuration that we construct in some space of dimension d. The reason is that the *dimension* of a configuration $\mathcal{C}$ is defined as the largest dimension of the space that is spanned by $\mathcal{C}$ (see p. 24 and Section 5.6 in [16]). Thus, we can only state that each configuration in our theorem above *spans* the given space (sometimes we also say, equivalently, that the configuration is *full-dimensional* in the given space). Investigating the actual dimension of our configurations is beyond the scope of this paper.

We think that the symmetry group is an essential property of our configurations realized in some Euclidean space. Although we did not investigate, we believe that several of these configurations have the maximal symmetry which is possible in the given space. (We note that e.g. for convex polytopes a question like this is far from trivial [12, 17].) It would also be interesting to know that for $d' < d$, whether there is a configuration in $\mathbb{E}^d$ whose symmetry group is a subgroup of the orthogonal group of $\mathbb{E}^{d'}$ and which is not symmetrically realizable in $\mathbb{E}^{d'}$. We consider these and other related questions as a possible subject of future study.

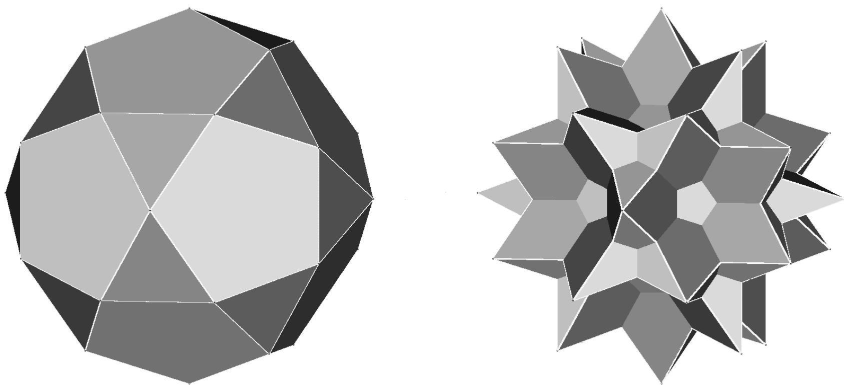
Finally, in Section 5 we present an incidence conjecture. It is suggested by one of the new configurations that we found. This is a (100_4) configuration in projective 3-space consisting of four quintuples of complete pentalaterals (thus there are altogether 400 incidences, 100 for each quintuple). Informally, the conjecture states that the incidences belonging to three quintuples of the complete pentalaterals imply the remaining 100 incidences.

2 First examples of spatial configurations

Before presenting our more detailed constructions, we remark that there are spatial point-line configurations which need little or no construction. They are there in the zoo of geometric figures which have been known for a long time; one just has to realize them. The first configuration presented here is just such an example. It is a nice

$$(60_4) \tag{2.1}$$

configuration, provided by a polyhedron called the *great icosidodecahedron*. The lines of the configuration are spanned by the edges, and half of its points are the vertices of this polyhedron.



(a) (b)

Figure 1: The icosidodecahedron (a), and the great icosidodecahedron (b).

The great icosidodecahedron is one of the 53 non-regular non-convex uniform polyhedra [8, 18, 22] (we note that a polyhedron is called *uniform* if its faces are regular polygons and its symmetry group is transitive on its vertices). It also occurs in [6], given by its Coxeter's symbol $\begin{Bmatrix} 5/2 \\ 3 \end{Bmatrix}$; this indicates that it has triangles and pentagrams (i.e. $\{5/2\}$ *star-polygons*, see [6]) as faces. The number of these faces is 20 and 12, respectively. Its name refers to its close relationship with its convex hull, the icosidodecahedron (one of the Archimedean solids). In particular, its 30 vertices coincide with those of the icosidodecahedron.

The mutual position of its two pentagram faces in non-parallel planes is of two kinds (just like that of the faces of the regular dodecahedron): the angle between them is either $\arctan 2$, or $\pi - \arctan 2$. The angle between the planes of two such faces sharing a common vertex is the acute angle. In addition to its vertices, the edges of a pentagram have five other intersection points (these points can be called "internal vertices" if, instead of a pentagram, we speak of a—complete—pentalateral, cf. Section 5 below). These "internal vertices" of the pentagrams do not belong to the vertex set of the polyhedron (and, in strict sense, not even to that of the pentagram). But two pentagrams in planes with obtuse angle between them share such an "internal vertex". Taking into account these latter points as

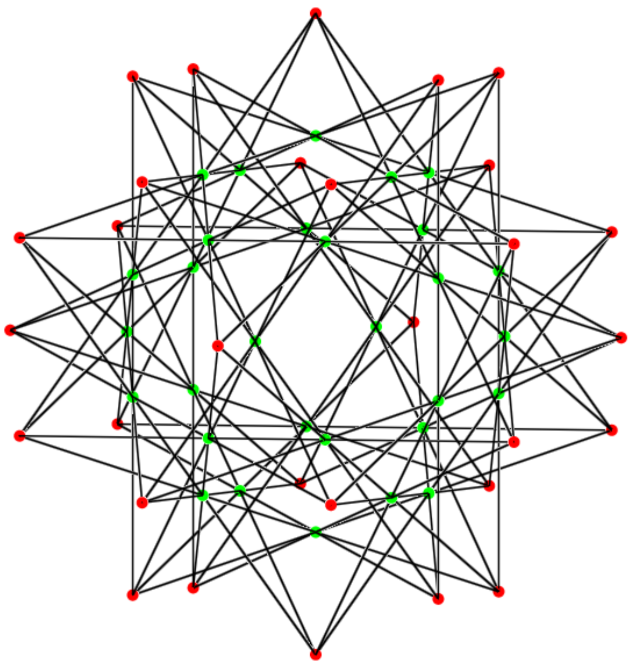


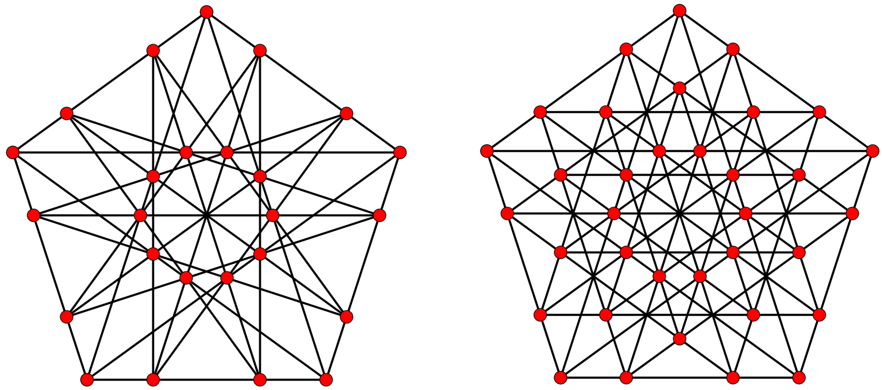
Figure 2: The (60_4) configuration associated with the great icosidodecahedron.

well, we have a system consisting of $12 \times 5 = 60$ edges, and 30+30 points, the latter all tetravalent. Replacing the edges by the lines that are spanned by them, we obtain directly the configuration (2.1). The symmetry properties of the underlying polyhedron imply that this configuration has two orbits of points and a single orbit of lines. This relatively high degree of symmetry makes it particularly interesting.

The same configuration is also provided by two other types of polyhedra in the same natural way. Namely, both the *great icosihemidodecahedron* $\begin{Bmatrix} 10/3 \\ 3 \end{Bmatrix}$ and the *great dodecahemidodecahedron* $\begin{Bmatrix} 10/3 \\ 5/2 \end{Bmatrix}$ has a system of vertices and edges coinciding with that of the great icosidodecahedron; and, the other 30 points of the configuration are provided likewise. With the details omitted, we just remark that all three types of these polyhedra can be derived from the regular dodecahedron.

We note that a much more simple example can be obtained from the ordinary cube, in the following way. Let the points be the 8 vertices of the cube, the 6 centres of the faces of the cube and the centre of the cube. As lines, choose the 12 diagonals of the faces of the cube, plus the 3 lines between the centres of two opposite faces of the cube. Thus we obtain a (15_3) configuration.

Our next two examples require some more steps of construction. We start from two planar configurations. The first is a (25_4) configuration, due to Jürgen Bokowski ([15], Figure 4; see also [16], Figure 3.3.13). The other is closely related to this and is due to Branko Grünbaum ([15], Figure 9). These configurations are shown in our Figure 3.



(a) Jürgen Bokowski's (25_4) (b) Branko Grünbaum's (35_4)

Figure 3: Two planar configurations.

Put 12 copies of the (25_4) configuration onto the faces of a regular dodecahedron so that the vertices of the pentagonal "frame" of the configuration coincide with the midpoints of edges of the dodecahedron. Then delete the edges of the dodecahedron and all the external edges of the 12 pentagonal frames. Thus we obtained a system, which is not a configuration; however, for each of its lines there are precisely four points incident with it. A system like this deserves to be the subject of a new definition.

Definition 2.1. A set consisting of p points and n lines is called a *semiconfiguration* if either of the following conditions hold:

(1) each point is incident with precisely q lines; or

(2) each line is incident with precisely k points.

The type of a semiconfiguration is denoted by (p_q, n_*) or (p_*, n_k), respectively.

If one wants to specify which version is actually used, one may call it a P-semiconfiguration or an L-semiconfiguration; thus, the abbreviation refers to the fact that the incidences are uniformly distributed among the points or the lines, respectively. Clearly, a system is a configuration if and only if it is both P-semiconfiguration and L-semiconfiguration.

Using this notion, we see that the system we obtained in the present step is an L-semiconfiguration of type $(270_*, 240_4)$. It contains a class of 120 trivalent points, a class of 120 tetravalent points, and a class of 30 tetravalent points. These classes are distinguished by the position of their points; in fact, they are transitivity classes with respect to the symmetry group of the dodecahedron. To obtain a balanced configuration, take a second, concentric and homothetic copy of this semiconfiguration; thus we have an "outer shell" and an "inner shell" of points. Finally, connect the trivalent points of these shells by radial lines. Thus we have 60 new lines, all incident with four points. At the same time, the 2×120 trivalent points turned into tetravalent. Hence we obtained a configuration of type

$$(540_4). \tag{2.2}$$

It has 6 orbits of points and 7 orbits of lines. It is shown in Figure 4. Although this figure, due to the large number of its elements, is necessarily somewhat crowded, the two shells can be distinguished; the radial lines are indicated by orange colour.

Remark 2.2. The relative size of the outer and the inner shell must be chosen carefully in order to avoid unintended incidences. Clearly, there are infinitely many choices.

Using the (35_4) configuration, and proceeding analogously, we obtain a

$$(780_4) \tag{2.3}$$

configuration. It has 8 orbits of points and 13 orbits of lines.

We note that one may find several analogous cases on the basis of our examples (2.2) and (2.3). For example, a geometrically different but completely analogous (540_4) configuration can be obtained by starting from another planar (25_4) configuration that is shown in [2], Figure 4a. Furthermore, starting from the same planar configurations $\mathcal{P}$ with pentagonal symmetry as above, one can also obtain geometrically different examples if one chooses other points of $\mathcal{P}$ to tack onto the edges of the dodecahedron's face, and delete the appropriate lines (the radial lines will also be different in these cases).

The full icosahedral symmetry can also be reduced so as to obtain a chiral configuration in this construction as well. One just has to replace the starting planar configuration by a suitable chiral one. What is more, even movable spatial examples can be obtained in

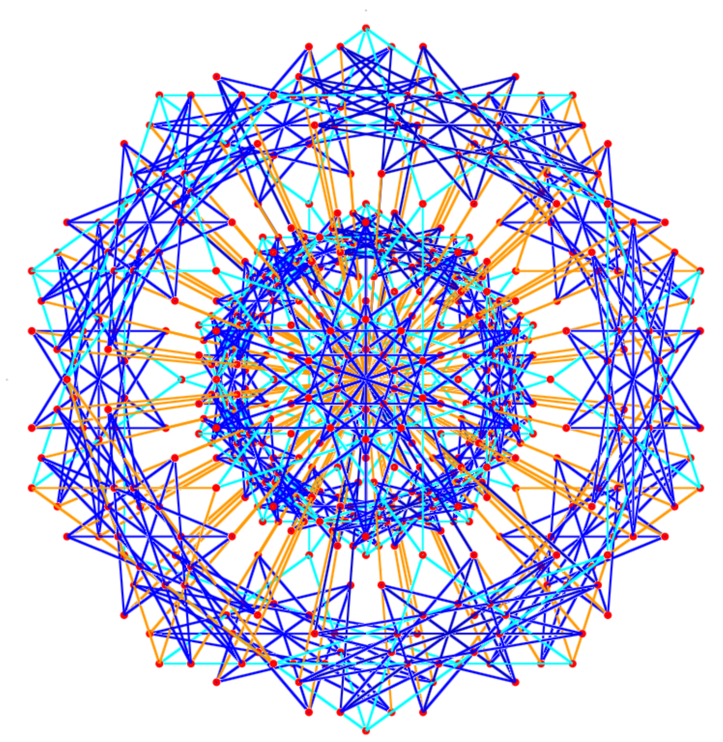


Figure 4: A (540_4) configuration.

this way, too; see a beautiful construction in [1] for movable planar (n_4) configurations (the example with 10-fold rotational symmetry given there in Figure 4 may be a possible candidate to this purpose). We do not pursue this idea here.

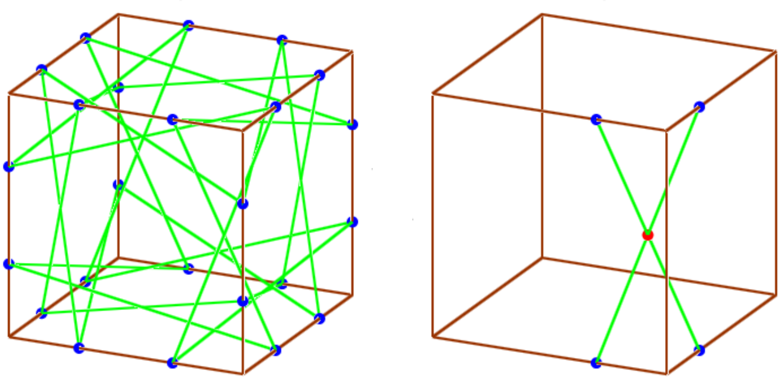
3 Classes of configurations based on regular polytopes

First we construct three infinite series of balanced configurations, so that we make use of the structure and symmetry properties of Platonic solids. (We note that these series have already been mentioned in [13], p. 327).

In what follows, TP denotes a Platonic solid whose 1-skeleton is a trivalent graph, i.e. TP is a tetrahedron, cube, or dodecahedron. It is well known that the Petrie polygon of these polytopes is a (regular, skew) quadrangle, hexagon, or decagon, respectively. (We recall that the Petrie polygon of a regular 3-polytope is a skew polygon such that any two consecutive edges, but no three, belong to a face of the polytope [6].) Given a Petrie polygon, consider for each of its vertices the third edge emanating from it but not belonging the to Petrie polygon. Take a point on each of these edges such that it subdivides, but not bisects, the edge in an arbitrary but fixed ratio; moreover, it is closer to the endpoint of the edge belonging to the Petrie polygon than to the other endpoint. Connect these points by straight line segments in the cylic order induced by the Petrie polygon. What is obtained is again a regular skew polygon, clearly having the same number of edges as the Petrie polygon we started from. We shall call it a *P-polygon*. The vertices of a P-polygon, together with the lines spanned by their edges, form a (p_2) configuration, where p is 4, 6 or 10 according as TP is the tetrahedron, the cube or the dodecahedron, respectively. Clearly, the number of Petrie polygons and P-polygons is the same in a given TP, that is 3, 4 or 6, respectively. It follows that taking the disjoint union of all these P-polygons (more precisely, the corresponding (p_2) configurations), one obtains a (non-connected) configuration whose type is

$$(12_2), \quad (24_2) \quad \text{or} \quad (60_2). \tag{3.1}$$

We call this configuration a *P-system* (see Figure 5).



(a) The whole system. (b) Edge crossing of two P-polygons.

Figure 5: The P-system (24_2) in the cube.

We emphasize that when constructing the P-system, the same ratio is used in the definition of each P-polygon. Thus it follows from the construction that the P-system inherits the (geometric) symmetry of TP, i.e. it has the same symmetry group. Moreover, note that both its points and its lines form a single transitivity class.

Looking at this configuration more closely, we find that it can be extended to form a non-balanced but connected configuration. For, observe that each edge e of TP belongs to precisely two distinct Petrie polygons. Moreover, these Petrie polygons are mirror images of each other with respect to the mirror plane of TP containing the edge e. It follows that the edges of the corresponding P-polygons cross each other in the vicinity of e in a point lying in that mirror plane. Furthermore, this point also lies in the mirror plane that is perpendicular to the former plane and bisects e. This amounts to saying that this point is on the line connecting the centre of TP with the midpoint of e. (For an example of such a crossing point in the cube, see Figure 5b, where it is indicated by red colour).

The number of these crossing points equals the number of the edges of TP. Hence, adding them to the P-system, we obtain a new configuration, which we shall call a *P-configuration*. Its type is

$$(18_2, 12_3), \quad (36_2, 24_3) \quad \text{or} \quad (90_2, 60_3). \tag{3.2}$$

The special position of the crossing points provides the possibility of a further extension of this configuration, as follows. Shrink the 1-skeleton of the original TP until the midpoints of its edges coincide with the crossing points, and add it to the configuration. Then, remove the vertices of this skeleton, and replace each of its edges by the line spanned by it. The new structure that is obtained is a subfiguration.

A *subfiguration* (p_q, n_k) is defined as a set of points and lines with incidences as in the definition of configurations, but with the difference that each of the p points is incident with at most q of the n lines, and each line is incident with at most k of the points [16]. If we want to emphasize that the number of the missing incidences is s (in comparison to a (p_q, n_k) configuration), we say that it is an *$\#s$-subfiguration*. (We note that in the converse case the notion of a *superfiguration* has also been introduced, in a similar way, in [16]).

Thus, the subfiguration that is obtained is of type

$$(18_3)^-, \quad (36_3)^- \quad \text{or} \quad (90_3)^-, \tag{3.3}$$

with s equal to 24, 48 or 120, respectively. (Here the superscript refers to the missing incidences; we use this notation to avoid confusion with a configuration of type (n_k).) One half of the missing incidences of this subfiguration belong to the one subsystem, and the other half of them belongs to the other subsystem, of which it is composed. For example, in the cubic case there are 24 missing incidences because the points of the P-system are of valency two, instead of three; and there are 24 other missing incidences, since all the 8 vertices of the cube skeleton (which are of valency three, and have been removed) are missing. We shall call these two kinds of defective points type A and type B, respectively. Likewise, the corresponding subsystems will be referred to as type A and B, respectively.

Due to the equal number of the defective points of the two types in the two subsystems, this subfiguration can serve as a repetitive unit; hence we shall call it an *R-unit*. The repetition is meant in the following way. Take a copy of an R-unit R_1, and shrink it so as to obtain a homothetic copy R_2, such that the points of type A of R_2 fit onto the corresponding lines in the subsystem of type B of R_1; then take the union $R_1 \sqcup R_2$. Using Figure 5a, in the cubic case this can simply be conceived as if the brown lines belonged to R_1 and the green lines belonged to R_2.

Observe that the new figure obtained in this way is again an $\#s$-subfiguration such that s remains the same; for, half of the defects both in R_1 and R_2 have been repaired, but the other half in both of them remained. At the same time, both the number of points and lines have been doubled.

The operation that we applied here is not simply a disjoint union; for, new incidences occured, and (in our particular case) the result is a connected structure. Thus we think it is appropriate to fix these properties in a separate definition. As that will refer not only to configurations, first we give a common name for all the four related types of structures used in this paper: we shall call such a structure an *X-figuration*, where "X-" may mean either "con", "semicon", "sub" or "super".

Definition 3.1. By the *incidence sum*[1] of X-figurations $\mathcal{F}_1$ and $\mathcal{F}_2$ we mean the X-figuration $\mathcal{F}$ which is the disjoint union of $\mathcal{F}_1$ and $\mathcal{F}_2$, together with a specified set $I \subseteq \mathcal{P}_1 \times \mathcal{L}_2 \cup \mathcal{P}_2 \times \mathcal{L}_1$ of incident point-line pairs, where $\mathcal{P}_i$ denotes the point set and $\mathcal{L}_i$ denotes the line set of $\mathcal{F}_i$, for $i = 1, 2$. We denote it by $\mathcal{F}_1 \oplus_I \mathcal{F}_2$.

Note that $\mathcal{F}_1$ and $\mathcal{F}_2$ may form distinct incidence sums depending on the set I; we do not consider here such cases; on the other hand, if the set I is fixed and is clear from the context (as in our present case), it can be omitted from the operation symbol.

Accordingly, in the present step of our construction we obtained the subfiguration of the form $R_1 \oplus R_2$. Furthermore, it is clearly seen that the process by which we obtained $R_1 \oplus R_2$ from R_1 can be repeated arbitrary many times. Thus, let λ be a shrinking factor defined by the equality $R_2 = \lambda R_1$, and set $R = R_1$. Then, starting with R, we obtain after $t - 1$ steps the subfiguration

$$\bigoplus_{i=1}^{t} \lambda^{i-1} R, \tag{3.4}$$

which is still an $\#s$-subfiguration with $s = 24, 48$ or 120, and is of type

$$((18t)_3)^-, \quad ((36t)_3)^- \quad \text{or} \quad ((90t)_3)^-, \tag{3.5}$$

respectively.

Finally, we have to extend this subfiguration, so as to obtain a configuration. First we construct a unit which, when added, closes the structure "outside". This construction also is analogous for each of the three types of TP; we explain it in the case of the tetrahedron. Start from the 1-skeleton of a regular tetrahedron, and take the midpoints of its edges. Add these points to the structure, so that one obtains a spatial graph with 10 vertices and 12 edges, such that four vertices are trivalent and six vertices are bivalent. For each bivalent vertex, take a line connecting it to the centre of the tetrahedron, and shift the vertex along this line outwards, each to the same extent; simultaneously, the edges incident to these vertices are stretched, and remain straight line segments. Although any ratio would serve our purpose, we note that if the distance of these shifted vertices from the centre is twice that of the original, then the angle between any two adjacent edges is $\arccos(-1/3)$. This is the famous "tetrahedral bond angle" in organic chemistry, and the figure that we obtained is precisely the carbon skeleton of a hydrocarbon molecule called *adamantane*[2], well known

[1] The present (improved) version of this definition was proposed by Tomaž Pisanski.

[2] The name refers to the fact that this is a repetitive unit of the diamond crystal lattice.

to chemists. This skeleton is shown in Figure 6 (by courtesy of H. Ramezani, from [24]). A related figure can be obtained from the 1-skeleton of either kind of TP in an analogous way; so we shall call each of them an *adamantane skeleton*. (Note that they have the symmetry of the type of TP from which it has been derived).

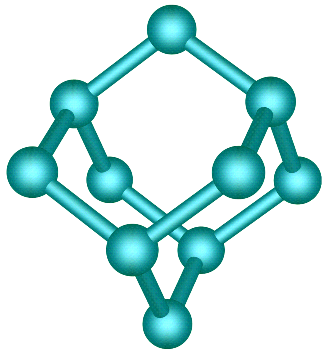


Figure 6: The adamantane skeleton.

We extend the adamantane skeleton in the following way. Take the 1-skeleton of a TP of suitable size and position (and of the corresponding type) such that the midpoints of its edges coincide with the bivalent vertices of the adamantane skeleton. Form the union of these figures, then replace each of its edges by the line spanned by it. We obtain a P-semiconfiguration of type

$$(14_3, 18_*), \quad (28_3, 36_*) \quad \text{or} \quad (70_3, 90_*). \tag{3.6}$$

This semiconfiguration will serve as a closure unit so as to close our construction "outside". In fact, observe that it has defective lines, i.e., lines that are incident with two points, instead of three (the lines corresponding to the original half-edges of TP). The number of these lines is 12, 24 or 60, respectively, and this is the same as the number of the missing incidences. On the other hand, it is half of the number of missing incidences of our subfiguration (3.4) (these latter come from points of type A). Furthermore, by taking a copy of a suitable size of this closure unit, one can form the incidence sum of it with the subfiguration (3.4). The result is a semiconfiguration of type

$$\begin{gathered}\big((18t+14)_3, (18(t+1))_*\big), \quad \big((36t+28)_3, (36(t+1))_*\big) \\ \text{or} \quad \big((90t+70)_3, (90(t+1))_*\big).\end{gathered} \tag{3.7}$$

The very last step of our construction is to close our system "inside". This is very simple, since 4, 8 or 20 points are missing from the smallest R-unit of the subfiguration (3.4) (points of type B, each representing three incidences). These are nothing else than the vertices of a tetrahedron, cube or dodecahedron, respectively. We just add these points to the system, and our construction is ready, resulting in three infinite series of balanced configurations, whose type is

$$\big(18(t+1)_3\big), \quad \big(36(t+1)_3\big) \quad \text{and} \quad \big(90(t+1)_3\big), \quad (t=1,2,\dots). \tag{3.8}$$

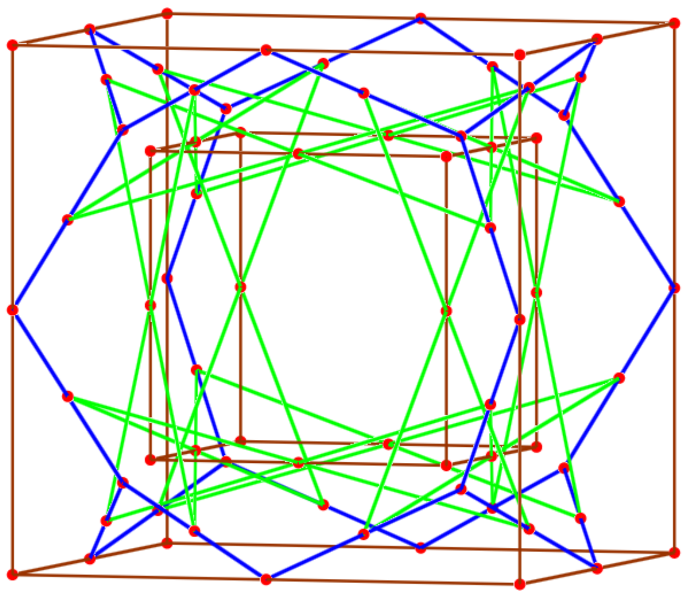


Figure 7: A (72_3) configuration: the cubic case with $t = 1$ of (3.8).

We note that the latter "closure units", consisting merely of points (but with fixed mutual position), can also be conceived as semiconfigurations. Denoting them by C_I, and those in (3.6) by C_O, we see that our configurations of type (3.8) can be described (and in fact, have been constructed) in the following form:

$$C_O \oplus \left(\bigoplus_{i=1}^{t} \lambda^{i-1} R \right) \oplus C_I, \tag{3.9}$$

where the middle term is the subfiguration (3.4).

We emphasize that throughout the construction, the original symmetry of the TP we started from is preserved. Thus the (geometric) symmetry group of all of the configurations which we obtained here is equal to that of the corresponding Platonic solid.

Our next construction provides a sporadic example. In this construction we apply some structural elements that have already been constructed above.

Start from the *compound* of five tetrahedra, which can be obtained by inscribing these tetrahedra in a regular dodecahedron [6] (see Figure 8). This is made possible by the property that the set of vertices of the dodecahedron can be partitioned into five quadruples such that within a quadruple, the vertices are at a distance 3 from each other (regarded in the graph of the dodecahedron). The same compound also determines a partition of the set of edges of the dodecahedron into five sextuples in the following way. Consider a compound inscribed in a dodecahedron of a fixed size. Apply a dilation to this compound. It is chosen so that the following condition holds. Let $ABCD$ be any path of length 3 in the graph of the dodecahedron, and let M be the midpoint of the corresponding tetrahedron edge AD in the compound. Then the dilate M' of M coincides with the midpoint of the edge BC of the dodecahedron. Thus, each sextuple of the dodecahedron edges corresponds

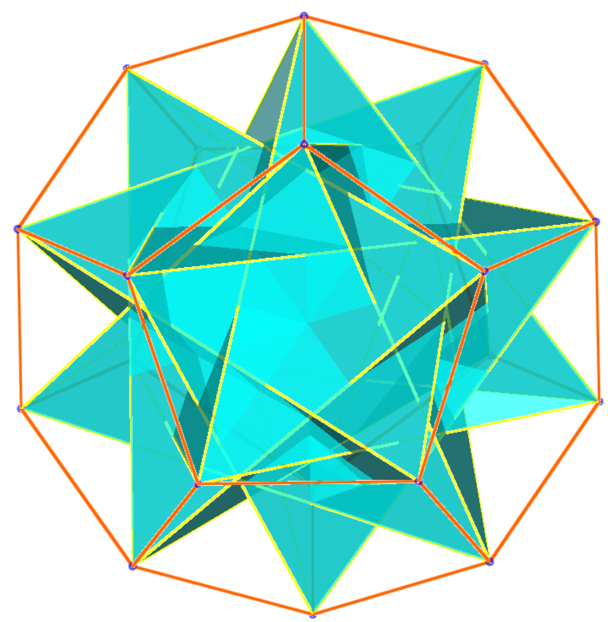


Figure 8: The compound of five tetrahedra inscribed in a dodecahedron.

to the set of edges of a tetrahedron in the compound. A consequence is that one can inscribe a (tetrahedral) adamantane skeleton in each sextuple of the edge-midpoints of the dodecahedron so that the bivalent vertices of the adamantane skeleton coincide with these edge-midpoints. In such an inscribed adamantane skeleton, we inscribe a P-configuration (3.2) constructed previously in this section, in the sense that the points of the latter fit onto the edges of former; moreover, this is performed so that the original local tetrahedral symmetry is preserved. Then we add a tetrahedron skeleton to this structure, so that the midpoints of the tetrahedron edges coincide with the "crossing points" of the P-configuration (cf. Figure 5b). Replace each edge by the corresponding line; thus, taking into account all the points, lines and incidences, we obtained a $(32_*, 30_3)$ semiconfiguration inscribed in a sextuple of edges of the dodecahedron.

By inscribing altogether five copies of this semiconfiguration into the 1-skeleton of the dodecahedron in the same way (and replacing its 30 edges by lines), a configuration of type

$$(180_3) \tag{3.10}$$

is obtained (it is mentioned in [13], too). The symmetry group of the compound of tetrahedra which we started from is the rotation group T of the tetrahedron; thus this compound is a *chiral* figure, i.e. it has no mirror symmetry. Our new configuration inherited this symmetry group, so it is a chiral configuration. Its set of points decomposes into 6 orbits, while there are 4 orbits of lines. These latter orbits are indicated with different colours in Figure 9. Note, in addition, that the structure of this configuration is closely related to those described above in this section. In fact, the orbits correspond to those of the tetrahedral case of (3.8), with $t = 1$. The only difference is that the outermost tetrahedral orbit is replaced by a dodecahedral orbit, and the others are multiplied by five.

Our last class constructed here is based on certain regular 4-polytopes. We start from a TP that we used above. Take a P-system inscribed in it (that is, inscribed in the sense that the points of the configuration lie on the edges of TP). Then take a smaller homothetic copy of TP in concentric position, and also inscribe a P-system in this copy. The smaller P-system is chosen so that it is not the homothetic copy of the larger one, but each triple of their vertices in the vicinity of a vertex of the TP (determining it) is relatively at a smaller distance from that vertex, than the triple of vertices of the larger P-system is from the corresponding vertex of the larger TP.

If the two P-systems were homothetic copies of each other (with respect to their common centre), then the lines connecting their corresponding points would meet all in a common point (in fact, in the centre). However, due to our particular choice, these lines meet now in threes, forming altogether 4, 8 or 20 points of intersection (depending on the type of TP). This is a consequence of the threefold rotational symmetries of TP. Thus for each such triples of lines in the vicinity of a given vertex v of TP there is a point of intersection which lies on the axis of rotation connecting v with the centre, and this point also is located in the vicinity of v. We shall not use these points later, they just served to explain the location of the connecting lines. On the contrary, we need the connecting lines in the following, so we shall call them c-lines. Another condition for the c-lines is that they are not perpendicular to the edges of TP (this can also be ensured by a suitable choice of the P-systems).

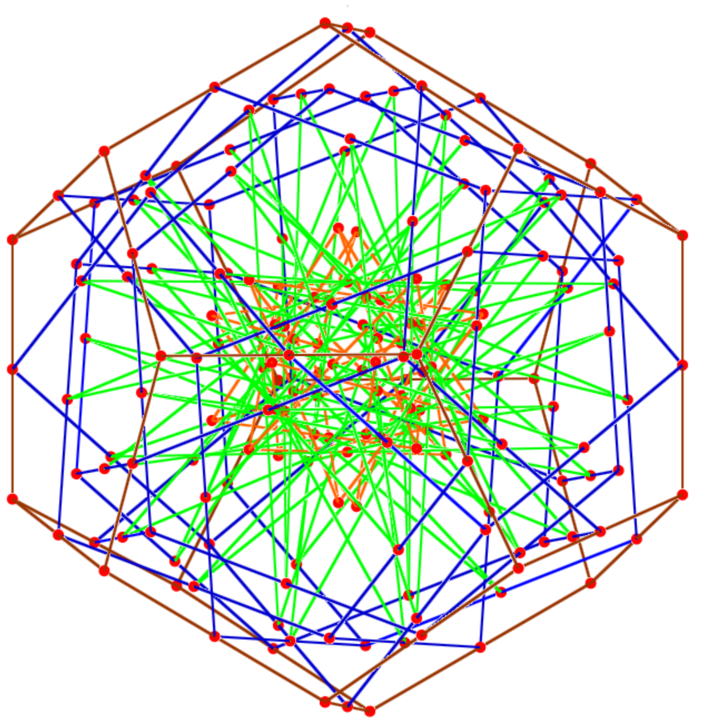


Figure 9: A (180_3) configuration derived from the compound of five tetrahedra.

The two copies of the P-system, together with the c-lines, form a configuration whose type is

$$(24_3, 36_2), \quad (48_3, 72_2) \quad \text{or} \quad (120_3, 180_2) \tag{3.11}$$

(see (3.1) above in this section for the type of the P-system).

By adding the crossing points of the lines of the P-systems, discussed in the first construction of this section, one obtains an s-subfiguration of type

$$(36_3)^-, \quad (72_3)^- \quad \text{or} \quad (180_3)^- \tag{3.12}$$

with $s = 24, 48$ or 120, respectively (note that the missing incidences are equally distributed between the points and the lines).

Let now P be a regular 4-polytope whose facets are of type TP. Thus P is a regular 4-simplex, a regular 4-cube, or a regular 120-cell. It has 5, 8, or 120 facets, respectively [6]. We put a copy of the subfiguration (3.12) in each of the facets of P, so that each of the points of such a subfiguration is in the interior of the facet, and the whole system preserves the original symmetry of P. This results in a (non-connected) subfiguration of type

$$(180_3)^-, \quad (576_3)^- \quad \text{or} \quad (21\,600_3)^-, \tag{3.13}$$

respectively. We convert it to a connected structure as follows.

Consider a facet F of P, and a c-line connecting two points of the P-systems whithin F. Clearly, this c-line intersects the edge of F, which is in the vicinity of these points (since it is within the local mirror plane of the facet lying on that edge). There are three facets of P meeting in a common edge; thus the point of intersection of the c-lines is trivalent. There are two such points on each edge of P. Thus, by adding all these points to our structure (3.13), the number of (trivalent) points increases by 20, 64 or 2400, respectively.

Note that with this completion the number of the missing incidences has been halved. The other half is supplied as follows. Take the 1-skeleton of TP, and put a pair of its copies in each facet of P. The size and location of these copies is such that for each of them there is a P-system in (3.13) in which the crossing points coincide with the midpoints of the edges in TP. Finally, replace each of the edges by the line spanned by it. In this way we supplied not only the rest of the missing incidences, but completed the structure by 40, 128 or 4800 points, and by 60, 192 or 7200 lines, respectively. As a result, we obtained three new balanced configurations in $\mathbb{E}^4$ whose type is

$$(240_3), \quad (768_3) \quad \text{and} \quad (28\,800_3). \tag{3.14}$$

Note that in each step of the construction the original symmetry was preserved, thus the symmetry group of these configurations is equal to that of the regular 4-polytope we started from. In each of these three configurations of type (n_3) the number n is twice the order of the corresponding symmetry group. Furthermore, in all three cases, there are 7 orbits of points and 5 orbits of lines.

4 Cartesian product of point-line configurations

We explore here the following notion.

Definition 4.1. Let $\mathcal{C}_1$ be a (p_q, m_k) configuration in an Euclidean space $\mathbb{E}_1$ and $\mathcal{C}_2$ be an (r_s, n_k) configuration in an Euclidean space $\mathbb{E}_2$. Observe that these two configurations have the same number k of points on each line. The *Cartesian product* of $\mathcal{C}_1$ and $\mathcal{C}_2$ is the $\big((pr)_{(q+s)}, (pn+rm)_k\big)$ configuration $\mathcal{C}_1 \times \mathcal{C}_2$ in $\mathbb{E}_1 \times \mathbb{E}_2$ whose point set is the Cartesian product of the point sets of $\mathcal{C}_1$ and $\mathcal{C}_2$ and where there is a line incident to two points (x_1, x_2) and (y_1, y_2) if and only if either $x_1 = y_1$ and there is a line incident to x_2 and y_2 in $\mathcal{C}_2$, or $x_2 = y_2$ and there is a line incident to x_1 and y_1 in $\mathcal{C}_1$.

We emphasize that the incidence degree of the lines of the two configurations $\mathcal{C}_1$ and $\mathcal{C}_2$ have to coincide. Therefore, in terms of abstract algebra, this product is merely a partial operation on the set of configurations (it is not defined for any pair of configurations). This shows that, when applied to configurations, the analogy of this kind of product with the classical Cartesian product of other objects (like polytopes, graphs, etc.) is not complete, in strict sense. On the other hand, one observes that if the incidence degrees differ, then this product can still be defined, and it results in a *semiconfiguration* (see Definition 2.1). Furthermore, the definition of the product can also be extended to semiconfigurations. Thus, the larger set of semiconfigurations will be closed under this product, and the partial operation extends to a total operation. Hence using the term *Cartesian product* is still justified, in this sense.

A consequence of the definition that if both $\mathcal{C}_1$ and $\mathcal{C}_2$ is full-dimensional in $\mathbb{E}_1$ resp., in $\mathbb{E}_2$, then $\mathcal{C}_1 \times \mathcal{C}_2$ is also full-dimensional in $\mathbb{E}_1 \times \mathbb{E}_2$. We note, however, that one cannot say that in the product the dimensions of $\mathcal{C}_1$ and $\mathcal{C}_2$ are added (see the remark on the *dimension* of a configuration in the Introduction). Thus, we do not think that our definition of product would automatically imply the additivity of dimension of configurations.

We remark that our motivating example is the spatial version of the Gray configuration consisting of 27 points and 27 lines. Actually, it provided the intuitive idea for the definition above, see Figure 10. We note that the (27_3) Gray configuration can in fact be decomposed into the product of three $(3_1, 1_3)$ configurations; however, to visualize the intuitive idea we think the decomposition given in Figure 10 is better. More generally, the $\big((k^k)_k\big)$ generalized Gray configuration is the kth power of the $(k_1, 1_k)$ configuration. (For the Gray configuration and the generalized Gray configuration, see [23]).

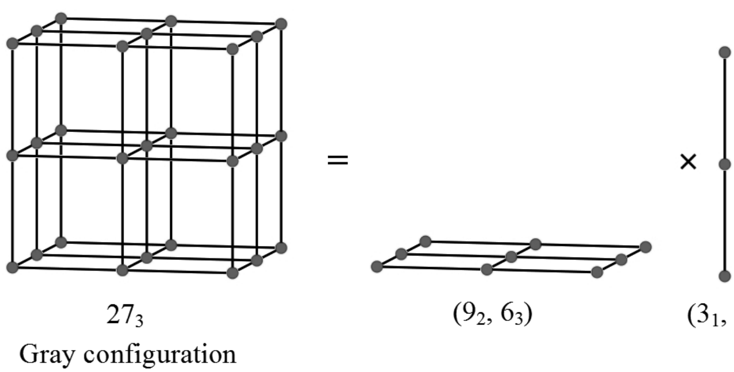


Figure 10: The Gray configuration as a product.

Accordingly, we formulated the definition above in the context of Euclidean geometry. However, an analogous construction also works in projective spaces, which can be described as follows. First recall that a d-dimensional (real) projective space $\mathbb{P}^d$ can be defined as the set of one-dimensional (linear) subspaces of $\mathbb{R}^{d+1}$. Given $\mathbb{P}^d$ in this way, let $\{e_1, \ldots, e_k, e_{k+1}, \ldots, e_{d+1}\}$ be a basis in the corresponding vector space $\mathbb{R}^{d+1}$. Now we have a projective space $\mathbb{P}^k$ given as the set of one-dimensional subspaces of the vector space spanned by the basis $\{e_1, \ldots, e_k, e_{k+1}\}$, and a projective space $\mathbb{P}^{d-k}$ determined analogously by the basis $\{e_{k+1}, \ldots, e_{d+1}\}$. In this case we say that $\mathbb{P}^d$ is decomposed to the *direct sum* of the spaces $\mathbb{P}^k$ and $\mathbb{P}^{d-k}$. More generally, let $\mathbb{P}^k$ and $\mathbb{P}^l$ be two projective spaces. If there are projective isomorphisms $\mathbb{P}^k \cong \overline{\mathbb{P}^k}$ and $\mathbb{P}^l \cong \overline{\mathbb{P}^l}$ such that $\overline{\mathbb{P}^k}$ and $\overline{\mathbb{P}^l}$ form a direct sum decomposition of a space $\mathbb{P}^{k+l}$, then $\mathbb{P}^{k+l}$ is said to be the direct sum of the spaces $\mathbb{P}^k$ and $\mathbb{P}^l$. It is not hard to see that this definition determines a unique bijection from the Cartesian product $\mathbb{P}^k \times \mathbb{P}^l$ to $\mathbb{P}^{k+l}$; thus the points in $\mathbb{P}^{k+l}$ can uniquely be represented by pairs (P, Q) with $P \in \mathbb{P}^k$, $Q \in \mathbb{P}^l$.

Now given the configurations $\mathcal{C}_1$ and $\mathcal{C}_2$, embedded in $\mathbb{P}_1$ and $\mathbb{P}_2$, respectively, both full-dimensional, the point set of their product consists of pairs (P_1, P_2) with $P_1 \in \mathbb{P}_1$, $P_2 \in \mathbb{P}_2$; furthermore, two points (P_1, P_2) and (Q_1, Q_2) are connected by a line in the product if and only if either $P_1 = Q_1$ and P_2 and Q_2 are connected in $\mathcal{C}_2$, or $P_2 = Q_2$ and P_1 and Q_1 are connected in $\mathcal{C}_1$. Clearly the product configuration is full-dimensional in the direct sum of $\mathbb{P}_1$ and $\mathbb{P}_2$, and its type is determined by the types of $\mathcal{C}_1$ and $\mathcal{C}_2$ in the same way as before.

It is clear that the product of a configuration with itself can be repeated, i.e. it can be raised to a power; given a configuration of a suitable type, this may provide a balanced configuration (as we have seen above in the case of generalized Gray configurations). In what follows we give some classes of such examples.

EXAMPLES: CLASS 1.

Consider n lines in the projective plane $\mathbb{P}^2$ in general position, i.e. such that no more than two of them intersect in one point. Together with all their points of intersection, they form a configuration $\left(\binom{n}{2}_2, n_{n-1}\right)$, which we call a *complete n-lateral*. (We note that it has already appeared in this context in [21], see p. 85, Satz 21.)

Taking $(2k+1)$-laterals ($k = 2, 3, \ldots$), we have the following infinite series of balanced configurations obtained as powers:

- complete 5-lateral: $(10_2, 5_4)^2 = (100_4) \subset \mathbb{P}^4$
- complete 7-lateral: $(21_2, 7_6)^3 = (9261_6) \subset \mathbb{P}^6$
- complete 9-lateral: $(36_2, 9_8)^4 = (1\,679\,616_8) \subset \mathbb{P}^8$
- complete 11-lateral: $(55_2, 11_{10})^5 = (503\,284\,375_{10}) \subset \mathbb{P}^{10}$

⋮

The general element of this series can be given as

$$\left(\binom{2k+1}{2}_2, (2k+1)_{2k}\right)^k = \left(\left(\binom{2k+1}{2}^k\right)_{2k}\right), \tag{4.1}$$

and is full-dimensional in the projective space $\mathbb{P}^{2k}$.

EXAMPLES: CLASS 2.

Again, let $k = 2, 3, \ldots$, and start from the simple configuration $((2k)_1, 1_{2k})$ consisting of $2k$ points and a single projective line. Raise it to the power $2k - 2$ so as to obtain a configuration of type

$$\left(\left((2k)^{2k-2}\right)_{2k-2}, \left((2k)^{2k-3}(2k-2)\right)_{2k}\right).$$

Then form the product of this configuration with the complete $(2k+1)$-lateral. The result is a balanced configuration of type

$$\left(\left((2k)^{2k-2}(2k+1)k\right)_{2k}\right), \tag{4.2}$$

which spans the projective space $\mathbb{P}^{2k}$.

We note that in this series the number of points grows faster than in the former one. For comparison, we give the type of the first four members:

$$(160_4),\ (27\,216_6),\ (9\,437\,184_8),\ \left(\left(5.5 \cdot 10^9\right)_{10}\right).$$

EXAMPLES: CLASS 3.

The method of scaffolding polytopes and raising to powers can also be combined to obtain balanced configurations. Here we construct in this way a finite class of examples in Euclidean space.

We start from the well-known Archimedean solid, the rhombicosidodecahedron [9] (see Figure 11). It can obtained from the regular dodecahedron by truncation [5, 11]; thus it is bounded by 12 pentagons, 20 triangles and 30 squares, originating from the faces, vertices and edges of the dodecahedron, respectively. Its 60 vertices can be given in the following form:

$$\left(\pm 1, \pm 1, \pm\tau^3\right)^c,\ \left(\pm\tau, \pm 2\tau, \pm\tau^2\right)^c,\ \left(0, \pm(2+\tau), \pm\tau^2\right)^c, \tag{4.3}$$

where the superscript denotes that all cyclic permutations of the coordinates are to be taken, and τ denotes the golden mean: $\tau = \frac{1}{2}\left(1+\sqrt{5}\right)$.

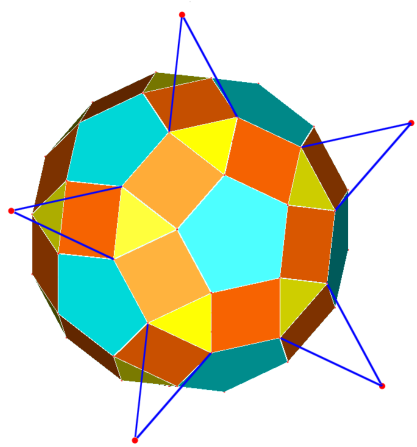


Figure 11: The rhombicosidodecahedron with a regular pentalateral.

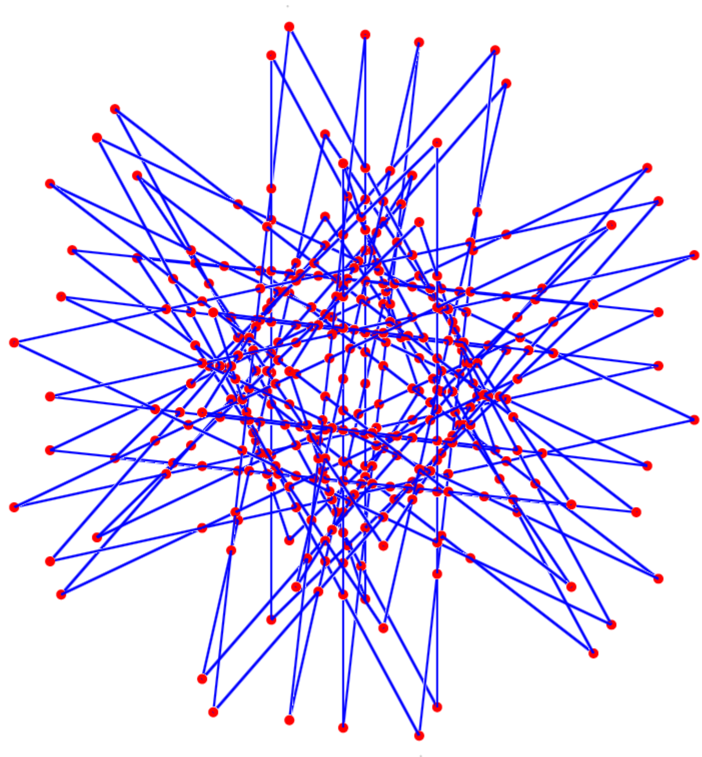


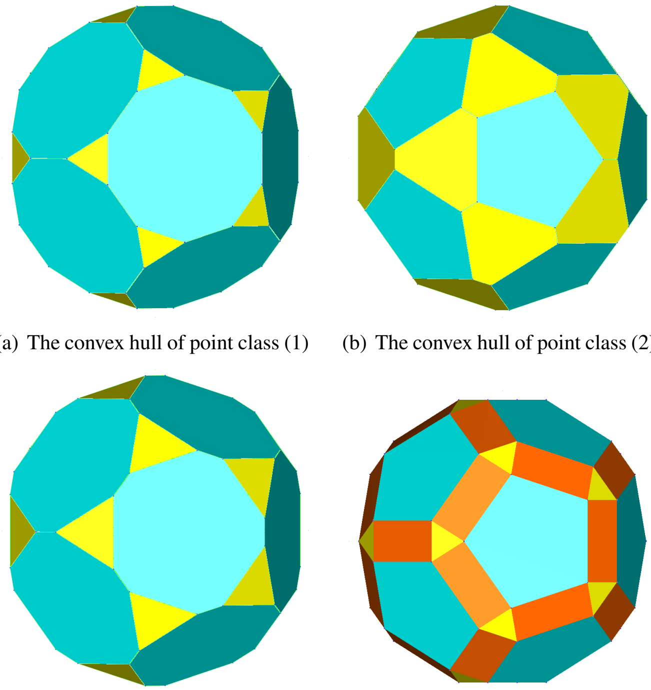
Figure 12: The $(360_2, 60_{12})$ configuration based on the rhombicosidodecahedron.

We use the rhombicosidodecahedron (briefly, RID) as a scaffolding to construct a class of configurations whose types are as follows:

$$(120_2, 60_4)\,; \quad (180_2, 60_6)\,; \quad (240_2, 60_8)\,; \quad (300_2, 60_{10})\,; \quad (360_2, 60_{12})\,. \qquad (4.4)$$

Observe that in the boundary complex of the RID, the *link* (we use this term following e.g. [26], p. 237) of a pentagonal face forms a regular decagon. Connecting by straight lines the vertices of this decagon that are pairwise at a distance 3 from each other, one obtains a $(10_2, 5_4)$ configuration, which is a regular complete pentalateral (see the definition of a complete n-lateral in our examples of Class 1; now we are in $\mathbb{E}^3$, and this figure is regular in Euclidean sense, i.e. its symmetry group is the dihedral group D_5). Figure 11 shows which one of the two possible positions of such a pentalateral is chosen (it can also be seen that five of its points are inside the RID). Clearly there are altogether 12 such regular pentalaterals, and they form a single orbit under the action of the symmetry group of the RID (this group is obviously the full icosahedral group I_h). Hence we obtain a system of 60 lines, which together with the vertices of the pentalaterals form a $(120_2, 60_4)$ configuration (see the first type of (4.4)).

It turns out, however, that there are altogether 360 intersection points of these 60 lines, so that the whole set of points and lines forms a $(360_2, 60_{12})$ configuration (the last type of (4.4)). This can be explained using the symmetry properties of the regular dodecahedron or, equivalently, of the RID. First, recall that a regular dodecahedron has altogether 15 mirror planes. For each edge of the dodecahedron, there are precisely three mirror planes in special position: one lies on it, one is its perpendicular bisector and one is parallel to it. The others intersect it obliquely. The 60 lines are parallel in pairs to the 30 edges of the dodecahedron (and, none of them lie on a mirror plane). Hence, for each of these lines, too, there are precisely 12 mirror planes in oblique position. Because these planes are mirror planes, their intersections with the lines provide points in which precisely two of the 60 lines meet. This is equivalent to the fact that on a given line no two of the 12 intersection points coincide. For, the coincidence means that more than two planes (not perpendicular to each other) meet in such a point, which implies that more than two lines meet in that point. But such multiple intersection does not occur here; this can be visually checked in a model constructed by a dynamic geometry software[3]. Figure 12 shows a screenshot of this model.



(a) The convex hull of point class (1) (b) The convex hull of point class (2)

(c) The convex hull of point class (5) (d) The convex hull of point class (6)

Figure 13: Supporting polytopes for the $(360_2, 60_{12})$ configuration.

The 360 points can be partitioned into 6 classes with respect to their distance from the origin, each containing 60 points; they also form orbits under the action of the group I_h. Because of this latter property, the convex hull of each of them is a vertex-transitive polytope (in fact, in each case it is combinatorially equivalent to an Archimedean solid, see Figure 13). In this way these polytopes form a nested sequence, and are particularly suitable for visualizing the structure of our $(360_2, 60_{12})$ configuration. Hence we call them

[3] *Euler 3D* developed by Tamás Petró.
`http://www.mozaik.info.hu/Homepage/Mozaportal/MPeuler3d.php`

supporting polytopes of this configuration. In Figure 13 they are shown in the order of growing size. The convex hull of class (3) is just the RID we started from; and class (4) is a homothetic copy of class (2) (actually, τ times larger), so we did not repeat the corresponding polytope in the figure. Note that these six polytopes fall by two into three combinatorial types.

Observe that these 6 classes of points can be switched in and out independently of each other; hence, all but last of the five types in list (4.4) above can be realized as more than one geometrically distinct configuration. (Among them, isomorphism may occur; we did not investigate this possibility.) By raising them to a suitable power, one obtains balanced configurations of the following types:

$$\begin{aligned}
&\bullet\ (120_2, 60_4)^2 &&= (14\,400_4) &&\subset \mathbb{E}^6\\
&\bullet\ (180_2, 60_6)^3 &&= (5\,832\,200_6) &&\subset \mathbb{E}^9\\
&\bullet\ (240_2, 60_8)^4 &&= (3\,317\,760\,000_8) &&\subset \mathbb{E}^{12}\\
&\bullet\ (300_2, 60_{10})^5 &&= \left(\left(2.43 \cdot 10^{10}\right)_{10}\right) &&\subset \mathbb{E}^{15}\\
&\bullet\ (360_2, 60_{12})^6 &&= \left(\left(2^{18} \cdot 3^{12} \cdot 5^6\right)_{12}\right) &&\subset \mathbb{E}^{18}.
\end{aligned} \tag{4.5}$$

Due to the geometric differences we mentioned just above, a number of geometrically distinct cases occur here as well. For example, even for the $(14\,400)_4$ configuration, this amounts to 125 geometrically distinct cases (possibly not all combinatorially distinct).

5 An incidence conjecture

Recall our definition of a complete n-lateral in the preceding section (examples of Class 1). For the case $n = 5$ we use the term *complete pentalateral*. The points of this configuration we shall also call *vertices*. The following properties of complete pentalaterals are well known (cf. [21], pp. 85–86, Satz 21 and Aufgabe 3b).

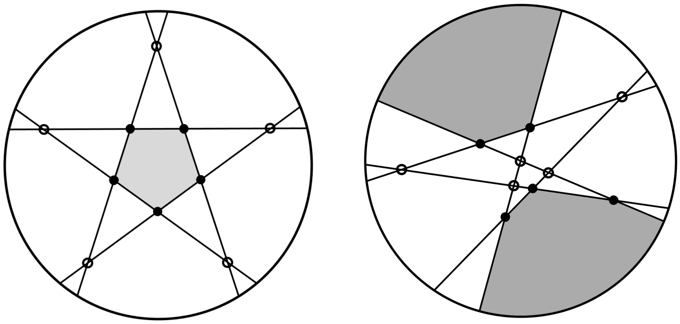
Proposition 5.1. *There is a unique complete pentalateral in the projective plane $\mathbb{P}^2$ up to combinatorial equivalence. It decomposes $\mathbb{P}^2$ into one pentagonal, five quadrangular and five triangular regions.*

We shall call the vertices of the complete pentalateral belonging to the pentagonal region *internal vertices*, while the other *external vertices*. The existence and uniqueness of the pentagon guarantees that such a distinction is indeed possible:

Proposition 5.2. *The partition of the set of vertices of the complete pentalateral to internal and external vertices is well-defined.*

The structure of the tiling of $\mathbb{P}^2$ just described is shown in Figure 14. Figure 14a also illustrates that the group of a complete pentalateral is isomorphic to D_5, i.e. to the symmetry group of a regular pentagon (the latter in Euclidean sense). Recall that the *group of a configuration* is defined as the group of the permutations (both the points and lines among themselves) preserving incidences [7].

We have seen that squaring a complete pentalateral results in a configuration (100_4) in projective 4-space (cf. Class 1 in the preceding section). This configuration can nicely be visualized by projecting it into three dimensions and restricting ourselves to Euclidean space. To this end, a useful tool is the Schlegel diagram [14, 26] . In fact, there are 10 copies



(a) (b)

Figure 14: The decomposition of the projective plane by a complete pentalateral, in two versions, with the pentagonal region shaded. The internal and external vertices are indicated by black and white vertices, respectively.

of the complete pentalateral in the configuration $(10 0_4)$ such that they can be inscribed in the 10 pentagonal 2-faces of the Cartesian product of two pentagons, which is a 4-polytope. The Schlegel diagram of this latter polytope is depicted in Figure 15, while the image of the (100_4) configuration is shown in Figure 16.

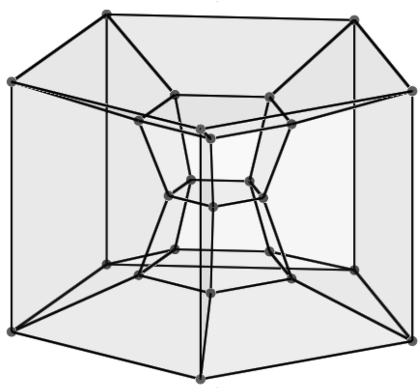


Figure 15: Schlegel diagram of the Cartesian product of two pentagons.

The following conjecture is motivated by the three-dimensional image of the (100_4) configuration. We will denote a complete pentalateral determined by lines $l_1, \ldots, l_5$ by $P(l_1, \ldots, l_5)$.

Conjecture 5.3. *Let be given in the projective space* $\mathbb{P}^3$ *25 lines,* a_{ij} $(i, j = 1, \ldots, 5)$ *such that they form five complete pentalaterals:*

$$A_1 = P(a_{11}, \ldots, a_{15}),\ \ldots,\ A_5 = P(a_{51}, \ldots, a_{55}).$$

Assume that the following conditions hold:

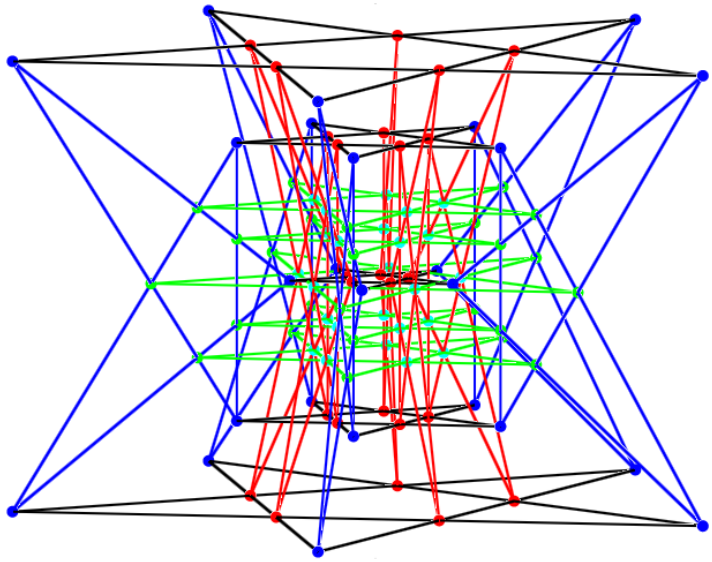


Figure 16: Twenty pentalaterals in $\mathbb{E}^3$: a (100_4) configuration.

1. *the external vertices of the pentalaterals A_i form the external vertices of complete pentalaterals $B_j = P(b_{1j}, \ldots, b_{5j})$, as follows:*

$$a_{ij} \cap a_{i,j+2} = b_{ij} \cap b_{i+2,j};$$

2. *the internal vertices of the pentalaterals A_i form the external vertices of complete pentalaterals $C_j = P(c_{1j}, \ldots, c_{5j})$, as follows:*

$$a_{ij} \cap a_{i,j+1} = c_{ij} \cap c_{i+2,j}$$

(indexing is meant modulo 5).

Then there is a quintuple of complete pentalaterals D_i such that their vertices coincide with the internal vertices of the pentalaterals B_j and C_j, as follows:

$$b_{ij} \cap b_{i+1,j} = d_{ij} \cap d_{i,j+2} \quad \text{and} \quad c_{ij} \cap c_{i+1,j} = d_{ij} \cap d_{i,j+1}.$$

In some particular cases this conjecture is known to be true. In these cases the pentalaterals are embedded in Euclidean 3-space, every A_i is in distinct and pairwise parallel planes (these planes can simply be conceived as "horizontal planes"), while every B_j and C_j are in planes perpendicular to the former ones (thus they can be conceived as being in "vertical position"). In addition, every D_i is in "horizontal" planes, too. The cases are as follows:

Case A.

The pentagons determined by the A_is and D_is are all regular (in Euclidean sense), and they have a common axis of rotation (of order five). In this case the conditions of the conjecture can easily be satisfied by suitably scaling the A_is and by suitably chosen shapes and sizes of the B_js. Just this case is shown in our Figure 16 above. Here the lines of the pentalaterals A_i, B_j, C_j and D_i, are distinguished by black, blue, red and green colour, respectively. Observe that each of these colour classes represents 100 incidences. Thus, our conjecture can also be formulated that the incidences belonging to any three of the colour classes imply the remaining 100 incidences.

Case B.

All the pentalaterals A_i are homothetic copies of a pentalateral A_0. Furthermore, the external vertices of A_0 (hence those of all the A_is) are inscribed in a circle. This case is visualized in an interactive model made using *Mathematica* [20]. In this model it is possible to move the external vertices of A_0 (and simultaneously, all the corresponding vertices of the A_is) along a circle, while all the incidences required by the conjecture are preserved.

These cases provide some support for the conjecture. We remark that any projective transformation preserves the conjecture. We also remark that Case A also illustrates the fact that the automorphism group of this configuration is larger than or isomorphic to the group $D_5 \times D_5 \times C_2$. Here the first factor corresponds to the group of the pentalaterals A_i and D_i, the second factor to that of the pentalaterals B_j and C_j, while the last term is responsible for interchanging the "horizontal" and "vertical" quintuples of pentalaterals.

More generally, one expects that given a configuration $\mathcal{C}$ with group G, the group of its pth power is larger than or isomorphic to the semi-direct product $G^p \rtimes S_p$, where the first term is a direct power of G, and the second term is the symmetric group of degree p.

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Distinguishing graphs with infinite motion and nonlinear growth

Johannes Cuno *

Institut für Mathematische Strukturtheorie, Technische Universität Graz
Steyrergasse 30/III, 8010 Graz, Austria

Wilfried Imrich

Department Mathematik und Informationstechnologie
Montanuniversität Leoben, 8700 Leoben, Austria

Florian Lehner

Institut für Geometrie, Technische Universität Graz
Kopernikusgasse 24/IV, 8010 Graz, Austria

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Abstract

The distinguishing number $\mathrm{D}(G)$ of a graph G is the least cardinal d such that G has a labeling with d labels which is only preserved by the trivial automorphism. We show that the distinguishing number of infinite, locally finite, connected graphs G with infinite motion and growth $o(n^2/\log_2 n)$ is either 1 or 2, which proves the Infinite Motion Conjecture of Tom Tucker for this type of graphs. The same holds true for graphs with countably many ends that do not grow too fast. We also show that graphs G of arbitrary cardinality are 2-distinguishable if every nontrivial automorphism moves at least uncountably many vertices $\mathrm{m}(G)$, where $\mathrm{m}(G) \geq |\mathrm{Aut}(G)|$. This extends a result of Imrich et al. to graphs with automorphism groups of arbitrary cardinality.

Keywords: Distinguishing number, automorphisms, infinite graphs.

Math. Subj. Class.: 05C25, 05C63, 05C15, 03E10.

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E-mail addresses: cuno@math.tugraz.at (Johannes Cuno), imrich@unileoben.ac.at (Wilfried Imrich), f.lehner@tugraz.at (Florian Lehner)

1 Introduction

Albertson and Collins [1] introduced the distinguishing number $\mathrm{D}(G)$ of a graph G as the least cardinal d such that G has a labeling with d labels which is only preserved by the trivial automorphism.

This seminal concept spawned many papers on finite and infinite graphs. We are mainly interested in infinite, locally finite, connected graphs of polynomial growth, see [8], [15], [13], and in graphs of higher cardinality, see [9], [11]. In particular, there is one conjecture on which we focus our attention, the Infinite Motion Conjecture of Tom Tucker.

Before stating it, we introduce the notation $\mathrm{m}(\phi)$ for the number of elements moved by an automorphism ϕ, and call $\mathrm{m}(\phi)$ the *motion* of ϕ. In other words, $\mathrm{m}(\phi)$ is the size of the set of vertices which are not fixed by ϕ, that is, the size of its *support*, $\mathrm{supp}(\phi)$.

The Infinite Motion Conjecture of Tom Tucker. *Let G be an infinite, locally finite, connected graph. If every nontrivial automorphism of G has infinite motion, then the distinguishing number* $\mathrm{D}(G)$ *of G is either 1 or 2.*

For the origin of the conjecture and partial results compare [13]. The conjecture is true if $\mathrm{Aut}(G)$ is countable, hence we concentrate on graphs with uncountable group.

The validity of the conjecture for graphs with countable group follows from either one of two different results in [10]. One of them replaces the requirement of infinite motion by a lower and upper bound on the size of the automorphism group. It asserts that every infinite, locally finite, connected graph G whose automorphism group is infinite, but strictly smaller than $2^{\aleph_0}$, has countable group, infinite motion, and distinguishing number 2. For a precise formulation see Theorem 4.1. The proof is not easy and follows from results of either Halin [6], Trofimov [14], or Evans [3].

The other one relaxes the condition of local finiteness and requires that the group is at most countable. It asserts that countably infinite, connected graphs with finite or countably infinite group and infinite motion are 2-distinguishable, no matter whether they are locally finite or not, see Theorem 4.2. The proof is short and elementary.

For uncountable connected graphs with countable motion the Infinite Motion Conjecture need not be true. We turn to this case in Section 4, suggest a version of the conjecture for uncountable connected graphs, and prove its validity under a bound on the size of the automorphism group.

2 Preliminaries

Throughout this paper the symbol $\mathbb{N}$ denotes the set $\{1, 2, 3, \ldots\}$ of positive integers, whereas the symbol $\mathbb{N}_0$ refers to the set $\{0, 1, 2, 3, \ldots\}$ of non-negative integers.

Let G be a graph with vertex set $\mathrm{V}(G)$. Let X be a set. An *X-labeling* l of G is a mapping $l : \mathrm{V}(G) \to X$. For us X will mostly be the set $\{\text{black}, \text{white}\}$. In this case, we speak of a *2-coloring* of G.

Let l be an X-labeling of G. Consider an automorphism $\phi \in \mathrm{Aut}(G)$. If, for every $v \in \mathrm{V}(G)$, $l(\phi(v)) = l(v)$, we say that *l is preserved by ϕ*. If this is not the case, we say that *l breaks ϕ*. An X-labeling l of G is called *distinguishing* if it is only preserved by the trivial automorphism. The *distinguishing number* $\mathrm{D}(G)$ of G is the least cardinal d such that there exists a distinguishing X-labeling of G with $|X| = d$.

Given a group A equipped with a homomorphism $\phi : A \to \mathrm{Aut}(G)$, we say that *$A$ acts on G*. Moreover, we say that *A acts nontrivially* on G if there is an $a \in A$ such that $\phi(a)$

moves at least one vertex of G. By abuse of language we write $a(v)$ instead of $\phi(a)(v)$ and say that an X-labeling l of G *is preserved by* $a \in A$ if it is preserved by $\phi(a) \in \operatorname{Aut}(G)$.

The *ball with center* $v_0 \in \mathrm{V}(G)$ *and radius* r is the set of all vertices $v \in \mathrm{V}(G)$ with $d_G(v_0, v) \leq r$ and is denoted by $B_{v_0}^G(r)$, whereas $S_{v_0}^G(r)$ stands for the set of all vertices $v \in \mathrm{V}(G)$ with $d_G(v_0, v) = r$. We call it the *sphere with center* $v_0 \in \mathrm{V}(G)$ *and radius* r. If G is clear from the context, we just write $B_{v_0}(r)$ and $S_{v_0}(r)$ respectively. For terms not defined here we refer to [7].

Although our graphs are infinite, as long as they are locally finite, all balls and spheres of finite radius are finite. The number of vertices in $B_{v_0}^G(r)$ is a monotonically increasing function of r, because

$$\left|B_{v_0}^G(r)\right| = \sum_{i=0}^{r} \left|S_{v_0}^G(i)\right| \quad \text{and} \quad \left|S_{v_0}^G(i)\right| \geq 1\,.$$

Nonetheless, the growth of $\left|B_{v_0}^G(r)\right|$ depends very much on G, and it is helpful to define the growth rate of a graph.

We say that an infinite, locally finite, connected graph G has *polynomial growth* if there is a vertex $v_0 \in \mathrm{V}(G)$ and a polynomial p such that

$$\forall\, r \in \mathbb{N}_0\colon\ \left|B_{v_0}^G(r)\right| \leq p(r)\,.$$

It is easy to see that this implies that all functions $\left|B_v^G(r)\right|$ are bounded by polynomials of the same degree as p, independent of the choice of $v \in \mathrm{V}(G)$. In this context it should be clear what we mean by *linear* and *quadratic growth*. Observe that the two-sided infinite path has linear growth, and that the growth of the grid of integers in the plane is quadratic.

We say that G has *exponential growth* if there is a constant $c > 1$ such that

$$\forall\, r \in \mathbb{N}_0\colon\ \left|B_{v_0}^G(r)\right| \geq c^r\,.$$

Notice that homogeneous trees of degree $d > 2$, that is, infinite trees where every vertex has the same degree d, have exponential growth. For the distinguishability of such trees and tree-like graphs, see [16] and [9].

We are mainly interested is the distinguishability of infinite, locally finite, connected graphs of polynomial growth. For us, the following lemma will be helpful.

Lemma 2.1. *Let A be a finite group acting on a graph G. If a coloring of G breaks some element of A, then it breaks at least half of the elements of A.*

Proof. The elements of A that preserve a given coloring form a subgroup. If some element of A is broken, then this subgroup is proper and thus, by Lagrange's theorem, cannot contain more than half of the elements of A. □

If the action is nontrivial, then we can always find a coloring that breaks at least one element. Hence, we have the following result.

Lemma 2.2. *Let G be a graph. If A is a finite group acting nontrivially on G, then there exists a 2-coloring of G that breaks at least half of the elements of A.*

The proof of Lemma 2.2 is based on the fact that A is a group. But a very similar result holds for any finite family of nontrivial automorphisms, as the following lemma shows.

Lemma 2.3. *Let G be a finite graph. If A is a finite set equipped with a mapping $\phi : A \to \mathrm{Aut}(G) \setminus \{\mathrm{id}\}$, then there exists a 2-coloring of G that breaks $\phi(a)$ for at least half of the elements of A.*

Proof. Let $\mathrm{V}(G) = \{v_1, v_2, \ldots, v_n\}$. For every $k \in \{1, 2, \ldots, n\}$, let A_k be the set of all $a \in A$ with $\mathrm{supp}(\phi(a)) \subseteq \{v_1, v_2, \ldots, v_k\}$. We show by induction that the assertion holds for all A_k and, in particular, for A. Because A_1 is the empty set, the assertion is true for A_1. Suppose it is true for A_{k-1}. Then we can choose a 2-coloring of G that breaks $\phi(a)$ for at least half of the elements of A_{k-1}. This remains true, even when we change the color of v_k. Notice that, for every $a \in A_k \setminus A_{k-1}$, $\phi(a)$ either maps v_k into a white vertex in $\{v_1, v_2, \ldots, v_{k-1}\}$ or into a black vertex in $\{v_1, v_2, \ldots, v_{k-1}\}$. Depending on which of the two alternatives occurs more often, we color v_k black or white such that this 2-coloring also breaks $\phi(a)$ for at least half of the elements of $A_k \setminus A_{k-1}$ and, hence, for at least half of the elements of A_k. □

If every nontrivial automorphism of a graph G has infinite motion, we say that G has *infinite motion*. For such graphs the following result from [10] will be of importance.

Lemma 2.4. *Let G be an infinite, locally finite, connected graph with infinite motion. If an automorphism $\phi \in \mathrm{Aut}(G)$ fixes a vertex $v_0 \in \mathrm{V}(G)$ and moves at least one vertex in $S_{v_0}(k)$, then, for every $i \geq k$, it moves at least one vertex in $S_{v_0}(i)$.*

3 Graphs of nonlinear growth

In [10], it was shown that infinite, locally finite, connected graphs with infinite motion and linear growth have countable automorphism group, and therefore distinguishing number either 1 or 2.

If the growth rate of such graphs becomes nonlinear, then the automorphism group can become uncountable. This holds, even if the growth rate becomes only slightly nonlinear.

Theorem 3.1. *Let $\varepsilon > 0$. Then there exists an infinite, locally finite, connected graph G with uncountable automorphism group, infinite motion, and nonlinear growth function $g : \mathbb{N}_0 \to \mathbb{N}_0$ such that, for sufficiently large $n \in \mathbb{N}_0$, $g(n)$ is bounded from above by $n^{1+\varepsilon}$.*

Proof. We construct G from T_3, that is, the tree in which every vertex has degree 3. First, choose an arbitrary vertex $v_0 \in \mathrm{V}(T_3)$. Our strategy is to replace the edges of T_3 by paths such that, for sufficiently large $n \in \mathbb{N}_0$, $g(n) = \left|B^G_{v_0}(n)\right| \leq n^{1+\varepsilon}$.

For every $i \in \mathbb{N}_0$, there are $3 \cdot 2^i$ edges from $S^{T_3}_{v_0}(i)$ to $S^{T_3}_{v_0}(i+1)$. If we replace them by paths of the same length, then the cardinality of the balls $B^G_{v_0}(n)$ grows linearly with slope $3 \cdot 2^i$ from $S^{T_3}_{v_0}(i)$ to $S^{T_3}_{v_0}(i+1)$.

Observe that, given any affine linear function $h : \mathbb{N}_0 \to \mathbb{N}_0$, there is a number $n_h \in \mathbb{N}$ such that, for all $n \geq n_h$, $h(n) \leq n^{1+\varepsilon}$. In particular, we may consider the functions $h_i : \mathbb{N}_0 \to \mathbb{N}_0$ defined by $h_i(x) = 3 \cdot 2^i \cdot x + 1$, and choose numbers $n_i \in \mathbb{N}$ such that, for every $n \geq n_i$, $h_i(n) \leq n^{1+\varepsilon}$.

As illustrated in Figure 1, for every $i \in \mathbb{N}_0$, we replace the edges from $S^{T_3}_{v_0}(i)$ to $S^{T_3}_{v_0}(i+1)$ by paths of length n_{i+1}. For every $i \in \mathbb{N}$ and every vertex $v \in \mathrm{V}(G)$ on such a path from $S^{T_3}_{v_0}(i)$ to $S^{T_3}_{v_0}(i+1)$, we have $d_G(v, v_0) \geq n_i$ and, hence,

$$g(d_G(v, v_0)) \leq 3 \cdot 2^i \cdot d_G(v, v_0) + 1 = h_i(d_G(v, v_0)) \leq d_G(v, v_0)^{1+\varepsilon}.$$

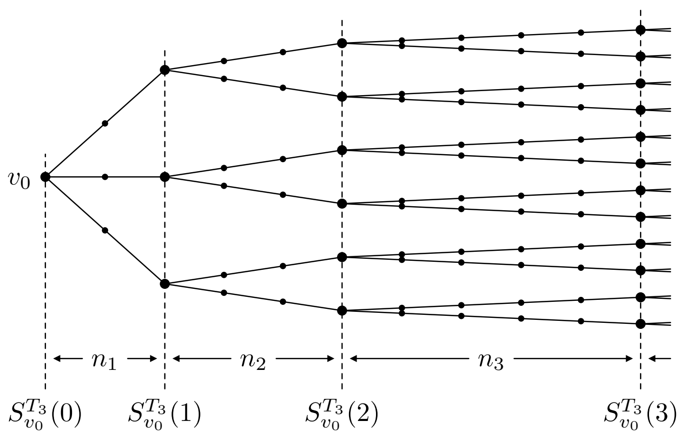

Figure 1: Replacing the egdes of T_3 by paths.

So, for every $n \geq n_1$, $g(n)$ is bounded from above by $n^{1+\varepsilon}$. Every automorphism of T_3 that fixes v_0 induces an automorphism of G. It is easy to see that this correspondence is bijective. Thus, $\mathrm{Aut}(G)$ is uncountable. Furthermore, G inherits infinite motion from T_3. Since $\mathrm{Aut}(G)$ is uncountable, the result of [10] mentioned at the beginning of Section 3 implies that G cannot have linear growth. □

Though we cannot assume that the automorphism groups of our graphs are countable, we prove that infinite, locally finite, connected graphs with infinite motion and nonlinear, but moderate, growth are still 2-distinguishable, that is, they have distinguishing number either 1 or 2.

Our construction of a suitable coloring consists of several steps. In Lemma 3.2 we color a part of the vertices in order to break all automorphisms that move a distinguished vertex v_0. In Lemma 3.3 we show how to color some of the remaining vertices in order to break more automorphisms. Iteration of this procedure yields a distinguishing coloring, as shown in Theorem 3.4.

Lemma 3.2. *Let G be an infinite, locally finite, connected graph with infinite motion and $v_0 \in \mathrm{V}(G)$. Then, for every $k \in \mathbb{N}$, one can 2-color all vertices in $B_{v_0}(k+3)$ and $S_{v_0}(\lambda k + 4)$, $\lambda \in \mathbb{N}$, such that, no matter how one colors the remaining vertices, all automorphisms that move v_0 are broken.*

Proof. If $k = 1$, then we color v_0 black and all $v \in \mathrm{V}(G) \setminus \{v_0\}$ white, whence all automorphisms that move v_0 are broken. So, let $k \geq 2$. First, we color all vertices in $S_{v_0}(0)$, $S_{v_0}(1)$, and $S_{v_0}(k+2)$ black and the remaining vertices in $B_{v_0}(k+3)$ white. Moreover, we color all vertices in $S_{v_0}(\lambda k+4)$, $\lambda \in \mathbb{N}$, black and claim that, no matter how we color the remaining vertices, v_0 is the only black vertex that has only black neighbors and only white vertices at distance $r \in \{2, 3, \ldots, k+1\}$, see Figure 2.

It clearly follows from this claim that this coloring breaks every automorphism that moves v_0. It only remains to verify the claim.

Consider a vertex $v \in \mathrm{V}(G) \setminus \{v_0\}$. If v is not in $S_{v_0}(1)$, then it is easy to see that v cannot have the aforementioned properties. So, let v be in $S_{v_0}(1)$ and assume it has only

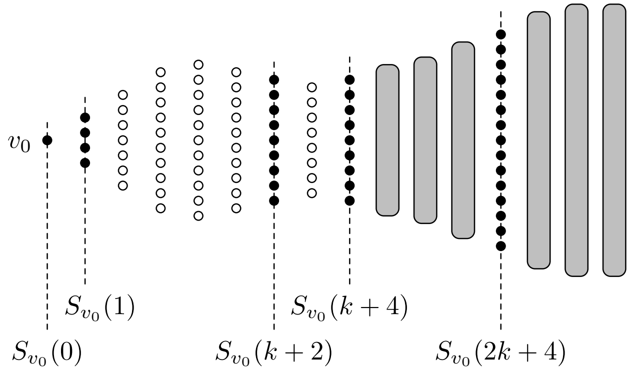

Figure 2: Breaking all automorphisms that move v_0.

black neighbors and only white vertices at distance 2. Then it cannot be neighbor to any vertex in $S_{v_0}(2)$, but must be neighbor to all vertices in $B_{v_0}(1)$ except itself. Therefore, the transposition of the vertices v and v_0 is a nontrivial automorphism of G with finite support. Since G has infinite motion, this is not possible. □

Lemma 3.3. *Let G be an infinite, locally finite, connected graph with infinite motion and $v_0 \in \mathrm{V}(G)$. Moreover, let $\varepsilon > 0$. Then there exists a $k \in \mathbb{N}$ such that, for every $m \in \mathbb{N}$ and for every $n \in \mathbb{N}$ that is sufficiently large and fulfills*

$$|S_{v_0}(n)| \leq \frac{n}{(1+\varepsilon)\log_2 n}, \tag{3.1}$$

one can 2-color all vertices in $S_{v_0}(m+1), S_{v_0}(m+2), \ldots, S_{v_0}(n)$, but not those in $S_{v_0}(\lambda k+4)$, $\lambda \in \mathbb{N}$, such that all automorphisms that fix v_0 and act nontrivially on $B_{v_0}(m)$ are broken.

The coloring and the meaning of the variables m, n, and k is illustrated by Figure 3.

Proof. First, choose a $k \in \mathbb{N}$ that is larger than $1 + \frac{1}{\varepsilon}$. Then

$$\frac{k-1}{k} > \frac{1}{1+\varepsilon}. \tag{3.2}$$

Let $m \in \mathbb{N}$. By (3.2), there is an $n_0 \in \mathbb{N}$ such that

$$\forall\, n \geq n_0 \colon (n-m) \cdot \frac{k-1}{k} \geq n \cdot \frac{1}{1+\varepsilon} + 1. \tag{3.3}$$

Let $n \in \mathbb{N}$ be sufficiently large, that is, $n \geq n_0$, and assume it fulfills (3.1). Then, the number of spheres $S_{v_0}(m+1), S_{v_0}(m+2), \ldots, S_{v_0}(n)$ that are not of the type $S_{v_0}(\lambda k+4)$, $\lambda \in \mathbb{N}$, is at least

$$\left\lfloor (n-m) \cdot \frac{k-1}{k} \right\rfloor \geq \left\lfloor n \cdot \frac{1}{1+\varepsilon} + 1 \right\rfloor > \frac{n}{1+\varepsilon}. \tag{3.4}$$

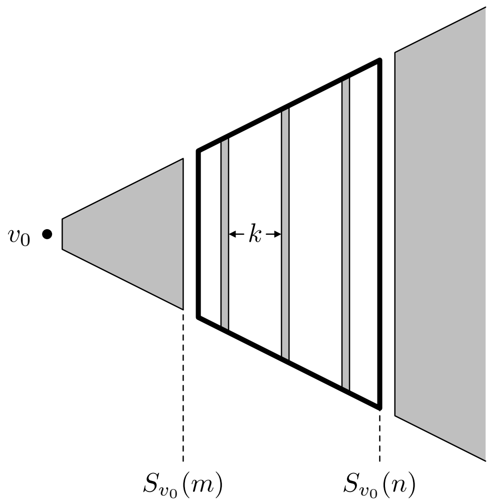

Figure 3: Breaking all automorphisms that fix v_0 and act nontrivially on $B_{v_0}(m)$.

Our goal is to 2-color the vertices in these spheres in order to break all automorphisms that fix v_0 and act nontrivially on $B_{v_0}(m)$.

Let $\mathrm{Aut}(G, v_0)$ be the group of all automorphisms that fix v_0. Every $\phi \in \mathrm{Aut}(G, v_0)$ induces a permutation $\phi|B_{v_0}(n)$ of the vertices in $B_{v_0}(n)$. These permutations form a group A. If σ and τ are different elements of A, then $\sigma\tau^{-1} \in A$ acts nontrivially on $B_{v_0}(n)$. By Lemma 2.4, it also does so on $S_{v_0}(n)$, which means that σ and τ do not agree on $S_{v_0}(n)$. Therefore, the cardinality of A is at most $|S_{v_0}(n)|!$, for which the following rough estimate suffices for our purposes:

$$\begin{aligned} |S_{v_0}(n)|! &\le |S_{v_0}(n)|^{|S_{v_0}(n)|-1} \le \left(\tfrac{n}{(1+\varepsilon)\log_2 n}\right)^{\frac{n}{(1+\varepsilon)\log_2 n}-1} \\ &\le n^{\frac{n}{(1+\varepsilon)\log_2 n}-1} = 2^{\left(\frac{n}{(1+\varepsilon)\log_2 n}-1\right)\log_2 n} \le 2^{\frac{n}{1+\varepsilon}-1}\,. \end{aligned} \tag{3.5}$$

It is clear that, if an element $\sigma \in A$ that acts nontrivially on $B_{v_0}(m)$ is broken by a suitable 2-coloring of some spheres in $B_{v_0}(n)$, then all $\phi \in \mathrm{Aut}(G, v_0)$ with $\phi|B_{v_0}(n) = \sigma$ are broken at once. So it suffices to break all $\sigma \in A$ that act nontrivially on $B_{v_0}(m)$ by a suitable 2-coloring of some spheres in $B_{v_0}(n)$ in order to ensure that all $\phi \in \mathrm{Aut}(G, v_0)$ that act nontrivially on $B_{v_0}(m)$ are broken.

Before doing this, let us remark that any element $\sigma \in A$ that acts nontrivially on the ball $B_{v_0}(m)$, also acts nontrivially on every sphere $S_{v_0}(m+1), \ldots, S_{v_0}(n)$. This is a consequence of Lemma 2.4, and implies that we can break σ by breaking the action of σ on any one of the spheres $S_{v_0}(m+1), \ldots, S_{v_0}(n)$.

Now, consider the subset $S \subseteq A$ of all elements that act nontrivially on $B_{v_0}(m)$. As already remarked, every $\sigma \in S$ acts nontrivially on every sphere $S_{v_0}(m+1), \ldots, S_{v_0}(n)$. Hence, we can apply Lemma 2.3 to break at least half of the elements of S by a suitable coloring of $S_{v_0}(m+1)$. What remains unbroken is a subset $S' \subseteq S$ of cardinality at most $|S|/2$. Now, we proceed to the next sphere. We can break at least half of the elements of S' by a suitable coloring of $S_{v_0}(m+2)$. What still remains unbroken, is a subset $S'' \subseteq S$

of cardinality at most $|S|/4$.

Iterating the procedure, but avoiding spheres of the type $S_{v_0}(\lambda k+4)$, $\lambda \in \mathbb{N}$, we end up with the empty subset $\varnothing \subseteq S$ after at most $\log_2|S|+1 \leq \log_2|A|+1 \leq \frac{n}{1+\varepsilon}$ steps, see (3.5). This is less than the number of spheres not of the type $S_{v_0}(\lambda k+4)$, $\lambda \in \mathbb{N}$, between $S_{v_0}(m+1)$ and $S_{v_0}(n)$, see (3.4). Thus, we remain within the ball $B_{v_0}(n)$. Hence, all $s \in S$ and, therefore, all $\phi \in \mathrm{Aut}(G, v_0)$ that act nontrivially on $B_{v_0}(m)$ are broken, and we are done. □

Theorem 3.4. *Let G be an infinite, locally finite, connected graph with infinite motion and $v_0 \in \mathrm{V}(G)$. Moreover, let $\varepsilon > 0$. If there exist infinitely many $n \in \mathbb{N}$ such that*

$$|S_{v_0}(n)| \leq \frac{n}{(1+\varepsilon)\log_2 n}\,, \tag{3.6}$$

then the distinguishing number $\mathrm{D}(G)$ of G is either 1 or 2.

Proof. Consider the $k \in \mathbb{N}$ provided by Lemma 3.3. First, we use Lemma 3.2 to 2-color all vertices in $B_{v_0}(k+3)$ and in $S_{v_0}(\lambda k+4)$, $\lambda \in \mathbb{N}$, such that, no matter how we color the remaining vertices, all automorphisms that move v_0 are broken.

Let $m_1 = k+3$. Among all $n \in \mathbb{N}$ that satisfy (3.6) we choose a number $n_1 \in \mathbb{N}$ that is larger than m_1 and sufficiently large to apply Lemma 3.3. Hence, we can 2-color all vertices in $S_{v_0}(m_1+1), S_{v_0}(m_1+2), \ldots, S_{v_0}(n_1)$, except those in $S_{v_0}(\lambda k+4)$, $\lambda \in \mathbb{N}$, such that all automorphisms that fix v_0 and act nontrivially on $B_{v_0}(m_1)$ are broken. Next, let $m_2 = n_1$ and choose an $n_2 \in \mathbb{N}$ to apply Lemma 3.3 again. Iteration of this procedure yields a 2-coloring of G.

If an automorphism $\phi \in \mathrm{Aut}(G) \setminus \{\mathrm{id}\}$ moves v_0, then it is broken by our coloring. If it fixes v_0, consider a vertex v with $\phi(v) \neq v$. Since G is connected and $m_1 < m_2 < m_3 < \ldots$, there is an $i \in \mathbb{N}$ such that v is contained in $B_{v_0}(m_i)$. Hence, ϕ acts nontrivially on $B_{v_0}(m_i)$ and is again broken by our coloring. □

Corollary 3.5. *Let G be an infinite, locally finite, connected graph with infinite motion and $v_0 \in \mathrm{V}(G)$. Moreover, let $\varepsilon > 0$. If there exist infinitely many $n \in \mathbb{N}$ such that*

$$|B_{v_0}(n)| \leq \frac{n^2}{(2+\varepsilon)\log_2 n}\,, \tag{3.7}$$

then the distinguishing number $\mathrm{D}(G)$ of G is either 1 or 2. In particular, the Infinite Motion Conjecture holds for all graphs of growth $o(n^2/\log_2 n)$.

Proof. Let $n_1 < n_2 < n_3 < \ldots$ be an infinite sequence of numbers that fulfill (3.7). Notice that, for every $k \in \mathbb{N}$,

$$\sum_{i=1}^{n_k} \frac{i}{(1+\frac{\varepsilon}{2})\log_2 i} > \frac{{n_k}^2}{(2+\varepsilon)\log_2 n_k} \geq |B_{v_0}(n_k)| > \sum_{i=1}^{n_k} |S_{v_0}(i)|\,. \tag{3.8}$$

Since

$$\lim_{k\to\infty}\left(\left(\sum_{i=1}^{n_k} \frac{i}{(1+\frac{\varepsilon}{2})\log_2 i}\right) - \frac{{n_k}^2}{(2+\varepsilon)\log_2 n_k}\right) = \infty\,, \tag{3.9}$$

we infer that

$$\lim_{k\to\infty}\sum_{i=1}^{n_k}\left(\frac{i}{(1+\frac{\varepsilon}{2})\log_2 i}-|S_{v_0}(i)|\right)=\infty\,, \tag{3.10}$$

and that, for infinitely many $i \in \mathbb{N}$,

$$|S_{v_0}(i)| < \frac{i}{(1+\frac{\varepsilon}{2})\log_2 i}\,. \tag{3.11}$$

Hence, we can apply Theorem 3.4 to show that the distinguishing number $\mathrm{D}(G)$ of G is either 1 or 2. □

A result similar to Theorem 3.4 can also be obtained for graphs with countably many ends[1], none of which grows too fast. Readers not familiar with the notion of ends may safely skip the rest of this section, as the result is not used elsewhere in the paper.

Theorem 3.6. *Let G be an infinite, locally finite, connected graph with countably many ends and infinite motion. Moreover, let $v_0 \in \mathrm{V}(G)$ and $\varepsilon > 0$. For an end ω of G let $S^{\omega}_{v_0}(n)$ be the set of vertices in $S_{v_0}(n)$ that lie in the same connected component of $G \setminus B_{v_0}(n-1)$ as ω. If, for every end ω, there are infinitely many $n \in \mathbb{N}$ such that*

$$\left|S^{\omega}_{v_0}(n)\right| \leq \frac{n}{(1+\varepsilon)\log_2 n}\,, \tag{3.12}$$

then the distinguishing number $\mathrm{D}(G)$ of G is either 1 or 2.

Proof. Basically the proof consists of three steps. First we color part of the vertex set in order to break all automorphisms that move v_0. In the second step we break all automorphisms in $\mathrm{Aut}(G, v_0)$ that do not fix all ends of the graph by coloring some other vertices. Finally, we color the remaining vertices to break the rest of the automorphisms.

In order to break all automorphisms that move v_0 we apply Lemma 3.2, just as in the proof of Theorem 3.4. The only difference is that we choose k twice as large as proposed by Lemma 3.3, because we would like to color some additional spheres in the second step of the proof before applying an argument similar to that in Lemma 3.3.

For the second step consider the spheres $S_{v_0}(\frac{2\lambda+1}{2}k+4)$, $\lambda \in \mathbb{N}$. We wish to color those spheres such that every automorphism that fixes v_0 and preserves the coloring also fixes every end of G.

It is not hard to see that the sets $S^{\omega}_{v_0}(\frac{2\lambda+1}{2}k+4)$, ω an end of G, $\lambda \in \mathbb{N}$, carry the following tree structure. Consider v_0, the root, which is connected by an edge to $S^{\omega}_{v_0}(\frac{3}{2}k+4)$ for each end ω. For every end ω of G and every $\lambda \in \mathbb{N}$, draw an edge from $S^{\omega}_{v_0}(\frac{2\lambda+1}{2}k+4)$ to $S^{\omega}_{v_0}(\frac{2\lambda+3}{2}k+4)$. To see that this is indeed a tree just notice that if $S^{\omega_1}_{v_0}(n) = S^{\omega_2}_{v_0}(n)$, then, for every $m \leq n$, $S^{\omega_1}_{v_0}(m) = S^{\omega_2}_{v_0}(m)$. So there cannot be any circles. By construction, this tree structure is infinite, locally finite, and does not have any endpoints.

Next, notice that every automorphism $\phi \in \mathrm{Aut}(G, v_0)$ that does not fix all ends also acts as an automorphism on this tree structure. By [16], the distinguishing number of infinite, locally finite trees without endpoints is at most 2. Therefore it is possible to 2-color the sets $S^{\omega}_{v_0}(\frac{2\lambda+1}{2}k+4)$, ω an end of G, $\lambda \in \mathbb{N}$, such that every such automorphism

[1] Ends were first introduced by Freudenthal [4] in a topological setting, but here the definition of Halin [5] is more appropriate. For an accessible introduction to ends of infinite graphs see [2].

is broken. It is also worth noting that so far we did not use the countability of the end space of G, nor did we use the growth condition on the ends.

Let us turn to the third step of the proof. So far we have colored the ball $B_{v_0}(k+3)$ and the spheres $S_{v_0}(\frac{\lambda}{2}k+4)$, $\lambda \geq 2$, in a way that color preserving automorphisms fix v_0 and move every $S^{\omega}_{v_0}(n)$ into itself. Consider such an automorphism ϕ, which acts nontrivially on G. If we remove the fixed points of ϕ from G, then the infinite motion of G implies that the resulting graph has only infinite components. Hence, there is a ray in G which contains no fixed point of ϕ. The image of this ray must lie in the same end ω. Thus, there is an index n_0, such that, for every $n \geq n_0$, ϕ acts nontrivially on $S^{\omega}_{v_0}(n)$.

Let $(\omega_i)_{i\in\mathbb{N}}$ be an enumeration of the ends of G. Choose a function $f\colon \mathbb{N} \to \mathbb{N}$ such that, for every $i \in \mathbb{N}$, $f^{-1}(i)$ is infinite. Assume that all spheres up to $S_{v_0}(m)$ have been colored in the first $i-1$ steps. In the i-th step we would like to color some more spheres in order to continue breaking all automorphisms in $\operatorname{Aut}(G, v_0)$ that act nontrivially on each of the spheres $S^{\omega_{f(i)}}_{v_0}(n)$, $n \geq m$. This can be done by exactly the same argument as the one used in the proof of Lemma 3.3.

As we already mentioned, every automorphism that was not broken in the first two steps acts by nontrivially on the rays of some end. Since, in the procedure described above, every end is considered infinitely often, it is clear that every such automorphism will eventually be broken. This completes the proof. □

4 Graphs with higher cardinality

If a graph G has trivial automorphism group, then G is obviously 1-distinguishable, that is, $\mathrm{D}(G) = 1$. From now on we assume that our graphs G have nontrivial automorphism group. In this case, the *motion* $\mathrm{m}(G)$ of G is defined as

$$\mathrm{m}(G) = \min_{\phi\in\operatorname{Aut}(G)\setminus\{\mathrm{id}\}} \mathrm{m}(\phi)\,. \tag{4.1}$$

As already mentioned, the Infinite Motion Conjecture does not hold for graphs of higher cardinality. An example is the Cartesian product $G = K_{\mathfrak{n}} \,\square\, K_{\mathfrak{m}}$ of two complete graphs on infinitely many vertices $\mathfrak{n}$ and $\mathfrak{m}$ with $2^{\mathfrak{n}} < \mathfrak{m}$. By [9], G has motion $\mathfrak{n}$, but $\mathrm{D}(G) > \mathfrak{n}$.

The question arises whether one can adapt the Infinite Motion Conjecture to graphs of higher cardinality. The starting point is [12, Theorem 1]. It asserts that a finite graph G is 2-distinguishable if $\mathrm{m}(G) > 2\log_2|\operatorname{Aut}(G)|$. However, a second look at the proof shows that the inequality sign can be replaced by $\geq$. For details see Section 5. For finite graphs we thus infer that

$$\mathrm{m}(G) \geq 2\log_2|\operatorname{Aut}(G)| \quad \text{implies} \quad \mathrm{D}(G) = 2, \tag{4.2}$$

which can also be written in the form

$$|\operatorname{Aut}(G)| \leq 2^{\frac{\mathrm{m}(G)}{2}} \quad \text{implies} \quad \mathrm{D}(G) = 2\,.$$

Notice that $2^{\frac{\mathrm{m}(G)}{2}} = 2^{\mathrm{m}(G)}$ if $\mathrm{m}(G)$ is infinite. We are thus tempted to conjecture for graphs G with infinite motion that $|\operatorname{Aut}(G)| \leq 2^{\mathrm{m}(G)}$ implies $D(G) = 2$. We formulate this conjecture as the

Motion Conjecture. *Let G be a connected graph with infinite motion $\mathrm{m}(G)$ and $|\operatorname{Aut}(G)| \leq 2^{\mathrm{m}(G)}$. Then $\mathrm{D}(G) = 2$.*

How does this compare with the Infinite Motion Conjecture? It asserts that the distinguishing number of a locally finite, connected graph G is 2 if $\mathrm{m}(G)$ is infinite. Since locally finite graphs are countable, the condition that $\mathrm{m}(G)$ is infinite is equivalent to $\mathrm{m}(G) = \aleph_0$. Furthermore, for countable graphs we have

$$|\mathrm{Aut}(G)| \leq \aleph_0^{\aleph_0} = 2^{\aleph_0}.$$

Hence, for countable graphs, and thus also for locally finite, connected graphs with infinite motion, the inequality of the Motion Conjecture is automatically satisfied, which means that the Infinite Motion Conjecture is a special case of the Motion Conjecture.

Now, let us focus on the two results from [10] that imply the validity of the Infinite Motion Conjecture for graphs with countable group.

Theorem 4.1. *Let G be a locally finite, connected graph that satisfies $\aleph_0 \leq |\mathrm{Aut}(G)| < 2^{\aleph_0}$. Then $|\mathrm{Aut}(G)| = \aleph_0$, $\mathrm{m}(G) = \aleph_0$, and $\mathrm{D}(G) = 2$.*

Notice that the only thing that is required here, besides local finiteness and connectedness, is an upper and a lower bound on the size of $\mathrm{Aut}(G)$. And it turns out, that $\mathrm{Aut}(G)$ is countable, even without the continuum hypothesis. Even infinite motion and $\mathrm{D}(G) = 2$ are consequences of this restriction on the size of the automorphism group.

Theorem 4.2. *Let G be a countably infinite, connected graph that satisfies the conditions $|\mathrm{Aut}(G)| \leq \mathrm{m}(G)$ and $\mathrm{m}(G) = \aleph_0$. Then $\mathrm{D}(G) = 2$.*

Here, without local finiteness, one cannot drop the assumption of infinite motion. If we assume that $\mathrm{Aut}(G)$ has smaller cardinality than the continuum, then we can ensure 2-distinguishability if the continuum hypothesis holds, but we do not know whether this is really necessary.

Corollary 4.3. *Let G be a countably infinite, connected graph with infinite motion. If the continuum hypothesis holds, and if $|\mathrm{Aut}(G)| < 2^{\mathrm{m}(G)}$, then $D(G) = 2$.*

The next theorem shows that Theorem 4.2 also holds for graphs of higher cardinality and uncountable motion.

Theorem 4.4. *Let G be a connected graph with uncountable motion. Then $|\mathrm{Aut}(G)| \leq \mathrm{m}(G)$ implies $\mathrm{D}(G) = 2$.*

Proof. Set $\mathfrak{n} = |\mathrm{Aut}(G)|$, and let ζ be the smallest ordinal number whose underlying set has cardinality $\mathfrak{n}$. Furthermore, choose a well ordering $\prec$ of $A = \mathrm{Aut}(G) \setminus \{\mathrm{id}\}$ of order type ζ, and let α_0 be the smallest element with respect to $\prec$. Then the cardinality of the set of all elements of A between α_0 and any other $\alpha \in A$ is smaller than $\mathfrak{n} \leq \mathrm{m}(G)$.

Now we color all vertices of G white and use transfinite induction to break all automorphisms by coloring selected vertices black.

INDUCTION BASE By the assumptions of the theorem, there exists a vertex v_0 that is not fixed by α_0. We color it black. This coloring breaks α_0.

INDUCTION STEP Let $\beta \in A$. Suppose we have already broken all $\alpha \prec \beta$ by pairs of distinct vertices $(v_\alpha, \alpha(v_\alpha))$, where v_α is black and $\alpha(v_\alpha)$ white. Clearly, the cardinality of the set R of all $(v_\alpha, \alpha(v_\alpha))$, $\alpha \prec \beta$, is less than $\mathrm{m}(G) \geq \mathfrak{n}$. By assumption, β moves at least $\mathrm{m}(G)$ vertices. Since there are still $\mathfrak{n}$ vertices not in R, there must be a pair of vertices $(v_\beta, \beta(v_\beta))$ that does not meet R. We color v_β black. This coloring breaks β. $\square$

Corollary 4.5. *Let G be a connected graph with uncountable motion. If the general continuum hypothesis holds, and if $|\mathrm{Aut}(G)| < 2^{\mathrm{m}(G)}$, then $D(G) = 2$.*

Proof. Under the assumption of the general continuum hypothesis $2^{\mathrm{m}(G)}$ is the successor of $\mathrm{m}(G)$. Hence $|\mathrm{Aut}(G)| \leq \mathrm{m}(G)$, and the assertion of the corollary follows from Theorem 4.4. □

5 The Motion Lemma of Russell and Sundaram

In order to show that a finite graph G is 2-distinguishable if $\mathrm{m}(G) > 2\log_2|\mathrm{Aut}(G)|$, Russell and Sundaram [12] first defined the cycle norm of an automorphism ϕ. If

$$\phi = (v_{11}v_{12}\dots v_{1l_1})(v_{21}\dots v_{2l_2})\dots(v_{k1}\dots v_{kl_k}),$$

then the *cycle norm* $\mathrm{c}(\phi)$ of ϕ is

$$\mathrm{c}(\phi) = \sum_{i=1}^{k}(l_i - 1).$$

The cycle norm $\mathrm{c}(\phi)$ is related to graph distinguishability as follows: Let G be randomly 2-colored by independently assigning each vertex a color uniformly from $\{\text{black}, \text{white}\}$. Then the probability that every cycle of ϕ is monochromatic is $2^{-\mathrm{c}(\phi)}$. In this case, ϕ preserves the coloring so chosen.

Further, they define the cycle norm $\mathrm{c}(G)$ of a graph G as

$$\mathrm{c}(G) = \min_{\phi\in\mathrm{Aut}(G)\setminus\{\mathrm{id}\}} \mathrm{c}(\phi).$$

We now reprove Theorem 2 of [12] with $\geq$ instead of $>$. Because $\mathrm{c}(\phi) \geq \mathrm{m}(\phi)/2$ and thus $\mathrm{c}(G) \geq \mathrm{m}(G)/2$ we infer from Theorem 5.1 below that G is 2-distinguishable if $\mathrm{m}(G) \geq 2\log_2|\mathrm{Aut}(G)|$. We propose to call this result "Motion Lemma of Russell and Sundaram". Actually, the only difference from the original proof is the insertion of the middle term in (5.2).

Theorem 5.1. *Let G be a finite graph, and $\mathrm{c}(G)\log d \geq \log|\mathrm{Aut}(G)|$. Then G is d-distinguishable, that is, $\mathrm{D}(G) \leq d$.*

Proof. Let χ be a random d-coloring of G, the probability distribution being given by selecting the color of each vertex independently and uniformly in the set $\{1, \dots, d\}$. For a fixed automorphism $\phi \in \mathrm{Aut}(G) \setminus \{\mathrm{id}\}$ consider the probability that the random coloring χ is preserved by ϕ:

$$\Pr_\chi[\forall\, v : \chi(\phi(v)) = \chi(v)] = \left(\frac{1}{d}\right)^{\mathrm{c}(\phi)} \leq \left(\frac{1}{d}\right)^{\mathrm{c}(G)}. \tag{5.1}$$

Collecting these events yields the inequality

$$\begin{aligned}\Pr_\chi[\exists\,\phi \in \mathrm{Aut}(G)\setminus\{\mathrm{id}\}\,\forall\, v : \chi(\phi(v)) = \chi(v)] &\leq (|\mathrm{Aut}(G)| - 1)\left(\tfrac{1}{d}\right)^{\mathrm{c}(G)}\\ &< |\mathrm{Aut}(G)|\left(\tfrac{1}{d}\right)^{\mathrm{c}(G)}.\end{aligned} \tag{5.2}$$

By hypothesis the last term is at most 1. Thus there exists a coloring χ such that, for every $\phi \in \mathrm{Aut}(G) \setminus \{\mathrm{id}\}$, there is a v for which $\chi(\phi(v)) \neq \chi(v)$, as desired. □

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Isomorphic tetravalent cyclic Haar graphs

Hiroki Koike
University of Primorska, UP IAM, Muzejski trg 2, SI6000 Koper, Slovenia

István Kovács *
University of Primorska, UP IAM and UP FAMNIT
Muzejski trg 2, SI6000 Koper, Slovenia

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Abstract

Let S be a subset of the cyclic group $\mathbb{Z}_n$. The cyclic Haar graph $H(\mathbb{Z}_n, S)$ is the bipartite graph with color classes $\mathbb{Z}_n^+$ and $\mathbb{Z}_n^-$, and edges $\{x^+, y^-\}$, where $x, y \in \mathbb{Z}_n$ and $y - x \in S$. In this paper we give sufficient and necessary conditions for the isomorphism of two connected cyclic Haar graphs of valency 4.

Keywords: Graph isomorphism, cyclic Haar graph, 4*-BCI-group.*

Math. Subj. Class.: 20B25, 05C25, 05C60

1 Introduction

Let S be a subset of a finite group G. The *Haar graph* $H(G, S)$ is the bipartite graph with color classes identified with G and written as G^+ and G^-, and the edges are $\{x^+, y^-\}$, where $x, y \in G$ and $yx^{-1} \in S$. Haar graphs were introduced for abelian groups by Hladnik, Marušič and Pisanski [6], and were redefined under the name *bi-Cayley graphs* in [17]. A Haar graph $H(G, S)$ is called *cyclic* if G is a cyclic group. In this paper we consider the problem of giving sufficient and necessary conditions for the isomorphism of two cyclic Haar graphs. This is a natural continuation of the isomorphism problem of circulant digraphs which has been solved by Muzychuk [12]. It appears in the context of circulant matrices under the name bipartite Ádám problem [16], and also in the context of cyclic configurations [2, 6].

The symbol $\mathbb{Z}_n$ denotes the additive group of the ring $\mathbb{Z}/n\mathbb{Z}$ of residue classes modulo n, and $\mathbb{Z}_n^*$ denotes the multiplicative group of units in $\mathbb{Z}/n\mathbb{Z}$. Two Haar graphs $H(\mathbb{Z}_n, S)$ and $H(\mathbb{Z}_n, T)$ are called *affinely equivalent,* written as $H(\mathbb{Z}_n, S) \cong_{\mathrm{aff}} H(\mathbb{Z}_n, T)$, if S

*Corresponding author.
E-mail addresses: hiroki.koike@upr.si (Hiroki Koike), istvan.kovacs@upr.si (István Kovács)

can be mapped to T by an affine transformation, i.e., $aS + b = T$ for some $a \in \mathbb{Z}_n^*$ and $b \in \mathbb{Z}_n$. It is an easy exercise to show that two affinely equivalent cyclic Haar graphs are isomorphic as usual graphs. The converse implication is not true in general, and this makes the following definition interesting (see [17]): we say that a subset $S \subseteq \mathbb{Z}_n$ is a *BCI-subset* if for each $T \subseteq G$, $H(\mathbb{Z}_n, S) \cong H(\mathbb{Z}_n, T)$ if and only if $H(\mathbb{Z}_n, S) \cong_{\text{aff}} H(\mathbb{Z}_n, T)$. Wiedemann and Zieve proved in [16, Theorem 1.1] that any subset S of $\mathbb{Z}_n$ is a BCI-subset if $|S| \leq 3$ (a special case was proved earlier in [3]). However, this is not true if $|S| \geq 4$ (see [6, 16]), hence the first nontrivial case of the isomorphism problem occurs when $|S| = 4$. In this paper we settle this case by proving the following theorem:

Theorem 1.1. *Two connected Haar graphs $H(\mathbb{Z}_n, S)$ and $H(\mathbb{Z}_n, T)$ with $|S| = |T| = 4$ are isomorphic if and only if there exist $a_1, a_2 \in \mathbb{Z}_n^*$ and $b_1, b_2 \in \mathbb{Z}_n$ such that*

(1) $a_1S + b_1 = T$; *or*

(2) $a_1S + b_1 = \{0, u, v, v + m\}$ *and* $a_2T + b_2 = \{0, u + m, v, v + m\}$, *where* $n = 2m$, $\mathbb{Z}_n = \langle u, v\rangle$, $2 \mid u$, $2u \mid m$ *and* $u/2 \not\equiv v + m/(2u) (\text{mod } m/u)$.

Remark 1.2. A group G is called an *m-BCI-group* if every subset S of G with $|S| \leq m$ is a BCI-subset (see [7, 17]). In this context [16, Theorem 1.1] can be rephrased as $\mathbb{Z}_n$ is a 3-BCI-group for any number n; and Theorem 1.1 says that $\mathbb{Z}_n$ is not a 4-BCI-group if and only if n is divisible by 8. This refines [16, Theorem 7.2] in which it is proved that, if $\mathbb{Z}_n$ contains a non-BCI-subset of size k, $k \in \{4, 5\}$, then n has a prime divisor less or equal to $2k(k - 1)$.

Our approach towards Theorem 1.1 is group theoretical, we adopt the ideas of [1, 11]. In short terms the initial problem is transformed to a problem about the automorphism group of the graphs in question. Theorem 1.1 is proven in two steps: first it is settled for graphs $H(\mathbb{Z}_n, S)$ with S satisfying additional conditions (see Theorem 3.1); then it is shown that, if S is not a BCI-subset, then it is affinely equivalent to a set satisfying the conditions of Theorem 3.1 (see Theorem 4.1).

We conclude the introduction with the following modification of Theorem 1.1:

Theorem 1.3. *Two connected Haar graphs $H(\mathbb{Z}_n, S)$ and $H(\mathbb{Z}_n, T)$ with $|S| = |T| = 4$ are isomorphic if and only if there exist $a_1, a_2 \in \mathbb{Z}_n^*$ and $b_1, b_2 \in \mathbb{Z}_n$ such that*

(1) $a_1S + b_1 = T$; *or*

(2) $a_1S + b_1 = \{0, u, v, v + m\}$ *and* $a_2T + b_2 = \{0, u + m, v, v + m\}$, *where* $n = 2m$, $\mathbb{Z}_n = \langle u, v\rangle$, $2 \mid u$, $2u \mid m$.

Proof. In view of Theorem 1.1 it is sufficient to prove that $H(\mathbb{Z}_n, S) \cong H(\mathbb{Z}_n, T)$ if

$$a_1S + b_1 = \{0, u, v, v + m\} \text{ and } a_2T + b_2 = \{0, u + m, v, v + m\},$$

where $n = 2m$, $\mathbb{Z}_n = \langle u, v\rangle$, $2 \mid u$, $2u \mid m$ and $u/2 \equiv v + m/(2u) (\text{mod } m/u)$. In fact, we are going to show below that there exist $a \in \mathbb{Z}_n^*$ and $b \in \mathbb{Z}_n$ such that

$$a \cdot \{0, u, v, v + m\} + b = \{0, u + m, v, v + m\}.$$

Then $(a_2^{-1}aa_1) \cdot S + a_2^{-1}(ab_1 + b - b_2) = T$, and so $H(\mathbb{Z}_n, S) \cong H(\mathbb{Z}_n, T)$.

Let us consider the following system of congruences:

$$ux \equiv -u + m (\bmod n) \text{ and } vx \equiv -u + v (\bmod n). \tag{1.1}$$

By the first congruence, using also that $2u \mid m$, x may be written in the form $x = (n/u)y - 1 + m/u$. Plugging this in the second one, we obtain $(vn/u)y \equiv 2v - u - vm/u (\bmod n)$, which has an integer solution in y exactly when $\gcd(vn/u, n) \mid (2v - u - vm/u)$. Then $\gcd(vn/u, n) = n/u \gcd(u, v)$, and since $\mathbb{Z}_n = \langle u, v\rangle$, n/u and $\gcd(u, v)$ are coprime. Since $\gcd(u, v)$ is clearly a divisor of $2v - u - vm/u$, a solution in y exists if and only if $n/u \mid (2v-u-vm/u)$, i.e., $u \equiv 2v-vm/u (\bmod 2m/u)$ (recall that $n = 2m$). On the other hand, one of the initial assumptions is $u/2 \equiv v+m/(2u) (\bmod m/u)$, and so $u \equiv 2v + m/u (\bmod 2m/u)$. We conclude that (1.1) has an integer solution if $-vm/u \equiv m/u (\bmod 2m/u)$. Now, the latter congruence holds because of the conditions $2 \mid u$, $2 \mid n$, and $\mathbb{Z}_n = \langle u, v\rangle$. Let a be a solution of (1.1). It follows from the above argument that $\gcd(a, m/u) = 1$. Notice that, since $2u \mid m$, $2 \nmid a$. Let $d = \gcd(a, u)$. By (1.1), $av \equiv -u + v (\bmod n)$, implying that $d \mid v$, and so $d = 1$. We see that $\gcd(a, 2m) = 1$, i.e., $a \in \mathbb{Z}_n^*$. Choosing $b = u + m$, we finally get by (1.1), $a \cdot 0 + b = u + m$, $au + b = 0$, $av + b = v + m$, and $a(v + m) + b = v$, as required. □

Theorem 1.3 becomes especially interesting when we compare it with the solution of the isomorphism problem of trivalent circulant digraphs. In fact, the same conditions can be derived from Muzychuk's general algorithm presented in [12]: two connected Cayley digraphs $\mathrm{Cay}(\mathbb{Z}_n, S)$ and $\mathrm{Cay}(\mathbb{Z}_n, T)$ with $|S| = |T| = 3$ are isomorphic if and only if there exist $a_1, a_2 \in \mathbb{Z}_n^*$ such that

- $a_1S = T$; or
- $a_1S = \{u, v, v + m\}$ and $a_2T = \{u + m, v, v + m\}$, where $n = 2m$, $\mathbb{Z}_n = \langle u, v\rangle$, $2 \mid u$, and $2u \mid m$.

The natural question arises whether this phenomenon holds also for graphs of larger valencies.

2 A Babai type theorem

In this paper every group, graph and digraph is finite. For a (di)graph Γ, the symbols $V(\Gamma)$, $E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ denote the set of its vertices, (directed) edges and the full group of its automorphisms, respectively. Regarding terminology and notation in permutation group theory we follow [5].

Let S be a subset of a group G. The *Cayley digraph* $\mathrm{Cay}(G, S)$ is the digraph with vertex set G, and its directed edges are (x, y), where $x, y \in G$ and $yx^{-1} \in S$. Two digraphs $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, T)$ are called *Cayley isomorphic*, written as $\mathrm{Cay}(G, S) \cong_{\mathrm{cay}} \mathrm{Cay}(G, T)$, if $T = S^{\varphi}$ for some group automorphism $\varphi \in \mathrm{Aut}(G)$. It is clear that such an automorphism induces an isomorphism between $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, T)$, and thus Cayley isomorphic digraphs are isomorphic as usual digraphs. It is also well-known that the converse implication is not true, and this makes sense for the following definition (see [1]): a subset $S \subseteq G$ is a *CI-subset* if for each $T \subseteq G$, $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$ if and only if $\mathrm{Cay}(G, S) \cong_{\mathrm{cay}} \mathrm{Cay}(G, T)$. The following equivalence was proved by Babai [1, 3.1 Lemma].

Theorem 2.1. *The following are equivalent for every Cayley digraph* $\mathrm{Cay}(G, S)$.

(1) S is a CI-subset.

(2) Every two regular subgroups of $\mathrm{Aut}(\mathrm{Cay}(G,S))$ *isomorphic to G are conjugate in* $\mathrm{Aut}(\mathrm{Cay}(G,S))$.

Theorem 2.1 essentially says that the CI-property of a given subset S depends entirely on the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G,S))$. In this section we prove analogous results for cyclic Haar graphs.

Let $V = V(H(\mathbb{Z}_n, S))$ be the vertex set of the Haar Graph $H(\mathbb{Z}_n, S)$. Throughout this paper c and d denote the permutations of V defined by

$$c : x^\varepsilon \mapsto (x+1)^\varepsilon \text{ and } d : x^\varepsilon \mapsto \begin{cases} (n-x)^- & \text{if } \varepsilon = +, \\ (n-x)^+ & \text{if } \varepsilon = -, \end{cases} \tag{2.1}$$

where $x \in \mathbb{Z}_n$ and $\varepsilon \in \{+,-\}$. It follows immediately that both c and d are automorphisms of any Haar graph $H(\mathbb{Z}_n, S)$. Denote by C the group generated by c, and by D the group generated by c and d. The group D acts regularly on V, and D is isomorphic to D_{2n}. Thus $H(\mathbb{Z}_n, S)$ is isomorphic to a Cayley graph over D, and so Theorem 2.1 can be applied. The following corollary is obtained.

Corollary 2.2. *The implication (1) ⇐ (2) holds for every Haar graph $H(\mathbb{Z}_n, S)$.*

(1) S is a BCI-subset.

(2) Every two regular subgroups of $\mathrm{Aut}(H(\mathbb{Z}_n,S))$ *isomorphic to D are conjugate in* $\mathrm{Aut}(H(\mathbb{Z}_n,S))$.

However, we do not have equivalence in Corollary 2.2 as it is shown in the following example.

Example 2.3. Let $\Gamma = H(\mathbb{Z}_{10}, \{0,1,3,4\})$. Using the computer package MAGMA [4] we compute that Γ is edge-transitive and its automorphism group $\mathrm{Aut}(\Gamma) \cong D_{20} \rtimes \mathbb{Z}_4$. Furthermore, $\mathrm{Aut}(\Gamma)$ contains a regular subgroup X which is isomorphic to D_{20} but $X \neq D_{20}$, hence (2) in Corollary 2.2 does not hold.

On the other hand, we find that for every subset $T \subseteq \mathbb{Z}_{10}$ with $0 \in T$ and $|T| = 4$, the corresponding Haar graph $H(\mathbb{Z}_{10}, T) \cong \Gamma$ exactly when $H(\mathbb{Z}_{10}, T) \cong_{\mathrm{aff}} \Gamma$. Thus $\{0,1,3,4\}$ is a BCI-subset, so (1) in Corollary 2.2 holds. □

Example 2.3 shows that the isomorphism problem of cyclic Haar graphs is not a particular case of the isomorphism problem of Cayley graphs over dihedral groups. We remark that the latter problem is still unsolved, for partial solutions, see [1, 13, 14]. Nonetheless, the idea of Babai works well if instead of the regular subgroup D we consider its index 2 cyclic subgroup C.

We say that a permutation group $G \le \mathrm{Sym}(\mathbb{Z}_n^+ \cup \mathbb{Z}_n^-)$ is *bicyclic* if G is a cyclic group which has two orbits: $\mathbb{Z}_n^+$ and $\mathbb{Z}_n^-$. By a bicyclic group of a Haar graph $\Gamma = H(\mathbb{Z}_n, S)$ we simply mean a bicyclic subgroup $X \le \mathrm{Aut}(\Gamma)$. Obviously, C is a bicyclic group of any cyclic Haar graph, and being so it will be referred to as the *canonical bicyclic group*.

Let $\mathrm{Iso}(\Gamma)$ denote the set of all isomorphisms from Γ to any other Haar graph $H(\mathbb{Z}_n, T)$, i.e.,

$$\mathrm{Iso}(\Gamma) = \{f \in \mathrm{Sym}(V) : \Gamma^f = H(\mathbb{Z}_n, T) \text{ for some } T \subseteq \mathbb{Z}_n\}.$$

And let $\mathcal{C}_{\mathrm{iso}}(\Gamma)$ denote the *isomorphism class* of cyclic Haar graphs which contains Γ, i.e., $\mathcal{C}_{\mathrm{iso}}(\Gamma) = \{\Gamma^f : f \in \mathrm{Iso}(\Gamma)\}$.

Lemma 2.4. *Let* $\Gamma = H(\mathbb{Z}_n, S)$ *be a connected Haar graph and* f *be in* $\mathrm{Sym}(V)$*. Then* $f \in \mathrm{Iso}(\Gamma)$ *if and only if* fCf^{-1} *is a bicyclic group of* Γ*.*

Proof. Let $f \in \mathrm{Iso}(\Gamma)$. Then $fCf^{-1} \leq \mathrm{Aut}(\Gamma)$. Clearly, fCf^{-1} is a cyclic group. Since the sets $\mathbb{Z}_n^+$ and $\mathbb{Z}_n^-$ are the color classes of the connected bipartite graph Γ, f preserves these color classes, implying that $\mathrm{Orb}(fCf^{-1}, V) = \{\mathbb{Z}_n^+, \mathbb{Z}_n^-\}$. The group fCf^{-1} is a bicyclic group of Γ.

Conversely, suppose that fCf^{-1} is a bicyclic group of Γ. Then $C = f^{-1}(fCf^{-1})f \leq \mathrm{Aut}(\Gamma^f)$. Because that $\mathrm{Orb}(fCf^{-1}, V) = \{\mathbb{Z}_n^+, \mathbb{Z}_n^-\}$, the graph Γ^f is connected and bipartite with color classes $\mathbb{Z}_n^+$ and $\mathbb{Z}_n^-$. We conclude that $\Gamma^f = H(\mathbb{Z}_n, T)$ for some $T \subseteq \mathbb{Z}_n$, so $f \in \mathrm{Iso}(\Gamma)$. The lemma is proved. □

Lemma 2.4 shows that the normalizer $N_{\mathrm{Sym}(V)}(C) \subseteq \mathrm{Iso}(H(\mathbb{Z}_n, S))$. The group $N_{\mathrm{Sym}(V)}(C)$ is known to consist of the following permutations:

$$\varphi_{r,s,t} : x^\varepsilon \mapsto \begin{cases} (rx+s)^+ & \text{if } \varepsilon = +, \\ (rx+t)^- & \text{if } \varepsilon = -, \end{cases} \qquad \psi_{r,s,t} : x^\varepsilon \mapsto \begin{cases} (rx+s)^- & \text{if } \varepsilon = +, \\ (rx+t)^+ & \text{if } \varepsilon = -, \end{cases} \tag{2.2}$$

where $r \in \mathbb{Z}_n^*$ and $s, t \in \mathbb{Z}_n$. Note that, two Haar graphs $H(\mathbb{Z}_n, S)$ and $H(\mathbb{Z}_n, T)$ are from the same orbit under $N_{\mathrm{Sym}(V)}(C)$ exactly when $H(\mathbb{Z}_n, S) \cong_{\mathrm{aff}} H(\mathbb{Z}_n, T)$. Let $\mathcal{C}_{\mathrm{aff}}(\Gamma)$ denote the *affine equivalence class* of cyclic Haar graphs which contains the graph $\Gamma = H(\mathbb{Z}_n, S)$, i.e., $\mathcal{C}_{\mathrm{aff}}(\Gamma) = \{\Gamma^\varphi : \varphi \in N_{\mathrm{Sym}(V)}(C)\}$. It is clear that the isomorphism class $\mathcal{C}_{\mathrm{iso}}(\Gamma)$ splits into affine equivalence classes:

$$\mathcal{C}_{\mathrm{iso}}(\Gamma) = \mathcal{C}_{\mathrm{aff}}(\Gamma_1) \,\dot{\cup}\, \cdots \,\dot{\cup}\, \mathcal{C}_{\mathrm{aff}}(\Gamma_k)^1.$$

Our next goal is to describe the above decomposition with the aid of bicyclic groups. Let X be a bicyclic group of a connected graph $\Gamma = H(\mathbb{Z}_n, S)$. Then $g^{-1}Xg$ is also a bicyclic group for every $g \in \mathrm{Aut}(\Gamma)$, hence the full set of bicyclic groups of Γ is the union of $\mathrm{Aut}(\Gamma)$-conjugacy classes. We say that a subset $\Xi \subseteq \mathrm{Iso}(\Gamma)$ is a *bicyclic base* of Γ if the subgroups $\xi C \xi^{-1}, \xi \in \Xi$, form a complete set of representatives of the corresponding conjugacy classes. Thus every bicyclic group X can be expressed as

$$X = g\xi C(g\xi)^{-1} \text{ for a unique } \xi \in \Xi \text{ and } g \in \mathrm{Aut}(\Gamma).$$

Remark 2.5. Our definition of a bicyclic base copies in a sense the definition of a cyclic base introduced by Muzychuk [11, Definition, page 591].

Theorem 2.6. *Let* $\Gamma = H(\mathbb{Z}_n, S)$ *be a connected Haar graph with a bicyclic base* Ξ*. Then* $\mathcal{C}_{\mathrm{iso}}(\Gamma) = \dot{\bigcup}_{\xi \in \Xi} \mathcal{C}_{\mathrm{aff}}(\Gamma^\xi)$.

Proof. It follows immediately that,

$$\mathcal{C}_{\mathrm{iso}}(\Gamma) \supseteq \bigcup_{\xi \in \Xi} \mathcal{C}_{\mathrm{aff}}(\Gamma^\xi). \tag{2.3}$$

[1]Here we mean that $\mathcal{C}_{\mathrm{iso}}(\Gamma) = \mathcal{C}_{\mathrm{aff}}(\Gamma_1) \cup \cdots \cup \mathcal{C}_{\mathrm{aff}}(\Gamma_k)$ and $\mathcal{C}_{\mathrm{aff}}(\Gamma_i) \cap \mathcal{C}_{\mathrm{aff}}(\Gamma_j) = \emptyset$ for every $i, j \in \{1, \ldots, k\}$, $i \neq j$.

We prove that equality holds in (2.3). Pick $\Sigma \in \mathcal{C}_{\rm iso}(\Gamma)$. Then $\Sigma = \Gamma^f$ for some $f \in \mathrm{Iso}(\Gamma)$. By Lemma 2.4, fCf^{-1} is a bicyclic group of Γ, hence

$$fCf^{-1} = g\xi C(g\xi)^{-1},\ \xi \in \Xi, g \in \mathrm{Aut}(\Gamma).$$

Thus $f^{-1}g\xi = h$, where $h \in N_{\mathrm{Sym}(V)}(C)$. Then

$$\Sigma = \Gamma^f = \Gamma^{g\xi h^{-1}} = \left(\Gamma^{\xi}\right)^{h^{-1}}.$$

This shows that $\Sigma \in \mathcal{C}_{\rm aff}(\Gamma^{\xi})$, and so

$$\mathcal{C}_{\rm iso}(\Gamma) \subseteq \bigcup_{\xi \in \Xi} \mathcal{C}_{\rm aff}(\Gamma^{\xi}).$$

In view of (2.3) the two sides are equal.

Moreover, if $\mathcal{C}_{\rm aff}(\Gamma^{\xi_1}) \cap \mathcal{C}_{\rm aff}(\Gamma^{\xi_2}) \neq \emptyset$ for $\xi_1, \xi_2 \in \Xi$, then $\Gamma^{\xi_1} = \Gamma^{\xi_2 h}$ for some $h \in N_{\mathrm{Sym}(V)}(C)$. Hence $\xi_2 h \xi_1^{-1} = g$ for some $g \in \mathrm{Aut}(\Gamma)$, and so

$$\xi_1 C \xi_1^{-1} = g^{-1}\xi_2 h C h^{-1} \xi_2^{-1} g = g^{-1}(\xi_2 C \xi_2^{-1})g.$$

The bicyclic subgroups $\xi_1 C \xi_1^{-1}$ and $\xi_2 C \xi_2^{-1}$ are conjugate in $\mathrm{Aut}(\Gamma)$, hence $\xi_1 = \xi_2$ follows from the definition of the bicyclic base Ξ. We obtain that $\mathcal{C}_{\rm aff}(\Gamma^{\xi_1}) \cap \mathcal{C}_{\rm aff}(\Gamma^{\xi_2}) = \emptyset$ whenever $\xi_1, \xi_2 \in \Xi$, $\xi_1 \neq \xi_2$, and so $\mathcal{C}_{\rm iso}(\Gamma) = \dot{\bigcup}_{\xi \in \Xi} \mathcal{C}_{\rm aff}(\Gamma^{\xi})$. The theorem is proved. □

As a direct consequence of Theorem 2.6 we obtain the following corollary, analog of Theorem 2.1.

Corollary 2.7. *The following are equivalent for every connected Haar graph $H(\mathbb{Z}_n, S)$.*

(1) S is a BCI-subset.

(2) Any two bicyclic groups of $H(\mathbb{Z}_n, S)$ are conjugate in $\mathrm{Aut}(H(\mathbb{Z}_n, S))$.

In our last proposition we connect the BCI-property with the CI-property. For $a^{\varepsilon} \in V$, in what follows $\mathrm{Aut}(H(\mathbb{Z}_n, S))_{a^{\varepsilon}}$ denotes the vertex stabilizer of a^{ε} in $\mathrm{Aut}(H(\mathbb{Z}_n, S))$.

Proposition 2.8. *Suppose that $\Gamma = H(\mathbb{Z}_n, S)$ is a connected Haar graph such that for some $a \in \mathbb{Z}_n$, $\mathrm{Aut}(\Gamma)_{0^+} = \mathrm{Aut}(\Gamma)_{a^-}$. Then the following are equivalent.*

(1) S is a BCI-subset.

(2) $S - a = \{s - a : s \in S\}$ is a CI-subset.

Proof. For sake of simplicity we put $A = \mathrm{Aut}(\Gamma)$ and $G = \mathrm{Aut}(\Gamma)_{\{\mathbb{Z}_n^+\}}$, i.e., the setwise stabilizer of the color class $\mathbb{Z}_n^+$ in $\mathrm{Aut}(\Gamma)$. Obviously, $X \leq G$ for every bicyclic group X of Γ. Since $A = G \rtimes \langle d \rangle$ and d normalizes C, it follows that the conjugacy class of subgroups of A containing C is equal to the conjugacy class of subgroups of G containing C. Using this and Theorem 2.6, we obtain that S is a BCI-subset if and only if every bicyclic group is conjugate to C in G.

Let $W = \{0^+, a^-\}$ and consider the setwise stabilizer $A_{\{W\}}$. Since $A_{0^+} = A_{a^-}$, $A_{0^+} \le A_{\{W\}}$. By [5, Theorem 1.5A], the orbit of 0^+ under $A_{\{W\}}$ is a block of imprimitivity (for short a block) for A. Denote this block by Δ and the induced system of blocks by δ (i.e., $\delta = \{\Delta^g : g \in G\}$). Consider the element $g = dc^a$ from D. We see that g switches 0^+ and a^-, hence $A_{\{W\}} = A_{0^+}\langle g\rangle$. Therefore, $\Delta = (0^+)^{A_{\{W\}}} = (0^+)^{A_{0^+}\langle g\rangle} = (0^+)^{\langle g\rangle} = W$, and so

$$\delta = \big\{ \{x^+, (x+a)^-\} : x \in \mathbb{Z}_n \big\}.$$

Define the mapping $\varphi : \delta \to \mathbb{Z}_n$ by $\varphi : \{x^+, (x+a)^-\} \mapsto x$, $x \in \mathbb{Z}_n$. Now, an action of A on $\mathbb{Z}_n$ can be defined by letting $g \in A$ act as

$$x^g = x^{\varphi^{-1} g \varphi},\ x \in \mathbb{Z}_n.$$

For $g \in A$ we write $\bar{g}$ for the image of g under the corresponding permutation representation, and for a subgroup $X \le A$ we let $\bar{X} = \{\bar{x} : x \in X\}$. In this action of A the subgroup $G < A$ is faithful. Also notice that, a subgroup $X \le G$ is a bicyclic group of Γ if and only if $\bar{X}$ is a regular cyclic subgroup of $\bar{G}$. In particular, for the canonical bicyclic group C, $\bar{C} = (\mathbb{Z}_n)_{\text{right}}$, where $(\mathbb{Z}_n)_{\text{right}}$ denotes the group generated by the affine transformation $x \mapsto x+1$, $x \in \mathbb{Z}_n$.

Pick $g \in G$ and $(x, x+s-a) \in \mathbb{Z}_n \times \mathbb{Z}_n$, where $s \in S$. Then g maps the directed edge $(x^+, (x+s)^-)$ to a directed edge $(y^+, (y+q)^-)$ for some $y \in \mathbb{Z}_n$ and $q \in S$. Since δ is a system of blocks for G, g maps $(x+s-a)^+$ to $(y+q-a)^+$, and so $\bar{g}$ maps the pair $(x, x+s-a)$ to the pair $(y, y+q-a)$. We have just proved that $\bar{g}$ leaves the set $\big\{ (x, x+s-a) : x \in \mathbb{Z}_n, s \in S \big\}$ setwise fixed. As the latter set is the set of all directed edges of the digraph $\mathrm{Cay}(\mathbb{Z}_n, S-a)$, $\bar{G} \le \mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_n, S-a))$. For an automorphism h of $\mathrm{Cay}(\mathbb{Z}_n, S-a)$, define the permutation g of V by

$$g : x^\varepsilon \mapsto \begin{cases} (x^h)^+ & \text{if } \varepsilon = +, \\ ((x-a)^h + a)^- & \text{if } \varepsilon = -, \end{cases} \quad x \in \mathbb{Z}_n,\ \varepsilon \in \{+,-\}.$$

The reader is invited to check that the above permutation g is an automorphism of Γ. It is clear that $g \in G$ and $\bar{g} = h$; we conclude that $\bar{G} = \mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_n, S-a))$.

Now, the proposition follows along the following equivalences:

$$\begin{aligned} (1) \quad &\iff \quad \text{Every bicyclic group of } \Gamma \text{ is conjugate to } C \text{ in } G \\ &\iff \quad \text{Every regular cyclic subgroup of } \bar{G} \text{ is conjugate to } \bar{C} \text{ in } \bar{G} \\ &\iff \quad (2). \end{aligned}$$

The last equivalence is Theorem 2.1. □

Remark 2.9. Let us remark that the equality $\mathrm{Aut}(\Gamma)_{0^+} = \mathrm{Aut}(\Gamma)_{a^-}$ does not hold in general. For example, take Γ as the incidence graph of the projective space $\mathrm{PG}(d,q)$ where $d \ge 2$ and q is a prime power (i.e., Γ is the bipartite graph whose color classes are identified by the set of points and the set of hyperplanes, respectively, and the edges are defined by the incidence relation of the space). It is well-known that $\mathrm{PG}(d,q)$ admits a cyclic group of automorphisms (called a Singer subgroup) acting regularly on both the points and the hyperplanes. This shows that Γ is isomorphic to a cyclic Haar graph, and we may identify the set of points with $\mathbb{Z}_n^+$, and the set of hyperplanes with $\mathbb{Z}_n^-$, where

$n = (q^d - 1)/(q - 1)$. The automorphism group $\mathrm{Aut}(\Gamma) = \mathrm{P\Gamma L}(d+1, q) \rtimes \mathbb{Z}_2$; and as $\mathrm{P\Gamma L}(d+1, q)$ acts inequivalently on the points and the hyperplanes, $\mathrm{Aut}(\Gamma)_{0^+}$ cannot be equal to $\mathrm{Aut}(\Gamma)_{a^-}$ for any $a \in \mathbb{Z}_n$.

3 Haar graphs $H(\mathbb{Z}_{2m}, \{0, u, v, v+m\})$

In this section we prove Theorem 1.1 for Haar graphs $H(\mathbb{Z}_n, S)$ satisfying certain additional conditions.

Theorem 3.1. *Let $n = 2m$ and $S = \{0, u, v, v+m\}$ such that*

(a) $\mathbb{Z}_n = \langle u, v \rangle$;

(b) $1 < u < m$, $u \mid m$;

(c) $\mathrm{Aut}(H(\mathbb{Z}_n, S))_{0^+}$ *leaves the set* $\{0^-, u^-\}$ *setwise fixed.*

Then $H(\mathbb{Z}_n, S) \cong H(\mathbb{Z}_n, T)$ if and only if there exist $a \in \mathbb{Z}_n^*$ and $b \in \mathbb{Z}_n$ such that

(1) $aT + b = S$; *or*

(2) $aT + b = \{0, u+m, v, v+m\}$, *and* $2 \mid u$, $2u \mid m$, $u/2 \not\equiv v + m/(2u) (\mathrm{mod}\ m/u)$.

By (b) of Theorem 3.1 we have $2u \le m$. We prove the extremal case, when $2u = m$, separately. Notice that, in this case the conditions in (2) of Theorem 3.1 that $2 \mid u$, $2u \mid m$ and $u/2 \not\equiv v + m/(2u) (\mathrm{mod}\ m/u)$ can be replaced by one condition: $u \equiv 2 (\mathrm{mod}\ 4)$.

Lemma 3.2. *Let S be the set defined in Theorem 3.1. If $2u = m$, then $H(\mathbb{Z}_n, S) \cong H(\mathbb{Z}_n, T)$ if and only if there exist $a \in \mathbb{Z}_n^*$ and $b \in \mathbb{Z}_n$ such that*

(1) $aT + b = S$; *or*

(2) $aT + b = \{0, u+m, v, v+m\}$ *and* $u \equiv 2 (\mathrm{mod}\ 4)$.

Proof. Let $d = \gcd(n, v)$. Because of $\langle u, v \rangle = \mathbb{Z}_n$ we have that $\gcd(u, v, n) = 1$, i.e., $\gcd(n/4, v) = 1$, and this gives that $d \in \{1, 2, 4\}$. Note that, if $d \neq 1$, then necessarily $2 \nmid u$. Let us write $v = v_1 d$, where $\gcd(v_1, n) = 1$. Let v_1^{-1} denote the inverse of v_1 in the group $\mathbb{Z}_n^*$. Then the following hold in $\mathbb{Z}_n$ (here we use that $u = n/4$):

$$v_1^{-1} v = d,\ v_1^{-1}(v+m) = d + m \text{ and } v_1^{-1} u \in \{u, 3u\}.$$

We conclude that S is affinely equivalent to one of the sets $S_i(d)$, $i \in \{1, 2\}$ and $d \in \{1, 2, 4\}$, where

$$S_1(d) = \{0, u, d, d+2u\} \text{ or } S_2(d) = \{0, 3u, d, d+2u\}.$$

The lemma follows from the following claims:

(i) $H(\mathbb{Z}_n, S_1(1)) \cong H(\mathbb{Z}_n, S_2(1))$.

(ii) $H(\mathbb{Z}_n, S_1(1)) \cong_{\mathrm{aff}} H(\mathbb{Z}_n, S_1(d))$ for $d \in \{2, 4\}$;

(iii) $H(\mathbb{Z}_n, S_1(d)) \cong_{\mathrm{aff}} H(\mathbb{Z}_n, S_2(d)) \iff d \in \{2, 4\}$ or ($d = 1$ and $u \not\equiv 2 (\mathrm{mod}\ 4)$);

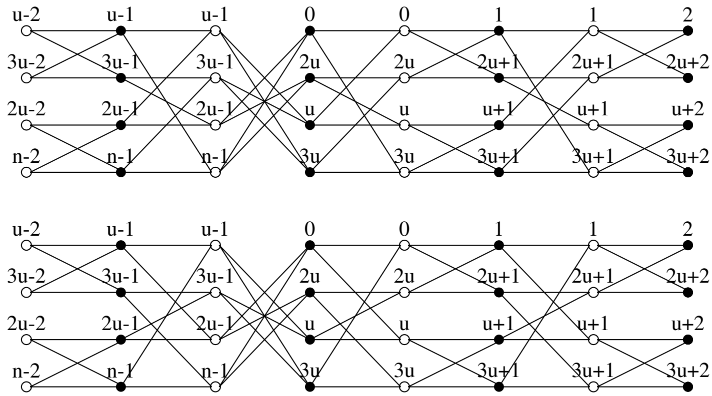

Figure 1: Haar graphs $H(\mathbb{Z}_n, S_1(1))$ and $H(\mathbb{Z}_n, S_2(1))$.

(i): Define the mapping $f : V \mapsto V$ by

$$f : x^{\varepsilon} \mapsto \begin{cases} x^{\varepsilon} & \text{if } x \in \{0, 1, \ldots .u-1\} \cup \{2u, \ldots .3u-1\}, \\ (x+2u)^{\varepsilon} & \text{otherwise.} \end{cases}$$

We leave for the reader to verify that f is in fact an isomorphism from $H(\mathbb{Z}_n, S_1(1))$ to $H(\mathbb{Z}_n, S_2(1))$ (compare the graphs in Figure 1; here the white vertices represent the color class $\mathbb{Z}_n^+$, while the black ones represent the color class $\mathbb{Z}_n^-$).

(ii): Since $d \in \{2, 4\}$, u is an odd number. For $d \in \{2, 4\}$ define $r_d \in \mathbb{Z}_n^*$ as follows:

$$r_2 = \begin{cases} 2+u & \text{if } u \equiv 1 (\text{mod } 4), \\ 2+3u & \text{if } u \equiv 3 (\text{mod } 4), \end{cases} \quad r_4 = \begin{cases} 4+u & \text{if } u \equiv 3 (\text{mod } 4), \\ 4+3u & \text{if } u \equiv 1 (\text{mod } 4). \end{cases}$$

It can be directly checked that $r_d S_1(1) + u = S_1(d)$, so $H(\mathbb{Z}_n, S_1(1)) \cong_{\text{aff}} H(\mathbb{Z}_n, S_1(d))$ for $d \in \{2, 4\}$.

(iii): If u is odd, then $(2u+1)S_1(d) = S_2(d)$, hence $H(\mathbb{Z}_n, S_1(d)) \cong_{\text{aff}} H(\mathbb{Z}_n, S_2(d))$. Since u is odd whenever $d \in \{2, 4\}$, we are left with the case that $d = 1$ and u is even. If also $u \equiv 0 (\text{mod } 4)$, then $(u+1)S_1(1) + 3u = S_2(1)$, and again $H(\mathbb{Z}_n, S_1(1)) \cong_{\text{aff}} H(\mathbb{Z}_n, S_2(1))$.

Suppose that $d = 1$ and $u \equiv 2 (\text{mod } 4)$. We finish the proof by showing that in this case $H(\mathbb{Z}_n, S_1(1)) \not\cong_{\text{aff}} H(\mathbb{Z}_n, S_2(1))$. Suppose that, there is an affine transformation $\psi : x \mapsto rx + s$, $r \in \mathbb{Z}_n^*$ and $s \in \mathbb{Z}_n$, which maps the set $S_1(1)$ to $S_1(2)$. Then $1^{\psi} - (1+2u)^{\psi} = 2u$ in $\mathbb{Z}_n$. This implies that $\{1, 1+2u\}^{\psi} = \{1, 1+2u\}$ and $\{0, u\}^{\psi} = \{0, 3u\}$, and hence

$$r + s \in \{1, 1+2u\} \text{ and } r\{0, u\} + s = \{0, 3u\}.$$

A direct analysis shows that the above equations cannot hold if $u \equiv 2 (\text{mod } 4)$. Thus $H(\mathbb{Z}_n, S_1(1)) \not\cong_{\text{aff}} H(\mathbb{Z}_n, S_2(1))$, this completes the proof of (iii). □

Now, we turn to the case when $2u \neq m$. Recall that the canonical bicyclic group C is generated by the permutation c defined in (1). For a divisor $\ell \mid n$, C^{ℓ} will denote the

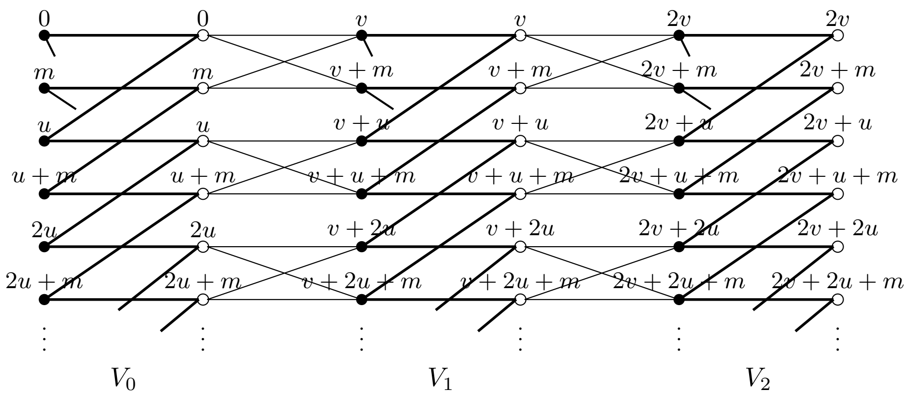

Figure 2: The Haar graph $H(\mathbb{Z}_n, S)$

subgroup of C generated by c^ℓ. It will be convenient to denote by δ_ℓ the partition of V into the orbits of C^ℓ, i.e., $\delta_\ell = \mathrm{Orb}(C^\ell, V)$. Furthermore, we set $\eta_{n,\ell}$ for the homomorphism $\eta_{n,\ell} : \mathbb{Z}_n \to \mathbb{Z}_\ell$ defined by $\eta_{n,\ell}(1) = 1$.

Observe that, if δ_ℓ is, in addition, a system of blocks for the group $A = \mathrm{Aut}(H(\mathbb{Z}_n, S))$, then an action of A can be defined on $H(\mathbb{Z}_\ell, \eta_{n,\ell}(S))$ by letting $g \in A$ act as for $x \in \mathbb{Z}_\ell$ and for $\varepsilon, \varepsilon' \in \{+, -\}$,

$$(x^\varepsilon)^g = y^{\varepsilon'} \iff \big\{z^\varepsilon : z \in \eta_{n,\ell}^{-1}(x)\big\}^g = \big\{z^{\varepsilon'} : z \in \eta_{n,\ell}^{-1}(y)\big\}. \tag{3.1}$$

We denote by $A_{(\delta_\ell)}$ the corresponding kernel, and by g^{δ_ℓ} the image of an element $g \in G$. Note that, if X is a bicyclic group of $H(\mathbb{Z}_n, S)$, then $X^{\delta_\ell} = \{x^{\delta_\ell} : x \in X\}$ is a bicyclic group of $H(\mathbb{Z}_\ell, \eta_{n,\ell}(S))$.

Let $S = \{0, u, v, v + m\}$ be the subset of $\mathbb{Z}_n$ defined in Theorem 3.1. Let δ be the partition of V defined by

$$\delta = \big\{X \cup X^{\psi_{1,0,0}} : X \in \mathrm{Orb}(C^u, V)\big\}, \tag{3.2}$$

where $\psi_{1,0,0}$ is defined in (2.2). We write $\delta = \{V_0, \ldots, V_{u-1}\}$, where

$$V_i = \big\{(iv + ju)^+, (iv + ju)^- : j \in \{0, 1, \ldots, (n/u) - 1\}\big\}.$$

A part of $H(\mathbb{Z}_n, S)$ is drawn in Figure 2 using the partition δ. White and black colors represent again the color classes $\mathbb{Z}_n^+$ and $\mathbb{Z}_n^-$, respectively. For $i \in \{0, 1, \ldots, u-1\}$, let e_i be the involution of V defined by

$$e_i : x^\varepsilon \mapsto \begin{cases} (x+m)^\varepsilon & \text{if } x^\varepsilon \in V_i, \\ x^\varepsilon & \text{otherwise.} \end{cases}$$

It is clear that each $e_i \in \mathrm{Aut}(H(\mathbb{Z}_n, S))$, and also that $e_i e_j = e_j e_i$ for all $i, j \in \{0, 1, \ldots, u-1\}$. Let $E = \langle e_0, e_1, \ldots, e_{u-1}\rangle$. Thus $E \le \mathrm{Aut}(H(\mathbb{Z}_n, S))$ and $E \cong \mathbb{Z}_2^u$. For a subset $I \subseteq \{0, 1, \ldots, u-1\}$ let e_I be the element in E defined by $e_I = \prod_{i \in I} e_i$.

The following lemma about imprimitivity systems of blocks (systems of blocks for short) will be used throughout the paper.

Lemma 3.3. *Let* $\Gamma = H(\mathbb{Z}_n, R)$ *be a Haar graph and suppose that* $R_* \subseteq R$ *such that the point stabilizer* $\mathrm{Aut}(\Gamma)_{0^+}$ *fixes setwise* R_*^-, *and let* $d = |\langle R_* - R_* \rangle|$, *where* $R_* - R_* = \{r_1 - r_2 : r_1, r_2 \in R_*\}$. *Then the partition* π *of* V *defined by*

$$\pi = \{X \cup X^{\psi_{1,r,-r}} : X \in \mathrm{Orb}(C^{n/d}, V)\}, \text{ where } r \in R_*,$$

is a system of blocks for $\mathrm{Aut}(\Gamma)$.[2]

Proof. For short we set $A = \mathrm{Aut}(\Gamma)$. Since R_*^- is fixed setwise by A_{0^+}, we may write

$$R_* = R_1 \cup \cdots \cup R_k,$$

where R_i^- is an A_{0^+}-orbit for every $i \in \{1, 2, \ldots, k\}$. Choose an arc $(0^+, r_i^-)$ of Γ where we fix an element $r_i \in R_i$ for every $i \in \{1, \ldots, k\}$. We claim that, the orbital graph of A containing $(0^+, r_i^-)$ is self-paired, and in fact it is equal to the Haar graph $H(\mathbb{Z}_n, R_i)$ (for a definition of an orbital graph, see [5]).

Define $\bar{A}$ as the color preserving subgroup of A. Then $A = \bar{A} \rtimes \langle \psi_{-1,0,0} \rangle$. Also, $\bar{A} = A_{0^+}C$, as C is transitive on $\mathbb{Z}_n^+$. Then the orbit of the arc $(0^+, r_i^-)$ under A is

$$\begin{aligned}
(0^+, r_i^-)^A &= (0^+, r_i^-)^{A_{0^+}C\langle\psi_{-1,0,0}\rangle} = \{(0^+, r_i'^-) : r_i' \in R_i\}^{C\langle\psi_{-1,0,0}\rangle} \\
&= \{(j^+, (j + r_i')^-) : r_i' \in R_i, j \in \mathbb{Z}_n\}^{\langle\psi_{-1,0,0}\rangle} \\
&= \{(j^+, (j + r_i')^-) : r_i' \in R_i, j \in \mathbb{Z}_n\} \cup \\
&\quad\ \{((-j)^-, (-j - r_i')^+) : r_i' \in R_i, j \in \mathbb{Z}_n\}, \\
&= \{(j^+, (j + r_i')^-) : r_i' \in R_i, j \in \mathbb{Z}_n\} \cup \\
&\quad\ \{((j + r_i')^-, j^+) : r_i' \in R_i, j \in \mathbb{Z}_n\},
\end{aligned}$$

which is clearly equal to the set of arcs of $H(\mathbb{Z}_n, R_i)$. The claim is proved.

Since $H(\mathbb{Z}_n, R_i)$ is an orbital graph, $A \le \mathrm{Aut}(H(\mathbb{Z}_n, R_i))$. Combining this with $H(\mathbb{Z}_n, R_*) = \cup_{i=1}^k H(\mathbb{Z}_n, R_i)$, we have that $A \le \mathrm{Aut}(H(\mathbb{Z}_n, R_*))$. Let Σ be the connected component of $H(\mathbb{Z}_n, R_*)$ which contains 0^+. Obviously, the set W of vertices contained in Σ is a block for A. It is easy to verify that $W = X \cup X^{\psi_{1,r,-r}}$ where X is the orbit of 0^+ under $C^{n/d}$. The lemma follows. □

Lemma 3.4. *Let* S *be the set defined in Theorem 3.1. If* $2u \neq m$, *then the stabilizer* $\mathrm{Aut}(H(\mathbb{Z}_n, S))_{0^+}$ *is given as follows.*

(1) If $u \not\equiv 2v (\mathrm{mod}\ m/u)$, *then* $\mathrm{Aut}(H(\mathbb{Z}_n, S))_{0^+} = E_{0^+}$.

(2) If $u \equiv 2v (\mathrm{mod}\ m/u)$, *then* $\mathrm{Aut}(H(\mathbb{Z}_n, S))_{0^+} = E_{0^+} \times F$ *for a subgroup* $F \le \mathrm{Aut}(H(\mathbb{Z}_n, S))_{0^+}$, $|F| = 2$.

Proof. For short we set $\Gamma = H(\mathbb{Z}_n, S)$ and $A = \mathrm{Aut}(\Gamma)$. Consider the partition δ defined in (3.2). Applying Lemma 3.3 with $R = S$, $R_* = \{0, u\}$ and $r = 0$, we obtain that δ is a system of blocks for A. The quotient graph Γ/δ is a u-circuit if $u > 2$ and a 2-path if $u = 2$. Let $g \in A_{0^+}$. Then g fixes the directed edge (V_0, V_1) of Γ/δ, hence it must fix all sets V_i. Thus $A_{0^+} \le A_{(\delta)}$, where $A_{(\delta)}$ is the kernel of the action of A on δ.

Consider the action of A_{0^+} on V_0. The corresponding kernel is $A_{(V_0)}$, the pointwise stabilizer of V_0 in A, and the corresponding image is a subgroup of $\mathrm{Aut}(\Gamma[V_0])$, where

[2] Notice that, π does not depend of the choice of the element $r \in R_*$.

$\Gamma[V_0]$ is the subgraph of Γ induced by V_0. Using that $2u \neq m$, we show next that $A_{(V_0)} = E_{0^+}$. It is clear that $A_{(V_0)} \geq E_{0^+}$. We are going to prove that $A_{(V_0)} \leq E_{0^+}$ also holds. Let $g \in A_{(V_0)}$. Then for a suitable element $e \in \langle e_1 \rangle$, the product ge fixes pointwise V_0 and fix the vertex v^- from block V_1 (see Figure 2). Thus ge acts on V_1 as the identity or the unique reflection of the circuit $\Gamma[V_1]$ that fixes v^-. If this action is not the identity, then ge switches v^+ and $(v+n-u)^+$, and so it must switch $(v+u)^-$ and $(v+n-u)^-$. On the other hand, since $(v+u)^-$ is connected to $u^+ \in V_0$, it follows that $(v+u)^-$ can only be mapped to $(v+u+m)^-$, and so $(v+n-u)^- = (v+u+m)^-$, contradicting that $2u \neq m$. We conclude that ge acts as the identity also on V_1. Continuing in this way, we find that ge' is the identity with a suitable choice of $e' \in E_{0^+}$, hence $g = e'$.

The equality $A_{(V_0)} = E_{0^+}$ together with $\mathrm{Aut}(\Gamma[V_0]) \cong D_{4u}$ imply that $|A_{0^+} : E_{0^+}| \leq 2$. Moreover, $|A_{0^+} : E_{0^+}| = 2$ holds exactly when A_{0^+} contains an involution g for which $g : 0^- \leftrightarrow u^-$. In the latter case $A_{0^+} = E_{0^+} \times \langle g \rangle$ as g centralizes E (to see this, observe that g is in the kernel $A_{(\delta)}$, and acts on every block V_i as an element of $D_{2n/u}$, whereas E acts on V_i as the center $Z(D_{2n/u})$.) We settle the lemma by proving the following equivalence :

$$A_{0^+} \cong E_{0^+} \times \mathbb{Z}_2 \iff u \equiv 2v (\mathrm{mod}\; m/u). \tag{3.3}$$

Suppose first that $A_{0^+} = E_{0^+} \times \langle g \rangle$, where $g \in A_{0^+}$ and $g : 0^- \leftrightarrow u^-$. By (c) of Theorem 3.1, $\{v^-, (v+m)^-\}^{A_{0^+}} = \{v^-, (v+m)^-\}$. Applying Lemma 3.3 with $R = S$, $R_* = \{v, v+m\}$ and $r = v$, we obtain that the set $B = \{0^+, m^+, v^-, (v+m)^-\}$ is a block for A. The induced graph $\Gamma[B]$ is a 4-circuit (again, see Figure 2). Denote by $A_{\{B\}}$ the setwise stabilizer of B in A, and by $A_{\{B\}}^B$ the permutation group of B induced by $A_{\{B\}}$. As $\Gamma[B]$ is a 4-circuit, $A_{\{B\}}^B \leq D_8$. This gives that $\{0^+, m^+\}$ is a block for $A_{\{B\}}^B$, and therefore it is also a block for A. We conclude that $\delta_m = \{X : X \in \mathrm{Orb}(C^m, V)\}$ is a system of blocks for A. Consider the action of A on $H(\mathbb{Z}_m, \eta_{n,m}(S))$ defined in (3.1). Then $E \leq A_{(\delta_m)}$, while $g \notin A_{(\delta_m)}$. This implies that g^{δ_m} is an automorphism of $H(\mathbb{Z}_m, \eta_{n,m}(S))$ which normalizes its canonical bicyclic group. This means that $g^{\delta_m} = \varphi_{r,s,t}$ for some $r \in \mathbb{Z}_m^*$ and $s, t \in \mathbb{Z}_m$. Using that $g^{\delta_m} : 0^+ \mapsto 0^+$ and $0^- \mapsto \eta_{n,m}(u)^-$, we find that $s = 0$ and $t = \eta_{n,m}(u)$, and so

$$A^{\delta_m} = \langle D^{\delta_m}, \varphi_{r,0,\eta_{n,m}(u)} \rangle. \tag{3.4}$$

Also, $g^{\delta_m} : \eta_{n,m}(u)^- \mapsto 0^-$ and $\eta_{n,m}(v)^- \mapsto \eta_{n,m}(v)^-$, hence $r\eta_{n,m}(u) = -\eta_{n,m}(u)$ and $r\eta_{n,m}(v) = \eta_{n,m}(v-u)$ hold in $\mathbb{Z}_m$. From these $r \equiv -1 (\mathrm{mod}\; m/u)$ and $rv \equiv v - u (\mathrm{mod}\; m/u)$, i.e., $u \equiv 2v (\mathrm{mod}\; m/u)$. The implication "⇒" in (3.3) is now proved.

Suppose next that $u \equiv 2v (\mathrm{mod}\; m/u)$. Define the permutation g of V by

$$g : (iv + ju)^\varepsilon \mapsto \begin{cases} \big(iv - (i+j)u\big)^+ & \text{if } \varepsilon = +, \\ \big(iv - (i+j-1)u\big)^- & \text{if } \varepsilon = -, \end{cases}$$

where $i \in \{0, 1, \ldots, u-1\}$ and $j \in \{0, 1, \ldots, n/u - 1\}$. We complete the proof by verifying that $g \in A_{0^+}$. Since ${0^+}^g = 0^+$ and $g : 0^- \leftrightarrow u^-$, this will imply that $A_{0^+} = E_{0^+} \times \langle g \rangle$. Thus part "⇐" of (3.3) is also proved.

Choose an arbitrary vertex $w \in \mathbb{Z}_n^+$ such that $w = (iv + ju)^+$, $i \in \{0, 1, \ldots, u-1\}$ and $j \in \{0, 1, \ldots, n/u - 1\}$, and suppose for the moment that $i < u - 1$. Then w has the following neighbors:

$$(iv + ju)^-, (iv + (j+1)u)^-, ((i+1)v + ju)^-, ((i+1)v + (j + m/u)u)^-,$$

where $v+1 \in \{0,1,\dots,u-1\}$, and $j+1$ and $j+m/u$ are from $\{0,1,\dots,n/u-1\}$. Thus these vertices are mapped by g to

$$(iv-(i+j-1)u)^-, (iv-(i+j)u)^-, ((i+1)v-(i+j)u)^-, ((i+1)v-(i+j+m/u)u)^-.$$

A direct check shows that these are just the neighbors of $w^g = (iv-(i+j)u)^+$. Let $i = u-1$. Then the neighbors of w are:

$$(iv+ju)^-, (iv+(j+1)u)^-, ((j+v)u)^-, ((j+v+m/u)u)^-,$$

where $j+v$ and $j+v+m/u$ are from $\{0,...,n/u-1\}$. Then these vertices are mapped by g to

$$(iv-(i+j-1)u)^-, (iv-(i+j)u)^-, (-(j+v-1)u)^-, (-(j+v+m/u-1)u)^-.$$

The first two are clearly connected with $w^g = (iv-(i+j)u)^+$; whereas the rest two are connected with w^g if and only if the following equality holds in $\mathbb{Z}_n$:

$$\{iv-(i+j)u+v, iv-(i+j)u+v+m\} = \{-(j+v-1)u, -(j+v+m/u-1)u\}.$$

Using that $v = u-1$, this reduces to $\{-(u-v)u, -(u-v)u+m\} = \{-vu, -vu+m\}$. Finally, observe that this equality holds if $(u-v)u \equiv vu (\mathrm{mod}\ m)$, and the latter congruence follows from the initial assumption that $u \equiv 2v (\mathrm{mod}\ m/u)$. □

Lemma 3.5. *Let S be the set defined in Theorem 3.1. If $2u \neq m$, then for the normalizer $N_{\mathrm{Aut}(H(\mathbb{Z}_n,S))}(C)$ of C in $\mathrm{Aut}(H(\mathbb{Z}_n,S))$,*

$$\big|\mathrm{Aut}(H(\mathbb{Z}_n,S)) : N_{\mathrm{Aut}(H(\mathbb{Z}_n,S))}(C)\big| = \begin{cases} 2^{u-2} & \textit{if } 2 \mid u \textit{ and } \big(u \not\equiv 2v (\mathrm{mod}\ m/u) \textit{ or} \\ & u/2 \equiv v (\mathrm{mod}\ m/u)\big), \\ 2^{u-1} & \textit{otherwise.} \end{cases} \tag{3.5}$$

Proof. For short we set $A = \mathrm{Aut}(H(\mathbb{Z}_n,S))$ and $N = N_A(C)$. Since $A = DA_{0^+}$ and $D \leq N$, $N = D(N \cap A_0^+)$. The cases (1) and (2) in Lemma 3.4 are considered separately.

CASE 1. $u \not\equiv 2v (\mathrm{mod}\ m/u)$.

In this case, from Lemma 3.4, $A_{0^+} = E_{0^+}$, hence $|A| = 2^u n$. Let $g \in N \cap A_0^+$. Since $g \in E_{0^+}$, it follows quickly that $g = 1_A$ or $2 \mid u$ and $g = e_1 e_3 \cdots e_{u-1}$. Combining this with $N = D(N \cap G_0^+)$ we find that $|N| = 4n$ if $2 \mid u$, and $|N| = 2n$ if $2 \nmid u$. Formula (3.5) follows.

CASE 2. $u \equiv 2v (\mathrm{mod}\ m/u)$.

From Lemma 3.4, $A_{0^+} = E_{0^+} \times F$ for a subgroup $F \leq A_{0^+}$, $|F| = 2$, hence $|A| = 2^{u+1}n$. It follows from the proof of Lemma 3.4 that, there exists $r \in \mathbb{Z}_m^*$ such that the following hold:

$$r\eta_{n,m}(u) = -\eta_{n,m}(u) \text{ and } r\eta_{n,m}(v) = \eta_{n,m}(v-u).$$

Let $s \in \mathbb{Z}_n^*$ such that $\eta_{n,m}(s) = r$. Then

$$su \in \{-u, -u+m\} \text{ and } sv \in \{v-u, v-u+m\}. \tag{3.6}$$

Suppose that $2 \nmid u$. Then we get as before that $N \cap E_{0^+}$ is trivial. Notice also that, $u \equiv 2v + m/u (\mathrm{mod}\ n/u)$, which follows from the assumption that $u \equiv 2v (\mathrm{mod}\ m/u)$ and that $2 \nmid u$. Thus $2 \nmid m$ and $2 \mid (u+m)$, implying that in (3.6) we have $su = -u$. We obtain that $\varphi_{s,u,0} \in N \cap (A_{0^+} \setminus E_{0^+})$, and so $|N \cap A_{0^+}| = 2$.

Suppose next that $2 \mid u$. Then $|N \cap E_{0^+}| = 2$. It is easily seen that $|N \cap A_{0^+}| = 4$ if and only if there exists $r \in \mathbb{Z}_n^*$ such that $ru = -u$ and $rv = v - u$ hold in $\mathbb{Z}_n$. Consider the following system of linear congruences:

$$xu \equiv u (\mathrm{mod}\ n),\ xv \equiv v - u (\mathrm{mod}\ n). \tag{3.7}$$

From the first congruence we can write x in the form $x = yn/u - 1$. Substitute this into the second congruence. We obtain that $yvn/u \equiv 2v - u (\mathrm{mod}\ n)$. This has a solution if and only if $\gcd(vn/u, n) \mid (2v - u)$. Suppose that $\gcd(v, n) \neq 1$. Using that $\langle u, v\rangle = \mathbb{Z}_n$ and that $2 \mid u$, we obtain that $\gcd(v, m/u) \neq 1$. However, then from the assumption that $u \equiv 2v (\mathrm{mod}\ m/u)$ it follows that also $\gcd(v, u) \neq 1$, which contradicts that $\langle u, v\rangle = \mathbb{Z}_n$. Hence $\gcd(v, n) = 1$, $\gcd(vn/u, n) = n/u$, and so (3.7) has a solution if and only if $u \equiv 2v (\mathrm{mod}\ n/u)$, or equivalently, $u/2 \equiv v (\mathrm{mod}\ m/u)$ (recall that $2 \mid u$ and $u \mid m$). It is not hard to show that any solution to (3.7) is necessarily prime to n, hence is in $\mathbb{Z}_n^*$. The above arguments can be summarized as follows: $|N| = 8n$ if $2 \mid u$ and $u/2 \equiv v (\mathrm{mod}\ m/u)$, and $|N| = 4n$ otherwise. This is consistent with (3.5). The lemma is proved. □

Lemma 3.6. *Let $r \in \mathbb{Z}_n^*$, $r \neq 1$ and $s \in \mathbb{Z}_n$ such that the permutation $\varphi_{r,0,s}$ is of order 2. Then the group $\langle D, \varphi_{r,0,s}\rangle$ contains a bicyclic subgroup different from C if and only if $8 \mid n$, $r = n/2 + 1$, and $s = 0$ or $s = n/2$.*

Proof. Suppose that $\langle D, \varphi_{r,0,s}\rangle$ contains a bicyclic subgroup X such that $X \neq C$. Then X is generated by a permutation in the form $c^i \varphi_{r,0,s}$. Since $\varphi_{r,0,s}^2 = id_V$, $r^2 = 1$ in $\mathbb{Z}_n$, and we calculate that $(c^i \varphi_{r,0,s})^2$ sends x^+ to $(x + r(r+1)i)^+$ for every $x \in \mathbb{Z}_n$. That $\mathbb{Z}_n^+$ is an orbit of X is equivalent to the condition that $\gcd(n, r+1) = 2$. Using this and that $r^2 - 1 = (r-1)(r+1) \equiv 0 (\mathrm{mod}\ n)$, we find that $n/2$ divides $r - 1$, so $r = 1$ or $r = n/2 + 1$. Since $r \neq 1$, we have that $r = n/2 + 1$ and $8 \mid n$. Then $(\varphi_{r,0,s})^2$ sends x^- to $(x + (n/2 + 2)s)^-$. Since $(\varphi_{r,0,s})^2 = id_V$, we obtain that $s = 0$ or $s = n/2$.

On the other hand, it can be directly checked that, if $8 \mid n$, $r = n/2 + 1$ and $s \in \{0, n/2\}$, then the permutation $c\varphi_{r,0,s}$ generates a bicyclic subgroup of $\langle D, \varphi_{r,0,s}\rangle$. Obviously, this bicyclic subgroup cannot be C. The lemma is proved. □

Everything is prepared to prove the main result of the section.

Proof of Theorem 3.1. The case that $2u = m$ is settled already in Lemma 3.2, hence let $2u \neq m$. We consider the action of $A = \mathrm{Aut}(H(\mathbb{Z}_n, S))$ on the system of blocks δ_m defined in (3.1). We claim that the corresponding image A^{δ_m} has a unique bicyclic subgroup (which is, of course, C^{δ_m}).

This is easy to see if $A_{0^+} = E_{0^+}$, because in this case $A^{\delta_m} = (DA_{0^+})^{\delta_m} = D^{\delta_m}$.

Let $A_{0^+} \neq E_{0^+}$. Then $A_{0^+} = E_{0^+} \times F$ for some subgroup F, $|F| = 2$. By (3.4), $A^{\delta_m} = \langle D^{\delta_m}, \varphi_{r,0,\eta_{n,m}(u)}\rangle$. Also, $r \equiv -1 (\mathrm{mod}\ m/u)$, hence $r \neq 1$ in $\mathbb{Z}_m$. By Lemma 3.5, A^{δ_m} contains more than one bicyclic subgroup if and only if $8 \mid m$, $r = m/2 + 1$ and

$\eta_{n,m}(u) \in \{0, m/2\}$. In the latter case $u \in \{m, m/2\}$, which is impossible as $u < m/2$. Hence A^{δ_m} contains indeed a unique bicyclic subgroup.

We calculate next the number of bicyclic groups of $H(\mathbb{Z}_n, S)$; we denote this number by $\mathbb{B}$. In fact, we are going to derive the following formula:

$$\mathbb{B} = \begin{cases} 2^{u-2} & \text{if } 2 \mid u \text{ and } 2 \nmid (m/u), \\ 2^{u-1} & \text{otherwise.} \end{cases} \tag{3.8}$$

Let $g \in G$ such that $\langle g \rangle$ is a bicyclic group of $H(\mathbb{Z}_n, S)$. Since $G = DA_{0+}$, g can be written as $g = xy$ with $x \in D$ and $y \in A_{0+}$. Since $\langle g \rangle$ is a bicyclic group, g fixes the color classes setwise, implying that $x \in C$. The image $\langle g \rangle^{\delta_m}$ is also a bicyclic subgroup of A^{δ_m}, hence by the previous paragraph, $\langle g \rangle^{\delta_m} = C^{\delta_m}$. Now, since $x \in C$, $y^{\delta_m} \in C^{\delta_m}$, from which $y^{\delta_m} = id_{\delta_m}$. We conclude that $x = c^i \in C$ for some $i \in \{1, \ldots, n-1\}$ with $\gcd(i, m) = 1$, and $y \in E_{0+}$, and so $y = e_I$ for a subset $I \subseteq \{1, \ldots, u-1\}$. Obviously, the product $\phi(n)\mathbb{B}$ calculates the number of elements $g \in G$ such that $\langle g \rangle$ is a bicyclic group of $H(\mathbb{Z}_n, S)$, where ϕ denotes the Euler's totient function. Therefore, $\phi(n)\mathbb{B}$ is equal to the number of elements in the form $c^i e_I$ that $i \in \{1, \ldots, n-1\}, \gcd(i, m) = 1$, $I \subseteq \{1, \ldots, u-1\}$, and $\langle c^i e_I \rangle$ is a bicyclic group of $H(\mathbb{Z}_n, S)$.

Let us pick $c^i e_I$ with $i \in \{1, \ldots, n-1\}, \gcd(i, m) = 1$, and $I \subseteq \{1, \ldots, u-1\}$. It is easily seen that $e_I c^i = c^i e_{I+i}$, where $I + i = \{x + i : x \in I\}$, here the addition is taken modulo u. Using this and induction on u, it follows that

$$(c^i e_I)^u = c^{ui} e_I e_{I+i} \cdots e_{I+(u-1)i}.$$

Since $\gcd(i, m) = 1$ and $u \mid m$, $\gcd(i, u) = 1$, from which

$$e_I e_{I+i} \cdots e_{I+(u-1)i} = (e_0 e_1 \cdots e_{u-1})^{|I|} = c^{m|I|}.$$

Thus $(c^i e_I)^u = c^{u(\,i + \frac{m}{u}|I|\,)}$. This and $\gcd(i, u) = 1$ show that $\langle c^i e_I \rangle$ is a semiregular group. Therefore, $\langle c^i e_I \rangle$ is a bicyclic group if and only if $c^i e_I$ is of order n, or equivalently,

$$\gcd\left(i + \frac{m}{u}|I|,\ \frac{2m}{u}\right) = 1. \tag{3.9}$$

Notice that, since $\gcd(i, m) = 1$, the greatest common divisor above is always equal to 1 or 2. Suppose at first that $2 \mid (m/u)$. Then $2 \mid m$ and i is odd. Hence (3.9) always holds. We obtain that the number of elements in A which generate a bicyclic group is $\phi(n)2^{u-1}$, and so $\mathbb{B} = 2^{u-1}$, as claimed in (3.8). Suppose next that $2 \nmid (m/u)$. Now, if $2 \mid u$, then $2 \mid m$, hence $2 \nmid i$, and so (3.9) holds if and only if $|I|$ is even. We deduce from this that $\mathbb{B} = 2^{u-2}$, as claimed in (3.8). Finally, if $2 \nmid u$, then $2 \nmid m$, and in this case (3.9) holds if and only if $\gcd(i, n) = 1$ and $|I|$ is even, or $\gcd(i, n) = 2$ and $|I|$ is odd. We calculate that $\mathbb{B} = 2^{u-1}$, and this completes the proof of (3.8).

Let Ξ be a bicyclic base of $H(\mathbb{Z}_n, S)$. By (3.5) and (3.8) we obtain that, $|\Xi| > 1$ if and only if

$$|A : N_A(C)| = 2^{u-2} \text{ and } \mathbb{B} = 2^{u-1}.$$

This happens exactly when

$$\big(\,2 \mid u \text{ and } (u \not\equiv 2v (\text{mod } m/u) \text{ or } u/2 \equiv v (\text{mod } m/u))\,\big) \text{ and } \big(2 \nmid u \text{ or } 2u \mid m\big).$$

After some simplification,

$$|\Xi| > 1 \iff 2 \mid u,\ 2u \mid m \text{ and } u/2 \not\equiv v + m/(2u) (\operatorname{mod} m/u).$$

Suppose that $|\Xi| > 1$. Then A contains exactly 2^{n-1} bicyclic subgroups, 2^{n-2} of which are conjugate to C. These 2^{n-1} subgroups are enumerated as: $\langle ce_I \rangle,\ I \subseteq \{1, \ldots, u-1\}$. For $i \in \{1, \ldots, u-2\}$, $e_i c e_i = ce_{\{i,i+1\}}$. We can conclude that the bicyclic subgroups split into two conjugacy classes:

$$\big\{\langle ce_I \rangle : I \subseteq \{1, \ldots, u-1\}, |I| \text{ is even } \big\} \text{ and } \big\{\langle ce_I \rangle : I \subseteq \{1, \ldots, u-1\}, |I| \text{ is odd } \big\}.$$

In particular, $|\Xi| = 2$. Choose ξ from $\operatorname{Sym}(V)$ which satisfies

$$\xi c \xi^{-1} = ce_1 \text{ and } \xi : 0^+ \mapsto 0^+,\ 0^- \mapsto 0^-.$$

Then Ξ can be chosen as $\Xi = \{id_V, \xi\}$. Also, $\{v^-, (v+m)^-\}^\xi = \{v^-, (v+m)^-\}$, and since $(ce_1)^{u+m} = c^u$, $(u^-)^\xi = (0^-)^{(ce_1)^{u+m}\xi} = (0^-)^{\xi c^{u+m}} = (u+m)^-$. Thus $H(\mathbb{Z}_n, S)^\xi = H(\mathbb{Z}_n, \{0, u+m, v, v+m\})$. The theorem follows from Theorem 2.6. □

4 Proof of Theorem 1.1

Theorem 1.1 follows from Theorem 3.1 and the following theorem.

Theorem 4.1. *Let $H(\mathbb{Z}_n, S)$ be a connected Haar graph such that $|S| = 4$ and S is not a BCI-subset. Then $n = 2m$, and there exist $a \in \mathbb{Z}_n^*$ and $b \in \mathbb{Z}_n$ such that $aS + b = \{0, u, v, v+m\}$ and the conditions (a)-(c) in Theorem 3.1 hold.*

Before we prove Theorem 4.1 it is necessary to give three preparatory lemmas. For an element $i \in \mathbb{Z}_n$, we denote by $|i|$ the order of i viewed as an element of the additive group $\mathbb{Z}_n$. Thus we have $|i| = n/\gcd(n, i)$.

Lemma 4.2. *If $R = \{i, n-i, j\}$ is a generating subset of $\mathbb{Z}_n$ with $|i|$ odd, then R is a CI-subset.*

Proof. For short we set $A = \operatorname{Aut}(\operatorname{Cay}(\mathbb{Z}_n, R))$ and denote by A_0 the stabilizer of $0 \in \mathbb{Z}_n$ in A. Clearly, A_0 leaves R setwise fixed. If A_0 acts on R trivially, then $A \cong \mathbb{Z}_n$, and the lemma follows by Theorem 2.1. If A_0 acts on R transitively, then $\operatorname{Cay}(\mathbb{Z}_n, R)$ is edge-transitive. This condition forces that R is a CI-subset (see [10, page 320]).

We are left with the case that R consists of two orbits under A_0. These orbits must be $\{i, n-i\}$ and $\{j\}$. It is clear that A_0 leaves the subgroups $\langle i \rangle$ and $\langle j \rangle$ fixed; moreover, the latter set is fixed pointwise, and since $|i|$ is odd, $\langle i \rangle$ consists of $(|i|-1)/2$ orbits under A_0, each of length 2, and one orbit of length 1. We conclude that $\mathbb{Z}_n = \langle i \rangle \times \langle j \rangle$, and also that A is permutation isomorphic to the permutation direct product $\big((\mathbb{Z}_{|i|})_{\text{right}} \rtimes \langle \pi \rangle\big) \times (\mathbb{Z}_{|j|})_{right}$. For $\ell \in \{|i|, |j|\}$, $(\mathbb{Z}_\ell)_{\text{right}}$ is generated by the affine transformation $x \mapsto x+1$, and π is the affine transformation $x \mapsto -x$. We leave for the reader to verify that the above group has a unique regular cyclic subgroup. The lemma follows by Theorem 2.1. □

Lemma 4.3. *Let $n = 2m$ and $R = \{i, n-i, j, j+m\}$ be a subset of $\mathbb{Z}_n$ such that*

(a) R generates $\mathbb{Z}_n$;

(b) $|i|$ is odd;

(c) the stabilizer $\mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_n, R))_0$ *leaves the set* $\{i, n-i\}$ *setwise fixed.*

Then R *is a CI-subset.*

Proof. For short we set $A = \mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_n, R))$. Let T be a subset of $\mathbb{Z}_n$ such that $\mathrm{Cay}(\mathbb{Z}_n, R) \cong \mathrm{Cay}(\mathbb{Z}_n, T)$ and let f be an isomorphism from $\mathrm{Cay}(\mathbb{Z}_n, R)$ to $\mathrm{Cay}(\mathbb{Z}_n, T)$ such that $f(0) = 0$. Let us consider the subgraphs

$$\Gamma_1 = \mathrm{Cay}(\mathbb{Z}_n, \{i, n-i\}) \text{ and } \Gamma_2 = \mathrm{Cay}(\mathbb{Z}_n, \{j, j+m\}).$$

By condition (c), the group A preserves both of these subgraphs, that is, $A \le \mathrm{Aut}(\Gamma_\ell)$ for $\ell \in \{1, 2\}$. As f is an isomorphism between two Cayley graphs, $f(\mathbb{Z}_n)_{right} f^{-1} \le A$. Then $f(\mathbb{Z}_n)_{right} f^{-1} \le A \le \mathrm{Aut}(\Gamma_\ell)$, implying that f maps Γ_ℓ to a Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, T_\ell)$ for both $\ell \in \{1, 2\}$. Clearly, $T = T_1 \cup T_2$. It was proved by Sun [15] (see also [10]) that every subset of $\mathbb{Z}_n$ of size 2 is a CI-set. Using this, it follows from $\mathrm{Cay}(\mathbb{Z}_n, \{i, n-i\}) \cong \mathrm{Cay}(\mathbb{Z}_n, T_1)$ that $T_1 = a\{i, n-i\}$ for some $a \in \mathbb{Z}_n^*$. Letting $t_1 = ai$, we have $T_1 = \{t_1, n-t_1\}$ such that $|i| = |t_1|$. In the same way, $T_2 = a'\{j, j+m\}$ for some $a' \in \mathbb{Z}_n^*$, and letting $t_2 = a'j$, we have $T_2 = \{t_2, t_2+m\}$ with $|t_2| = |j|$. Since $f(0) = 0$, f maps $\{i, n-i\}$ to $T_1 = \{t_1, n-t_1\}$ and $\{j, j+m\}$ to $T_2 = \{t_2, t_2+m\}$.

We claim that the partition of $\mathbb{Z}_n$ into the cosets of $\langle m \rangle$ is a system of blocks for $\mathrm{Aut}(\Gamma_2)$, hence also for the group $A \le \mathrm{Aut}(\Gamma_2)$. Let us put $\bar{A} = \mathrm{Aut}(\Gamma_2)$. Then $\bar{A}_0$ leaves the set $T = \{j, j+m\}$ setwise fixed. Thus the setwise stabilizer $\bar{A}_{\{T\}}$ of the set T in $\bar{A}$ can be written as $\bar{A}_{\{T\}} = \bar{A}_{\{T\}} \cap \bar{A} = \bar{A}_{\{T\}} \cap \bar{A}_0(\mathbb{Z}_n)_{\mathrm{right}} = \bar{A}_0(\bar{A}_{\{T\}} \cap (\mathbb{Z}_n)_{\mathrm{right}}) = \bar{A}_0\langle m_{\mathrm{right}} \rangle$. Here $(\mathbb{Z}_n)_{\mathrm{right}}$ is generated by the affine transformation $x \mapsto x+1$, and m_{right} is the permutation $x \mapsto x+m$ for every $x \in \mathbb{Z}_n$. Thus $\bar{A}_0\langle m_{\mathrm{right}} \rangle$ is a subgroup of $\bar{A}$ which clearly contains $\bar{A}_0$. By [5, Theorem 1.5A], the orbit of 0 under the group $\bar{A}_0\langle m_{\mathrm{right}} \rangle$ is a block for $\bar{A}$. Now, the required statement follows as the latter orbit is equal to $0^{\bar{A}_0\langle m_{\mathrm{right}} \rangle} = 0^{\langle m_{\mathrm{right}} \rangle} = \langle m \rangle$.

Since the partition of $\mathbb{Z}_n$ into the cosets of $\langle m \rangle$ is a system of blocks for A, the isomorphism f induces an isomorphism from $\mathrm{Cay}(\mathbb{Z}_m, \eta_{n,m}(R))$ to $\mathrm{Cay}(\mathbb{Z}_m, \eta_{n,m}(T))$, we denote this isomorphism by $\bar{f}$. Note that, $\bar{f}(0) = 0$ for the identity element $0 \in \mathbb{Z}_m$.

The set $\eta_{n,m}(R)$ satisfies the conditions (a)-(c) of Lemma 4.2, hence it is a CI-subset. This means that $\bar{f}$ is equal to a permutation $x \mapsto rx$ for some $r \in \mathbb{Z}_m^*$. Let $s \in \mathbb{Z}_n^*$ such that $\eta_{n,m}(s) = r$. Then $\eta_{n,m}(si) = \eta_{n,m}(s)\eta_{n,m}(i) = \eta_{n,m}(t_1)$, and so the following holds in $\mathbb{Z}_n$:

$$si = t_1 \text{ or } si = t_1 + m. \tag{4.1}$$

The order $|t_1| = |i|$ is odd by (b), implying that $|t_1| \ne |t_1 + m|$, and so $si = t_1$ holds in (4.1). We conclude that $sR = T$, so R is a CI-subset. The lemma is proved. □

Lemma 4.4. *Let* $n = 2m$ *and* $S = \{0, u, v, v+m\}$ *such that*

(a) S *generates* $\mathbb{Z}_n$;

(b) $1 < u < n$, $u \mid n$ *but* $u \nmid m$;

(c) $\mathrm{Aut}(H(\mathbb{Z}_n, S))_{0^+}$ *leaves the set* $\{0^-, u^-\}$ *setwise fixed.*

Then S *is a BCI-subset.*

Proof. Let δ be the partition of V defined in (3.2). Applying Lemma 3.3 with $R = S$, $R_* = \{0, u\}$ and $r = 0$, we obtain that δ is a system of blocks for $A = \mathrm{Aut}(H(\mathbb{Z}_n, S))$. Thus the stabilizer A_{0^+} leaves the set V_0 setwise fixed, and we may consider the action of A_{0^+} on V_0. The subgraph of $H(\mathbb{Z}_n, S)$ induced by the set V_0 is a circuit of length $2n/u$, thus A_{0^+} fixes also the vertex on this circuit antipodal to 0^+. We find that this antipodal vertex is $(u/2 + m)^-$. Therefore, $A_{0^+} = A_{(m+u/2)^-}$, and thus S is a BCI-subset if and only if $S - u/2 + m$ is a CI-subset of $\mathbb{Z}_n$, see Proposition 2.8. The latter set is

$$S - u/2 + m = \{\, u/2 + m, -u/2 + m, v - u/2, v - u/2 + m \,\}.$$

Since $u \nmid m$, u is even and the order $|u/2 + m|$ is odd. Lemma 4.3 is applicable to the set $S - u/2 + m$ (choose $i = u/2 + m$ and $j = v - u/2$), it gives us that $S - u/2 + m$ is a CI-subset. This completes the proof. □

Proof of Theorem 4.1. Let S be the subset of $\mathbb{Z}_n$ given in Theorem 4.1. We deal first with the case when the canonical bicyclic group C is normal in $A = \mathrm{Aut}(H(\mathbb{Z}_n, S))$.

CASE 1. $C \trianglelefteq A$.

By Theorem 2.6, there is a bicyclic group X of $H(\mathbb{Z}_n, S)$ such that $X \neq C$. Since $C \trianglelefteq A$, X is generated by a permutation in the form $c^i\varphi_{r,0,s}$, $r \in \mathbb{Z}_n^*$, $s \in \mathbb{Z}_n$, and $\mathrm{ord}(\varphi_{r,0,s}) \geq 2$. The permutation $\varphi_{r,0,s}$ acts on both $\mathbb{Z}_n^+$ and $\mathbb{Z}_n^-$ as an affine transformation. This fact together with the connectedness of $H(\mathbb{Z}_n, S)$ imply that, $\varphi_{r,0,s}$ acts faithfully on S^-. Thus $\mathrm{ord}(\varphi_{r,0,s}) \leq 4$.

Suppose that $\mathrm{ord}(\varphi_{r,0,s}) = 4$. We may assume without loss of generality that S^- can be obtained as $S^- = \{\, (0^-)^{\varphi^j_{r,0,s}} : j \in \{0, 1, 2, 3\} \,\}$, and so $S = \{0, s, (r + 1)s, (r^2 + r + 1)s\}$ and $(r^3 + r^2 + r + 1)s = 0$. Since $H(\mathbb{Z}_n, S)$ is connected, $\gcd(s, n) = 1$, and $(r + 1)(r^2 + 1) = 0$. We find that $(c^i\varphi_{r,0,s})^4$ sends x^+ to $(x + r(r + 1)(r^2 + 1)i)^+ = x^+$. Since $X = \langle c^i\varphi_{r,0,s} \rangle$ is bicyclic, $n = 4$, and so $H(\mathbb{Z}_n, S) \cong K_{4,4}$. This, however, contradicts that $C \trianglelefteq A$.

Now, suppose that $\mathrm{ord}(\varphi_{r,0,s}) = 3$. If A_{0^+} is transitive on S^-, then it must be regular [9, Theorem 4.3]. This implies that S^- splits into two orbits under A_{0^+} with length 1 and 3, respectively. Let $s \in S$ such that $\{s^-\}$ is an orbit under A_{0^+}. Then $A_{0^+} = A_{s^-}$, and by Proposition 2.8, $S - s$ is not a CI-subset of $\mathbb{Z}_n$. However, in this case the graph $\mathrm{Cay}(\mathbb{Z}_n, S - s)$ is edge-transitive, and thus $S - a$ is a CI-subset (see [10, page 320]), which is a contradiction.

Finally, suppose that $\mathrm{ord}(\varphi_{r,0,s}) = 2$. If $r = 1$, then $2 \mid n$ and $s = m$, where $n = 2m$. This implies that S^- is a union of two orbits of C^m, we may write $S = \{0, m, s, s + m\}$. The graph $H(\mathbb{Z}_n, S)$ is then isomorphic to the lexicographical product $C_n[K_2^c]$ of an n-circuit C_n with the graph K_2^c, see Figure 3. It is easily seen that then A_{0^+} is not faithful on the set S^-, which is a contradiction.

Let $r \neq 1$. By Lemma 3.6, $8 \mid n$, $r = m + 1$ and $s \in \{0, m\}$, where $n = 2m$. We consider only the case when $s = 0$ (the case when $s = m$ can be treated in the same manner). Then $\mathbb{Z}_n^-$ splits into the following orbits under $\varphi_{r,0,s}$:

$$\{(2i)^-\},\ \{(2i + 1)^-, (2i + 1 + m)^-\},\ \text{where } i \in \{0, 1, \ldots, m - 1\}.$$

Since $H(\mathbb{Z}_n, S)$ is connected and cannot be the union C^m-orbits (see above), S^- contains one orbit under $\varphi_{r,0,s}$ of length 2, and two orbits of length 1. Let S_1 denote the orbit of

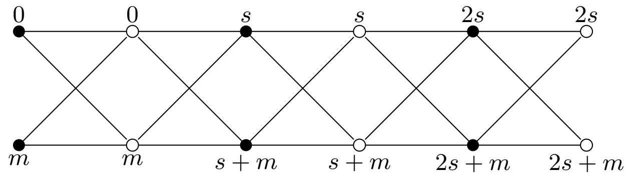

Figure 3: The lexicographical product $C_n[K_2^c]$.

length 2 and let $S_2 = S \setminus S_1$. Then we may write $S_1 = \{s, s+m\}$, and $S_2 = \{s', s''\}$, where both s' and s'' are even. Let $u = \gcd(s' - s'', n)$. Then u is a divisor of n and also $2 \mid u$. There exist $a \in \mathbb{Z}_n^*$ such that $a(s' - s'') \equiv u (\bmod\, n)$. Choosing $b = -as''$ (all arithmetic is done in $\mathbb{Z}_n$), we find that $aS_2 + b = \{u, 0\}$. Now, letting $v = as + b$, we get $aS_1 + b = \{v, v+m\}$. We finish the proof of this case by showing that the set $R = aS + b = \{0, u, v, v+m\}$ satisfies the conditions (a)-(c) of Theorem 3.1.

(a): As $H(\mathbb{Z}_n, S)$ is connected, $H(\mathbb{Z}_n, R)$ is also connected. This implies that $\{u, v\}$ is a generating set of $\mathbb{Z}_n$.

(c): Since $C \trianglelefteq A$, $C \trianglelefteq \mathrm{Aut}(H(\mathbb{Z}_n, R))$. To the contrary assume that the stabilizer $\mathrm{Aut}(H(\mathbb{Z}_n, R))_{0^+}$ does not leave $\{0^-, u^-\}$ setwise fixed. Thus there exists some $g \in A_{0^+}$ which maps v^- into $\{0^-, u^-\}$. Letting $w_1^- = (v^-)^g$ and $w_2^- = ((v+m)^-)^g$, we find that $w_1 - w_2 = m$, and from this that $u = m$. However, then $H(\mathbb{Z}_n, R) \cong C_n[K_2^c]$, which we have already excluded above. Thus $\mathrm{Aut}(H(\mathbb{Z}_n, R))_{0^+}$ fixes setwise $\{0^-, u^-\}$.

(b): We have already showed (see previous paragraph) that $u \neq m$ and $1 < u$. Since S is not a BCI-subset, R is also a not a BCI-subset. This also implies that $u \mid m$ by Lemma 4.4, and we conclude that $1 < u < m$ and $u \mid m$, as required.

CASE 2. $C \ntrianglelefteq A$.

Let A_{0^+} act transitively on S^-. This gives that $H(\mathbb{Z}_n, S)$ is edge-transitive. Since $C \ntrianglelefteq A$, $D \ntrianglelefteq A$, in other words, $H(\mathbb{Z}_n, S)$ is non-normal as a Cayley graph over the dihedral group D. We apply [8, Theorem 1.2], and obtain that $H(\mathbb{Z}_n, S)$ is either isomorphic to $K_n[K_2^c]$, or to one of 5 graphs of orders $10, 14, 26, 28$ and 30, respectively. Suppose that the former case holds. Then $n = 2m$, and we obtain quickly that S consists of two C^m-orbits. Then S can be mapped by an affine transformation to a set $\{0, m, v, v+m\}$, where $\langle m, v \rangle \cong \mathbb{Z}_n$. Then v or $v+m$ is a generating element of $\mathbb{Z}_n$, and so S can actually be mapped by an affine transformation to $\{0, m, 1, 1+m\}$. Now, the same holds for any set T with $H(\mathbb{Z}_n, T) \cong H(\mathbb{Z}_n, S) \cong K_n[K_2^c]$, contradicting that S is not a BCI-subset. In the latter case, a direct computation by the computer package MAGMA [4] shows that none of these graphs is possible (in fact, in each case the corresponding subset S is a BCI-subset).

The set S^- cannot split into two orbits under A_{0^+} of size 1 and 3, respectively (see the argument above). We are left with the case that $S = S_1 \cup S_2$, $|S_1| = |S_2|$, and A_{0^+} leaves both sets S_1 and S_2 setwise fixed. For $i \in \{1, 2\}$, let $n_i = |\langle S_i - S_i \rangle|$, $n_1 \leq n_2$, where $S_i - S_i = \{a - b \,:\, a, b \in S_i\}$.

We claim that $n_1 = 2$. To the contrary assume that $n_1 > 2$. We prove first that $C^{n/n_1} \trianglelefteq A$. Applying Lemma 3.3 with $R = S$, $R_* = S_1$ and $r = s_1 \in S_1$, we obtain that the partition

$$\delta = \{X \cup X^{\psi_{1,s_1,-s_1}} \,:\, X \in \mathrm{Orb}(C^{n/n_1}, V)\},$$

is a system of blocks for A. Let us consider the action of $A_{(\delta)}$ (the kernel of A acting on δ) on the block of δ which contains 0^+. Denote this block by Δ, and by Δ' the block which contains s^- for some $s \in S_2$. Notice that, the subgraph of $H(\mathbb{Z}_n, S)$ induced by any block of δ is a circuit of length $2n_1$, and when deleting these circuits, the rest splits into pairwise disjoint circuits of length $2n_2$. Let Σ denote the unique $(2n_2)$-circuit through s^-. Now, suppose that $g \in A_{(\delta)}$ which fixes Δ pointwise. If $V(\Sigma) \cap \Delta = \{0^+\}$, then g must fix the edge $\{0^+, s^-\}$, and so fixes also s^-. If $V(\Sigma) \cap \Delta \neq \{0^+\}$, then $|V(\Sigma) \cap \Delta| = n_2 > 2$. This implies that g fixes every vertex on Σ, in particular, also s^-. The block Δ' has at least n_1 vertices having a neighbor in Δ, hence by the previous argument we find that all are fixed by g. Since $n_1 > 2$, Δ' is fixed pointwise by g. It follows, using the connectedness of $H(\mathbb{Z}_n, S)$, that $g = id_V$, hence that $A_{(\delta)}$ is faithful on Δ. Thus C^{n/n_1} is a characteristic subgroup of $A_{(\delta)}$, and since $A_{(\delta)} \trianglelefteq A$, $C^{n/n_1} \trianglelefteq A$.

Let G be the unique normal subgroup of A that fixes the color classes $\mathbb{Z}_n^+$ and $\mathbb{Z}_n^-$. We consider $N = G \cap C_A(C^{n/n_1})$. Then $C \leq N$ and $N \trianglelefteq A$. Pick $g \in N_{0^+}$ such that g acts non-trivially on S^-. Since N centralizes C^{n/n_1}, g fixes pointwise the orbit of 0^+ under C^{n/n_1}, and hence also Δ. Then g^2 fixes S^- pointwise, and so also Δ'. We conclude that $g^2 = id_V$, and that either $N = C$, or $N = C \rtimes \langle g \rangle$. The case $N = C$ is impossible because $C \ntrianglelefteq A$. Let $N = C \rtimes \langle g \rangle$. Then $(S_i^-)^g = S_i^-$ (for both $i \in \{1, 2\}$), hence S_i is a union of orbits of g. As g normalizes C and fixes 0^+, $g = \varphi_{r,0,s}$. Recall that $\text{ord}(g) = 2$. If $r \neq 1$, then by Lemma 3.6, either C is the unique cyclic subgroup of N, or $8 \mid n$, $r = n/2 + 1$ and $s = 0$ or $s = n/2$. In the former case C is characteristic in N, and since $N \trianglelefteq A$, $C \trianglelefteq A$, a contradiction. Therefore, we are left with the case that $r = 1$ (and so $s = n/2$), or $8 \mid n$, $r = n/2 + 1$ and $s = 0$ or $s = n/2$. Then every orbit of g is of length 1 or 2, and if it is of length 2, then is in the form $\{j^\varepsilon, (j + m)^\varepsilon\}$ as we proved in Case 1. Since $n_i > 2$, we see that S_i^- must be fixed pointwise by g for both $i \in \{1, 2\}$. This, however, contradicts that g was assumed to act non-trivially on S^-; and so $n_1 = 2$.

This means that $2 \mid n$, say $n = 2m$, and the group generated by the set $S_1 - S_1 = \{x - y : x, y \in S_1\}$ is equal to $\{0, m\}$. Then we can write $S_1 = \{s, s + m\}$. It can be proved as before that there exist $a \in \mathbb{Z}_n^*$ and $b \in \mathbb{Z}_n$ such that $aS_2 + b = \{0, u\}$ for some divisor u of n. Then, letting $v = as + b$, we get $aS_1 + b = \{v, v + m\}$. We finish the proof of this case by showing that the set $R = aS + b = \{0, u, v, v + m\}$ satisfies the conditions (a)-(c) of Theorem 3.1.

(a): As $H(\mathbb{Z}_n, S)$ is connected, $H(\mathbb{Z}_n, R)$ is also connected. This implies that $\{u, v\}$ is a generating set of $\mathbb{Z}_n$.

(c): Since S_1 and S_2 are left fixed setwise by A, $\text{Aut}(H(\mathbb{Z}_n, R))_{0^+}$ leaves the set $\{0^-, u^-\}$ setwise fixed.

(b): If $u = 1$, then $\text{Aut}(H(\mathbb{Z}_n, \{0, u\})) \leq D_{4n}$. But then $C \trianglelefteq A$, which is a contradiction. We conclude that $1 < u$, and by Lemma 4.4, $u \mid m$ also holds, i.e., $1 < u < m$ and $u \mid m$, as required. □

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Small cycles in the Pancake graph

Elena Konstantinova *
Sobolev Institute of Mathematics, Novosibirsk, 630090, Russia
Department of Mathematics, Yeungnam University, 712-749, South Korea

Alexey Medvedev
Sobolev Institute of Mathematics, Novosibirsk, 630090, Russia
Central European University, Budapest, 1051, Hungary

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Abstract

The Pancake graph is well known because of the open Pancake problem. It has the structure that any l–cycle, $6 \leqslant l \leqslant n!$, can be embedded in the Pancake graph $P_n, n \geqslant 3$. Recently it was shown that there are exactly $\frac{n!}{6}$ independent 6–cycles and $n!(n-3)$ distinct 7–cycles in the graph. In this paper we characterize all distinct 8–cycles by giving their canonical forms as products of generating elements. It is shown that there are exactly $\frac{n!(n^3+12n^2-103n+176)}{16}$ distinct 8–cycles in $P_n, n \geqslant 4$. A maximal set of independent 8–cycles contains $\frac{n!}{8}$ of these.

Keywords: Cayley graphs, Pancake graph, cycle embedding, small cycles.

Math. Subj. Class.: 05C15, 05C25, 05C38, 90B10

1 Introduction

The Pancake graph $P_n = (Sym_n, PR), n \geqslant 2$, is the Cayley graph on the symmetric group Sym_n of permutations $\pi = [\pi_1\pi_2 \dots \pi_n]$, where $\pi_i = \pi(i)$ for any $1 \leqslant i \leqslant n$, with the generating set $PR = \{r_i \in Sym_n : 2 \leqslant i \leqslant n\}$ of all prefix–reversals r_i reversing the order of any substring $[1, i], 2 \leqslant i \leqslant n$, of a permutation π when multiplied on the right, i.e. $[\pi_1 \dots \pi_i\pi_{i+1} \dots \pi_n]r_i = [\pi_i \dots \pi_1\pi_{i+1} \dots \pi_n]$. It is a connected vertex–transitive $(n-1)$-regular graph of order $n!$. This graph is well known because of the combinatorial *Pancake problem* which was posed in [4] as the problem of finding its diameter. The problem is still

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E-mail addresses: e_konsta@math.nsc.ru (Elena Konstantinova), an_medvedev@yahoo.com (Alexey Medvedev)

open. Some upper and lower bounds [5, 6] as well as exact values for $2 \leqslant n \leqslant 19$ [1, 2] are known. One of the main difficulties in solving this problem is the complicated cycle structure of the Pancake graph. This graph is hamiltonian [11] and the following result is also known.

Theorem 1.1. *[7, 10] All cycles of length l, where $6 \leqslant l \leqslant n!$, can be embedded in the Pancake graph $P_n, n \geqslant 3$, but there are no cycles of length $3, 4$ or 5.*

An explicit description of cycles is gradually being developed. The first results concerning cycle characterization in the Pancake graph were obtained in [8] where the following cycle representation via a product of generating elements was used. A sequence of prefix–reversals $C_l = r_{i_0} \ldots r_{i_{l-1}}$, where $2 \leqslant i_j \leqslant n$, and $i_j \neq i_{j+1}$ for any $0 \leqslant j \leqslant l-1$, such that $\pi r_{i_0} \ldots r_{i_{l-1}} = \pi$, where $\pi \in Sym_n$, is called *a form of a cycle C_l of length l*. A cycle C_l of length l is also called an l–cycle, and a vertex of P_n is identified with the permutation which corresponds to this vertex. It is evident that any l–cycle can be presented by $2l$ forms (not necessarily different) with respect to a vertex and a direction. *The canonical form C_l of an l–cycle* is called a form with a lexicographically maximal sequence of indices $i_0 \ldots i_{l-1}$. By using this description, the results characterizing 6– and 7–cycles were obtained in [8].

Theorem 1.2. *[8] The Pancake graph $P_n, n \geqslant 3$, has $\frac{n!}{6}$ independent 6–cycles of the canonical form $C_6 = r_3 r_2 r_3 r_2 r_3 r_2$ and $n!(n-3)$ distinct 7–cycles of the canonical form $C_7 = r_k r_{k-1} r_k r_{k-1} r_{k-2} r_k r_2$, where $4 \leqslant k \leqslant n$. Each of the vertices of P_n belongs to exactly one 6–cycle and $7(n-3)$ distinct 7–cycles.*

The main result of this paper is the following theorem.

Theorem 1.3. *Each of vertices of $P_n, n \geqslant 4$, belongs to $N = \frac{n^3+12n^2-103n+176}{2}$ distinct 8–cycles of the following canonical forms:*

$$C_8^1 = r_k\, r_j\, r_i\, r_j\, r_k\, r_{k-j+i}\, r_i\, r_{k-j+i}, \qquad 2 \leqslant i < j \leqslant k-1,\ 4 \leqslant k \leqslant n, \tag{1.1}$$

$$C_8^2 = r_k\, r_{k-1}\, r_2\, r_{k-1}\, r_k\, r_2\, r_3\, r_2, \qquad 4 \leqslant k \leqslant n, \tag{1.2}$$

$$C_8^3 = r_k\, r_{k-i}\, r_{k-1}\, r_i\, r_k\, r_{k-i}\, r_{k-1}\, r_i, \qquad 2 \leqslant i \leqslant k-2,\ 4 \leqslant k \leqslant n, \tag{1.3}$$

$$C_8^4 = r_k\, r_{k-i+1}\, r_k\, r_i\, r_k\, r_{k-i}\, r_{k-1}\, r_{i-1}, \qquad 3 \leqslant i \leqslant k-2,\ 5 \leqslant k \leqslant n, \tag{1.4}$$

$$C_8^5 = r_k\, r_{k-1}\, r_{i-1}\, r_k\, r_{k-i+1}\, r_{k-i}\, r_k\, r_i, \qquad 3 \leqslant i \leqslant k-2,\ 5 \leqslant k \leqslant n, \tag{1.5}$$

$$C_8^6 = r_k\, r_{k-1}\, r_k\, r_{k-i}\, r_{k-i-1}\, r_k\, r_i\, r_{i+1}, \qquad 2 \leqslant i \leqslant k-3,\ 5 \leqslant k \leqslant n, \tag{1.6}$$

$$C_8^7 = r_k\, r_{k-j+1}\, r_k\, r_i\, r_k\, r_{k-j+1}\, r_k\, r_i, \qquad 2 \leqslant i < j \leqslant k-1,\ 4 \leqslant k \leqslant n, \tag{1.7}$$

$$C_8^8 = r_4\, r_3\, r_4\, r_3\, r_4\, r_3\, r_4\, r_3. \tag{1.8}$$

There are two corollaries of Theorem 1.3, which will be proven in the final section of this paper.

Corollary 1.4. *There are $\frac{n!(n^3+12n^2-103n+176)}{16}$ distinct 8–cycles in $P_n, n \geqslant 4$.*

Corollary 1.5. *A maximal set of independent 8–cycles in $P_n, n \geqslant 4$, contains $\frac{n!}{8}$ of these.*

The proof of Theorem 1.3 is based on the hierarchical (recursive) structure of the Pancake graph which can be presented as follows. The graph $P_n, n \geqslant 3$, is constructed from n copies of $P_{n-1}(i), 1 \leqslant i \leqslant n$, such that each $P_{n-1}(i)$ has the vertex set $V_i = \{[\pi_1 \dots \pi_{n-1} i]$, where $\pi_k \in \{1, \dots, n\} \backslash \{i\} : 1 \leqslant k \leqslant n-1\}$, $|V_i| = (n-1)!$, and the edge set $E_i = \{\{[\pi_1 \dots \pi_{n-1} i], [\pi_1 \dots \pi_{n-1} i]\, r_j\} : 2 \leqslant j \leqslant n-1\}$, $|E_i| = \frac{(n-1)!(n-2)}{2}$. Any two copies $P_{n-1}(i), P_{n-1}(j), i \neq j$, are connected by $(n-2)!$ edges presented as $\{[i\pi_2 \dots \pi_{n-1} j],\ [j\pi_{n-1} \dots \pi_2 i]\}$, where $[i\pi_2 \dots \pi_{n-1} j] r_n = [j\pi_{n-1} \dots \pi_2 i]$. Prefix–reversals $r_j, 2 \leqslant j \leqslant n-1$, define *internal edges* in all n copies $P_{n-1}(i), 1 \leqslant i \leqslant n$, and the prefix–reversal r_n defines *external edges* between copies. Copies $P_{n-1}(i)$, or just P_{n-1} when it is not important to specify the last element of permutations belonging to the copy, are also called $(n-1)$*–copies*.

Since $P_3 \cong C_6$ and due to the hierarchical structure, P_4 has four copies of P_3, each of which obviously cannot contain 8–cycles. However, P_4 has 8–cycles consisting of paths within copies of P_3 together with external edges between these copies. In general, any 8–cycle of $P_n, n \geqslant 4$, must consist of paths within subgraphs that are isomorphic to P_{k-1} for some $4 \leqslant k \leqslant n$, joined by external edges between these subgraphs. Hence, all 8–cycles of $P_n, n \geqslant 4$, could be found recursively by considering 8–cycles within each $P_k, 4 \leqslant k \leqslant n$, consisting of vertices from some copies of P_{k-1}. This approach is used in the proof of Theorem 1.3. To get the main result, we also need some technical lemmas concerning paths of length three between vertices of a given form. So, in the next section we introduce additional notations and prove two small lemmas. In Section 3 we prove Theorem 1.3 and its corollaries.

2 Technical lemmas

A segment $[\pi_i \dots \pi_j]$ of a permutation $\pi = [\pi_1 \dots \pi_i \dots \pi_j \dots \pi_n]$ consists of all elements that lie between π_i and π_j inclusive. Any permutation can be written as a sequence of singleton and multiple segments. We use characters from $\{p, q, s, t\}$ to denote singletons and characters from $\{\alpha, \beta, \gamma, A, B, C\}$ to denote multiple segments. If $\pi = [\alpha\beta]$, where $\alpha = [\pi_1 \pi_2 \dots \pi_i]$ and $\beta = [\pi_{i+1} \dots \pi_n]$, then $\pi r_{|\alpha|} = [\overline{\alpha}\beta]$, where $|\alpha|$ is the number of elements in a segment α, and $\overline{\alpha}$ is the inversion of a segment α. It is obvious that $\overline{\overline{\alpha}} = \alpha$. Note that we allow empty segments where this does not contradict the initial definitions.

An independent set D of vertices in a graph is called an *efficient dominating set* if each vertex not in D is adjacent to exactly one vertex in D [3]. It is known [9] that $D_p = \{[p\pi_2 \dots \pi_n] : \pi_j \in \{1, \dots, n\} \backslash \{p\}, 2 \leqslant j \leqslant n\}$, $|D_p| = (n-1)!$, $1 \leqslant p \leqslant n$, are efficient dominating sets in $P_n, n \geqslant 3$. Let us note that external edges of P_n are incident to vertices from different efficient dominating sets of P_n. The *distance* $d = d(\pi, \tau)$ between two vertices π, τ in P_n is defined as the least number of prefix–reversals transforming π into τ, i.e. $\pi r_{i_1} r_{i_2} \dots r_{i_d} = \tau$.

The next lemma gives a full list of paths of length three between two vertices of the same efficient dominating set.

Lemma 2.1. *Two permutations $\pi, \tau \in D_p, 1 \leqslant p \leqslant n$, are at distance three from each other in $P_n, n \geqslant 3$, if and only if:*
1) either $\tau = \pi r_j r_i r_j$, $2 \leqslant i < j \leqslant n$, where $\pi = [AB\gamma]$, $\tau = [A\overline{B}\gamma]$;
2) or $\tau = \pi r_j r_i r_{i-j+1}$, $2 \leqslant j < i \leqslant n$, where $\pi = [pAB\gamma]$, $\tau = [pBA\gamma]$.

Proof. We consider $\pi \in D_p$ such that $\pi = [p\alpha q\beta k], \pi_j = q$. Let us find other vertices from D_p being at the distance three from π. Let $\pi^1 = \pi r_j = [q\overline{\alpha} p\beta k]$, where $\pi^1_j = p, 2 \leqslant j \leqslant n$. An application of a prefix–reversal $r_i, 2 \leqslant i \leqslant n, i \neq j$, to the permutation π^1 gives us two cases: either $i < j$ or $i > j$.

1) If $i < j$ then $\pi^2 = \pi^1 r_i = [\alpha_2 q \overline{\alpha_1} p \beta k]$, where $\pi_j^2 = p$, $\alpha = \alpha_1\alpha_2$ and $|\alpha_2| = i - 1$, and then $\tau = \pi^2 r_j = [p\alpha_1 q \overline{\alpha_2} \beta k]$. Hence $\tau = \pi r_j r_i r_j$ and we get $\pi = [AB\gamma]$, $\tau = [A\overline{B}\gamma]$ by setting $A = p\alpha_1$, $B = \alpha_2 q$, $\gamma = \beta k$. Note, that using r_j is the only way to restore p to the first position and thus to end at an element of D_p after reaching π^2.

2) If $i > j$ then $\pi^2 = \pi^1 r_i = [\overline{\beta_1} p \alpha q \beta_2 k]$, where $\pi^2_{i-j+1} = p$, $\beta = \beta_1\beta_2$ and $|\beta_1| = i - j$, and then $\tau = \pi^2 r_{i-j+1} = [p\beta_1\alpha q\beta_2 k]$. Hence $\tau = \pi r_j r_i r_{i-j+1}$ and we get $\pi = [pAB\gamma]$, $\tau = [pBA\gamma]$ by setting $A = \alpha q$, $B = \beta_1$, $\gamma = \beta_2 k$. Note, that using r_{i-j+1} is the only way to restore p to the first position and thus to end at an element of D_p after reaching π^2. □

The next lemma gives a description of paths of length three defined on internal edges of the graph between vertices of given forms.

Lemma 2.2. *Let permutations π and τ be presented as:*
1) $\pi = [\gamma_1 AB\gamma_2]$ and $\tau = [\gamma_1 \overline{A} B\gamma_2]$, where $|\gamma_1| \geqslant 1$, $|A| \geqslant 2$. Then:

a) there exists a unique path of length three:

$$\tau = \pi r_{|\gamma_1|+|A|} r_{|A|} r_{|\gamma_1|+|A|}, \tag{2.1}$$

provided that either $|\gamma_1| \geqslant 2$ and $|A| \geqslant 2$, or $|\gamma_1| = 1$ and $|A| \geqslant 3$;

b) there are two paths of length three:

$$\tau = \pi r_2 r_3 r_2, \quad \tau = \pi r_3 r_2 r_3, \tag{2.2}$$

provided that $|\gamma_1| = 1$ and $|A| = 2$;
2) $\pi = [\gamma_1 AB\gamma_2]$ and $\tau = [\gamma_1 BA\gamma_2]$, where $|\gamma_1| \geqslant 0$, $|A| \geqslant 1$, $|B| \geqslant 1$. Then:

a) there is a unique path of length three:

$$\tau = \pi r_{|\gamma_1|+2} r_2 r_{|\gamma_1|+2}, \tag{2.3}$$

provided that $|\gamma_1| \geqslant 2$, and $|A| = |B| = 1$;

b) there is a unique path of length three:

$$\tau = \pi r_{|\gamma_1|+|A|} r_{|\gamma_1|+|A|+|B|} r_{|\gamma_1|+|B|}, \tag{2.4}$$

provided that $|\gamma_1| = 1$, and $|A| \neq 1$ or $|B| \neq 1$;

c) there are two paths of length three:

$$\tau = \pi r_2 r_3 r_2 = \pi r_3 r_2 r_3, \tag{2.5}$$

provided that $|\gamma_1| = |A| = |B| = 1$;

d) there is a unique path of length three:

$$\tau = \pi r_{|A|} r_{|A|+|B|} r_{|B|}, \tag{2.6}$$

provided that $|\gamma_1| = 0$ and $|A| \geqslant 2$, $|B| \geqslant 2$.
There are no other paths of length three between π and τ of these types.

Proof. 1) If $\pi = [\gamma_1 AB\gamma_2]$ and $\tau = [\gamma_1 \overline{A} B\gamma_2]$, then (2.1) is checked by a direct verification: $[\gamma_1 AB\gamma_2] \overset{r_{|\gamma_1|+|A|}}{\to} [\overline{A}\overline{\gamma_1}B\gamma_2] \overset{r_{|A|}}{\to} [A\overline{\gamma_1}B\gamma_2] \overset{r_{|\gamma_1|+|A|}}{\to} [\gamma_1\overline{A}B\gamma_2]$. Suppose that there is one more path of length three. Then these two paths should form a 6–cycle. In part (a), either $|\gamma_1| \geqslant 2$ and $|A| \geqslant 2$, or $|\gamma_1| = 1$ and $|A| \geqslant 3$, so $r_{|\gamma_1|+|A|} = r_m$ for some $m \geqslant 4$, but by Theorem 1.2, no 6–cycle includes r_m with $m \geqslant 4$ in its form. Therefore, the given path is the only one in this case. In part (b), $|\gamma_1| = 1$ and $|A| = 2$, so $m = 3$ and the condition of Theorem 1.2 holds, hence we obtain two distinct paths of stated forms (2.2).

2) If $\pi = [\gamma_1 AB\gamma_2]$ and $\tau = [\gamma_1 BA\gamma_2]$, and $|\gamma_1| \geqslant 2$, $|A| \geqslant 1$, $|B| \geqslant 1$, then there is the following path of length four between these vertices:

$$\pi = [\gamma_1 AB\gamma_2] \overset{r_{|\gamma_1|+|A|}}{\to} [\overline{A}\overline{\gamma_1}B\gamma_2] \overset{r_{|\gamma_1|+|A|+|B|}}{\to} [\overline{B}\gamma_1 A\gamma_2] \overset{r_{|\gamma_1|+|B|}}{\to}$$

$$[\overline{\gamma_1}BA\gamma_2] \overset{r_{|\gamma_1|}}{\to} [\gamma_1 BA\gamma_2] = \tau. \tag{2.7}$$

Suppose there is also a path of length three between π and τ. By Theorem 1.1, there are no cycles of length 3 or 5, and hence no paths of lengths 3 and 4 exist between two fixed vertices unless the paths are disjoint. This means that these two paths should form a 7–cycle, including the sequence $r_{m+a}r_{m+a+b}r_{m+b}r_m$, where $|\gamma_1| = m$, $|A| = a$ and $|B| = b$. By Theorem 1.2, this is possible only in the case when $m = k-2, k \geqslant 4$, and $a = b = 1$, which implies that a unique path of length three has the form $r_{m+2}r_2r_{m+2}$ that corresponds to (2.3).

Putting $|\gamma_1| = 1$ in (2.7), a path $\tau = \pi r_{|\gamma_1|+|A|}r_{|\gamma_1|+|A|+|B|}r_{|\gamma_1|+|B|}$, corresponding to (2.4), is obtained. Taking $|\gamma_1| = m$, $|A| = a$ and $|B| = b$, the obtained path is presented as $r_{m+a}r_{m+a+b}r_{m+b}$. Suppose that there is one more path of length three between π and τ. Then these two paths should form a 6–cycle. By Theorem 1.2, this is possible only in the case when $m = a = b = 1$, which gives us the paths $\tau = \pi r_2 r_3 r_2$ and $\tau = \pi r_3 r_2 r_3$, corresponding to (2.5).

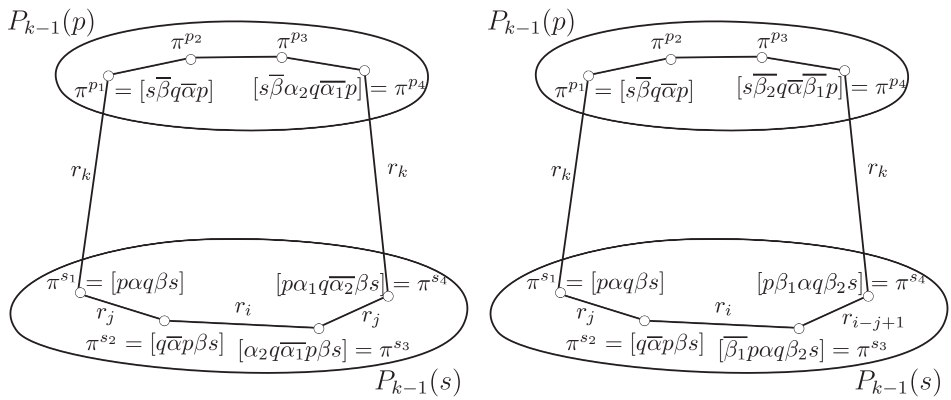
Putting $|\gamma_1| = 0$ in (2.7), a path $\tau = \pi r_{|A|}r_{|A|+|B|}r_{|B|}$, corresponding to (2.6) with $|A| \geqslant 2$, $|B| \geqslant 2$, is obtained. Suppose there is one more path of length three between π and τ. Then these two paths should form a 6–cycle. By the conditions of Lemma, $|A| + |B| \geqslant 4$, hence $r_{|A|+|B|} = r_m$ for some $m \geqslant 4$, but by Theorem 1.2, no 6–cycle includes r_m with $m \geqslant 4$ in its form. Therefore, the given path is the only one in this case. If $|A| = 1$ or $|B| = 1$, then the path above is transformed into a 2–path or an edge. This completes the proof of the lemma. □

3 Proof of Theorem 1.3

To find all 8–cycles passing through the same vertex in $P_n, n \geqslant 4$, we use its hierarchical structure by considering recursively 8–cycles within each copy $P_k, 4 \leqslant k \leqslant n$, consisting of vertices from copies of P_{k-1}. It is assumed that any copy of P_{k-1} has at least two vertices, since each vertex has a unique external edge. We obtain canonical forms of 8–cycles and count their numbers.

Case 1: an 8–cycle within P_k has vertices from two copies of P_{k-1}

Suppose that an 8–cycle is formed on vertices from copies $P_{k-1}(p)$ and $P_{k-1}(s)$, $1 \leqslant p \neq s \leqslant k$. It was shown in [8] that if two vertices π and τ, belonging the same $(k-1)$–copy, are at the distance at most two, then their external neighbours $\overline{\pi}$ and $\overline{\tau}$ should belong to distinct $(k-1)$–copies. Hence, an 8–cycle cannot occur in situations when its two (three)



case 1 case 2

Figure 1: $(4+4)$–situation.

vertices belong to one copy and six (five) vertices belong to another one. Therefore, such an 8–cycle must have four vertices in each of the two copies.

$(4+4)$–situation. Suppose that four vertices of such an 8–cycle belong to a copy $P_{k-1}(s)$, and other four vertices belong to a copy $P_{k-1}(p)$. Herewith, four vertices of $P_{k-1}(s)$ should form a path of length three whose endpoints should be adjacent to vertices from $P_{k-1}(p)$, which means both vertices should belong to the efficient dominating set D_p. So, one vertex of $P_{k-1}(s)$ that is adjacent to a vertex of $P_{k-1}(p)$ must have the form $[p\alpha q\beta s]$. By Lemma 2.1, it is not hard to see that this gives rise to two possible forms for the remaining vertices of $P_{k-1}(s)$. These are given in Figure 1, where $|\alpha| = j-2$ and so $|\beta| = k-j-1$. In the first case we also have $\alpha = \alpha_1\alpha_2$ and $|\alpha_2| = i-1 \geqslant 1$, while in the second case we also have $|\beta_1| = i-j \geqslant 1$ and $|\beta_2| = k-i-1$.

Denote $\gamma_1 = s\overline{\beta}$, $A = q\overline{\alpha_2}$, $B = \alpha_1$, $\gamma_2 = p$, where $|\gamma_1| = |\beta|+1 \geqslant 1$, $|A| \geqslant 2$, $|B| \geqslant 0$, then in the first case π^{p_1}, π^{p_4} have the forms $[\gamma_1 AB\gamma_2]$ and $[\gamma_1\overline{A}B\gamma_2]$. By Lemma 2.2 (case 1a), there is a unique path of length three between these permutations if $|\gamma_1| = |\beta|+1 = k-j \geqslant 1$ and $|A| = |\alpha_2|+1 = i \geqslant 3$, or $k-j \geqslant 2$ and $i \geqslant 2$, and by Lemma 2.2 (case 1b), there are two distinct paths if $k-j=1$ and $i=2$. Hence, such an 8–cycle has the form $C_8^1 = r_{k-j+i}r_ir_{k-j+i}r_kr_jr_ir_jr_k$, with $2 \leqslant i < j \leqslant k-1$, $4 \leqslant k \leqslant n$, the canonical form of which corresponds to (1.1). The case of $k-j=1$ and $i=2$ by symmetry gives one additional form $C_8^2 = r_{k-1}r_2r_{k-1}r_kr_2r_3r_2r_k$, the canonical form of which corresponds to (1.2).

Denote $\gamma_1 = s\overline{\beta_2}$, $A = \overline{\beta_1}$, $B = q\overline{\alpha}$, $\gamma_2 = p$, where $|\gamma_1| = |\beta_2|+1 \geqslant 1$, $|A| = |\beta_1| \geqslant 1$, $|B| = |\alpha|+1 \geqslant 1$, then in the second case we have $\pi^{p_1} = [\gamma_1 AB\gamma_2]$, $\pi^{p_4} = [\gamma_1 BA\gamma_2]$. By Lemma 2.2 (case 2a), there is a unique path of length three between π^{p_1} and π^{p_4} if $|\gamma_1| = k-i \geqslant 2$, $|A| = |\beta_1| = i-j = 1$ and $|B| = |\alpha|+1 = j-1 = 1$. Hence, $j=2$, $i=3$, and for $k \geqslant 5$ an 8–cycle has the form $r_{k-1}r_2r_{k-1}r_kr_2r_3r_2r_k$, corresponding again to the canonical form (1.2).

By Lemma 2.2 (case 2b), there also exists a unique path of length three between π^{p_1} and π^{p_4} if $|\gamma_1| = k-i = 1$, $|A| = |\beta_1| = i-j \geqslant 1$, $|B| = |\alpha|+1 = j-1 \geqslant 1$. This means that $i = k-1$, and such an 8–cycle has the form $C_8^4 = r_kr_{k-j}r_{k-1}r_jr_kr_{k-j}r_{k-1}r_j$, $2 \leqslant j \leqslant k-2$, the canonical form of which corresponds to (1.3), if we set $j=i$. So, there

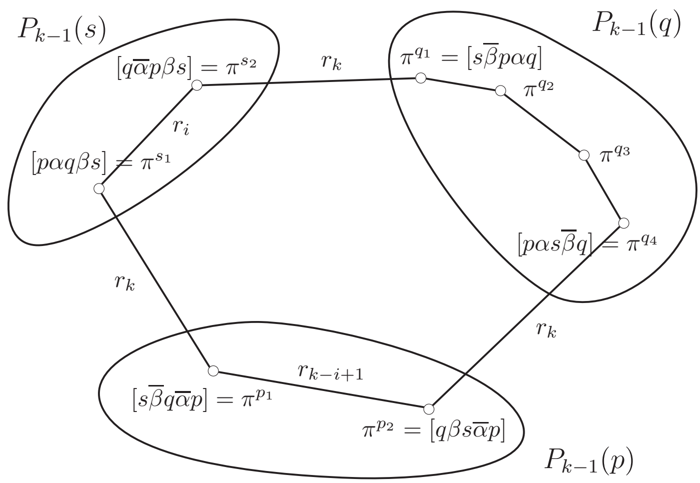


Figure 2: $(2+2+4)$–situation.

is a unique path of length three under the conditions listed, unless $|A| = |B| = 1$ when by Lemma 2.2 (case 2c) this path is not unique. So, $k = 4, j = 2, i = 3$ and 8–cycles take forms $r_2r_3r_2r_4r_3r_2r_3r_4$ and $r_3r_2r_3r_4r_3r_2r_3r_4$, corresponding to forms (1.2) and (1.1).

Thus, all 8–cycles occurring in the case of two copies are found.

Case 2: an 8–cycle within P_k has vertices from three copies of P_{k-1}

Suppose an 8–cycle is formed on vertices from copies $P_{k-1}(p)$, $P_{k-1}(q)$, $P_{k-1}(s)$, where $1 \leqslant p \neq q \neq s \leqslant k$. There are following possible situations in this case.

$(2+2+4)$–situation. The distribution of vertices among the copies is presented by Figure 2. Let $\pi^{s_1} = [p\alpha q\beta s]$ where $\pi_i^{s_1} = q$ with $|\alpha| = i-2$, $|\beta| = k-i-1$. Then π^{s_2}, π^{p_1}, π^{p_2}, π^{q_1} and π^{q_4} are straightforward to define. Vertices π^{q_1} and π^{q_4} differ in the order of segments $s\overline{\beta}$ and $p\alpha$, hence they have the forms $[\gamma_1 AB\gamma_2]$ and $[\gamma_1 BA\gamma_2]$, where γ_1 is empty, $A = s\overline{\beta}$, $B = p\alpha$, $\gamma_2 = q$ and $|A| = |\beta| + 1 \geqslant 1$, $|B| = |\alpha| + 1 \geqslant 1$. By Lemma 2.2 (case 2d), between π^{q_1} and π^{q_4} there exists a unique path of length three provided that $|A| = k-i \geqslant 2$, $|B| = i-1 \geqslant 2$, and no path of this length if $|A| = 1$ or $|B| = 1$. Thus, an 8–cycle has the form $C_8^4 = r_k r_{k-i+1} r_k r_{i-1} r_{k-1} r_{k-i} r_k r_i$, where $3 \leqslant i \leqslant k-2$, $k \geqslant 5$, the canonical form of which corresponds to (1.4).

$(2+3+3)$–situation. The distribution of vertices among the copies is presented by Figure 3. Let $\pi^{s_1} = [p\alpha q\beta s]$, where $|\alpha| = i-2$ and $|\beta| = k-i-1$. Then π^{s_2}, π^{p_1} and π^{q_1} are straightforward to define. Since π^{p_3} and π^{q_3} are joined by an external edge, $\pi_1^{p_3} = q$. Moreover, π^{p_1} and π^{p_3} should be joined by a path of length two that can be obtained by two ways:

$$\pi^{p_1} = [s\overline{\beta}q\overline{\alpha}p] \rightarrow \begin{cases} [\beta_2 s\overline{\beta_1}q\overline{\alpha}p] \rightarrow [q\beta_1 s\overline{\beta_2}\overline{\alpha}p] = \pi^{p_3^1}, & \text{where } |\beta_2| \neq 0. \\ [\alpha_2 q\beta s\overline{\alpha_1}p] \rightarrow [q\overline{\alpha_2}\beta s\overline{\alpha_1}p] = \pi^{p_3^2}, & \text{where } |\alpha_2| \neq 0. \end{cases}$$

From the other side, π^{q_3} and π^{p_3} are joined by an external edge, hence $\pi_1^{q_3} = p$, and there

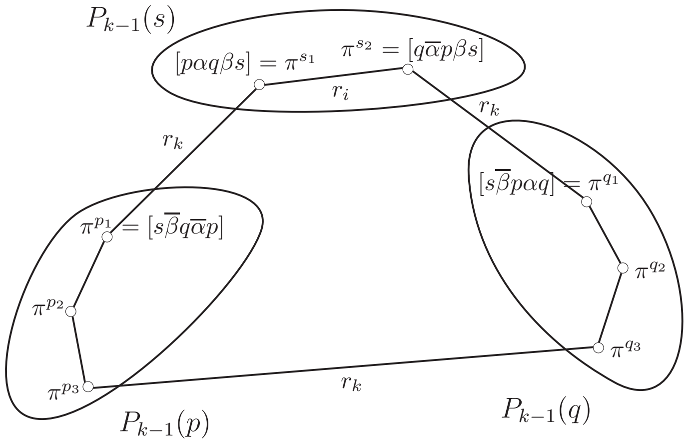


Figure 3: $(2+3+3)$–situation.

should be a path of length two between π^{q_1} and π^{q_3} such that:

$$\pi^{q_1} = [s\overline{\beta}p\alpha q] \to \begin{cases} [\beta_2 s\overline{\beta_1}p\alpha q] \to [p\beta_1 s\overline{\beta_2}\alpha q] = \pi^{q_3^1}, & \text{where } |\beta_2| \neq 0. \\ [\overline{\alpha_1}p\beta s\alpha_2 q] \to [p\alpha_1\beta s\alpha_2 q] = \pi^{q_3^2}, & \text{where } |\alpha_1| \neq 0. \end{cases}$$

Analysis of non–empty segments in these permutations shows that external edges occur between: $\pi^{p_3^1}$ and $\pi^{q_3^2}$, if $|\alpha_2| = 0$, $|\beta_1| = 0$; $\pi^{p_3^2}$ and $\pi^{q_3^1}$, if $|\alpha_1| = 0$, $|\beta_1| = 0$; $\pi^{p_3^2}$ and $\pi^{q_3^2}$, if $|\beta| = 0$. There is no external edge between $\pi^{p_3^1}$ and $\pi^{q_3^1}$ since they have the same order of elements in the segment $s\overline{\beta_2}$.

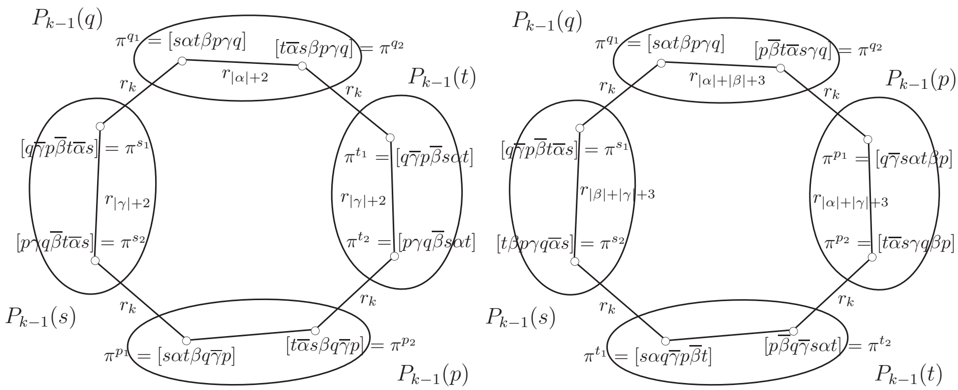
Since $|\alpha| = i-2$, $|\beta| = k-i-1$, then using the edge between $\pi^{p_3^1}$ and $\pi^{q_3^2}$, where $|\alpha_2| = 0$, $|\beta_1| = 0$, we have $|\alpha| \geqslant 1$, $|\beta| \geqslant 1$, and such an 8–cycle has the form $C_8^5 = r_k r_{k-i} r_{k-i+1} r_k r_{i-1} r_{k-1} r_k r_i$, with $3 \leqslant i \leqslant k-2$, $5 \leqslant k \leqslant n$, the canonical form of which corresponds to (1.5). Using the external edge between $\pi^{p_3^2}$ and $\pi^{q_3^1}$, where $|\alpha_1| = 0$, $|\beta_1| = 0$, we have $|\alpha| \geqslant 1$, $|\beta| \geqslant 1$, and such an 8–cycle has the form $r_k r_{k-1} r_{i-1} r_k r_{k-i+1} r_{k-i} r_k r_i$, where $3 \leqslant i \leqslant k-2$, the canonical form of which also corresponds to the form (1.5). Finally, using the external edge between $\pi^{p_3^2}$ and $\pi^{q_3^2}$, where $|\beta| = 0$, we have $i = k-1$, $|\alpha_1| = j \geqslant 1$, $|\alpha_2| = k-3-j \geqslant 1$, so there is one more 8–cycle of the form $C_8^6 = r_k r_{k-j-1} r_{k-j-2} r_k r_{j+1} r_{j+2} r_k r_{k-1}$, where $1 \leqslant j \leqslant k-4$, $5 \leqslant k \leqslant n$, the canonical form of which corresponds to (1.6), if we put $j = i-1$.

Thus, all 8–cycles occurring in the case of three copies are found.

Case 3: an 8–cycle within P_k has vertices from four copies of P_{k-1}

The distribution of vertices among the copies is presented by Figure 4. Let $\pi^{q_1} = [s\alpha t\beta p\gamma q]$, where $|\alpha| \geqslant 0$, $|\beta| \geqslant 0$, $|\gamma| \geqslant 0$. There are two cases.

1) Suppose that π^{q_1} is adjacent to π^{s_1}, and π^{q_2} is adjacent to π^{t_1}. Since there is only one cycle edge within each copy, hence this edge is uniquely defined and all vertices' labels are straightforward to obtain (see Figure 4, case 1). Thus, we end up with $\pi^{p_1} = [s\alpha t\beta q\overline{\gamma}p]$, $\pi^{p_2} = [t\overline{\alpha}s\beta q\overline{\gamma}p]$. If an 8–cycle does exist, then π^{p_1}, π^{p_2} should be incident to the same



case 1 case 2

Figure 4: $(2+2+2+2)$–situation.

internal edge, and hence, there should exist a prefix–reversal transforming π^{p_1} into π^{p_2}, namely, $r_{|\alpha|+2}$. If we set $|\alpha| = i-2$, $|\beta| = j-i-1$, $|\gamma| = k-j-1$, where $2 \leqslant i < j < k$, then such an 8–cycle is presented by the form $C_8^7 = r_k r_{k-j+1} r_k r_i r_k r_{k-j+1} r_k r_i$, where $2 \leqslant i < j \leqslant k-1$, $4 \leqslant k \leqslant n$, the canonical form of which corresponds to (1.7).

2) Suppose that π^{q_1} is adjacent to π^{s_1}, and π^{q_2} is adjacent to π^{p_1} (see Figure 4, case 2), then we end up with $\pi^{t_1} = [s\alpha q\overline{\gamma}p\overline{\beta}t]$, $\pi^{t_2} = [p\overline{\beta}q\overline{\gamma}s\alpha t]$. In this case, an internal edge between vertices π^{t_1} and π^{t_2} does exist only if $|\alpha| = |\beta| = |\gamma| = 0$, which means that $k = 4$ and such an 8–cycle takes the form (1.8).

Therefore, all canonical forms for 8–cycles in $P_n, n \geqslant 4$, are obtained.

Now we count the total number $N = \sum_{i=1}^{8} N_{C_8^i}$ of distinct 8–cycles passing through a given vertex, where $N_{C_8^i}$ corresponds to the number of distinct 8–cycles described by the canonical form $C_8^i, 1 \leqslant i \leqslant 8$. Let us note that any canonical form of an l–cycle describes l cycles (not necessarily distinct) for a given vertex. Among all canonical forms (1.1)–(1.8), there is the only one, namely the form (1.5), which describes eight distinct 8–cycles. In other cases, identical forms occur. For example, from the canonical form $C_8^8 = r_4r_3r_4r_3r_4r_3r_4r_3$ one can get two forms, namely, $r_4r_3r_4r_3r_4r_3r_4r_3$ and $r_3r_4r_3r_4r_3r_4r_3r_4$ which are identical because they describe the same 8–cycle. Thus, the canonical form C_8^8 gives the only 8–cycle, hence, $N_{C_8^8} = 1$. In other cases, it can be shown in the same manner (by taking into account identical forms) that the numbers $N_{C_8^i}, 1 \leqslant i \leqslant 7$, are given as follows: $N_{C_8^1} = \frac{(n-3)(n-2)(n-1)}{3}$, $N_{C_8^2} = 4(n-3)$, $N_{C_8^3} = (n-2)(n-3)$, $N_{C_8^4} = N_{C_8^6} = 2(n-3)(n-4)$, $N_{C_8^5} = 4(n-3)(n-4)$, $N_{C_8^7} = \frac{(n-3)(n-2)(n-1)}{6}$. Thus, the total number is given by

$$N = \frac{n^3 + 12n^2 - 103n + 176}{2},$$

which completes the proof of the theorem. □

The total number of distinct 8–cycles in $P_n, n \geqslant 4$, is given by $\frac{n!(n^3+12n^2-103n+176)}{16}$ since there are $n!$ vertices in the graph each of which belongs to N distinct 8–cycles. This proves Corollary 1.4.

A maximal set of independent 8–cycles in $P_n, n \geqslant 4$, contains $\frac{n!}{8}$ of these, since P_4 has three independent 8–cycles, and there are $\frac{n!}{24}$ copies of P_4, each of which consists of exactly three independent 8–cycles. This proves Corollary 1.5.

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Fat Hoffman graphs with smallest eigenvalue at least $-1-\tau$

Akihiro Munemasa
Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan

Yoshio Sano
Division of Information Engineering, Faculty of Engineering, Information and Systems University of Tsukuba, Ibaraki 305-8573, Japan

Tetsuji Taniguchi
Matsue College of Technology, Matsue 690-8518, Japan

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Abstract

In this paper, we show that all fat Hoffman graphs with smallest eigenvalue at least $-1-\tau$, where τ is the golden ratio, can be described by a finite set of fat $(-1-\tau)$-irreducible Hoffman graphs. In the terminology of Woo and Neumaier, we mean that every fat Hoffman graph with smallest eigenvalue at least $-1-\tau$ is an $\mathcal{H}$-line graph, where $\mathcal{H}$ is the set of isomorphism classes of maximal fat $(-1-\tau)$-irreducible Hoffman graphs. It turns out that there are 37 fat $(-1-\tau)$-irreducible Hoffman graphs, up to isomorphism.

Keywords: Hoffman graph, line graph, graph eigenvalue, special graph.

Math. Subj. Class.: 05C50, 05C75

1 Introduction

P. J. Cameron, J. M. Goethals, J. J. Seidel, and E. E. Shult [1] characterized graphs whose adjacency matrices have smallest eigenvalue at least -2 by using root systems. Their results revealed that graphs with smallest eigenvalue at least -2 are generalized line graphs, except a finite number of graphs represented by the root system E_8. Another characterization for generalized line graphs were given by D. Cvetković, M. Doob, and S. Simić [3]

E-mail addresses: munemasa@math.is.tohoku.ac.jp (Akihiro Munemasa), sano@cs.tsukuba.ac.jp (Yoshio Sano), tetsuzit@matsue-ct.ac.jp (Tetsuji Taniguchi)

by determinig minimal forbidden subgraphs (see also [4]). Note that graphs with smallest eigenvalue greater than -2 were studied by A. J. Hoffman [5].

Hoffman [6] also studied graphs whose adjacency matrices have smallest eigenvalue at least $-1-\sqrt{2}$ by using a technique of adding cliques to graphs. R. Woo and A. Neumaier [12] formulated Hoffman's idea by introducing the notion of Hoffman graphs. A Hoffman graph is a simple graph with a distinguished independent set of vertices, called fat vertices, which can be considered as cliques of size infinity in a sense (see Definition 2.1, and also [8, Corollary 2.15]). To deal with graphs with bounded smallest eigenvalue, Woo and Neumaier introduced a generalization of line graphs by considering decompositions of Hoffman graphs. They gave a characterization of graphs with smallest eigenvalue at least $-1-\sqrt{2}$ in terms of Hoffman graphs by classifying fat indecomposable Hoffman graphs with smallest eigenvalue at least $-1-\sqrt{2}$. This led them to prove a theorem which states that every graph with smallest eigenvalue at least $-1-\sqrt{2}$ and sufficiently large minimum degree is a subgraph of a Hoffman graph admitting a decomposition into subgraphs isomorphic to only four Hoffman graphs. In the terminology of [12], this means that every graph with smallest eigenvalue at least $-1-\sqrt{2}$ and sufficiently large minimum degree is an $\mathcal{H}$-line graph, where $\mathcal{H}$ is the set of four isomorphism classes of Hoffman graphs. For further studies on graphs with smallest eigenvalue at least $-1-\sqrt{2}$, see the papers by T. Taniguchi [10, 11] and by H. Yu [13].

Recently, H. J. Jang, J. Koolen, A. Munemasa, and T. Taniguchi [8] made the first step to classify the fat indecomposable Hoffman graphs with smallest eigenvalue -3. However, it seems that there are so many such Hoffman graphs. A key to solve this problem is the notion of *special graphs* introduced by Woo and Neumaier. A special graph is an edge-signed graph defined for each Hoffman graph. Although non-isomorphic Hoffman graphs may have isomorphic special graphs, it is not difficult to recover all the Hoffman graphs with a given special graph in some cases.

In this paper, we introduce irreducibility of Hoffman graphs and classify fat $(-1-\tau)$-irreducible Hoffman graphs, where $\tau := \frac{1+\sqrt{5}}{2}$ is the golden ratio. This is a somewhat more restricted class of Hoffman graphs than those considered in [8], and there are only 37 such Hoffman graphs. As a consequence, every fat Hoffman graph with smallest eigenvalue at least $-1-\tau$ is a subgraph of a Hoffman graph admitting a decomposition into subgraphs isomorphic to only 18 Hoffman graphs. In the terminology of [12], this means that every fat Hoffman graph with smallest eigenvalue at least $-1-\tau$ is an $\mathcal{H}$-line graph, where $\mathcal{H}$ is the set of 18 isomorphism classes of maximal fat $(-1-\tau)$-irreducible Hoffman graphs.

2 Preliminaries

2.1 Hoffman graphs and eigenvalues

Definition 2.1. A *Hoffman graph* $\mathfrak{H}$ is a pair (H,μ) of a graph H and a vertex labeling $\mu : V(H) \to \{\mathbf{slim}, \mathbf{fat}\}$ satisfying the following conditions: (i) every vertex with label **fat** is adjacent to at least one vertex with label **slim**; (ii) the vertices with label **fat** are pairwise non-adjacent.

Let $V(\mathfrak{H}) := V(H)$, $V^s(\mathfrak{H}) := \mu^{-1}(\mathbf{slim})$, $V^f(\mathfrak{H}) := \mu^{-1}(\mathbf{fat})$, and $E(\mathfrak{H}) := E(H)$. We call a vertex in $V^s(\mathfrak{H})$ a *slim vertex*, and a vertex in $V^f(\mathfrak{H})$ a *fat vertex* of $\mathfrak{H}$. We represent a Hoffman graph $\mathfrak{H}$ also by the triple $(V^s(\mathfrak{H}), V^f(\mathfrak{H}), E(\mathfrak{H}))$.

For a vertex x of a Hoffman graph $\mathfrak{H}$, we define $N^f_{\mathfrak{H}}(x)$ (resp. $N^s_{\mathfrak{H}}(x)$) to be the set of fat (resp. slim) neighbors of x in $\mathfrak{H}$. The set of all neighbors of x is denoted by $N_{\mathfrak{H}}(x)$, that is, $N_{\mathfrak{H}}(x) := N^f_{\mathfrak{H}}(x) \cup N^s_{\mathfrak{H}}(x)$.

A Hoffman graph $\mathfrak{H}$ is said to be *fat* if every slim vertex of $\mathfrak{H}$ has a fat neighbor. A Hoffman graph is said to be *slim* if it has no fat vertex.

Two Hoffman graphs $\mathfrak{H} = (H, \mu)$ and $\mathfrak{H}' = (H', \mu')$ are said to be *isomorphic* if there exists an isomorphism from H to H' which preserves the labeling.

A Hoffman graph $\mathfrak{H}' = (H', \mu')$ is called an *induced Hoffman subgraph* (or simply a *subgraph*) of another Hoffman graph $\mathfrak{H} = (H, \mu)$ if H' is an induced subgraph of H and $\mu(x) = \mu'(x)$ holds for any vertex x of $\mathfrak{H}'$.

The subgraph of a Hoffman graph $\mathfrak{H}$ induced by $V^s(\mathfrak{H})$ is called the *slim subgraph* of $\mathfrak{H}$.

Definition 2.2. For a Hoffman graph $\mathfrak{H}$, let A be its adjacency matrix,

$$A = \begin{pmatrix} A_s & C \\ C^T & O \end{pmatrix}$$

in a labeling in which the fat vertices come last. The *eigenvalues* of $\mathfrak{H}$ are the eigenvalues of the real symmetric matrix $B(\mathfrak{H}) := A_s - CC^T$. We denote by $\lambda_{\min}(\mathfrak{H})$ the smallest eigenvalue of $B(\mathfrak{H})$.

Remark 2.3. An ordinary graph H without vertex labeling can be regarded as a slim Hoffman graph $\mathfrak{H}$. Then the matrix $B(\mathfrak{H})$ coincides with the ordinary adjacency matrix of the graph H. Thus the eigenvalues of H as a slim Hoffman graph are the same as the eigenvalues of H as an ordinary graph in the usual sense.

Example 2.4. Let $\mathfrak{H}_{\mathrm{I}}$, $\mathfrak{H}_{\mathrm{II}}$, and $\mathfrak{H}_{\mathrm{III}}$ be the Hoffman graphs defined by

$$\begin{array}{lll} V^s(\mathfrak{H}_{\mathrm{I}}) = \{v_1\}, & V^f(\mathfrak{H}_{\mathrm{I}}) = \{f_1\}, & E(\mathfrak{H}_{\mathrm{I}}) = \{\{v_1, f_1\}\}, \\ V^s(\mathfrak{H}_{\mathrm{II}}) = \{v_1\}, & V^f(\mathfrak{H}_{\mathrm{II}}) = \{f_1, f_2\}, & E(\mathfrak{H}_{\mathrm{II}}) = \{\{v_1, f_1\}, \{v_1, f_2\}\}, \\ V^s(\mathfrak{H}_{\mathrm{III}}) = \{v_1, v_2\}, & V^f(\mathfrak{H}_{\mathrm{III}}) = \{f_1\}, & E(\mathfrak{H}_{\mathrm{III}}) = \{\{v_1, f_1\}, \{v_2, f_1\}\} \end{array}$$

(see Figure 1). Note that $\lambda_{\min}(\mathfrak{H}_{\mathrm{I}}) = -1$ and $\lambda_{\min}(\mathfrak{H}_{\mathrm{II}}) = \lambda_{\min}(\mathfrak{H}_{\mathrm{III}}) = -2$.

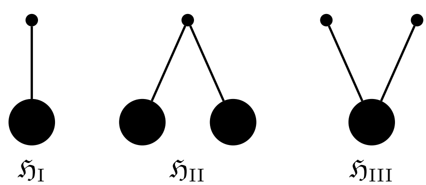


Figure 1: The Hoffman graphs $\mathfrak{H}_{\mathrm{I}}$, $\mathfrak{H}_{\mathrm{II}}$, and $\mathfrak{H}_{\mathrm{III}}$

Lemma 2.5 ([12, Lemma 3.4])**.** *The diagonal entry* $B(\mathfrak{H})_{xx}$ *of the matrix* $B(\mathfrak{H})$ *is equal to* $-|N^f_{\mathfrak{H}}(x)|$*, and the off-diagonal entry* $B(\mathfrak{H})_{xy}$ *is equal to* $A_{xy} - |N^f_{\mathfrak{H}}(x) \cap N^f_{\mathfrak{H}}(y)|$.

Lemma 2.6 ([12, Corollary 3.3])**.** *If* $\mathfrak{G}$ *is an induced Hoffman subgraph of a Hoffman graph* $\mathfrak{H}$*, then* $\lambda_{\min}(\mathfrak{G}) \geq \lambda_{\min}(\mathfrak{H})$ *holds. In particular, if* Γ *is the slim subgraph of* $\mathfrak{H}$*, then* $\lambda_{\min}(\Gamma) \geq \lambda_{\min}(\mathfrak{H})$.

2.2 Decompositions of Hoffman graphs

Definition 2.7. A *decomposition* of a Hoffman graph $\mathfrak{H}$ is a family $\{\mathfrak{H}^i\}_{i=1}^n$ of Hoffman subgraphs of $\mathfrak{H}$ satisfying the following conditions:

(i) $V(\mathfrak{H}) = \bigcup_{i=1}^n V(\mathfrak{H}^i)$;

(ii) $V^s(\mathfrak{H}^i) \cap V^s(\mathfrak{H}^j) = \emptyset$ if $i \neq j$;

(iii) if $x \in V^s(\mathfrak{H}^i)$, $y \in V^f(\mathfrak{H})$, and $\{x, y\} \in E(\mathfrak{H})$, then $y \in V(\mathfrak{H}^i)$;

(iv) if $x \in V^s(\mathfrak{H}^i)$, $y \in V^s(\mathfrak{H}^j)$, and $i \neq j$, then $|N^f_{\mathfrak{H}}(x) \cap N^f_{\mathfrak{H}}(y)| \leq 1$, and $|N^f_{\mathfrak{H}}(x) \cap N^f_{\mathfrak{H}}(y)| = 1$ if and only if $\{x, y\} \in E(\mathfrak{H})$.

If a Hoffman graph $\mathfrak{H}$ has a decomposition $\{\mathfrak{H}^i\}_{i=1}^n$, then we write $\mathfrak{H} = \biguplus_{i=1}^n \mathfrak{H}^i$.

Example 2.8. The (slim) complete graph K_n is precisely the slim subgraph of the Hoffman graph $\mathfrak{H} = \biguplus_{i=1}^n \mathfrak{H}^i$ where each $\mathfrak{H}^i$ is isomorphic to $\mathfrak{H}_{\mathrm{I}}$, sharing the unique fat vertex.

Ordinary line graphs are precisely the slim subgraphs of Hoffman graphs $\mathfrak{H} = \biguplus_{i=1}^n \mathfrak{H}^i$, where each $\mathfrak{H}^i$ is isomorphic to $\mathfrak{H}_{\mathrm{II}}$.

The (slim) cocktail party graph $CP(n) = K_{n\times 2}$ is precisely the slim subgraph of the Hoffman graph $\mathfrak{H} = \biguplus_{i=1}^n \mathfrak{H}^i$ where each $\mathfrak{H}^i$ is isomorphic to $\mathfrak{H}_{\mathrm{III}}$, sharing the unique fat vertex.

Generalized line graphs are precisely the slim subgraphs of Hoffman graphs $\mathfrak{H} = \biguplus_{i=1}^n \mathfrak{H}^i$, where each $\mathfrak{H}^i$ is isomorphic to $\mathfrak{H}_{\mathrm{II}}$ or $\mathfrak{H}_{\mathrm{III}}$ (see [12]).

Definition 2.9. A Hoffman graph $\mathfrak{H}$ is said to be *decomposable* if $\mathfrak{H}$ has a decomposition $\{\mathfrak{H}^i\}_{i=1}^n$ with $n \geq 2$. We say $\mathfrak{H}$ is *indecomposable* if $\mathfrak{H}$ is not decomposable.

Example 2.10. A disconnected Hoffman graph is decomposable.

Definition 2.11. Let α be a negative real number and let $\mathfrak{H}$ be a Hoffman graph with $\lambda_{\min}(\mathfrak{H}) \geq \alpha$. The Hoffman graph $\mathfrak{H}$ is said to be *α-reducible* if there exists a Hoffman graph $\mathfrak{H}'$ containing $\mathfrak{H}$ as an induced Hoffman subgraph such that there is a decomposition $\{\mathfrak{H}^i\}_{i=1}^2$ of $\mathfrak{H}'$ with $\lambda_{\min}(\mathfrak{H}^i) \geq \alpha$ and $V^s(\mathfrak{H}^i) \cap V^s(\mathfrak{H}) \neq \emptyset$ $(i = 1, 2)$. We say $\mathfrak{H}$ is *α-irreducible* if $\lambda_{\min}(\mathfrak{H}) \geq \alpha$ and $\mathfrak{H}$ is not α-reducible. A Hoffman graph $\mathfrak{H}$ is said to be *reducible* if $\mathfrak{H}$ is $\lambda_{\min}(\mathfrak{H})$-reducible. We say $\mathfrak{H}$ is *irreducible* if $\mathfrak{H}$ is not reducible.

Lemma 2.12 ([8, Lemma 2.12])**.** *If a Hoffman graph $\mathfrak{H}$ has a decomposition $\{\mathfrak{H}^i\}_{i=1}^n$, then $\lambda_{\min}(\mathfrak{H}) = \min\{\lambda_{\min}(\mathfrak{H}^i) \mid 1 \leq i \leq n\}$. In particular, an irreducible Hoffman graph is indecomposable.*

Example 2.13. For a non-negative integer t, let $\mathfrak{K}_{1,t}$ be the connected Hoffman graph having exactly one slim vertex and t fat vertices, i.e.,

$$\mathfrak{K}_{1,t} = (V^s(\mathfrak{K}_{1,t}), V^f(\mathfrak{K}_{1,t}), E(\mathfrak{K}_{1,t})) = (\{v\}, \{f_1, \ldots, f_t\}, \{\{v, f_i\} \mid i = 1, \ldots t\}).$$

Then $\mathfrak{K}_{1,t}$ is irreducible and $\lambda_{\min}(\mathfrak{K}_{1,t}) = -t$.

Example 2.14. By Example 2.13, the Hoffman graphs $\mathfrak{H}_{\mathrm{I}}$ $(\cong \mathfrak{K}_{1,1})$ and $\mathfrak{H}_{\mathrm{II}}$ $(\cong \mathfrak{K}_{1,2})$ are irreducible. The Hoffman graph $\mathfrak{H}_{\mathrm{III}}$ is also irreducible.

Example 2.15. Let $\mathfrak{H}_{\rm IV}$ be the Hoffman graph defined by

$$V^s(\mathfrak{H}_{\rm IV}) = \{v_1, v_2\}, V^f(\mathfrak{H}_{\rm IV}) = \{f_1, f_2\} \quad \text{and}$$

$$E(\mathfrak{H}_{\rm IV}) = \{\{v_1, v_2\}, \{v_1, f_1\}, \{v_2, f_2\}\}.$$

The Hoffman graph $\mathfrak{H}_{\rm IV}$ is indecomposable but reducible. Indeed, it is clear that $\mathfrak{H}_{\rm IV}$ is indecomposable. Let $\mathfrak{H}'$ be the Hoffman graph obtained from $\mathfrak{H}_{\rm IV}$ by adding a new fat vertex f_3 and two edges $\{v_1, f_3\}$ and $\{v_2, f_3\}$. The Hoffman graph $\mathfrak{H}'$ is the sum of two copies of $\mathfrak{H}_{\rm II}$, where the newly added fat vertex is shared by both copies, that is, $\mathfrak{H}'$ is decomposable. Furthermore, $\lambda_{\min}(\mathfrak{H}_{\rm II}) = \lambda_{\min}(\mathfrak{H}_{\rm IV}) = -2$. Hence $\mathfrak{H}_{\rm IV}$ is reducible.

Proposition 2.16. *Let G be a slim graph with at least two vertices. If G has maximum degree k, then G is $(-k)$-reducible.*

Proof. Let G be a slim graph with maximum degree k. We define a Hoffman graph $\mathfrak{H}$ by adding a fat vertex for each edge e of G and joining it to the two end vertices of e. Note that G is the slim subgraph of $\mathfrak{H}$. For each slim vertex $x \in V^s(\mathfrak{H})$, let $\mathfrak{H}^x$ be the Hoffman subgraph of $\mathfrak{H}$ induced by $\{x\} \cup N^f_{\mathfrak{H}}(x)$. Then $\mathfrak{H}^x$ is isomorphic to the Hoffman graph $\mathfrak{K}_{1,\deg_G(x)}$ defined in Example 2.13, and we can check that $\mathfrak{H} = \biguplus_{x \in V^s(\mathfrak{H})} \mathfrak{H}^x$. Since the maximum degree of G is k, $\lambda_{\min}(\mathfrak{H}^x) = -\deg_G(x) \geq -k$. Thus G is $(-k)$-reducible. ☐

Definition 2.17. Let $\mathcal{H}$ be a family of isomorphism classes of Hoffman graphs. An *$\mathcal{H}$-line graph* is an induced Hoffman subgraph of a Hoffman graph which has a decomposition $\{\mathfrak{H}^i\}_{i=1}^n$ such that the isomorphism class of $\mathfrak{H}^i$ belongs to $\mathcal{H}$ for all $i = 1, \ldots, n$.

2.3 The special graphs of Hoffman graphs

Definition 2.18. An *edge-signed graph* $\mathcal{S}$ is a pair (S, sgn) of a graph S and a map sgn : $E(S) \to \{+, -\}$. Let $V(\mathcal{S}) := V(S)$, $E^+(\mathcal{S}) := \mathrm{sgn}^{-1}(+)$, and $E^-(\mathcal{S}) := \mathrm{sgn}^{-1}(-)$. Each element in $E^+(\mathcal{S})$ (resp. $E^-(\mathcal{S})$) is called a *$(+)$-edge* (resp. a *$(-)$-edge*) of $\mathcal{S}$. We represent an edge-signed graph $\mathcal{S}$ also by the triple $(V(\mathcal{S}), E^+(\mathcal{S}), E^-(\mathcal{S}))$.

An edge-signed graph $\mathcal{S}' = (S', \mathrm{sgn}')$ is called an *induced edge-signed subgraph* of an edge-signed graph $\mathcal{S} = (S, \mathrm{sgn})$ if S' is an induced subgraph of S and $\mathrm{sgn}(e) = \mathrm{sgn}'(e)$ holds for any edge e of S'.

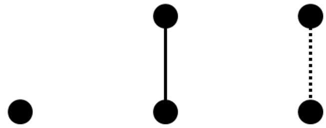
Two edge-signed graphs $\mathcal{S}$ and $\mathcal{S}'$ are said to be *isomorphic* if there exists a bijection $\phi : V(\mathcal{S}) \to V(\mathcal{S}')$ such that $\{u, v\} \in E^+(\mathcal{S})$ if and only if $\{\phi(u), \phi(v)\} \in E^+(\mathcal{S}')$ and that $\{u, v\} \in E^-(\mathcal{S})$ if and only if $\{\phi(u), \phi(v)\} \in E^-(\mathcal{S}')$.

An edge-sined graph $\mathcal{S}$ is said to be *connected* (resp. *disconnected*) if the graph $(V(\mathcal{S}), E^+(\mathcal{S}) \cup E^-(\mathcal{S}))$ is connected (resp. disconnected).

Example 2.19. A connected edge-signed graph with at most two vertices is isomorphic to one of the edge-signed graphs $\mathcal{S}_{1,1}$, $\mathcal{S}_{2,1}$, and $\mathcal{S}_{2,2}$, where

$$\begin{array}{lll}
V(\mathcal{S}_{1,1}) = \{v_1\}, & E^+(\mathcal{S}_{1,1}) = \emptyset, & E^-(\mathcal{S}_{1,1}) = \emptyset, \\
V(\mathcal{S}_{2,1}) = \{v_1, v_2\}, & E^+(\mathcal{S}_{2,1}) = \{\{v_1, v_2\}\}, & E^-(\mathcal{S}_{2,1}) = \emptyset, \\
V(\mathcal{S}_{2,2}) = \{v_1, v_2\}, & E^+(\mathcal{S}_{2,2}) = \emptyset, & E^-(\mathcal{S}_{2,2}) = \{\{v_1, v_2\}\}.
\end{array}$$

(see Figure 2 in which we draw an edge-signed graph by depicting $(+)$-edges as full lines and $(-)$-edges as dashed lines).



$\mathcal{S}_{1,1}$ $\mathcal{S}_{2,1}$ $\mathcal{S}_{2,2}$

Figure 2: The connected edge-signed graphs with at most two vertices

Definition 2.20. The *special graph* of a Hoffman graph $\mathfrak{H}$ is the edge-signed graph

$$\mathcal{S}(\mathfrak{H}) := (V(\mathcal{S}(\mathfrak{H})), E^+(\mathcal{S}(\mathfrak{H})), E^-(\mathcal{S}(\mathfrak{H})))$$

where $V(\mathcal{S}(\mathfrak{H})) := V^s(\mathfrak{H})$ and

$$E^+(\mathcal{S}(\mathfrak{H})) := \{\{u,v\} \mid u,v \in V^s(\mathfrak{H}), u \neq v, \{u,v\} \in E(\mathfrak{H}), N^f_{\mathfrak{H}}(u) \cap N^f_{\mathfrak{H}}(v) = \emptyset\},$$
$$E^-(\mathcal{S}(\mathfrak{H})) := \{\{u,v\} \mid u,v \in V^s(\mathfrak{H}), u \neq v, \{u,v\} \notin E(\mathfrak{H}), N^f_{\mathfrak{H}}(u) \cap N^f_{\mathfrak{H}}(v) \neq \emptyset\}.$$

Lemma 2.21 ([8, Lemma 3.4])**.** *A Hoffman graph $\mathfrak{H}$ is indecomposable if and only if its special graph $\mathcal{S}(\mathfrak{H})$ is connected.*

Definition 2.22. For an edge-signed graph $\mathcal{S}$, we define its *signed adjacency matrix $M(\mathcal{S})$* by

$$(M(\mathcal{S}))_{uv} = \begin{cases} 1 & \text{if } \{u,v\} \in E^+(\mathcal{S}), \\ -1 & \text{if } \{u,v\} \in E^-(\mathcal{S}), \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\lambda_{\min}(\mathcal{S})$ the smallest eigenvalue of $M(\mathcal{S})$.

We remark that P. J. Cameron, J. J. Seidel, and S. V. Tsaranov studied the eigenvalues of edge-signed graphs in [2].

Lemma 2.23. *If $\mathcal{S}'$ is an induced edge-signed subgraph of an edge-signed graph $\mathcal{S}$, then $\lambda_{\min}(\mathcal{S}') \geq \lambda_{\min}(\mathcal{S})$.*

Proof. Since $M(\mathcal{S}')$ is a principal submatrix of $M(\mathcal{S})$, the lemma holds. □

Lemma 2.24. *Let $\mathfrak{H}$ be a Hoffman graph in which any two distinct slim vertices have at most one common fat neighbor. Then*

$$M(\mathcal{S}(\mathfrak{H})) = B(\mathfrak{H}) + D(\mathfrak{H}),$$

where $D(\mathfrak{H})$ is the diagonal matrix defined by $D(\mathfrak{H})_{xx} := |N^f_{\mathfrak{H}}(x)|$ for $x \in V^s(\mathfrak{H})$.

Proof. This follows immediately from the definitions and Lemma 2.5. □

Lemma 2.25. *If $\mathfrak{H}$ is a fat Hoffman graph with smallest eigenvalue greater than -3, then $\lambda_{\min}(\mathcal{S}(\mathfrak{H})) \geq \lambda_{\min}(\mathfrak{H}) + 1$.*

Proof. If some two distinct slim vertices of $\mathfrak{H}$ have two common fat neighbors, then $\mathfrak{H}$ contains an induced subgraph with smallest eigenvalue at most -3. This contradicts the assumption by Lemma 2.6. Thus the hypothesis of Lemma 2.24 is satisfied. Since $\mathfrak{H}$ is fat, the smallest eigenvalue of $M(\mathcal{S}(\mathfrak{H})) = B(\mathfrak{H}) + D(\mathfrak{H})$ is at least $\lambda_{\min}(\mathfrak{H}) + 1$ by [7, Corollary 4.3.3], proving the desired inequality. □

3 Main Results

3.1 The edge-signed graphs with smallest eigenvalue at least $-\tau$

Definition 3.1. Let p, q, and r be non-negative integers with $p+q \leq r$. Let V_p, V_q, and V_r be mutually disjoint sets such that $|V_i| = i$ where $i \in \{p, q, r\}$. Let U_p and U_q be subsets of V_r satisfying $|U_p| = p$, $|U_q| = q$, and $U_p \cap U_q = \emptyset$. Let $\mathcal{Q}_{p,q,r}$ be the edge-signed graph defined by

$$\begin{aligned} V(\mathcal{Q}_{p,q,r}) &:= V_p \cup V_q \cup V_r, \\ E^+(\mathcal{Q}_{p,q,r}) &:= \{\{u,v\} \mid u \in U_p, v \in V_p\} \cup \{\{v,v'\} \mid v, v' \in V_r, v \neq v'\}, \\ E^-(\mathcal{Q}_{p,q,r}) &:= \{\{u,v\} \mid u \in U_q, v \in V_q\} \end{aligned}$$

(see Figure 3 for an illustration).

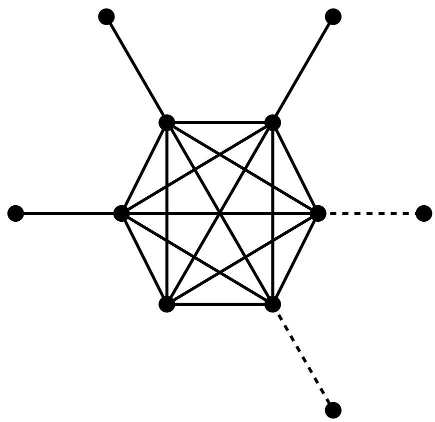


Figure 3: $\mathcal{Q}_{3,2,6}$

Lemma 3.2. *For any non-negative integers p, q, and r with* $p+q \leq r$, $\lambda_{\min}(\mathcal{Q}_{p,q,r}) \geq -\tau$.

Proof. Let

$$M(\mathcal{Q}_{r,r,2r}) = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \\ I & 0 & J-I & J \\ 0 & -I & J & J-I \end{bmatrix}.$$

Multiplying

$$\begin{bmatrix} I & 0 & xI & 0 \\ 0 & I & 0 & -xI \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

from the left to $xI - M(\mathcal{Q}_{r,r,2r})$, we find

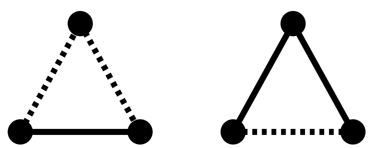
$$\begin{aligned}|xI - M(\mathcal{Q}_{r,r,2r})| &= (-1)^r \begin{vmatrix}(x^2+x-1)I & -(x^2+x-1)I \\ xJ & -(x^2+x-1)I+xJ\end{vmatrix} \\ &= (x^2+x-1)^r \begin{vmatrix} I & I \\ xJ & (x^2+x-1)I - xJ\end{vmatrix} \\ &= (x^2+x-1)^r |(x^2+x-1)I - 2xJ| \\ &= (x^2+x-1)^{2r-1}(x^2-(2r-1)x-1).\end{aligned}$$

In particular, we obtain $\lambda_{\min}(\mathcal{Q}_{r,r,2r}) = -\tau$. Since $p \le r$ and $q \le r$, $\mathcal{Q}_{r,r,2r}$ has an induced edge-signed subgraph isomorphic to $\mathcal{Q}_{p,q,r}$. By Lemma 2.23, $\lambda_{\min}(\mathcal{Q}_{p,q,r}) \ge \lambda_{\min}(\mathcal{Q}_{r,r,2r}) = -\tau$. □

Example 3.3. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the edge-signed triangles having exactly one $(+)$-edge and exactly two $(+)$-edges, respectively, i.e., $V(\mathcal{T}_1) = V(\mathcal{T}_2) = \{v_1, v_2, v_3\}$, $E^+(\mathcal{T}_1) = E^-(\mathcal{T}_2) = \{\{v_1, v_2\}\}$, and $E^-(\mathcal{T}_1) = E^+(\mathcal{T}_2) = \{\{v_1, v_3\}, \{v_2, v_3\}\}$ (see Figure 4).

For $\epsilon_1, \epsilon_2, \epsilon_3 \in \{1, -1\}$ and $\delta \in \{0, \pm 1\}$, let $\mathcal{S}_1(\epsilon_1, \epsilon_2, \epsilon_3)$, $\mathcal{S}_2(\epsilon_1, \epsilon_2, \delta)$, $\mathcal{S}_3(\epsilon_1, \epsilon_2, \delta)$, and $\mathcal{S}_4(\epsilon_1, \epsilon_2)$ be the edge-signed graphs in Figure 5, where an edge with label 1 (resp. -1) represents a $(+)$-edge (resp. a $(-)$-edge) and an edge with label 0 represents a non-adjacent pair.

It can be checked that each of the edge-signed graphs $\mathcal{T}_2$, $\mathcal{S}_1(\epsilon_1, \epsilon_2, \epsilon_3)$, $\mathcal{S}_2(\epsilon_1, \epsilon_2, \delta)$, $\mathcal{S}_3(\epsilon_1, \epsilon_2, \delta)$, $\mathcal{S}_4(\epsilon_1, \epsilon_2)$ has the smallest eigenvalue less than $-\tau$.



$\mathcal{T}_1$ $\mathcal{T}_2$

Figure 4: The edge-signed triangles $\mathcal{T}_1$ and $\mathcal{T}_2$

Theorem 3.4. *Let $\mathcal{S}$ be a connected edge-signed graph with $\lambda_{\min}(\mathcal{S}) \ge -\tau$. Assume that $\mathcal{S}$ does not contain an induced edge-signed subgraph isomorphic to $\mathcal{T}_1$. Then either $\mathcal{S}$ is isomorphic to $\mathcal{Q}_{p,q,r}$ for some non-negative integers p, q, r with $p + q \le r$, or $\mathcal{S}$ has at most* 6 *vertices and is isomorphic to one of the* 15 *edge-signed graphs in Figure 6.*

Proof. By using computer [14], we checked that the theorem holds when $|V(\mathcal{S})| \le 7$. We prove the assertion by induction on $|V(\mathcal{S})|$. Assume that the assertion holds for $|V(\mathcal{S})| = n$ (≥ 7). Suppose that $|V(\mathcal{S})| = n + 1$. It follows from Problem 6(a) in Section 6 of [9] that there exists a vertex v which is not a cut vertex of $\mathcal{S}$. Then $\mathcal{S} - v$ is connected, where $\mathcal{S} - v$ is the edge-signed subgraph induced by $V(\mathcal{S}) \setminus \{v\}$. Since $\lambda_{\min}(\mathcal{S} - v) \ge \lambda_{\min}(\mathcal{S}) \ge -\tau$, the inductive hypothesis implies that $\mathcal{S} - v$ is isomorphic to $\mathcal{Q}_{p,q,r}$ for some p, q, r with $p + q + r = n$. Thus $\mathcal{S}$ is the edge-signed graph obtained from $\mathcal{Q}_{p,q,r}$ by adding the vertex v and signed edges between v and some vertices in $\mathcal{Q}_{p,q,r}$. Note that $r \ge 4$ since $n = p + q + r \ge 7$ and $p + q \le r$.

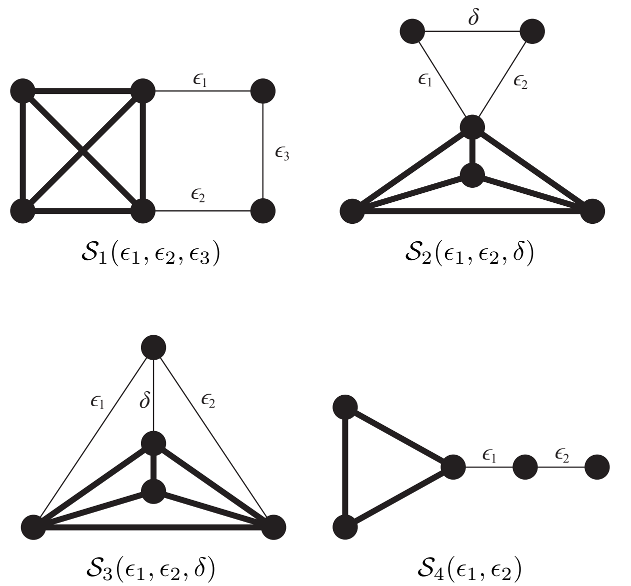


Figure 5: Edge-signed graphs with smallest eigenvalue less than $-\tau$

We claim that either v is adjacent to only one vertex of V_r, or to all the vertices of V_r. Note that $\mathcal{S}$ cannot contain any of the edge-signed graphs $\mathcal{T}_2$, $\mathcal{S}_1(\epsilon_1, \epsilon_2, \epsilon_3)$, $\mathcal{S}_2(\epsilon_1, \epsilon_2, \delta)$, $\mathcal{S}_3(\epsilon_1, \epsilon_2, \delta)$, $\mathcal{S}_4(\epsilon_1, \epsilon_2)$ in Example 3.3. If v is adjacent to none of the vertices of V_r, then $\mathcal{S}$ contains $\mathcal{S}_4(\epsilon_1, \epsilon_2)$ as an induced edge-signed subgraph, a contradiction. If the number of neighbors of v in V_r is at least 2 and less than r, then $\mathcal{S}$ contains $\mathcal{S}_3(\epsilon_1, \epsilon_2, \delta)$ as an induced edge-signed subgraph, a contradiction. Thus the claim holds.

Now, if v is adjacent to only one vertex of V_r, then the unique neighbor of v in V_r is in $V_r \setminus (U_p \cup U_q)$. Indeed, otherwise we would find $\mathcal{S}_2(\epsilon_1, \epsilon_2, \delta)$ as an induced edge-signed subgraph, a contradiction. Also, v is adjacent to none of the vertices of $V_p \cup V_q$ since otherwise we would find $\mathcal{S}_1(\epsilon_1, \epsilon_2, \epsilon_3)$ as an induced edge-signed subgraph, a contradiction. Thus $\mathcal{S}$ is isomorphic to $\mathcal{Q}_{p+1,q,r}$ or $\mathcal{Q}_{p,q+1,r}$.

Suppose that v is adjacent to all the vertices of V_r. Since V_r is a clique consisting of $(+)$-edges only, the assumption implies that v is incident with at most one $(-)$-edge to V_r. If there is a vertex of V_r joined to v by a $(-)$-edge, then we find $\mathcal{T}_2$ as an induced edge-signed subgraph, a contradiction. Thus all the edges from v to V_r are $(+)$-edges. Now v is adjacent to none of the vertices of $V_p \cup V_q$ since otherwise we would find $\mathcal{S}_3(\epsilon_1, \epsilon_2, 0)$ as an induced edge-signed subgraph, a contradiction. Thus $\mathcal{S}$ is isomorphic to $\mathcal{Q}_{p,q,r+1}$. Hence the theorem holds. □

Lemma 3.5. *The smallest eigenvalues of the signed adjacency matrices of the edge-signed graphs in Figure 6 are given as follows:*

$$\lambda_{\min}(\mathcal{S}) = \begin{cases} -\sqrt{2} & \approx -1.414213 & \text{if } \mathcal{S} \in \{\mathcal{S}_{3,1}, \mathcal{S}_{4,4}\}, \\ \frac{1-\sqrt{17}}{2} & \approx -1.561553 & \text{if } \mathcal{S} \in \{\mathcal{S}_{4,5}, \mathcal{S}_{5,5}, \mathcal{S}_{5,6}\}, \\ 1+t & \approx -1.601679 & \text{if } \mathcal{S} = \mathcal{S}_{6,3}, \\ -\tau & \approx -1.618034 & \text{otherwise}, \end{cases}$$

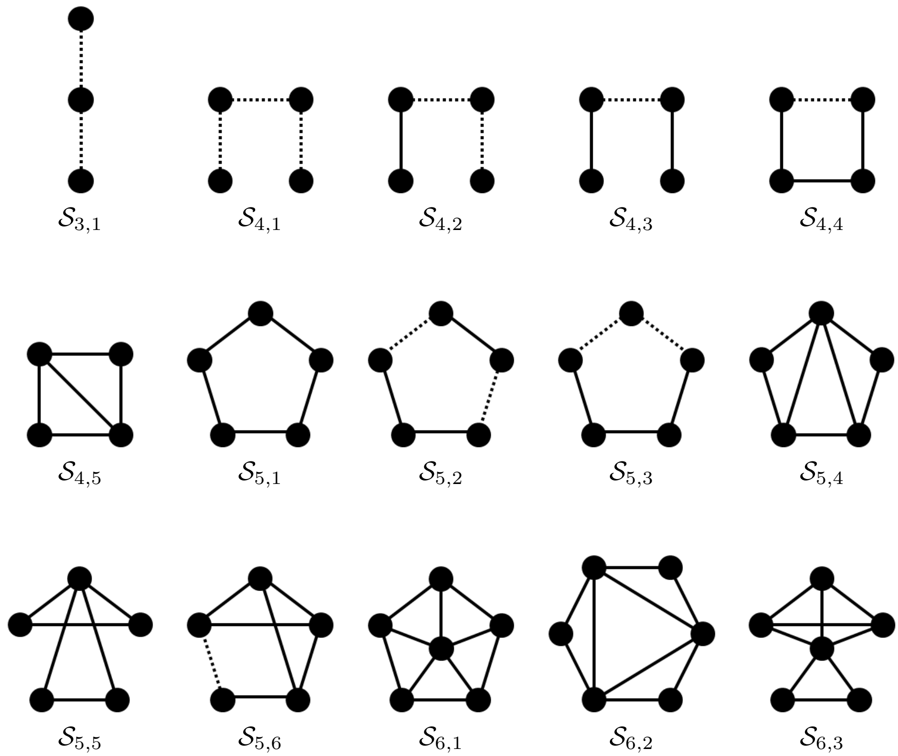


Figure 6: The 15 connected edge-signed graphs with smallest eigenvalue at least $-\tau$ not containing $\mathcal{T}_1$, other than $\mathcal{Q}_{p,q,r}$ where p, q, r are some non-negative integers with $p+q \leq r$

where t is the smallest zero of the polynomial $x^3 - 6x + 2$.

Proof. This can be checked by a direct calculation. □

Remark 3.6. Among edge-signed graphs in Figure 6, the maximal ones with respect to taking induced edge-signed graphs are $\mathcal{S}_{4,1}$, $\mathcal{S}_{5,2}$, $\mathcal{S}_{5,3}$, $\mathcal{S}_{5,6}$, $\mathcal{S}_{6,1}$, $\mathcal{S}_{6,2}$, $\mathcal{S}_{6,3}$.

3.2 The special graphs of fat $(-1-\tau)$-irreducible Hoffman graphs

Lemma 3.7. *Let $\mathfrak{H}$ be a Hoffman graph with smallest eigenvalue at least $-1-\tau$. Then every slim vertex of $\mathfrak{H}$ has at most two fat neighbors.*

Proof. If a slim vertex v of $\mathfrak{H}$ has at least 3 fat neighbors, then $\mathfrak{H}$ contains an induced Hoffman subgraph isomorphic to the Hoffman graph $\mathfrak{K}_{1,3}$ (see Example 2.13). By Lemma 2.6, we have $\lambda_{\min}(\mathfrak{H}) \leq \lambda_{\min}(\mathfrak{K}_{1,3}) = -3$, which is a contradiction to $\lambda_{\min}(\mathfrak{H}) \geq -1-\tau$. □

Lemma 3.8. *Let $\mathcal{S}$ be a connected edge-signed graph with three vertices. Let D be a 3×3 diagonal matrix with diagonal entries 1 or 2 such that at least one of the diagonal entries is 2. Then $M(\mathcal{S}) - D$ has the smallest eigenvalue less than $-1-\tau$.*

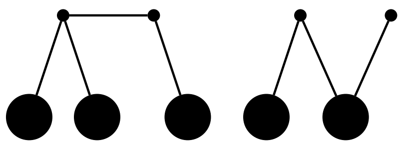
Proof. This can be checked by a direct calculation. □

Lemma 3.9. *Let* $\mathfrak{H}$ *be a fat indecomposable Hoffman graph with smallest eigenvalue at least* $-1-\tau$*. If some slim vertex of* $\mathfrak{H}$ *has at least two fat neighbors, then the special graph* $\mathcal{S}(\mathfrak{H})$ *of* $\mathfrak{H}$ *is isomorphic to* $\mathcal{Q}_{0,0,1}$*,* $\mathcal{Q}_{1,0,1}$*, or* $\mathcal{Q}_{0,1,1}$*.*

Proof. In view of Lemma 2.6, it suffices to show that every fat indecomposable Hoffman graph with three slim vertices, in which some slim vertex has two fat neighbors, has the smallest eigenvalue less than $-1-\tau$.

Let $\mathfrak{H}$ be such a Hoffman graph. Then $\mathcal{S}(\mathfrak{H})$ is connected by Lemma 2.21 and $B(\mathfrak{H}) = M(\mathcal{S}(\mathfrak{H})) - D$ for some diagonal matrix D with diagonal entries 1 or 2 such that at least one of the diagonal entries is 2. Then we have a contradiction by Lemma 3.8. □

Example 3.10. Let $\mathfrak{H}_{\mathrm{XVI}}$ and $\mathfrak{H}_{\mathrm{XVII}}$ be the Hoffman graphs in Figure 7. The special graphs of $\mathfrak{H}_{\mathrm{XVI}}$ and $\mathfrak{H}_{\mathrm{XVII}}$ are $\mathcal{Q}_{1,0,1}$ and $\mathcal{Q}_{0,1,1}$, respectively, and $\lambda_{\min}(\mathfrak{H}_{\mathrm{XVI}}) = \lambda_{\min}(\mathfrak{H}_{\mathrm{XVII}}) = -1-\tau$.



$\mathfrak{H}_{\mathrm{XVI}}$ $\mathfrak{H}_{\mathrm{XVII}}$

Figure 7: Fat indecomposable Hoffman graphs

Lemma 3.11. *Let* $\mathfrak{H}$ *be a Hoffman graph in which every slim vertex has at most one fat neighbor. Then the special graph* $\mathcal{S}(\mathfrak{H})$ *of* $\mathfrak{H}$ *does not contain an induced edge-signed subgraph isomorphic to* $\mathcal{T}_1$*.*

Proof. Suppose that the special graph $\mathcal{S}(\mathfrak{H})$ of $\mathfrak{H}$ contains $\mathcal{T}_1 = (\{v_1, v_2, v_3\}, \{\{v_1, v_2\}\}, \{\{v_1, v_3\}, \{v_2, v_3\}\})$ as an induced edge-signed subgraph. Since v_3 is incident to a $(-)$-edge, v_3 must have a fat neighbor. Since every slim vertex of $\mathfrak{H}$ has at most one fat neighbor, v_3 has a unique fat neighbor f. Then f is adjacent to v_1 and v_2. This is a contradiction to $\{v_1, v_2\} \in E^+(\mathcal{T}_1)$. □

Lemma 3.12. *Let* $\mathfrak{H}$ *be a Hoffman graph in which every slim vertex has exactly one fat neighbor. If the special graph* $\mathcal{S}(\mathfrak{H})$ *of* $\mathfrak{H}$ *is isomorphic to* $\mathcal{Q}_{p,q,r}$ *for some non-negative integers* p, q, r*, then* $\mathfrak{H}$ *is an induced Hoffman subgraph of a Hoffman graph* $\mathfrak{H}'$ *with* $V^s(\mathfrak{H}') = V^s(\mathfrak{H})$ *which has a decomposition* $\{\mathfrak{H}^i\}_{i=1}^r$ *such that* $\mathfrak{H}^i$ *is isomorphic to* $\mathfrak{H}_{\mathrm{XVI}}$*,* $\mathfrak{H}_{\mathrm{XVII}}$*, or* $\mathfrak{H}_{\mathrm{II}}$ *for all* $i = 1, \dots, r$*. In particular, if* $r \geq 2$*, then* $\mathfrak{H}$ *is* $(-1-\tau)$*-reducible.*

Proof. By the assumption, $V^s(\mathfrak{H}) = V(\mathcal{S}(\mathfrak{H}))$ is partitioned into $V_p \cup V_q \cup V_r$ as Definition 3.1. Consider the Hoffman graph $\mathfrak{H}'$ defined by $V^s(\mathfrak{H}') := V^s(\mathfrak{H})$, $V^f(\mathfrak{H}') := V^f(\mathfrak{H}) \cup \{f^*\}$, and $E(\mathfrak{H}') := E(\mathfrak{H}) \cup \{\{v, f^*\} \mid v \in V_r\}$, where f^* is a new fat vertex. Note that $\mathfrak{H}$ is an induced Hoffman subgraph of $\mathfrak{H}'$ with $V^s(\mathfrak{H}) = V^s(\mathfrak{H}')$. Then $\mathfrak{H}'$ has a decomposition $\{\mathfrak{H}^i\}_{i=1}^r$ with $\mathfrak{H}^i \cong \mathfrak{H}_{\mathrm{XVI}}$ for $1 \leq i \leq p$, $\mathfrak{H}^i \cong \mathfrak{H}_{\mathrm{XVII}}$ for $p < i \leq p+q$, and $\mathfrak{H}^i \cong \mathfrak{H}_{\mathrm{II}}$ for $p+q < i \leq p+q+r$ (see Examples 2.4 and 3.10). Since $r \geq 2$ and each of the Hoffman graphs $\mathfrak{H}^i$ has the smallest eigenvalue at least $-1-\tau$, it follows that $\mathfrak{H}$ is $(-1-\tau)$-reducible. □

Theorem 3.13. *Let $\mathfrak{H}$ be a fat indecomposable Hoffman graph with smallest eigenvalue at least $-1-\tau$. Then the following hold:*

(i) *If some slim vertex of $\mathfrak{H}$ has at least two fat neighbors, then the special graph $\mathcal{S}(\mathfrak{H})$ of $\mathfrak{H}$ is isomorphic to $\mathcal{Q}_{0,0,1}$, $\mathcal{Q}_{1,0,1}$, or $\mathcal{Q}_{0,1,1}$.*

(ii) *If every slim vertex of $\mathfrak{H}$ has exactly one fat neighbor, then the special graph $\mathcal{S}(\mathfrak{H})$ of $\mathfrak{H}$ is isomorphic to $\mathcal{Q}_{p,q,r}$ for some non-negative integers p, q, r with $p+q \le r$ or one of the 15 edge-signed graphs in Figure 6.*

Proof. The statement (i) follows from Lemma 3.9. We show (ii). Suppose that every slim vertex of $\mathfrak{H}$ has exactly one fat neighbor. By Lemma 2.21, $\mathcal{S}(\mathfrak{H})$ is connected, and by Lemma 2.25, $\mathcal{S}(\mathfrak{H})$ has smallest eigenvalue at least $-\tau$. Moreover, by Lemma 3.11, $\mathcal{S}(\mathfrak{H})$ does not contain an induced edge-signed subgraph isomorphic to $\mathcal{T}_1$. Now Theorem 3.4 implies that $\mathcal{S}(\mathfrak{H})$ is isomorphic $\mathcal{Q}_{p,q,r}$ or one of the 15 edge-signed graphs in Figure 6. □

Corollary 3.14. *Let $\mathfrak{H}$ be a fat $(-1-\tau)$-irreducible Hoffman graph. Then the special graph $\mathcal{S}(\mathfrak{H})$ of $\mathfrak{H}$ is isomorphic to $\mathcal{Q}_{0,0,1}$, $\mathcal{Q}_{1,0,1}$, $\mathcal{Q}_{0,1,1}$, or one of the 15 edge-signed graphs in Figure 6.*

Proof. Since $\mathfrak{H}$ is $(-1-\tau)$-irreducible, $\mathfrak{H}$ is indecomposable. If some slim vertex of $\mathfrak{H}$ has at least two fat neighbors, then the statement holds by Theorem 3.13 (i). Suppose that every slim vertex of $\mathfrak{H}$ has exactly one fat neighbor. By Theorem 3.13 (ii), $\mathcal{S}(\mathfrak{H})$ is isomorphic to $\mathcal{Q}_{p,q,r}$ for some non-negative integers p, q, r, or one of the 15 edge-signed graphs in Figure 6. Since $\mathfrak{H}$ is $(-1-\tau)$-irreducible, the former case occurs only for $r = 1$ by Lemma 3.12. Hence the corollary holds. □

3.3 The classification of fat Hoffman graphs with smallest eigenvalue at least $-1-\tau$

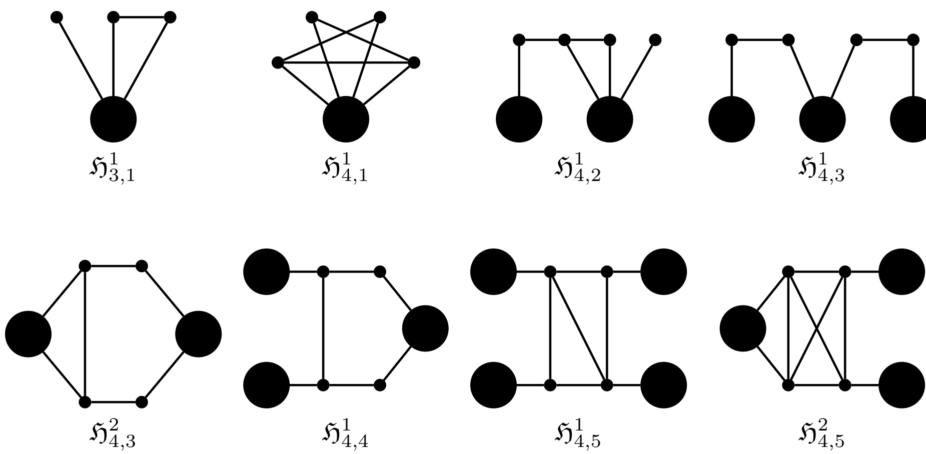


Figure 8: The fat $(-1-\tau)$-irreducible Hoffman graphs with 3 or 4 slim vertices

Lemma 3.15. *Let $\mathfrak{H}$ be a fat indecomposable Hoffman graph with smallest eigenvalue at least $-1-\tau$. If the number of slim vertices of $\mathfrak{H}$ is at most two, then $\mathfrak{H}$ is isomorphic to one of $\mathfrak{H}_{\mathrm{I}}$, $\mathfrak{H}_{\mathrm{II}}$, $\mathfrak{H}_{\mathrm{III}}$, $\mathfrak{H}_{\mathrm{IV}}$, $\mathfrak{H}_{\mathrm{XVI}}$, and $\mathfrak{H}_{\mathrm{XVII}}$.*

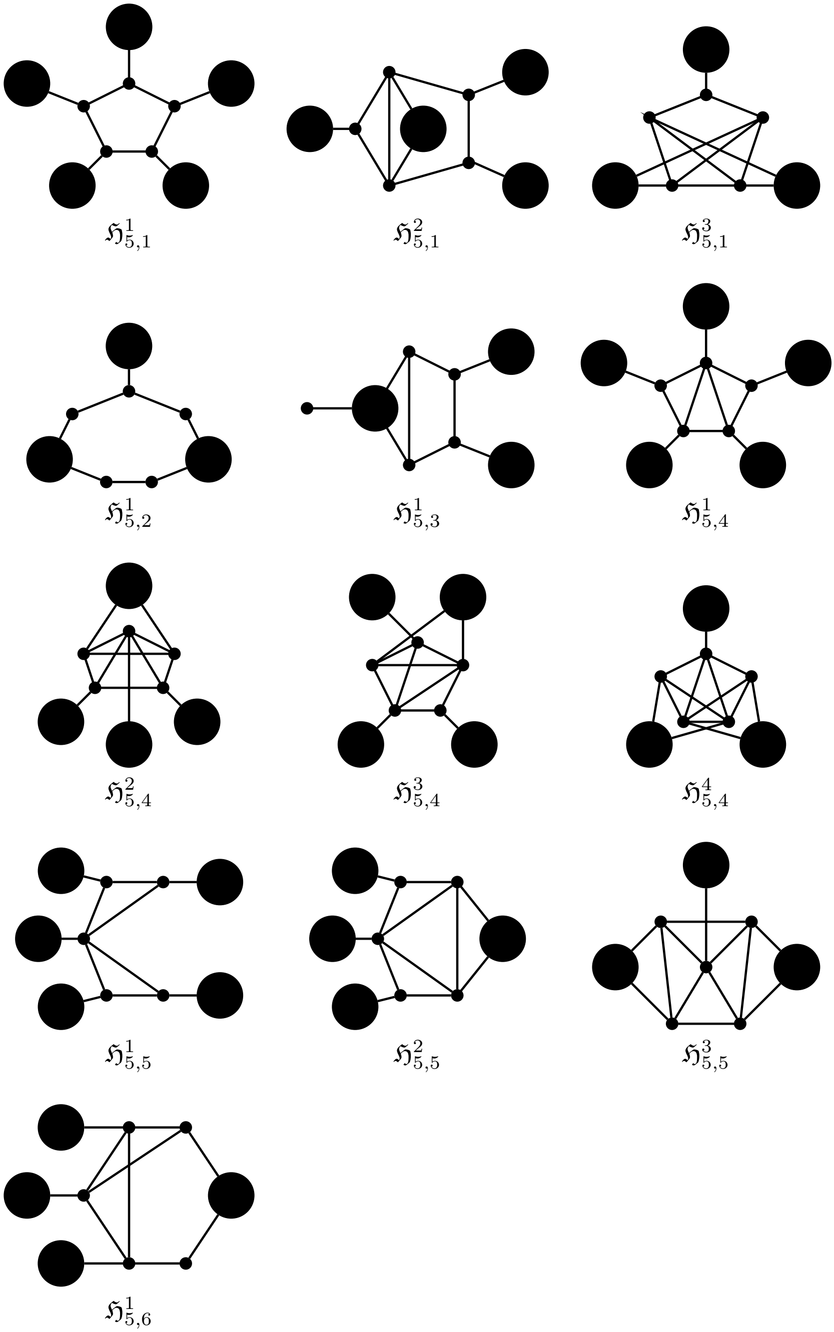


Figure 9: The fat $(-1-\tau)$-irreducible Hoffman graphs with 5 slim vertices

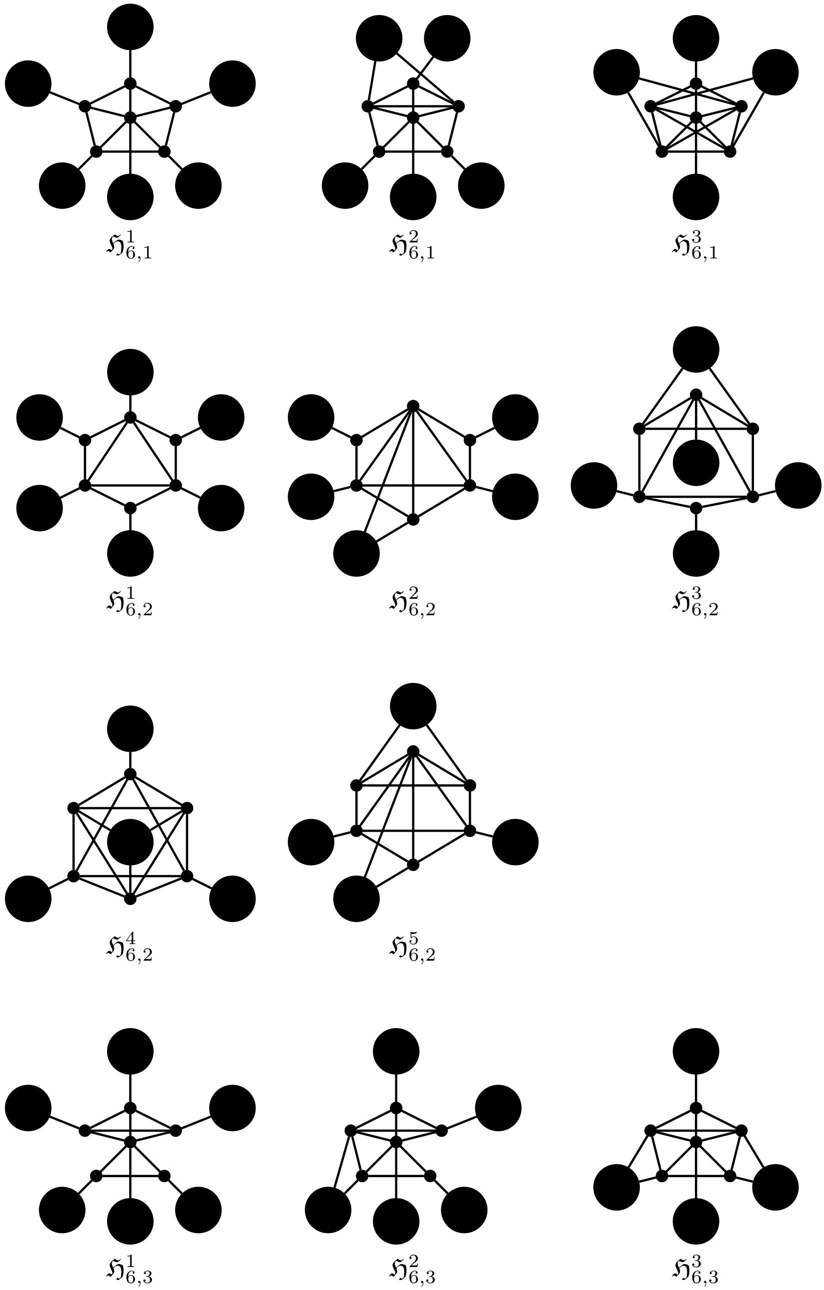


Figure 10: The fat $(-1-\tau)$-irreducible Hoffman graphs with 6 slim vertices

Proof. Straightforward. □

Lemma 3.16. *Let $\mathfrak{H}$ be a fat $(-1-\tau)$-irreducible Hoffman graph. If $\mathcal{S}(\mathfrak{H})$ is isomorphic to $\mathcal{S}_{i,j}$ in Figure 6, then $\mathfrak{H}$ is isomorphic to $\mathfrak{H}^k_{i,j}$ for some k in Figures 8, 9, and 10.*

Proof. By Lemma 3.9, every slim vertex has exactly one fat neighbor. It is then straightforward to establish the lemma. □

Theorem 3.17. *Let $\mathfrak{H}$ be a fat $(-1-\tau)$-irreducible Hoffman graph. Then $\mathfrak{H}$ is isomorphic to one of $\mathfrak{H}_{\mathrm{I}}$, $\mathfrak{H}_{\mathrm{II}}$, $\mathfrak{H}_{\mathrm{III}}$, $\mathfrak{H}_{\mathrm{XVI}}$, $\mathfrak{H}_{\mathrm{XVII}}$, and the 32 Hoffman graphs given in Figures 8, 9, and 10.*

Proof. By Corollary 3.14, the special graph $\mathcal{S}(\mathfrak{H})$ of $\mathfrak{H}$ is isomorphic to $\mathcal{Q}_{0,0,1}$, $\mathcal{Q}_{1,0,1}$, $\mathcal{Q}_{0,1,1}$, or one of the 15 edge-signed graphs in Figure 6. If the number of slim vertices of $\mathfrak{H}$ is at most two, then the statement holds by Lemma 3.15 and Example 2.15. If the number of slim vertices of $\mathfrak{H}$ is at least three, then the statement holds by Lemma 3.16. □

Theorem 3.18. *Let $\mathcal{H}$ be the set of isomorphism classes of the maximal members of the 37 fat $(-1-\tau)$-irreducible Hoffman graphs given in Theorem 3.17, with respect to taking induced Hoffman subgraphs. More precisely, $\mathcal{H}$ is the set of isomorphism classes of $\mathfrak{H}_{\mathrm{XVI}}$, $\mathfrak{H}_{\mathrm{XVII}}$, $\mathfrak{H}^1_{4,1}$, $\mathfrak{H}^2_{4,3}$, $\mathfrak{H}^1_{5,2}$, $\mathfrak{H}^1_{5,3}$, $\mathfrak{H}^1_{5,6}$, and the 11 Hoffman graphs in Figure 10. Then every fat Hoffman graph with smallest eigenvalue at least $-1-\tau$ is an $\mathcal{H}$-line graph.*

Proof. It suffices to show that every fat indecomposable Hoffman graph $\mathfrak{H}$ with smallest eigenvalue at least $-1-\tau$ is an $\mathcal{H}$-line graph. First suppose that some slim vertex of $\mathfrak{H}$ has two fat neighbors. Then by Lemma 3.9, $\mathfrak{H}$ has at most two slim vertices, and by Lemma 3.15, $\mathfrak{H}$ is isomorphic to one of $\mathfrak{H}_{\mathrm{I}}$, $\mathfrak{H}_{\mathrm{II}}$, $\mathfrak{H}_{\mathrm{III}}$, $\mathfrak{H}_{\mathrm{IV}}$, $\mathfrak{H}_{\mathrm{XVI}}$, and $\mathfrak{H}_{\mathrm{XVII}}$. Since $\mathfrak{H}_{\mathrm{I}}$ and $\mathfrak{H}_{\mathrm{II}}$ are induced Hoffman subgraphs of $\mathfrak{H}_{\mathrm{XVI}}$, and $\mathfrak{H}_{\mathrm{III}}$ is an induced Hoffman subgraph of $\mathfrak{H}_{\mathrm{XVII}}$, they are $\mathcal{H}$-line graphs. Note that $\mathfrak{H}_{\mathrm{IV}}$ is also an $\mathcal{H}$-line graph since Example 2.15 shows that $\mathfrak{H}_{\mathrm{IV}}$ is an induced Hoffman subgraph of a Hoffman graph having a decomposition into two induced Hoffman subgraphs isomorphic to $\mathfrak{H}_{\mathrm{I}}$. Thus the result holds in this case.

Next suppose that every slim vertex of $\mathfrak{H}$ has exactly one fat neighbor. Then by Theorem 3.13 (ii), $\mathcal{S}(\mathfrak{H})$ is isomorphic to $\mathcal{Q}_{p,q,r}$ for some non-negative integers p, q, r with $p+q \le r$ or one of the 15 edge-signed graphs in Figure 6. In the former case, $\mathfrak{H}$ is an $\mathcal{H}$-line graph by Lemma 3.12. In the latter case, Lemma 3.16 implies that $\mathfrak{H}$ is an $\mathcal{H}$-line graph since $\mathcal{H}$ contains all the maximal members of the isomorphism classes of Hoffman graphs $\mathfrak{H}^k_{i,j}$. □

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Mathematical Chemistry Issue

Dedicated to the Memory of Ante Graovac

Split, 15.7.1945 – Zagreb, 13.11.2012

In memory of our dear colleague, member of the advisory board of our journal, and a leading expert and promotor of mathematical chemistry Ante Graovac, we will publish a special issue of AMC dedicated to topics from mathematical chemistry with special emphasis on areas related to the work of Ante.

Papers will be subject to our standard editorial procedure. In the pre-screening phase, one or two experts will be asked for a quick overall assessment of the contribution. If the opinion is reached that the paper does not fall within the scope of this special issue, the authors may be advised to submit it to a regular issue of the AMC or to some more appropriate journal. For papers that pass the initial screening, two referees will be assigned. At least one of them will be tasked to judge the mathematical content of the contribution. If for one reason or another the refereeing process for a particular paper requires more time than would allow the issue to be completed on schedule, that paper may be transferred to a regular issue of our journal.

We are seeking high-quality research articles or substantial surveys of significant topics of mathematical chemistry. The deadline for submission of papers is 31st December 2014. The special issue will be published by the end of 2015.

Patrick Fowler, Sheffield, UK
Tomaž Pisanski, Ljubljana and Koper, SI

Guest Editors

ANNOUNCEMENT

PhD Fellowship ("Young Researcher" position) at the University of Primorska, Slovenia

The University of Primorska announces three "Young researcher" positions under the supervision of

- Dragan Marušič (Algebra and Discrete Mathematics);
- Štefko Miklavič (Algebra and Discrete Mathematics);
- Enes Pašalić (Cryptography).

Applicants should have a BSc or equivalent training (by September 2014). Applicants for "Young researcher" positions in mathematics are expected to enroll in the PhD program at UP FAMNIT.

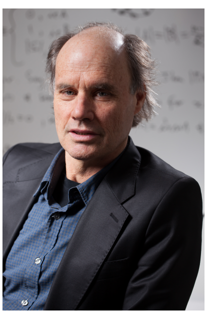
The positions are for 3 and 1/2 years and include a tuition fee waiver. The holder is expected to complete his/hers PhD training in 4 years.

The deadline for applications will be in June 2014 (the exact date yet to be confirmed). Applicants should send a letter of interest with CV and two recommendation letters to:

"Young Researcher position"
University of Primorska, UP IAM
Muzejski trg 2
6000 Koper
Slovenia

The application should also be sent by e-mail to the address olga.kaliada@upr.si.

For any additional information please contact Olga Kaliada at
Phone: +386 5 611 7599)
Fax: +386 5 611 7592
Email: olga.kaliada@upr.si



Photos: Miha Peroša

Symmetries of Graphs and Networks (SYGN IV) and 2014 PhD Summer School in Discrete Mathematics

Rogla, Slovenia, June 29 – July 5, 2014

http://www.famnit.upr.si/sl/konference

INVITED SPEAKERS AT SYGN IV WORKSHOP:

Marston Conder (University of Auckland, New Zealand),
Shaofei Du (Capital Normal University, China),
Yan Quan Feng (Beijing Jiaotong University, China),
Michael Giudici (University of Western Australia, Australia),
Isabel Hubard (IMATE-UNAM, Mexico),
György Kiss (Eötvös Loránd University, Hungary),
Jin Ho Kwak (POSTECH, South Korea),
Young Soo Kwon (Yeungnam University, South Korea),
Cai-Heng Li (University of Western Australia, Australia)
Luis Martínez Fernández (University of the Basque Country, Spain),
Joy Morris (University of Lethbridge, Canada),
Tamás Szönyi (Eötvös Loránd University, Hungary),
Paul Terwilliger (University of Wisconsin-Madison, USA),
Steve Wilson (Northern Arizona University, USA),
Dave Witte Morris (University of Lethbridge, Canada).

SUMMER SCHOOL PROGRAMME: Aimed at bringing PhD students to several open problems in the active research areas, four minicourses (6 hour of lectures each) will be given on the following topics:

- *Construction techniques for graph embeddings* (Mark Ellingham, Vanderbilt University, USA),
- *Combinatorial designs* (Mariusz Meszka, AGH University of Science and Technology, Poland),
- *Some topics in the theory of finite groups* (Primož Moravec, University of Ljubljana, Slovenia),
- *Symmetric key cryptography and its relation to graph theory* (Enes Pasalic, University of Primorska, Slovenia).

In addition to lectures, time will also be devoted to workshop sessions and students presentations.

VENUE: Both events will take place at Rogla, a highland in the north-eastern part of Slovenija, located 130 km by road from Slovenian capital Ljubljana. At around 1500m above sea level, the beautiful natural scenery of Rogla provides pleasant climate conditions and stimulating working environment.

ORGANIZED BY University of Primorska, UP IAM and UP FAMNIT, in collaboration with Centre for Discrete Mathematics UL PeF.

SCIENTIFIC COMMITTEE: K. Kutnar, A. Malnič, D. Marušič, Š. Miklavič, P. Šparl.

ORGANIZING COMMITTEE: I. Antončič, A. Hujdurović, B. Frelih, B. Kuzman.

Sponsored by Slovenian National Research Agency (ARRS) and Ministry of Education, Science and Sport (MIZS).

For more information, visit our website or email your inquiry to sygn@upr.si.

First Announcement

8th Slovenian International Conference on Graph Theory

Organized by

IMFM – Institute of Mathematics, Physics and Mechanics

in collaboration with

UL FMF – University of Ljubljana, Faculty of Mathematics and Physics
UL Pef – University of Ljubljana, Faculty of Education
UM FNM – University of Maribor, Faculty of Natural Sciences and Mathematics
UP FAMNIT – University of Primorska, Faculty of Mathematics, Natural Sciences and Information Technologies

Important dates

Duration: June 21 – 27, 2015

Conference venue

The conference will take place in Kranjska Gora, Slovenia

Additional information

Latest information on the conference
will be available at the conference web page

kg15.imfm.si

Inquiries about the conference should be addressed by e-mail to

kranjska.gora2015@gmail.com

Scientific committee

Sandi Klavžar • Dragan Marušič • Bojan Mohar • Tomaž Pisanski

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