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Complete forcing numbers of graphs*

Xin He, Heping Zhang †

School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, P.R. China

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Abstract

The complete forcing number of a graph G with a perfect matching is the minimum cardinality of an edge set of G on which the restriction of each perfect matching M is a forcing set of M. This concept can be view as a strengthening of the concept of global forcing number of G. Došlić in 2007 obtained that the global forcing number of a connected graph is at most its cyclomatic number. Motivated from this result, we obtain that the complete forcing number of a graph is no more than 2 times its cyclomatic number and characterize the matching covered graphs whose complete forcing numbers attain this upper bound and minus one, respectively. Besides, we present a method of constructing a complete forcing set of a graph. By using such method, we give closed formulas for the complete forcing numbers of wheels and cylinders.

Keywords: Perfect matching, global forcing number, complete forcing number, cyclomatic number, wheel, cylinder.

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1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *matching* of G is a set of disjoint edges of G. A *perfect matching* M of G is a matching that covers all vertices of G. An edge of G is termed *allowed* if it lies in some perfect matching of G and *forbidden* otherwise. A *forcing set* of M is a subset of M contained in no other perfect matching of G. The *forcing number* of M is the minimum possible cardinality of forcing sets of M. We may refer to a survey [6] on this topic. A subset $S \subseteq E(G) \setminus M$ is called an *anti-forcing set* of M [14] if $G - S$ has a unique perfect matching M. The *anti-forcing number* of M is the smallest cardinality of anti-forcing sets of M.

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†Corresponding author.

E-mail addresses: hex2015@lzu.edu.cn (Xin He), zhanghp@lzu.edu.cn (Heping Zhang)

Let G be a graph with a perfect matching. Concerning all perfect matchings of G, Vukičević et al. [21, 22] introduced the concept of *global (or total) forcing set*, which is defined as a subset of $E(G)$ on which there are no two distinct perfect matchings coinciding. The minimum possible cardinality of global forcing sets is called the *global forcing number* of G. For more about the global forcing number of a graph, the reader is referred to [3, 7, 20, 27].

As a strengthening of the concept of global forcing set of G, Xu et al. [24] proposed the concept of the *complete forcing set* of G, which is defined as a subset of $E(G)$ on which the restriction of each perfect matching M is a forcing set of M. A complete forcing set with the minimum cardinality is called a *minimum complete forcing set* of G, and its cardinality is called the *complete forcing number* of G, denoted by $cf(G)$. If G has at least two perfect matchings, then $cf(G)$ is larger than the global forcing number of G [24].

A subgraph G_0 of G is said to be *nice* if $G - V(G_0)$ has a perfect matching. Obviously, an even cycle C of G is nice if and only if there is a perfect matching M of G such that $E(C) \cap M$ is a perfect matching of C. We call each of the two perfect matchings of C a *frame* (or a typeset [24]) of C, which was ever used in [1] to obtain a min-max theorem for the Clar problem on 2-connected plane bipartite graphs.

Xu et al. established the following equivalent condition for a subset of edges of a graph to be a complete forcing set.

Theorem 1.1 ([24]). *Let G be a graph with a perfect matching. Then $S \subseteq E(G)$ is a complete forcing set of G if and only if for any nice cycle C of G, the intersection of S and each frame of C is nonempty.*

Let S be a complete forcing set of G. For a perfect matching M of G, from Theorem 1.1, $S \setminus M$ contains at least one edge of every M-alternating cycle of G. By Lemma 2.1 of [14], $S \setminus M$ is an anti-forcing set of M. So a complete forcing set of G both forces and antiforces each perfect matching. Further, Chan et al. [4] obtained that the complete forcing number of a catacondensed hexagonal system is equal to the number of hexagons plus the Clar number and presented a linear-time algorithm for computing it. Besides, some certain explicit formulas for the complete forcing numbers of rectangular polynominoes, polyphenyl systems, spiro hexagonal systems and primitive coronoids have been derived [5, 15, 16, 23]. In recent papers [11, 12], we established a sufficient condition for an edge set of a hexagonal system (HS) to be a complete forcing set in terms of elementary edge-cut cover, which yields a tight upper bound on the complete forcing numbers of HSs. For a normal HS, we gave two lower bounds on its complete forcing number by the number of hexagons and matching numbers of some subgraphs of its inner dual graph, respectively. In addition, we showed that the complete forcing numbers of catacondensed HSs, normal HSs without 2×3 subsystems, parallelogram, regular hexagon- and rectangle-shaped HSs attain one of the two above lower bounds.

Let $c(G) = |E(G)| - |V(G)| + \omega(G)$ denote the *cyclomatic number* of G, where $\omega(G)$ is the number of components of G. In 2007, Došlić [7] obtained that the global forcing number of a connected graph is at most its cyclomatic number and gave a characterization: a connected (bipartite) graph has the global forcing number attaining its cyclomatic number if and only if each cycle is nice (such graphs are called 1-cycle resonant graphs; see [9]). As a corollary, the global forcing number of any catacondensed HS is equal to the number of hexagons.

Motivated by Došlić's result, in this paper we obtain that the complete forcing number of a graph is no more than 2 times its cyclomatic number by presenting a method of constructing a complete forcing set of a graph (see the next section). Moreover, in Section 3, we show that the complete forcing number of a matching covered graph attains the above upper bound if and only if such graph is either K_2 (a complete graph with 2 vertices) or an even cycle. Besides, we characterize the matching covered graphs whose complete forcing numbers are equal to 2 times their cyclomatic numbers minus 1 in terms of ear decomposition. In the last section, we present some lower bounds on the complete forcing numbers of some types of graphs including plane elementary bipartite graphs and cylinders. Combining such methods, we give closed formulas for the complete forcing numbers of wheels and cylinders.

2 An upper bound on complete forcing number

All graphs considered in this paper are simple and all the bipartite graphs are given a proper black and white coloring: any two adjacent vertices receive different colors.

Let G be a graph. Suppose that V' is a nonempty subset of $V(G)$. The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both end-vertices in V' is called the subgraph of G *induced* by V' and is denoted by $G[V']$. The induced subgraph $G[V \setminus V']$ is denoted by $G - V'$. For $E' \subseteq E(G)$, the spanning subgraph $(V(G), E(G) \setminus E')$ is denoted by $G - E'$ [2]. For a nonempty proper subset V' of $V(G)$, the set of all edges of G having exactly one end-vertex in V' is called an *edge cut* of G and denoted by $\partial_G(V')$ (or simply $\partial(V')$).

For $v \in V(G)$ and $e \in E(G)$, for simplicity we use $G - v$, $G - e$ and $\partial(v)$ to represent $G - \{v\}$, $G - \{e\}$ and $\partial(\{v\})$ respectively. Further the cardinality of $\partial_G(v)$ is called the *degree* of v in G and is denoted by $d_G(v)$ (or simply $d(v)$).

In this section, we will present a method of constructing a complete forcing set of G in terms of elementary edge cut, which was introduced in [19, 26] to show the existence of perfect matchings in HS and plays an important role in resonance theory of graphs [13, 25, 28] especially in the computation of Clar number of HSs [10]. Elementary edge cut was previously defined in bipartite graphs, we extend this concept to general graphs as follows.

We call an edge cut D of G an *elementary edge cut* (e-cut for short) if it satisfies the following three conditions:

(1) $\omega(G - D) = \omega(G) + 1$, that is, there are exactly two components O_1 and O_2 of $G - D$ that are different from of all components of G.

(2) At least one of O_1 and O_2 is a bipartite graph.

(3) All edges of D are incident with the vertices of the same color in one bipartite component of O_1 and O_2 (for example, the bold edges of G_1 in Figure 10 form an e-cut of G_1).

A *bridge* of G is an edge cut of G consisting of exactly one edge. A *cut-vertex* of G is a vertex whose deletion increases the number of components. A *block* of G is a maximal connected subgraph of G that has no cut-vertices. Each block with at least 3 vertices is 2-connected. The blocks of a loopless graph are its isolated vertices, bridges, and maximal 2-connected subgraph. A block of G that contains exactly one cut-vertex of G is called an *end-block* of G.

Let D be an e-cut of a graph G with at least two edges. Then we define an *e-cut deletion operation* (simply ED operation) of G in the following steps:

(1) Delete D from G,

(2) Delete the set B consisting of all bridges of $G - D$, and

(3) Delete the isolated vertices of $G - D - B$.

Let G' be the subgraph obtained from G by an ED operation. Then G' has neither isolated vertices nor bridges. If G' is not empty, then each block of G' is 2-connected. Let v' be a non-cut-vertex of a block of G' with at most one cut-vertex of G'. Then $\omega(G' - \partial_{G'}(v')) = \omega(G') + 1$ and v' is a component of $G' - \partial_{G'}(v')$, so $\partial_{G'}(v')$ is an e-cut of G' with at least 2 edges and we can do an ED operation on G'. If we can do l ED operations from G and obtain the following subgraph sequence $G = G_1 \supset G_2 \supset \cdots \supset G_{l+1}$, where G_i is not empty graph and G_{i+1} is obtained by doing an ED operation from G_i for $i = 1, 2, \ldots, l$, then we call this procedure an *e-cut decomposition* from G_1 to G_{l+1}. Let D_i be the deleted e-cut from G_i. Then

$$c(G_i - D_i) = |E(G_i)| - |D_i| - |V(G_i)| + \omega(G_i) + 1 = c(G_i) - (|D_i| - 1).$$

Let B_i be the set of bridges deleted from $G_i - D_i$. Then we have

$$c((G_i - D_i) - B_i) = c(G_i - D_i) = c(G_i) - (|D_i| - 1).$$

Since deleting the isolated vertices from $G_i - D_i - B_i$ keeps its cyclomatic number unchanged,

$$c(G_{i+1}) = c((G_i - D_i) - B_i) = c(G_i) - (|D_i| - 1).$$

So, we have

$$c(G_i) - c(G_{i+1}) = |D_i| - 1. \tag{2.1}$$

Since $|D_i| \geq 2$, Equation (2.1) implies that an ED operation on G_i decrease the cyclomatic number by at least 1.

From the above discussion, we have the following result.

Lemma 2.1. *If a graph G has an e-cut with at least two edges, then there exists an e-cut decomposition from G to empty graph.*

Lemma 2.2. *Let G be a graph without isolated vertices or bridges. If H is a 2-connected induced subgraph of G, then there exists an e-cut decomposition from G to H.*

Proof. If G is not 2-connected, then there is a block B of G such that H is not an induced subgraph of B and B contains at most one cut-vertex of G. Since G has neither isolated vertices nor bridges, B is 2-connected. Let v_1 be a vertex of B that is not a cut-vertex of G. Then $\partial_G(v_1)$ is an e-cut of G with at least two edges. We can use $\partial_G(v_1)$ to do an ED operation on G. If G is 2-connected and $G \neq H$, let v_2 be a vertex of $V(G) \setminus V(H)$. Then $\partial_G(v_2)$ is an e-cut of G with at least two edges. We can use $\partial_G(v_2)$ to do an ED operation on G. In either of the above two cases, we can see that H is still an induced subgraph of the resulting graph. Clearly, we can do ED operations repeatedly like the above until the resulting subgraph is H. □

Lemma 2.3. *Let G be a graph that admits a perfect matching and F be the set of all forbidden edges of G. If there exists an e-cut decomposition from $G - F = G_1$ to G_{l+1} $(l \geq 1)$ such that G_{l+1} is empty graph or each cycle of G_{l+1} is not a nice cycle of G, then $D_1 \cup D_2 \cup \cdots \cup D_l$ is a complete forcing set of G, where D_i $(i = 1, 2, \ldots, l)$ is the e-cut deleted from G_i in the e-cut decomposition. Further, $cf(G) \leq c(G) + l - c(G_{l+1})$.*

Proof. For $i = 1, 2, \ldots, l$, let B_i be the set of bridges deleted from $G_i - D_i$ in the e-cut decomposition. Then $E(G)$ can be partitioned into $F \cup D_1 \cup B_1 \cup D_2 \cup B_2 \cup \cdots \cup D_l \cup B_l \cup E(G_{l+1})$. Since every edge of a nice cycle C is allowed, $E(C) \cap F = \emptyset$.

Claim. For $i = 1, 2, \ldots, l$, if there is a nice cycle C of G that has an edge in $D_i \cup B_i$, then each frame of C intersects $D_1 \cup D_2 \cup \cdots \cup D_i$.

Proof. We shall proceed by induction on i. For $i = 1$, if $E(C) \cap B_1 \neq \emptyset$, then $E(C) \cap D_1 \neq \emptyset$. Choose an edge e_1 in $E(C) \cap D_1$. Since D_1 is an e-cut of G_1, there is a bipartite component O_1 of $G_1 - D_1$ such that all edges of D_1 are incident with the same colored vertices of O_1 (say black). After C passes through e_1 to black end-vertex in O_1, C must pass through another edge of D_1 from black end-vertex in O_1. Let e_2 be the first such edge. Then the path of C_1 between black vertices of e_1 and e_2 in O_1 has even length. This yields that edges e_1 and e_2 in D_1 belong to different frames of C and the claim holds for $i = 1$. Suppose that the claim holds for $i \leq l - 1$. We shall prove it for $i + 1$. If C has an edge in $E(G) \setminus E(G_{i+1})$, that is, C has some edge in $D_1 \cup B_1 \cup D_2 \cup B_2 \cup \cdots \cup D_i \cup B_i$, by the induction hypothesis, the intersection of each frame of C and $D_1 \cup D_2 \cup \cdots \cup D_i$ is nonempty. So we may assume that $E(C) \subseteq E(G_{i+1})$. Since C has an edge in $D_{i+1} \cup B_{i+1}$, similarly we have that each frame of C must have an edge in D_{i+1}. Consequently, the claim holds. □

Since G_{l+1} is empty graph or every cycle of G_{l+1} is not a nice cycle of G, every nice cycle of G contains an edge in $D_1 \cup B_1 \cup D_2 \cup B_2 \cup \cdots \cup D_l \cup B_l$. By the claim, each frame of each nice cycle of G intersects $D_1 \cup D_2 \cup \cdots \cup D_l$. So, by Theorem 1.1, $D_1 \cup D_2 \cup \cdots \cup D_l$ is a complete forcing set of G.

From Equation (2.1), we have

$$c(G_1) - c(G_{l+1}) = \sum_{i=1}^{l} (|D_i| - 1), \tag{2.2}$$

and then

$$\sum_{i=1}^{l} |D_i| = c(G_1) + l - c(G_{l+1}).$$

Since $G_1 = G - F$, $c(G_1) \leq c(G)$ and

$$cf(G) \leq |D_1 \cup D_2 \cup \cdots \cup D_l| = c(G_1) + l - c(G_{l+1}) \leq c(G) + l - c(G_{l+1}). \quad \square$$

From Lemma 2.3, we have the following upper bound on the complete forcing number.

Theorem 2.4. *Let G be a graph that admits a perfect matching. Then $cf(G) \leq 2c(G)$.*

Proof. If G has a unique perfect matching, then $cf(G) = 0$, and the conclusion holds. If G has at least two perfect matchings, let F be the set of all forbidden edges of G. Then each K_2 block of $G - F$ is a component, so there is a 2-connected block B of $G - F$ with at most one cut-vertex of $G - F$. Let v be a vertex of B that is not a cut-vertex of $G - F$. Then $\partial_{G-F}(v)$ is an e-cut of $G - F$ with at least two edges. By Lemma 2.1 there exists an e-cut decomposition from $G - F$ to empty graph: $G_1 \supset G_2 \supset \cdots \supset G_{l+1}$, where $G_1 = G - F$ and $G_{l+1} = \emptyset$. For $i = 1, 2, \ldots, l$, let D_i be the e-cut deleted from G_i in this e-cut decomposition. From Equation (2.2), since $|D_i| \geq 2$ and $c(G_1) \leq c(G)$, we have $l \leq \sum_{i=1}^{l}(|D_i| - 1) = c(G_1) \leq c(G)$. Combining with Lemma 2.3, we have $cf(G) \leq c(G) + l - c(G_{l+1}) \leq 2c(G)$. □

3 Some extremal matching covered graphs

A connected graph G is said to be *matching covered* if it has at least two vertices and each edge is allowed. Every matching covered graph with at least four vertices is 2-connected [18].

In this section, we will characterize the matching covered graphs whose complete forcing numbers attain the upper bound given in Theorem 2.4 and minus one, respectively.

Theorem 3.1. *Let G be a matching covered graph. Then $cf(G) = 2c(G)$ if and only if G is either K_2 or an even cycle.*

Proof. The sufficiency is obvious. So we consider the necessity. If $c(G) = 0$, then G is a tree. Since G is matching covered, G can only be K_2. For $c(G) \geq 1$, suppose to the contrary that G is not an even cycle. Then G has a vertex v with degree at least 3. Let $D_1 = \partial(v)$. Then since G is 2-connected, $G - D_1$ has exactly two components and D_1 is an e-cut with $|D_1| \geq 3$. We use D_1 to do an ED operation on $G_1 = G$ and obtain G_2, and then we do ED operations from G_2 repeatedly until the empty graph is obtained. Consequently, we obtain an e-cut decomposition $G = G_1 \supset G_2 \supset \cdots \supset G_{l+1} = \emptyset$ ($l \geq 1$). For $i = 2, 3, \ldots, l$, let D_i be the e-cut deleted from G_i in this e-cut decomposition. By Equation (2.1), $c(G_2) - c(G_1) = |D_1| - 1 \geq 2$ and $c(G_{i+1}) - c(G_i) = |D_i| - 1 \geq 1$ ($i = 2, 3, \ldots, l$). Combining with Equation (2.2), we have $l + 1 \leq \sum_{i=1}^{l}(|D_i| - 1) = c(G) - c(G_{l+1})$, and thus $l \leq c(G) - 1$. By Lemma 2.3, $c(G) \leq 2c(G) - 1$, a contradiction. □

Corollary 3.2. *Let G be a graph with a perfect matching. Then $cf(G) = 2c(G)$ if and only if*

(i) *each forbidden edge of G is a bridge, and*

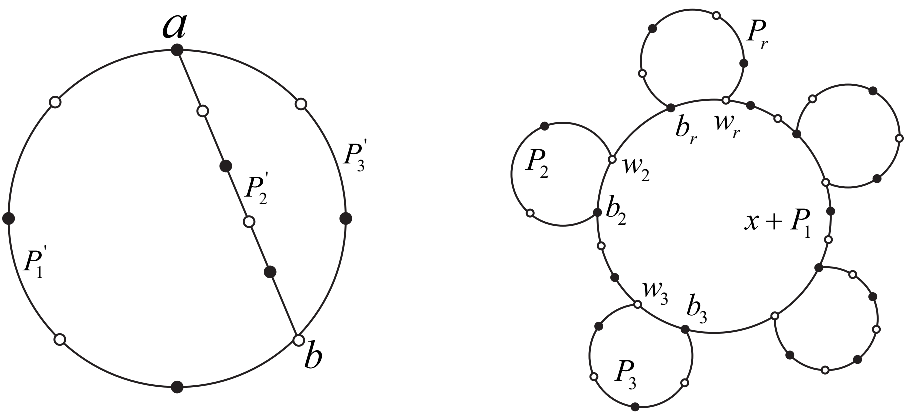
(ii) *each component of the graph obtained by deleting all forbidden edges from G is either K_2 or an even cycle.*

Proof. Let G_0 be the graph obtained from G by deleting all forbidden edges. Since each forbidden edge of G does not appear in any minimum complete forcing set of G, $cf(G) = cf(G_0)$. Let $O_1, O_2, \ldots, O_t$ ($t \geq 1$) be the components of G_0. By Theorem 2.4 we have

$$cf(G) = cf(G_0) = \sum_{i=1}^{t} cf(O_i) \leq 2\sum_{i=1}^{t} c(O_i) = 2c(G_0) \leq 2c(G).$$

In the above expression, Theorem 3.1 implies that the third equality holds if and only if each O_i is either K_2 or an even cycle, and the fifth equality holds if and only if G_0 and G have the same cyclomatic number, that is, each forbidden edge of G is a bridge. □

A connected graph G is said to be *elementary* if all its allowed edges form a connected subgraph of G. A connected bipartite graph is elementary if and only if each edge is allowed [17]. An elementary bipartite graph has the so-called "bipartite ear decomposition". Let x be an edge. Join the end vertices of x by a path P_1 of odd length (the so-called "first ear"). We proceed inductively to build a sequence of bipartite graphs as follows: If $G_{r-1} = x + P_1 + P_2 + \cdots + P_{r-1}$ has already been constructed, add the r-th ear P_r (a path of odd length) by joining any two vertices in different colors of G_{r-1} such that P_r has no other vertices in common with G_{r-1}. The decomposition $G_r = x + P_1 + P_2 + \cdots + P_r$ will be called an *(bipartite) ear decomposition* of G_r. It is known that a bipartite graph G is elementary if and only if G has a bipartite ear decomposition [17]. We can see that the number r of ears is equal to the cyclomatic number of G.



(*a*) (*b*)

Figure 1: Two examples for graphs G with $cf(G) = 2c(G) - 1$.

Theorem 3.3. *Let G be a matching covered graph. Then $cf(G) = 2c(G) - 1$ if and only if G is a bipartite graph and one of the following holds :*

(i) $c(G) = 2$ *(see* Figure 1(a)*);*

(ii) *G has an ear decomposition $G = x + P_1 + P_2 + \cdots + P_r$ $(r \geq 3)$ such that one frame of $x + P_1$ contains at least $r-1$ edges $w_2b_2, w_3b_3, \ldots, w_rb_r$ and the two ends of $P_2, P_3, \ldots, P_r$ are the two end-vertices of $w_2b_2, w_3b_3, \ldots, w_rb_r$, respectively (see* Figure 1(b)*).*

Proof. Sufficiency. (i) If $c(G) = 2$, by the ear decomposition of G, G contains two 3-degree vertices, denoted by a and b. Let P_1', P_2', P_3' be the 3 internally disjoint paths from a to b (see Figure 1(a)) and S be a complete forcing set of G. If $|S| \leq 2$, then one of P_1', P_2' and P_3' has no edges in S, say P_1'. We can see that one of the two nice cycles $P_1' \cup P_2'$ and $P_1' \cup P_3'$ has a frame containing no edges of S, which contradicts that S is a complete forcing set by Theorem 1.1. So $cf(G) \geq 3$. Conversely, we can see that $\partial_G(a)$ is a complete forcing set of G, which means that $cf(G) \leq 3$. Consequently, we have $cf(G) = 3 = 2c(G) - 1$.

(ii) For $2 \leq i \leq r$, let $C_i = P_i \cup \{w_i b_i\}$. Then we can see that $C_2, C_3, \ldots, C_r$ are $r-1$ vertex-disjoint nice cycles of G. Let S be a complete forcing set of G. By Theorem 1.1, each frame of C_i $(2 \leq i \leq r)$ has at least one edge of S, so each C_i contains 2 edges of S. Further the frame of the nice cycle $x+P_1$ that does not contain $\{w_2b_2, w_3b_3, \ldots, w_rb_r\}$ has an edge in S. So $|S| \geq 2r-1$. Conversely, let $D_1 = \partial_G(b_2)$ and D_i $(i = 2, 3, \ldots, r-1)$ be any two adjacent edges of C_{i+1}. We use $D_1, D_2, \ldots, D_{r-1}$ to do ED operations from G in turn and obtain empty graph finally. By Lemma 2.3, $D_1 \cup D_2 \cup \cdots \cup D_{r-1}$ is a complete forcing set of G and $cf(G) \leq |D_1 \cup D_2 \cup \cdots \cup D_{r-1}| = 3 + 2(r-2) = 2r-1$. Consequently, we have $cf(G) = 2r-1 = 2c(G)-1$.

Necessity. If $c(G) = 0$ or 1, then G is K_2 or an even cycle. By Theorem 3.1, $cf(G) = 2c(G)$, contradicting $cf(G) = 2c(G)-1$. So $c(G) \geq 2$ and $|V(G)| \geq 4$. Since G is matching covered, G is 2-connected.

Claim 1. For an e-cut decomposition from $G = G_1$ to $G_{l+1} = \emptyset$, if there is an integer k $(1 \leq k \leq l)$ such that $|D_k| \geq 4$ or there are two integers m and n $(1 \leq m < n \leq l)$ such that $|D_m| \geq 3$ and $|D_n| \geq 3$, then $cf(G) \leq 2c(G) - 2$.

Proof. If $|D_k| \geq 4$ $(1 \leq k \leq l)$, then since $|D_i| \geq 2$ for $i = 1, 2, \ldots, k-1, k+1, \ldots, l$, $\sum_{i=1}^{l}(|D_i|-1) \geq l+2$. From Equation (2.2), we have $l \leq c(G)-2$, and thus $cf(G) \leq 2c(G)-2$ by Lemma 2.3.

If $|D_m| \geq 3$ and $|D_n| \geq 3$ $(1 \leq m < n \leq l)$. Then since $|D_i| \geq 2$ $(i = 1, 2, \ldots, m-1, m+1, \ldots, n-1, n+1, \ldots, l)$, $\sum_{i=1}^{l}(|D_i|-1) \geq l+2$. From Equation (2.2), we have $l \leq c(G)-2$, and thus $cf(G) \leq 2c(G)-2$ by Lemma 2.3. □

If G has a vertex v_0 with degree at least 4, let $D_1 = \partial(v_0)$. Since G is 2-connected, $G - \partial(v_0)$ has exactly two components and D_1 is an e-cut of G. Then we can give an e-cut decomposition from G to empty graph by taking D_1 as the first e-cut. By Claim 1, $cf(G) \leq 2c(G)-2$, a contradiction. Thus $d_G(v) \leq 3$ for each vertex $v \in V(G)$. In addition, since $c(G) \geq 2$, G has a 3-degree vertex v_1. Since G is 2-connected, $G - v_1$ is connected.

Claim 2. G is a bipartite graph.

Proof. Suppose to the contrary that G is not a bipartite graph. Let v be a vertex of G. Then $G - v$ is not a bipartite graph as well. Otherwise, $G - v$ has a bipartition (W, B) $(|W| < |B|)$. If v is adjacent to a vertex w of W in G, then vw is a forbidden edge of G, which contradicts that G is matching covered. So v can only be adjacent to vertices of B in G, and thus G is a bipartite graph, a contradiction to the supposition. Hence, $G - v_1$ has an odd cycle C_1.

Let $D_1 = \partial(v_1)$. Since G is 2-connected, $G - D_1$ has exactly two components and D_1 is an e-cut of G with $|D_1| = 3$. We obtain G_2 by doing an ED operation on $G_1 = G$ via D_1. Since $G[V(C_1)]$ is 2-connected and $G[V(C_1)]$ is still a subgraph of G_2, from Lemma 2.2, there exists an e-cut decomposition from G_2 to $G[V(C_1)] = G_m$. For $i = 2, 3, \ldots, m-1$, we denote by D_i the deleted e-cut from G_i in this e-cut decomposition. If G_m has a 3-degree vertex v_m, let $D_m = \partial_{G[V(C_1)]}(v_m)$. We can give an e-cut decomposition from G_m to empty graph by taking D_m as the first e-cut. Combining the above two e-cut decompositions, we have an e-cut decomposition from G_1 to empty graph with $|D_1| = |D_m| = 3$. By Claim 1, $cf(G) \leq 2c(G)-2$, a contradiction. If G_m is an odd cycle, by Lemma 2.3, $D_1 \cup D_2 \cup \cdots \cup D_{m-1}$ is a complete forcing set of G and $cf(G) \leq$

$c(G)+(m-1)-c(G_m)$. Since $|D_1|=3$ and $|D_i|\geq 2$ $(i=2,3,\ldots,m-1)$, $c(G)-c(G_m)=\sum_{i=1}^{m-1}(|D_i|-1)\geq m$, so we have $m\leq c(G)-c(G_m)=c(G)-1$. Hence, $cf(G)\leq c(G)+(m-1)-c(G_m)\leq 2c(G)-3$, a contradiction. □

Claim 3. Each block of $G-v_1$ is either K_2 or an even cycle.

Proof. Let B be a block of $G-v_1$. Then $d_B(v)\leq 2$ for each $v\in V(B)$. Otherwise, let v' be a vertex of B with $d_B(v')=3$. By Lemma 2.2, there exists an e-cut decomposition from $G=G_1$ to $B=G_m$ by taking $D_1=\partial_G(v_1)$ as the first e-cut. Let $D_m=\partial_{G_m}(v')$. Then D_m is an e-cut of B and we can give an ED decomposition from B to empty graph by taking D_m as the first e-cut. Combining with the above two e-cut decompositions, we have an e-cut decomposition from G to empty graph with $|D_1|=|D_m|=3$. By Claim 1, $cf(G)\leq 2c(G)-2$, a contradiction. Since G is a bipartite graph by Claim 2, each block of $G-v_1$ is K_2 or an even cycle. □

In the following we may assume that v_1 is a black vertex of G.

Claim 4. If each block of $G-v_1$ is K_2, then $c(G)=2$ and (i) holds.

Proof. Obviously $G-v_1$ is a tree. If $G-v_1$ has no 3-degree vertices, then it is a path P. Since G is 2-connected, the end-vertices of P are adjacent to v_1 and receive white. Further, since $d_G(v_1)=3$, v_1 has third white neighbor as an internal vertex of P (see Figure 2(a)). So $c(G)=2$.

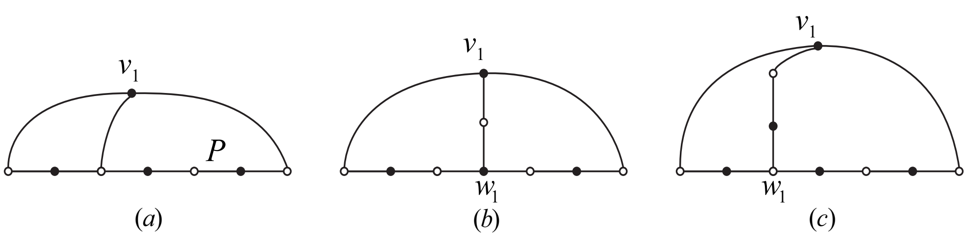


Figure 2: Illustration for Claim 4.

If $G-v_1$ has a 3-degree vertex, then $G-v_1$ has only one 3-degree vertex, denoted by w_1. Otherwise, $G-v_1$ has at least four 1-degree vertices, but just three of them is adjacent to v_1 in G, so G has a 1-degree vertex, a contradiction. Thus $G-v_1$ has three 1-degree vertices which are adjacent to v_1 in G. It follows that w_1 is a white vertex (see Figure 2(c)); Otherwise, G has an odd number of vertices (see Figure 2(b)), a contradiction. So $c(G)=2$. □

In what follows we suppose that $G-v_1$ has a block that is an even cycle.

Claim 5. Each even cycle block of $G-v_1$ has at most two 3-degree vertices in G.

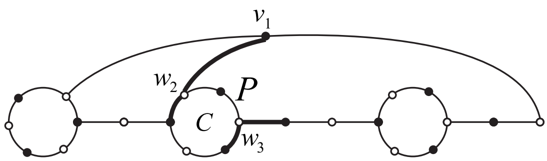


Figure 3: An even cycle block C of $G - v_1$ has exactly three 3-degree vertices of G.

Proof. If $G - v_1$ has an even cycle block that has four 3-degree vertices in G, then it has at least one end-block that has no vertices that are adjacent to v_1 in G. This causes G to have a cut-vertex, which contradicts that G is 2-connected. If there is an even cycle block C of $G - v_1$ that has exactly three 3-degree vertices of G, then two of such three vertices w_2 and w_3 have the same color in G. Let P be a path contained in C with ends w_2 and w_3 (see Figure 3). Then each internal vertex of P is still a 2-degree vertex of G. Further, since G has an e-cut $\partial(V(P))$ of four edges, we can give an e-cut decomposition from G to empty graph by taking $\partial(V(P))$ as the first e-cut. By Claim 1, we have $cf(G) \leq 2c(G) - 2$, a contradiction. □

Claim 6. $G - v_1$ is not 2-connected, and has no vertices contained in three K_2 blocks.

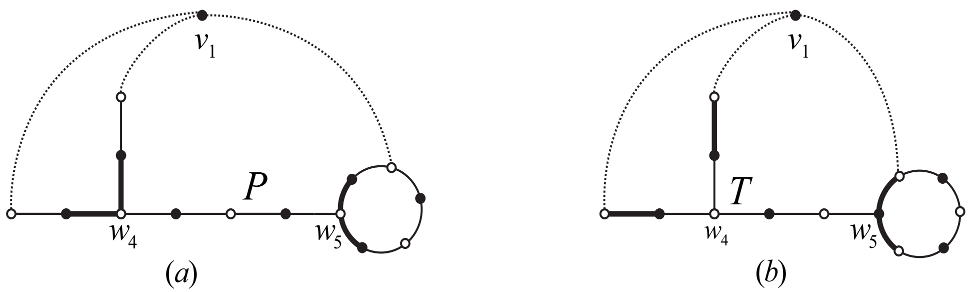


Figure 4: (a) w_4 and w_5 have the same color; (b) w_4 and w_5 have different colors.

Proof. If $G - v_1$ is 2-connected, then by Claim 3, $G - v_1$ is an even cycle and v_1 is adjacent to three vertices of this cycle in G, which contradicts Claim 5. So, $G - v_1$ is not 2-connected.

Suppose to the contrary that $G - v_1$ has a vertex w_4 incident with three K_2 blocks. Then $G - v_1$ has at least 3 end-blocks. Since G is 2-connected and $d_G(v_1) = 3$, $G - v_1$ has exactly three end-blocks. Let P be a shortest path between w_4 and a 3-degree vertex w_5 of $G - v_1$ in an even cycle block so that each internal vertex of P is a 2-degree vertex in G. If w_4 and w_5 have the same color, then $\partial(V(P))$ is an e-cut of G (see Figure 4(a)). There exists an e-cut decomposition from G to empty graph by taking $\partial(V(P))$ as the first e-cut. By Claim 1, we have $cf(G) \leq 2c(G) - 2$, a contradiction. If w_4 and w_5 have different colors, let T be the tree consisting of P and the remaining two K_2 blocks of $G - v_1$ that has an end-vertex w_4. Then $\partial(V(T))$ is an e-cut of G (see Figure 4(b)). Similarly we have $cf(G) \leq 2c(G) - 2$, a contradiction. □

By Claims 3, 5 and 6, $G - v_1$ has exactly two end-blocks which each has a white non-cut-vertex of $G - v_1$ adjacent to v_1 in G, and $G - v_1$ can be constructed as follows: $r - 1$ disjoint paths $P'_1, P'_2, \ldots, P'_{r-1}$ connect $r-2$ disjoint even cycles $C_1, C_2, \ldots, C_{r-2}$ in turn so that P'_i only connects C_{i-1} and C_i for $i = 2, 3, \ldots, r-2$, where $r \geq 3$, and P'_1 and P'_{r-1} connect only C_1 and C_{r-2} respectively (see Figure 5(a)). Let v_2 be the third neighbor of v_1 in $G - v_1$.

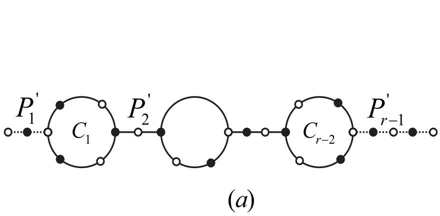


Figure 5: (a) The construction of $G - v_1$; (b) Illustration for Claim 7.

Claim 7. v_2 must be an internal vertex of paths P'_1 and P'_{r-1}.

Proof. If v_2 belongs to some even cycle C_k $(1 \leq k \leq r-2)$ in $G - v_1$, then C_k has three 3-degree vertices of G, which contradicts Claim 5. If v_2 is an internal vertex of P'_i $(2 \leq i \leq r-2)$ (see Figure 5(b)), let the ends of P'_i be v_3 and v_4. Then there exists an e-cut decomposition from G to empty graph by taking $\partial(v_3)$ and $\partial(v_4)$ as the first two e-cuts. Since $|\partial(v_3)| = |\partial(v_4)| = 3$, by Claim 1, $cf(G) \leq 2c(G) - 2$, a contradiction. Hence v_2 is an internal vertex of P'_1 or P'_{r-1} and the claim holds. □

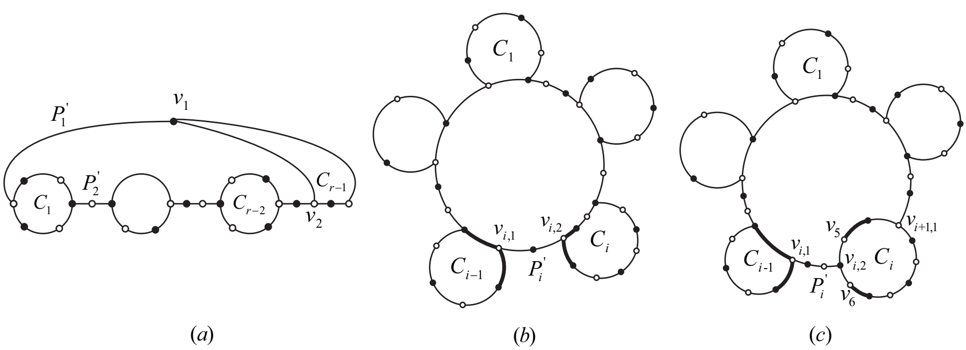


Figure 6: (a) The construction of G; (b) e-cut (bold edges) leaving from P'_i; (c) e-cut (bold edges) leaving from three T.

By Claim 7 we may suppose v_2 is an internal vertex of P'_{r-1} that has length at least 3. Then the subpath of P'_{r-1} between both neighbors of v_1 with two incident edges forms a cycle, denoted by C_{r-1}. Thus G can be constructed from $r-1$ disjoint even cycles $C_1, C_2, \ldots, C_{r-1}$ by using $r-1$ disjoint paths to connect them in a cyclic way. More precisely, each P_i' connects vertex $v_{i,1}$ of C_{i-1} and vertex $v_{i,2}$ of C_i, where $i = 1, 2, \ldots, r-1$ and $C_0 = C_{r-1}$ (see Figure 6(a)). Note that $P_2', \ldots, P_{r-2}'$ remain unchanged, but P_1' is lengthened by one edge and P_{r-1}' is shorten. Then $v_{i,1}$ and $v_{i,2}$ have different colors in G. Otherwise, there exists an e-cut decomposition from G to empty graph by taking $\partial(V(P_i'))$ (see Figure 6(b)) as the first e-cut that has four edges. By Claim 1, $cf(G) \leq 2c(G) - 2$, a contradiction. Further, $v_{i,2}$ and $v_{i+1,1}$ have different colors in G, where $v_{r,1} = v_{1,1}$. Otherwise, two edges leaving from even cycle C_i are forbidden edges of G, which contradicts that G is matching covered. Finally we claim that $v_{i,2}$ and $v_{i+1,1}$ are adjacent in C_i. Otherwise, since $v_{i,2}$ and $v_{i+1,1}$ have different colors, two paths between $v_{i,2}$ and $v_{i+1,1}$ in C_i have length at least 3 (see Figure 6(c)). Let v_5 and v_6 be the two neighbors of $v_{i,2}$ in C_i. Then v_5, v_6 and $v_{i,1}$ are all of the same color in G. Let T be the tree induced by $\{v_5, v_6\} \cup V(P_i')$. Then $\partial(V(T))$ is an e-cut of G of four edges. So there exists an e-cut decomposition from G to empty graph by taking $\partial(V(T))$ as the first e-cut. By Claim 1, we have $cf(G) \leq 2c(G) - 2$, a contradiction.

Let $x + P_1$ be an even cycle formed by the paths P_i' and edges $v_{i,2}v_{i+1,1}$, $i = 1, 2, \ldots, r-1$, and let P_{i+1} be the path between $v_{i,2}$ and $v_{i+1,1}$ in C_i of length at least three. Then the edges $v_{i,2}v_{i+1,1}$, $i = 1, 2, \ldots, r-1$, are contained in a frame of $x + P_1$ and $G = x + P_1 + P_2 + \cdots + P_r$ is an ear decomposition of G described as in (ii). □

4 Wheels and cylinders

In this section, we first present some lower bounds on the complete forcing numbers of some special types of graphs. We then derive some closed formulas for the complete forcing numbers of wheels and cylinders, respectively. Our main idea is to apply an e-cut decomposition on a given graph to construct a complete forcing set whose cardinality attains a lower bound on the complete forcing number.

Lemma 4.1. *Let G be a graph that admits a perfect matching. If there is a set $\mathcal{C}$ of nice cycles of G such that every edge of G lies in exactly two nice cycles of $\mathcal{C}$, then $cf(G) \geq |\mathcal{C}|$.*

Proof. For a nice cycle C of $\mathcal{C}$, let $T_1(C)$ and $T_2(C)$ be the two frames of C. Let S be a minimum complete forcing set of G. By Theorem 2.1, we have

$$|S \cap T_i(C)| \geq 1, i = 1, 2, \text{ for each nice cycle } C \text{ of } \mathcal{C}.$$

Summing all the above inequalities together, we have

$$2|S| = \sum_{C \in \mathcal{C}} (|S \cap T_1(C)| + |S \cap T_2(C)|) \geq 2|\mathcal{C}|,$$

because each edge of S belongs to exactly two nice cycles of $\mathcal{C}$. Then we have

$$cf(G) = |S| \geq |\mathcal{C}|.$$ □

For a plane elementary bipartite graph G, all facial cycles (including the exterior facial cycle) of G are nice cycles [28]. Since each edge of G lies in exactly two of these facial cycles, by Lemma 4.1, we have

Corollary 4.2. *Let G be a plane elementary bipartite graph with n faces. Then $cf(G) \geq n$.*

This result is a generalization of a lower bound on the complete forcing numbers of normal hexagonal systems (see [11]).

A *wheel* W_n ($n \geq 4$) is a graph formed by connecting a single vertex (called the *hub*) to all vertices of a cycle (called the *rim*) with $n-1$ vertices. We can check that W_{2n} ($n \geq 2$) is matching covered by the definition.

Theorem 4.3. *For $n \geq 2$, $cf(W_{2n}) = 2n-1$.*

Proof. We denote by v_0 the hub of W_{2n} and by $v_1, v_2, \ldots, v_{2n-1}$ the vertices in the rim of W_{2n} along one of two directions of it. We can see that the set $\mathcal{C}$ of 4-cycles $\{v_i v_0 v_{i+2} v_{i+1} v_i | i = 1, 2, \ldots, 2n-1\}$ consisting of $2n-1$ nice cycles of W_{2n}, where $v_{2n} = v_1$ and $v_{2n+1} = v_2$. Moreover, each edge of W_{2n} lies in exactly two nice cycles of $\mathcal{C}$. By Lemma 4.1, $cf(W_{2n}) \geq |\mathcal{C}| = 2n-1$. On the other hand, let $D_1 = \partial_{W_{2n}}(v_0)$. Then D_1 is an e-cut of W_{2n} with $2n-1$ edges. We use D_1 to do an e-cut operation on $G_1 = W_{2n}$ and obtain G_2. Since G_2 is an odd cycle, by Lemma 2.3, D_1 is a complete forcing set. So $cf(W_{2n}) \leq |D_1| = 2n-1$. Consequently, $cf(W_{2n}) = 2n-1$. □

The *cartesian product* $G \times H$ of two graphs G and H is a graph with vertex set $V(G) \times V(H)$ specified by putting (u, v) adjacent to (u', v') if and only if (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$. Let $P_m = u_1 u_2 \cdots u_m$ be a path with m vertices. Recently, Chang et al. [5] obtained that $cf(P_m \times P_n) = \lfloor \frac{n}{2} \rfloor (m-1) + \lfloor \frac{m}{2} \rfloor (n-1)$. It is natural to consider the complete forcing numbers of $m \times n$ cylinders. Let $C_n = v_1 v_2 \cdots v_n v_1$ be a cycle with n vertices. An $m \times n$ cylinder $P_m \times C_n$ consists of $m-1$ concentric layers of quadrangles (i.e. each layer is a cyclic chain of n quadrangles), capped on each end by an n-polygon (see G_1 of Figure 7 for an example). If both m and n are odd, then $P_m \times C_n$ has an odd number of vertices and thus has no perfect matchings. So we only consider the complete forcing number of $P_m \times C_n$ with even mn. The operation of inserting a new vertex of degree two on an edge of a graph is called a *subdivision* of the edge.

Lemma 4.4. *If m is even, then*

$$cf(P_m \times C_n) \geq \begin{cases} mn - \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{2mn+m-n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. For $1 \leq i \leq m-1$, let R_i be the subgraph of $P_m \times C_n$ induced by $\{(u_i, v_j), (u_{i+1}, v_j) | \, j = 1, 2, \ldots, n\}$ and $E_{1i} = \{(u_i, v_j)(u_{i+1}, v_j) | j = 1, 2, \ldots, n\}$. For $1 \leq j \leq n$, let L_j be the subgraph of $P_m \times C_n$ induced by $\{(u_i, v_j), (u_i, v_{j+1}) | i = 1, 2, \ldots, m\}$ and $E_{2j} = \{(u_i, v_j)(u_i, v_{j+1}) | i = 1, 2, \ldots, m\}$, where $v_{n+1} = v_1$. Let S be a minimum complete forcing set of $P_m \times C_n$. Since R_i has n nice quadrangles of $P_m \times C_n$ and each quadrangle of R_i has a frame completely contained in E_{1i}, by Theorem 1.1, each quadrangle of R_i has an edge in $S \cap E_{1i}$. If n is even, then $|S \cap E_{1i}| \geq \frac{n}{2}$. And if n is odd, then $|S \cap E_{1i}| \geq \frac{n+1}{2}$. Since L_j has $m-1$ nice quadrangles of $P_m \times C_n$ and each quadrangle of L_j has a frame completely contained in E_{2j}, by Theorem 1.1, each quadrangle of L_j has an edge in $S \cap E_{2j}$. Since m is even, $|S \cap E_{2j}| \geq \frac{m}{2}$. Thus we have if n is even, then $cf(P_m \times C_n) = |S| \geq \sum_{i=1}^{m-1} |S \cap E_{1i}| + \sum_{j=1}^{n} |S \cap E_{2j}| \geq \frac{n(m-1)}{2} + \frac{mn}{2} = mn - \frac{n}{2}$. And if n is odd, then $cf(P_m \times C_n) = |S| \geq \sum_{i=1}^{m-1} |S \cap E_{1i}| + \sum_{j=1}^{n} |S \cap E_{2j}| \geq \frac{(m-1)(n+1)}{2} + \frac{mn}{2} = \frac{2mn+m-n-1}{2}$. □

Lemma 4.5 (Pick's theorem [8]). *Let P be a simple polygon constructed on a polyomino such that all the polygon's vertices are polyomino's vertices. Let the number of polyomino's vertices in the interior of P be i and the number of polyomino's vertices on the boundary of P be b. Then the area of P is given by $A = \frac{b}{2} + i - 1$.*

Theorem 4.6.

$$cf(P_m \times C_n) = \begin{cases} mn - n + 2, & \text{if } m \text{ is odd and } n \text{ is even } (m \geq 1, n \geq 4), \\ mn - \frac{n}{2}, & \text{if both } m \text{ and } n \text{ are even } (m \geq 2, n \geq 4), \\ \frac{2mn+m-n-1}{2}, & \text{if } m \text{ is even and } n \text{ is odd } (m \geq 2, n \geq 3). \end{cases}$$

Proof. Since mn is even, we can see that each edge of $P_m \times C_n$ is allowed, so $P_m \times C_n$ is matching covered. To construct a complete forcing set of $P_m \times C_n$, by Lemma 2.3, we can directly apply e-cut decomposition on $P_m \times C_n$.

We divide our proof into the following three cases.

Case 1. m is odd and n is even $(m \geq 1, n \geq 4)$.

If $m = 1$, then $P_m \times C_n$ is an even cycle and $cf(P_m \times C_n) = 2$ by Theorem 3.1, and the conclusion holds. In the following, we suppose that $m \geq 3$.

By Corollary 4.2, $cf(P_m \times C_n) \geq mn - n + 2$. So it suffices to construct a complete forcing set of $P_m \times C_n$ of size $mn - n + 2$.

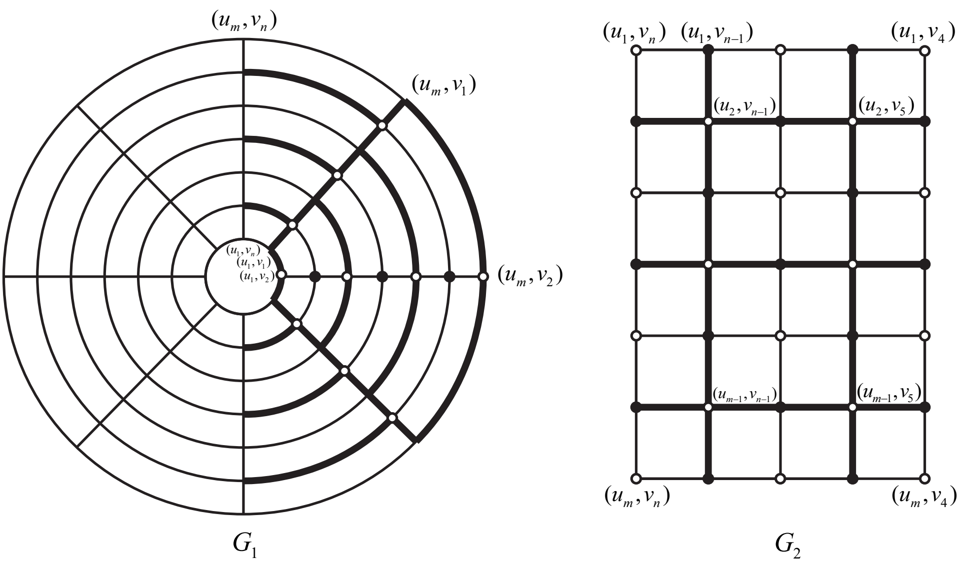


Figure 7: m is odd and n is even.

Let

$$\begin{aligned}D_1 =&\{(u_{2i+1},v_1)(u_{2i+1},v_2),(u_{2i+1},v_2)(u_{2i+1},v_3) \mid i=0,1,2,\ldots,\frac{m-1}{2}\}\cup\\ &\{(u_j,v_1)(u_{j+1},v_1),(u_j,v_3)(u_{j+1},v_3) \mid j=1,2,\ldots,m-1\}\cup\\ &\{(u_{2k},v_n)(u_{2k},v_1),(u_{2k},v_3)(u_{2k},v_4) \mid k=1,2,\ldots,\frac{m-1}{2}\}\end{aligned}$$

(see bold edges of G_1 of Figure 7). Then D_1 is an e-cut of $G_1 = P_m \times C_n$. We use D_1 to do an ED operation on G_1 and obtain $G_2 = P_m \times P_{n-3}$ (see Figure 7). Let $D_2, D_3, \ldots, D_{\frac{(m-1)(n-4)}{4}+1}$ be $\partial_{G_2}((u_{2i},v_5))$, $\partial_{G_2}((u_{2i},v_7))$,..., $\partial_{G_2}((u_{2i},v_{n-1}))$ ($i=1,2,\ldots,\frac{m-1}{2}$), respectively. Then we continue to do ED operations from G_2 by $D_2, D_3, \ldots, D_{\frac{(m-1)(n-4)}{4}+1}$ in turn and obtain $G_{\frac{(m-1)(n-4)}{4}+2}$. Note that D_i is an e-cut of G_i for $i=1,2,\ldots,\frac{(m-1)(n-4)}{4}+1$. We find that $G_{\frac{(m-1)(n-4)}{4}+2}$ can be obtained by subdividing every edge of $P_{\frac{m-1}{2}+1} \times P_{\frac{n-4}{2}+1}$ as shown in the thin edges of G_2 in Figure 7. Let C be a cycle of $G_{\frac{(m-1)(n-4)}{4}+2}$. Suppose that C encloses some region R in the plane, let A be the area of R, b be the number of vertices of G_2 on C, and i be the number of vertices of G_2 in the interior of C. Then A is divisible by 4. We can see that C is obtained by subdividing every edge of a cycle C' of $P_{\frac{m-1}{2}+1} \times P_{\frac{n-4}{2}+1}$. Since C' is a cycle of even length and $|V(C)| = 2|V(C')|$, b is divisible by 4. By Lemma 4.5, i is odd. Then $G_1 - V(C)$ has no perfect matchings. So each cycle of $G_{\frac{(m-1)(n-4)}{4}+2}$ is not a nice cycle of G_1. By Lemma 2.3, $D_1 \cup D_2 \cup \ldots D_{\frac{(m-1)(n-4)}{4}+1}$ is a complete forcing set of G_1. Since $|D_1| = (m+1)+2(m-1)+(m-1) = 4m-2$ and $|D_i| = 4$ for $i=2,3,\ldots,\frac{(m-1)(n-4)}{4}+1$, $cf(G_1) \leq |D_1 \cup D_2 \cup \ldots D_{\frac{(m-1)(n-4)}{4}+1}| = mn-n+2$. Consequently, $cf(P_m \times C_n) = mn-n+2$.

Case 2. Both m and n are even ($m \geq 2, n \geq 4$).

By Lemma 4.4, it suffices to construct a complete forcing set of $P_m \times C_n$ of size $mn - \frac{n}{2}$. Let

$$\begin{aligned}D_1 =&\{(u_{2i+1},v_1)(u_{2i+1},v_2),(u_{2i+1},v_2)(u_{2i+1},v_3) \mid i=0,1,2,\ldots,\frac{m-2}{2}\}\cup\\ &\{(u_j,v_1)(u_{j+1},v_1),(u_j,v_3)(u_{j+1},v_3) \mid j=1,2,\ldots,m-1)\}\cup\\ &\{(u_{2k},v_n)(u_{2k},v_1),(u_{2k},v_3)(u_{2k},v_4) \mid k=1,2,\ldots,\frac{m}{2}\}.\end{aligned}$$

Then D_1 is an e-cut of $G_1 = P_m \times C_n$. We use D_1 to do an ED operation on $G_1 = P_m \times C_n$ and obtain $G_2 = P_m \times P_{n-3}$ (see Figure 8). Let $D_2, D_3, \ldots, D_{\frac{m(n-4)}{4}+1}$ be $\partial_{G_2}((u_{2i},v_5))$, $\partial_{G_2}((u_{2i},v_7))$,..., $\partial_{G_2}((u_{2i},v_{n-1}))$ ($i=1,2,\ldots,\frac{m}{2}$), respectively. Continuously doing ED operations from G_2 by $D_2, D_3, \ldots, D_{\frac{m(n-4)}{4}+1}$ in turn, we obtain $G_{\frac{m(n-4)}{4}+2}$. Note that $G_{\frac{m(n-4)}{4}+2}$ can be obtained by subdividing every edge of $P_{\frac{m-2}{2}+1} \times P_{\frac{n-4}{2}+1}$ as shown in Figure 8. Let C be a cycle of $G_{\frac{m(n-4)}{4}+2}$. Suppose that C encloses some region R in the plane, let A be the area of R, b the number of vertices of G_2 on C, and i be the number of vertices of G_2 in the interior of C. Then A is divisible by 4. We can see that C is obtained by subdividing every edge of a cycle C' of $P_{\frac{m-2}{2}+1} \times P_{\frac{n-4}{2}+1}$. Since C' is a cycle of even length and $|V(C)| = 2|V(C')|$, b is divisible by 4. By Lemma 4.5, i is

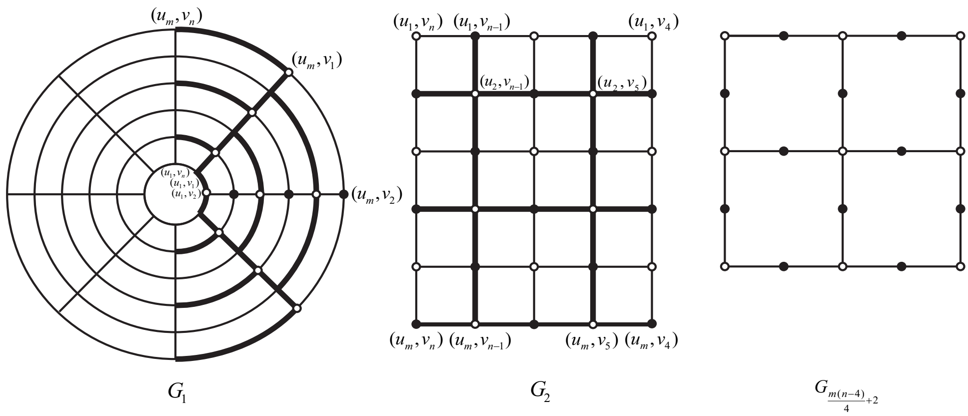


Figure 8: Both m and n are even.

odd. So $G_1 - V(C)$ has no perfect matchings. Thus each cycle of $G_{\frac{m(n-4)}{4}+2}$ is not a nice cycle of G_1. By Lemma 2.3, $D_1 \cup D_2 \cup \ldots D_{\frac{m(n-4)}{4}+1}$ is a complete forcing set of G_1. Since $|D_1| = m + 2(m-1) + m = 4m - 2$, $|D_i| = 4$ for $i = 2, 3, \ldots, \frac{(m-2)(n-4)}{4} + 1$ and $|D_j| = 3$ for $j = \frac{(m-2)(n-4)}{4} + 2, \frac{(m-2)(n-4)}{4} + 3, \ldots, \frac{m(n-4)}{4} + 1$, $cf(G_1) \leq |D_1 \cup D_2 \cup \ldots D_{\frac{(m-1)(n-4)}{4}+1}| = mn - \frac{n}{2}$. Consequently, $cf(P_m \times C_n) = mn - \frac{n}{2}$.

Case 3. m is even and n is odd $(m \geq 2, n \geq 3)$.

By Lemma 4.4, it suffices to prove $cf(G_1) \leq \frac{2mn+m-n-1}{2}$.

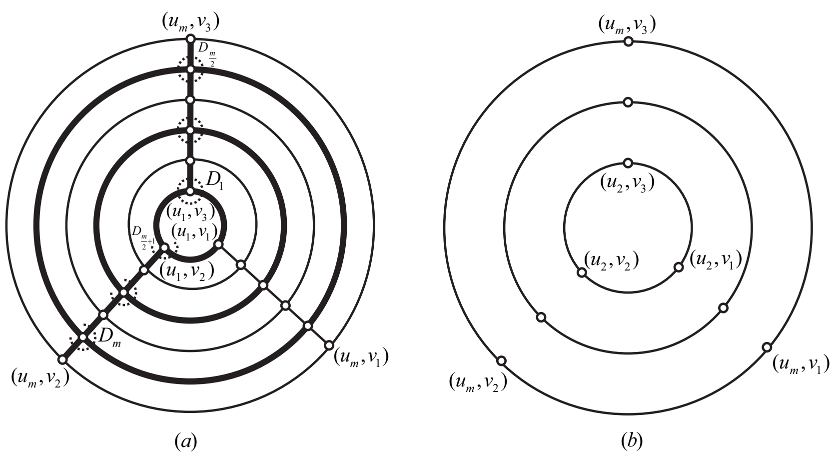


Figure 9: m is even and $n = 3$.

Subcase 3.1. $n = 3$.

Let $G_1 = P_m \times C_3$ and $D_1, D_2, \ldots, D_{\frac{m}{2}}$ be $\partial_{G_1}((u_{2i+1}, v_3))$ ($i = 0, 1, \ldots, \frac{m-2}{2}$), respectively (see Figure 9(a)). Then we use $D_1, D_2, \ldots, D_{\frac{m}{2}}$ to do ED operations on G_1 in turn and obtain $G_{\frac{m}{2}+1}$. Let $D_{\frac{m}{2}+1}, D_{\frac{m}{2}+2}, \ldots, D_m$ be $\partial_{G_{\frac{m}{2}+1}}((u_{2i+1}, v_2))$ ($i = 0, 1, \ldots, \frac{m-2}{2}$), respectively. Then we use $D_{\frac{m}{2}+1}, D_{\frac{m}{2}+2}, \ldots, D_m$ to do ED operations in turn and obtain G_{m+1}. We can see that G_{m+1} consists of $\frac{m}{2}$ disjoint cycles of length 3 and $c(G_{m+1}) = \frac{m}{2}$ as shown in Figure 9(b). Thus each cycle of G_{m+1} is not a nice cycle of G_1. By Lemma 2.3, $cf(G_1) \leq c(G_1) + m - c(G_{m+1}) = 3(m-1) + 1 + m - \frac{m}{2} = \frac{7m-4}{2}$. Consequently, $cf(P_m \times C_{2n}) = \frac{7m-4}{2}$.

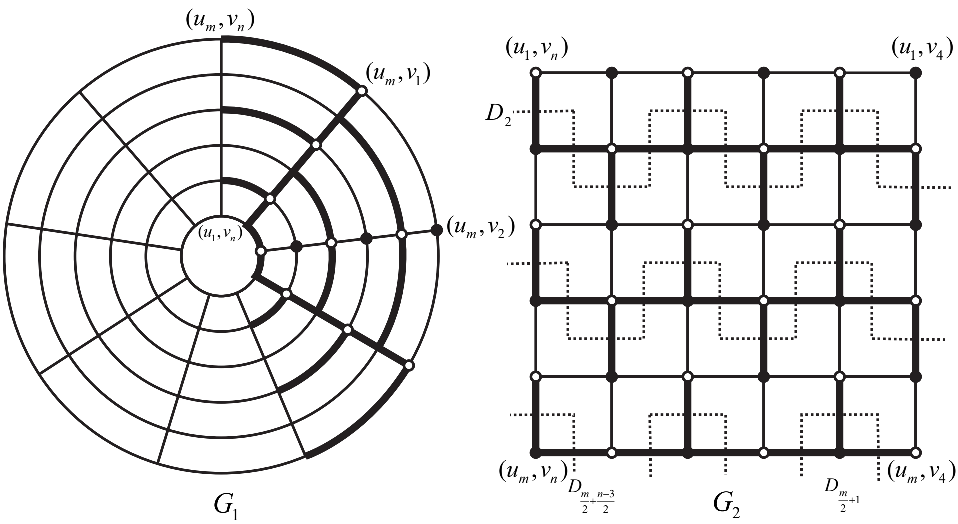


Figure 10: m is even and n is odd ($n \geq 5$).

Subcase 3.2. $n \geq 5$.

Let

$$\begin{aligned} D_1 = &\{(u_{2i+1}, v_1)(u_{2i+1}, v_2), (u_{2i+1}, v_2)(u_{2i+1}, v_3) \mid i = 0, 1, 2, \ldots, \frac{m-2}{2})\} \cup \\ &\{(u_j, v_1)(u_{j+1}, v_1), (u_j, v_3)(u_{j+1}, v_3) | j = 1, 2, \ldots, m-1\} \cup \\ &\{(u_{2k}, v_n)(u_{2k}, v_1), (u_{2k}, v_3)(u_{2k}, v_4) | k = 1, 2, \ldots, \frac{m}{2}\}. \end{aligned}$$

Then we use D_1 to do an ED operation on $G_1 = P_m \times C_{2n}$ and obtain $G_2 = P_m \times P_{n-3}$ (see Figure 10). Let D_t ($t = 2, 3, \ldots, \frac{m}{2}$) be

$$\begin{aligned} &\{(u_{2t-2}, v_{2i+2})(u_{2t-1}, v_{2i+2}) \mid i = 1, 2, \ldots, \frac{n-3}{2}\} \cup \\ &\{(u_{2t-2}, v_{j+3})(u_{2t-2}, v_{j+4}) \mid j = 1, 2, \ldots, n-4\} \cup \\ &\{(u_{2t-3}, v_{2k+3})(u_{2t-2}, v_{2k+3}) \mid k = 1, 2, \ldots, \frac{n-3}{2}\}. \end{aligned}$$

Then we use $D_2, D_3, \ldots, D_{\frac{m}{2}}$ to do ED operations on G_1 in turn and obtain $G_{\frac{m}{2}+1}$ which is $P_2 \times P_{n-3}$. Let $D_{\frac{m}{2}+1}, D_{\frac{m}{2}+2}, \ldots, D_{\frac{m}{2}+\frac{n-3}{2}}$ be $\partial_{G_{\frac{m}{2}+1}}((u_m, v_{2s+3}))$ ($s = 1, 2, \ldots, \frac{n-3}{2}$), respectively. Then we use $D_{\frac{m}{2}+1}, D_{\frac{m}{2}+2}, \ldots, D_{\frac{m}{2}+\frac{n-3}{2}}$ to do ED operations on $G_{\frac{m}{2}+1}$ in turn and obtain $G_{\frac{m}{2}+\frac{n-3}{2}+1}$ which is the empty graph. By Lemma 2.3, $D_1 \cup D_2 \cup \cdots \cup D_{\frac{m}{2}+\frac{n-3}{2}}$ is a complete forcing set of G_1 and $cf(G_1) \leq c(G_1) + \frac{m}{2} + \frac{n-3}{2} - 0 = \frac{2mn+m-n-1}{2}$. Consequently, $cf(P_m \times C_n) = \frac{2mn+m-n-1}{2}$. □

At the end of this paper, by some simple calculations, we present the relationship between the cyclomatic number and complete forcing number for wheels and cylinders. For a wheel W_{2n}, $c(W_{2n}) = |E(W_{2n})| - |V(W_{2n})| + 1 = 2(2n-1) - 2n + 1 = 2n - 1$. By Theorem 4.3, $cf(W_{2n}) = c(W_{2n})$. For a cylinder $P_m \times C_n$, $c(P_m \times C_n) = |E(P_m \times C_n)| - |V(P_m \times C_n)| + 1 = n(m-1) + mn - mn + 1 = mn - n + 1$. By Theorem 4.6, we can see that $cf(P_m \times C_n) = c(P_m \times C_n) + 1$ if m is odd and n is even, $cf(P_m \times C_n) = c(P_m \times C_n) + \frac{n}{2} - 1$ if both m and n are even, and $cf(P_m \times C_n) = c(P_m \times C_n) + \frac{m+n-3}{2}$ if m is even and n is odd.

ORCID iDs

Xin He https://orcid.org/0000-0002-5853-9653
Heping Zhang https://orcid.org/0000-0001-5385-6687

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