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A note on homomorphisms of matrix semigroup

Matjaž Omladič
*Faculty of Mathematics and Physics, University of Ljubljana
Jadranska 19, SI-1000, Ljubljana Slovenia*

Bojan Kuzma
*University of Primorska, FAMNIT, Glagoljaška 8, SI-6000 Koper, Slovenia
and
Institute of Mathematics, Physics and Mechanics, Department of Mathematics,
Jadranska 19, SI-1000 Ljubljana, Slovenia*

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Abstract

Let $\mathbb{F}$ be a field. We classify multiplicative maps from $\mathcal{M}_n(\mathbb{F})$ to $\mathcal{M}_{\binom{n}{k}}(\mathbb{F})$ which annihilate a zero matrix and map rank-k matrix into a rank-one matrix.

Keywords: Matrix semigroup, Homomorphism, Representation.

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1 Introduction and preliminaries

Let $\mathcal{M}_n(\mathbb{F})$ denote the semigroup of all n–by–n matrices with coefficients in a field $\mathbb{F}$, let E_{ij} be its matrix units, and let $\mathrm{Id} = \mathrm{Id}_n := \sum E_{ii}$ be its identity. In [5], Jodeit and Lam classified nondegenerate semigroup homomorphisms $\pi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$, that is, maps which are (i) multiplicative $\pi(AB) = \pi(A)\pi(B)$ and (ii) their restriction on singular matrices is nonconstant. It was shown that the semigroup of such maps is generated by three simple types: (i) a similarity, (ii) a fixed field homomorphism applied entry-wise on a matrix, and (iii) the map which sends A to a matrix of its cofactors. We refer below for more precise definitions.

The complete classification of degenerate maps on $\mathcal{M}_n(\mathbb{F})$ is more involved. They are all of the type $A \mapsto \pi_1(A) \oplus \mathrm{Id}_{n-m}$ for some integer $m \in \{0, \dots, n\}$ and some degenerate multiplicative $\pi_1 : M_n(\mathbb{F}) \to M_m(\mathbb{F})$ with $\pi_1(0) = 0$ [5]. When $m = 1$, Đoković [2, Theorem 1] proved the following.

E-mail addresses: matjaz.omladic@fmf.uni-lj.si (Matjaž Omladič), bojan.kuzma@famnit.upr.si (Bojan Kuzma)

Lemma 1.1. Let $\mathbb{F}$ be a field, and $n \geq 2$. If $\pi : \mathcal{M}_n(\mathbb{F}) \to \mathbb{F}$ is multiplicative, then there exists multiplicative $\phi : \mathbb{F} \to \mathbb{F}$ so that $\pi(X) = \phi(\det X)$. □

When $m < n$ and the characteristic of $\mathbb{F}$ differs from 2, Đoković [2, Theorem 2] also showed π_1 factors through determinant so that $\pi_1 = f \circ \det$ for some multiplicative $f : \mathbb{F} \to M_m(\mathbb{F})$. The classification of those seems to be difficult, and as far as we know they are known only in case $\mathbb{F} = \mathbb{C}$ is the filed of complex numbers, by the work of Omladič, Radjavi, and Šemrl [8]. Later, Guralnick, Li, and Rodman [4], extended the result of Đoković to include also the case $n = m$.

Semigroup homomorphisms mapping into higher dimensional algebras are less known. Kokol-Bukovšek [6, 7] classified them in case they are nondegenerate and map 2–by–2 matrices into 3–by–3 or into 4–by–4. Under additional assumption that a degenerate homomorphism is a polynomial in matrix entries, the classification is well-known, see a book by Weyl [9] (see also Fulton and Harris [3] for holomorphic homomorphisms over a field of complex numbers).

It is our aim to show that all homomorphisms from n–by–n matrices to $\binom{n}{k}$–by–$\binom{n}{k}$ matrices which map a rank-k matrix into a rank-one come from exterior product. Both assumptions on the dimension of the target space as well as on the rank of the matrices are essential; otherwise there are many more maps as we show in Remark 1.4 below. We remark that the main idea, that rank-k idempotents are mapped into rank-1 idempotents, is essentially due to Jodeit and Lam [5].

To be self-contained, we briefly repeat the basics about exterior products. Let $\mathbf{e}_1, \ldots,$ $\mathbf{e}_n$ be the standard basis of column vectors in $\mathbb{F}^n$. Given a linear operator X on $\mathbb{F}^n$, denote by $\bigwedge^k(X)$ the k-th exterior product of X, acting on $\bigwedge^k(\mathbb{F}^n)$, i.e., a k-th exterior product of $\mathbb{F}^n$. Recall [3] that, as a vector space, $\bigwedge^k(\mathbb{F}^n)$ has a basis consisting of $\binom{n}{k}$ elements $\{\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k};\ \ 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$, where $x \wedge y = -y \wedge x$ and $x \wedge x = 0$ is the alternating tensor. Then by definition, $\bigwedge^k(X) : \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k} \mapsto (X\mathbf{e}_{i_1}) \wedge \cdots \wedge (X\mathbf{e}_{i_k})$. It follows easily that $\bigwedge^k(AB) = \bigwedge^k(A) \bigwedge^k(B)$. Also, in lexicographic order of a basis $\left(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}\right)_{1 \leq i_1 < \cdots < i_k \leq n}$, the matrix of $\bigwedge^k(X)$ equals the $\binom{n}{k}$–by–$\binom{n}{k}$ matrix of all k–by–k minors of X, where the element at position corresponding to $(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}\,,\ \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_k})$ is the minor obtained by taking columns $i_1, \ldots, i_k$ and rows $j_1, \ldots, j_k$ of a matrix X. In particular, $\bigwedge^n(X) = \det X$ and $\bigwedge^{n-1}(X)$ is similar to a matrix of cofactors under similarity $S = \sum_{i=1}^{n} (-1)^{i+1} E_{i\,(n-i+1)}$.

Besides the $(n-1)$-st exterior product there are at least two additional multiplicative maps from $\mathcal{M}_n(\mathbb{F})$ to itself. One is an inner automorphism $X \mapsto SXS^{-1}$ where $S \in \mathcal{M}_n(\mathbb{F})$ is fixed, invertible. The other is induced by a field homomorphism $\phi : \mathbb{F} \to \mathbb{F}$ (i.e. an additive multiplicative map) applied entry-wise, that is, it maps a matrix $\sum x_{ij} E_{ij}$ into $\sum \phi(x_{ij}) E_{ij}$. With a slight abuse of notation, we denote this map again by $\phi : X \mapsto \phi(X)$.

Theorem 1.2. Let $\mathbb{F}$ be a field, let $n \geq 2$ be an integer, and let $m = \binom{n}{k}$ for some integer $k = 1, \ldots, n$. If $\pi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$, $\pi(0) = 0$, is a multiplicative map such that $\operatorname{rk}(\pi(A_0)) = 1$ for some matrix A_0 of rank-k, then

$$\pi(X) = S\phi(\bigwedge^k(X))S^{-1}$$

where $\phi : \mathbb{F} \to \mathbb{F}$ is a multiplicative map and $S \in \mathcal{M}_m(\mathbb{F})$ is invertible.

Moreover, if $k < n$ then ϕ is also additive, hence a field embedding.

Remark 1.3. Without the assumption $\pi(0) = 0$, there are more possibilities. Say, $\pi(A) = 1 \oplus \mathrm{Sym}^2(\bigwedge^{n-1} A) \oplus f(\det A)$, where Sym^2 is the second symmetric power (see [3]) and $f : \mathbb{F} \to \mathcal{M}_{m-1-\binom{n+1}{2}}(\mathbb{F})$ is multiplicative.

However, we remark that to classify multiplicative maps π it suffices to assume $\pi(0) = 0$. In fact, if $\pi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ is multiplicative, then $P := \pi(0)$ is an idempotent, and from $\pi(X)P = \pi(X0) = \pi(0) = P = \pi(0X) = P\pi(X)$ we deduce that, relative to decomposition $\mathbb{F}^m = \mathrm{Ker}\, P \oplus \mathrm{Im}\, P$ we have

$$\pi(X) = \pi_1(X) \oplus \mathrm{Id}_r,$$

where $r := \mathrm{rk}\, P$ and $\pi_1 : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{m-r}(\mathbb{F})$ is multiplicative with $\pi_1(0) = 0$.

Remark 1.4. If $m \neq \binom{n}{k}$ there are more possibilities, say $\pi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{\binom{n^2}{4}}(\mathbb{F})$, defined by $A \mapsto \bigwedge^4(A \otimes A)$, is multiplicative and maps a rank-2 matrix $E_{11} + E_{22}$ into matrix of rank-one but is not of the form in the Theorem. This is because if $\mathrm{rk}\, A = r$ then $\mathrm{rk}(\bigwedge^k A) = \binom{r}{k}$, while π maps a rank-3 matrix $E_{11} + E_{22} + E_{33}$ into a matrix whose rank equals 126.

If $\mathrm{rank}(\pi(A_0)) \neq 1$ there are more possibilities as can be seen by the map $\pi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{\binom{n}{k}}(\mathbb{F})$, defined by $A \mapsto A \oplus 0_{\binom{n}{k}-n}$.

Proof of Theorem 1.2. If $k = n$ then $m = 1$, so $\pi : \mathcal{M}_n(\mathbb{F}) \to \mathbb{F}$. Such multiplicative maps were proven to be in accordance with our results by Lemma 1.1.

Hence, we may assume in the sequel that $k < n$. Clearly, $\pi(\mathrm{Id})$ is an idempotent, and from $\pi(X)\pi(\mathrm{Id}) = \pi(X \cdot \mathrm{Id}) = \pi(X) = \pi(\mathrm{Id})\pi(X)$ we deduce that, relative to decomposition $\mathbb{F}^m = \mathrm{Im}\,\pi(\mathrm{Id}) \oplus \mathrm{Ker}\,\pi(\mathrm{Id})$ we have $\pi(X) = \pi_1(X) \oplus 0_{m-r} \in \mathcal{M}_r(\mathbb{F}) \oplus 0_{m-r}$, where $r := \mathrm{rk}\,\pi(\mathrm{Id})$ and π_1 is multiplicative with $\pi_1(0) = 0$ and $\pi_1(\mathrm{Id}) = \mathrm{Id}_r$. As such, if X is invertible, then $\mathrm{Id}_r = \pi_1(\mathrm{Id}) = \pi_1(XX^{-1}) = \pi_1(X)\pi_1(X^{-1})$, so $\pi_1(X)$ is also invertible and $\pi_1(X)^{-1} = \pi_1(X^{-1})$.

Let X be any matrix of rank-k. Then, there exist invertible $S, T \in \mathcal{M}_n(\mathbb{F})$ with $SXT = \mathrm{Id}_k \oplus 0_{n-k}$, and in particular, there exist invertible S_1, T_1 such that $X = S_1 A_0 T_1$. Consequently,

$$1 = \mathrm{rk}\,\pi(A_0) = \mathrm{rk}\big(\pi_1(S_1^{-1}XT_1^{-1}) \oplus 0_{m-r}\big) = \mathrm{rk}\big(\pi_1(S_1)^{-1}\pi_1(X)\pi_1(T_1)^{-1} \oplus 0_{m-r}\big),$$

wherefrom $\mathrm{rk}\,\pi(X) = 1$ for every X of rank-k. Consequently, $\pi(\mathrm{Id}_k \oplus 0_{n-k})$ is an idempotent of rank-1, and by appropriate similarity we may assume it equals E_{11}.

Given $X = \hat{X} \oplus 0_{n-k} \in \mathcal{M}_k(\mathbb{F}) \oplus 0_{n-k}$, we have $X = (\mathrm{Id}_k \oplus 0)X(\mathrm{Id}_k \oplus 0)$, wherefrom $\pi(X) = E_{11}\pi(X)E_{11} \in \mathbb{F}E_{11}$. Hence, π induces a multiplicative map $\hat{\pi} : \mathcal{M}_k(\mathbb{F}) \to \mathbb{F}$ by

$$\hat{\pi}(\hat{X})E_{11} := \pi(\hat{X} \oplus 0_{n-k}).$$

It follows by Lemma 1.1 that there exists a nonzero multiplicative map $\phi_1 : \mathbb{F} \to \mathbb{F}$ such that

$$\hat{\pi}(\hat{X}) = \phi_1(\det \hat{X}).$$

In particular, if the rank of $X = \hat{X} \oplus 0_{n-k}$ is smaller than k, then $\pi(X) = \hat{\pi}(\hat{X})E_{11} = 0$. By multiplicativity, $\pi(X) = 0$ for every $X \in \mathcal{M}_n(\mathbb{F})$ with $\mathrm{rk}\, X \leq k - 1$. Moreover, given any $A \in \mathcal{M}_n(\mathbb{F})$, letting $\hat{A}$ be the compression of A to the upper-left k–by–k block, we have

$$\pi\big((\mathrm{Id}_k \oplus 0)A(\mathrm{Id}_k \oplus 0)\big) = \pi(\hat{A} \oplus 0_{n-k}) = \phi_1(\det \hat{A}).$$

Next, among diagonal idempotent matrices, there exists exactly $m := \binom{n}{k}$ of them that have k ones and $n-k$ zeros on diagonal. We order them lexicographically according to position of ones on diagonal, and denote them

$$P_1 = (\mathrm{Id}_k \oplus 0_{n-k}), \dots, P_m = (0_{n-k} \oplus \mathrm{Id}_k).$$

Given two such diagonal idempotents P_i, P_j, we have $\mathrm{rk}(P_iP_j) \le k$ and the equality holds only if $P_i = P_j$. Hence $\pi(P_1), \dots, \pi(P_m)$ are pairwise orthogonal idempotents of rank-one. It is well-known (say, [1, Lemma 2.2]) that, by applying appropriate similarity, we can achieve $\pi(P_i) = E_{ii}$ for $i = 1, \dots, m$. Combined with $\pi(P_i) = \pi_1(P_i) \oplus 0_{m-r} \in \mathcal{M}_r(\mathbb{F}) \oplus 0_{n-r} \subseteq \mathcal{M}_m(\mathbb{F})$ we see that $r = 0$. Hence, $\pi = \pi_1$ is already unital.

As above for $\pi(P_1AP_1) = \phi_1(\det \hat{A})$ we see that for each $i = 1, \dots, m$ there exist nonzero multiplicative map $\phi_i : \mathbb{F} \to \mathbb{F}$ so that

$$\pi(P_iAP_i) = \phi_i(\det A^{(ii)})E_{ii}, \tag{1.1}$$

where, for a matrix $X \in \mathcal{M}_n(\mathbb{F})$, we denote $X^{(ij)}$ the k–by–k submatrix of X which lies on the rows where P_i has nonzero entries and on the columns where P_j has nonzero entries. Observe that a nonzero multiplicative ϕ_i satisfies $\phi_i(1) = 1$.

Consider any $A \in \mathcal{M}_n(\mathbb{F})$. Then,

$$\pi(A) = \mathrm{Id}\,\pi(A)\,\mathrm{Id} = \Big(\sum_{i=1}^{m} E_{ii}\Big)\pi(A)\Big(\sum_{j=1}^{m} E_{jj}\Big) = \sum_{i,j} E_{ii}\pi(A)E_{jj} = \sum_{i,j} \pi(P_iAP_j).$$

Given indices $i \ne j$, there exists $B_{ji} \in \mathcal{M}_n$ of rank-k such that $B_{ji} = P_jB_{ji}P_i$ and $\det{(B_{ji})}^{(ji)} = 1$; for instance, if $P_i = \sum_{t\in\{t_1,\dots,t_k\}} E_{tt}$ and $P_j = \sum_{s\in\{s_1,\dots,s_k\}} E_{ss}$, with $t_1 < \dots < t_k$ and $s_1 < \dots < s_k$, we can take

$$B_{ji} = \sum_{i=1}^{k} E_{s_it_i} \tag{1.2}$$

and then $(B_{ji})^{(ji)} = \mathrm{Id}_k$. In particular then, $\pi(B_{ji}) = \gamma_{ji}E_{ji} \ne 0$. It follows that

$$\pi(P_iAP_j)\pi(B_{ji}) = \pi(P_i(AP_jP_jB_{ji})P_i) = \phi_i\big(\det{(P_iAP_jP_jB_{ji}P_i)}^{(ii)}\big)E_{ii}. \tag{1.3}$$

Observe that

$$(P_iAP_jP_jB_{ji}P_i)^{(ii)} = A^{(ij)}B_{ji}^{(ji)}.$$

(This follows easily by writing $P_i = \sum_{t\in\{t_1,\dots,t_k\}} E_{tt}$ and $P_j = \sum_{s\in\{s_1,\dots,s_k\}} E_{ss}$, $t_1 < t_2 < \dots < t_k$ and $s_1 < s_2 < \dots < s_k$, and observing that

$$P_iXP_j = \sum_{(t,s)\in\{t_1,\dots,t_k\}\times\{s_1,\dots,s_k\}} x_{ts}E_{ts}$$

and

$$P_jYP_i = \sum_{(t,s)\in\{t_1,\dots,t_k\}\times\{s_1,\dots,s_k\}} y_{st}E_{st},$$

and hence $P_iXP_j \cdot P_jYP_i = \sum_{t,t' \in \{t_1,\dots,t_k\}} \sum_{s \in \{s_1,\dots,s_k\}} x_{ts}y_{st'}E_{tt'}$.)
Hence,

$$\begin{aligned}\phi_i(\det(P_iAP_jP_jB_{ji}P_i)^{(ii)}) &= \phi_i(\det A^{(ij)})\phi_i(\det(B_{ji})^{(ji)}) = \phi_i(\det A^{(ij)}) \cdot \phi_i(1) \\ &= \phi_i(\det A^{(ij)}).\end{aligned}$$

On the other hand, $\pi(P_iAP_j) = \pi(P_i)\pi(A)\pi(P_j) = E_{ii}\pi(A)E_{jj} = \alpha_{ij}(A)E_{ij}$ wherefrom $\pi(P_iAP_j) \cdot \pi(B_{ji}) = \alpha_{ij}(A)E_{ij} \cdot \gamma_{ji}E_{ji} = \alpha_{ij}(A)\gamma_{ji}E_{ii}$, and hence, by (1.3)

$$\alpha_{ij}(A) = \frac{1}{\gamma_{ji}}\phi_i(\det A^{(ij)}).$$

By similar arguments we also have that

$$\begin{aligned}\gamma_{ji}\,\alpha_{ij}(A)E_{jj} &= \pi(B_{ji})\pi(P_iAP_j) = \pi(B_{ji}P_iAP_j) = \pi(P_jB_{ji}P_iAP_j) \\ &= \phi_j(\det A^{(ij)})E_{jj}\end{aligned}$$

and since A was arbitrary, we see that $\phi_i = \phi_j =: \phi$ is independent of i, j. Hence,

$$\pi(X) = \sum_{i,j} \alpha_{ij}(X)E_{ij} = \sum_{i,j} \frac{1}{\gamma_{ij}}\phi(\det X^{(ij)})E_{ij},$$

where, in accordance with (1.1), we define $\gamma_{ii} = 1$ for $i = 1, \dots, m$. Recall also that $\phi(1) = \phi_i(1) = 1$.

We only need to show that multiplicativity of π forces that ϕ is additive and that $\gamma_{ij}\gamma_{jv} = \gamma_{iv}$. To prove additivity of ϕ, choose a scalar α and consider a rank-$(k+1)$ matrix $A_\alpha := \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right) \oplus \mathrm{Id}_{k-1} \oplus 0_{n-k-1}$. It is easy to see that in A_α the number of k–by–k submatrices of rank-k equals $(k+2)$, and they are all obtained from the principal $(k+1)$–by–$(k+1)$ block by deleting one of the following (i) the same row and column, or (ii) second row and first column. Under (i) the resulting submatrix equals Id_k, while under (ii) it equals $\alpha \oplus \mathrm{Id}_{k-1}$. Thus, there exist indices $i_1, \dots, i_{k+1}$ and $i, j \in \{i_1, \dots, i_{k+1}\}$, $i \neq j$, so that

$$\pi(A_\alpha) = \sum_{t=1}^{k+1} \phi(1)E_{i_ti_t} + \tfrac{1}{\gamma_{ij}}\phi(\alpha)E_{ij}.$$

(A deeper analysis reveals that, in lexicographic order, $i = \binom{n-2}{k-2} + 1$ and $j = \binom{n-1}{k-1} + 1$). As $A_\alpha A_\beta = A_{\alpha+\beta}$, the multiplicativity of π together with $\phi(1) = 1$ yields

$$\sum_t E_{i_ti_t} + \tfrac{1}{\gamma_{ij}}\phi(\alpha+\beta)E_{ij} = \pi(A_\alpha A_\beta) = \pi(A_\alpha)\pi(A_\beta) = \sum_t E_{i_ti_t} + \tfrac{\phi(\alpha)+\phi(\beta)}{\gamma_{ij}}E_{ij},$$

wherefrom ϕ is additive.

It remains to prove $\gamma_{ij}\gamma_{jv} = \gamma_{iv}$. Take matrices B_{ij} and B_{jv} defined in (1.2). Then, $\det\big((B_{ij}B_{jv})^{(iv)}\big) = \det((B_{ij})^{(ij)})\det((B_{jv})^{(jv)}) = 1 \cdot 1 = 1$. Hence,

$$\frac{1}{\gamma_{iv}}E_{iv} = \frac{1}{\gamma_{iv}}\phi(\det(B_{ij}B_{jv})^{(iv)})E_{iv} = \pi(B_{ij}B_{jv}) = \pi(B_{ij})\pi(B_{jv}) = \frac{1}{\gamma_{ij}\gamma_{jv}}E_{ij} \cdot E_{jv},$$

wherefrom $\gamma_{iv} = \gamma_{ij}\gamma_{jv}$.

Consider now an invertible diagonal matrix $D = \mathrm{diag}(\gamma_{11}, \dots, \gamma_{1m})$. Then, $\pi(X) = \sum_{i,j} \frac{1}{\gamma_{ij}}\phi(\det X^{(ij)})E_{ij} = D^{-1}\sum_{i,j}\phi(\det X^{(ij)})E_{ij}D = D^{-1}\phi(\bigwedge^k(X))D$. □

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