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Decompositions of the automorphism groups of edge-colored graphs into the direct product of permutation groups

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Abstract

In the paper *Graphical complexity of products of permutation groups*, M. Grech, A. Jeż, and A. Kisielewicz have proved that the direct product of automorphism groups of edge-colored graphs is itself the automorphism groups of an edge-colored graph. In this paper, we study the direct product of two permutation groups such that at least one of them fails to be the automorphism group of an edge-colored graph. We find necessary and sufficient conditions for the direct product to be the automorphism group of an edge-colored graph. The same problem is settled for the edge-colored digraphs.

Keywords: Colored graph, automorphism group, permutation group, direct product.

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1 Introduction

For permutation groups $(A, V), (B, W)$, the *direct product* of A and B (with product action) is a permutation group $(A \times B, V \times W)$ with the action given by

$$(a, b)(x, y) = (a(x), b(y)).$$

The study of the direct product of automorphism groups of graphs was initiated by G. Sabidussi [19] in 1960. The problem was taken up in 1971 by M. Watkins [20]. In 1972, L. Nowitz and M. Watkins [21], and independently W. Imrich [13], have described the conditions under which the direct product of *regular* permutation groups that are automorphism groups of graphs is itself the automorphism group of a graph. This result was

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a contribution to the description of all *regular* automorphism groups of graphs, which has been completed in 1978 by C. Godsil [5] for graphs, and in 1980 by L. Babai [1] for digraphs. The above results in [13, 21] have been extended to arbitrary permutation groups in [6], where the description of the conditions, under which the direct product of automorphism groups of graphs is itself an automorphism group of a graph, is given. In [8], the same is done for digraphs.

All the above results are motivated more or less directly by trying to make a contribution to the solution of the concrete version of König problem asking about a characterization of those permutation groups that are the automorphism groups of graphs (see [14]). There are a number of results (see e.g. [9, 10, 18] and [14]) showing that it is more natural and effective to study the automorphism groups of (edge-)colored graphs (rather than simple graphs), which is essentially the approach suggested by Wielandt [23].

In [14], A. Kisielewicz has introduced the notion of graphical complexity of permutation groups and suggested the study of constructions of permutation groups in this context. By $G(k)$, we denote the class of the automorphism groups of k-edge-colored graphs (those using at most k colors), and by GR, the union of all the classes $G(k)$, which in Wielandt's terminology [23] is the class of 2^*-closed groups. Similarly, by $DG(k)$ we denote the class of the automorphism groups of k-edge-colored digraphs, and by DGR the union of all the classes $DG(k)$ (which in Wielandt's terminology is the class of 2-closed groups). Clearly, $GR \subseteq DGR$, and $G(k) \subseteq DG(k)$, for any k.

The main general problem is to determine which permutation groups are the automorphism groups of edge-colored graphs. Various aspects of this general problem are investigated. For example, it leads to the concept of colored totally symmetric graphs, that was described in [11, 12]. This coincides to a large extent with the research on homogeneous factorization of graphs (c.f., [4, 15, 16]). One direction of research is to consider various constructions of permutation groups and to ask the following question: is it true that if the components of the construction belong to a particular class $G(k)$, then the result belongs to $G(k)$, as well? And if not, how many colors one must add to make sure that the result of the construction belong to $G(k + r)$?

For the direct product the problem has been solved in [9, Theorem 2.2].

Theorem 1.1 (Grech, Jeż, Kisielewicz)**.** If permutation groups $A, B \in GR$, then $A \times B \in GR$. Also, if $A, B \in DGR$, then $A \times B \in DGR$.

Note that the second part of this theorem was also shown in [3, Theorem 5.1]

This result, with some exceptions, is also true for particular classes $G(k)$ and $DG(k)$ (for details see [7]). In this paper we consider the converse of the theorem above. We show that while for DGR the converse also holds (Theorem 3.1), for GR it is not generally true. The main results is Theorem 3.2 characterizing the conditions under which the direct product of two arbitrary permutation groups belongs to GR.

2 Preliminaries

We assume that the reader has basic knowledge in the areas of graphs and permutation groups, so we omit an introduction to standard terminology. If necessary, additional details can be found in [2, 24].

By a *k-edge-colored graph G*, we mean a pair $G = (V, E)$, where V is the set of vertices of G, and E the *edge-color function* from the set $P_2(V)$ of unordered pairs of vertices into the set of colors $\{0, \ldots, k-1\}$ ($E : P_2(V) \to \{0, \ldots, k-1\}$). Thus, G is a complete simple graph with colored edges. Similarly, by a *k-edge-colored digraph* G, we mean a pair (V, E) where E is a color function from the set of ordered pairs of different elements of V to the set of colors $\{0, \ldots, k - 1\}$ ($E : ((V \times V) \setminus \{(v, v); v \in V\}) \to \{0, \ldots, k-1\}$).

An automorphism of an edge-colored graph G is a permutation a of the set V preserving the edge function: $E(\{v, w\}) = E(\{a(v), a(w)\})$, for all $v, w \in V$. The group of automorphisms of G will be denoted by $Aut(G)$, and considered as a permutation group $(Aut(G), V)$ acting on the set of the vertices V. Edge-colored digraphs are defined similarly.

All groups considered in this paper are groups of permutations. They are considered up to permutation group isomorphism. Generally, a permutation group A acting on a set V is denoted (A, V) or just A, if the set V is clear from the context or not important. By S_n we denote the symmetric group on n elements, and by I_n, the one element group acting on n elements (consisting of the identity only, denoted by id).

We shall consider the natural actions of a given permutation group $A = (A, V)$ on the sets of ordered and unordered pairs of V, $V \times V$ and $P_2(V)$, respectively. Let $a \in A$ and $v, w \in V$. Then, the first action of a is given by the formula

$$a((v, w)) = (a(v), a(w)),$$

while the second action is given by

$$a(\{v, w\}) = \{a(v), a(w)\}.$$

The orbits of A in the action on $V \times V$ are called *orbitals* of A. Since in this paper we concider graphs (digraphs) without loops, we exclude trivial orbitals consisting of pairs of the form (v, v). For two orbitals O_1, O_2 we say that O_1 is *paired* with O_2 if and only if $O_2 = \{(w, v) : (v, w) \in O_1\}$. We call an orbital O *self-paired* if it is paired with itself. Moreover, we say that a permutation a *transposes* O_1 and O_2, if $a(O_1) = O_2$.

In addition, the orbits of A in the action on $P_2(V)$ will be called here *2^*-orbitals*. Note that we can think of a 2^*-orbital either as a self paired orbital or as a pair of paired orbitals.

Since $A \times I_1 = I_1 \times A = A$ (up to permutation isomorphism), in this paper, we consider only the direct products $A \times B$ with both the permutation groups A, B different from I_1.

Let $A = (A, V)$ be a permutation group, and let $O_1^*, \ldots O_k^*$ be all the 2^*-orbitals of A. We define an edge-colored graph $G^*(A)$ (called *2^*-orbital graph*) as follows.

$$G^*(A) = (V, E), \text{ where } E : P_2(V) \to \{0, \ldots k-1\}.$$

$$E(\{v, w\}) = i \text{ if and only if the edge } \{v, w\} \text{ belongs to the } 2^*\text{-orbital } O_i^*.$$

Now, we define $A^* = Aut(G^*(A))$. Obviously, $A \subseteq A^*$. It should be clear that A^* is the smallest permutation group on V that contains A and belongs to GR. (Indeed, if G' is a colored graph whose automorphism group contains A, then edges in each 2^*-orbital of A have to have the same color. Hence, each permutation in $Aut(G^*(A))$ belongs to $Aut(G')$.) In particular, we have that $A \in GR$ if and only if $A = A^*$.

Similarly we define the *orbital digraph* $G(A)$ replacing 2^*-orbitals by orbitals. In the same way, denoting $\overline{A} = Aut(G(A))$, we have that $\overline{A}$ is the smallest permutation group on X that contains A and belongs to DGR. Moreover, $A \in DGR$ if and only if $A = \overline{A}$. In addition, $A \subseteq \overline{A} \subseteq A^*$.

For direct products of permutation groups we have the following inclusions

Lemma 2.1.

(i) $A \times B \subseteq Aut(G^*(A \times B)) \subseteq A^* \times B^*$,

(ii) $A \times B \subseteq Aut(G(A \times B)) \subseteq \overline{A} \times \overline{B}$,

Proof. The first inclusion holds for all permutation groups, as it was remarked above. We prove the second inclusion.

Consider the edges of the form $\{(v_1, w), (v_2, w)\}$, which we may refer as edges belonging to the rows. Obviously, they form a union of 2^*-orbitals, and therefore the edges $\{(v_1, w_1), (v_2, w_2)\}$ with $w_1 \neq w_2$ in $Aut(G^*(A \times B))$ have different colors than those belonging to the rows. The same is true for columns, i.e. the edges of the form $\{(w, v_1), (w, v_2)\}$. Thus, rows can be mapped only onto rows by automorphisms of $G^*(A \times B)$, and columns can be mapped only onto columns. This implies that $Aut(G^*(A \times B)) \subseteq A_1 \times B_1$, for some A_1 and B_1. Now let $(a, b) \in Aut(G^*(A \times B))$. Then, the edges $(a, b)(\{(v_1, w), (v_2, w)\})$ and $\{(v_1, w), (v_2, w)\}$ have the same color. Therefore, there is $(a_1, b_1) \in A \times B$ such that $(a_1, b_1)(\{(v_1, w), (v_2, w)\}) = \{(v_1, w), (v_2, w)\}$. Hence, $(a_1^{-1}a, b_1^{-1}b) \in Aut(G^*(A \times B))$ preserves the row with the edge $\{(v_1, w), (v_2, w)\}$. Since every row in $Aut(G^*(A \times B))$ is a copy of $G^*(A)$ (up to recoloring), we have that $a_1^{-1}a \in A^*$, which implies that $a \in A^*$. In a similar way, $b \in B^*$, which completes the proof of the first part of the theorem. The second part is proved similarly. □

We observe that if $C = Aut(G^*(A \times B))$, then C^* may be a proper subgroup of $A^* \times B^*$. The smallest example is $I_2 \times I_2$, where $Aut(G^*(I_2 \times I_2)) = I_2 \times I_2$, while ${I_2}^* \times {I_2}^* = S_2 \times S_2$.

We observe also that if $a \in A^*$, then it not only preserves 2^*-orbitals of A (by definition), but it also preserves orbits of A.

Lemma 2.2. Let $A \neq I_2$ be a permutation group. If $a \in A^*$, then a preserves the orbits of A.

Proof. Let Q_t, $t \in \{1, \ldots, m\}$ be the orbits of A. The claim is obvious if $A = I_t$ for any $t > 2$, so we may assume that there is an orbit Q_i that has at least two elements. Then, the set $P_2(Q_i)$ is nonempty. Moreover, it is clear that $P_2(Q_i)$ is the union of 2^*-orbitals of A. Hence, the edges of $G^*(A)$ that belong to $P_2(Q_i)$ have different colors than the remaining edges. This implies that a preserves the orbit Q_i.

Now, if there is another orbit Q_t, $t \neq i$, then obviously, the edges $\{v, w\}$ with $v \in Q_i$ and $w \in Q_t$ have different colors than the remaining edges. Consequently, every orbit is preserved by a. □

3 Results

We proceed to the main problem of this paper to describe conditions under which $A \times B$ belongs to GR or DGR. The case of directed graphs is pretty easy.

Theorem 3.1. Let A and B be permutation groups. Then, $A \times B \in DGR$ if and only if both A and B are in DGR.

Proof. In view of the Theorem 1.1 quoted in the introduction we need to prove merely the “only if” part. It is enough to prove, without loss of generality, that if $A \notin DGR$, then

$A \times B \notin DGR$. Let $A = (A, V)$ and $B = (B, W)$. We assume that $A \notin DGR$. Then, $A \neq I_2$ (since $I_2 \in DGR$). Moreover, we may choose $a \in \overline{A} \setminus A$. By definition, it preserves all orbitals of A.

Let id_B be the identity in the permutation group B. We show that the permutation (a, id_B) belongs to $Aut(G(A \times B))$. To this end, we show that for every directed edge $e = ((v_1, w_1), (v_2, w_2))$, where $v_1, v_2 \in V$, $w_1, w_2 \in W$, the image $(a, id_B)(e)$ has the same color as e.

Assume first that $v_1 \neq v_2$. Since a preserves orbitals of A, for every pair (v_1, v_2), there is a permutation $a_2 \in A$ such that $a(v_1) = a_2(v_1)$ and $a(v_2) = a_2(v_2)$. We have $(a, id_B)(e) = (a_2, id_B)(e)$, and therefore the directed edges $(a, id_B)(e)$ and e belong to the same orbital of $A \times B$. So, by the definition of the edge-colored digraph $G(A \times B)$, $(a, id_B)(e)$ and e have the same color in $G(A \times B)$.

If $v_1 = v_2$, then since $A \neq I_2$, we may use Lemma 2.2 and find a permutation $a_1 \in A$ such that $a_1(v_1) = a(v_1)$. We have $(a, id_B)(e) = (a_1, id_B)(e)$, and therefore the directed edges $(a, id_B)(e)$ and e belong to the same orbital of $A \times B$. So, they have the same color.

Thus, in all the cases $(a, id_B) \in Aut(G(A \times B))$, but (a, id_B) does not belong to $A \times B$. Therefore, $A \times B \notin DGR$. □

This settles the problem for the case of edge-colored digraphs. The case of edge-colored graphs is different and more complex.

Theorem 3.2. Let A and B be permutation groups. Then, $A \times B \in GR$, except for the following cases:

(i) $A \times B \notin DGR$, that is, either $A \notin DGR$ or $B \notin DGR$,

(ii) either every orbital of $A \in GR$ is self-paired and $B \notin GR \cup \{I_2\}$ or every orbital of $B \in GR$ is self-paired and $A \notin GR \cup \{I_2\}$,

(iii) $A, B \in DGR \setminus (GR \cup \{I_2\})$, and there exist $a \in A^* \setminus A$ and $b \in B^* \setminus B$, such that a transposes every pair of paired orbitals in A, and b transposes every pair of paired orbitals in B.

Proof. We consider a few cases. An obvious consequence of Theorem 3.1 is the following

Corollary 3.3. Let $A \notin DGR$ and B be an arbitrary permutation group. Then, $A \times B \notin GR$.

Accordingly to this corollary, we will assume further that both the components of $A \times B$ belongs to DGR. The next three lemmas deal with the case when one of the groups belongs to GR or is equal to I_2.

Lemma 3.4. Let $A \in DGR \setminus (GR \cup \{I_2\})$ and $B \in GR$. If every orbital of B is self-paired, then $A \times B \notin GR$.

Proof. Denote $A = (A, V)$ and $B = (B, W)$. Let $a \in A^* \setminus A$, and id_B be the identity in the permutation group B. Let $e = \{(v_1, w_1), (v_2, w_2)\}$, where $v_1, v_2 \in V$, $w_1, w_2 \in W$. We show that the edges e and $(a, id_B)(e)$ have the same color. To this end it is enough to prove that $(a, id_B)(e)$ belongs to the same 2^*-orbital of $A \times B$ as e.

If $w_1 = w_2$, then the statement holds by the fact that a preserves all 2*-orbitals of A. Assume $v_1 = v_2$. Since $A \neq I_2$, by Lemma 2.2, a preserves all orbits of A (in its action on V). Hence, there is $a_1 \in A$ such that $a(v_1) = a_1(v_1)$. We have,

$$\begin{aligned}(a, id_B)(\{(v_1, w_1), (v_1, w_2)\}) &= \{(a(v_1), w_1), (a(v_1), w_2)\} \\ &= (a_1, id_B)(\{(v_1, w_1), (v_1, w_2)\}).\end{aligned}$$

Thus, e and $(a, id_B)(e)$ belong to the same 2*-orbital of $A \times B$.

Now let $v_1 \neq v_2$ and $w_1 \neq w_2$. If the pair $a((v_1, v_2))$ belongs to the same orbital of A as the pair (v_1, v_2), then there is $a_1 \in A$ such that $a_1(v_1) = a(v_1)$ and $a_1(v_2) = a(v_2)$. Similarly as above, we have,

$$\begin{aligned}(a, id_B)(\{(v_1, w_1), (v_2, w_2)\}) &= \{(a(v_1), w_1), (a(v_2), w_2)\} \\ &= (a_1, id_B)(\{(v_1, w_1), (v_2, w_2)\}).\end{aligned}$$

Assume, finally, that $v_1 \neq v_2$, $w_1 \neq w_2$ and the pairs $a((v_1, v_2))$, (v_1, v_2) belong to different orbitals of A. Since $a \in A^*$, we know that a preserves all 2*-orbitals of A. This implies that, the pairs $a((v_1, v_2))$ and (v_2, v_1) belong to the same orbital of A. Hence, there is $a_1 \in A$ such that $a_1((v_2, v_1)) = a((v_1, v_2))$. Moreover, since all orbitals of B are self-paired, there is $b \in B$ such that $b((w_1, w_2)) = (w_2, w_1)$. Consequently,

$$(a, id_B)(e) = \{(a_1(v_2), b(w_2)), (a_1(v_1), b(w_1))\} = (a_1, b)(e).$$

Thus $(a, id_B)(e)$ and e belongs to the same 2*-orbital of $A \times B$, and consequently, (a, id_B) does not change the color of the edges.

It follows that $(a, id_B) \in Aut(G^*(A \times B)) = (A \times B)^*$. Since $a \in A^* \setminus A$, $(a, id_B) \notin A \times B$, and therefore $A \times B \neq (A \times B)^*$, which completes the proof. □

Lemma 3.5. Let $A \in DGR \setminus (GR \cup \{I_2\})$ and let $B \in GR$ have at least one not-self-paired orbital. Then, $A \times B \in GR$.

Proof. Let $A = (A, V)$ and $B = (B, W)$. We know, by Lemma 2.1(1), that $Aut(G^*(A \times B)) \subseteq A^* \times B$. Therefore, every $c \in Aut(G^*(A \times B))$ has the form (a, b), where $a \in A^*$ and $b \in B$. We show that, in fact, a always belongs to A. Assume, to the contrary, that $a \in A^* \setminus A$. In this case, since $A \in DGR \setminus (GR \cup \{I_2\})$, there is an (ordered) pair $(v_1, v_2), v_1, v_2 \in V$ such that $a((v_1, v_2)) \neq a_1((v_1, v_2))$, for every $a_1 \in A$. Since B has an orbital which is not-self-paired, there are $w_1, w_2 \in W$ such that $b((w_1, w_2)) \neq (w_2, w_1)$ for every $b \in B$. Now, observe that the edges $(a, b)(\{(v_1, w_1), (v_2, w_2)\})$ and $\{(v_1, w_1), (v_2, w_2)\}$ belong to different 2*-orbitals of $A \times B$. Indeed, if the edges $(a, b)(\{(v_1, w_1), (v_2, w_2)\})$ and $\{(v_1, w_1), (v_2, w_2)\}$ belong to the same 2*-orbital of $A \times B$, then either there are $a_1 \in A$ and $b_1 \in B$ such that $a((v_1, v_2)) = a_1((v_1, v_2))$ and $b((w_1, w_2)) = b_1((w_1, w_2))$ or there are $a_2 \in A$ and $b_2 \in B$ such that $a((v_1, v_2)) = a_2((v_2, v_1))$ and $b((w_1, w_2)) = b_2((w_2, w_1))$. The first case is impossible by the assumption on a. In the second case, we get $b_2^{-1} b((w_1, w_2)) = (w_2, w_1)$, which contradicts the assumption. This implies that $E((a, b)(\{(v_1, w_1), (v_2, w_2)\})) \neq E(\{(v_1, w_1), (v_2, w_2)\})$, which contradicts the fact that $(a, b) \in Aut(G^*(A \times B))$. Consequently, we have $Aut(G^*(A \times B)) \subseteq A \times B$, which completes the proof. □

We summarize Lemma 3.4 and Lemma 3.5.

Corollary 3.6. Let $A \in DGR \setminus (GR \cup \{I_2\})$ and $B \in GR$. Then, $A \times B \in GR$ if and only if there exists a non-self-paired orbital of B.

The following special case must be considered separately.

Lemma 3.7. Let $B \in GR$. Then, $B \times I_2 \in GR$.

Proof. By Lemma 2.1(1), $Aut(G^*(B \times I_2))$ is equal either to $B \times I_2$ or to $B \times S_2$. By our general assumption $B \neq I_1$, hence, in $G^*(B \times I_2)$, there is at least one edge of the form $\{(v, 0), (w, 0)\}$, and being in different orbitals, it has a different color than $\{(v, 1), (w, 1)\}$. Thus, $Aut(G^*(B \times I_2)) = B \times I_2$. Therefore, $B \times I_2 \in GR$. □

This completes the description in all the cases where at least one of the components belongs to GR.

The remaining case occurs where $A, B \in (DGR \setminus GR)$. We start with the following.

Lemma 3.8. Let $A, B \in (DGR \setminus GR)$. If for every $b \in B^*$ there exists a pair of paired orbitals $O_1 \neq O_2$ of B such that b does not transpose O_1 and O_2, then $A \times B \in GR$.

Proof. Let $A = (A, V)$ and $B = (B, W)$. Assume to the contrary that there exists $(a, b) \in Aut(G^*(A \times B)) \setminus (A \times B)$.

First, assume that $a \in A$; then, $b \notin B$. Since $A \in (DGR \setminus GR)$, there is an (ordered) pair (v_1, v_2), where $v_1, v_2 \in V$, which belongs to a non-self paired orbital of A. Since $B \in DGR$, there is an (ordered) pair (w_1, w_2) where $w_1, w_2 \in W$, for which there is no $b_1 \in B$ such that $b_1((w_1, w_2)) = b((w_1, w_2))$. We prove that the edge $\{(v_1, w_1), (v_2, w_2)\}$ belongs to a different 2^*-orbital than the edge $(a, b)(\{(v_1, w_1), (v_2, w_2)\})$. Indeed, if the edges $(a, b)(\{(v_1, w_1), (v_2, w_2)\})$ and $\{(v_1, w_1), (v_2, w_2)\}$ belong to the same 2^*-orbital, then either there are $a_1 \in A$ and $b_1 \in B$ such that $a((v_1, v_2)) = a_1((v_1, v_2))$ and $b((w_1, w_2)) = b_1((w_1, w_2))$ or there are $a_2 \in A$ and $b_2 \in B$ such that $a((v_1, v_2)) = a_2((v_2, v_1))$ and $b((w_1, w_2)) = b_2((w_2, w_1))$. In the former, by assumption on b and w_1, w_2, this is impossible. In the latter, since $a \in A$ it is also impossible. Hence, the edges $(a, b)(\{(v_1, w_1), (v_2, w_2)\})$ and $\{(v_1, w_1), (v_2, w_2)\}$ have different colors in $G^*(A \times B)$. This contradicts the assumption that $(a, b) \in Aut(G^*(A \times B))$.

Next, consider the case where $a \notin A$. Since $A \in DGR$, there is an ordered pair (v_1, v_2), where $v_1, v_2 \in V$, for which there is no permutation $a_1 \in A$ such that $a_1((v_1, v_2)) = a((v_1, v_2))$. Let O_1, O_2 be orbital from the statement of the lemma. By assumption, there are $w_1, w_2 \in W$ such that $\{w_1, w_2\} \in O_1$ and $b((w_1, w_2)) \in O_1$. Thus, $b((w_1, w_2)) = b_1((w_1, w_2))$ for some $b_1 \in B$. A similar proof as above shows that the edge

$$(a, b)(\{(v_1, w_1), (v_2, w_2)\}) = (a, b_1)(\{(v_1, w_1), (v_2, w_2)\})$$

belongs to a different 2^*-orbital than the edge $\{(v_1, w_1), (v_2, w_2)\}$. Again, this contradicts the assumption that $(a, b) \in Aut(G^*(A \times B))$. □

Now, we consider the case where one of the groups is equal to I_2.

Lemma 3.9. Let $A \in (DGR \setminus GR)$. Then, $A \times I_2 \in GR$.

Proof. Let $A = (A, V)$ and $I_2 = (I_2, \{w_1, w_2\})$. Assume to the contrary that there is $(a, b) \in Aut(G^*(A \times I_2)) \setminus (A \times I_2)$. Since, for any $v_1, v_2, v_3, v_4 \in V$, the edges $\{(v_1, w_1), (v_2, w_1)\}$ and $\{(v_3, w_2), (v_4, w_2)\}$ have different colors, $b = id$. In the same way as in the second case of the proof of the Lemma 3.8, we get a contradiction. □

Now, we consider the last case.

Lemma 3.10. Let $A, B \in DGR \setminus (GR \cup I_2)$. If there exists $a \in A^* \setminus A$ which transposes all the pairs of the paired orbitals of A and there exists $b \in B^* \setminus B$ which transposes all the pairs of the paired orbitals of B, then $A \times B \notin GR$. Moreover, $A \times B$ is transitive.

Proof. Let $A = (A, V)$ and $B = (B, W)$. Since $A \neq I_2$ and $B \neq I_2$, by Lemma 2.2, every permutation $a \in A^* \setminus A$ preserves the orbits of A (in its action on V) and every permutation $b \in B^* \setminus B$ preserves the orbits of B (in its action on W). Hence, we obtain immediately, under the assumptions on A and B, that the permutation groups A and B have to be transitive. Consequently, for every $a \in A^*, b \in B^*$, $v, v_1, v_2 \in V$, and $w, w_1, w_2 \in W$, the edge $(a, b)(\{(v, w_1), (v, w_2)\})$ has the same color in $G^*(A \times B)$ as the edge $\{(v, w_1), (v, w_2)\}$, and moreover, the edge $(a, b)(\{(v_1, w), (v_2, w)\})$ has the same color as the edge $\{(v_1, w), (v_1, w)\}$.

We choose a and b as in the statement of the lemma, and fix the elements $v_1 \neq v_2 \in V$ and $w_1 \neq w_2 \in W$. Since a and b preserves no non-self-paired orbital, the ordered pair $a((v_1, v_2))$ belongs to the orbital of the ordered pair (v_2, v_1) and the ordered pair $b((w_1, w_2))$ belongs to the orbital of the ordered pair (w_2, w_1). Hence, there are $a_1 \in A$ and $b_1 \in B$ such that $a((v_1, v_2)) = a_1((v_2, v_1))$ and $b((w_1, w_2)) = b_1((w_2, w_1))$. Therefore, we have

$$\begin{aligned} E((a, b)(\{(v_1, w_1), (v_2, w_2)\})) &= E(\{(a(v_1), b(w_1)), (a(v_2), b(w_2))\}) \\ &= E(\{(a_1(v_2), b_1(w_2)), (a_1(v_1), b_1(w_1))\}) \\ &= E((a_1, b_1)(\{(v_1, w_1), (v_2, w_2)\})) \\ &= E(\{(v_1, w_1), (v_2, w_2)\}). \end{aligned}$$

The vertices v_1, v_2, w_1, and w_2 are arbitrary. Hence, the permutation (a, b) preserves all colors. Consequently, $(a, b) \in Aut(G^*(A \times B) \setminus (A \times B))$. □

This exhausts all cases and ends the proof of the theorem. □

4 Corollaries and problems

First, it is worth noting that for some subclasses the result may be stated in a nice simple form. Since all intransitive permutation groups have a non-self-paired orbital, we have the following.

Corollary 4.1. Let $A \in DGR$, and $B \in GR$ be intransitive. Then, $A \times B \in GR$.

Also, it is easy to observe that the only regular groups with all self-paired orbitals are $S_2^n, n \geq 1$. This implies that:

Corollary 4.2. Let $A \in DGR$, and $B \in GR$ be regular. Then, $A \times B \in GR$ if and only if $B \neq S_2^n$, for every n.

Next, we give an alternative proof of the known fact, that was first observed in [22, Example 3.15]

Corollary 4.3. Every regular permutation group belongs to DGR.

Proof. Let U be an nonsolvable regular group. Then, for every regular group A, the group $A \times U$ is nonsolvable. By [5], we have $A \times U \in G(2) \subseteq DGR$. By Theorem 3.1, $A \in DGR$. □

The next fact, it seems, was not recognized so far.

Corollary 4.4. Except for the abelian groups of exponent greater than two and generalized dicyclic groups, all the finite regular permutation groups belong to the class GR.

Proof. Let A be an abelian group of exponent greater than two or a generalized dicyclic group. It is proved in [5], that in such a case $A \notin G(2)$. The proof shows, in fact, that $A \notin GR$. Assume that A is not as those groups mentioned above. Then, it is well known (see [5]) that $A \times S_2^4 \in G(2)$. Since $S_2^4 \in GR$ and it has all orbitals self-paired, then by Theorem 3.2 (ii), $A \in GR$. □

Theorem 3.2 suggests a few open problems.

Problem 4.5. Describe the permutation groups that have all orbitals self-paired.

This does not seem to be an easy problem. Examples of groups whose all orbitals are self-paired are S_n and their transitive products (direct product, wreath product, etc.). In particular, all groups of the form S_2^k (the direct power) belong to this class. Yet, there are other examples, like the automorphism groups of totally symmetric graphs described in [11]. Note that if a permutation group A having all orbitals self-paired is an automorphism group of a colored digraph D, $A = Aut(D)$, then D is, in fact, an undirected colored graph, and so $A \in GR$.

It would be also desirable to have a description of permutation groups with the property given in Theorem 3.2(iii).

Problem 4.6. Describe all transitive permutation groups A having a permutation $\sigma \in A^* \setminus A$ transposing all pairs of paired orbitals.

We note that all regular abelian group of exponent greater than two and regular generalized dicyclic groups have this property. However, there are also many other examples. For instance, the group $A = \langle (0, 1, 2, 3, 4, 5, 6), (1, 2, 4)(3, 6, 5) \rangle$ is one of them. This group is a subgroup of Frobenius group F_7 generated by translations and multiplication by 2 (which is a permutation of order 3). This suggest the following.

Problem 4.7. Let A be a subgroup of the permutation group $AGL_n(p)$ generated by translations and ω^{2k}, where ω is a generator of the the multiplicative group $F_{p^n}^*$, and k divides n. Moreover, let -1 be not quadratic in F_{p^n}. Is it true that for each such group there is an element $a \in A^* \setminus A$ transposing all pairs of paired orbitals?

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