ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 5 (2022) #P4.03 / 561–565
https://doi.org/10.26493/1855-3974.2600.dcc
(Also available at http://amc-journal.eu)

# A note on the $k$-tuple domination number of graphs*

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Received 9 April 2021, accepted 19 December 2021, published online 3 August 2022

**Abstract**

In a graph $G$, a vertex dominates itself and its neighbours. A set $D \subseteq V(G)$ is said to be a $k$-tuple dominating set of $G$ if $D$ dominates every vertex of $G$ at least $k$ times. The minimum cardinality among all $k$-tuple dominating sets is the $k$-tuple domination number of $G$. In this note, we provide new bounds on this parameter. Some of these bounds generalize other ones that have been given for the case $k = 2$.

*Keywords: $k$-domination, $k$-tuple domination.*

*Math. Subj. Class. (2020): 05C69*

## 1 Introduction

Throughout this note we consider simple graphs $G$ with vertex set $V(G)$. Given a vertex $v \in V(G)$, $N(v)$ denotes the *open neighbourhood* of $v$ in $G$. In addition, for any set $D \subseteq V(G)$, the *degree* of $v$ in $D$, denoted by $\deg_D(v)$, is the number of vertices in $D$ adjacent to $v$, i.e., $\deg_D(v) = |N(v) \cap D|$. The *minimum* and *maximum degrees* of $G$ will be denoted by $\delta(G)$ and $\Delta(G)$, respectively. Other definitions not given here can be found in standard graph theory books such as [12].

Domination theory in graphs have been extensively studied in the literature. For instance, see the books [9, 10, 11]. A set $D \subseteq V(G)$ is said to be a *dominating set* of $G$ if $\deg_D(v) \geq 1$ for every $v \in V(G) \setminus D$. The *domination number* of $G$ is the minimum cardinality among all dominating sets of $G$ and it is denoted by $\gamma(G)$. We define a $\gamma(G)$-set as a dominating set of cardinality $\gamma(G)$. The same agreement will be assumed for optimal parameters associated to other characteristic sets defined in the paper.

*We are grateful to the anonymous reviewers for their useful comments on this note that improved its presentation.

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In 1985, Fink and Jacobson [4, 5] extended the idea of domination in graphs to the more general notion of $k$-domination. A set $D \subseteq V(G)$ is said to be a *$k$-dominating set* of $G$ if $\deg_D(v) \geq k$ for every $v \in V(G) \setminus D$. The *$k$-domination number* of $G$, denoted by $\gamma_k(G)$, is the minimum cardinality among all $k$-dominating sets of $G$. Subsequently, and as expected, several variants for $k$-domination were introduced and studied by the scientific community. In two different papers published in 1996 and 2000, Harary and Haynes [7, 8] introduced the concept of double domination and, more generally, the concept of $k$-tuple domination. Given a graph $G$ and a positive integer $k \leq \delta(G) + 1$, a $k$-dominating set $D$ is said to be a *$k$-tuple dominating set* of $G$ if $\deg_D(v) \geq k - 1$ for every $v \in D$. The *$k$-tuple domination number* of $G$, denoted by $\gamma_{\times k}(G)$, is the minimum cardinality among all $k$-tuple dominating sets of $G$. The case $k = 2$ corresponds to double domination, in such a case, $\gamma_{\times 2}(G)$ denotes the double domination number of graph $G$.

In this note, we provide new bounds on the $k$-tuple domination number. Some of these bounds generalize other ones that have been given for the double domination number.

## 2 New bounds on the $k$-tuple domination number

Recently, Hansberg and Volkmann [6] put into context all relevant research results on multiple domination that have been found up to 2020. In that chapter, they posed the following open problem.

**Problem 2.1** ([6, Problem 5.8, p. 194])**.** Give an upper bound for $\gamma_{\times k}(G)$ in terms of $\gamma_k(G)$ for any graph $G$ of minimum degree $\delta(G) \geq k - 1$.

A fairly simple solution for the problem above is given by the straightforward relationship $\gamma_{\times k}(G) \leq k\gamma_k(G)$, which can be derived directly by constructing a set of vertices $D' \subseteq V(G)$ of minimum cardinality from a $\gamma_k(G)$-set $D$ such that $D \subseteq D'$ and $\deg_{D'}(x) \geq k - 1$ for every vertex $x \in D$. From this construction above, it is easy to check that $D'$ is a $k$-tuple dominating set of $G$ and so,

$$\gamma_{\times k}(G) \leq |D'| = |D| + |D' \setminus D| \leq |D| + (k-1)|D| = k\gamma_k(G).$$

This previous inequality was surely considered by Hansberg and Volkmann and, in that sense, they have established the previous problem assuming that $\gamma_{\times k}(G) < k\gamma_k(G)$ for every graph $G$ with $\delta(G) \geq k - 1$.

We next confirm their suspicions and provide a solution to Problem 2.1.

**Theorem 2.2.** *Let $k \geq 2$ be an integer. For any graph $G$ with $\delta(G) \geq k - 1$,*

$$\gamma_{\times k}(G) \leq k\gamma_k(G) - (k-1)^2.$$

*Proof.* Let $D$ be a $\gamma_k(G)$-set. As $\gamma_{\times k}(G) \leq |V(G)|$ we assume, without loss of generality, that $k|D| - (k-1)^2 \leq |V(G)|$. Now, let $U = \{u_1, \ldots, u_{k-1}\} \subseteq V(G) \setminus D$, $D' = D \cup U$ and $D_0 = \{v \in D : \deg_{D'}(v) < k - 1\}$. The following inequalities arise from counting arguments on the number of edges joining $U$ with $D_0$ and $U$ with $D \setminus D_0$, respectively.

$$\sum_{v \in D_0} \deg_{D'}(v) \geq \sum_{i=1}^{k-1} \deg_{D_0}(u_i) \quad \text{and} \quad |D \setminus D_0|(k-1) \geq \sum_{i=1}^{k-1} \deg_{D \setminus D_0}(u_i).$$

By the previous inequalities and the fact that $D$ is a $k$-dominating set of $G$, we deduce that

$$\begin{aligned}\sum_{v\in D_0}\deg_{D'}(v)+|D\setminus D_0|(k-1) &\geq \sum_{i=1}^{k-1}\deg_{D_0}(u_i)+\sum_{i=1}^{k-1}\deg_{D\setminus D_0}(u_i)\\ &=\sum_{i=1}^{k-1}\deg_D(u_i)\\ &\geq k(k-1).\end{aligned}$$

Now, we define $D''\subseteq V(G)$ as a set of minimum cardinality among all supersets $W$ of $D'$ such that $\deg_W(x)\geq k-1$ for every vertex $x\in D$. Since $\deg_{D'}(x)\geq k-1$ for every $x\in D\setminus D_0$, the condition on $W$ is equivalent to that every vertex $v\in D_0$ has at least $k-1-\deg_{D'}(v)$ neighbours in $W\setminus D$. Hence, by the minimality of $D''$ and the inequality chain above, we deduce that

$$\begin{aligned}|D''\setminus D'| &\leq |D_0|(k-1)-\sum_{v\in D_0}\deg_{D'}(v)\\ &=|D|(k-1)-\left(\sum_{v\in D_0}\deg_{D'}(v)+|D\setminus D_0|(k-1)\right)\\ &\leq |D|(k-1)-k(k-1).\end{aligned}$$

Moreover, it is easy to check that $D''$ is a $k$-tuple dominating set of $G$ because each vertex in $V(G)\setminus D$ is dominated $k$ times by vertices of $D\subseteq D''$ (recall that $D$ is a $k$-dominating set of $G$) and the construction of $D''$ ensures that each vertex in $D$ is dominated $k$ times by vertices of $D''$. Hence,

$$\begin{aligned}\gamma_{\times k}(G)\leq |D''| &= |D'|+|D''\setminus D'|\\ &\leq |D|+k-1+|D|(k-1)-k(k-1)\\ &=k\gamma_k(G)-(k-1)^2,\end{aligned}$$

which completes the proof. □

The bound above is tight. For instance, it is achieved by any complete bipartite graph $K_{k,k'}$ with $k'\geq k$, as $\gamma_{\times k}(K_{k,k'})=2k-1$ and $\gamma_k(K_{k,k'})=k$. When $k=2$, Theorem 2.2 leads to the relationship $\gamma_{\times 2}(G)\leq 2\gamma_2(G)-1$ given in 2018 by Bonomo et al. [1].

A set $D\subseteq V(G)$ is a 2-packing of a graph $G$ if $N[u]\cap N[v]=\emptyset$ for every pair of different vertices $u,v\in D$. The 2-packing number of $G$, denoted by $\rho(G)$, is the maximum cardinality among all 2-packings of $G$.

The next theorem relates the $k$-tuple domination number with the 2-packing number of a graph. Note that the bounds given in this result are generalizations of the bounds $\gamma_{\times 2}(G)\geq 2\rho(G)$ due to Chellali et al. [3], and $\gamma_{\times 2}(G)\leq |V(G)|-\rho(G)$ due to Chellali and Haynes [2].

**Theorem 2.3.** *Let $k\geq 2$ be an integer. For any graph $G$ of order $n$ and $\delta(G)\geq k$,*

$$k\rho(G)\leq\gamma_{\times k}(G)\leq n-\rho(G).$$

*Proof.* Let $D$ be a $\rho(G)$-set and $S$ a $\gamma_{\times k}(G)$-set. Since $\deg_S(v) \geq k$ for every $v \in D \setminus S$, and $\deg_S(v) \geq k-1$ for every $v \in D \cap S$, we deduce that

$$\gamma_{\times k}(G) = |S| \geq \sum_{v \in D \setminus S} \deg_S(v) + \sum_{v \in D \cap S} (\deg_S(v) + 1) \geq k|D| = k\rho(G),$$

and the lower bound follows.

Next, let us proceed to prove that $V(G) \setminus D$ is a $k$-tuple dominating set of $G$. Since $\delta(G) \geq k$, $N(D) \cap D = \emptyset$ and $\deg_D(x) \leq 1$ for every $x \in V(G) \setminus D$, we deduce that $\deg_{V(G)\setminus D}(v) \geq k$ for every $v \in D$ and $\deg_{V(G)\setminus D}(v) \geq k-1$ for every $v \in V(G) \setminus D$. Hence, $V(G) \setminus D$ is a $k$-tuple dominating set of $G$, as desired.

Therefore, $\gamma_{\times k}(G) \leq |V(G) \setminus D| = n - \rho(G)$, which completes the proof. □

Let $\mathcal{H}$ be the family of graphs $H_{k,r}$ defined as follows. For any pair of integers $k, r \in \mathbb{Z}$, with $k \geq 2$ and $r \geq 1$, the graph $H_{k,r}$ is obtained from a complete graph $K_{kr}$ and an empty graph $rK_1$ such that $V(H_{k,r}) = V(K_{kr}) \cup V(rK_1)$, $V(K_{kr}) = \{v_1, \dots, v_{kr}\}$ and $V(rK_1) = \{u_1, \dots, u_r\}$ and $E(H_{k,r}) = E(K_{kr}) \cup (\bigcup_{i=0}^{r-1} \{u_{i+1}v_{ki+1}, \dots, u_{i+1}v_{ki+k}\})$. Figure 1 shows a graph of this family. Observe that $|V(H_{k,r})| = r(k+1)$, $\gamma_{\times k}(H_{k,r}) = kr$ and $\rho(H_{k,r}) = r$ for every $H_{k,r} \in \mathcal{H}$. Therefore, for these graphs the bounds given in Theorem 2.3 are tight, i.e., $\gamma_{\times k}(H_{k,r}) = k\rho(H_{k,r}) = |V(H_{k,r})| - \rho(H_{k,r})$.

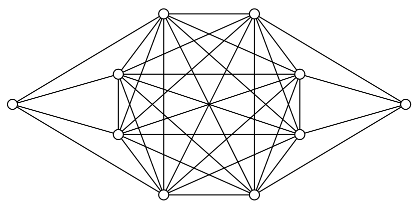

Figure 1: The graph $H_{4,2} \in \mathcal{H}$.

In [8], Harary and Haynes showed that $\gamma_{\times k}(G) \geq \frac{2kn-2m}{k+1}$ for any graph $G$ of order $n$ and size $m$ with $\delta(G) \geq k-1$. The next result is a partial refinement of the bound above because it only considers graphs with minimum degree at least $k$.

**Proposition 2.4.** *Let $k \geq 2$ be an integer. For any graph $G$ of order $n$ and size $m$ with $\delta(G) \geq k$,*

$$\gamma_{\times k}(G) \geq \frac{(\delta(G)+k)n - 2m}{\delta(G)+1}.$$

*Proof.* Let $S$ be a $\gamma_{\times k}(G)$-set and $\overline{S} = V(G) \setminus S$. Hence,

$$\begin{aligned}
2m &= \sum_{v \in S} \deg_S(v) + 2\sum_{v \in \overline{S}} \deg_S(v) + \sum_{v \in \overline{S}} \deg_{\overline{S}}(v) \\
&= \sum_{v \in S} \deg_S(v) + \sum_{v \in \overline{S}} \deg_S(v) + \sum_{v \in \overline{S}} \deg_{V(G)}(v) \\
&\geq (k-1)|S| + k(n-|S|) + \delta(G)(n-|S|) \\
&= (k-1)|S| + (\delta(G)+k)(n-|S|) \\
&= (\delta(G)+k)n - (\delta(G)+1)|S|,
\end{aligned}$$

which implies that $|S| \geq \frac{(\delta(G)+k)n-2m}{\delta(G)+1}$. Therefore, the proof is complete. □

The bound above is tight. For instance, it is achieved for the join graph $G = K_k + C_k$ obtained from the complete graph $K_k$ and the cycle graph $C_k$, with $k \geq 3$. For this case, we have that $\gamma_{\times k}(G) = k$, $|V(G)| = 2k$, $\delta(G) = k + 2$ and $2|E(G)| = 3k^2 + k$. Also, it is achieved for the complete graph $K_n$ ($n \geq 3$) and any $k \in \{2, \ldots, n-1\}$.

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