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One-point extensions in n_3 configurations

William L. Kocay *

Computer Science Department and St. Paul's College,
University of Manitoba, Winnipeg, Manitoba, R3T 2N2 Canada

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Abstract

Given an n_3 configuration, a 1-point extension is a technique that constructs an $(n+1)_3$ configuration from it. It is proved that all $(n + 1)_3$ configurations can be constructed from an n_3 configuration using a 1-point extension, except for the Fano, Pappus, and Desargues configurations, and a family of Fano-type configurations. A 3-point extension is also described. A 3-point extension of the Fano configuration produces the Desargues and anti-Pappian configurations.

The significance of the 1-point extension is that it can frequently be used to construct real and/or rational coordinatizations in the plane of an $(n + 1)_3$ configuration, whenever it is geometric, and the corresponding n_3 configuration is also geometric.

Keywords: Fano configuration, Pappus, Desargues, $(n, 3)$*-configuration.*

Math. Subj. Class.: 51E20, 51E30

1 Projective Configurations

A *projective configuration* consists of a set Σ of points and lines, and an incidence relation Π, such that two lines intersect in at most one point. We denote this by (Σ, Π). For example, a triangle with points A, B, C and lines a, b, c can be represented by the pair $(\{A, B, C, a, b, c\}, \{Ab, Ac, Ba, Bc, Ca, Cb\})$. A configuration (Σ, Π) can also be viewed as a bipartite incidence graph of points versus lines. We will always assume that the incidence graph of a configuration is connected. Excellent references on configurations are the recent books by Grünbaum [7], and by Pisanski and Servatius [11].

An n_3-configuration is a projective configuration with n points and n lines such that every line is incident on 3 points, and every point is incident on 3 lines. There is a unique 7_3-configuration, the Fano configuration, and a unique 8_3-configuration, the Möbius-Kantor

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E-mail address: bkocay@cs.umanitoba.ca (William L. Kocay)

configuration. In 1887, Martinetti [10] presented a method to construct the $(n+1)_3$ configurations from the n_3 configurations. This is described in [7, 6]. Boben [1, 2] has analysed and extended Martinetti's construction significantly. Important related work has also been done by Carstens, Dinski and Steffen [4]. See also [12]. A recent paper [13] by Stokes studies extensions of configurations in a very general setting. The 1-point extension presented here can be related to Stokes's construction, but does not follow directly from it.

An n_3 configuration which can be represented by a collection of points and straight lines in the real or rational plane, such that all incidences are respected, and no two points or two lines coincide, and no unwanted incidences occur, is termed a *geometric* n_3 configuration. In order to show that an n_3 configuration is geometric, the usual method is to assign suitable homogeneous coordinates to its points and lines. We call this a *coordinatization* of the configuration. Some n_3 configurations are not geometric configurations, although it is currently an unsolved problem to determine which n_3 configurations are geometric.

The purpose of this paper is to present a theorem, the 1-point extension theorem, which describes another method to construct an $(n+1)_3$-configuration from an n_3-configuration; and to characterize which configurations can be obtained in this way. The significance of this construction is that if the n_3 configuration is geometric, with a given coordinatization, then there is usually a simple method to extend the coordinatization to the $(n+1)_3$ configuration, that is, the $(n+1)_3$ configuration will also be geometeric. This is too long to include here, it will be the subject of another paper, currently in preparation [8].

In particular the following theorem is proved.

Theorem 1.1. *Let (Σ, Π) be an $(n+1)_3$-configuration. Then (Σ, Π) can be constructed by a 1-point extension from an n_3-configuration if and only if (Σ, Π) is not one of the following configurations:*

a) the Fano configuration,

b) the Pappus configuration,

c) the Desargues configuration,

d) a Fano-type configuration (to be described).

We begin with the idea of a 1-point extension in an n_3-configuration.

Theorem 1.2. (1-Point Extension) *Let (Σ, Π) be an n_3-configuration. Let a_1, a_2, a_3 be 3 distinct points in Σ, and let ℓ_1, ℓ_2, ℓ_3 be 3 distinct lines in Σ such that $a_1 = \ell_1 \cap \ell_2$, $a_2 = \ell_2 \cap \ell_3$ and $a_3 \in \ell_3$, where $a_3 \notin \ell_1$. We can represent this in tabular form as*

$$\begin{array}{ccccc} (\Sigma, \Pi) & \ell_1 & \ell_2 & \ell_3 & \cdots \\ & a_1 & a_1 & a_2 & \cdots \\ & \cdot & a_2 & a_3 & \cdots \\ & \cdot & \cdot & \cdot & \cdots \end{array}$$

where the dots indicate other points of the configuration. Let ℓ' be the third line containing a_1. Suppose further that if $\ell' \cap \ell_3 \neq \emptyset$, then $\ell' \cap \ell_3 = a_3$. Construct a new configuration (Σ', Π') as follows. $\Sigma' = \Sigma \cup \{a_0, \ell_0\}$ where a_0 is a new point and ℓ_0 is a new line. $\Pi' = \Pi - \{a_1\ell_1, a_2\ell_2, a_3\ell_3\} \cup \{a_1\ell_3, a_2\ell_0, a_3\ell_0, a_0\ell_0, a_0\ell_1, a_0\ell_2\}$. We can represent this in tabular form as

(Σ',Π')	ℓ_0	ℓ_1	ℓ_2	ℓ_3	$\cdots$
	a_2	a_0	a_1	a_1	$\cdots$
	a_3	$\cdot$	a_0	a_2	$\cdots$
	a_0	$\cdot$	$\cdot$	$\cdot$	$\cdots$

Here the dots represent exactly the same points as in the previous table. Then (Σ',Π') is an $(n+1)_3$-configuration.

Proof. The only incidences in which (Σ',Π') and (Σ,Π) differ are those involving $\ell_0,\ell_1,\ell_2,\ell_3$. It is easy to verify from the tables that each of a_1,a_2 and a_3 occurs in exactly 3 lines in both (Σ',Π') and (Σ,Π), and that a_0 also occurs in exactly 3 lines. We must still verify that any two lines of (Σ',Π') intersect in at most one point. Notice that ℓ_0 intersects ℓ_1 and ℓ_2 in exactly one point, since $a_3 \notin \ell_1,\ell_2$. Also, ℓ_0 intersects ℓ_3 in exactly one point. If $\ell \neq \ell_1,\ell_2,\ell_3$ is any line of (Σ,Π) intersecting ℓ_1, then in (Σ',Π'), it intersects ℓ_1 in either 0 or 1 point. If ℓ intersects ℓ_2 in (Σ,Π), then in (Σ',Π'), it intersects ℓ_2 in either 0 or 1 point. If $\ell=\ell'$, the third line of (Σ,Π) containing a_1, then in (Σ',Π'), ℓ intersects ℓ_3 in only a_1, because of the condition concerning ℓ'. If $\ell\neq\ell'$ and ℓ intersects ℓ_3 in (Σ,Π), then then since $a_1\notin\ell_3$ in (Σ,Π), it follows that ℓ intersects ℓ_3 in 0 or 1 point in (Σ',Π'). Finally, if ℓ is any line of (Σ,Π) not intersecting ℓ_1,ℓ_2, then it does not intersect ℓ_1,ℓ_2 in (Σ',Π'). If ℓ does not intersect ℓ_3 in (Σ,Π), it may intersect ℓ_3 in a_1 in (Σ',Π'). This completes the proof of the theorem.

□

Example 1.3. The Fano configuration can be represented by the following table.

Fano	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7
	1	2	3	4	5	6	7
	2	3	4	5	6	7	1
	4	5	6	7	1	2	3

Choose ℓ_1,ℓ_2,ℓ_3 as indicated, and choose $a_1=2$, $a_2=3$, $a_3=6$, and let $a_0=8$. Notice that the third line containing a_1 is $\ell'=\ell_6$, which intersects ℓ_3 in $a_3=6$. Then by Theorem 1.2, the following table represents an 8_3-configuration, which is known to be unique.

8_3-config	ℓ_0	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7
	3	1	2	2	4	5	6	7
	6	4	5	3	5	6	7	1
	8	8	8	4	7	1	2	3

The 8_3-configuration can be viewed as a double cover of the cube [9]. It is possible to apply a 1-point extension to this configuration in two possible ways, resulting in two distinct 9_3-configurations. The third 9_3-configuration, known as the Pappus configuration, cannot be obtained in this way.

The 1-point extension theorem can be illustrated by the diagram of Figure 1. In (Σ,Π), we have a substructure consisting of 3 points a_1,a_2,a_3, and 3 lines, ℓ_1,ℓ_2,ℓ_3, sequentially incident, forming a self-dual substructure contained in the n_3-configuration. After the extension, we find that (Σ',Π') contains a triangle with vertices a_1,a_2,a_0 and sides ℓ_2,ℓ_3,ℓ_0, where the third point on ℓ_0 is a_3, and the third line through a_0 is ℓ_1. This is again a self-dual substructure in the configuration.

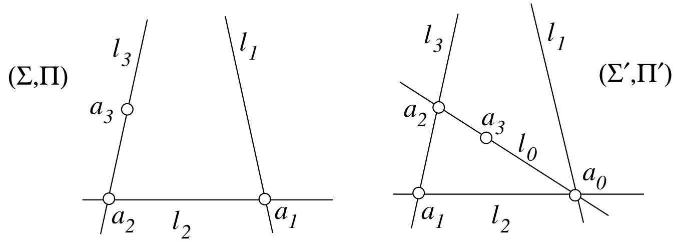


Figure 1: A 1-point extension with 3 points

Corollary 1.4. *In (Σ',Π'), the third line through a_1 does not intersect ℓ_1; the third point on ℓ_3 is not collinear with a_3; and the third line through a_2 does not intersect ℓ_2.*

Proof. If there were a line ℓ in (Σ',Π') through a_1 which intersected ℓ_1 in a point u, then in (Σ,Π), ℓ would intersect ℓ_1 in u and a_1, which is impossible. If there were a point x in (Σ',Π') on ℓ_3 collinear with a_3, then the line ℓ containing a_3 and x would also be a line in (Σ,Π), where it would intersect ℓ_3 in two points. Finally, if there were a line ℓ in (Σ',Π') through a_2 which intersected ℓ_2 in a point u, then in (Σ,Π), ℓ would intersect ℓ_2 in a_2 and u, which is impossible. □

The purpose of this paper is to characterize the configurations that can be obtained using 1-point extensions. In practice, the 1-point extensions are very easy to find and apply, and can easily be done by computer. However, the characterization of which configurations can be obtained by them is very long and tedious. We shall refer to the Fano, Pappus, and Desargues configurations, illustrated in Figure 1.1.

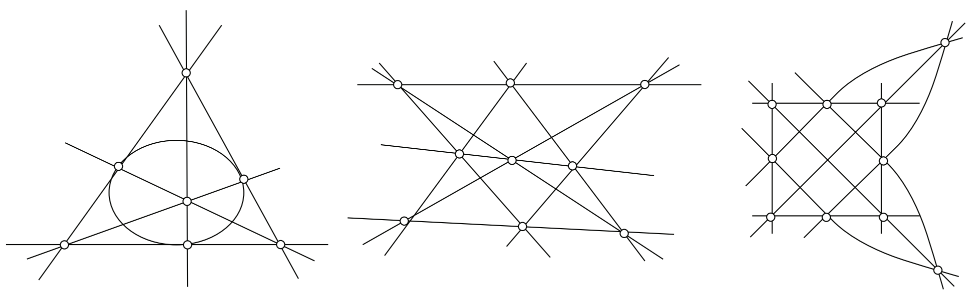


Figure 2: The Fano, Pappus, and Desargues configurations

The conditions of Corollary 1.4 will be used frequently in the characterization. We state them here. We are concerned with an ordered triangle, denoted $\Delta(i,j,k)$, where i, j and k are the first, second, and third vertices, respectively, of the triangle. The line containing i and j is denoted ℓ_{ij}, etc.

Definition 1.5. Let (Σ,Π) be a configuration containing an ordered triangle $\Delta(i,j,k)$. We define the following 3 conditions:

A) The third line through k intersects ℓ_{ij};

B) The third line through i intersects the third line through j;

C) The third point on ℓ_{ik} is collinear with the third point on ℓ_{jk}.

The definition is illustrated in Figure 3.

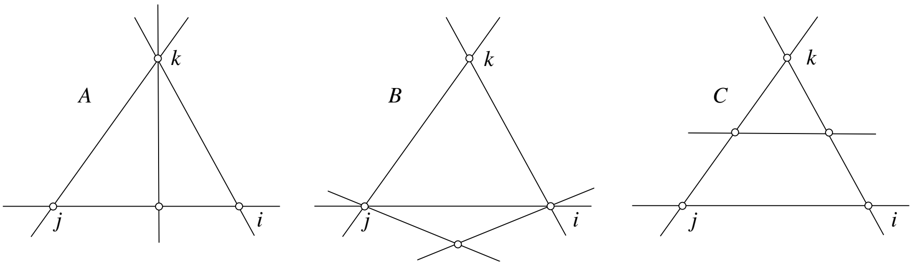


Figure 3: Conditions A, B and C for triangle $\Delta(i, j, k)$

Theorem 1.6. *Let (Σ', Π') be an $(n + 1)_3$-configuration containing a triangle Δ. If conditions A, B and C do not apply to some ordering of the triangle, then (Σ', Π') can be derived from an n_3-configuration by a 1-point extension.*

Proof. Let the ordered triangle to which conditions A, B and C do not apply be $\Delta(a_0, a_1, a_2)$, and let the sides of the triangle be ℓ_0, ℓ_2, ℓ_3, where $a_0 = \ell_0 \cap \ell_2$, $a_1 = \ell_2 \cap \ell_3$, $a_2 = \ell_3 \cap \ell_0$. Let a_3 be the third point on ℓ_0, and let ℓ_1 be the third line through a_0. Observe that $a_3 \notin \ell_1$. These incidences are characterized by the following table.

(Σ, Π)	ℓ_0	ℓ_1	ℓ_2	ℓ_3
	a_2	a_0	a_1	a_1
	a_3	$\cdot$	a_0	a_2
	a_0	$\cdot$	$\cdot$	$\cdot$

We can then delete a_0 and ℓ_0, and change the incidences to the following.

(Σ', Π')	ℓ_1	ℓ_2	ℓ_3
	a_1	a_1	a_2
	$\cdot$	a_2	a_3
	$\cdot$	$\cdot$	$\cdot$

Call the result (Σ', Π'). If ℓ is the third line through a_2 in (Σ, Π), then since condition A does not apply, we know that in (Σ', Π'), ℓ and ℓ_2 intersect in just one point. If ℓ is the third line through a_1 in (Σ, Π), then since condition B does not apply, we know that in (Σ', Π'), ℓ and ℓ_1 intersect in just one point, a_1. Since $\ell \cap \ell_3 = a_1$ in (Σ, Π), it follows that in (Σ', Π'), if ℓ and ℓ_3 intersect, they intersect in a_3.

If ℓ is any line other than ℓ_0 through a_3 in (Σ, Π), then since condition C does not apply, we know that in (Σ', Π'), ℓ and ℓ_3 intersect in just one point. The result is an n_3-configuration to which Theorem 1.2 applies.

□

Given an ordered triangle $\Delta(i, j, k)$, the dual is an ordered triangle whose sides are lines which can be denoted i', j', k'. The dual of condition A is that the third point on k' is collinear with $i' \cap j'$. But this is just condition A again applied to the triangle $\Delta(i' \cap k', j' \cap k', i' \cap j')$. So condition A is self-dual. The dual of condition B is that the third point on i' is collinear with the third point on j'. This is just condition C applied to the triangle $\Delta(i' \cap k', j' \cap k', i' \cap j')$. So B and C are dual conditions.

Theorem 1.6 is the main tool which we will use to characterize the extensions. We will find all configurations such that at least one of conditions A, B, and C apply to every ordering of every triangle. We will also need longer cycles than triangles.

2 The General Extension Theorem

Before beginning the characterization of the n_3-configurations that can be obtained by 1-point extensions, we generalize Theorem 1.2 to m points and m lines, sequentially incident.

Theorem 2.1. (General 1-Point Extension) *Let (Σ, Π) be an n_3-configuration. Let $a_1, a_2, \ldots, a_m$ be m distinct points in Σ, where $3 \le m \le n$, and let $\ell_1, \ell_2, \ldots, \ell_m$ be m distinct lines in Σ such that $a_1 = \ell_1 \cap \ell_2$, $a_2 = \ell_2 \cap \ell_3$, …, $a_{m-1} = \ell_{m-1} \cap \ell_m$, and $a_m \in \ell_m$. Suppose that $a_{m-1}, a_m \notin \ell_1, \ell_2$, and that $a_i \notin \ell_{i+3}$, where $i = 1, 2, \ldots, m-3$. We can represent this in tabular form as*

$$\begin{array}{lcccccc}
(\Sigma,\Pi) & \ell_1 & \ell_2 & \ell_3 & \ldots & \ell_{m-1} & \ell_m \\
 & a_1 & a_1 & a_2 & \ldots & a_{m-2} & a_{m-1} \\
 & \cdot & a_2 & a_3 & \ldots & a_{m-1} & a_m \\
 & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot
\end{array}$$

where the dots indicate other points of the configuration. Let ℓ'_i be the third line containing a_i, where $1 \le i \le m-2$. Suppose further that if $\ell'_i \cap \ell_{i+2} \neq \emptyset$, then $\ell'_i \cap \ell_{i+2} = a_{i+2}$. Construct a new configuration (Σ', Π') as follows. $\Sigma' = \Sigma \cup \{a_0, \ell_0\}$ where a_0 is a new point and ℓ_0 is a new line. $\Pi' = \Pi - \{a_1\ell_1, a_2\ell_2, \ldots, a_m\ell_m\} \cup \{a_1\ell_3, a_2\ell_4, \ldots, a_{m-2}\ell_m, a_{m-1}\ell_0, a_m\ell_0, a_0\ell_0, a_0\ell_1, a_0\ell_2\}$. We can represent this in tabular form as

$$\begin{array}{lccccccc}
(\Sigma',\Pi') & \ell_0 & \ell_1 & \ell_2 & \ell_3 & \ldots & \ell_{m-1} & \ell_m \\
 & a_{m-1} & a_0 & a_0 & a_1 & \ldots & a_{m-3} & a_{m-2} \\
 & a_m & \cdot & a_1 & a_2 & \ldots & a_{m-2} & a_{m-1} \\
 & a_0 & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot
\end{array}$$

Here the dots represent exactly the same points as in the previous table. Then (Σ', Π') is an $(n+1)_3$-configuration.

Proof. The only incidences in which (Σ', Π') and (Σ, Π) differ are those involving $\ell_0, \ell_1, \ell_2, \ldots, \ell_m$. It is easy to verify from the tables that each of $a_1, a_2, \ldots, a_m$ occurs in exactly 3 lines in both (Σ', Π') and (Σ, Π), and that a_0 also occurs in exactly 3 lines. We must still verify that any two lines of (Σ', Π') intersect in at most one point. Notice that ℓ_0 intersects ℓ_1 and ℓ_2 in exactly one point, since $a_{m-1}, a_m \notin \ell_1, \ell_2$. It does not intersect $\ell_3, \ldots, \ell_{m-1}$, and it intersects ℓ_m in exactly one point.

Let $\ell \neq \ell_1, \ell_2, \ldots, \ell_m$ be a line of (Σ, Π). If ℓ intersects ℓ_1 in (Σ, Π), then in (Σ', Π'), it intersects ℓ_1 in either 0 or 1 point. If ℓ intersects ℓ_2 in (Σ, Π), then in (Σ', Π'), it intersects

ℓ_2 in either 0 or 1 point. Suppose that ℓ intersects ℓ_3 in (Σ, Π). If $\ell = \ell_1'$, then $\ell \cap \ell_3 = a_3$ in (Σ, Π) according to the condition of the theorem concerning ℓ_i'. It follows that $\ell \cap \ell_3 = a_1$ in (Σ', Π'). If $\ell \neq \ell_1'$, then ℓ intersects ℓ_3 in either 0 or 1 point in (Σ', Π'). An identical argument holds if ℓ intersects one of $\ell_4, \ldots, \ell_m$ in (Σ, Π).

Suppose that ℓ does not intersect ℓ_1 in (Σ, Π). Then it also does not intersect ℓ_1 in (Σ', Π'). Similarly, if ℓ does not intersect ℓ_2 in (Σ, Π), then it also does not intersect ℓ_2 in (Σ', Π'). Suppose that ℓ does not intersect ℓ_3 in (Σ, Π). Then in (Σ', Π'), it may intersect ℓ_3 only in a_1. A similar argument holds if ℓ does not intersect $\ell_4, \ldots, \ell_m$.

Finally, let ℓ_i and ℓ_j, where $1 \leq i < j \leq m$, be two lines of (Σ, Π). If $j = i + 1$, then ℓ_i and ℓ_j intersect in one point in both (Σ, Π) and (Σ', Π'). Suppose that $j = i + 2$. If $\ell_i \cap \ell_j = \emptyset$ in (Σ, Π), then it is also $\emptyset$ in (Σ', Π'). Now $\ell_i \cap \ell_j \neq a_{i-1}$ in (Σ, Π) (when $i > 1$), because of the hypothesis that $a_k \notin \ell_{k+3}$. Also, $\ell_i \cap \ell_j \neq a_i$, because ℓ_{i+1} contains a_i and a_{i+1}. It follows that $|\ell_i \cap \ell_j|$ is the same in (Σ, Π) and (Σ', Π') when $j = i + 2$. Suppose now that $j \geq i + 3$. It is easy to see that $|\ell_i \cap \ell_j| \leq 1$ in (Σ', Π'). This completes the proof of the theorem. □

Theorem 2.1 is illustrated in Figure 4, with $m = 4$. This general form of Theorem 2.1 is stated separately from Theorem 1.2, because the form with $m = 3$ is simpler, and because we shall mostly only require Theorems 1.2 and 1.6 when characterizing extensions.

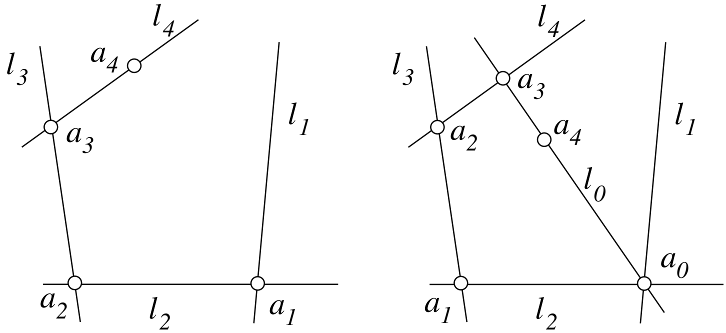


Figure 4: A 1-point extension with 4 points

An ordered *cycle* in a configuration is a sequence of distinct points and lines which are cyclicly incident, for example $C = (a_1, \ell_1, a_2, \ell_2, \ldots, a_m, \ell_m)$, where $a_i = \ell_{i-1} \cap \ell_i$ for $i = 2, 3, \ldots, m$, and $a_1 = \ell_m \cap \ell_1$. Here $m \geq 3$. Each point of C is incident on two lines of C, and vice versa.

Corollary 2.2. *Let (Σ, Π) and (Σ', Π') be as in Theorem 2.1, so that $C = (a_0, \ell_2, a_1, \ell_3, \ldots, a_{m-2}, \ell_m, a_{m-1}, \ell_0)$ is an ordered cycle in (Σ', Π'). Then in (Σ', Π'):*

i) the third points of ℓ_m and ℓ_0 are not collinear;

ii) the third point on ℓ_i is not contained in the third line through a_i, for $i = 2, \ldots, m - 1$;

iii) the third lines through a_0 and a_1 do not intersect.

Proof. The third point of ℓ_0 is a_m. If there were a line ℓ in (Σ', Π') containing a_m and the third point of ℓ_m, then in (Σ, Π), ℓ and ℓ_m would intersect in two points, which is impossible.

Let ℓ be the third line through a_i in (Σ', Π'), for some $i = 2, \ldots, m-1$, and let u be the third point on ℓ_i. Suppose that $u \in \ell$. In (Σ', Π'), a_i is contained in ℓ_{i+1} and ℓ_{i+2}, but in (Σ, Π), a_i is contained in ℓ_i and ℓ_{i+1}. We then find that in (Σ, Π), $\ell \cap \ell_i = \{u, a_i\}$, which is impossible.

The third line through a_0 is ℓ_1. Let ℓ be the third line through a_1. If $\ell \cap \ell_1 = u$ in (Σ', Π'), then in (Σ, Π), $\ell \cap \ell_1 = \{u, a_1\}$, which is impossible. $\square$

Observe that a triangle is a set of three distinct points and lines that are cyclically incident. Similarly, we define a *quadrangle* to be a set of four distinct points and lines that are cyclically incident. We will also need conditions similar to A, B, C for quadrangles. An ordered quadrangle with vertices i, j, k, m is denoted $\square(i, j, k, m)$. In analogy with Definition 1.5 and Corollary 2.2, we make the following definition for a quadrangle.

Definition 2.3. Let (Σ, Π) be a configuration containing an ordered quadrangle $\square(i, j, k, m)$. We define the following 4 conditions:

D) The third point on ℓ_{im} is collinear with the third point on ℓ_{km};

E) The third line through m intersects ℓ_{jk};

F) The third line through k intersects ℓ_{ij};

G) The third line through j intersects the third line through i.

These conditions are illustrated in Figure 5.

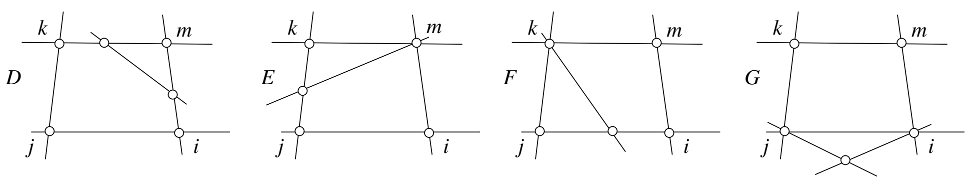


Figure 5: Conditions D, E, F, G for quadrangle $\square(i, j, k, m)$

The analog of Theorem 1.6 for general 1-point extensions is the following.

Theorem 2.4. *Let (Σ', Π') be an $(n+1)_3$-configuration containing an ordered cycle $C = (a_0, \ell_2, a_1, \ell_3, a_2, \ell_4, \ldots, a_{m-2}, \ell_m, a_{m-1}, \ell_0)$, where $m \geq 4$; $a_0, a_1, \ldots, a_{m-1}$ are distinct points; and $\ell_0, \ell_2, \ell_3, \ldots, \ell_{m-1}$ are distinct lines. Let ℓ_1 denote the third line containing a_0 and let a_m denote the third point on ℓ_0. Suppose that ℓ_1 is distinct from $\ell_0, \ell_2, \ell_3, \ldots, \ell_{m-1}$ and that $a_2 \notin \ell_1$. Let ℓ'_i denote the third line containing a_i, for $i = 1, 2, \ldots, m-1$. Suppose that ℓ'_i does not not contain the third point of ℓ_i, for $i = 2, \ldots, m-1$; that $\ell'_1 \cap \ell_1 = \emptyset$; and that a_m is not collinear with the third point of ℓ_m. Then (Σ', Π') can be derived from an n_3-configuration by a 1-point extension.*

Proof. The incidences of the ordered cycle can be represented by the following table.

(Σ,Π)	ℓ_0	ℓ_1	ℓ_2	ℓ_3	$\dots$	ℓ_{m-1}	ℓ_m
	a_{m-1}	a_0	a_0	a_1	$\dots$	a_{m-3}	a_{m-2}
	a_m	$\cdot$	a_1	a_2	$\dots$	a_{m-2}	a_{m-1}
	a_0	$\cdot$	$\cdot$	$\cdot$	$\dots$	$\cdot$	$\cdot$

We can then delete a_0 and ℓ_0, and change the incidences to the following.

(Σ',Π')	ℓ_1	ℓ_2	ℓ_3	$\dots$	ℓ_{m-1}	ℓ_m
	a_1	a_1	a_2	$\dots$	a_{m-2}	a_{m-1}
	$\cdot$	a_2	a_3	$\dots$	a_{m-1}	a_m
	$\cdot$	$\cdot$	$\cdot$	$\dots$	$\cdot$	$\cdot$

Call the result (Σ,Π). It is clear that each point of (Σ,Π) is contained in exactly three lines. We have to show that any two lines intersect in at most one point in (Σ,Π), and that $\ell_1,\ell_2,\ell_3,\dots,\ell_m$ are distinct lines in (Σ,Π). Any two of $\ell_1,\ell_2,\dots,\ell_m$ intersect in at most one point because we began with an ordered cycle of distinct points, and because $a_2 \notin \ell_1$. Let ℓ be any line not in this set. Suppose that ℓ intersects ℓ_i in two points, for some $i = 2,\dots,m-1$. Now ℓ_i contains a_{i-1}, a_i and a third point z. If ℓ contained a_i, then $\ell = \ell_i'$, which does not intersect ℓ_i in (Σ',Π'), by assumption. Therefore $a_i \notin \ell$. Otherwise ℓ must contain a_{i-1} and z. But these points are in ℓ_i in (Σ',Π'), and ℓ is unchanged. It follows that ℓ intersects $\ell_2,\dots,\ell_{m-1}$ in at most one point each.

Suppose that ℓ intersects ℓ_1 in two points in (Σ,Π). Now ℓ_1 contains a_1 and two other points u,v. As u and v are both on ℓ_1 in (Σ',Π'), it follows that ℓ does not contain both u and v. Therefore $\ell = \ell_1'$. But by assumption, $\ell_1' \cap \ell_1 = \emptyset$ in (Σ',Π').

Suppose that ℓ intersects ℓ_m in two points in (Σ,Π). The two points cannot be a_{m-1}, a_m, because these points occur on ℓ_0 in (Σ',Π'). They cannot be a_{m-1} and a third point w, because these points occur on ℓ_m in (Σ',Π'). And they cannot be a_m and the third point w, because by assumption, a_m is not collinear with the third point of ℓ_m in (Σ',Π'). We conclude that (Σ,Π) is an n_3-configuration to which the conditions of Theorem 2.1 apply. □

Corollary 2.5. *Let (Σ',Π') be an $(n+1)_3$-configuration containing a quadrangle $\square(i,j,k,m)$. If conditions D,E,F and G do not apply to some ordering of the quadrangle, and if the third line through i does not contain k, then (Σ',Π') can be derived from an n_3-configuration by a 1-point extension.*

Proof. The conditions D,E,F,G, and $a_2 = k \notin \ell_1$ are the conditions of Theorem 2.4 applied to an ordered quadrangle. □

Theorem 2.6. *Let (Σ',Π') be an $(n+1)_3$-configuration. If (Σ',Π') does not contain a triangle, then it can be derived by a 1-point extension from an n_3-configuration.*

Proof. Choose a cycle of smallest possible length in (Σ',Π'). Denote the cycle by

$$(a_0,\ell_2,a_1,\ell_3,a_2,\ell_4,\dots,a_{m-2},\ell_m,a_{m-1},\ell_0),$$

where $m \geq 4$. Let ℓ_1 be the third line containing a_0, and let a_m be the third point on ℓ_0. This can be denoted in tabular from by

$$\begin{array}{llllllll}
(\Sigma,\Pi) & \ell_0 & \ell_1 & \ell_2 & \ell_3 & \dots & \ell_{m-1} & \ell_m \\
 & a_{m-1} & a_0 & a_0 & a_1 & \dots & a_{m-3} & a_{m-2} \\
 & a_m & \cdot & a_1 & a_2 & \dots & a_{m-2} & a_{m-1} \\
 & a_0 & \cdot & \cdot & \cdot & \dots & \cdot & \cdot
\end{array}$$

Let ℓ'_i denote the third line containing a_i, where $i = 1, 2, \dots, m-1$. If ℓ'_i were to intersect ℓ_i in a point z, where $i = 2, \dots, m-1$, this would create a triangle $\Delta(a_{i-1}, a_i, z)$. If ℓ'_1 were to intersect ℓ_1 in a point u, this would create a triangle $\Delta(a_0, a_1, u)$. If a_m were collinear with the third point w of ℓ_m, this would create a triangle $\Delta(a_{m-1}, a_m, w)$. If ℓ_1 contained a_2, this would create a triangle $\Delta(a_0, a_1, a_2)$. It follows that the conditions of Theorem 2.4 apply, so that (Σ', Π') can be derived by a 1-point extension from an n_3-configuration. □

3 Fano-Type Configurations

Let F denote the Fano configuration, the unique 7_3 configuration. We will use three sub-configurations to build a family of n_3 configurations which cannot be obtained by 1-point extensions.

Definition 3.1. Denote by F' the unique configuration obtained from F by removing a single incidence. Denote by F_ℓ the unique configuration obtained from F by removing a line. Denote by F_p the unique configuration obtained from F by removing a point. Note that F_ℓ and F_p are dual configurations.

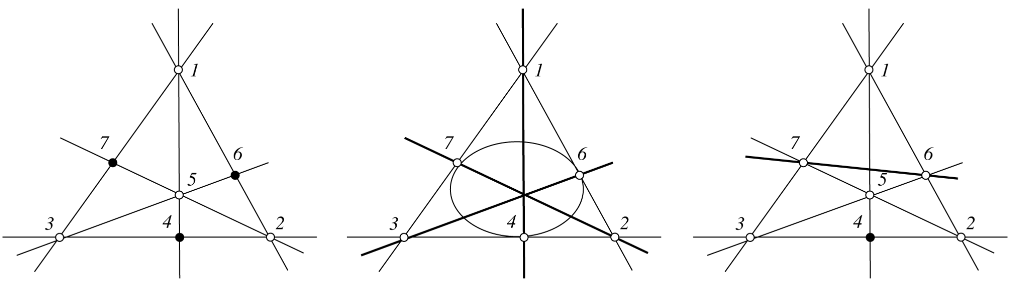


Figure 6: The configurations F_ℓ, F_p and F'

The configurations F_ℓ, F_p and F' are not n_3-configurations. They can be used as building blocks of n_3 configurations, which we call *Fano-type* configurations. F' has one point on only two lines, and one line containing only two points. F_p has three lines containing only two points. Every point is in three lines. F_ℓ has three points in only two lines. Every line contains three points. These are illustrated schematically in Figure 7, where the points missing a line are indicated as black circles, and the lines missing a point are indicated as lines.

These sub-configurations can be used as modules, which can be connected together like vertices of a graph, to create graphs representing n_3 configurations. For example, two or more copies of F' can be connected into a cycle or path of arbitrary length. If only F_ℓ and F_p are used, the resulting structure is a bipartite graph.

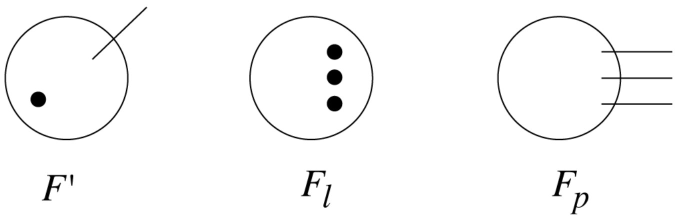


Figure 7: F', F_ℓ and F_p schematically

Theorem 3.2. *Let G be a multigraph which is isomorphic to either a cycle of length ≥ 2, or a subdivision of a 3-regular bipartite multigraph, with bipartition (X, Y). Replace each vertex of X by a configuration F_p, replace each vertex of Y by a configuration F_ℓ, and replace each vertex of degree two by a configuration F'. The result is an n_3 configuration which can not be obtained by a 1-point extension.*

Proof. Refer to Figure 8, showing a cycle of length four, and a configuration constructed from the unique 3-regular bipartite multigraph on four vertices.

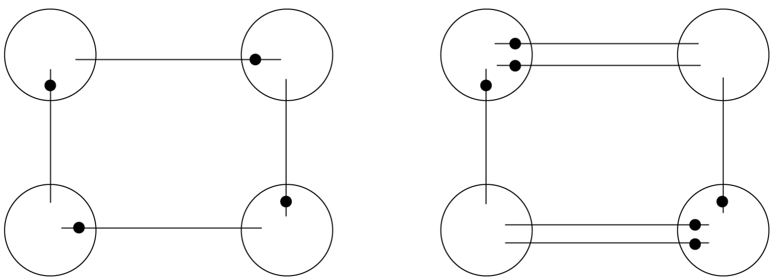


Figure 8: Configurations constructed from F', F_ℓ and F_p

We must show that the n_3 configurations constructed like this cannot be obtained by a 1-point extension. Observe first that the Fano configuration F is a projective plane, so that every two points are contained in a line, and every two lines intersect in a point. Consequently, every triangle contained in F', F_ℓ or F_p has an ordering which satisfies one of conditions A, B or C. By Corollary 1.4, a Fano-type configuration cannot be obtained by a triangular 1-point extension (Theorem 1.2). Suppose that it can be obtained by a general 1-point extension (Theorem 2.1). By Corollary 2.2, there must be an ordered cycle C of length ≥ 4 satisfying certain conditions. Let $C = (a_0, \ell_2, a_1, \ell_3, \ldots, a_{m-2}, \ell_m, a_{m-1}, \ell_0)$ be as in Corollary 2.2, and let ℓ_i' denote the third line containing a_i, where $i = 1, 2, \ldots, m-1$. Let ℓ_1 denote the third line containing a_0, and let a_m denote the third point on ℓ_0. If C were contained within an F', F_ℓ or F_p, then C would have length 4, because any 5 points of F necessarily contain three collinear points. But in F', F_ℓ or F_p, every ordered quadrangle satisfies at least one of conditions D, E, F, G, since the Fano configuration is a projective plane.

It follows that C is not contained within an F', F_ℓ or F_p. Consider the portion of C contained within some F', F_ℓ or F_p. It is a sequence of sequentially incident points and lines. Suppose first that it is contained within an F'. Referring to Figure 6 we see that the shortest possible portion of C contained within an F' is $(a_i, \ell_{i+2}, a_{i+1}, \ell_{i+3}, a_{i+2}, \ell_{i+4})$, for some $i = 0, 1, \ldots, m-1$ where subscripts are reduced modulo m. If $a_{i+2} \neq a_0, a_1$, then ℓ'_{i+2} contains the third point of ℓ_{i+2}, which is in F'. If $a_{i+2} = a_0$, then $a_{i+1} = a_{m-1}$ and $\ell_{i+2} = \ell_m$, so that a_m is collinear in F' with the third point of ℓ_m. If $a_{i+2} = a_1$, then $a_{i+1} = a_0$, so that ℓ_1 and ℓ'_1 are in F' and $\ell'_1 \cap \ell_1 \neq \emptyset$. Thus, the conditions of Corollary 2.2 are never satisfied if a portion of C is contained within an F'.

Suppose next that a portion of C is contained within an F_ℓ. Referring to Figure 6 we see that the shortest possible portion of C contained within an F_ℓ is $(a_i, \ell_{i+2}, a_{i+1}, \ell_{i+3}, a_{i+2})$, for some $i = 0, 1, \ldots, m-1$ where subscripts are reduced modulo m. If $a_{i+2} \neq a_0, a_1$, then ℓ'_{i+2} contains the third point of ℓ_{i+2}, which is in F_ℓ. If $a_{i+2} = a_0$, then $a_{i+1} = a_{m-1}$ and $\ell_{i+2} = \ell_m$, so that a_m is collinear in F_ℓ with the third point of ℓ_m. If $a_{i+2} = a_1$, then $a_{i+1} = a_0$, so that ℓ_1 and ℓ'_1 are in F_ℓ and $\ell'_1 \cap \ell_1 \neq \emptyset$. Thus, the conditions of Corollary 2.2 are never satisfied if a portion of C is contained within an F_ℓ. A similar result holds for F_p, which is the dual of F_ℓ. We conclude that the Fano-type configurations can not be obtained by a 1-point extension. □

4 The Characterization Theorem

In this section we will assume that (Σ, Π) is an n_3-configuration which cannot be derived by a 1-point extension. It follows from Theorem 2.6 that we can assume that (Σ, Π) has a triangle. Let the points of (Σ, Π) be numbered $1, 2, \ldots, n$. Without loss of generality, we can assume that $\Delta(2, 3, 1)$ is a triangle in (Σ, Π). This is illustrated in Figure 9. It will be convenient to omit the commas and brackets from expressions like $\Delta(2, 3, 1)$, and write simply $\Delta 231$.

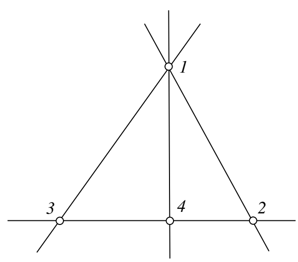


Figure 9: Triangle $\Delta 231$ with condition A

We divide the analysis into two cases according to whether or not (Σ, Π) has a triangle satisfying condition A. The theorem obtained will be the following.

Theorem 4.1. *If (Σ, Π) is an n_3-configuration which cannot be obtained from a 1-point extension, then either:*

i) (Σ, Π) *is one of the Fano, Pappus, or Desargues configurations; or*

ii) (Σ, Π) *is a Fano-type configuration.*

Proof. The proof of this theorem is very long, involving an analysis of many possible cases.

Case A. (Σ, Π) has a triangle satisfying condition A.

Let the ordered triangle be $\Delta 231$, as above. Condition A tells us that the third line through 1 intersects ℓ_{23}. Call the point of intersection 4. This is shown in Figure 9. We will show that any n_3 configuration that cannot be obtained by a 1-point extension, and which satisfies Condition A, is either a Fano-type configuration, or the Fano configuration. Now consider $\Delta 142$. It currently does not satisfy conditions A, B, or C. Since every triangle must satisfy at least one of these conditions, there are three possibilities, which we indicate by $\Delta 142A$, $\Delta 142B$, and $\Delta 142C$. These are shown in Figure 10. In $\Delta 142A$, the third line through 4 intersects ℓ_{12} (in point 5). In $\Delta 142B$, the third lines through 1 and 4 intersect (in point 5). In $\Delta 142C$, the third points on ℓ_{12} (point 5) and ℓ_{24} (point 3) are collinear.

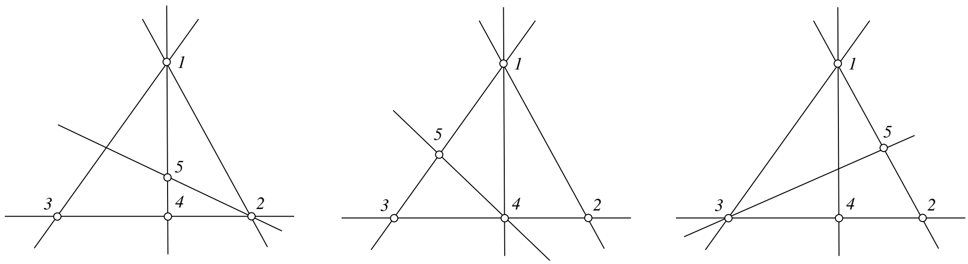


Figure 10: $\Delta 142A$, $\Delta 142B$, and $\Delta 142C$

These three structures are easily seen to be isomorphic, by relabelling the points. Each structure is self-dual, having two points incident on 3 lines each, and two lines each containing 3 points. Thus, without loss of generality, we can assume that the subconfiguration $\Delta 142A$ exists in (Σ, Π) in Case A. Consider triangle $\Delta 124$. It currently does not satisfy condition A, B, or C. Since it must satisfy at least one of these conditions, there are three possibilities, which we indicate by $\Delta 142A\Delta 124A$, $\Delta 142A\Delta 124B$, and $\Delta 142A\Delta 124C$. These are shown in Figure 11.

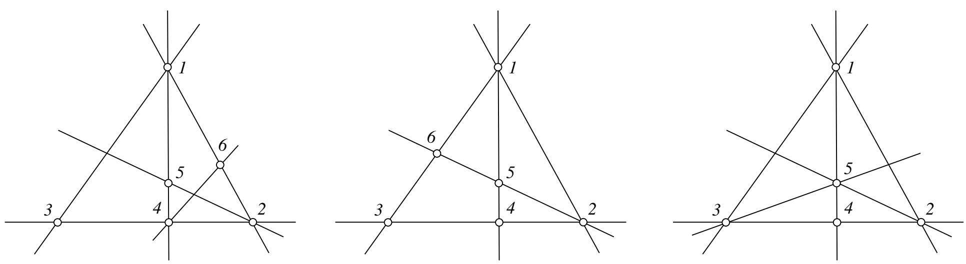


Figure 11: $\Delta 142A\Delta 124A$, $\Delta 142A\Delta 124B$, and $\Delta 142A\Delta 124C$

The structures $\Delta 142A\Delta 124B$ and $\Delta 142A\Delta 124C$ are duals of each other. The first has 6 points and 5 lines, while the other has 5 points and 6 lines. It can be verified by exhaustion that every ordered triangle in these structures satisfies at least one of conditions A, B, or C.

Case $\mathbf{\Delta 142A\Delta 124A}$.

Consider the quadrangle $\square 6431$ in $\Delta 142A\Delta 124A$. It must satisfy at least one of conditions D, E, F, G (see Figure 5). Condition D is possible only if ℓ_{25} intersects ℓ_{13}. Condition E is not possible. Condition F is possible only if the third line through 3 intersects ℓ_{46}. Condition G is possible only if there is a line ℓ_{56}. These cases are illustrated in Figure 12.

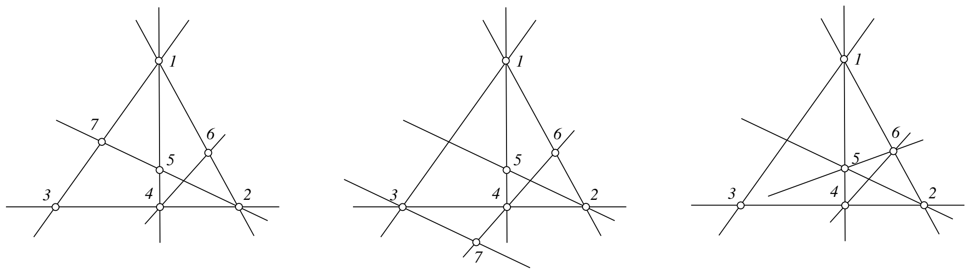


Figure 12: $\Delta 142A\Delta 124A\square 6431D$, $\Delta 142A\Delta 124A\square 6431F$, $\Delta 142A\Delta 124A\square 6431G$

Now the diagrams $\Delta 142A\Delta 124A\square 6431D$ and $\Delta 142A\Delta 124A\square 6431G$ are duals of each other, for the mapping which sends points $1, 2, 3, 4, 5, 6, 7$ of D to $\ell_{15}, \ell_{16}, \ell_{25}, \ell_{24}, \ell_{46}, \ell_{13}, \ell_{56}$ of G is an isomorphism. Therefore we need only consider cases D and F.

Case $\mathbf{\Delta 142A\Delta 124A\square 6431D}$.

It can be verified that all triangles of the diagram satisfy one of conditions A, B, C. Consider the quadrangle $\square 3164$. Condition D is only possible if point 7 lies on line ℓ_{46}. Condition E is not possible. Condition F is only possible if there is a line ℓ_{67}. Condition G is only possible if there is a line ℓ_{35}. These cases are illustrated in Figure 13.

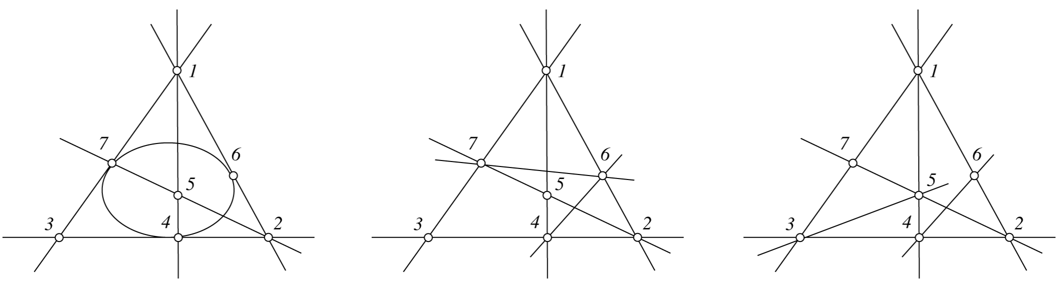


Figure 13: $\Delta 142A\Delta 124A\square 6431D\square 3164D$, F, and G

Case $\mathbf{\Delta 142A\Delta 124A\square 6431D\square 3164D}$.

It can be verified that every triangle satisfies at least one of conditions A, B, C, and every quadrangle satisfies at least one of conditions D, E, F, G. This configuration is isomorphic to the **Fano configuration, with one line removed** (ℓ_{356}), which we denote as F_ℓ. The dual configuration is the **Fano configuration, with one point removed**, which we denote as F_p.

Case $\mathbf{\Delta 142A\Delta 124A\square 6431D\square 3164F}$.

Consider the quadrangle $\square 2376$. Condition D requires that ℓ_{15} intersects ℓ_{67}, which is impossible. Condition E requires that ℓ_{46} contains point 1, which is impossible. Condition F requires that ℓ_{75} contains point 4, which is impossible. Condition G requires a line ℓ_{35}. The result is illustrated in Figure 14.

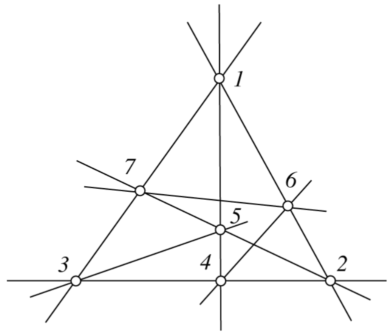


Figure 14: Case **Δ142AΔ124A□6431D□3164F□2376G**

We then consider quadrangle □6237. Condition D requires that ℓ_{15} intersects ℓ_{67}, which is impossible. Condition E requires that ℓ_{75} contains point 4, which is impossible. Condition F requires that ℓ_{35} contains point 1, which is impossible. Condition G requires that ℓ_{46} and ℓ_{25} intersect in point 5, which is impossible. We conclude that case $\Delta 142A\Delta 124A\square 6431D\square 3164F$ is not possible.

Case Δ142AΔ124A□6431D□3164G.
Consider the quadrangle □4316. Condition D requires that ℓ_{25} intersects ℓ_{46}. The point of intersection can only be 7. Condition E requires that ℓ_{75} contains point 6, which is impossible. Condition F requires that ℓ_{15} contains point 2, which is impossible. Condition G requires a line ℓ_{356}. These cases are illustrated in Figure 15.

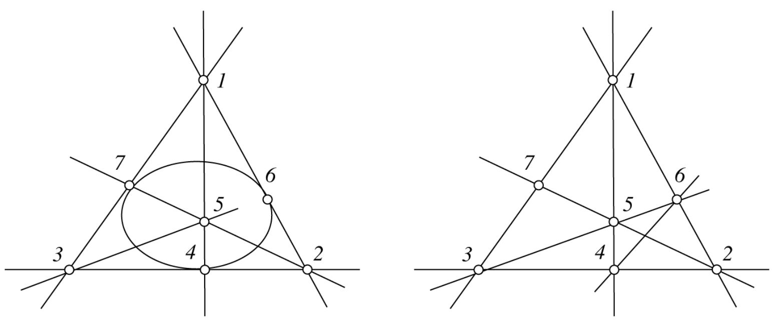


Figure 15: Cases $\Delta 142A\Delta 124A\square 6431D\square 3164G\square 4316D$ and G

These two configurations are easily seen to be isomorphic, by the permutation of the points given by $(2,3,4)(5,6,7)$, mapping D onto G. They are both isomorphic to the **Fano configuration, with one incidence removed**, denoted by F'. Every triangle satisfies at least one of conditions A, B, C, and every quadrangle satisfies at least one of conditions D, E, F, G.

Note that we can complete F' to the Fano configuration, which can not be constructed by a 1-point extension.

We summarise Case A as follows:

Consider an n_3 configuration (Σ, Π), where $n > 7$, which cannot be constructed by a 1-point extension. Every triangle satisfying condition A is contained in a unique sub-configuration isomorphic to one of F_ℓ, F_p or F'.

Case B. (Σ, Π) has no triangle satisfying condition A.

We begin with triangle $\Delta 231$. It must satisfy condition B or C. These two possibilities are shown in Figure 16.

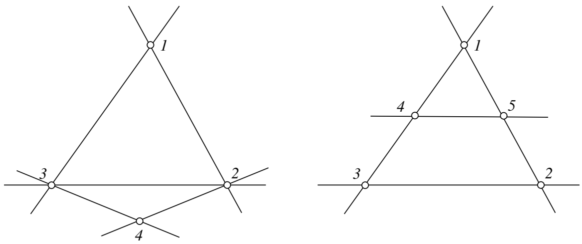


Figure 16: $\Delta 231B$ and $\Delta 231C$

These two structures are duals of each other. Hence we can assume without loss of generality that (Σ, Π) contains the structure $\Delta 231B$.

Consider the triangle $\Delta 123$. It must satisfy condition B or C. We must take these as two separate cases, Case $B\Delta 123B$ and Case $B\Delta 123C$. They are shown in Figure 17. It will be necessary to examine a great many subcases.

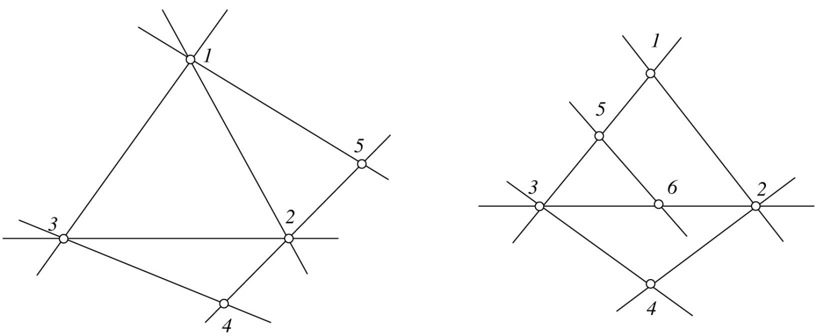


Figure 17: Cases $B\Delta 123B$ and $B\Delta 123C$

Case BΔ123B.

Consider triangle $\Delta 132$. There are two possibilities, cases $B\Delta 123B\Delta 132B$ and $B\Delta 123B\Delta 132C$, which must both be considered. They are shown in Figure 18.

Case BΔ123BΔ132B.

Consider triangle $\Delta 243$. There are two choices $B\Delta 123B\Delta 132B\Delta 243B$ and

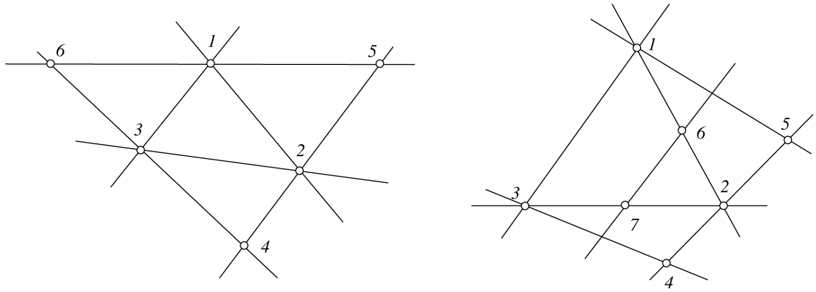


Figure 18: Cases $B\Delta 123B\Delta 132B$ and $B\Delta 123B\Delta 132C$

$B\Delta 123B\Delta 132B\Delta 243C$. They are shown in Figure 19. These structures both have 7 points $\{1, 2, \ldots, 7\}$, so that a mapping from the first to the second can be denoted by a permutation. It is easy to see that the permutation $(1, 2, 3)(4, 6, 5)(7)$ maps the first to the second. Thus, without loss of generality, we can suppose that (Σ, Π) contains the structure $B\Delta 123B\Delta 132B\Delta 243B$.

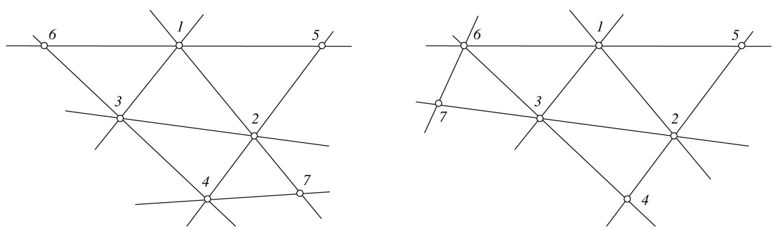


Figure 19: Isomorphic cases $B\Delta 123B\Delta 132B\Delta 243$ B and C

Consider triangle $\Delta 342$. There are two possibilities, $B\Delta 123B\Delta 132B\Delta 243B$ $\Delta 342B$ and $B\Delta 123B\Delta 132B\Delta 243B\Delta 342C$. They are shown in Figure 20. We must consider both possibilities.

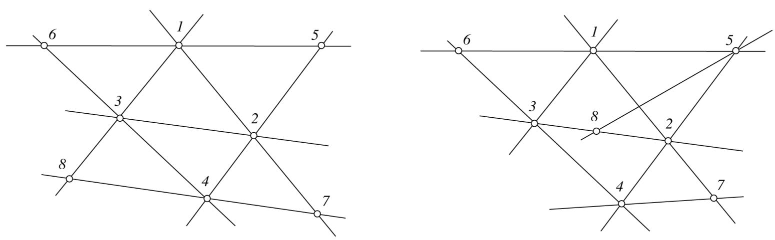


Figure 20: Cases $B\Delta 123B\Delta 132B\Delta 243B\Delta 342B$ and $B\Delta 123B\Delta 132B\Delta 243B\Delta 342C$

This is beginning to look remarkably like the Pappus configuration.

Case B$\boldsymbol{\Delta}$123B$\boldsymbol{\Delta}$132B$\boldsymbol{\Delta}$243B$\boldsymbol{\Delta}$342B.

Consider the quadrangle $\square 1248$. At least one of conditions D, E, F, G must be satisfied. Of these, it is only possible to satisfy condition E, namely the third line through 8 must intersect ℓ_{24}. The point of intersection can only be 5. Therefore the left diagram of Figure 21 must exist in (Σ, Π).

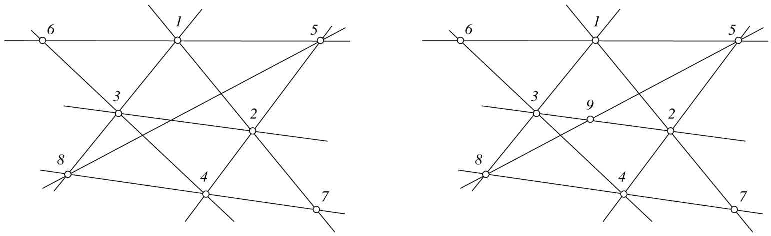


Figure 21: Cases $B\square 1248E$ and $B\square 1248E\square 7238E$

Consider the quadrangle $\square 7238$. At least one of conditions D, E, F, G must be satisfied. Of these, it is only possible to satisfy condition E, namely the third line through 8 must intersect ℓ_{23}. Therefore the right diagram of Figure 21 must exist in (Σ, Π).

Consider the quadrangle $\square 3159$. It is only possible to satisfy condition E, namely the third line through 9 must intersect ℓ_{15} in point 6. Therefore the following structure (Figure 22) must exist in (Σ, Π).

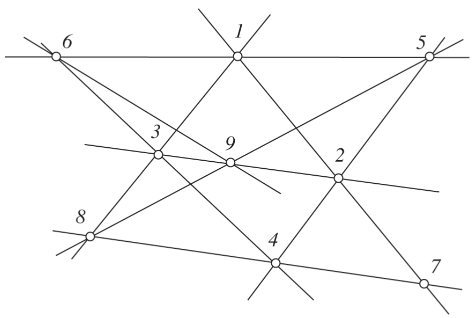


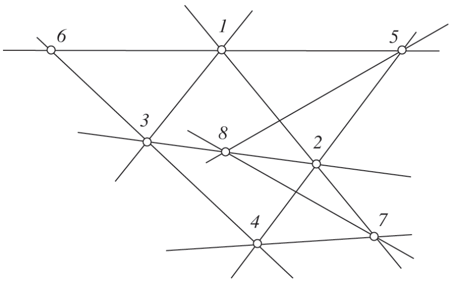
Figure 22: Case $B\square 1248E\square 7238E\square 3159E$

Consider the quadrangle $\square 1347$. It is only possible to satisfy condition E, namely the third line through 7 must intersect ℓ_{34}. The point of intersection must be 6, so that point 7 is incident with ℓ_{69}. Therefore the diagram is completed to a 9_3-configuration, so that (Σ, Π) can only be the **Pappus configuration.**

Case BΔ123BΔ132BΔ243BΔ342C.

This case is illustrated in Figure 20. Consider the triangle $\Delta 274$. There are two possibilities, $\Delta 274B$ and $\Delta 274C$, shown in Figure 23. These are duals of each other. The mapping which sends the points $1, 2, \ldots, 8$ of $\Delta 274B$ to the lines $\ell_{15}, \ell_{25}, \ell_{34}, \ell_{32}, \ell_{12}, \ell_{13}, \ell_{58}, \ell_{47}$ of $\Delta 274C$ is an isomorphism. Hence we only need to consider one of them, the first, say.

Consider the quadrangle $\square 1783$. It is only possible to satisfy condition E, namely the third line through 3 must intersect ℓ_{78}. The point of intersection must be 6, so that ℓ_{78} must be extended to include point 6. Consider next quadrangle $\square 1745$. It is



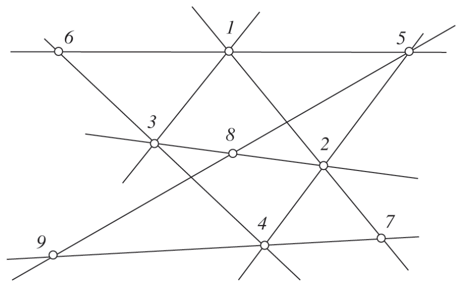


Figure 23: Case $B\Delta 123 B\Delta 132 B\Delta 243 B\Delta 342 C\Delta 274$, B and C

only possible to satisfy condition E, namely the third line through 5 must intersect ℓ_{47}. The result is illustrated in Figure 24.

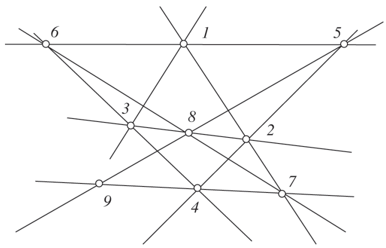


Figure 24: Case $B\Delta 123 B\Delta 132 B\Delta 243 B\Delta 342 C\Delta 274 B\square 1783\square 1745$

Finally, consider quadrangle $\square 7138$. It is only possible to satisfy condition E, namely the third line through 8 must intersect ℓ_{13}. The point of intersection must be 9, so that ℓ_{13} must be extended to include point 9. Once again we have the **Pappus configuration.**

Case B$\boldsymbol{\Delta}$123B$\boldsymbol{\Delta}$132C.

This case is illustrated in Figure 18. Consider the triangle $\Delta 267$. There are two possible ways to satisfy condition B, namely the third line through 6 could contain either 4 or 5. The first of these choices is illustrated in Figure 25. The second is not allowed, as it would create a triangle $\Delta 125$ satisfying condition A. There are two possible ways to satisfy condition C, namely ℓ_{67} could intersect ℓ_{13} or ℓ_{34}. Call these two results C_1 and C_2, respectively, also shown in Figure 25.

Case B$\boldsymbol{\Delta}$123B$\boldsymbol{\Delta}$132C$\boldsymbol{\Delta}$267B.

Consider the quadrangle $\square 1673$. It is not possible to satisfy conditions D or F. Condition E can only be satisfied if ℓ_{34} intersects ℓ_{67}. Condition G can only be satisfied if ℓ_{15} intersects ℓ_{46}. These cases are shown in Figure 26.

Now case G (the right diagram) leads to a contradiction, for consider the quadrangle $\square 3167$. Conditions E, F, G are not possible. Condition D is only possible if $5 \in \ell_{67}$. But this creates a triangle $\Delta 156$ satisfying condition A, a contradiction. Therefore we consider case E (the left diagram). Consider the quadrangle $\square 3761$. Conditions D, F, G cannot be satisfied. Condition E can only be satisfied if ℓ_{15} intersects ℓ_{67} in point 8, as shown in Figure 27. Consider next the quadrangle $\square 6137$. Conditions

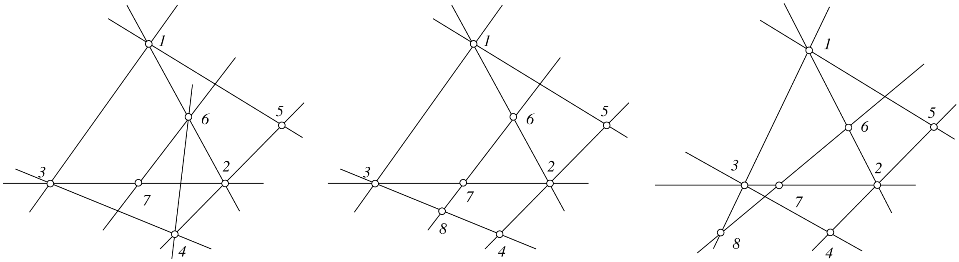


Figure 25: Cases $B\Delta 123B\Delta 132C\Delta 267$ B, C_1, and C_2

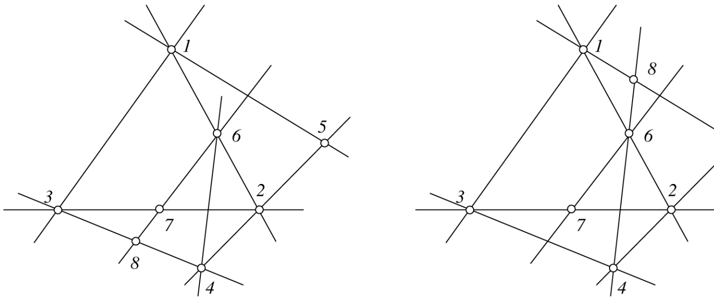


Figure 26: Cases $B\Delta 123B\Delta 132C\Delta 267B\Box 1673$ E and G

D, F, G cannot be satisfied. Condition E can only be satisfied if the third line through 7 intersects ℓ_{13} in a point 9, also illustrated in Figure 27.

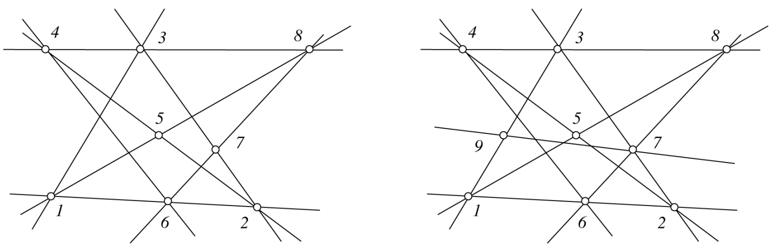


Figure 27: Cases $E\Box 1673E$ and $E\Box 1673E\Box 6137E$

Consider now the quadrangle $\Box 2685$ in the right diagram of Figure 27. Conditions D, F, G cannot be satisfied. Condition E can only be satisfied if the third line through 5 contains point 7, which is only possible if $5 \in \ell_{79}$. The result is isomorphic to the diagram of Figure 24. Once again, we obtain the **Pappus configuration**.

Case $\mathbf{B\Delta 123B\Delta 132C\Delta 267C_1}$.

Refer to Figure 25. Consider the quadrangle $\Box 2784$. Conditions D and F cannot be satisfied. Condition E can only be satisfied if there is a line ℓ_{46}, which gives a result identical to the left diagram of Figure 26. Condition G can only be satisfied if the third line through 7 intersects ℓ_{26} in point 1, but this creates a triangle $\Delta 127$

satisfying condition A, which is not allowed. This completes this case.

Case B$\boldsymbol{\Delta}$123B$\boldsymbol{\Delta}$132C$\boldsymbol{\Delta}$267C$_{\mathbf{2}}$.

Refer to Figure 25. Consider the quadrangle $\square 1376$. Conditions D, E, F are not possible. Condition G is only possible if ℓ_{15} and ℓ_{34} intersect, shown in Figure 28. Consider now the quadrangle $\square 1872$. Conditions D, E, F are not possible. Condition G is possible if ℓ_{15} intersects the third line through 8. The point of intersection can be either 5 or 9, resulting in G_1 and G_2, also shown in Figure 28.

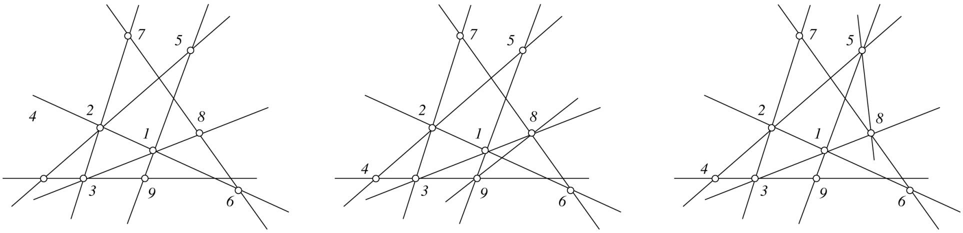


Figure 28: Cases $C_2\square 1376G$, $G\square 1872G_1$ and $G\square 1872G_2$

Consider the quadrangle $\square 7218$ in diagram $G\square 1872G_1$. Conditions D, E, F cannot be satisfied. Condition G can only be satisfied if the third line through 7 intersects ℓ_{24}. The point of intersection can be 4 or 5. But 4 creates a triangle $\Delta 734$ satisfying condition A, a contradiction. Therefore the intersection must be point 5, as shown in Figure 29. Then consider quadrangle $\square 7812$. Conditions D, E, F cannot be satisfied. Condition G can only be satisfied if ℓ_{15} and ℓ_{89} intersect, also shown in Figure 29. Next, consider quadrangle $\square 1572$. Conditions D, E, F, G cannot be satisfied, a contradiction. This completes this case.

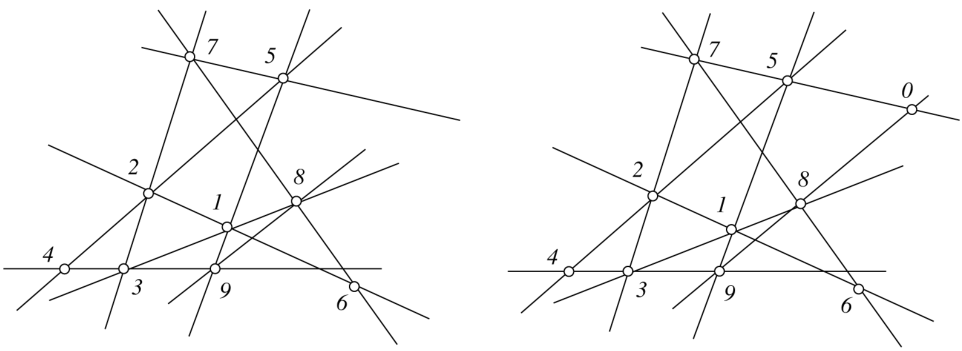


Figure 29: Cases $G_1:\quad \square 7218G$ and $\square 7218G\square 7812G$

Consider next $G\square 1872G_2$, and quadrangle $\square 7218$. Conditions D, E, F cannot be satisfied. Condition G can only be satisfied if the third line through 7 intersects ℓ_{24}. The point of intersection must be 4. But this creates a triangle $\Delta 734$ satisfying condition A, a contradiction. This completes this case, and also case $B\Delta 123B\Delta 132C\Delta 267C_2$, and case $B\Delta 123B\Delta 132C$ and case $B\Delta 123B$.

Case B$\boldsymbol{\Delta}$123C.

Refer to Figure 17. Consider the triangle $\Delta 132$. Condition B can be satisfied if the third line through 1 intersects ℓ_{34}. There are two ways this can occur – the intersection can be point 4, or a new point. This gives B_1 and B_2, shown in Figure 30. Condition C can be satisfied if point 6 is collinear with the third point on ℓ_{12}. There are two ways this can occur. The line through 6 intersecting ℓ_{12} can be ℓ_{56} or a new line. This gives C_1 and C_2, shown in Figure 31.

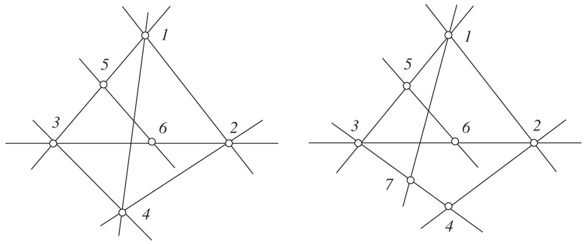


Figure 30: Case $B\Delta 123C\Delta 132$ B_1 and B_2

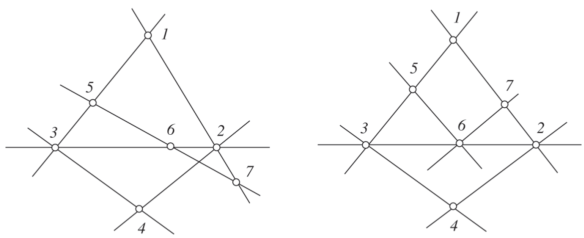


Figure 31: Case $B\Delta 123C\Delta 132$ C_1 and C_2

It can be observed that C_1 is isomorphic to the dual of B_1. If we map points $1, 2, 3, 4, 5, 6, 7$ of C_1 to lines $\ell_{12}, \ell_{23}, \ell_{13}, \ell_{56}, \ell_{14}, \ell_{34}, \ell_{24}$, respectively, of B_1, we have an isomorphism. Similarly, C_2 is isomorphic to the dual of B_2. An isomorphism maps points $1, 2, 3, 4, 5, 6, 7$ of C_2 to lines $\ell_{12}, \ell_{13}, \ell_{23}, \ell_{56}, \ell_{24}, \ell_{34}, \ell_{17}$, respectively, of B_2. Consequently, we have only cases B_1 and B_2 to deal with.

Case $\mathbf{B\Delta 123C\Delta 132B_1}$.

Consider the quadrangle $\square 1562$. Condition D can only be satisfied if the third point on ℓ_{12} is collinear with point 3. But then triangle $\Delta 123$ would satisfy condition A, which is not allowed. Condition E can be satisfied if ℓ_{24} intersected ℓ_{56}. This is shown in Figure 32. Condition F can only be satisfied if the third line through 6 intersected ℓ_{15} in point 3. However, 6 and 3 are already collinear. Condition G can be satisfied if the third line through 5 intersected ℓ_{14}. The third line through 5 cannot be ℓ_{24}, for $\Delta 124$ would then satisfy condition A. Thus, the third line through 5 must be a new line, as shown also in Figure 32.

Case $\mathbf{B\Delta 123C\Delta 132B_1\square 1562E}$.

Consider the triangle $\Delta 267$. Condition B can be satisfied if the third line through 6 intersected ℓ_{12}. The third line through 6 cannot be ℓ_{14}, as the triangle $\Delta 123$ would then satisfy condition A. Hence, the third line through 6 must be a new line, as shown in Figure 33. Condition C can only be satisfied if points 4 and 5 are collinear.

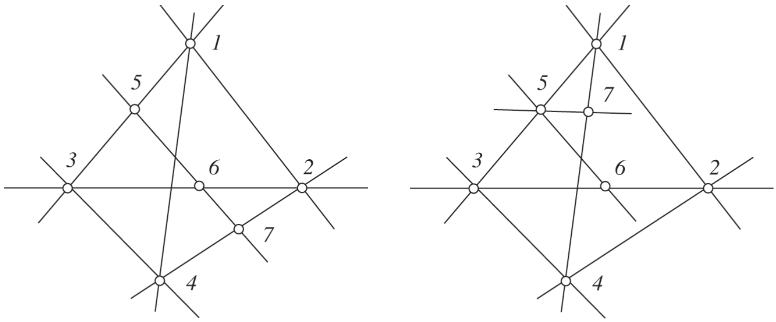


Figure 32: Case $B\Delta123C\Delta132B_1\Box1562$ E and G

The line containing 4 and 5 cannot be ℓ_{14} and it cannot be ℓ_{34}. Therefore Condition C is impossible, and we must have $B\Delta123C\Delta132B1\Box1562E\Delta267B$, shown in Figure 33.

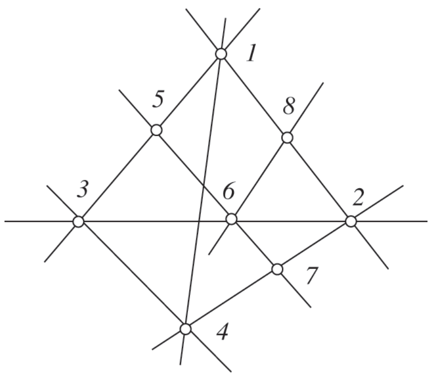


Figure 33: Case $B\Delta123C\Delta132B_1\Box1562E\Delta267B$

This structure is found to be isomorphic to the dual of $B\Delta123B\Delta132C\Delta267B$ $\Box1673G$, shown in Figure 26. The isomorphism maps points $1, 2, 3, 4, 5, 6, 7, 8$ to lines $\ell_{24}, \ell_{26}, \ell_{56}, \ell_{15}, \ell_{34}, \ell_{18}, \ell_{68}, \ell_{14}$. This completes case $B\Delta123C\Delta132B1$ $\Box1562E$.

Case B$\boldsymbol{\Delta}$123C$\boldsymbol{\Delta}$132B$_1\Box$1562G.

Consider the quadrangle $\Box2651$. Condition D can only be satisfied if the third point on ℓ_{23} is collinear with point 3. However triangle $\Delta132$ would then satisfy condition A. Condition E can only be satisfied if ℓ_{14} intersected ℓ_{56}. The point of intersection cannot be 7. If it were point 4, then $\Delta563$ would then satisfy condition A. Hence condition E is not possible. Condition F can only be satisfied if ℓ_{57} intersected ℓ_{26} in point 3. However 5 and 3 are already collinear. Condition G can be satisfied if the third line through 6 intersected ℓ_{24}. The point of intersection cannot be 4. The only possibility is a new line through 6, as shown in Figure 34.

Consider the quadrangle $\Box4863$. Condition D can only be satisfied if the third point on ℓ_{34} is collinear with point 2. The triangle $\Delta342$ would then satisfy condition A,

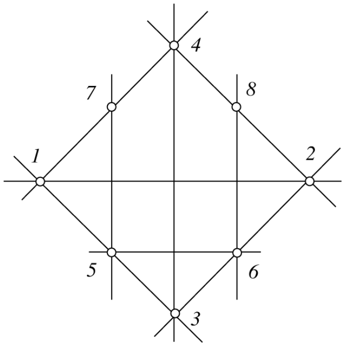


Figure 34: Cases $B\Delta 123C\Delta 132B_1\square 1562G : \square 2651G$ and $\square 2651G\square 4863G$

which is not allowed. Condition E can only be satisfied if ℓ_{13} intersected ℓ_{68} in either 1 or 5. However, 1 and 5 are already each on 3 lines. Condition F can only be satisfied if ℓ_{56} intersected ℓ_{48} in 2. However 6 and 2 are already collinear. Condition G can be satisfied if the third line through 8 intersected ℓ_{14}. The point of intersection can only be 7, shown in the right diagram of Figure 34.

Consider the quadrangle $\square 6512$. Condition D can only be satisfied if the third point on ℓ_{12} were collinear with point 3. But triangle $\Delta 123$ would then satisfy condition A. Condition E can only be satisfied if ℓ_{24} intersected ℓ_{15} in 3. This is not possible. Condition F can only be satisfied if ℓ_{14} intersected ℓ_{56}. This is not possible. Condition G can only be satisfied if ℓ_{57} intersected ℓ_{68}. This is shown in Figure 35.

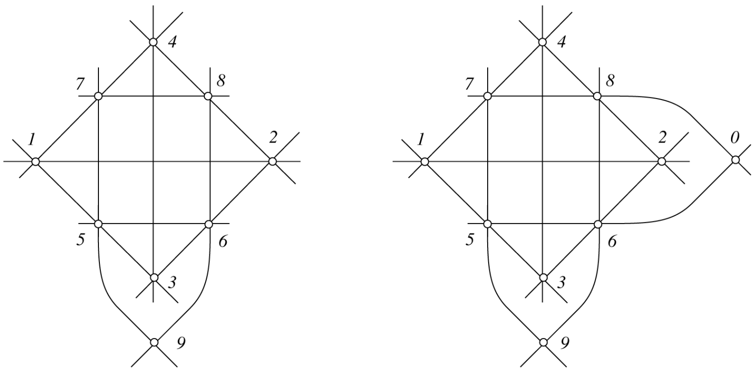


Figure 35: Cases $\square 6512G$ and $\square 6512G\square 5743G$

Consider the quadrangle $\square 5743$. Condition D can only be satisfied if the third point on ℓ_{34} were collinear with point 1. But then triangle $\Delta 341$ would satisfy condition A. Condition E can only be satisfied if ℓ_{23} intersected ℓ_{47} in point 1. This is not possible. Condition F can only be satisfied if ℓ_{24} intersected ℓ_{57} in 9. This is not possible. Condition G can only be satisfied if ℓ_{78} intersected ℓ_{56} in a new point, also shown in Figure 35.

Consider the triangle $\Delta 157$. Condition B can only be satisfied if ℓ_{12} intersected

ℓ_{56}. The point of intersection must be point 0. Condition C can only be satisfied if points 4 and 9 are collinear. The line of collinearity must be ℓ_{34}. The resulting two structures are both isomorphic to the **Desargues configuration**, with one incidence missing, as can be seen from Figure 1.1. If we then consider $\Delta 268$, the remaining incidence is forced. This completes case $B\Delta 123C\Delta 132B_1\square 1562G$ and also case $B\Delta 123C\Delta 132B_1$.

Case $\mathbf{B\Delta 123C\Delta 132B_2}$.

Refer to Figure 30. Consider the triangle $\Delta 173$. Condition B can be satisfied if the third line through 7 intersected ℓ_{12}. The point of intersection cannot be point 2. Therefore it is a new point, as shown in Figure 36. Condition C can be satisfied if points 4 and 5 are collinear. The line of collinearity cannot be ℓ_{56}, for triangle $\Delta 453$ would then satisfy condition A. Hence ℓ_{45} is a new line, also shown in Figure 36. be satisfied if ℓ_{57} intersected ℓ_{68}. This is shown in Figure 35.

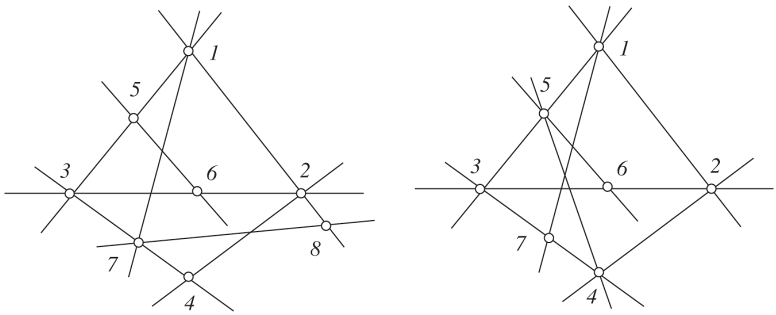


Figure 36: Cases $B\Delta 123C\Delta 132B_2\Delta 173$ B and C

Now case $B\Delta 123C\Delta 132B_2\Delta 173C$ is isomorphic to case $B\Delta 123B\Delta 132C\Delta 267B$, shown in Figure 25. As both diagrams have 7 points, the isomorphism can be given by a permutation, $(1,5,6)(2,3,4)$, which maps diagram $B\Delta 123B\Delta 132C\Delta 267B$ to $B\Delta 123C\Delta 132B_2\Delta 173C$. Thus we need only consider case $B\Delta 123C\Delta 132B_2$ $\Delta 173B$.

Consider the triangle $\Delta 781$ in the left diagram of Figure 36. Condition B can be satisfied if the third line through 8 intersected ℓ_{37}. The point of intersection cannot be 3. Therefore there must be a line ℓ_{48}, as shown in Figure 37. Condition C can be satisfied if the third point on ℓ_{17} is collinear with point 2. The line of collinearity cannot be ℓ_{26}, for if point 6 were on ℓ_{17}, triangle $\Delta 173$ would satisfy condition A. Hence ℓ_{24} must intersect ℓ_{17} in a new point. This is also shown in Figure 37.

Case $\mathbf{B\Delta 123C\Delta 132B_2\Delta 173B\Delta 781B}$.

Consider the triangle $\Delta 365$. Condition B can be satisfied if the third line through 6 intersected ℓ_{37}. The point of intersection cannot be 4, because ℓ_{48} would then contain 6, causing a triangle $\Delta 682$ satisfying condition A. Line ℓ_{17} cannot contain 6, for then triangle $\Delta 136$ would satisfy condition A. Therefore condition B requires that ℓ_{78} contain 6, shown in Figure 38. Condition C can be satisfied if the third point on ℓ_{56} were collinear with point 1. The line of collinearity must be ℓ_{17}, also shown in Figure 38.

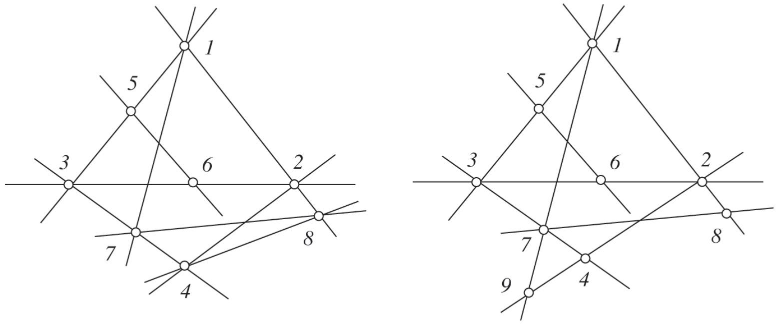


Figure 37: Cases $B\Delta 123 C\Delta 132 B_2 \Delta 173 B\Delta 781$ B and C

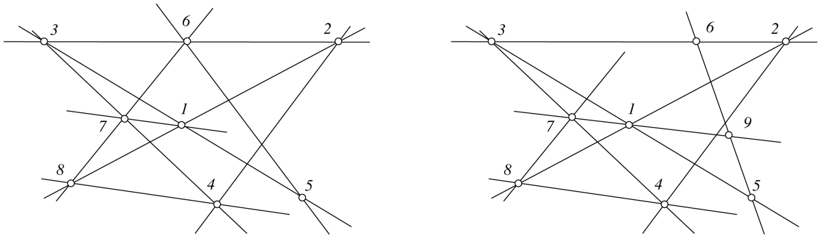


Figure 38: Case $B\Delta 123 C\Delta 132 B_2 \Delta 173 B\Delta 781 B\Delta 365$ B and C

Case $\mathbf{B\Delta 123 C\Delta 132 B_2 \Delta 173 B\Delta 781 B\Delta 365 B}$.

Refer to the left diagram of Figure 38. Consider the quadrangle $\square 2176$. Condition D can only be satisfied if points 3 and 8 were collinear. This is not possible as 3 and 8 are already incident on 3 lines each. Condition E can only be satisfied if ℓ_{56} intersected ℓ_{17}, shown in Figure 39. Condition F can only be satisfied if ℓ_{37} intersected ℓ_{12} in 8. However, 7 and 8 are already collinear. Condition G can only be satisfied if ℓ_{15} and ℓ_{24} intersected. The point of intersection must be 5, making triangle $\Delta 132$ satisfy condition A. We conclude that only E is possible.

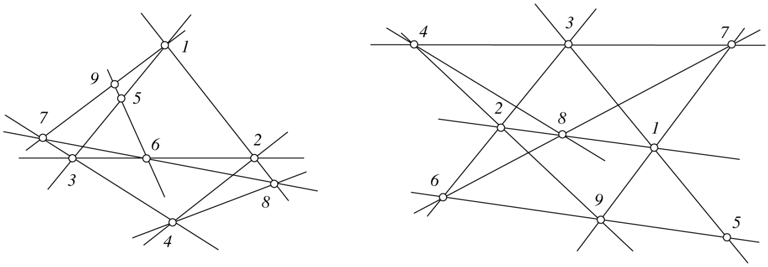


Figure 39: Case $B\Delta 123 C\Delta 132 B_2 \Delta 173 B\Delta 781 B\Delta 365 B\square 2176 E$

Consider the quadrangle $\square 2156$. Condition D can only be satisfied if points 3 and 9 were collinear, which is impossible. Condition E can only be satisfied if ℓ_{67} intersected ℓ_{15} in point 3, which is impossible. Condition F can only be satisfied if the third line through 5 intersected ℓ_{12} in point 8, which is impossible. Condition G can only be satisfied if ℓ_{17} and ℓ_{24} intersected. The point of intersection must be point 9, also shown in Figure 39. As can be seen from the diagram, this is the Pappus configuration with one incidence missing. We conclude that this case results in the **Pappus configuration**.

Case B$\boldsymbol{\Delta}$123C$\boldsymbol{\Delta}$132B$_2\boldsymbol{\Delta}$173B$\boldsymbol{\Delta}$781B$\boldsymbol{\Delta}$365C.

Refer to the right diagram of Figure 38. Consider the quadrangle $\square 7123$. Condition D can only be satisfied if points 4 and 6 are collinear, which is impossible. Condition E can only be satisfied if ℓ_{13} contains 8, which is impossible. Condition F can only be satisfied if ℓ_{24} contains point 9. Condition G can only be satisfied if ℓ_{78} intersected ℓ_{13}. The point of intersection must be 5, creating a triangle $\Delta 195$ satisfying condition A, a contradiction. We conclude that only condition F is possible, shown in Figure 40.

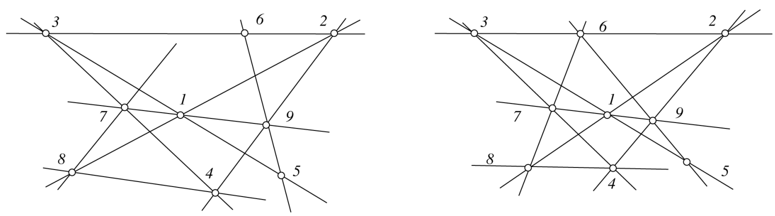


Figure 40: Cases $B\Delta 123C\Delta 132B_2\Delta 173B\Delta 781B\Delta 365C\square 7123F$ and $\square 2371F$

Consider the quadrangle $\square 2371$. Condition D can only be satisfied if points 8 and 9 are collinear, which is impossible. Condition E is only possible if ℓ_{13} contains 4, which is impossible. Condition F is possible only if ℓ_{78} contains 6. Condition G is only possible if ℓ_{29} and ℓ_{35} intersected, which is impossible. We conclude that condition F is necessary.

We next consider quadrangle $\square 4862$. Condition D can only be satisfied if points 9 and 3 are collinear, which is impossible. Condition E can only be satisfied if ℓ_{21} contains point 7, which is impossible. Condition F is possible only if ℓ_{69} and ℓ_{48} intersect in point 5. Condition G is only possible if ℓ_{47} and ℓ_{81} intersected, which is impossible. We conclude that condition F is necessary, giving the **Pappus configuration**. This completes case $B\Delta 123C\Delta 132B_2\Delta 173B\Delta 781B$.

Case B$\boldsymbol{\Delta}$123C$\boldsymbol{\Delta}$132B$_2\boldsymbol{\Delta}$173B$\boldsymbol{\Delta}$781C.

Refer to the right diagram of Figure 37. Consider triangle $\Delta 243$. Condition B can only be satisfied if the third line through 4 intersected ℓ_{28}. The point of intersection can only be 8, as shown in Figure 41. Condition C can only be satisfied if points 6 and 7 are collinear. The line of collinearity cannot be ℓ_{17}, as triangle $\Delta 231$ would then satisfy condition A. Hence, the line can only be ℓ_{78}, which must contain 6, as shown in Figure 41.

Case C is isomorphic to the dual of $B\Delta 123C\Delta 132B2\Delta 173B\Delta 178B\Delta 365B$, shown in Figure 38. An isomorphism maps points $1, 2, \ldots, 9$ of C to lines ℓ_{67}, ℓ_{34}, $\ell_{23}, \ell_{24}, \ell_{56}, \ell_{13}, \ell_{12}, \ell_{17}, \ell_{48}$, respectively, of B. Thus we only need consider case B.

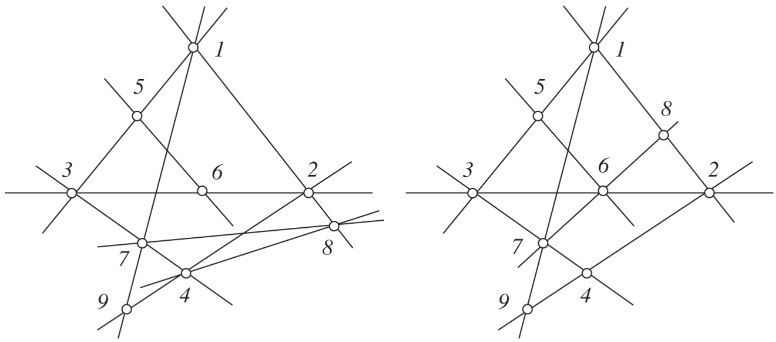


Figure 41: Case $B\Delta 123C\Delta 132B2\Delta 173B\Delta 178C\Delta 243B$ and C

Consider the quadrangle $\square 8731$. Condition D can only be satisfied if points 2 and 6 are collinear, which is impossible, as the line of collinearity could only be ℓ_{24}. Condition E cannot be satisfied. Condition F can only be satisfied if ℓ_{36} intersects ℓ_{87}. The point of intersection must be 6, as shown in Figure 42. Condition G can only be satisfied if ℓ_{84} and ℓ_{79} intersect, which is impossible. Thus, only condition F is possible. But this diagram is isomorphic to case $B\Delta 123B\Delta 132C\Delta 267B\square 1673E\square 6137E$, shown in Figure 27. An isomorphism is given by $(5, 9)(6, 7, 8)$.

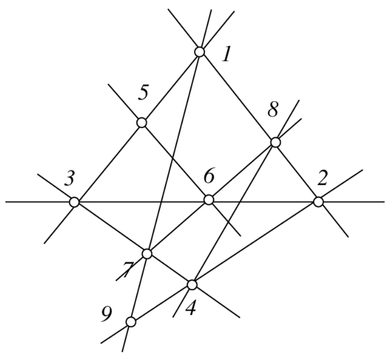


Figure 42: Case $B\Delta 123C\Delta 132B2\Delta 173B\Delta 178C\Delta 243B\square 8731F$

We summarise Case B as follows:

An n_3 configuration (Σ, Π), which cannot be constructed by a 1-point extension, and having no triangle satisfying condition A, is one of the Pappus or Desargues configurations.

We still must show that the Fano, Pappus, and Desargues configurations cannot be obtained by 1-point extensions. This is clearly so for the Fano configuration, as there are no 6_3 configurations. Consider the Pappus configuration. One way to show that it cannot be obtained by a 1-point extension is to start with the unique 8_3 configuration and to show that the possible 1-point extensions do not produce the Pappus configuration. Another way is to show that every ordering of every triangle and quadrilateral in the Pappus configuration satisfies one of conditions A, B, C, D, E, F, G, so that the Pappus configuration does not arise by a 1-point extension. The collineation group of the Pappus configuration has order 108. It is transitive on points, lines, triangles, and quadrangles, so that only one triangle and one quadrilateral need be tested. We omit the proof.

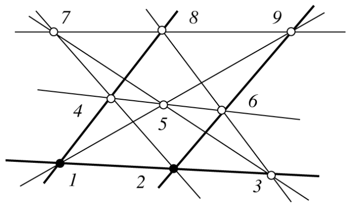


Figure 43: The Pappus configuration

Consider next the Desargues configuration. Its collineation group has order 120. It is transitive on points, lines, triangles, quadrangles, and also on quadruples $(a_0, \ell_2, a_1, \ell_3)$, where $a_0, a_1 \in \ell_2$, $a_0 \neq a_1$, $a_1 \in \ell_3$, and $\ell_2 \neq \ell_3$. It is not transitive on pentagons, hexagons, etc. Refer to Figure 44. We look for a cycle beginning $(a_0, \ell_2, a_1, \ell_3, \ldots, \ell_0) = (1, \ell_{13}, 3, \ell_{34}, \ldots)$, satisfying the conditions of Theorem 2.4. Since $\ell_1' \cap \ell_1 = \emptyset$, where $\ell_1' = \ell_{37}$, and ℓ_1 is the third line through $a_0 = 1$, we must have $\ell_1 = \ell_{15}$, so that $\ell_0 = \ell_{17}$. Since $a_2 \notin \ell_1$, by Theorem 2.4, we cannot have $a_2 = 5$. Hence, $a_2 = 4$.

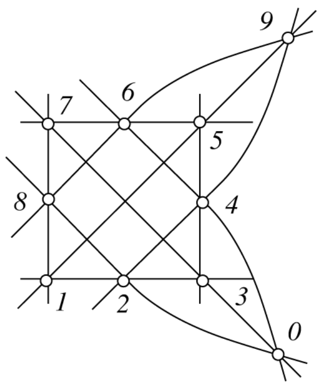


Figure 44: The Desargues configuration

Then since $\ell_2' \cap \ell_2 = \emptyset$, we cannot have $\ell_2' = \ell_{42}$, as ℓ_{42} intersects $\ell_2 = \ell_{13}$ in 2. Therefore $\ell_4 = \ell_{49}$, from which we have $a_3 = 9$, and the cycle is $(1, \ell_{13}, 3, \ell_{34}, 4, \ell_{49}, 9, \ldots, \ell_{17})$. Since $\ell_3' \cap \ell_3 = \emptyset$, we cannot have $\ell_3' = \ell_{59}$, as ℓ_{59} intersects $\ell_3 = \ell_{34}$ in 5. It follows that $\ell_5 = \ell_{59}$. But then a_4 must be either 1 or 5, both of which are impossible. We conclude that the Desargues configuration cannot be obtained by a 1-point extension. This completes the proof of Theorem 4.1. □

Observe that we have only used 1-point extensions based on triangles and quadrangles in the proof of Theorem 4.1. Hence we have proved that if an $(n+1)_3$ configuration cannot be obtained using a 1-point extensions based on triangles or quadrangles, then it is the Fano, Pappus, Desargues, or a Fano-type configuration. Therefore we have the following corollary.

Corollary 4.2. *Every $(n+1)_3$ configuration that can be obtained from an n_3 configuration by a 1-point extension, can be obtained using a 1-point extension based on triangles or quadrangles.*

A consequence of this corollary is that the $(n+1)_3$ configurations can be constructed from the n_3 configurations by constructing all sequences of sequentially incident points and lines of length at most 4, and testing whether they satisfy the conditions required for a 1-point extension. Isomorphism testing of the resulting $(n+1)_3$ configurations then gives all configurations that can be constructed by 1-point extensions. Those which cannot be constructed in this way are the Fano-type configurations, which can be constructed from cycles and subdivisions of bipartite 3-regular multigraphs, using Theorem 3.2.

One of the central problems in the theory of n_3 configurations is to determine whether they are geometric, that is, whether they can be *coordinatized* over the reals and/or rationals. See [3, 14, 15, 16]. This means to assign homogeneous coordinates in the real and/or rational projective plane, so that the lines are straight lines, and all incidences and non-incidences are respected. The application of 1-point extensions to geometric configurations will be described in another article (in preparation).

5 The 3-Point Extension

Let (Σ, Π) be an n_3-configuration. Choose a line ℓ, and let its points be a_1, a_2, a_3. Construct a new configuration (Σ', Π') as follows. $\Sigma' = \Sigma \cup \{b_1, b_2, b_3, \ell_1, \ell_2, \ell_3\}$, where b_1, b_2, b_3 are new points and ℓ_1, ℓ_2, ℓ_3 are new lines. The incidences Π' are constructed as follows. ℓ_1 contains the points a_1, b_2, b_3. ℓ_2 contains the points b_1, a_2, b_3, and ℓ_3 contains the points b_1, b_2, a_3. Choose 3 lines $\ell'_1, \ell'_2, \ell'_3 \neq \ell$ such that ℓ'_i contains a_i. Remove a_i from ℓ'_i and place b_i on ℓ'_i. This is illustrated in the following table. Then Π' contains all remaining incidences of Π, except for the incidences $a_1\ell'_1, a_2\ell'_2, a_3\ell'_3$.

ℓ	ℓ_1	ℓ_2	ℓ_3	ℓ'_1	ℓ'_2	ℓ'_3
a_1	a_1	b_1	b_1	b_1	b_2	b_3
a_2	b_2	a_2	b_2	$\cdot$	$\cdot$	$\cdot$
a_3	b_3	b_3	a_3	$\cdot$	$\cdot$	$\cdot$

Theorem 5.1. (Σ', Π') *is an* $(n+3)_3$*-configuration.*

Proof. Note that each b_i is incident on exactly 3 lines, and that each of $\ell'_1, \ell'_2, \ell'_3$ is incident on exactly 3 points. We must verify that any 2 lines of (Σ', Π') intersect in at most one point. Clearly $\ell, \ell_1, \ell_2, \ell_3$ intersect each other in at most one point. Similarly for $\ell, \ell'_1, \ell'_2, \ell'_3$. The same is true for all other lines of Σ', because it is true for (Σ, Π). □

Example 5.2. The Fano configuration has 7 points and 7 lines, all of which are equivalent under automorphisms. There is one way to choose 3 points a_1, a_2, a_3. The incidences of $\ell, \ell_1, \ell_2, \ell_3$ are uniquely determined. The choice of $\ell'_1, \ell'_2, \ell'_3$ is not unique, as each a_i is incident on two lines other than ℓ. There results two possible 3-point extensions of the Fano configuration. One of these is the Desargues configuration. The other is known as the "anti-Pappian" configuration [5].

A *complete quadrilateral* in an n_3 configuration is a set of four distinct lines intersecting in six distinct points. Notice that the extended configuration (Σ', Π') always contains a complete quadrilateral $\ell, \ell_1, \ell_2, \ell_3$, intersecting in the six points $a_1, a_2, a_3, b_1, b_2, b_3$. The

3-point extension can also be constructed from the dual point of view – rather than beginning with 3 collinear points a_1, a_2, a_3, we begin with 3 concurrent lines, and so forth. This is equivalent to using the 3-point extension in the dual of (Σ, Π), and then dualizing (Σ', Π'). In this case, the 3-point extension will always contain a complete quadrangle, that is, the dual of a complete quadrilateral.

Theorem 5.3. *The Fano-type configurations cannot be obtained by a 3-point extension.*

Proof. Suppose that a Fano-type configuration (Σ, Π) were obtained by a 3-point extension. It would then contain a complete quadrilateral $\ell, \ell_1, \ell_2, \ell_3$, intersecting in the six points $a_1, a_2, a_3, b_1, b_2, b_3$. These four lines and six points must all be part of a single F', F_p, or F_ℓ. Refer to Figure 6. Now the points a_1, a_2, a_3 must be collinear. Furthermore, there must be a line containing a_1, b_2, b_3, and so forth. This determines the labelling of an F', F_p, or F_ℓ. But we then find there is a line containing at least one of the pairs a_1, b_1; a_2, b_2; a_3, b_3, which is not possible in a 3-point extension. □

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