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The expansion of a chord diagram and the Genocchi numbers

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Abstract

A chord diagram E is a set of chords of a circle such that no pair of chords has a common endvertex. Let $v_1, v_2, \ldots, v_{2n}$ be a sequence of vertices arranged in clockwise order along a circumference. A chord diagram $\{v_1 v_{n+1}, v_2 v_{n+2}, \ldots, v_n v_{2n}\}$ is called an n-crossing and a chord diagram $\{v_1 v_2, v_3 v_4, \ldots, v_{2n-1} v_{2n}\}$ is called an n-necklace. For a chord diagram E having a 2-crossing $S = \{x_1 x_3, x_2 x_4\}$, the expansion of E with respect to S is to replace E with $E_1 = (E \setminus S) \cup \{x_2 x_3, x_4 x_1\}$ or $E_2 = (E \setminus S) \cup \{x_1 x_2, x_3 x_4\}$. Beginning from a given chord diagram E as the root, by iterating chord expansions in both ways, we have a binary tree whose all leaves are nonintersecting chord diagrams. Let $\mathcal{NCD}(E)$ be the multiset of the leaves. In this paper, the multiplicity of an n-necklace in $\mathcal{NCD}(E)$ is studied. Among other results, it is shown that the multiplicity of an n-necklace generated from an n-crossing equals the Genocchi number when n is odd and the median Genocchi number when n is even.

Keywords: Chord diagram, chord expansion, Genocchi number, Seidel triangle.

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1 Introduction

A set of chords of a circle is called a *chord diagram*, if they have no common endvertex. If a chord diagram consists of a set of n mutually crossing chords, it is called an n-*crossing*. A 2-crossing is simply called a crossing as well. If a chord diagram contains no crossing, it is called *nonintersecting*.

Let V be a set of $2n$ vertices on a circle, and let E be a chord diagram of order n, where each chord has endvertices of V. In this situation, V is called a *support* of

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E. We denote the family of all chord diagrams having V as a support by $\mathcal{CD}(V)$. Let $x_1, x_2, x_3, x_4 \in V$ be placed on a circle in clockwise order. Let $E \in \mathcal{CD}(V)$. For a crossing $S = \{x_1x_3, x_2x_4\} \subset E$, let $S_1 = \{x_2x_3, x_4x_1\}$, and $S_2 = \{x_1x_2, x_3x_4\}$. The *expansion* of E with respect to S is defined as a replacement of E with $E_1 = (E \setminus S) \cup S_1$ or $E_2 = (E \setminus S) \cup S_2$ (see Figure 1).

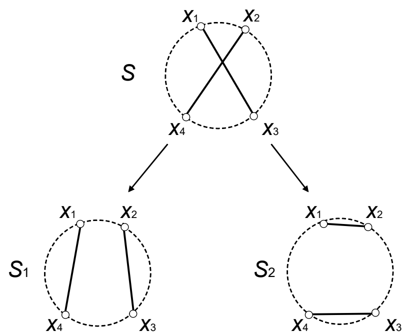


Figure 1: The expansion of a chord diagram with respect to a 2-crossing S. Other chords except those in S are not shown.

Let $E \in \mathcal{CD}(V)$ be a chord diagram. Form a binary tree as follows. Begin with E as the root, arbitrarily choose a crossing of E, and expand E in both ways, adding the results as children of E. Choose crossings in each child if any exists, expand them each in both ways, and repeat the procedure until all leaves are nonintersecting. This procedure terminates and the multiset of leaves is independent of the choices made at each step ([14]). Let us denote the multiset of nonintersecting chord diagrams generated from E by $\mathcal{NCD}(E)$. For a chord diagram $E \in \mathcal{CD}(V)$, let us define the *chord expansion number* $f(E)$ as the cardinality of $\mathcal{NCD}(E)$ as a multiset.

For a chord diagram E, the *circle graph*, also called the *interlace graph* G_E of E, is a graph such that a vertex of G_E corresponds to a chord of E and two vertices of G_E are joined by an edge if their corresponding chords of E are mutually crossing. We say that two chord diagrams E_1 and E_2 with a common support are isomorphic if G_{E_1} and G_{E_2} are isomorphic as graphs. It is proved that $f(E)$ equals $t(G_E; 2, -1)$, where $t(G; x, y)$ is the *Tutte polynomial* of a graph G ([15]).

In the case E is an n-crossing C_n, its associated circle graph is a complete graph K_n with n vertices. In [13], Merino proved that $t(K_n; 2, -1) = Eul_{n+1}$ for $n \geq 1$, where $(Eul)_{n\geq 1} = (1, 1, 2, 5, 16, 61, 272, \ldots)$ is the *Euler number*. Hence, we have $f(C_n) = Eul_{n+1}$ for $n \geq 1$. See also [12] for the evaluation of $t(G; 2, -1)$ for a graph G.

For two nonnegative integers k and n with $k \leq n$, we define $A(n, k)$ as a chord diagram of order $n + 1$, in which there is an n-crossing E_0 with an extra chord e such that e crosses exactly k chords of E_0. (See Figure 2.) Note that $A(n - 1, n - 1)$ is simply an n-crossing, and that $A(n, 0)$ is a union of an n-crossing and an isolated chord.

Let us denote $\{1, 2, \ldots, n\}$ by $[n]$. A permutation σ on $[n]$ is called an *alternating permutation* if $(\sigma(i) - \sigma(i - 1))(\sigma(i + 1) - \sigma(i)) < 0$ for $2 \leq i \leq n - 1$. An alternating permutation σ is called an *up-down permutation* (resp. *down-up permutation*) if $\sigma(1) < \sigma(2)$ (resp. $\sigma(1) > \sigma(2)$). For $0 \leq k \leq n$, the *Entringer number* $Ent_{n,k}$ is defined as the number of down-up permutations on $[n + 1]$ with the first term $k + 1$ ([11]). For $n \geq 1$,

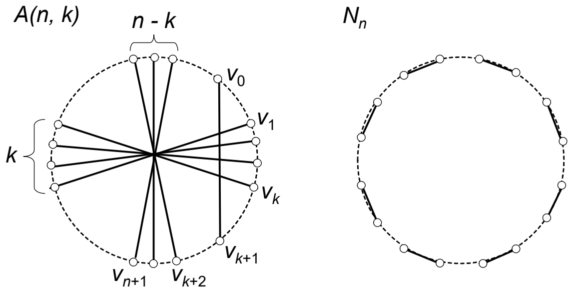


Figure 2: $A(n,k)$ with $n=7$ and $k=4$ (left), and N_n with $n=8$ (right).

$Ent_{n+1,1}$ equals Eul_n, the number of all down-up permutations on $[n]$. In [14], it is proved that $f(A(n,k)) = Ent_{n+2,k+1}$.

For a chord diagram E and for a nonintersecting chord diagram F with a common support, let us denote the *multiplicity* of F in $\mathcal{NCD}(E)$ by $m(E,F)$. For a nonintersecting chord diagram E, a chord $e \in E$ is called an *ear*, if there is no other chord of E on at least one side of e. In [15], it is shown that for an n-crossing C_n and a nonintersecting chord diagram F with a common support, $m(C_n, F) = 1$ if and only if F has at most 3 ears. A nonintersecting chord diagram E with n chords is called an n-*necklace*, denoted by N_n, if all chords of E are ears. (See Figure 2.) The main purpose of the paper is to show that $m(C_n, N_n)$ equals the *Genocchi number* when n is odd and the *median Genocchi number* when n is even. The Genocchi numbers and the median Genocchi numbers will be introduced in the following section.

Recently, Bigeni showed a relation between a weight system of sl_2 of chord diagrams and the median Genocchi numbers ([2]). In Definition 1 of [2], followed from [3], a weight system of sl_2 is defined inductively by applying an operation for chord diagrams. The operation and the chord expansion are closely related to each other, although our main results in the paper do not seem directly followed from the results in [2].

The rest of this paper is organized as follows. In Section 2, the Genocchi numbers and the median Genocchi numbers are introduced. In Section 3, the main results of the paper are proved. In Section 4, another combinatorial interpretation for the multiplicity of n-necklaces is exhibited. Finally, in Section 5, some open problems are discussed.

2 The Genocchi numbers and the median Genocchi numbers

According to [10], but with slightly different indices, let us recursively define the entry $S(n,k)$ in row $n \geq 1$ and column $k \geq 0$ of the *Seidel triangle* ([17]):

$$S(1,1) = 1,$$

$$S(n,k) = 0 \quad \text{for } k = 0 \text{ or } n \leq 2(k-1),$$

$$S(2n,k) = \sum_{i \geq k} S(2n-1,i) \quad \text{for } 1 \leq k \leq n, \tag{2.1}$$

$$S(2n+1,k) = \sum_{i \leq k} S(2n,i) \quad \text{for } 1 \leq k \leq n+1. \tag{2.2}$$

Table 1: The Seidel triangle $S(n, k)$.

$n \setminus k$	1	2	3	4	5
1	1				
2	1				
3	1	1			
4	2	1			
5	2	3	3		
6	8	6	3		
7	8	14	17	17	
8	56	48	34	17	
9	56	104	138	155	155
10	608	552	448	310	155

(See Table 1.) By the equations (2.1) and (2.2), we have the following recurrence relations.

$$S(2n, k) = S(2n-1, k) + S(2n, k+1) \quad \text{for } 1 \le k \le n, \tag{2.3}$$
$$S(2n+1, k) = S(2n, k) + S(2n+1, k-1) \quad \text{for } 1 \le k \le n+1. \tag{2.4}$$

The *Genocchi numbers* (or Genocchi numbers of the first kind) $G(2n)$ are defined as $S(2n-1, n)$, the numbers on the right edge of the Seidel triangle, and the *median Genocchi numbers* (or Genocchi numbers of the second kind) $H(2n+1)$ are defined as $S(2n+2, 1)$, the numbers on the left edge of the Seidel triangle. Note that $(G(2n))_{n\ge1} = (1, 1, 3, 17, 155, \ldots)$ and $(H(2n+1))_{n\ge0} = (1, 2, 8, 56, 608, \ldots)$.

Combinatorial properties of the Genocchi numbers have been extensively studied ([1, 4, 5, 6, 7, 8, 9, 10, 16, 19]). It is known that the Genocchi number $G(2n)$ counts the number of permutations σ on $[2n-1]$ such that $\sigma(i) < \sigma(i+1)$ if $\sigma(i)$ is odd, and $\sigma(i) > \sigma(i+1)$ if $\sigma(i)$ is even ([6]). It is also known that the median Genocchi number $H(2n+1)$ counts the number of permutations σ on $[2n+1]$ such that $\sigma(i) > i$ if i is odd and $i \ne 2n+1$, and $\sigma(i) < i$ if i is even ([6]).

In the on-line encyclopedia of integer sequences [18], we can find more information for the sequences A001469 (Genocchi numbers), A005439 (median Genocchi numbers), A099960 (An interleaving of the Genocchi numbers of the first and second kind) and A014781 (Seidel triangle).

3 Main results

Our aim is to show a new combinatorial interpretation for the values of the Seidel triangle by using chord expansions.

Let $v_0, v_1, \ldots, v_{2n+1}$ be a sequence of vertices in clockwise order along a circumference. Let $V = \{v_i : 0 \le i \le 2n+1\}$. As one of chord diagrams $E \in \mathcal{CD}(V)$ isomorphic to $A(n, k)$, introduced in the previous section, we have $E = \{v_0 v_{k+1}\} \cup \{v_i v_{n+i+1} : 1 \le i \le k\} \cup \{v_i v_{n+i} : k+2 \le i \le n+1\}$. (See Figure 2.) Now let us define $(n+1)$-necklaces $N^+_{n+1,k}$ and $N^-_{n+1,k} \in \mathcal{CD}(V)$ such that $N^+_{n+1,k}$ contains an ear $v_k v_{k+1}$ and $N^-_{n+1,k}$ contains an ear $v_{k+1} v_{k+2}$. The values of $m(A(n, k), N^+_{n+1,k})$ for n and k small are shown in Table 2.

Table 2: $m(A(n,k), N^+_{n+1,k})$ for $0 \le k \le n \le 8$.

$n \setminus k$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	1	1						
3	1	1	2	2					
4	2	2	3	3	3				
5	3	3	6	6	8	8			
6	8	8	14	14	17	17	17		
7	17	17	34	34	48	48	56	56	
8	56	56	104	104	138	138	155	155	155

Let us define $b^+_{n,k} = m(A(n,k), N^+_{n+1,k})$ and $b^-_{n,k} = m(A(n,k), N^-_{n+1,k})$. We also simply denote $b^+_{n,k}$ by $b_{n,k}$. The main result of the paper is the following theorem.

Theorem 3.1. *Let $n \ge 1$. Then we have*

$$b_{2n-1,k} = S(2n, n - \lfloor k/2 \rfloor) \quad \textit{for } 0 \le k \le 2n-1, \tag{3.1}$$

and

$$b_{2n,k} = S(2n+1, \lfloor k/2 \rfloor + 1) \quad \textit{for } 0 \le k \le 2n. \tag{3.2}$$

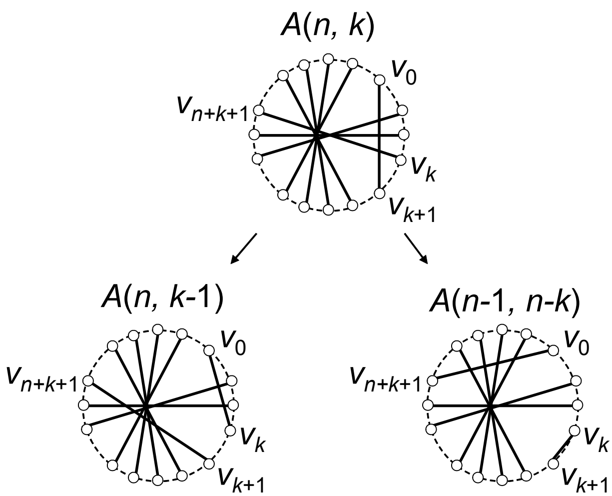


Figure 3: A chord expansion of $A(n,k)$ with respect to $\{v_0v_{k+1}, v_kv_{n+k+1}\}$ with $n = 7$ and $k = 3$.

Firstly, we show a relation between $b^-_{n,k}$ and $b^+_{n,k}$.

Lemma 3.2. $b^-_{n,k} = b^+_{n,k-1}$ *for $1 \le k \le n$.*

Proof. Let E be a chord diagram isomorphic to $A(n,k)$, as shown in Figure 3. By the chord expansion of E with respect to $\{v_0v_{k+1}, v_kv_{n+k+1}\}$, we have two successors E_1 and E_2, which are isomorphic to $A(n,k-1)$ and $A(n-1,n-k)$, respectively. Since E_2 contains a chord v_kv_{k+1}, it does not generate $N^-_{n+1,k}$. Furthermore, since $N^-_{n+1,k}$ is a necklace having a chord $v_{k-1}v_k$, we have $b^-_{n,k} = m(A(n,k), N^-_{n+1,k}) = m(A(n,k-1), N^+_{n+1,k-1}) = b^+_{n,k-1}$, as required. □

In order to prove Theorem 3.1, let us show a recurrence relation for $b_{n,k}$.

Lemma 3.3. *We have $b_{0,0} = 1$ and for $n \geq 1$, we have*

$$\begin{aligned}
&b_{n,0} = b_{n,1} = b_{n-1,n-1},\\
&b_{n,k} = \begin{cases} b_{n,k-2} + b_{n-1,n-k} & \text{for } 2 \leq k \leq n \text{ and } n \text{ is odd},\\ b_{n,k-2} + b_{n-1,n-k-1} & \text{for } 2 \leq k \leq n-1 \text{ and } n \text{ is even},\end{cases}\\
&b_{n,n} = b_{n,n-2} \quad \text{for } n \text{ is even}.
\end{aligned}$$

Proof. When $k = 0, 1$ or n, equations $b_{n,0} = b_{n,1} = b_{n-1,n-1}$ can be proved easily. Let us consider the case $2 \leq k \leq n$. As in the proof of Lemma 3.2, we use the expansion of $A(n,k)$ with respect to $\{v_0v_{k+1}, v_kv_{n+k+1}\}$.

If n is odd, we have

$$\begin{aligned}
b^+_{n,k} &= b^-_{n,k-1} + b^+_{n-1,n-k}\\
&= b^+_{n,k-2} + b^+_{n-1,n-k}.
\end{aligned}$$

If n is even and $k < n$, we have

$$\begin{aligned}
b^+_{n,k} &= b^-_{n,k-1} + b^-_{n-1,n-k}\\
&= b^+_{n,k-2} + b^+_{n-1,n-k-1}.
\end{aligned}$$

Finally, if n is even and $k = n$, since $b^-_{n-1,0} = 0$, we have

$$\begin{aligned}
b^+_{n,n} &= b^-_{n,n-1} + b^-_{n-1,0}\\
&= b^+_{n,n-2},
\end{aligned}$$

as needed. □

Proof of Theorem 3.1. We proceed by induction on n and k. For (3.1) with $n = 1$, we have $b_{1,0} = 1$ and $b_{1,1} = 1$. On the other hand, we have $S(2,1) = 1$. For (3.2) with $n = 1$, we have $b_{2,0} = 1$, $b_{2,1} = 1$ and $b_{2,2} = 1$. On the other hand, we have $S(3,1) = S(3,2) = 1$.

Let $n \geq 2$. For $k = 0$, we have

$$\begin{aligned}
b_{2n-1,0} &= b_{2n-2,2n-2}\\
&= S(2n-1,n)\\
&= S(2n,n),
\end{aligned}$$

and

$$\begin{aligned} b_{2n,0} &= b_{2n-1,2n-1} \\ &= S(2n,1) \\ &= S(2n+1,1). \end{aligned}$$

For $k = 1$, we have

$$\begin{aligned} b_{2n-1,1} &= b_{2n-1,0} \\ &= S(2n,n), \end{aligned}$$

and

$$\begin{aligned} b_{2n,1} &= b_{2n,0} \\ &= S(2n+1,1). \end{aligned}$$

For (3.1) with $2 \le k \le 2n-1$, we have

$$\begin{aligned} b_{2n-1,k} &= b_{2n-1,k-2} + b_{2n-2,2n-1-k} \\ &= S(2n, n - \lfloor (k-2)/2 \rfloor) + S(2n-1, \lfloor (2n-1-k)/2 \rfloor + 1) \\ &= S(2n, n+1-\lfloor k/2 \rfloor) + S(2n-1, n - \lfloor k/2 \rfloor) \\ &= S(2n, n - \lfloor k/2 \rfloor), \end{aligned}$$

and for (3.2) with $2 \le k \le 2n-1$, we have

$$\begin{aligned} b_{2n,k} &= b_{2n,k-2} + b_{2n-1,2n-1-k} \\ &= S(2n+1, \lfloor (k-2)/2 \rfloor + 1) + S(2n, n - \lfloor (2n-1-k)/2 \rfloor) \\ &= S(2n+1, \lfloor k/2 \rfloor) + S(2n, 1 + \lfloor k/2 \rfloor) \\ &= S(2n+1, 1 + \lfloor k/2 \rfloor), \end{aligned}$$

and for (3.2) with $k = 2n$, we have

$$\begin{aligned} b_{2n,2n} &= b_{2n,2n-2} \\ &= S(2n+1,n) \\ &= S(2n+1,n+1). \end{aligned}$$

□

By Theorem 3.1, we have the following corollary.

Corollary 3.4. *$m(C_{2n}, N_{2n}) = H(2n-1)$ and $m(C_{2n-1}, N_{2n-1}) = G(2n)$ for $n \ge 1$.*

Proof. By Theorem 3.1, we have $m(C_{2n}, N_{2n}) = b_{2n-1,2n-1} = S(2n,1) = H(2n-1)$, and $m(C_{2n-1}, N_{2n-1}) = b_{2n-2,2n-2} = S(2n-1,n) = G(2n)$. □

4 Multiplicity of an N-necklace and the number of perfect matchings of an associated graph

In this section, we will exhibit a combinatorial interpretation of $m(E, N_n)$ for a given chord diagram E. For a set V of vertices on the circumference, $\mathcal{C}(V)$ denotes the set of all chords whose endvertices are in V. A *Ptolemy weight* w on $\mathcal{C}(V)$ is defined as a function that satisfies

$$w(x_1x_3)w(x_2x_4) = w(x_2x_3)w(x_1x_4) + w(x_1x_2)w(x_3x_4) \tag{4.1}$$

for all vertices $x_1, x_2, x_3, x_4 \in V$ placed along the circle. If $w(e)$ is the Euclidean length of a chord e, then (4.1) holds by the Ptolemy's theorem in Euclidean geometry. Let w be a Ptolemy weight on $\mathcal{C}(V)$. If a chord diagram $E \in \mathcal{CD}(V)$ has a 2-crossing S, by the chord expansion of E with respect to S, we have two successors E_1 and E_2. Then by (4.1), we have

$$\prod_{e\in E} w(e) = \prod_{e\in E_1} w(e) + \prod_{e\in E_2} w(e). \tag{4.2}$$

We denote the left-hand side of (4.2) by $w(E)$. By iterating chord expansions with (4.2), we have

$$w(E) = \sum_{F\in\mathcal{NCD}(E)} w(F). \tag{4.3}$$

Let $V = \{v_1, v_2, \dots, v_{2n}\}$, where $v_1, v_2, \dots, v_{2n}$ are placed along the circumference in this order. A Ptolemy weight w on $\mathcal{C}(V)$ is called *rectilinear* if $w(v_iv_j) = \sum_{i\le k<j} w(v_kv_{k+1})$ for all $1 \le i < j \le 2n$. For example, if the vertices are placed on a straight line and the weight $w(v_iv_j)$ is defined as the Euclidean distance between v_i and v_j, then w is indeed a rectilinear Ptolemy weight.

In order to analyze $m(E, N_n)$, let us consider the rectilinear Ptolemy weight w on $\mathcal{C}(V)$ such that $w(v_{2k-1}v_{2k}) = x_k$ for $1 \le k \le n$ and $w(v_{2k}v_{2k+1}) = 0$ for $1 \le k \le n-1$. In this weight, since for every chord e, $w(e)$ corresponds to a first degree polynomial of a multiple variables $x_1, x_2, \dots, x_n$ or $w(e) = 0$, for all chord diagrams E, $w(E)$ is a homogeneous polynomial of degree n or $w(E) = 0$. From this point until the end of this section, we fix this weight. Let us define an n-necklace $N_n = \{v_{2k-1}v_{2k} : 1 \le k \le n\}$.

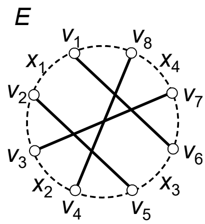
Lemma 4.1. *In the rectilinear Ptolemy weight w as defined in the above, for a chord diagram E, $m(E, N_n)$ equals the coefficient of $x_1x_2 \dots x_n$ of the polynomial $w(E)$.*

Proof. Since $w(N_n) = x_1x_2 \dots x_n$, what we need to show is that if $F \in \mathcal{NCD}(E)\setminus\{N_n\}$, a polynomial $w(F)$ contains no monomial $x_1x_2 \dots x_n$. Suppose to a contradiction that $F \in \mathcal{NCD}(E) \setminus \{N_n\}$ and F has a monomial $x_1x_2 \dots x_n$. Since $F \ne N_n$, there exists a chord $v_{2k-1}v_{2\ell}$ of F with $1 \le k < \ell \le n$ such that $\ell - k \ge 1$ is maximal. Then the two variables x_k and x_ℓ do not appear together in the weight of any chord of F, otherwise such a chord would either intersect $v_{2k-1}v_{2\ell}$ or contradict $\ell - k$ being maximal. It follows that the product x_kx_ℓ never appears in $w(F)$. This contradicts to that $w(F)$ contains a monomial $x_1x_2 \dots x_n$. □

For a chord diagram E having n chords $e_1, e_2, \dots, e_n$ with the rectilinear Ptolemy weight w as defined in the above, let us define a balanced bipartite graph $G(E, X)$ with partite sets $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_n\}$ as follows. For $1 \le i \le n$ and $1 \le j \le n$, a_i and b_j are adjacent if and only if a polynomial $w(e_i)$ contains a monomial x_j.

Theorem 4.2. *For a chord diagram E with n chords and its associated balanced bipartite graph $G(E, X)$ as defined in the above, $m(E, N_n)$ equals the number of perfect matchings of $G(E, X)$.*

Proof. We have $w(E) = \prod_{e \in E} w(e)$, and for all chords e, $w(e) = 0$ or $w(e) = x_i + x_{i+1} + \cdots + x_j$ for some $1 \le i \le j \le n$. Hence, the coefficient of $x_1 x_2 \ldots x_n$ of $w(E)$, which is $m(E, N_n)$ by Lemma 4.1, is the number of possible combinations to choose a variable $x \in X$ from each $w(e)$ without repetition. This is the number of perfect matchings of $G(E, X)$. $\square$



G(E, X)

	x_1	x_2	x_3	x_4
e_1	1	1	1	0
e_2	0	1	0	0
e_3	0	1	1	0
e_4	0	0	1	1

Figure 4: A chord diagram E (left) and its biadjacency matrix of a corresponding bipartite graph $G(E, X)$ (right).

Example 4.3. Let $n = 4$. Let $V = \{v_i : 1 \le i \le 2n\}$, where $v_1, v_2, \ldots, v_{2n}$ are placed on the circumference in this order. Let us consider a rectilinear Ptolemy weight w on $\mathcal{C}(V)$ such that $w(v_{2i-1}v_{2i}) = x_i$ for $1 \le i \le n$ and $w(v_{2i}v_{2i+1}) = 0$ for $1 \le i \le n-1$. Let $E = \{e_i : 1 \le i \le 4\}$ be a chord diagram, where $e_1 = v_1 v_6, e_2 = v_2 v_5, e_3 = v_3 v_7, e_4 = v_4 v_8$. (See Figure 4.) Since

$$w(E) = \prod_{1 \le i \le n} w(e_i) = (x_1 + x_2 + x_3) x_2 (x_2 + x_3)(x_3 + x_4),$$

the coefficient of $x_1 x_2 x_3 x_4$ of $w(E)$ is 1, and the number of perfect matchings of $G(E, X)$ is also 1. Hence, we have $m(E, N_n) = 1$.

By Corollary 3.4 and Theorem 4.2 for n-crossings C_n, we have the following corollary.

Corollary 4.4. *The number of perfect matchings of the following bipartite graphs G and H corresponds to Genocchi numbers $G(2n)$ and median Genocchi numbers $H(2n-1)$ as follows:*

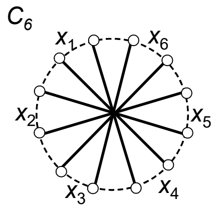
$$\begin{aligned} V(G) &= E \cup X, \quad \textit{where } E = \{e_1, e_2, \ldots, e_{2n-1}\}, X = \{x_1, x_2, \ldots, x_{2n-1}\}, \\ E(G) &= \{e_i x_j : 1 \le i \le 2n-1, \lfloor i/2 \rfloor + 1 \le j \le \lfloor (i-1)/2 \rfloor + n\}. \\ V(H) &= E \cup X, \quad \textit{where } E = \{e_1, e_2, \ldots, e_{2n}\}, X = \{x_1, x_2, \ldots, x_{2n}\}, \\ E(H) &= \{e_i x_j : 1 \le i \le 2n, \lfloor i/2 \rfloor + 1 \le j \le \lfloor i/2 \rfloor + n\}. \end{aligned}$$

Example 4.5. As shown in Figure 5,

$$w(C_6) = (x_1 + x_2 + x_3)(x_2 + x_3 + x_4)^2 (x_3 + x_4 + x_5)^2 (x_4 + x_5 + x_6).$$

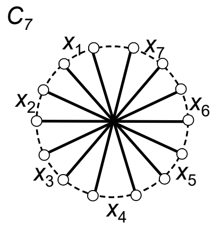
The coefficient of $x_1 x_2 x_3 x_4 x_5 x_6$ of $w(C_6)$ is 8, and the number of perfect matchings of $G(C_6, X)$ is also 8. Hence, we have $H(5) = m(C_6, N_6) = 8$.

$$\begin{aligned} w(C_7) = (x_1 + x_2 + x_3 + x_4)(x_2 + x_3 + x_4)(x_2 + x_3 + x_4 + x_5)(x_3 + x_4 + x_5) \\ (x_3 + x_4 + x_5 + x_6)(x_4 + x_5 + x_6)(x_4 + x_5 + x_6 + x_7). \end{aligned}$$



$G(C_6, X)$

	x_1	x_2	x_3	x_4	x_5	x_6
e_1	1	1	1	0	0	0
e_2	0	1	1	1	0	0
e_3	0	1	1	1	0	0
e_4	0	0	1	1	1	0
e_5	0	0	1	1	1	0
e_6	0	0	0	1	1	1



$G(C_7, X)$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
e_1	1	1	1	1	0	0	0
e_2	0	1	1	1	0	0	0
e_3	0	1	1	1	1	0	0
e_4	0	0	1	1	1	0	0
e_5	0	0	1	1	1	1	0
e_6	0	0	0	1	1	1	0
e_7	0	0	0	1	1	1	1

Figure 5: n-crossings (upper left, lower left) and their biadjacency matrices of corresponding bipartite graphs $G(C_n, X)$ (upper right, lower right).

The coefficient of $x_1x_2x_3x_4x_5x_6x_7$ of $w(C_7)$ is 17, and the number of perfect matchings of $G(C_7, X)$ is also 17. Hence, we have $G(8) = m(C_7, N_7) = 17$.

5 Further discussions

There are a lot of unknown things for the multiplicity in $\mathcal{NCD}(E)$. One ambitious problem is to find a formula for $m(E, F)$ in general.

In Section 4, we represent $m(E, N_n)$ by the number of perfect matchings of a corresponding bipartite graph. It is interesting if we can find an efficient method to calculate the number of perfect matchings in a graph of this kind.

As is shown in [15], there is a relation between the chord expansion number and the evaluation of the Tutte polynomial at the point $(2, -1)$. As a future research subject, it is considered to find a relation between the multiplicity $m(E, F)$ in general, or $m(E, N_n)$, and some counting polynomials of graphs.

References

[1] A. Bigeni, Combinatorial interpretations of the Kreweras triangle in terms of subset tuples, *Electron. J. Combin.* **25** (2018), #P4.44 (11 pages), doi:10.37236/7531.

[2] A. Bigeni, A generalization of the Kreweras triangle through the universal sl_2 weight system, *J. Comb. Theory Ser. A* **161** (2019), 309–326, doi:10.1016/j.jcta.2018.08.005.

[3] S. V. Chmutov and A. N. Varchenko, Remarks on the Vassiliev knot invariants coming from sl_2, *Topology* **36** (1997), 153–178, doi:10.1016/0040-9383(95)00071-2.

[4] M. Domaratzki, Combinatorial interpretations of a generalization of the Genocchi numbers, *J. Integer Seq.* **7** (2004), Article 04.3.6 (11 pages), https://cs.uwaterloo.ca/journals/JIS/VOL7/Domaratzki/doma23.html.

[5] D. Dumont, Sur une conjecture de Gandhi concernant les nombres de Genocchi, *Discrete Math.* **1** (1972), 321–327, doi:10.1016/0012-365x(72)90039-8.

[6] D. Dumont, Interprétations combinatoires des nombres de Genocchi, *Duke Math. J.* **41** (1974), 305–318, doi:10.1215/s0012-7094-74-04134-9.

[7] D. Dumont and D. Foata, Une propriété de symétrie des nombres de Genocchi, *Bull. Soc. Math. France* **104** (1976), 433–451, doi:10.24033/bsmf.1839.

[8] D. Dumont and A. Randrianarivony, Dérangements et nombres de Genocchi, *Discrete Math.* **132** (1994), 37–49, doi:10.1016/0012-365x(94)90230-5.

[9] D. Dumont and G. Viennot, A combinatorial interpretation of the Seidel generation of Genocchi numbers, in: J. Srivastava (ed.), *Combinatorial Mathematics, Optimal Designs and Their Applications*, Elsevier, volume 6 of *Annals of Discrete Mathematics*, pp. 77–87, 1980, doi:10.1016/S0167-5060(08)70696-4.

[10] R. Ehrenborg and E. Steingrímsson, Yet another triangle for the Genocchi numbers, *European J. Combin.* **21** (2000), 593–600, doi:10.1006/eujc.1999.0370.

[11] R. C. Entringer, A combinatorial interpretation of the Euler and Bernoulli numbers, *Nieuw Arch. Wisk.* **14** (1966), 241–246.

[12] A. J. Goodall, C. Merino, A. de Mier and M. Noy, On the evaluation of the Tutte polynomial at the points $(1, -1)$ and $(2, -1)$, *Ann. Comb.* **17** (2013), 311–332, doi:10.1007/s00026-013-0180-y.

[13] C. Merino, The number of 0-1-2 increasing trees as two different evaluations of the Tutte polynomial of a complete graph, *Electron. J. Combin.* **15** (2008), #N28 (5 pages), doi:10.37236/903.

[14] T. Nakamigawa, Expansions of a chord diagram and alternating permutations, *Electron. J. Combin.* **23** (2016), #P1.7 (8 pages), doi:10.37236/5120.

[15] T. Nakamigawa and T. Sakuma, The expansion of a chord diagram and the Tutte polynomial, *Discrete Math.* **341** (2018), 1573–1581, doi:10.1016/j.disc.2018.02.015.

[16] J. Riordan and P. R. Stein, Proof of a conjecture on Genocchi numbers, *Discrete Math.* **5** (1973), 381–388, doi:10.1016/0012-365x(73)90131-3.

[17] L. Seidel, Über eine einfache entstehungsweise der Bernoullischen zahlen und einiger verwandten reihen, *Sitzungsber. Münch. Akad.* **4** (1877), 157–187, http://publikationen.badw.de/de/003384831.

[18] N. J. A. Sloane (ed.), The On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org.

[19] G. Viennot, Interprétations combinatoires des nombres d'Euler et de Genocchi, in: *Seminar on Number Theory, 1981/1982*, Université de Bordeaux I, Talence, Exp. No. 11 (94 pages), 1982, held at the Université de Bordeaux I, Talence, 1981/1982.