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Osnove za konstrukcijo algoritmov minimalne porabe goriva pri vleki

Basic Background for Minimum Fuel Consumption Algorithms in Traction

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Opisana je konstrukcija algoritmov, namenjenih za rešitev nekaterih problemov najnižjega nivoja pri načrtovanju voznih redov na Slovenskih železnicah. Ključni kriterij je minimalna poraba goriva. Algoritmi so zasnovani na rešitvi variacijskega problema z razširitvami glede na različnost robnih pogojev. Pri konstrukciji algoritmov ni bila uporabljena Hamiltonova teorija, rešitve slone na preprostejših izpeljavah.

(Ključne besede: Slovenske železnice, načrtovanje voznih redov, algoritmi, poraba goriva)

The design of algorithms for solving some low level problems in timetable planning for Slovenian railways is described. The main aim is the use of the minimum amount of fuel. The algorithms are based on a solution of a variational problem with extensions to various types of boundary conditions. The use of the Hamiltonian theory has been avoided and the solutions are based on more simple derivations.

(Keywords: Slovenian Railways, timetables planning, algorithms, fuel consumption)

0 UVOD

Standardni problem minimalne porabe goriva (MPG) se pojavlja v železniški vleki in pri drugih načinih transporta kot naloga prepeljati vlak z enega konca na drugega, v predpisanem času in s predpisano začetno in končno hitrostijo. V normalnih razmerah pride do dveh tipičnih situacij. V prvi situaciji opazi vlakovodja varnostni signal in pospešuje do predpisane hitrosti, ki jo mora doseči na glavnem signalu. V drugi situaciji mora prevoziti dano razdaljo s konstantno hitrostjo, mogoče pa je delati z majhnimi popravki hitrosti, da bi se izboljšala gospodarnost rabe goriva, ne da bi poprave vplivale na vozni red.

Gotovo je pri različnih načinih uporabe v železniški vleki MPG manj pomemben od drugih vlakovnih problemov glede na zelo svojsko in zapleteno tehnologijo. Predstavljeni pregled algoritmov MPG je zato treba gledati v širšem pomenu kakor nabor metod, ki jih lahko uporabimo tam, kjer se zdi vpliv MPG pomemben.

Pod nekimi pogoji (npr. nespremenljiva strmina tira) obstajajo analitične rešitve problema MPG, ki slonijo na načelu največje vrednosti. V praksi vendar pričakujemo uporabo izključno numeričnih algoritmov.

V drugem poglavju je MPG predstavljen formalno, v prvem delu so formulirane osnovne enačbe. V drugem delu so dane posebne rešitve na

0 INTRODUCTION

The classic minimum fuel problem (MFP) as posed by railway traction, and other similar forms of transportation, is to bring the train from the initial to the final position on the track, within the prescribed travel time, initial and final velocities. In normal traffic conditions, the following two typical situations occur. First, the operator observes the warning signal and must accelerate to reach the prescribed velocity at the main signal. Second, the given distance must be covered with constant velocity, although small velocity corrections are allowed to improve the fuel economy without affecting the time schedule.

It is clear that the MFP in railway traction is of less importance than other problems of traction which concern very specific and complex technology. This present review of MFP algorithms must then be regarded, in a broader sense, as a toolbox which could be used if the MFP becomes significant.

Under certain circumstances (e.g. constant track slope), analytic solutions of the MFP problem exist, based on the principle of maximality. In practice however, only the use of numerical solutions is to be expected.

In section 2, the MFP is presented in formal terms, in the first part, basic equations are formulated. In the second part, a particular solution is given podlagi variacijskega računa. Tretji del obravnava vpliv strmine tira. Do optimalne rešitve je v vseh primerih bilo mogoče priti brez uporabe teorema maksimuma v krmilni teoriji.

based on the variational calculus. In the third part change in the rail inclination is addressed. The optimum solution can be achieved without the use of the maximum principle in the control theory.

1 FORMALIZACIJA PROBLEMA MINIMALNE PORABE GORIVA

1 FORMALIZATION OF MINIMUM FUEL CINSUMPTION PROBLEM

1.1 Osnovne enačbe

Osnovni izraz za gibanje vlaka kot zgoščene masne točke je [1]:

1.1 Basic statements

The general expression for the movement of a train with mass concentrated at a single point is [1]:

$$(m_l + m_t)\frac{d^2x}{dt^2} = u - R_t - R_l - (m_l + m_t)gi_g \quad (1),$$

kjer pomenijo: x - razdalja od startne točke, t - čas, m_l in m_t - ekvivalentne mase lokomotive in vlaka, vključno rotacijske mase, u - vlečno silo, R_t - povprečeni upor vlaka, R_l - povprečeni upor lokomotive, i_g - vpliv strmine tira. Upor vlaka lahko aproksimiramo s polinomom drugega reda v odvisnosti od hitrosti. S pozitivnimi polinomskimi koeficienti a,b,c lahko (1) prepišemo kot:

where x is the distance from the starting point, t is the time, m_l and m_t are the equivalent masses of the locomotive and train including rotating masses, u is the tractive force, R_t is the average resistance of the train, R_l is the average resistance of the locomotive, and i_g represents the slope of the track. The resistance of the train can be approximated with a 2nd order polynomial in terms of velocity. With positive polynomial coefficients a,b,c eq. (1) can be rewritten as:

$$m\frac{d^2x}{dt^2} = u - \left[a\left(\frac{dx}{dt}\right)^2 + b\frac{dx}{dt} + c\right] - mgi_g \quad (2).$$

Robni pogoji so dani z:

The boundary conditions are given as:

$$\begin{aligned}(t=0): x = x_0;\quad & \frac{dx}{dt} = v = v_0\\ (t=t_f): x = x_f;\quad & \frac{dx}{dt} = v = v_f\end{aligned} \quad (3).$$

Zaradi enostavnosti naj velja vedno predlog 1, razen če njegova veljava ni neposredno odpravljena v spremnem besedilu.

For the sake of simplicity, proposition 1 is valid unless stated otherwise.

Predlog 1:
Hitrost je nepadajoča funkcija v t in x.

Proposition 1:
Velocity v is a nondecreasing function of t and x.

Celotna masa vlaka m je enaka vsoti $m_l + m_t$. Glede na eksperimentalne podatke [1] lahko v normalnem področju hitrosti zanemarimo konstante b,c,i_g v (2) in tako omogočimo, da se analitični izrazi zapišejo bolj kompaktno. Če je vlečna sila u in velja predlog 1, je rešitev enačbe (2) $v=dx/dt$ dana s (4):

The total mass of the train m equals the sum $m_l + m_t$. According to the experimental data in [1], the constants b,c,i_g in (2) can be neglected for normal velocities, thus enabling analytic solutions to be written in a more compact form. If the traction force u is constant and proposition 1 is true, the solution of eq. (2) is $v=dx/dt$ from eq.(4):

$$k = \sqrt{\frac{u}{a}};\quad w = \frac{2ka}{m};\quad e_1 = \frac{v_0 + k}{v_0 - k}e^{w.t};\quad v(t;v_0,u) = -k\frac{1+e_1}{1-e_1} \quad (4).$$

Prostor rešitev (4) mora biti dopolnjen z ustaljeno hitrostjo $v = \sqrt{u/a}$. Razdalja x je integral enačbe (4):

The velocity $v = \sqrt{u/a}$ must be added to the solutions of eq. (4). The distance x is derived by integrating eq. (4):

$$p = \frac{v_0 + k}{v_0 - k};\quad p_1 = ln(1 - e^{wpt})$$

$$x(t;v_0,u) = -kt + \frac{2kp_1}{w} - \frac{2kln(1-p)}{w} \quad (5).$$

Z vložitvijo $v= dx/dt$ je mogoče rešiti (2) v obliki:

After substituting $v= dx/dt$ eq. (2) can be expressed as:

$$v(x;v_0,u)=\left[\frac{u}{a}-(\frac{u}{a}-v_0^2)\exp(-2\frac{a}{m}x)\right]^{1/2} \quad (6).$$

Inverzni rešitvi enačb (6) in (4) sta enačbi:

Inverse solutions for eqns. (6) and (4) are:

$$x(v;v_0,u)=\frac{m}{2a}ln\left(\frac{u_0/a-v_0^2}{u/a-v^2}\right) \quad (7),$$

$$t(v;v_0,u)=\frac{m}{2\sqrt{ua}}ln\left[\frac{-v-k}{-v+k}\frac{v_0-k}{v_0+k}\right] \quad (8).$$

Z vstavljanjem (6) v (8) dobimo analitični izraz za $t=t(x;v_0,u)$.

Za rešitev MPG je treba najti vlečno silo u kot funkcijo t, ki minimizira integral v (9):

If eq. (6) is inserted into (8) the analytical expression for $t=t(x;v_0,u)$ can be obtained.

To solve the MFP, the traction force u must be determined found as a function of t which minimizes the integral in eq.(9):

$$\int_0^{t_f} udt=\min \quad (9).$$

Enačbe (2), (3) in (9) predstavljajo problem minimalne porabe goriva [2].

Eqns. (2), (3) and (9) represent the minimum fuel problem [2].

1.2 Problem minimalne porabe goriva v variacijskem računu

1.2 Minimum fuel consumption problem in variational calculus

Iz enačb (1) ali (2) je mogoče u izraziti kot funkcijo $u=u(t,x,x',x'')$ neodvisne spremenljivke t, odvisne spremenljivke x, in njenih derivacij po času x',x''. Funkcional v (9) je mogoče pisati v obliki:

From eqs.(1) or (2) u can be expressed as a function $u=u(t,x,x',x'')$ of independent variable t, dependent variable x, and its time derivatives x',x''. The functional in eq.(9) can then be expressed in the form:

$$J(x)=\int_0^{t_f} f(t,x,x',x'')dt=\int_0^{t_f}(\alpha.x''+\beta.x'^2).dt=\min \quad (10)$$

s primernimi konstantami α,β in robnimi pogoji v (3).

Enačbi (10) in (3) sta dobro znani v variacijskem računu. Prirejena Eulerjeva enačba je:

with suitable constants α,β and the boundary conditions in eq.(3)

Eqns. (10) and (3) are a well known problem in variational calculus. The adjoint Euler equation is:

$$\frac{\partial f}{\partial x}-\frac{d}{dt}\frac{\partial f}{\partial x'}+\frac{d^2}{dt^2}\frac{df}{dx''}=0 \quad (11).$$

Enačba (11) je diferencialna enačba četrtega reda s štirimi integracijskimi konstantami, ki jih je mogoče izračunati iz (3). Z vstavitvijo funkcije f iz enačbe (10) v enačbo (11) dobimo:

Eq. (11) is a 4th order differential equation with 4 integration constants, which can be calculated from eq. (3). If the function f from eq. (10) is inserted into eq. (11) the result is:

$$0-\frac{d}{dt}(2\beta x')+\frac{d^2}{dt^2}(\alpha)=0;\Rightarrow x''=0;\Rightarrow x'=konst \quad (12).$$

Iz izraza za u v (10) in ker je $x''=0$ in $x'=konst$, izhaja, da je optimalno krmiljenje u konstantno. Fizikalna interpretacija tega naj bo izražena v naslednji lemi.

It follows from the expression for u in eq. (10) when $x''=0$ and $x' = const.$ that the optimum control of u is constant. The physical interpretation of this fact shall be formulated in the next lemma.

Lema 1:

Če je enačba gibanja podana v (1) in če objekt preide pot med 0 in x_f s konstantno hitrostjo (kar implicira $v_0=v_f$ in $u=konst$) in so pri tem robni

Lemma 1:

If eq. (1) is the equation of motion and the object traverses the path between 0 and x_f with constant velocity (implying that $v_0=v_f$ and u = const.) with *pogoji v (3) že izpolnjeni, potem je funkcional porabe goriva v (10) minimalen v množici sočasnih rešitev.*

Naj se fizikalni model v lemi 1 spremeni v toliko, da v intervalu $[0, x_f]$ ostane tak kakršen je , v dodatnem diferenčnem časovnem koraku $\lfloor t_f, t_f + \Delta t_f \rfloor$ pa objekt pospešuje s konstantnim vlekom u in doseže končno hitrost $v_f + \Delta v_f$ v času $t_f + \Delta t$. Enačbo (2) pri $t = t_f$ potem lahko napišemo v diferenčni formi:

$$m\frac{\Delta v}{\Delta t} + av^2 = u \qquad (13).$$

Iz te enačbe se izračuna impulz goriva $u\Delta t$:

$$u\Delta t = \frac{m\Delta v u}{u - av^2} \qquad (14).$$

V primeru pospeševanja je seveda $u > av^2$ in potem funkcija v (14) monotono pada z naraščajočim u . Z drugimi besedami: poraba goriva se zmanjšuje, če je za vleko uporabljena večja sila. Potemtakem je optimalno krmiljenje neskončni pozitivni u - impulz z omejeno vrednostjo za uporabo goriva, ki trenutno poveča hitrost objekta od v_f do $v_f + \Delta v$. Naravno je, da je velikost vlečne sile tehnološko omejena.

Ker je postavka o u impulzu bistvenega pomena, naj bo argumentirana še na drug način. Naj se sistem po lemi 1 spremeni v toliko, da je sekcija $[0,x_f]$ razdeljena na dvoje intervalov, $[0,x]$ in $[x,x_f]$ in je $x \in [0,x_f]$. Časi pospeševanja naj bodo zanemarjeni in hitrosti objekta v prvem in drugem intervalu naj bosta konstantni in enaki v_1 oziroma v_2. Potem je MPG mogoče pisati v obliki:

$$\frac{x}{v_1} + \frac{x_f - x}{v_2} = t_f$$
$$v_1 x + v_2 (x_f - x) = Min \qquad (14b).$$

Prva enačba v (14b) je omejitev, druga je kriterijska funkcija. Po lemi 1 je rešitev (14b) enaka $v_1 = v_2 = v_a$ za poljuben $x \in [0,x_f]$. Sistemu naj bo potem dodana še ena omejitev $v_2 = v_{2f}$ z dodatnim pogojem $v_{2f} > v_a$. Lahko se je prepričati, da je rešitev tega novega sistema $x = x_f$, $v_1 = v_a$ in seveda $v_2 = v_{2f}$. Ta oblika rešitve zahteva, da v točki x_f hitrost trenutno naraste od v_1 na v_{2f}, kar je mogoče storiti samo z u impulzom. Potem zaradi obeh dokazov o pomenu u impulza drži sledeča lema:

Lema 2:

Sistem iz leme 1 naj bo razširjen s pospeševanjem objekta v trenutku t_f od v_f do $v_f + \Delta v_f$ s pozitivnim Δv_f .Potem je optimalno krmiljenje dvostopenjsko. V prvi fazi je optimalno krmiljenje podano z lemo 1 v intervalu $0 \le t \le t_f$. Krmiljenje

the boundary conditions in eq. (3) already fulfilled, then the functional in eq. (10) is minimised in the set of concurrent solutions.

Let the physical model in lemma 1 be changed such that in the interval $[0, x_f]$ it remains unchanged and in the additional differential time interval $\lfloor t_f, t_f + \Delta t_f \rfloor$ the object accelerates under constant u and reaches its final velocity $v_f + \Delta v_f$ at time $t_f + \Delta t$. Eq. (2) at $t = t_f$ can then be written in the differential form:

From this equation the fuel-impulse $u\Delta t$ is :

Clearly in the case of acceleration $u > av^2$, then the function in eq. (14) monotonically decreases with increasing *u*. In other words, the fuel consumption decreases when greater traction force is applied. Then the optimal control is the infinite positive u - impulse with limited fuel consumption value, which instantaneously changes the velocity of the object from v_f to $v_f + \Delta v$. Naturally the intensity of u is technologically restricted.

As the statement about *u* impulse is the essential one, it shall be proved otherwise else. Let the system in lemma 1 be changed in the sense, that the section $[0,x_f]$ is divided in two intervals $[0,x]$ in $[x,x_f]$ with $x \in [0,x_f]$. If the acceleration times are neglected, then the object velocities in the first rsp. second interval shall be constant v_1 rsp. v_2. Then the MFP can be written by:

The first equation in (14b) is the constraint, the second one is the objective function. By lemma 1 the solution of eq. (14b) is $v_1 = v_2 = v_a$ with arbitrary $x \in [0,x_f]$. Let then the additional constraint be adopted that $v_2 = v_{2f}$ and $v_{2f} > v_a$. It is easy to see that the solution of this new problem is $x = x_f$, $v_1 = v_a$ and clearly $v_2 = v_{2f}$. This form of solution requires, that at the point x_f the velocity instantly increases from v_1 to v_{2f} and this can just be done with the *u* impulse. Then with these findings about *u* impulse the next lemma holds:

Lemma 2:

Let the system from lemma 1 be extended with an acceleration of the object at the time instant t_f from v_f to $v_f + \Delta v_f$ with Δv_f positive. Then the optimal control is two-stage, and in the first stage optimal control is given by lemma 1 in the interval *v drugi stopnji je maksimalna vlečna sila* u_{max}*, ki traja, dokler objekt ne doseže hitrosti* $v_f + \Delta v$ *in interval* $t > t_f$.

Podobno lahko analiziramo gibanje objekta, če sistem iz leme 1 ekspandira proti levi z začetno hitrostjo $v_0 - \Delta_v$ ob času $-\Delta t$.

Optimalno krmiljenje za sistema v enačbah (2), (3) in (9) za $v_f > v_0$ lahko določimo takoj. Naj bodo rešitve enačb (4) oz. (5) napisane kot funkcije parametrov $v = v(t; v_0, u)$ in $x = x(t; v_0, u)$. Z uporabo označb s slike 1 je mogoče postaviti naslednji sistem enačb:

$0 \le t \le t_f$. *The control action in the second stage is the maximum traction force* u_{max}*, applied until the object reaches* $v_f + \Delta v$ *in the interval* $t > t_f$.

In a similar manner a motion analysis can be made if the system from lemma 1 is 'left expanded' with start velocity $v_0 - \Delta_v$ at the time instant $-\Delta t$.

The optimal control for the system in eqs.(2), (3) and (9) for $v_f > v_0$ can now be determined. Let the solutions of eqs. (4) and (5) respectively, be written as functions of parameters $v = v(t; v_0, u)$ and $x = x(t; v_0, u)$. Using the notations from fig. (1) the system of equations can be set up us follows:

$$
\begin{aligned}
x_1 &= x(t_1; v_0, u_{max}) \\
v_1 &= v(t_1; v_0, u_{max}) \\
x_2 &= v_1 . t_2 \\
x_3 &= x(t_3; u_{max}, v_1) \\
v_f &= v(t_3; v_1, u_{max}) \\
x_f &= x_1 + x_2 + x_3 \\
t_f &= t_1 + t_2 + t_3
\end{aligned} \qquad (15).
$$

Sistem sedmih nelinearnih enačb s sedmimi neznankami $(x_1, x_2, x_3, t_1, t_2, t_3, v_1)$ ima, kar je v splošnem dovolj nenavadno, analitično rešitev. Ker je $(t_1 + t_3) = t(v_f; v_0, u_{max})$ po enačbi (8) in $(x_1 + x_3) = x(v_f; v_0, u_{max})$ po enačbi (7), je ustaljena hitrost v_1 podana z:

The system of 7 nonlinear eqs. (15) with 7 unknown variables $(x_1, x_2, x_3, t_1, t_2, t_3, v_1)$ can easily be solved analytically. As $(t_1 + t_3) = t(v_f; v_0, u_{max})$ according to eq. (8) and $(x_1 + x_3) = x(v_f; v_0, u_{max})$ according to eq. (7) the constant velocity v_1 is:

$$v_1 = \frac{x_f - (x_1 + x_3)}{t_f - (t_1 + t_3)} \qquad (16).$$

Druge spremenljivke v (15) lahko izračunamo neposredno. Iz rešitvene sheme zgoraj sledi, da lahko sistem enačb (15) poenostavimo na enačbe za posamezne spremenljivke v eksplicitni obliki. Rešitve na sl.1 so bile računane s podatki iz preglednice 1 s faktorjem upora *a* kot parametrom. Rezultati za *a* = 0,6 so bili potrjeni tudi numerično z drugimi postopki in rezultati so prikazani na sliki 2. Na njej je izvlečena črta rešitev za primer uporabe kriterija minimalne energije po enačbi (19). Rešitve so v okviru toleranc različnih numeričnih postopkov identične.

Krmilno strategijo na slikah 1 in 2 lahko preprosto formuliramo v lemi 3.

Lema 3:
Krmilna strategija z namenom, da bi bila poraba goriva kar najmanjša, je za sistem opisana v enačbah (15) in ob veljavnosti predloga 1 je trostopenjska strategija. V prvi stopnji je krmiljenje največja vlečna sila u_{max}*, ki vleče toliko časa, da hitrost doseže* v_1 *iz enačbe (16). V drugi stopnji se objekt premika s stalno hitrostjo* v_1*. V tretji stopnji objekt pospešuje zaradi vlečne sile* u_{max} *od* v_1 *do končne hitrosti* v_f.

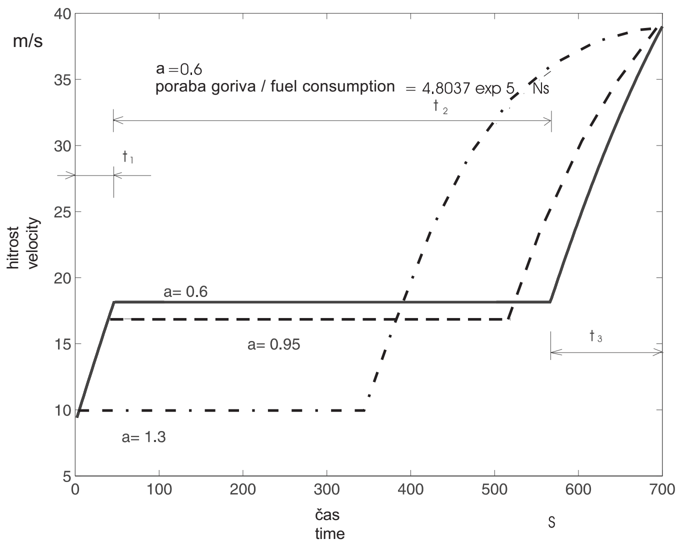
Other unknowns in eq. (15) can be obtained directly. From the above approach it follows, that the eqns. system (15) can be reduced to a single variable equation in its explicit form. Solutions in fig. (1) were calculated for the data in table 1 with resistance factor *a* as a parameter. These results for *a* =0.6 were verified numerically using other methods and the results appear in fig. 2, where the solid line symbolizes the solution if the minimum energy criterion in eq.(19) is applied. The results are identical within the limits of numerical accuracy.

The control strategy from fig. (1) and fig.(2) can be simply formulated in lemma 3.

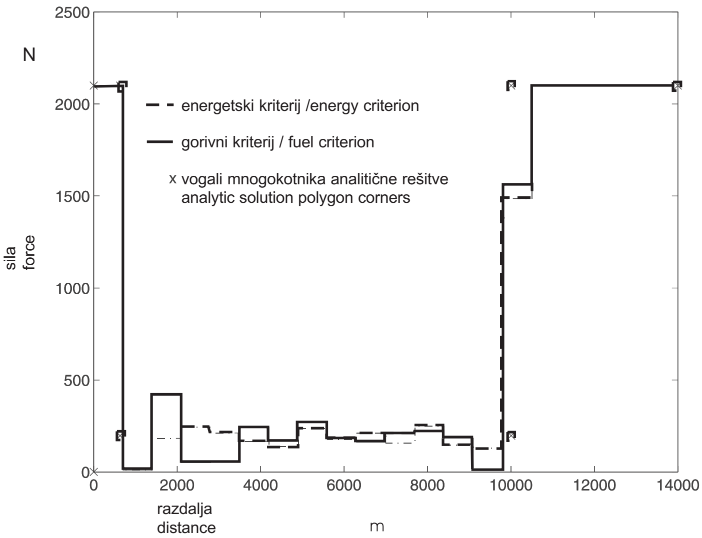
Lemma 3:
Given that proposition 1 is valid, the minimum fuel control strategy given by eq. (15) is three-staged. In the first stage the control variable equals the maximum of the traction force, u_{max}*, which is applied until the velocity reaches* v_1*, as given in eq.(16). In the second stage the object is moving with constant velocity* v_1*. In the third stage the object accelerates from* v_1 *under* u_{max} *until the final velocity* v_f *is reached.*

Preglednica 1. *Modelni parametri numeričnih rešitev*
Table 1. *Model parameters for numeric solutions*

opis description	oznaka symbol	vrednost value	enote units
masa vlaka train mass	m	10000	kg
kvadr. upor quadr. resist.	a	0,6	kg/m
začetna hitrost initial velocity	v_0	9	m/s
končna hitrost final velocity	v_f	39	m/s
največja sila vleke max.traction force	u_{max}	2100	N
dolžina poti path length	x_f	14000	m
čas prevoza crossing time	t_f	700	s



Sl. 1. *Optimalni hitrostni profili*
Fig.1. *Optimal velocity profiles*



Sl. 2. *Optimalno krmiljenje vlečne sile*
Fig. 2. *Tracking force optimal control*

Pravi razlog za uvedbo predloga 1 je dejstvo, da se lahko potem problem minimalne porabe goriva rešuje analitično po enačbi (16). Naj bo najprej predlog 1 obrnjen, tj. hitrost naj bo kvečjemu nemonotono padajoča funkcija. Figurativno sta sedaj funkcijska grafa na sl.1 in sl. 2 zrcaljena okoli osi *x*. Sila vleke (oz. zaviranja) v fazah negativnega pospeška je minimalna ali enaka 0. Analitična rešitev po (16) seveda tudi v tem primeru obstaja. Lahko se predlog 1 tudi kar opusti. Optimalna rešitev za tak splošen primer je dana v naslednji lemi:

Lema 4:
Optimalna strategija v MFP je trifazna strategija: v prvi fazi je treba uporabiti največjo ali najmanjšo vlečno moč, v drugi fazi je hitrost nespremenljiva in v tretji fazi mora biti vlečna moč najmanjša ali največja.

Če je vlečna sila u v lemi 4 v prvi fazi enaka tisti v tretji fazi, je mogoče izrabiti analitično rešitev po enačbi (16). V nasprotnem primeru je treba numerično reševati minimalni problem za določitev hitrosti v drugi fazi, tj. hitrosti v_1.

Optimalna strategija je invariantna glede na spremembo končne hitrosti v_f. Poraba goriva se zmanjšuje z v_f, dokler je $v_f > v_0$. Če se v_f izenači z ustaljeno hitrostjo v_1, se lahko izpusti robni pogoj v_f v (3) in je najmanjša poraba goriva dosežena za v_1, ki se računa po (17) z krmilno strategijo po lemi 4a:

The real reason for introducing proposition 1 is that the minimum fuel problem can be solved analytically using eq. (16). Let the first proposition be reversed, i.e. the velocity can only be a non-monotonically increasing function, then the function graphs in fig. 1 and fig. 2 are mirrored in the *x*-axis. The traction (or braking) force in the phases of deceleration is the minimum or equal to zero. The analytical solution according to eq. (16) exists also in this case and proposition 1 can be completely neglected. The optimum solution for such a general case can then be given as in the next lemma:

Lemma 4 :
The optimum strategy in the MFP is the three-phase strategy: in the first phase the maximum or minimum force u must be applied, in the second phase the velocity is constant and in the third phase the maximum or minimum force must be applied again.

If the traction force u, in lemma 4, in the first phase is equal to that of the third phase, then the analytic solution according to eq. (16) is possible, otherwise it should be solved numerically to determine the velocity in the second phase, v_1.

The optimum strategy is invariant according to the change in velocity, v_f. Fuel consumption decreases with v_f until $v_f > v_0$. If v_f becomes equal to the velocity v_1, then the boundary condition v_f in eq. (3) can be omitted and the minimum fuel consumption is achieved for v_1 as calculated in eq. (17) under the control strategy in lemma 4a:

$$x_f - x(v_1; v_0, u_{\max}) = (t_f - t(v_1; v_0, u_{\max}))v_1 \qquad (17).$$

Lema 4a:
Strategija najmanjše porabe goriva pri reduciranih robnih pogojih v_0, t_f, x_f, v_f, (tj. če je v_f ustaljena hitrost v_1) v (3) je dvofazna strategija. Najprej je treba z vlečno silo $u_{\max}$ doseči ustaljeno hitrost v_1 kot rešitev (17), v drugi fazi ostane ta hitrost stalna do konca intervala t_f oz. x_f.

Varianto MPG, kakor je formulirana v (2), (3) in (9) z veljavnim predlogom 1, je mogoče spremeniti v krmilni problem s prostim koncem, če t_f v (3) ni določen. Naravna rešitev problema je potem taka, da se objekt upočasni proti *u*=0, medtem ko se čas prehoda neomejeno povečuje. Za izločitev tega pojava je treba uvesti 'kazen' za počasno vožnjo, npr. s ponovno uvedbo negativne konstante - sile trenja c v (2). Vsekakor ostaneta veljavni lemi 1 in 2 z največjo vlečno silo $u_{\max}$ v fazah pospeševanja. Optimalna strategija je potem enaka tisti v lemi 3 s to spremembo, da je hitrost v_1 nadomeščena z ekonomično hitrostjo v_e. Le to je mogoče izračunati iz enačbe (18) , v kateri je

Lemma 4a:
The minimum fuel control strategy with a reduced set of boundary conditions v_0, t_f, x_f, v_f, (i.e if v_f is the constant velocity v_1) in eq. (3) is the two-phase strategy. The velocity v_1, as the solution to eq. (17), should be achieved first by applying the traction force $u_{\max}$, during the second phase the velocity remains constant until the end of the interval t_f or x_f.

The MFP as formulated in eqs.(2), (3) and (9) with the valid proposition 1 can be converted to the 'free end' control problem, if t_f in eq. (3) remains undetermined. The 'natural' solution then is that the object slows down under u = 0 while the travelling time increases infinitely. To block this behaviour, the slow motion must be 'penalized', e.g. by setting constant c in eq. (2) to be the negative constant friction force, and lemmas 1 and 2 remain valid with the maximum traction force $u_{\max}$ in the acceleration phases. Then the optimum strategy is the same as that in lemma 3, with the exception that the velocity v_1 is replaced with the economic velocity v_e. This velocity can be calculated from eq. (18)

postavljena zahteva, da je poraba goriva pri ustaljenem gibanju najmanjša:

providing that the fuel consumption at the constant velocity is a minimum:

$$ut_f = (v^2 a + c)\frac{x_f}{v} = \min \quad (18).$$

Če je odvod (18) po spremenljivki v enak ničli, je iz te nove enačbe mogoče izračunati optimalno vrednost $v_e = \sqrt{c/a}$. Krmilna strategija se opiše z naslednjo lemo:

Setting the velocity derivative of eq. (18) to zero gives the optimum value for $v_e = \sqrt{c/a}$. The control strategy can then be described in the following lemma:

Lema 4b:
Najmanjša strategija porabe goriva za problem s prostim koncem z neznanim končnim časom t_f je trofazna strategija, kakor je formulirana v lemi 3, če je v_1 zamenjan z ekonomično hitrostjo v_e. Držati mora predlog 1.

Lemma 4b:
The minimum fuel control strategy for the 'free end' with unknown finishing time t_f is a three-phase strategy as formulated in lemma 3, if v_1 is replaced by the economic velocity v_e and proposition 1 holds true.

Primerno rešitev za vse dosedaj obravnavane primere bi bilo mogoče dobiti tudi z uporabo Hamiltonove funkcije in Pontrjaginovega načela največje vrednosti, vendar se zdi izbrana pot bolj premočrtna in preprosta.

Poglavje sklenimo s pripombo, da so nekatere dobljene rešitve veljavne tudi za problem z najmanjšo energijo. V tem primeru funkcional v (10) zapišemo v naslednji obliki:

The proper solution in all the above cases could be obtained using the Hamiltonian function and Pontryagin's maximum principle, however, the method described above is much more straightforward and simpler.

This section can be brought to a close with the statement that some solutions obtained are also valid for the minimum energy problem. In this case the functional in eq. (10) is written as:

$$J_e(x) = \int_0^{t_f} f(t, x, x', x'')x'dt = \int_0^{t_f} f_e(t, x, x', x'')dt \quad (19).$$

Sklep, dobljen v (12), tj. $x' = konst$ ostane veljaven in zato drži lema 1. Nadalje je po analogiji s (13) in (14) treba pokazati, da je nujno potrebno uporabiti največjo vlečno silo $u_{\max}$, če naj se hitrost zveča za Δv ob najmanjši porabi energije. Porabo energije na skrajnem desnem intervalu poti po lemi 2 zapišemo v naslednji obliki:

The conclusions from eq. (12), i.e. $x' = const.$ remain valid and therefore lemma 1 holds. Next, from an analogy with eqs. (13) and (14) it must be shown that the maximum force $u_{\max}$ must be applied if one wishes to increase the velocity by Δv with the minimum energy consumption. The energy consumption in the extended interval by lemma 2 can be written as:

$$u\Delta s \cong u\Delta t\left(v + \frac{\Delta v}{2}\right) = \frac{m\Delta v u}{u - av^2}\left(v + \frac{\Delta v}{2}\right) \quad (14a).$$

Analogno z enačbo (14) se funkcija (14a) monotono znižuje z naraščajočim *u*. Torej držita tudi lemi 2 do 4.

Analogously to eq. (14) the function in eq.(14a) monotonically decreases with the increasing *u*. Consequently lemmas 2 to 4 are also valid.

1.3 Spremembe v strmini tira

V točkah 1.1 in 1.2 je bil gravitacijski faktor mgi_g v (2) enak ničli. Pri praktični uporabi MPG algoritmov se ta faktor spreminja s strmino tira in enačb (15) ni mogoče neposredno uporabiti. Da bi dobili občutek, kako sprememba strmine tira vpliva na na rešitev MPG, lahko začnemo z dinamičnim modelom M_v, v katerem je zanemarjena poraba goriva v fazah pospeševanja in je gravitacijski faktor koračno konstantna funkcija g_i na *i*-tem odseku dolžine x_f / n vzdolž osi x. Naj bosta potem robna pogoja le čas prevoza t_f in dolžina poti x_f. Problem MFP sedaj lahko zapišemo z enačbama (20) in (21), pri tem je J poraba goriva:

1.3 Change in track inclination

In sections 2.1 and 2.2 the gravitational factor mgi_g in eq.(2) was set to zero. For the practical use of the MFP algorithms, however, this factor changes with the track inclination and consequently the system of eq. (15) cannot be used directly. To better understand how this change affects the MFP solution one can start from the dynamic model M_v where the fuel consumption in the acceleration phases is neglected and the gravitation factor is a constant function g_i in the *i*-th section of the length x_f / n over x axis. Let the crossing time t_f and the path length x_f be the only boundary conditions. Eqs. (20) and (21) can then be written with J as the fuel consumption:

$$t_f = \sum_i \frac{x_f}{n}\frac{1}{v_i} \tag{20}$$

$$J = \sum_i (g_i + a v_i^2)\frac{x_f}{n}\frac{1}{v_i} \tag{21}$$

n v (20), (21) je celoštevilčna konstanta in v_i je nespremenljiva sekcijska hitrost. Optimizacijski problem z omejitvami v (20) in (21) je mogoče rešiti z Lagrangeovo funkcijo (22):

n in eqs.(20) and (21) is an integer constant and v_i is the constant section velocity. The constraint optimization problem in eqs. (20) and (21) can be solved with the Lagrangian in eq. (22):

$$L = \sum_i (g_i + a v_i^2)\frac{x_f}{n v_i} + \lambda\left[(\sum_i \frac{x_f}{n v_i}) - t_f\right] \tag{22}$$

Parcialno odvajanje enačbe (22) po spremenljivki v_i in izenačitev odvoda z ničlo da (23):

The partial derivative of eq. (22) with respect to v_i and setting the derivatives to zero gives eqs. (23):

$$v_i^2 = \frac{g_i + \lambda}{a} \tag{23}$$

Po odvajanju (22) po spremenljivki λ in izenačitvi odvoda z ničlo po uporabi (23) rezultira enačba (24), ki se lahko numerično razreši po λ:

After taking the derivative of eq. (22) with respect to λ and setting the derivative to zero and using eq. (23), then eq. (24) is obtained which can be numerically resolved in terms of λ:

$$-t_f + \sum_i \frac{x_f}{n}\frac{a}{\sqrt{g_i + \lambda}} = 0 \tag{24}$$

Iz fizikalne interpretacije (20) in (21) izhaja, da minimum sistema (20) in (21) vedno obstaja za $n = 2$ in lahko se pokaže veljavnost naslednje enačbe (25):

Further more, it follows from the physical interpretation of eqs. (20) and (21) that the minimum of the system, (21) and (22), always exists for $n = 2$ and that the following eq. (25) can be proved:

$$J(g_1, g_2, v_1, v_2) = J(0, g_2 - g_1, v_1, v_2) + 2 g_1 t_f \tag{25}$$

Iz (25) je očitno, da je lega minimuma (v_1^*, v_2^*) odvisna samo od razlike v gravitacijskih faktorjih g_1, g_2, saj sta odvoda po v_1, v_2 funkcij $J(g_1, g_2, v_1, v_2)$ in $J(0, g_2 - g_1, v_1, v_2)$ enaka. Potem je lahko za $n = 2$ gravitacijska komponenta g_1 kar enaka ničli in Lagrangeovo funkcijo po skaliranju in normalizaciji zapišemo z novimi koeficienti $a', g_2^{'}, t_f^{'}$ kot (26):

It is clear from eq. (25), that the position of the minimum (v_1^*, v_2^*) depends only on the difference in the gravity factors g_1, g_2, because the derivatives of $J(g_1, g_2, v_1, v_2)$ and $J(0, g_2 - g_1, v_1, v_2)$ with respect to v_1, v_2 are equal. Then for $n = 2$ the gravitational component g_1 can be zeroed and the Lagrangian written after scaling and normalization with new coefficients $a', g_2^{'}, t_f^{'}$ as in eq. (26):

$$L = a' v_1 + (\frac{g_2^{'}}{v_2} + a' \upsilon_2) + \lambda(\frac{1}{v_1} + \frac{1}{v_2} - t_f^{'}) \tag{26}$$

Lagrangeov koeficient λ lahko izrazimo neposredno iz $\partial L / \partial v_1 = 0$ in $\partial L / \partial v_2 = 0$ kot funkcijo v_1 oz. v_2. Po izenačitvi obeh izrazov sledi (27):

The Lagrangian coefficient λ can be directly expressed from $\partial L / \partial v_1 = 0$ and $\partial L / \partial v_2 = 0$ as a function of v_1 and v_2 respectively. After setting both the expressions to be equal it follows that:

$$v_1 = \sqrt{v_2^2 - \frac{g_2^{'}}{a'}} \tag{27}$$

Po vstavitvi (27) v izraz $\partial L / \partial \lambda = 0$ sledi posredna enačba (28) za račun v_2.

If eq. (27) is inserted into the expression $\partial L / \partial \lambda = 0$ the implicit eq.(28) follows for v_2.

$$v_2^2 = (t_f^{'} v_2 - 1)^2 (v_2^2 - \frac{g_2^{'}}{a'}) \tag{28}$$

Enačba (28) je polinom četrte stopnje po v_2 in jo lahko rešimo numerično ali z radikali. Tehnološko primerna množica rešitev MPG za

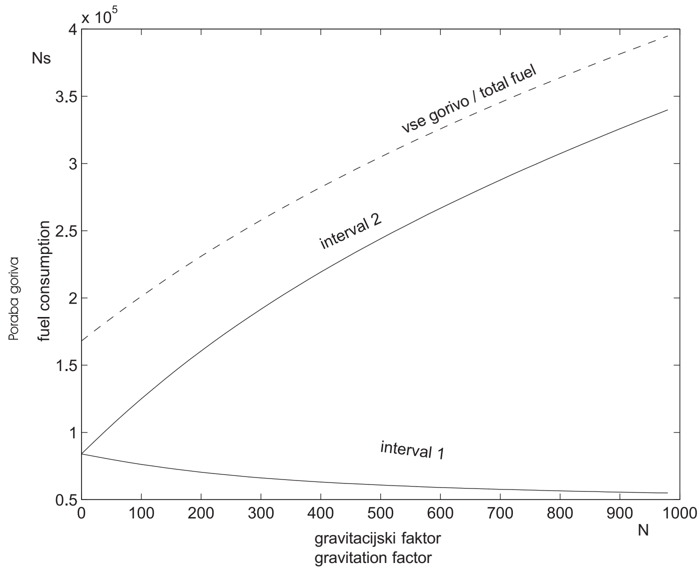
Eq. (28) above, is a fourth order polynomial of v_2 which can be solved numerically or with radicals. The technologically meaningful set of solutions of the MFP

Lagrangeov problem (26) je prikazana na sliki 3. Interpretacija izraza (27) je podana v lemi 5.

Lema 5:
Pri dinamičnem modelu M_v povečanje gravitacijskega faktorja v točki x_c v intervalu $[0, x_f]$ od vrednosti g_1 na $g_2 > g_1$ povzroči povečanje nespremenljive vlečne sile in povečanje hitrosti v intervalu $[x_c, x_f]$ in zmanjšanje nespremenljive vlečne sile in hitrosti v intervalu $[0, x_c]$, če naj bo količina porabljenega goriva najmanjša.

for the Lagrangian problem (26) is presented in fig . 3. The interpretation of eq. (27) is formulated in lemma 5.

Lemma 5:
For the dynamic model M_v the increase in gravity factor at point x_c in the interval $[0, x_f]$ from value g_1 to $g_2 > g_1$ induces an increase in constant driving force and velocity in the interval $[x_c, x_f]$ and induces a decrease in the constant driving force and velocity in the interval $[0, x_c]$, if the fuel consumption has to be held at the minimum.



Sl. 3. *Najmanjša poraba goriva v odvisnosti od gravitacijske komponente*
Fig. 3. *Minimum fuel consumption as function of gravitational component*

2 SKLEP

Opisani so nekateri algoritmi in strategije pri vleki s kriterijem najmanjše porabe goriva. Algoritmi so konstruirani na elementarnih dejstvih, tako da se je bilo mogoče izogniti Hamiltonovi teoriji optimalnega krmiljenja. Razviti so bili numerični algoritmi, ki temeljijo na računu najkrajših časov v grafih [3], na modificirani Newton - Raphson metodi in na kombinaciji Frechet-jevega odvoda z linearnim programiranjem. Od stohastičnih algoritmov je bila uporabljena enostavna varianta simuliranega ogrevanja.

Opis numeričnih algoritmov v članku ni vključen. Numerični rezultati potrjujejo kontrolne strategije kot so le te opisane v lemah 1 do 4. Niso pa bile uporabljene metode dinamičnega programiranja, kot v [4] in [5].

Kljub temu, da numerične tehnike niso posebej opisane, predstavljajo osnovni razlog za postavitev algoritmov optimalne porabe goriva. Tipično za njih je namreč krmiljenje tipa 'vklop - izklop', ki pri računski obdelavi lahko poslabša

2 CONCLUSION

The few traction algorithms and strategies are described with the minimum fuel consumption as the criterion function. Algorithms are derived from elementary ideas avoiding the Hamiltonian theory. Numerical algorithms were developped, based on the shortest time graph algorithm [3], the modified Newton-Raphson method and the combination of the Frechet derivative with the linear programming. From the stochastic algorithms the simplest variant of simulated annealing algorithm was used.

The numerical algorithms are not included into the paper. The numeric results confirm the control strategy as defined in lemmas 1 to 4. The dynamic programming algorithms which were used in [4], [5] are not applied .

Although the numerical procedures are not described in the paper, they present the actual reason why the optimal fuel strategy algorithms were evolved. Typically for them is even the 'on -off' control, which can be the cause of bad convergence in numeric algorithms. Then it was possible to check konvergenco postopka. Tako je bilo mogoče testirati trivialne rešitve MPG s splošnejšimi numeričnimi algoritmi, ki so potem uporabni za bolj posplošene modele vleke, npr. za model z upoštevanjem karakteristik električne vleke.

out the trivially solutions for MFP with more general numeric algorithms, which can be used for more adequate models of traction too, e.g. for the models where the characteristics of the electric traction are considered .

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