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ARS MATHEMATICA CONTEMPORANEA

Contents

Volume 19, Number 2, Fall/Winter 2020, Pages 173–374

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Distinguishing numbers of finite 4-valent vertex-transitive graphs*

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Abstract

The distinguishing number of a graph G is the smallest k such that G admits a k-colouring for which the only colour-preserving automorphism of G is the identity. We determine the distinguishing number of finite 4-valent vertex-transitive graphs. We show that, apart from one infinite family and finitely many examples, they all have distinguishing number 2.

Keywords: Vertex colouring, symmetry breaking in graph, distinguishing number, vertex-transitive graphs.

Math. Subj. Class. (2020): 05C15, 05E18

1 Introduction

All graphs in this paper will be finite. A *distinguishing colouring* of a graph is a colouring which is not preserved by any non-identity automorphism. The *distinguishing number* $D(G)$ of a graph G is the least number of colours needed for a distinguishing colouring of the vertices of G. These concepts were first introduced by Albertson and Collins [1] and have since received considerable attention.

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It is an easy observation that a graph has distinguishing number 1 if and only if its automorphism group is trivial. Hence, by [6] almost all graphs have distinguishing number 1. This obviously is not true for non-trivial vertex-transitive graphs which always have non-trivial automorphisms. However, it seems that the vast majority of vertex-transitive graphs still have the lowest possible distinguishing number, namely 2. Hence let us call a vertex-transitive graph *exceptional* if its distinguishing number is not equal to 2.

One of the most interesting results concerning distinguishing numbers of vertex-transitive graphs is that, apart from the complete and edgeless graphs, there are only finitely many exceptional vertex-primitive graphs [3, 14]. It is only natural to ask whether something similar holds for vertex-transitive graphs as well. As a first step, Hüning et al. recently determined the exceptional 3-valent vertex-transitive graphs and their distinguishing numbers.

Theorem 1.1 ([7, Corollary 2.2]). *The exceptional connected* 3*-valent vertex-transitive graphs are*

1. K_4 *and* $\mathrm{K}_{3,3}$*, with distinguishing number* 4*, and*
2. $\mathrm{Q}_3 \cong \mathrm{K}_4 \times \mathrm{K}_2$ *and the Petersen graph, with distinguishing number* 3*.*

This result shows that there are only finitely many connected 3-valent vertex-transitive exceptional graphs. This is not true for 4-valent graphs, as shown by the following family of graphs. For $n \geq 3$, the *wreath graph* W_n is the lexicographic product $\mathrm{C}_n[2\mathrm{K}_1]$ of a cycle of length n with an edgeless graph of order 2, see Figure 1.

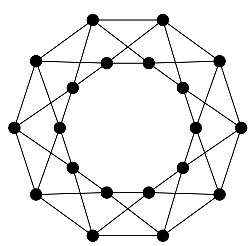

Figure 1: The wreath graph W_{10}.

It is easy to see that wreath graphs form an infinite family of connected exceptional 4-valent vertex-transitive graphs, thus providing a negative answer to [7, Question 2]. Our main result shows that this is the only such family, that is, apart from the wreath graphs, there are only finitely many connected exceptional 4-valent vertex-transitive graphs.

Theorem 1.2. *The exceptional connected* 4*-valent vertex-transitive graphs are*

1. K_5 *and* $\mathrm{K}_{4,4} \cong \mathrm{W}_4$*, with distinguishing number* 5*, and*
2. $\mathrm{K}_3 \,\square\, \mathrm{K}_3$*,* $\mathrm{K}_4 \,\square\, \mathrm{K}_2$*,* $\mathrm{K}_5 \times \mathrm{K}_2$ *and* W_n *for some* $n \geq 3$*,* $n \neq 4$*, with distinguishing number* 3*.*

In particular, there is no example with distinguishing number 4. This leads us to the following question.

Question 1.3. For $\Delta \geq 5$, is there a connected Δ-valent vertex-transitive graph G with $D(G) = \Delta$?

More generally, one could ask about "gaps" in the set of distinguishing numbers of connected Δ-valent vertex-transitive graphs, as a subset of $\{2, \ldots, \Delta + 1\}$.

Using lexicographic products, it is not hard to construct infinite families of connected exceptional vertex-transitive graphs with fixed valency.

Example 1.4. Let H_1 be a connected vertex-transitive graph of valency Δ_1 and let H_2 be a vertex-transitive graph of valency Δ_2 on n_2 vertices. Then the lexicographic product $H_1[H_2]$ is connected, has valency $\Delta_1 n_2 + \Delta_2$ and its distinguishing number is at least $D(H_2) + 1$. For an infinite family of examples that are not lexicographic products, note that, for every $n \geq 3$ and every $d \geq 2$, the graph $(\mathrm{C}_n[d^2\mathrm{K}_1]) \,\square\, \mathrm{K}_2$ has valency $2d^2 + 1$ and distinguishing number strictly greater than d.

We hence pose the following (informal) problem.

Problem 1.5. Is there a "natural small family" $\mathcal{F}$ of exceptional graphs such that, for every positive integer k, all but finitely many k-valent connected exceptional vertex-transitive graphs are contained in $\mathcal{F}$?

2 Definitions and auxiliary results

Throughout this paper, all graphs are assumed to be finite and simple. Graph theoretic notions that are not explicitly defined will be taken from [5].

An *automorphism* of a graph is an adjacency preserving permutation of its vertices. The group of all automorphisms of a graph G is denoted by $\operatorname{Aut} G$. We say that a graph is *vertex-transitive* if its automorphism group is transitive (that is, for every pair of vertices, there exists an automorphism mapping the first to the second).

An *arc* in a graph G is an ordered pair of adjacent vertices, or equivalently, a walk of length 1 in G. An *s-arc* is a non-backtracking walk of length s in G, i.e. a sequence of vertices $v_0, \ldots, v_s$ where v_i is adjacent to v_{i+1} for $0 \leq i \leq s-1$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. The automorphism group $\operatorname{Aut} G$ acts on the set of edges, arcs, and s-arcs of G in an obvious way. Call a graph *edge-transitive*, *arc-transitive*, or *s-arc-transitive*, if the action of $\operatorname{Aut} G$ on edges, arcs, or s-arcs is transitive, respectively. Analogously define *arc-regular* and *s-arc-regular*.

The *local group* at a vertex v is the permutation group induced by the stabiliser of v acting on its neighbourhood $N(v)$. Note that, for vertex-transitive graphs, this does not depend on the choice of v (up to permutation equivalence). We say that a graph is *locally* Γ, if the local group is isomorphic to Γ.

A graph G is called *k-connected* if it remains connected after removing any set of at most $k-1$ vertices and all incident edges, and *k-edge connected* if it remains connected after removing any set of k edges. The following result about the connectivity of vertex-transitive graphs is due to Watkins [16].

Lemma 2.1. *A vertex-transitive graph with valency r is at least $\frac{2r}{3}$-connected.*

If we impose additional properties on the set of vertices to be removed, then we can remove much larger sets without disconnecting the graph. The following lemma follows easily from results in [15].

Lemma 2.2. *If G is a k-valent vertex-transitive graph with $k \geq 4$ and girth $g \geq 5$, then there is a g-cycle C in G such that $G - C$ is 2-edge connected.*

Proof. By [15, Theorem 4.5], there is a g-cycle C such that the edges with one endpoint in C and the other endpoint in $H := G - C$ form a minimum (w.r.t. cardinality) cut separating two cycles in G. Assume that H was not 2-edge connected and let e be a cut-edge of H. Let A and B be the two components of $H - e$. By [15, Lemma 3.3], the minimum degree of H is 2, so A and B each contain at most one vertex of degree 1, and thus there are cycles in both components. Now either the cut separating $A \cup C$ from B, or the cut separating $B \cup C$ from A contains strictly fewer edges than the cut separating C from H, contradicting the minimality. □

We will also need the notion of *distinguishing index* $D'(G)$ of a graph G, which is the least number of colours needed for a distinguishing colouring of the edges of G. Here are a few results giving upper bounds on $D'(G)$. The first two are Theorems 2.8 and 3.2 in [11].

Theorem 2.3. *Let G be a connected graph that is neither a symmetric nor a bisymmetric tree. If the maximum degree $\Delta(G)$ of G is at least 3, then $D'(G) \leq \Delta(G) - 1$ unless G is K_4 or $\mathrm{K}_{3,3}$.*

Theorem 2.4. *If G is a graph of order at least 7 with a Hamiltonian path, then $D'(G) \leq 2$.*

Lemma 2.5. *If G is a connected graph on 5 or more vertices, then $\operatorname{Aut} \mathrm{L}(G)$ is permutationally equivalent to $\operatorname{Aut} G$ with its natural action on $\mathrm{E}(G)$. Furthermore, in this case $D'(G) \leq D(G)$, unless G is a tree.*

Proof. The first part is a variant of Whitney's theorem due to Jung [8], the second part follows by applying [10, Theorem 1.3] to a distinguishing colouring with $D(G)$ colours. □

In the remainder of this section, we discuss some known results on distinguishing numbers and determine the distinguishing numbers of several graphs that will occur in the proof of Theorem 1.2. The following lemma gives a general bound on distinguishing numbers and was independently proved in [4] and [9].

Lemma 2.6. *If G is a connected graph with maximum degree Δ, then $D(G) \leq \Delta + 1$, with equality if and only if G is either C_5, or K_n or $\mathrm{K}_{n,n}$ for some $n \geq 1$.*

For $n \geq 2$, we define a family of graphs $C_{n,\mathrm{K}_{3,3}}$ as follows. For $1 \leq i \leq n$, let H_i be disjoint copies of $\mathrm{K}_{3,3}$ with bipartition $\mathrm{V}(H_i) = X_i \cup Y_i$. Let $C_{n,\mathrm{K}_{3,3}}$ be the graph obtained from this collection by adding a matching between X_i and Y_{i+1} for $1 \leq i \leq n-1$, and between X_n and Y_1, see Figure 2.

Lemma 2.7. *The following graphs have distinguishing number at most 2:*

(1) The line graph of every non-exceptional 3-valent graph;

(2) The line graphs of the following graphs: the Petersen graph, Q_3, $\mathrm{K}_3 \,\square\, \mathrm{K}_3$, $\mathrm{K}_5 \times \mathrm{K}_2$, and W_n for every $n \geq 3$;

(3) The bipartite complement of the Heawood graph;

(4) The 4-dimensional hypercube Q_4;

(5) The $(4,6)$*-cage, and*

(6) The graph $C_{n,\mathrm{K}_{3,3}}$ *for* $n \geq 2$.

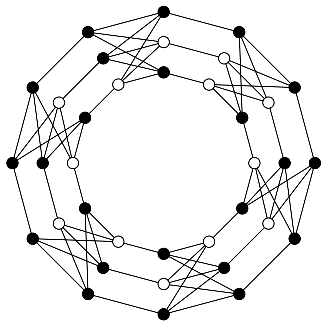

Figure 2: Distinguishing colouring of $C_{6,\mathrm{K}_{3,3}}$.

Proof. Lemma 2.5 immediately implies (1).

For (2), it suffices to observe that all the base graphs have at least 7 vertices and a Hamiltonian path, and then apply Theorem 2.4 and Lemma 2.5 .

For (3) note that the bipartite complement of the Heawood graph has the same automorphism group as the Heawood graph and thus also the same distinguishing number. By Theorem 1.1, this distinguishing number is 2.

(4) follows from [2], where distinguishing numbers of all hypercubes were determined.

For the proof of (5) first note that the $(4,6)$-cage is bipartite and any two vertices in each of its parts have exactly one neighbour in common. Let v be any vertex, let v_i for $1 \leq i \leq 4$ be the neighbours of v, and let v_{ij} $1 \leq i \leq 3$ be the neighbours of v_i.

Colour v white, for $1 \leq i \leq 4$ colour v_i black, and colour v_{ij} black if $i < j$ and white otherwise. Finally colour the common neighbours of v_{22} and v_{32}, and v_{22} and v_{33} black and all other vertices at distance 3 from v white, see Figure 3.

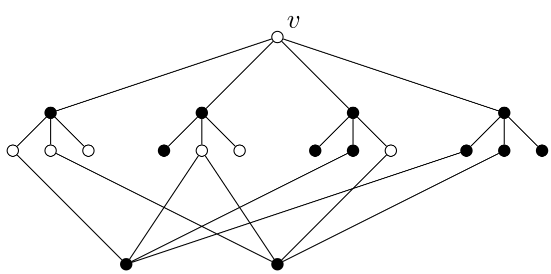

Figure 3: Colouring of the $(4,6)$-cage, all vertices at distance 3 from v not shown in the picture are coloured white.

Let γ be a colour preserving automorphism. Then γ must fix v, since it is the only white vertex with 4 black neighbours. Furthermore γ must fix all neighbours of v since they have a different number of black neighbours. It must also fix the two black vertices at distance

3 from v for the same reason. Now it is easy to see that γ has to fix all vertices at distance 2 from v and hence it is the identity.

For (6), consider the colouring shown in Figure 2. Note that the automorphism group has two orbits on edges: those that belong to a copy of $\mathrm{K}_{3,3}$, and those that don't, which we call matching edges. There is a unique matching edge both of whose endpoints are coloured white. Every colour preserving automorphism must fix this edge and the matching it is contained in. The colours on the remaining edges in this matching make sure that every colour preserving automorphism must fix this matching pointwise, and thus must fix every matching between two copies of $\mathrm{K}_{3,3}$ setwise. It is now easy to see that a colour preserving automorphism fixes all vertices of $C_{6,\mathrm{K}_{3,3}}$. Finally note that this colouring can be generalised to a colouring of $C_{n,\mathrm{K}_{3,3}}$ for any number $n \geq 2$. □

3 The proof of Theorem 1.2

In this section, we prove our main result. Determining the distinguishing numbers of the exceptional graphs is straightforward and will be left to the reader.

To show that the remaining graphs have distinguishing number 2, we distinguish cases according to the local group of $A := \operatorname{Aut} G$. Define the *type* of an edge uv as the size of the orbit of u under the action of the local group at v. By the orbit-stabiliser lemma, this is the index of A_{uv} in A_v. Since by vertex transitivity $|A_v| = |A_u|$, this also shows that the type is well-defined, i.e. it does not depend on the endpoint of the edge.

Note that since the orbits of the local group at v partition the neighbourhood of v the types of edges incident to v correspond to a partition of 4. Since G is vertex-transitive, this partition is the same for every vertex. Since the only partitions of 4 that do not contain a part of size 1 are $(2, 2)$ and (4), we split up the proof of Theorem 1.2 into the following three cases:

1. There are edges of type 1. This case is treated in Section 3.1.

2. All edges have type 2. This is treated in Section 3.2.

4. All edges have type 4, and hence G is arc-transitive. For this case, see Section 3.3.

3.1 Graphs with edges of type 1

Let $G_{t\geq 2}$ be the graph obtained from G by removing all edges of type 1. Note that the components of $G_{t\geq 2}$ form a system of imprimitivity for A. We will need the following results.

Lemma 3.1. *Assume that every vertex of G is incident to a unique type 1-edge, $G_{t\geq 2}$ is not connected, and any two components of $G_{t\geq 2}$ are connected by at most one type 1-edge. Then G has a distinguishing 2-colouring.*

Proof. Let k be the number of vertices in a component of $G_{t\geq 2}$. Consider the graph H obtained from G by contracting every component of $G_{t\geq 2}$ to a single vertex. By our assumptions, H is a k-regular graph and it follows from Lemma 2.6 that its distinguishing number is at most $k + 1$. Let c' be a distinguishing colouring of H with colours $\{0, 1, \dots, k\}$. We now colour G in the following way: in every component of $G_{t\geq 2}$, we colour as many vertices black as the colour of the corresponding vertex of H suggests.

Since c' is distinguishing, any automorphism which preserves the resulting colouring has to fix all components of $G_{t\geq 2}$ setwise. As every type 1 edge is uniquely identified by the components it connects, each type 1 edge and hence also every vertex must be fixed by every colour-preserving automorphism. □

Lemma 3.2. *Let G be a connected vertex-transitive graph. Assume that $G_{t\geq 2}$ is not connected, let H be a component of $G_{t\geq 2}$ and let $v \in H$. If H admits a 2-colouring c' such that the only automorphism of H fixing v and preserving c' is the identity, then G has a distinguishing 2-colouring.*

Proof. Denote the components of $G_{t\geq 2}$ by $H_1, \dots, H_R$. Note that each H_i is isomorphic to H. Let $v_1 \in H_1$. Note that the graph obtained from G by contracting the components $H_1, \dots, H_R$ is connected and vertex-transitive and thus at least 2-connected. Hence $G - H_1$ is connected, and thus $(G - H_1) + v_1$ is connected as well.

For $i \in \{2, \dots, R\}$, pick some shortest path from H_i to v_1 in $(G - H_1) + v_1$ and let v_i and e_i be the first vertex and edge of this path, respectively. Without loss of generality we may assume that the number of black vertices in c' is not exactly one—otherwise change the colour of v to obtain a colouring with this property. Let $\pi_i \colon H \to H_i$ be an isomorphism which maps v to v_i. Such an isomorphism exists because G (and thus also H) is vertex-transitive. Now define a colouring c of G by

$$c(x) = \begin{cases} \text{black} & \text{if } x = v_1, \\ \text{white} & \text{if } x \in H_1 - v_1, \\ c'(\pi_i^{-1}(x)) & \text{if } x \in H_i \text{ for } i \neq 1. \end{cases}$$

Let γ be an automorphism of G preserving c. We show that γ fixes every vertex and thus c is distinguishing.

First, note that γ must fix v_1, since v_1 is the only black vertex in H_1 which in turn is the only component with a unique black vertex.

Next we show that, for $i \neq 1$, every H_i must be fixed pointwise by γ. Assume not. Let H_i be a component such that the distance from H_i to v_1 is minimal, among the components that are not fixed pointwise. The endpoint u_i of e_i which does not lie in H_i is either v_1, or it lies in some component H_j which is closer to v_1. Hence u_i is fixed by γ. Since e_1 has type 1, γ must also fix v_i and thus induce an automorphism of H_i. By hypothesis, this induced automorphism is trivial and thus γ fixes H_i pointwise.

Finally, let $x \in H_1 - v_1$. Then x is incident to an edge of type 1 which connects H_1 to a different component H_i. Since the other endpoint of this edge is fixed by γ, the same must be true for x. □

Corollary 3.3. *Let G be a connected, vertex-transitive graph and let H be a component of $G_{t\geq 2}$. If H has a distinguishing 2-colouring, then so does G.*

Proof. If H is the only component of $G_{t\geq 2}$, then a distinguishing colouring of H is also distinguishing for G, otherwise apply Lemma 3.2. □

Theorem 3.4. *Let G be a connected 4-valent vertex-transitive graph containing edges of type 1. Then $D(G) = 2$, unless G is $\mathrm{K}_4 \,\square\, \mathrm{K}_2$.*

Proof. If all edges are of type 1, then $A_v = 1$ and thus colouring one vertex black and all other vertices white yields a distinguishing colouring.

Next assume that the local group has two orbits of size 1 and one orbit of size 2. In this case $G_{t\geq 2}$ is a union of cycles. If there is only one such cycle, then it must have length 6 or more, and hence G is 2-distinguishable by Corollary 3.3. If there is more than one, then the conditions of Lemma 3.2 are satisfied.

Finally consider the case where the local group has one orbit of size 1 and one orbit of size 3. All components of $G_{t\geq 2}$ are isomorphic to some 3-regular vertex-transitive graph G'. Also note that the induced action of A on G' is arc-transitive.

If G' has distinguishing number 2, then we can apply Corollary 3.3 to obtain a distinguishing 2-colouring of G. By Theorem 1.1, the only other possibility is that G' is isomorphic to one of $\mathrm{K}_4, \mathrm{K}_{3,3}, \mathrm{Q}_3$ or the Petersen graph.

If $G_{t\geq 2}$ is connected, then G is obtained from G' by adding edges of type 1. Since A is arc-transitive on G', no edge of type 1 can connect two neighbours (in G') of the same vertex. Otherwise any two neighbours of this vertex would have to be connected by an edge, contradicting the fact that each vertex of G is adjacent to only one edge of type 1. Hence an edge of type 1 can't connect vertices at distance at most 2 in G'. This rules out $\mathrm{K}_4, \mathrm{K}_{3,3}$ and the Petersen graph as possibilities for G', since they have diameter at most 2. The only way to add edges with respect to this constraint in the cube Q_3 yields $G = \mathrm{K}_{4,4}$ which does not contain edges of type 1.

Thus we can assume that $G_{t\geq 2}$ is not connected. Both the Petersen graph and Q_3 have colourings satisfying the condition of Lemma 3.2, see Figure 4. Hence if G' is one of them, then G has a distinguishing 2-colouring.

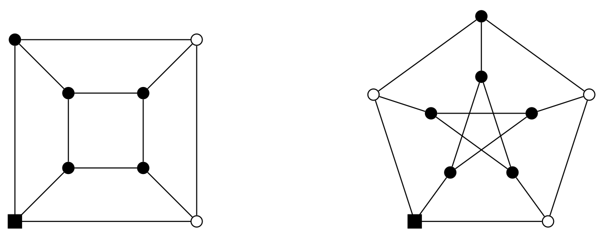

Figure 4: Colourings satisfying the condition of Lemma 3.2, v is the square vertex.

We may thus assume that G' is either K_4 or $\mathrm{K}_{3,3}$. By Lemma 3.1 we may assume that there is a pair of components of $G_{t\geq 2}$ connected by multiple type 1 edges. Since G is vertex-transitive and each vertex is incident to a unique edge of type 1, the number of type 1 edges between any pair of adjacent components of $G_{t\geq 2}$ is independent of the choice of the pair. Furthermore, recall that A acts arc-transitively on G'. Hence if two adjacent vertices in a component H are both adjacent to the same component H' (via type 1 edges), then all vertices of H are adjacent to H'. For $G' = \mathrm{K}_4$, this is the only possibility, and the resulting graph is $G = \mathrm{K}_4 \,\square\, \mathrm{K}_2$. For $G' = \mathrm{K}_{3,3}$, the above observation tells us that all vertices in the same bipartite class of a component send their type 1 edges to the same component, and hence $G = C_{n,\mathrm{K}_{3,3}}$ (see Figure 2) for some $n \geq 2$, which has distinguishing number 2. $\square$

3.2 Graphs with only edges of type 2

In this section, we assume that all edges of G are of type 2. This implies that A has two orbits on arcs and therefore at most two orbits on edges. We distinguish two subcases according to whether G is edge-transitive or not.

3.2.1 Edge-transitive case

Theorem 3.5. *Let G be a connected 4-valent graph that is vertex- and edge-transitive but not arc-transitive. Then $D(G) = 2$.*

Proof. In this case, A has two orbits on arcs and each arc is in a different orbit than its inverse arc. By removing one of the two orbits, G becomes an arc-transitive directed graph in which every vertex has in- and out-degree 2. There is some $s \geq 1$ such that A acts regularly on directed s-arcs (see for example [12, Lemma 5.4(v)]).

Let $P = (v_0, \ldots, v_s)$ be a directed s-arc in G. Suppose for a contradiction that there is an arc from v_s to v_0. Clearly, in this case $s \geq 2$, as G does not contain any 2-cycles. There is an automorphism fixing $(v_0, \ldots, v_{s-1})$ pointwise, but not fixing v_s. Therefore, the second out-neighbour $v'_s \neq v_s$ of v_{s-1} must also have v_0 as an out-neighbour. By directed 2-arc-transitivity we conclude that for any vertex v_i on P, the out-neighbours of v_i are exactly the in-neighbours of v_{i+2}, so the digraph is a directed wreath graph and G is arc-transitive, which gives the desired contradiction.

We may thus assume that there is no arc from v_s to v_0. Colour the vertices of P black and the remaining vertices white. Note that v_0 is the unique black vertex with no black in-neighbour. Hence v_0 and thus all of P must be fixed by any colour-preserving automorphism. By s-arc-regularity, this implies that the colouring is distinguishing and G has distinguishing number 2. □

3.2.2 Non-edge-transitive case

If G is not edge-transitive, then there must be 2 orbits on edges each of which forms a disjoint union of cycles. Denote the two subgraphs induced by the edge orbits by G_1 and G_2. By transitivity, all cycles in G_1 have the same length, the same is true for G_2.

We will inductively construct a distinguishing colouring from partial colourings of G. Let $\tilde{c}$ be a partial colouring of G with domain $\tilde{V} \subseteq \mathrm{V}(G)$, that is, $\tilde{c}$ is a function from $\tilde{V}$ to some set C of colours. An *extension* of $\tilde{c}$ is a colouring c of G such that c and $\tilde{c}$ coincide on $\tilde{V}$.

Lemma 3.6. *Let G be a connected 4-valent vertex-transitive but not edge-transitive graph and assume that all edges have type 2. Let G_1 and G_2 be the subgraphs induced by the two edge orbits. Let V' be a set of vertices of G and let C be a cycle in G_1 which is disjoint from V' and contains a neighbour v of some vertex in V'. Then there is a cycle D in G_1 which is disjoint from V' (possibly $D = C$) and a partial 2-colouring $\tilde{c}$ of G with domain $C \cup D$ such that*

- *C and D both contain either 1 or 2 black vertices, and*
- *if $\gamma \in \operatorname{Aut} G$ fixes V' pointwise and fixes any extension of $\tilde{c}$, then γ fixes $V' \cup C \cup D$ pointwise.*

Proof. Call a vertex u a twin of v if there is an automorphism in the stabiliser of V' that moves u to v. Note that v has at most one twin, since there is an edge in G_2 connecting v to some w in V', and w has only one other neighbour in G_2.

If v has no twin then every automorphism that fixes V' pointwise must fix v. Set $D = C$, colour v and one of its neighbours on C black and colour the remaining vertices of C white. Then every automorphism which fixes V' as well as an extension of this colouring must fix v and its black neighbour and thus also fixes C.

Next assume that v has a twin that lies on C. Again let $D = C$ and colour v and one of its neighbours in C black, but make sure that the black neighbour of v is not a twin of v. The same argument as above tells us that this colouring has the desired properties.

Finally assume that v has a twin u that lies outside of C. Let D be the cycle in G_1 containing u and observe that D is also disjoint from V'. Colour v and one of its neighbours in C black, colour one of the neighbours of u in D black, and colour the remaining vertices of $C \cup D$ white. Any automorphism that fixes V' as well as an extension of this colouring must fix u and v and their respective black neighbours, whence we have found the desired colouring. □

Theorem 3.7. *Let G be a connected 4-valent vertex-transitive but not edge-transitive graph and assume that all edges have type 2. Then $D(G) = 2$.*

Proof. Let G_1 and G_2 be the subgraphs induced by the two edge orbits respectively and without loss of generality assume that cycles in G_1 are at least as long as cycles in G_2.

If G_1 consists of a single cycle then this cycle must have length at least 6. Hence there is a distinguishing 2-colouring of G_1 which must also be distinguishing 2-colouring of G. Hence we may assume that G_1 consists of more than one cycle.

If cycles in G_1 have length at least 4, then let C_1 be a cycle in G_1 and let v_1 be a vertex on this cycle. Now inductively apply Lemma 3.6. For the first step, let $V' = \{v_1\}$. In each step, pick a cycle $C \neq C_1$ which contains a G_2-neighbour of V', colour it according to the lemma and add the vertices of $C \cup D$ to V'. The graph obtained from G by contracting every cycle in G_1 is connected and vertex-transitive. Hence, by Lemma 2.1 it is 2-connected and remains connected after removing C_1. In particular, the above colouring procedure assigns colours to all vertices except those in C_1. Finally colour v_1 and its neighbours on C_1 black, and colour the rest of C_1 white.

We claim that the resulting colouring is distinguishing. Clearly, every colour-preserving automorphism must fix v_1 since it is the only black vertex both of whose neighbours in G_1 are black (recall that C_1 is the only cycle in G_1 containing 3 black vertices). Using Lemma 3.6 inductively, we see that every colour-preserving automorphism must fix every cycle pointwise, except possibly C_1. Hence the colouring is distinguishing unless the two neighbours of v_1 in G_1 have the same G_2-neighbourhood. In this case, by vertex-transitivity any two vertices at distance 2 in G_1 have the same G_2-neighbourhood. If cycles in G_1 have length 5 or more, this implies that vertices have degree at least 3 in G_2 which is a contradiction. If cycles in G_1 have length 4, then so do cycles in G_2 and G is a graph obtained by identifying antipodal points of 4-cycles, i.e., a wreath graph, which contradicts the assumption that G is not edge transitive.

It remains to deal with the case when both G_1 and G_2 are disjoint unions of 3-cycles. Let H be the graph with vertices these 3-cycles, with two such 3-cycles being adjacent in H if they share a vertex in G. It is easy to see that H is regular of valency 3 and $G = \mathrm{L}(H)$.

By Theorem 2.3, we have $D(G) = D'(H) \leq 2$, unless H is K_4 or $\mathrm{K}_{3,3}$. Finally, note that $\mathrm{L}(\mathrm{K}_4) \cong \mathrm{W}_3$ while $\mathrm{L}(\mathrm{K}_{3,3}) \cong \mathrm{K}_3 \,\square\, \mathrm{K}_3$. □

3.3 Arc-transitive graphs

We first prove a few lemmas to show that we can restrict ourselves to graphs with girth 4.

Lemma 3.8. *Let G be a connected* 4*-valent arc-transitive graph. If G has girth* 3*, then G is either* K_5 *or* W_3*, or the line graph of a* 3*-valent arc-transitive graph.*

Proof. Follows from [13, Theorem 5.1(1)]). □

Lemma 3.9. *Let G be a connected graph of minimal valency at least* 3 *and girth $g \geq 5$. If G is s-arc-transitive, then $s \leq g-3$, unless G is a Moore graph of girth* 5*, or the incidence graph of a projective plane.*

Proof. Assume for a contradiction that G is $(g-2)$-arc-transitive. Let $C = (v_0, \dots, v_{g-1})$ be a cycle of length g. Note that $(v_0, \dots, v_{g-2})$ is a $(g-2)$-arc and that its endpoints have a common neighbour. By $(g-2)$-arc-transitivity, every $(g-2)$-arc has this property. Let v'_{g-2} be a neighbour of v_{g-3} outside of C. Then $(v_0, \dots, v_{g-3}, v'_{g-2})$ is a $(g-2)$-arc, whence v'_{g-2} and v_0 have a common neighbour v'_{g-1}. Now the closed walk $(v_0, v_{g-1}, v_{g-2}, v_{g-3}, v'_{g-2}, v'_{g-1}, v_0)$ shows that $g \leq 6$.

If $g = 5$, then the fact that the endpoints of every 3-arc have a common neighbour implies that G has diameter 2 and is thus a Moore graph.

If $g = 6$, then an analogous argument as above yields that G has diameter 3. If G was not bipartite, then for $v \in \mathrm{V}(G)$ there would be an edge connecting two vertices x and y at the same distance from v, and since $g = 6$ we have $d(x, v) = d(y, v) = 3$. But then there is a 4-arc from v to x whence by the above argument v and x have a common neighbour, contradicting $d(x, v) = 3$.

Hence G is bipartite and every vertex at distance 2 from a given vertex v has a unique common neighbour with v. It follows that G is the incidence graph of a projective plane. □

Lemma 3.10. *Let G be a connected* 4*-valent arc-transitive graph of girth at least* 5*, then $D(G) = 2$.*

Proof. Let g be the girth of G and let s be such that G is s-arc-transitive but not $(s+1)$-arc-transitive. Note that there is no 4-valent Moore graph, and that there is a unique 4-valent graph that is the incidence graph of a projective plane, namely the $(4, 6)$-cage. By Lemmas 2.7 and 3.9 we may thus assume that $s \leq g - 3$.

By Lemma 2.2, there is a cycle $C = (v_0, \dots, v_{g-1})$ such that $G - C$ is 2-edge connected. Let $P = (v_{s+1}, v_s, \dots, v_1)$ and let X be its pointwise stabiliser. Note that P is an s-arc and thus X is not transitive on $N(v_1) \setminus \{v_2\}$ (otherwise G would be $(s+1)$-arc-transitive). Let v'_0 be a neighbour of v_1 that is in a different orbit than v_0 under X.

Note that the subgraph induced by the vertices $\{v'_0, v_0, v_1, \dots, v_{g-2}\}$ is a tree since any additional edge between these vertices would give a cycle of length less than g. Denote this tree by T and let H be the subgraph obtained from G by removing all vertices of T. Observe that v'_0 has degree at most 3 in $G - C$. If H is not connected, then there is one component of H that is connected to v'_0 by a unique edge. Removing that edge from $G - C$

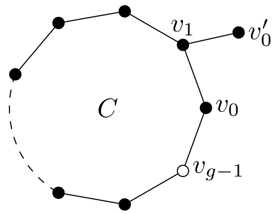

Figure 5: The tree T in the proof of Lemma 3.10.

would disconnect it, contradicting the fact that $G - C$ is 2-edge connected. It follows that H is connected.

Colour all vertices of T black and colour v_{g-1} white. Inductively colour the vertices of G as follows: Let x be a vertex at minimal distance to v_{g-1} in H that has not been coloured yet. If x is fixed by the pointwise stabiliser in A of all previously coloured points, then colour it white. Otherwise colour it black.

We claim that this colouring is distinguishing. First note that if an automorphism fixes two neighbours u and w of a vertex v, then it must also fix v, since otherwise the image of v would also be a common neighbour of u and w contradicting $g \geq 5$. Note that this implies that all vertices in H with a neighbour outside of H are coloured white. Indeed, at the time such a vertex x is considered for colouring, two of its neighbours are already coloured: its predecessor on a shortest v_{g-1}-x-path in H and its neighbour outside of H. Hence by the previous observation, x is coloured white.

Next we show that v_1 is the only black vertex with three black neighbours. By the above observations it is the only such vertex in T. Now let x be a black vertex in H. Then at most one neighbour of x was coloured before x (otherwise we would have coloured x white). Furthermore, if P is a shortest v_{g-1}-x-path in H, then $P \cup C$ contains an s-arc ending in x. Hence the pointwise stabiliser of x and all vertices coloured before x does not act transitively on the remaining neighbours of x, whence at most one of them will be coloured black.

Let γ be a colour preserving automorphism. The above discussion shows that γ must fix v_1. Furthermore all neighbours of T are white, so γ must preserve T setwise. Since there is no automorphism of G that fixes $(v_1, \dots, v_{g-2})$ and moves v_0 to v_0', γ must fix T pointwise. Finally assume that there is a vertex in H that is not fixed by γ and let x be the first such vertex that was coloured in the inductive procedure. Clearly, x is coloured black. Let y be the neighbour of x on a shortest v_{g-1}-x-path P, and let S be an s-arc contained in $C \cup P$. Then S is pointwise stabilised by γ, and since the orbit of x under the pointwise stabiliser of S is not a singleton, it contains exactly one other element x'. Every automorphism that fixes x and S also fixes x' and vice versa. Hence at most one of x and x' can be coloured black and thus neither of them can be moved by γ. $\square$

Next we give some results for the case when G has girth exactly 4. Note that in this case, there must be vertices at distance 2 from each other with 2 or more common neighbours. The following two lemmas follow from results in [13].

Lemma 3.11. *Let G be a connected 4-valent arc-transitive graph. If there are two vertices at distance 2 with 3 or more common neighbours, then G is isomorphic to either $\mathrm{K}_5 \times \mathrm{K}_2$ or W_n for some $n \geq 4$.*

Proof. If there are vertices with 4 common neighbours, then by [13, Lemma 4.3], G is a wreath graph. Otherwise, Subcase II.A of the proof [13, Theorem 3.3] implies that $G \cong \mathrm{K}_5 \times \mathrm{K}_2$. □

Lemma 3.12. *Let G be a connected 4-valent 2-arc-transitive graph. If G has girth 4 but no two vertices at distance 2 have more than 2 common neighbours, then G is isomorphic to either Q_4, or the bipartite complement of the Heawood graph.*

Proof. By 2-arc-transitivity, every edge is contained in at least three 4-cycles. Subcase II.B of the proof of [13, Theorem 3.3] then implies that G is isomorphic to one of the two graphs as claimed. □

The hardest case to deal with is when the graph is locally D_4. In this case, we take advantage of the following structural property. Note that D_4 in its natural action on 4 points admits a unique system of imprimitivity with 2 blocks of size 2. We say that a 2-arc (v_0, v_1, v_2) is *straight*, if $\{v_0, v_2\}$ is a block with respect to the local group at v_1, and *crooked* otherwise. Note that, of the three 2-arcs starting with a given arc, one is straight and two are crooked. Further note that fixing a crooked 2-arc fixes all neighbours of its midpoint. Finally, note that A acts transitively on crooked 2-arcs of G. Call a cycle in G *straight*, if all sub-arcs of length 2 are straight.

Theorem 3.13. *Let G be a connected 4-valent arc-transitive graph, then $D(G) = 2$ unless G is K_5, $\mathrm{K}_3 \,\Box\, \mathrm{K}_3$, $\mathrm{K}_5 \times \mathrm{K}_2$, or W_n for some $n \geq 3$.*

Proof. By Lemmas 3.8, 3.10, as well as Lemma 2.7, we can assume that G has girth 4. By Lemma 3.11, we can assume that no two vertices have more than two common neighbours.

Since G is arc-transitive, the local group must be a transitive subgroup of S_4. If the local group is 2-transitive, then G is 2-arc-transitive and this case is handled with Lemmas 3.12 and 2.7.

If the local group is C_4 or V_4, then G is arc-regular. One can then colour one vertex v and three of its neighbours black, and colour the remaining vertices white. Any colour preserving automorphism must fix the arc from v to its unique white neighbour, thus the colouring is distinguishing.

The last remaining case is that G is locally D_4. Suppose first that G contains a 4-cycle that is not straight. Let (u, v, w, x) be a 4-cycle of G such that (u, v, w) is a crooked 2-arc.

We claim that any automorphism fixing u and all of its neighbours must be the identity. By arc-transitivity and connectedness it is enough to show that such an automorphism must fix all neighbours of v. Since no pair of vertices has more than two common neighbours, u and w are the only two common neighbours of v and x. In particular, if an automorphism fixes w and all its neighbours, then it must also fix u. Hence it fixes a crooked 2-arc with midpoint v, and thus it fixes v and all of its neighbours, thus proving our claim.

Let y be the unique vertex such that (v, w, y) is a straight 2-arc, and let $P = (u, v, w, y)$. Suppose that y is adjacent to u. Let u' be the unique vertex other than u such that (u', v, w) is crooked. Note that there is an automorphism fixing v and w (and thus y) and mapping u to u', and thus y is adjacent to u', and v and y have at least 3 common neighbours (u, u', and w), contradicting an earlier hypothesis. We conclude that y is not adjacent to u and thus the induced subgraph on P is a path of length 3. Colour P black and colour the remaining vertices white. Since (u, v, w) is crooked, but (v, w, y) is straight, every colour preserving automorphism fixes P pointwise, and thus it fixes v and all its neighbours. Hence, by the above claim, this colouring is distinguishing.

From now on, we can assume that all 4-cycles of G are straight. Let $\mathcal{C}$ be the set of all 4-cycles. Note that every edge is contained in a unique straight 4-cycle, whence $\mathcal{C}$ forms a partition of $\mathrm{E}(G)$. Furthermore, any two elements of $\mathcal{C}$ intersect in at most one vertex, since otherwise there would be vertices with 3 or more common neighbours.

Now consider the auxiliary graph G' with vertex set $\mathcal{C}$ and an edge between two vertices if the 4-cycles have a vertex in common. Note that G' is a 4-valent graph on $|\mathcal{C}| = \frac{|\mathrm{E}(G)|}{4} = \frac{|\mathrm{V}(G)|}{2}$ vertices.

Note that A has a natural induced action on G', and this is easily seen to be locally D_4. Furthermore any distinguishing colouring of $\mathrm{L}(G')$ corresponds to a distinguishing colouring of G. By Lemma 2.5 and the above observations $D(G') \geq D(\mathrm{L}(G')) \geq D(G)$. Hence if $D(G') = 2$, then $D(G) = 2$ and we are done. By induction, we may thus assume that G' is one of K_5, $\mathrm{K}_3 \,\square\, \mathrm{K}_3$, $\mathrm{K}_5 \times \mathrm{K}_2$, or W_n for some $n \geq 3$. If $G' \neq \mathrm{K}_5$, then by Lemma 2.7(2), we have $D(\mathrm{L}(G')) = 2$ and we are done. Finally note that $G' = \mathrm{K}_5$ is not possible, since A induces a transitive, locally D_4 action on G', but K_5 admits no such action. $\square$

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Divergence zero quaternionic vector fields and Hamming graphs

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Dedicated to the memory of Marjan Jerman.

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Abstract

We give a possible extension of the definition of quaternionic power series, partial derivatives and vector fields in the case of two (and then several) non commutative (quaternionic) variables. In this setting we also investigate the problem of describing zero functions which are not null functions in the formal sense. A connection between an analytic condition and a graph theoretic property of a subgraph of a Hamming graph is shown, namely the condition that polynomial vector field has formal divergence zero is equivalent to connectedness of subgraphs of Hamming graphs $H(d, 2)$. We prove that monomials in variables z and w are always linearly independent as functions only in bidegrees $(p, 0), (p, 1), (0, q), (1, q)$ and $(2, 2)$.

Keywords: Quaternionic power series, bidegree full functions, Hamming graph, linearly independent quaternionic monomials.

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1 Introduction

Complex holomorphic vector fields with divergence zero represent an important tool for the description of the groups of volume preserving automorphisms of $\mathbb{C}^n$ with $n > 1$ (we refer the reader to [1] and [2] for a thorough description of this topic). In this paper we investigate generalizations of complex holomorphic vector fields in the quaternionic setting, and for this purpose we restrict our research to mappings represented by convergent quaternionic power series.

We introduce an alternative definition of partial derivative, namely as a first order approximation (which is not linear) and using this new notion of partial derivatives we define the corresponding divergence in the quaternionic setting. We show that quaternionic vector fields with divergence zero are bidegree full (see Section 2.2 for definition) and that the divergence zero condition on quaternionic vector fields is equivalent to finding connected subgraphs of Hamming graphs. The paper is structured as follows: Section 2 contains the description of our setting with basic definitions and notions, such as partial derivatives and divergence. Moreover, bidegree full functions are introduced together with some basic facts about Hamming graphs. Section 3 is devoted to vector fields and their properties, in particular it contains the main result, Theorem 3.4, on quaternionic vector fields with divergence zero and explains the connection between divergence zero vector fields and Hamming graphs. In Section 4 we prove the theorem on linear independence of monomials.

2 Preliminaries

2.1 Convergent quaternionic power series

In this section we introduce the basic concepts and notions to deal with generalizations of complex holomorphic power series in the quaternionic setting.

We denote by $\mathbb{H}$ the algebra of quaternions, $\mathbb{H} = \{z = x_0 + x_1 i + x_2 j + x_3 k,\ x_0, \dots, x_3 \in \mathbb{R}\}$, where i, j, k are imaginary units satisfying $i^2 = j^2 = k^2 = -1$, $ij = k, jk = i, ki = j$. Denote by $\mathbb{S}$ the sphere of imaginary unit quaternions, i.e. the set of quaternions I such that $I^2 = -1$; notice that for a quaternion z we have $z^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 + 2x_0(x_1 i + x_2 j + x_3 k)$, therefore the condition $z^2 = -1$ implies $z = x_1 i + x_2 j + x_3 k$ and $-x_1^2 - x_2^2 - x_3^2 = -1$. Given any nonreal quaternion z, there exist (and are uniquely determined) an imaginary unit I, and two real numbers x, y (with $y > 0$) such that $z = x + Iy$. With this notation, the conjugate of z will be $\bar{z} := x - Iy$. Each imaginary unit I generates (as a real algebra) a copy of a complex plane denoted by $\mathbb{C}_I$. We call such a complex plane a *slice*.

A product of nonzero quaternionic coefficients and the variables z, w of degree d is called *a generalized quaternionic monomial of degree* d. Let $\mathbb{H}_d[z, w]$ denote the set of all finite sums of generalized quaternionic monomials of degree d, which we call *generalized quaternionic homogenous polynomials of degree* d. For example, the generalized quaternionic polynomial $a_0 z a_1 w a_2 w a_3 + b_0 z^2 b_1 w b_3 + c_0 w c_1 z c_2 w c_3$ belongs to $\mathbb{H}_3[z, w]$. Let

$$\mathbb{H}[z, w] := \bigoplus_{d \geq 0} \mathbb{H}_d[z, w]$$

be the ring of generalized quaternionic polynomials in the variables z, w over the quaternions. We consider polynomials $P \in \mathbb{H}[z, w]$ as formal (left and right) linear combinations.

It turns out (see Section 4) that there are several polynomials defining the same polynomial function. We therefore identify a given polynomial function P with the equivalence class $[P]$ of all polynomials defining the same function. The set of all polynomial functions coincides with real polynomials in 8 variables with quaternionic coefficients (see [3]).

We consider the right–submodule $\mathbb{H}_{\mathrm{rhs}}[z,w]$ of $\mathbb{H}[z,w]$ which consists of all generalized quaternionic polynomials whose generalized monomials have coefficients on the *right-hand side*. To be precise, given the multiindex $\alpha=(\alpha_1,\ldots,\alpha_d)\in\{0,1\}^d$, called a *word* on *letters* $0,1$, we define the *length* of α to be $|\alpha|:=\sum_{l=1}^d \alpha_i$. Then we put

$$(z,w)^\alpha := (z^{\alpha_1}w^{1-\alpha_1})\cdots(z^{\alpha_d}w^{1-\alpha_d}).$$

For integers $p,q\geq 0, p+q=d$, denote by $\alpha^{p,q}$ a multiindex with $|\alpha^{p,q}|=p$. There are $\binom{d}{p}$ such multiindices. We call the pair (p,q) a *bidegree*. The (pure) monomials of degree d can be written in the form

$$(z,w)^{\alpha^{p,q}}$$

and hence define

$$\mathbb{H}_{\mathrm{rhs},(p,q)}[z,w] := \{P_{p,q}(z,w) = \sum_{\substack{\alpha^{p,q},\\ |\alpha^{p,q}|=p}} (z,w)^{\alpha^{p,q}} c_{\alpha^{p,q}}; c_{\alpha^{p,q}}\in\mathbb{H}\},$$

$$\mathbb{H}_{\mathrm{rhs},d}[z,w] := \{P_{p,q}(z,w) = \sum_{\substack{\alpha^{p,q},\\ |\alpha^{p,q}|=p}} (z,w)^{\alpha^{p,q}} c_{\alpha^{p,q}}; c_{\alpha^{p,q}}\in\mathbb{H}, p+q=d\}$$

so that $\mathbb{H}_{\mathrm{rhs}}[z,w]=\oplus_{d\geq 0}\mathbb{H}_{\mathrm{rhs},d}[z,w]$.

Our basic assumption on regularity, for the definition of the class of quaternionic series we are interested in, is that any such a series f

$$f(z,w) = \sum_{p,q\geq 0}\sum_{\lambda\in\Lambda_{p,q}} f_{p,q,\lambda}(z,w) \tag{2.1}$$

converges absolutely on $\mathbb{H}^2$. Notice that absolute convergence implies uniform convergence on compact sets of $\mathbb{H}^2$. The notation $f_{p,q,\lambda}(z,w)\in\mathbb{H}_d[z,w]$ stands for generalized monomials containing p copies of z and q copies of w with $p+q=d$ and the sets $\Lambda_{p,q}$ are supposed finite. The set of all such series f will be denoted by $\mathcal{H}[z,w]$. Putting

$$f_d(z,w) := \sum_{p=0}^{d}\sum_{\lambda\in\Lambda_{p,d-p}} f_{p,d-p,\lambda}(z,w),$$

any $f\in\mathcal{H}[z,w]$ also has a homogenous expansion $f(z,w)=\sum_{d\geq 0} f_d(z,w)$. Uniform convergence on compact sets of $\mathbb{H}^2$ means that given any $\varepsilon>0$ and a compact set $K\subset\mathbb{H}^2$, there exists a natural number $d_{\varepsilon,K}$ such that for any generalized polynomial of the form

$$P(z,w) = \sum_{d=0}^{d_{\varepsilon,K}} f_d(z,w) + \sum_{d>d_{\varepsilon,K}}\sum_{p=0}^{d}\sum_{\lambda\in\Lambda'_{p,d-p}\subset\Lambda_{p,d-p}} f_{p,d-p,\lambda}(z,w),$$

the uniform estimate $|f(z,w)-P(z,w)|_K<\varepsilon$ holds. Let the norm of the term $f_{p,q,\lambda}$ be $|f_{p,q,\lambda}(z,w)| = |z|^p|w|^q c_{p,q,\lambda}$ (with $c_{p,q,\lambda}\geq 0$) and define $c_{p,q}:=\sum_{\lambda\in\Lambda_{p,q}} c_{p,q,\lambda}$. The absolute convergence at the point (z_0, w_0) in the domain of definition of f means that

$$\sum_{p,q\geq 0}\sum_{\lambda\in\Lambda_{p,q}} |f_{p,q,\lambda}(z_0,w_0)| = \sum_{p,q\geq 0} |z_0|^p|w_0|^q c_{p,q} < \infty$$

and implies uniform convergence on compact sets of $B(0,|z_0|)\times B(0,|w_0|)$.

Any series $f\in\mathcal{H}[z,w]$ uniquely defines a function of two quaternionic variables, but as in the case of polynomials, there are many series defining the same function. We say that two quaternionic series are equivalent if each of them defines the same quaternionic function. This is an equivalence relation, and so we identify the function f with the corresponding equivalence class $[f]$ of all series in $\mathcal{H}[z,w]$ defining the same function. To avoid too many notations, we will say that a given function belongs to $\mathcal{H}[z,w]$ if it has a series representative in $\mathcal{H}[z,w]$. By abuse of notation, if $f\in\mathcal{H}[z,w]$, we also denote by $[f]$ the set of all series which determine the same function. Since uniqueness of the power series for a function f is not granted (see next paragraphs and (2.2)), the absolute convergence of a chosen power series for a given function $f\in\mathcal{H}[z,w]$ is not a consequence of uniform convergence on compact sets, as in the complex or real case, and has to be additionally required. In the sequel we focus our attention on the right $\mathbb{H}$-module $\mathcal{H}_{\mathrm{rhs}}[z,w]$ in $\mathcal{H}[z,w]$ of (absolutely convergent) power series with coefficients on the right. We extend all the above definitions also to series of three or more variables. Notice that in $\mathbb{H}_{\mathrm{rhs},d}[z_1,z_2,\dots,z_n]$ there are n^d different (pure) monomials. If we assume only uniform convergence of series in $\mathcal{H}_{\mathrm{rhs}}[z_1,z_2,\dots,z_n]$, given an uniformly convergent series $f(z_1,z_2,\dots,z_n)=\sum_{d\geq 0} f_d(z_1,z_2,\dots,z_n)$, for $R>0,\varepsilon>0$ there is a $d_0\in\mathbb{N}$ such that for each $d\geq d_0$ and $p=(p_1,\dots,p_n)\in\mathbb{N}_0^d$ with $|p|=\sum_1^n p_1=d$, where p denotes the multiindex, whose jth element p_j is the total degree of z_j in the corresponding monomial, the estimate $|f_{p,\lambda}(z_1,z_2,\dots,z_n)|<\varepsilon$ holds on the ball $B^n(0,R)\subset\mathbb{H}^n$. As a consequence, on the ball $B^n(0,R/(n+1))$, we have the estimate $|f_{p,\lambda}(z_1,z_2,\dots,z_n)|<\varepsilon/(n+1)^{|p|}$ (with $|p|=\sum_1^n p_1=d$), so that for $(z_1,z_2,\dots,z_n)\in B^n(0,R/(n+1))$ we have

$$\sum_{|p|=d} |f_{p,\lambda}(z_1,z_2,\dots,z_n)| < \varepsilon\left(\frac{n}{n+1}\right)^d,$$

which implies that the series $f(z_1,z_2,\dots,z_n)=\sum_{d\geq 0} f_d(z_1,z_2,\dots,z_n)$ is not just uniformly but also absolutely convergent. Once more, we observe that, in general, in $\mathcal{H}[z_1,z_2,\dots,z_n]$ one has to assume absolute convergence for a proper definition of series, since the number of different generalized monomials can grow faster than exponentially; for example, if the polynomial P is as in (2.2), then the sums $\sum_{k=0}^m P(z,a_k)$, with $a_k\in\mathbb{H}$, are identically 0 for any $m\in\mathbb{N}$ and they contain $6m$ different generalized monomials. Let us mention another right-submodule of $\mathcal{H}_{\mathrm{rhs}}[z,w]$, namely, the submodule of slice-regular functions in the sense of Ghiloni-Perotti (see [5]), denoted by $\mathcal{H}_{\mathrm{GP}}[z,w]$. It is generated by (pure) monomials of the form $z^k w^l$, $k,l\in\mathbb{N}_0$, with this precise order, so any element of $\mathcal{H}_{\mathrm{GP}}[z,w]$ has a unique power series expansion and uniform convergence on compact sets in $\mathbb{H}$ implies absolute convergence. Slice-regular functions in the sense of Ghiloni-Perotti can be also seen as the kernel of a suitable partial differential operator. Notice that $\mathcal{H}_{\mathrm{GP}}[z,w]\subset\mathcal{H}_{\mathrm{rhs}}[z,w]\subset\mathcal{H}[z,w]$.

Unfortunately, also in $\mathcal{H}_{\mathrm{rhs}}[z,w]$ there are several power series which define the same function. In general the monomials of a given bidegree are not (right) linearly independent as functions. As far as we know, very little is known about this question except for linear independence of monomials of bidegrees $(p, 1)$ and $(1, q)$ as proved in [6, Proposition 2.4]. In Section 4 we prove that monomials of bidegrees $(p, 0), (p, 1), (1, q), (0, q)$ and $(2, 2)$ are linearly independent but monomials of bidegree $(3, 2)$ (and all other bidegrees) are not necessarily: since the square of the commutator of z and w is real, i.e. $[z, w]^2 \in \mathbb{R}$, the polynomial of bidegree $(3, 2)$,

$$P(z, w) = -z^2wzw + z^2w^2z + zwz^2w - zw^2z^2 - wz^2wz + wzwz^2 = [[z, w]^2, z] \quad (2.2)$$

is identically zero as a function but it is not (formally) equal to the null polynomial. Therefore, even here there is no one-to-one correspondence between power series and functions. However, as we will see, this fact does not affect the generality of the problem we are interested in (see also Remark 3.8 and Example 4.4). We realized that there exists a submodule $\mathcal{H}^{\mathrm{BF}}[z, w]$ in $\mathcal{H}_{\mathrm{rhs}}[z, w]$ which gives rise to vector fields with nice analytic properties, but these vector fields could not in general be detected using just analytic tools, due to the fact that we are not able to describe formal properties of the series defining the zero class $[0]$. Nevertheless, it turns out that these vector fields have representatives in their corresponding classes of power series with specific symmetry properties and for them all the results stated are valid within a given bidegree up to adding a polynomial which defines the identically-zero function. Example 4.4 is a special case where analytic conditions imply the existence of this special type of representatives in the classes of power series and these representatives are unique.

We remark that $\mathcal{H}_{\mathrm{GP}}[z, w]$ contains, as a particular case, the right submodule of slice-regular functions in one variable denoted by $\mathcal{SR}$ as introduced in [4] (see also the monograph [3]): it is the class $\mathcal{H}_{\mathrm{rhs}}[z] := \mathcal{H}_{\mathrm{rhs}}[z, 1]$. Vaguely speaking it is defined to be the class of functions $f\colon \mathbb{H} \to \mathbb{H}$ such that the limit

$$\lim_{h\to 0} h^{-1}(f(z + h) - f(z))$$

exists if h and z belong to the same slice. These functions turn out to be quaternionic analytic and their power expansions are unique.

In general, there is no standard way of introducing a notion of (partial) derivative for quaternionic functions (see for instance [4, 5]). For example, for the slice-regular function $f(z) = z^2a$ the limit of the differential quotient

$$\lim_{h\to 0} h^{-1}(f(z + h) - f(z)) = \lim_{h\to 0}(h^{-1}zh + z + h)a$$

does not exist unless h and z belong to the same slice.

We introduce new differential operators $\widehat{\partial}_z, \widehat{\partial}_w\colon \mathcal{H}[z, w] \to \mathcal{H}[z, w, h]$, which can be interpreted as partial derivatives for a convergent power series as in (2.1) with respect to each of the variables z, w in a given direction h.

Definition 2.1. For a function $f \in \mathcal{H}[z, w]$ and $z_0, w_0, h_0 \in \mathbb{H}$ we define the quaternion $\widehat{\partial}_z f(z_0, w_0)[h_0]$ to be the limit

$$\widehat{\partial}_z f(z_0, w_0)[h_0] := \lim_{t\to 0} \frac{1}{t}(f(z_0 + th_0, w_0) - f(z_0, w_0)), \quad t \in \mathbb{R},$$

or equivalently

$$f(z_0 + th_0, w_0) - f(z_0, w_0) = t\widehat{\partial}_z f(z_0, w_0)[h_0] + o(|t|);$$

similarly

$$\widehat{\partial}_w f(z_0, w_0)[h_0] := \lim_{t\to 0} \frac{1}{t}(f(z_0, w_0 + th_0) - f(z_0, w_0)), \quad t \in \mathbb{R},$$

defines $\widehat{\partial}_w f(z_0, w_0)[h_0]$. The function $\widehat{\partial}_z f$ in three variables (z, w, h) is then defined to be

$$(\widehat{\partial}_z f)(z, w, h) := \widehat{\partial}_z f(z, w)[h],$$

and similarly

$$(\widehat{\partial}_w f)(z, w, h) := \widehat{\partial}_w f(z, w)[h].$$

We use the notation $\widehat{\partial}_z f(z, w)[h], \widehat{\partial}_w f(z, w)[h]$ also to denote the resulting functions of three variables in order to emphasize the special role the variable h plays.

Both the operators $\widehat{\partial}_z, \widehat{\partial}_w$ are additive and right–$\mathbb{H}$–linear, namely

$$\widehat{\partial}_z(f(z, w)a + g(z, w)b)[h] = \widehat{\partial}_z f(z, w)[h]a + \widehat{\partial}_z g(z, w)[h]b,$$
$$\widehat{\partial}_w(f(z, w)a + g(z, w)b)[h] = \widehat{\partial}_w f(z, w)[h]a + \widehat{\partial}_w g(z, w)[h]b.$$

The resulting functions are additive and real-homogenous in the variable h, but not linear in h. Furthermore, the Leibniz rule holds. In the language of analysis on manifolds, for a fixed h, the partial derivative $\widehat{\partial}_z f(z, w)[h]$ is the Lie derivative of the function f along the constant vector field $X = (h, 0)$ evaluated at (z, w) and $\widehat{\partial}_w f(z, w)[h]$ is the Lie derivative of the function f along the constant vector field $X = (0, h)$ evaluated at (z, w). In practice, for polynomial function represented by a polynomial, each of the operators $\widehat{\partial}_z, \widehat{\partial}_w$ acts by replacing one occurrence of the prescribed variable at a time in each monomial of f_d with $h \in \mathbb{H}$ as in the following example

$$\widehat{\partial}_z(zwz^2wa)[h] = (hwz^2w + zwhzw + zwzhw)a.$$

If $|f_{p,q,\lambda}(z, w)| = |z|^p|w|^q c_{p,q,\lambda}$, then we can estimate

$$|\widehat{\partial}_z f_{p,q,\lambda}(z, w)[h]| \le p|z|^{p-1}|w|^q|h|c_{p,q,\lambda}$$

and $|\widehat{\partial}_w f_{p,q,\lambda}(z, w)[h]| \le q|z|^p|w|^{q-1}|h|c_{p,q,\lambda}$, which, in view of the assumed absolute convergence of the power series, implies that the power series can be differentiated term by term. Therefore operators $\widehat{\partial}_z, \widehat{\partial}_w$ are well-defined as mappings from quaternionic analytic functions of two variables to quaternionic analytic functions of three variables. This motivates the following definition of partial derivatives for series:

Definition 2.2. Given a series $f \in \mathcal{H}[z, w]$,

$$f(z, w) = \sum_{p,q\ge 0} \sum_{\lambda\in\Lambda_{p,q}} f_{p,q,\lambda}(z, w)$$

the series $\widehat{\partial}_z f$ is defined as

$$(\widehat{\partial}_z f)(z, w, h) := \sum_{p,q\ge 0} \sum_{\lambda\in\Lambda_{p,q}} \widehat{\partial}_z f_{p,q,\lambda}(z, w)[h]. \tag{2.3}$$

The operator $\widehat{\partial}_w$ is defined similarly. Note that the operators $\widehat{\partial}_z, \widehat{\partial}_w$ map series in $\mathcal{H}[z, w]$ to series in $\mathcal{H}[z, w, h]$.

We also use the notation $\widehat{\partial}_z f(z,w)[h]$ for the series to indicate the special role the variable h plays.

Linearity of the derivation implies that if a function is represented by two different series f and g, then also the series $\widehat{\partial}_z f(z,w)[h]$ and $\widehat{\partial}_z g(z,w)[h]$ represent the same function.

The following result motivates the introduction of the differential operators $\widehat{\partial}_z, \widehat{\partial}_w$.

Lemma 2.3. *Let $f \in \mathcal{H}_{\mathrm{rhs}}[z,w]$ be a series. If $\widehat{\partial}_z f(z,w)[h]$ is the null-series, then $f(z,w)$ is (formally) independent of z and so is also the corresponding function. An analogous result holds for w.*

Proof. It suffices to prove the first assertion for polynomials $P_{(p,q)}$ of bidegree (p,q) for each (p,q). We proceed by induction on q. For $q=0$ and $P_{(p,0)}(z,w) = z^p c_p$ we have

$$\widehat{\partial}_z P_{(p,0)}(z,w)[h] = (hz^{p-1} + zhz^{p-2} + \cdots + z^{p-1}h)c_p = 0$$

formally, so $c_p = 0$. Moreover, by [6, Proposition 2.4] the same holds if $\widehat{\partial}_z P_{(p,0)}(z,w)[h] = 0$ as a function. If $q > 0$ write

$$P_{(p,q)}(z,w) = zP_{(p-1,q)}(z,w) + wP_{(p,q-1)}(z,w)$$

and then the formal identity

$$\widehat{\partial}_z P_{(p,q)}(z,w)[h] = hP_{(p-1,q)}(z,w) + z\widehat{\partial}_z P_{(p-1,q)}(z,w)[h] + w\widehat{\partial}_z P_{(p,q-1)}(z,w)[h] = 0$$

implies

$$P_{(p-1,q)}(z,w) = 0,\ \widehat{\partial}_z P_{(p-1,q)}(z,w)[h] = 0 \quad \text{and} \quad \widehat{\partial}_z P_{(p,q-1)}(z,w)[h] = 0$$

formally. By induction hypothesis, $\widehat{\partial}_z P_{(p,q-1)}(z,w)[h]$ being formally 0 implies $P_{(p,q-1)}(z,w) = w^{q-1}c_{q-1}$, so $P_{(p,q)} = w^q c_{q-1}$. □

Remark 2.4. In analogy to the one variable case one could also define the (differential) operator

$$\widetilde{\partial}_z f(z,w) := \widehat{\partial}_z f(z,w)[1].$$

In short, the operator $\widetilde{\partial}_z$ replaces one occurrence of the variable z at a time with 1. This operator is a derivation. Using the notation from the above Lemma, the expression $\widetilde{\partial}_z P_{(p,q)}(z,w)$ is a polynomial of bidegree $(p-1,q)$ (similarly for w). Furthermore, this operator coincides with the corresponding (Cullen) derivative, when f is a slice-regular function (see [4]).

However, a result like the one in Lemma 2.3 does not hold when considering $\widetilde{\partial}_z$ instead of $\widehat{\partial}_z$. Indeed,

$$\widetilde{\partial}_z(zw - wz) = w - w = 0$$

but the neither the series $f(z,w) = zw - wz$ nor the corresponding function do not depend on w only.

2.2 Bidegree full series

For p, q positive integers, consider the series

$$S_{p,q}(z,w) := \sum_{\substack{\alpha^{p,q}, \\ |\alpha^{p,q}|=p \\ p+q=d}} (z,w)^{\alpha^{p,q}}.$$

It is clear that $S_{p,q}(z,w) = S_{q,p}(w,z)$. We also have this important identity

$$\widehat{\partial}_z S_{p+1,q}(z,w)[h] = \widehat{\partial}_w S_{p,q+1}(z,w)[h]. \tag{2.4}$$

If z and w commute, then $S_{p,q}(z,w) = \binom{p+q}{p} z^p w^q$.

Definition 2.5. We define

$$\mathbb{H}_d^{BF}[z,w] := \left\{ \sum_{p+q=d} S_{p,q}(z,w) a_{p,q},\ a_{p,q} \in \mathbb{H} \right\}$$

and

$$\mathbb{H}^{BF}[z,w] := \bigoplus_{d \geq 0} \mathbb{H}_d^{BF}[z,w].$$

We say that $\mathbb{H}^{BF}[z,w]$ is the right module of *bidegree full* (in short BF) polynomials in the variables z, w. The equivalence class of BF polynomials is called a *bidegree polynomial function*. Similarly, we define the right module of *bidegree full series* to consist of all converging power series of the form

$$f(z,w) = \sum_{d=0}^{\infty} f_d(z,w),$$

with $f_d(z,w) \in \mathbb{H}_d^{BF}[z,w]$ and denote it by $\mathcal{H}^{BF}[z,w]$. The equivalence class of a BF series is called a *bidegree full function*.

The following result shows that bidegree full polynomials form an interesting class of polynomials.

Lemma 2.6. *For any real number μ and any $d \in \mathbb{N}$, the polynomial*

$$(z - \mu w)^d := \overbrace{(z - \mu w) \cdots (z - \mu w)}^{d \text{ times}}$$

is bidegree full. If

$$P(z,w) = \sum_{d=0}^{l} \sum_{\substack{p,q \geq 0, \\ p+q=d}} S_{p,q}(z,w) a_{p,q}$$

is a bidegree full polynomial of degree d*, then it also has a decomposition*

$$P(z,w) = \sum_{d=0}^{l} \sum_{p+q=d} \left(\sum_{n=0}^{d} (z - \mu w)^d r_{p,d}(n) \right) a_{p,q}, \quad \textit{with} \quad r_{p,d}(n) \in \mathbb{R}.$$

Proof. Indeed, from direct calculations, it follows that

$$(z-\mu w)^d = (z-\mu w)\cdots(z-\mu w) = \sum_{\substack{p,q\geq 0,\\ p+q=d}} S_{p,q}(z,w)(-\mu)^q.$$

The second statement follows from the fact (proved in [2] by induction on d with an argument which applies to our setting) that the polynomials $\{x^d, (x-1)^d, \dots, (x-d)^d\}$ form a basis of real polynomials of order less or equal to d and consequently polynomials $z^d, (z-w)^d, \dots, (z-dw)^d$ form a basis of $\mathbb{H}_d^{BF}[z,w]$ □

The term $(z-\mu w)^d$ is well-defined also for $\mu \in \mathbb{H}$. But the equality

$$\widehat{\partial}_w (z-\mu w)^d = -\mu \widehat{\partial}_z (z-\mu w)^d \tag{2.5}$$

holds if and only if $\mu \in \mathbb{R}$.

Remark 2.7. As a consequence of Lemma 2.6, from any $F \in \mathcal{SR}$, in the variable u

$$F(u) = \sum_{d\geq 0} u^d a_d,$$

one gets a bidegree full series by replacing u with $z-\mu w, \mu \in \mathbb{R}$ namely

$$f(z,w) = \sum_{d\geq 0} (z-\mu w)^d a_d \in \mathcal{H}^{BF}[z,w].$$

2.3 Basics on Hamming graphs

Since the monomials we are dealing with are described by words on two letters, the Hamming graphs are natural objects to associate with such monomials.

Definition 2.8. Given $d, q \in \mathbb{N}$, the graph (V,E) is a Hamming graph $H(d,q)$ if the set of vertices V consists of all words of length d on q different letters and there is an edge $e(v_1,v_2) \in E$ between two vertices v_1, v_2 if they differ in precisely one letter.

The Hamming graph $H(d,q)$ is, equivalently, the Cartesian product of d complete graphs K_q. We are interested in Hamming graphs on two letters, $0, 1$, i.e. on hypercubes. A *layer* L_p, $0 \leq p \leq d$, is a set of vertices which contain p copies of 1. It is easy to see that the following result holds:

Lemma 2.9. *Any two subsequent layers of the hypercube form a connected subgraph.*

Proof. The case $d=1$ is trivial since it consists of letters 0 and 1 and an edge connecting them. Assume that $d>1$ and take $p \in \{0,\dots,d-1\}$. Let L_{p+1} and L_p be two subsequent layers, and let $\alpha, \tilde{\alpha} \in L_{p+1}$ differ for one transposition of indices 0 and 1 on positions l, m. Without loss of generality we assume that $l=1$ and $m=2$. We may also assume that

$$\alpha = 01\alpha_1 \text{ and } \tilde{\alpha} = 10\alpha_1.$$

Define

$$\beta = 00\alpha_1.$$

Since α and β differ in precisely one letter, there is an edge between α and β and of course also an edge between β and $\tilde{\alpha}$, so there exist a path between any two vertices in L_{p+1}, since all other multiindices in L_{p+1} are permutations of letters of α. By the same reason there exist a path connecting any two vertices in L_p which proves the lemma. □

3 Quaternionic vector fields

In this section, using the partial derivatives $\widehat{\partial}_z, \widehat{\partial}_w$, we define an operator divergence for quaternionic vector fields in two variables. We show that there is a large class of vector fields with good properties of analyticity.

Definition 3.1. Given the series $f, g \in \mathcal{H}[z, w]$, then $X = (f, g)$ is called a *vector field* in $\mathbb{H}^2$, in short we write $X \in \mathcal{VH}$. If $f, g \in \mathcal{H}_{\mathrm{rhs}}[z, w]$, then we write $X \in \mathcal{VH}_{\mathrm{rhs}}$. In particular, we say that a vector field $X = (f, g)$ is *bidegree full* (in short BF) if f, g are bidegree full and we use the notation $X \in \mathcal{VH}^{BF}$. A vector field $X = (f, g)$ defines a vector mapping $[X] := ([f], [g]) \colon \mathbb{H}^2 \to \mathbb{H}^2$.

We assume from now on that the vector fields under consideration belong to $\mathcal{VH}_{\mathrm{rhs}}$. Next we introduce the following

Definition 3.2. Given the vector field $X = (f, g) \in \mathcal{VH}_{\mathrm{rhs}}$, we define the operator Div by

$$\operatorname{Div} X(z, w)[h] := \widehat{\partial}_z f(z, w)[h] + \widehat{\partial}_w g(z, w)[h],$$

where the partial differential operators are used in the sense of (2.3). A vector field $X(z, w)$ has *divergence zero* if $\operatorname{Div} X(z, w)[h]$ is the null series.

Clearly for a vector field, divergence zero implies divergence zero as a function.

Example 3.3. The vector field $X(z, w) = (zw + wz, -w^2)$ has divergence zero, since

$$\operatorname{Div}(zw + wz, -w^2)[h] = hw + wh - (hw + wh) = 0$$

and the divergence of the vector field $Y(z, w) = X(z, w) + (0, [[z, w]^2, z]) = (zw + wz, -w^2 + [[z, w]^2, z])$ is

$$\begin{aligned}\operatorname{Div}(zw + wz, -w^2 + [[z, w]^2, z])[h] = {} & hw + wh - (hw + wh) \\ & + [[z, h][z, w], z] + [[z, w][z, h], z].\end{aligned}$$

This shows that $\operatorname{Div} Y$ is not a null-series, but $\operatorname{Div} Y$, considered as a vector mapping, vanishes identically.

The vector field $(z^2 w, -zw^2)$ does not have divergence zero:

$$\operatorname{Div}(z^2 w, -zw^2)[h] = (hz + zh)w - z(hw + wh) = hzw - zwh \neq 0$$

and also $[hzw - zwh] \neq 0$. By identity (2.5) any vector field $(z - \mu w)^d(\mu, 1), \mu \in \mathbb{R}$ has divergence zero. Such vector fields are called *shear vector fields* and they generate a 1-parameter family of automorphisms of $\mathbb{H}^2$, namely

$$\Phi_t(z, w) = (z, w) + t(z - \mu w)^d(\mu, 1)a, \quad a \in \mathbb{H}, t \in \mathbb{R},$$

called *shears*. In the complex analytic case by a famous result due to Andersen (see [1]) every volume preserving automorphism of $\mathbb{C}^2$ (these are holomorphic automorphisms $f \colon \mathbb{C}^2 \to \mathbb{C}^2$ with determinant $\det Jf(z, w) = 1$) is approximable by a finite composition of shears. In search for analogous results in the quaternionic setting, it is then necessary to prove that any polynomial divergence zero vector field is generated by a shear vector field. Because of identity (2.4), any vector field

$$X_{p,q}(z,w) = (S_{p+1,q}(z,w), -S_{p,q+1}(z,w)) \tag{3.1}$$

has divergence zero. It can be shown using Lemma 2.6 that every vector field $X_{p,q}$ is a sum of shear vector fields. The interested reader can find the details in [6]. The next theorem shows that any divergence zero vector field is generated by such vector fields $X_{p,q}$.

Theorem 3.4. *Let $X = (f,g)$ be a vector field with divergence zero, then f and g are bidegree full.*

Remark 3.5. Example 4.4 shows that for any vector field X with components of bidegrees $(3,2)$ and $(2,3)$, the condition $\mathrm{Div}X(z,w)[h] = 0$ as a function of three variables implies that the mapping representing the vector field X has a bidegree full representative.

Corollary 3.6. *If X is a vector field with divergence zero, then X is of the form $X = \sum X_{p,q}a_{p,q}$, $a_{p,q} \in \mathbb{H}$ with $X_{p,q}$ as in (3.1).*

Before proceeding to the proof, let us show an example with vector fields of the form $X(z,w) = (f(z,w), g(z,w)) = (z^2wa_1 + zwza_2 + wz^2a_3, -w^2zb_1 - wzwb_2 - zw^2b_3)$. We first calculate the partial derivatives separately.

$$\begin{aligned}
\widehat{\partial}_z f(z,w)[h] &= (zhw + hzw)a_1 + (hwz + zwh)a_2 + (whz + wzh)a_3, \\
\widehat{\partial}_w g(z,w)[h] &= -(whz + hwz)b_1 - (hzw + wzh)b_2 - (zwh + zhw)b_3.
\end{aligned}$$

The sum of the partial derivatives is zero if and only if monomials of the same type cancel out, for example we have conditions $zhw(a_1 - b_3) = 0$ and $hzw(a_1 - b_2) = 0$ which imply $a_1 = b_3$ and $a_1 = b_2$ and similarly for other terms. We represent these equalities by means of a bipartite graph on $\{a_1, a_2, a_3\} \cup \{b_1, b_2, b_3\}$ in which there is an edge between a_i and b_j if and only if they are equal. The graph is given in Figure 1.

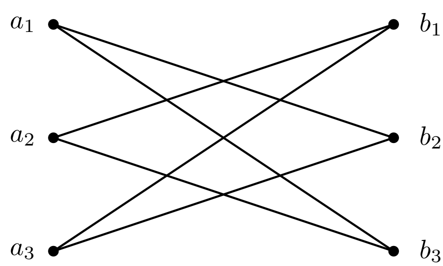

Figure 1: Bipartite graph.

Proof of Theorem 3.4. Let $f(z,w) = \sum f_{p,q}(z,w)$, $g(z,w) = \sum g_{p,q}(z,w)$ be the decompositions of series f and g with respect to the bidegrees. Then $X = (f,g)$ has divergence zero if and only if

$$\widehat{\partial}_z f_{p+1,q}(z,w)[h] + \widehat{\partial}_w g_{p,q+1}(z,w)[h] = 0.$$

Let $\mathcal{A} = \{\alpha \in \{0,1\}^{d+1}, |\alpha| = p+1\}$ and $\mathcal{B} = \{\beta \in \{0,1\}^{d+1}, |\beta| = p\}$. Write

$$f_{p+1,q}(z,w) = \sum_{\alpha\in\mathcal{A}} (z,w)^{\alpha} A_{\alpha}$$
$$g_{p,q+1}(z,w) = -\sum_{\beta\in\mathcal{B}} (z,w)^{\beta} B_{\beta}.$$

The monomials in the sum $\widehat{\partial}_z f_{p+1,q}(z,w)[h]$ (and similarly $\widehat{\partial}_z g_{p,q+1}(z,w)[h]$) are of the following form:

$$(z,w)^{\alpha_1} h (z,w)^{\alpha_2} A_{\alpha}$$

where $\alpha = \alpha_1 1 \alpha_2$. For any such α there is exactly one β, namely

$$\beta = \alpha_1 0 \alpha_2$$

such that in the sum $\widehat{\partial}_w (z,w)^{\beta} B_{\beta}(h)$ there is the monomial of the same type (but multiplied by a different constant)

$$-(z,w)^{\alpha_1} h (z,w)^{\alpha_2} B_{\beta}.$$

Zero divergence implies that $A_{\alpha} = B_{\beta}$ for any such pair α, β.

Define a bipartite graph on the vertices $V = \mathcal{A} \cup \mathcal{B}$. There is an edge between a word $\alpha \in A$ and a word $\beta \in B$ iff the word β is obtained from the word α by replacing one of the letters 1 by 0. So by definition, in this particular case, we are considering a subgraph of the Hamming graph $H(d+1,2)$, spanned on edges from the set $\mathcal{A}\cup\mathcal{B}$ which represent two subsequent layers in the corresponding hypercube, $\mathcal{A} = L_{p+1}$ and $\mathcal{B} = L_p$. By Lemma 2.9 this subgraph is connected. This implies that all $A_{\alpha} = A$ for some constant A and hence the same holds for all B_{β} so all the coefficients are the same and this means that $f_{p+1,q}, g_{p,q+1}$ are bidegree full. □

Remark 3.7. We should point out that the analytic condition on a vector field of two quaternionic variables having divergence zero is equivalent to connectedness of subgraphs of a Hamming graph. We could proceed analogously in higher dimensions. In three variables we would consider $H(d,3)$, graphs on three letters, where d is the degree, but in this case the divergence zero condition translates into looking for cycles of order 3 of a particular form. Its analysis turns out to be more complicated than the two-dimensional case and is not related to connectedness of subgraphs of Hamming graphs. For example, the divergence zero condition for the vector field $X(z,w,u) = (f(z,w,u), g(z,w,u), h(z,w,u))$ in the case

$$f(z,w,u) = zwua_1 + wzua_2 + wuza_3 + zuwa_4 + uzwa_5 + uwza_6,$$
$$g(z,w,u) = w^2ub_1 + wuwb_2 + uw^2b_3,$$
$$h(z,w,u) = wu^2c_1 + uwuc_2 + u^2wc_3,$$

gives equations $a_1 + b_1 + c_2 = 0$, $a_2 + b_1 + c_1 = 0, \dots$ and they can be represented as a 2-simplicial complex with 2-cells being triangles with vertices (a_1, b_2, c_2), (a_2, b_1, c_1) and so forth (see Figure 2).

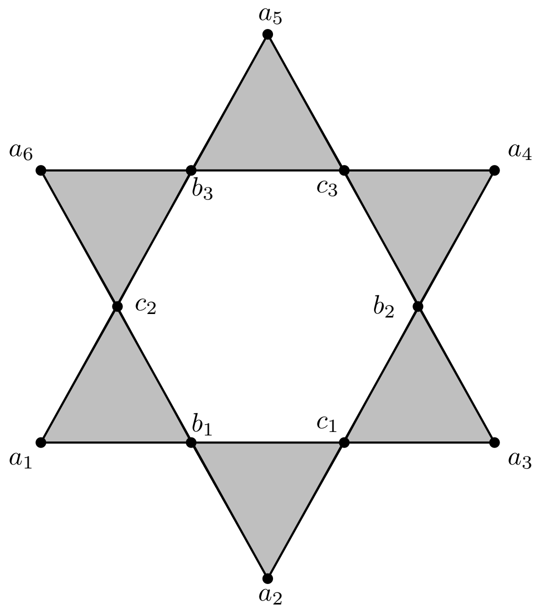

Figure 2: 2-simplicial complex describing the divergence 0 condition for 3 variables.

Remark 3.8. Notice that if we can write a vector mapping as a vector field $X = (f_1, g_1) + (f_2, g_2)$ such that (f_1, g_1) has divergence zero and such that each of f_2 and g_2 are not formally 0 but identically equal to 0 as functions, then the flow of $[X]$ coincides with the flow of $[(f_1, g_1)]$. Furthermore, the flow of $[(f_2, g_2)]$ exists and is the identity mapping, so it does not affect the problem of approximating a flow by shears.

4 Linear independence of monomials

In this section we consider the problem of linear independence of monomials in $\mathbb{H}_{\mathrm{rhs}}(z, w)$; in particular we exhibit an algorithm for determining linear independence of monomials in $\mathbb{H}_{\mathrm{rhs},(p,q)}[z, w]$. We point out that this approach does not involve the computation of independent monomials in 8 real variables of degree p in the first 4 variables and of degree q in the last 4 variables.

We prove the following result.

Theorem 4.1. *Given a bidegree (p, q), the set of all distinct monomials in $\mathbb{H}_{\mathrm{rhs},(p,q)}[z, w]$ is linearly independent if and only if (p, q) equals $(p, 0),\ (0, q),\ (p, 1),\ (1, q)$ or $(2, 2)$.*

The proof below gives an explicit algorithm for calculating the basis of the kernel of the linear mapping

$$\mathcal{A}_{p,2} \colon \mathbb{H}^{n_{p+2,2}} \to \mathcal{H}_{\mathrm{rhs}}[z, w], \quad \mathcal{A}_{p,2}(c_1, \dots, c_{n_{p+2,2}}) = \sum_{1}^{n_{p+2,2}} (z, w)^{\alpha_k} c_k,$$

where $\alpha_k \in \{0, 1\}^d$ are all distinct multiindices of length p, $|\alpha_k| = p$, with $k = 1, \dots, n_{p+2,2} = \binom{p+2}{2}$.

Proof. The cases in bidegrees $(p, 0)$, $(0, q)$ are trivial and the cases in bidegrees $(p, 1)$, $(1, q)$ were proved in [6, Proposition 2.4]. Assume that $z \in \mathbb{C}_I \setminus \mathbb{R}$ and choose any imaginary unit J orthogonal to I. Then we can write $w \in \mathbb{H}$ in the form $z_0 + z_1 J$, where $z_0, z_1 \in \mathbb{C}_I$ are uniquely determined. This choice of coordinates provides us with a frame which determines the identification $\mathbb{H} = \mathbb{C}_I \times \mathbb{C}_I$. In other words, if $w = z_0 + z_1 J \simeq (z_0, z_1) \in \mathbb{C}_I \times \mathbb{C}_I$, then, since $zJ = J\bar{z}$, we have

$$w^2 = z_0^2 - |z_1|^2 + (\bar{z}_0 z_1 + z_0 z_1)J;$$

similarly

$$[z, w] = z_1(z - \bar{z})J \quad \text{and} \quad [z, w]^2 = -|(z - \bar{z})z_1|^2.$$

We recall that the polynomial introduced in (2.2) is precisely $P(z, w) = [[z, w]^2, z]$.

We begin with polynomial $w^2 \in \mathbb{H}_{\mathrm{rhs},(0,2)}[z, w]$ and develop an algorithm for producing monomials in $\mathbb{H}_{\mathrm{rhs},(1,2)}[z, w]$ which we describe with respect to the above identification and then proceed inductively.

First of all, without loss of generality, we (may and will) assume that z is unitary, so $z^{-1} = \bar{z}$. Let $A_0 = B_0 = C_0 = \{w^2\}$. Define the sets $A_1 = \{zw^2\}, B_1 = \{wzw\}, C_1 = \{w^2 z\}$. The monomial in A_1 was obtained by adding z to the monomial w^2 on the left hand side, the one in B_1 by adding one z after the first w of the monomial w^2 and the monomial in C_1 z was obtained by adding a z on the right hand side of the monomial w^2.

If $w = z_0 + z_1 J$ (and then $w^2 = z_0^2 - |z_1|^2 + (\bar{z}_0 z_1 + z_0 z_1)J$) we have

$$\begin{aligned}
A_1 \ni zw^2 &= z(z_0^2 - |z_1|^2 + (\bar{z}_0 z_1, z_0 z_1)J)\bar{z}z \\
&= (z_0^2 - |z_1|^2 + (z^2 \bar{z}_0 z_1 + z^2 z_0 z_1)J)z = f_1(z, z_0, z_1)z, \\
B_1 \ni wzw &= (z_0 + z_1 J)z(z_0 + z_1 J)\bar{z}z \\
&= ((z_0^2 - \bar{z}^2 |z_1|^2 + (\bar{z}_0 z_1 + z^2 z_0 z_1)J)z = f_2(z, z_0, z_1)z, \\
C_1 \ni w^2 z &= (z_0^2 - |z_1|^2 + (\bar{z}_0 z_1, z_0 z_1)J)z = f_3(z, z_0, z_1)z.
\end{aligned}$$

We identify w^2 with the vector $(z_0^2, -|z_1|^2, \bar{z}_0 z_1, z_0 z_1) \in \mathbb{C}_I^4$ and identify the function f_1 with the vector $u_1 = (z_0^2, -|z_1|^2, z^2 \bar{z}_0 z_1, z^2 z_0 z_1) \in \mathbb{C}_I^4$, f_2 with the vector $u_2 = (z_0^2, -\bar{z}^2 |z_1|^2, \bar{z}_0 z_1, z^2 z_0 z_1) \in \mathbb{C}_I^4$ and the function f_3 with the vector $u_3 = (z_0^2, -|z_1|^2, \bar{z}_0 z_1, z_0 z_1) \in \mathbb{C}_I^4$. We notice that u_1 is obtained from u_3 by multiplying the first two components by 1 and the last two by z^2, i.e. $u_1 = u_3 * (1, 1, z^2, z^2)$, where $*$ denotes the componentwise multiplication in $\mathbb{C}_I^4$ defined as follows: if $(a, b, c, d) \in \mathbb{C}_I^4$ and $(x, y, u, v) \in \mathbb{C}_I^4$ then

$$(a, b, c, d) * (x, y, u, v) := (ax, by, cu, dv).$$

Notice that the componentwise multiplication $*$ in $\mathbb{C}_I^4$ is commutative, i.e.

$$(x, y, u, v) * (a, b, c, d) = (a, b, c, d) * (x, y, u, v)$$

for any $(a, b, c, d) \in \mathbb{C}_I^4$ and $(x, y, u, v) \in \mathbb{C}_I^4$ and has no zero divisors.

Similarly $u_2 = u_3 * (1, \bar{z}^2, 1, z^2)$ and $u_3 = u_3 * (1, 1, 1, 1)$. Consider a quaternionic (right-hand side) null linear combination of the monomials which generates $\mathbb{H}_{\mathrm{rhs},(1,2)}[z, w]$, namely $zw^2 a + wzwb + w^2 zc = 0$ with $a, b, c \in \mathbb{H}$.

In terms of the vectors u_1, u_2, u_3, we can write the same null linear combination as

$$\begin{aligned}
&u_3 * (1, 1, z^2, z^2)za + u_3 * (1, \bar{z}^2, 1, z^2)zb + u_3 * (1, 1, 1, 1)zc = \\
&\quad = u_3 * [(1, 1, z^2, z^2)za + (1, \bar{z}^2, 1, z^2)zb + (1, 1, 1, 1)zc] = 0.
\end{aligned} \tag{4.1}$$

If we write $a = a_0 + a_1 J$, $b = b_0 + b_1 J$ $c = c_0 + c_1 J$ (according to the adopted frame) and look at the first component in (4.1), since $u_3 \neq 0$, we get the equation

$$\begin{aligned} z(a_0 + a_1 J) + z(b_0 + b_0 J) + z(c_0 + c_1 J) &= z((a_0 + a_1 J + b_0 + b_0 J + c_0 + c_1 J) \\ &= z[(a_0 + b_0 + c_0) + (a_1 + b_1 + c_1)J] = 0 \end{aligned}$$

which implies $a_0 + b_0 + c_0 = 0$ and $a_1 + b_1 + c_1 = 0$. From the vanishing of the second component in (4.1) we get the equation

$$\begin{aligned} 1 \cdot z(a_0 + a_1 J) &+ \bar{z}^2 z(b_0 + b_0 J) + z(c_0 + c_1 J) \\ &= z(a_0 + a_1 J) + z\bar{z}^2(b_0 + b_1 J) + z(c_0 + c_1 J) \\ &= z[(a_0 + a_1 J) + \bar{z}^2(b_0 + b_1 J) + (c_0 + c_1 J)] \\ &= z[(a_0 + \bar{z}^2 b_0 + c_0) + (a_1 + b_1 \bar{z}^2 + c_1)J] = 0 \end{aligned}$$

which implies $a_0 + \bar{z}^2 b_0 + c_0 = 0$ and $(a_1 + \bar{z}^2 b_1 + c_1)J = 0$. From the vanishing of the third component in (4.1) we get the equation

$$\begin{aligned} z^2 Jz(a_0 + a_1 J) &+ Jz(b_0 + b_0 J) + Jz(c_0 + c_1 J) \\ &= Jz(\bar{z}^2(a_0 + a_1 J) + (b_0 + b_1 J) + (c_0 + c_1 J) \\ &= Jz[(\bar{z}^2 a_0 + b_0 + c_0) + (\bar{z}^2 a_1 + b_1 + c_1)J) = 0 \end{aligned}$$

which implies $\bar{z}^2 a_0 + b_0 + c_0 = 0$ and $\bar{z}^2 a_1 + b_1 + c_1 = 0$. Here we can replace $\bar{z}^2$ with z^2 since we are allowed to plug in any $z \in \mathbb{C}_I$, in particular we can plug in $\bar{z}$ and use the fact that $\bar{\bar{z}} = z$. From the vanishing of the last component in (4.1) we get the equation

$$\begin{aligned} z^2 Jz(a_0 + a_1 J) &+ z^2 Jz(b_0 + b_1 J) + Jz(c_0 + c_1 J) \\ &= Jz(\bar{z}^2(a_0 + a_1 J) + \bar{z}^2(b_0 + b_1 J) + (c_0 + c_1 J) \\ &= Jz[(\bar{z}^2 a_0 + \bar{z}^2 b_0 + c_0) + (\bar{z}^2 a_1 + \bar{z}^2 b_1 + c_1)J] = 0 \end{aligned}$$

which implies $\bar{z}^2 a_0 + \bar{z}^2 b_0 + c_0 = 0$ and $\bar{z}^2 a_1 + \bar{z}^2 b_1 + c_1 = 0$. Also in these equations, one can substitute $\bar{z}$ with z.[1] In the vectorial version, we can write the above-given equations as

$$\begin{aligned} (1, 1, z^2, z^2)a_0 + (1, \bar{z}^2, 1, z^2)b_0 + (1, 1, 1, 1)c_0 = 0, \\ (1, 1, z^2, z^2)a_1 + (1, \bar{z}^2, 1, z^2)b_1 + (1, 1, 1, 1)c_1 = 0. \end{aligned}$$

Therefore, the linear dependence of the monomials zw^2, wzw and $w^2 z$ (generators of $\mathbb{H}_{\mathrm{rhs},(1,2)}[z, w]$) is equivalent to the linear dependence of the vector functions $X_1(z) = (1, 1, z^2, z^2)$, $X_2(z) = (1, \bar{z}^2, 1, z^2)$ and $X_3(z) = (1, 1, 1, 1)$. In this case, the vector functions $X_1(z), X_2(z)$ and $X_3(z)$ are evidently linearly independent and so are the monomials zw^2, wzw and $w^2 z$.

To generalize the formalization of the above ideas we introduce the operator ad_z and adopt the identification $w^2 \simeq (z_0^2, -|z_1|^2, \bar{z}_0 z_1, z_0 z_1) \in \mathbb{C}_I^4$. For $q \in \mathbb{H} \setminus \{0\}$, $|q| = 1$, let

[1] In all equations we can plug in the variable $\bar{z}$ instead of z and we can consider the conjugated equation. Then the linear independence in question is equivalent to (one of) the above equations with *real* coefficients.

$\mathrm{ad}_q(w) := qw\bar{q}$. This transformation represents a rotation in $\mathbb{H}$ which keeps fixed the slice which contains q. Notice that if $z \in \mathbb{C}_I \setminus \mathbb{R}$ and $w = z_0 + z_1 J$ (with $z_0, z_1 \in \mathbb{C}_I$), then

$$\mathrm{ad}_z(w) = z_0 + z_1 z^2 J$$

and

$$zw^2 = (\mathrm{ad}_z(w))^2 z = \mathrm{ad}_z(w^2)z, \qquad \mathrm{ad}_z(\mathrm{ad}_z(w)) = \mathrm{ad}_{z^2}(w).$$

Then

$$\begin{aligned}
\mathrm{ad}_z(w^2) = zw^2\bar{z} &\simeq (z_0^2, -|z_1|^2, z^2\bar{z}_0 z_1, z^2 z_0 z_1)\\
&= (z_0^2, -|z_1|^2, \bar{z}_0 z_1, z_0 z_1) * (1, 1, z^2, z^2),\\
w\,\mathrm{ad}_z(w) = wzw\bar{z} &\simeq (z_0^2, -\bar{z}^2|z_1|^2, \bar{z}_0 z_1, z^2 z_0 z_1)\\
&= (z_0^2, -|z_1|^2, \bar{z}_0 z_1, z_0 z_1) * (1, \bar{z}^2, 1, z^2),\\
w^2\mathrm{ad}_z(1) = w^2 &\simeq (z_0^2, -|z_1|^2, \bar{z}_0 z_1, z_0 z_1)\\
&= (z_0^2, -|z_1|^2, \bar{z}_0 z_1, z_0 z_1) * (1, 1, 1, 1).
\end{aligned}$$

Define the functions $\varphi_z, \psi_z, \mathrm{id}_z \colon \mathbb{C}_I^4 \to \mathbb{C}_I^4$ by

$$\begin{aligned}
\varphi_z[(a,b,c,d)] &= (a,b,c,d) * X_1(z), \qquad \psi_z[(a,b,c,d)] = (a,b,c,d) * X_2(z),\\
\mathrm{id}_z[(a,b,c,d)] &= (a,b,c,d) * X_3(z).
\end{aligned}$$

Since φ_z, ψ_z and id_z are linear in $\mathbb{C}_I^4$ and can be represented by diagonal matrices, we identify the maps with the diagonals of the corresponding matrices: $\varphi_z \cong (1, 1, z^2, z^2) = X_1(z)$, $\psi_z \cong (1, \bar{z}^2, 1, z^2) = X_2(z)$, and $\mathrm{id}_z \cong (1, 1, 1, 1) = X_3(z)$. In this sense the functions φ_z, ψ_z and id_z are linearly independent since $X_1(z), X_2(z)$ and $X_3(z)$ are linearly independent as vectors.

Now one can write the monomial $zw^2 \in A_1$ as $\varphi_z[(1,1,1,1)] =: \mathbf{v}_1(z)$, the monomial $wzw \in B_1$ as $\psi_z[(1,1,1,1)] =: \mathbf{v}_2(z)$ and the monomial $w^2 z \in C_1$ as $\mathrm{id}_z[(1,1,1,1)] =: \mathbf{v}_3(z)$. In the sequel the functions like $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ will be called *vector functions*. With this identification we have $A_1 = \varphi_z(A_0)$, $B_1 = \psi_z(B_0)$ and $C_1 = \mathrm{id}_z(C_0)$. Notice that one can also write $A_0 = A_0 \cup B_0 \cup C_0$ and $B_0 = B_0 \cup C_0$.

We proceed by inductive construction and define

$$A_p = \varphi_z(A_{p-1} \cup B_{p-1} \cup C_{p-1}),\ B_p = \psi_z(B_{p-1} \cup C_{p-1}) \quad \text{and} \quad C_p = \mathrm{id}_z(C_{p-1}).$$

The set A_p contains all monomials, obtained by adding a z on the left hand side to all bidegree $(p-1, 2)$ monomials, the set B_p is obtained by adding a z after the first w of the monomials in B_{p-1} and in C_{p-1} and the set C_p is obtained by adding a z on the right hand side to the monomials in C_{p-1}.

Let us describe the sets A_p, B_p, C_p together with the corresponding vector functions and compute the kernels of $\mathcal{A}_{p,2}$ for $p = 2, 3$. Notice that with the adopted identifications, it turns out that $C_p = \{w^2 z^p\}$ for any $p \geq 0$ and this implies that the vector function associated with the unique monomial in C_p is the same, namely $(1,1,1,1)$ for any $p \geq 0$.

Here we list the sets A_p B_p and C_p (together with the description of monomials as vector functions) for $p = 2$:

$$
\begin{array}{rll}
 & \text{vector function} & \text{monomial} \\
A_2 = & \left\{\begin{array}{l} \mathbf{v}_1(z) = (1, 1, z^4, z^4) \\ \mathbf{v}_2(z) = (1, \bar{z}^2, z^2, z^4) \\ \mathbf{v}_3(z) = (1, 1, z^2, z^2) \end{array}\right. & \left.\begin{array}{l} \simeq z^2w^2 \\ \simeq zwzw \\ \simeq zw^2z \end{array}\right\} \\
B_2 = & \left\{\begin{array}{l} \mathbf{v}_4(z) = (1, \bar{z}^4, 1, z^4) \\ \mathbf{v}_5(z) = (1, \bar{z}^2, 1, z^2) \end{array}\right. & \left.\begin{array}{l} \simeq wz^2w \\ \simeq wzwz \end{array}\right\} \\
C_2 = & \left\{ \mathbf{v}_6(z) = (1, 1, 1, 1) \right. & \left. \simeq w^2z^2 \right\}.
\end{array}
$$

Notice that each of the components of the vector functions $\mathbf{v}_k(z)$ is generated by $\{1, \bar{z}^2, \bar{z}^4, z^2, z^4\}$. We look for the functional kernel of the linear mapping $\mathcal{A}_{2,2}(c_1, \ldots, c_6) = \sum_{k=1}^{6} \mathbf{v}_k(z)c_k$ where the vector functions $\mathbf{v}_k(z)$ are listed above; in other words we are imposing conditions on c_k's to have $\sum_{k=1}^{6} \mathbf{v}_k(z)c_k \equiv 0$ as a function of z. From the vanishing of the first component we only get one equation, from the vanishing of the second, third and fourth components the (linear) equations are obtained by imposing the vanishing of coefficients in the basis $\{1, \bar{z}^2, \bar{z}^4\, z^2, z^4\}$. In this way we obtain a homogeneous linear system whose corresponding matrix is

$$
M_2 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0
\end{pmatrix}.
$$

The matrix M_2 has trivial kernel and this proves the linear independence of sets of distinct monomials in $\mathbb{H}_{\mathrm{rhs},(2,2)}[z, w]$. For $p = 3$, using the same approach, we get

$$
\begin{array}{rll}
 & \text{vector function} & \text{monomial} \\
A_3 = & \left\{\begin{array}{l} \mathbf{v}_1(z) = (1, 1, z^6, z^6) \\ \mathbf{v}_2(z) = (1, \bar{z}^2, z^4, z^6) \\ \mathbf{v}_3(z) = (1, 1, z^4, z^4) \\ \mathbf{v}_4(z) = (1, \bar{z}^4, z^2, z^6) \\ \mathbf{v}_5(z) = (1, \bar{z}^2, z^2, z^4) \\ \mathbf{v}_6(z) = (1, 1, z^2, z^2) \end{array}\right. & \left.\begin{array}{l} \simeq z^3w^2 \\ \simeq z^2wzw \\ \simeq z^2w^2z \\ \simeq zwz^2w \\ \simeq zwzwz \\ \simeq zw^2z^2 \end{array}\right\} \\
B_3 = & \left\{\begin{array}{l} \mathbf{v}_7(z) = (1, \bar{z}^6, 1, z^6) \\ \mathbf{v}_8(z) = (1, \bar{z}^4, 1, z^4) \\ \mathbf{v}_9(z) = (1, \bar{z}^2, 1, z^2) \end{array}\right. & \left.\begin{array}{l} \simeq wz^3w \\ \simeq wz^2wz \\ \simeq wzwz^2 \end{array}\right\} \\
C_3 = & \left\{ \mathbf{v}_{10}(z) = (1, 1, 1, 1) \right. & \left. \simeq w^2z^3 \right\}
\end{array}
$$

Notice that each of the sets A_3, B_3, C_3 contains only linearly independent monomials. We look for the functional kernel of the linear mapping $\mathcal{A}_{3,2}(c_1, \dots, c_{10}) = \sum_{k=1}^{10} \mathbf{v}_k(z) c_k$ where the vector functions $\mathbf{v}_k(z)$ are listed above; in other words we are imposing conditions on c_k's to have $\sum_{k=1}^{10} \mathbf{v}_k(z) c_k \equiv 0$ as a function of z. We list the equations in the same order as in the previous case. The homogeneous linear system in this case has as corresponding matrix

$$M_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

whose kernel is spanned by the vector $(0, -1, 1, 1, 0, -1, 0, -1, 1, 0)$. The generator of the kernel represents precisely the polynomial $P(z, w) = [[z, w]^2, z]$ in (2.2).

In the same way one can verify that the kernel of the mapping $\mathcal{A}_{4,2}$ in $\mathbb{H}_{\text{rhs},(4,2)}[z, w]$ is three-dimensional with generators $P_1(z, w) = zP(z, w)$ and $P_2(z, w) = P(z, w)z$, obtained from the polynomial P, and the polynomial

$$Q(z, w) = [[z, w]z[z, w], z]. \tag{4.2}$$

The latter is a zero function since $[z, w]z[z, w] = -|(z - \bar{z})z_1|^2 \bar{z}$. The polynomial Q is formally linearly independent from the other two generators since it contains the term wz^3wz, which does not appear in P_1 or P_2. Then the kernel of the mapping $\mathcal{A}_{5,2}$ in $\mathbb{H}_{\text{rhs},(5,2)}[z, w]$ is six-dimensional, generated by $z^2P(z, w)$, $zP(z, w)z$, $P(z, w)z^2$, $zQ(z, w)$, $Q(z, w)z$ and $[[z^2, w], z]$. By a similar argument as in bidegree $(4, 2)$, the first five polynomials are formally linearly independent and the last one contains the term wz^4wz, which does not appear in the first five polynomials.

In fact it is easy to see that in general the first component of the vector functions in A_p, B_p and C_p is always 1, whereas in the second component terms containing $1, \bar{z}^2, \dots, \bar{z}^{2(p)}$ will appear; similarly, in the third and the fourth component only terms containing $1, z^2, \dots, z^{2p}$ will show up.

Let us count the number of equations obtained by imposing the vanishing of coefficients of $\mathcal{A}_{p,2}$. There is only one equation coming from the first component (which is redundant) and the last three components give $(2^2 - 1)(p + 1)$ equations, whereas we have $\binom{p+2}{2} = (p + 2)(p + 1)/2$ formally different monomials, so we see that the dimension of the kernel grows quadratically in the bidegree $(p, 2)$. If $p = 2$ we have 6 monomials and $9 + 1$ equations and if $p = 3$ there are 10 monomials and $12 + 1$ equations. If $p = 5$ we have for the first time that the number of equations (which is 19) is smaller than the number of monomials (which is 21).

By a similar procedure one would expect $(2^3-1)(p+1)$ equations for $\binom{p+3}{3}$ monomials in the submodule $\mathbb{H}_{\mathrm{rhs},(p,3)}[z,w]$ and so forth, but it turns out the for $q=3$ there are 7 linearly independent monomials of degree 3 in $z_0, z_1, \bar{z}, \bar{z_1}$ in the expression of w^3, with the first component giving a redundant equation as before, therefore we get less equations.

The same procedure applied to $\mathbb{H}_{\mathrm{rhs},(p,1)}[z,w]$ is equivalent to looking only at the sets A_p and C_p and their union, since B_p is empty. Moreover in $\mathbb{H}_{\mathrm{rhs},(p,1)}[z,w]$ the generating monomials have as corresponding vector functions $(1,1,z^{2k},z^{2k})$, $k=0,\dots,p$ and they are obviously linearly independent. This is an alternative proof of Proposition 2.4 in [6].

It is clear that if a set of distinct monomials $\{m_\lambda(z,w)\}_{\lambda\in\Lambda}$ is not linearly independent in the submodule $\mathbb{H}_{\mathrm{rhs},(p,q)}[z,w]$, so the set $\{z^n m_\lambda(z,w)\}_{\lambda\in\Lambda}$ is not linearly independent in $\mathbb{H}_{\mathrm{rhs},(p+n,q)}[z,w]$ for each $n\in\mathbb{N}$ and because of symmetry the set $\{m_\lambda(w,z)\}_{\lambda\in\Lambda}$ is not linearly independent in $\mathbb{H}_{\mathrm{rhs},(q,p)}[z,w]$. Putting this together, we see that a subset of all distinct monomials in $\mathbb{H}_{\mathrm{rhs},(3+n,2+m)}[z,w], m,n\in\mathbb{N}_0$ and in $\mathbb{H}_{\mathrm{rhs},(2+n,3+m)}[z,w]$, $m,n\in\mathbb{N}_0$ is not linearly independent. $\square$

Since $P(z,w)=[[z,w]^2,z]=0$ as a function and also the polynomial Q of bidegree $(4,2)$, $Q(z,w)=[[z,w]z[z,w],z]$ is identically 0 as a function (as explained in the last section, Equation (4.2)), we conjecture that all zero polynomial functions not formally 0 are obtained from polynomials P and Q after multiplying them by other polynomials and inserting variables z^k or w^l.

Remark 4.2. The described procedure can be interpreted as a complex Fourier series analysis with respect to the complex variables z, z_0 and z_1. We could have assumed that all the three variables z, z_0 and z_1 are unitary complex numbers, since the modulus is not relevant. In the expansion we considered, there are only 4 generators of the basis of the Fourier series in variables z_0 and z_1 and this is reflected in the vector functions having 4 components. With respect to the variable z, the number of the basic vector functions in question is $2p+1$ if bidegree is $(p,2)$.

Remark 4.3. After applying the partial derivative operator $\widehat{\partial}_z$ to the generators of the kernel of $\mathcal{A}_{p,q}$ in $S_{p,q}(z,w)$, one obtains polynomials in tridegree $(p-1,q,1)$ with respect to variables z,w,h, e.g. polynomials with $p-1$ copies z, q copies of w and one h. Analogous statement holds for $\widehat{\partial}_w$.

Example 4.4. Consider a vector field $X=(f(z,w),g(z,w))$, where f has bidegree $(3,2)$ and g has bidegree $(2,3)$, and let the vector field Y be defined by $Y(z,w)=X(z,w)+(P(z,w)a,\tilde{P}(z,w)b)$, (with $a,b,\in\mathbb{H}$), where P is the bidegree $(3,2)$ polynomial defined in (2.2) and the polynomial $\tilde{P}(z,w)=P(w,z)$ is then a bidegree $(2,3)$ polynomial. Obviously we have $\mathrm{Div}\,X(z,w)[h]=\mathrm{Div}\,Y(z,w)[h]$ as a function since P and $\tilde{P}$ are identically 0 as functions. Within this bidegree, the equivalence relation $X\sim Y$ if $[X-Y]=[0]$ means $X-Y=(Pa,\tilde{P}b)$ for some choice of $a,b\in\mathbb{H}$. After a careful study of linear independence of monomials in tridegree $(2,2,1)$, i.e. monomials with two copies of z-s, two copies of w-s and one copy of h – which, it should be mentioned, boils down to determining the kernel of a 80×30 linear system !!! – it turns out that in this particular case, $\mathrm{Div}\,X(z,w)[h]=0$ as a function if and only if $X=X_{2,2}+(Pa,\tilde{P}b)$, which means that the vector mapping has divergence 0 as a function if and only if it has a bidegree full representative in the sense of the above equivalence relation. Examples of bidegree full polynomial vector fields are given in (3.1).

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A Möbius-type gluing technique for obtaining edge-critical graphs*

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Abstract

Using a technique which is inspired by topology, we construct original examples of 3- and 4-edge critical graphs. The 3-critical graphs cover all even orders starting from 26; the 4-critical graphs cover all even orders starting from 20 and all the odd orders. In particular, the 3-critical graphs are not isomorphic to the graphs provided by Goldberg for disproving the Critical Graph Conjecture. Using the same approach we also revisit the construction of some fundamental critical graphs, such as Goldberg's infinite family of 3-critical graphs, Chetwynd's 4-critical graph of order 16 and Fiol's 4-critical graph of order 18.

Keywords: Edge-colouring, critical graph, Möbius strip.

Math. Subj. Class. (2020): 05C10, 05C15

1 Introduction

In the present paper, we deal with graphs that are not necessarily simple, so multiple (or parallel) edges are allowed but loops are excluded. We denote by $\chi'(G)$ the chromatic index of a graph G, namely, the minimum number of colours that are needed for an edge-colouring of G. Vizing, in [12], proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + \mu(G)$, where

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$\Delta(G)$ and $\mu(G)$ are the maximum degree and the maximum multiplicity (the number of parallel edges for two fixed vertices) respectively. A simple graph G is said to be class 1 or 2 according to whether $\chi'(G)$ is $\Delta(G)$ or $\Delta(G) + 1$, respectively. We will restrict our attention to graphs whose chromatic index is at most $\Delta + 1$. *Edge-critical* graphs will be our main object of study:

Definition 1.1. For a given graph G, let $G - e$ denote the graph obtained by removing an edge e; G is Δ-(*edge*)-*critical* if $\chi'(G) = \Delta + 1$ and $\chi'(G - e) = \Delta$ for any edge e.

In the literature, three small critical graphs of considerable importance appeared respectively in [9, 7] and [6]. The first graph (see the left side of Figure 10) was constructed by Goldberg as the first counterexample related to the "Critical Graph Conjecture" according to which all critical graphs should have an odd number of vertices (see [6]); such a graph had the smallest number of vertices (22) in an infinite family of graphs of even order constructed by Goldberg. The second graph – see the left side of Figure 1 – was found by Fiol as an example of critical, simple graph of smaller order, namely 18; the last graph – see the right side of the figure – is due to Chetwynd; it has order 16 but it is not simple because of one multiple edge.

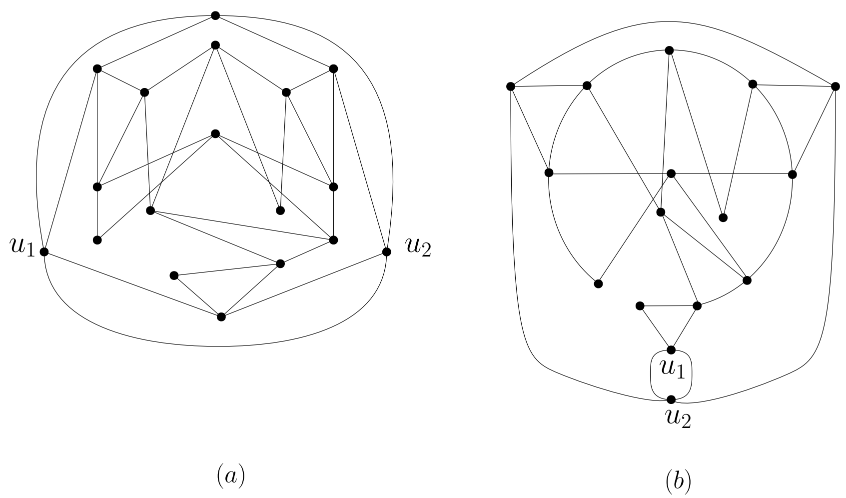

Figure 1: Two remarkable 4-critical graphs.

It is still unknown whether a simple, critical graph of order 16 exists. As to smaller orders, such a question was settled by a number of contributions over the years. In details, Jacobsen's work (see [10]) ruled out all graphs with $4, 6, 8,$ and 10 vertices; Fiorini and Wilson (see [8]) added the case 12 to the above list of non-admissible values; Bokal, Brinkmann, and Grünewald (see [2]) proved that also 14 is non-admissible.

In this paper, we push forward the analogy between non-orientable manifolds and class 2 graphs which was introduced in [11] and describe a new method for constructing critical graphs. We show the effectiveness of this method by constructing infinite families of critical simple graphs. The constructions cover all odd and even orders for 4-critical graphs, the odd order starting from 5, the even orders starting from 20, as well as all even orders for 3-critical graphs, including the orders of Goldberg's infinite family starting from 28 (the orders of Goldberg's graphs are all those numbers congruent to $8 \pmod{16}$, and the further value 22). The 3-critical graphs of even order that we construct are not isomorphic to the graphs of Goldberg's infinite family; the graphs are simple, except the 4-critical graph of order 16. According to the literature, our constructions provide in particular the first example of an infinite family of Δ-critical graphs for degree 4. The present approach is expected to yield infinite families also for larger degrees, in the next future, because the key definitions can be easily exported to the general case.

Our method allows to build up critical graphs starting from class 1 graphs with an elementary and "nice" shape (see for instance Figure 2). This is innovative with respect to well-know methods that construct Δ-critical graphs starting from critical graphs with maximum degree not exceeding Δ – see Theorem 4.6 and 4.9 in [14].

Following the mentioned approach in [11], we also show that the infinite family of Goldberg's graphs disproving the "Critical Graph Conjecture" and the other two counterexamples constructed by Fiol and Chetwynd can be obtained by a suitable identification of vertices which is pretty analogous to the topological identification yielding the Möbius strip from a rectangular strip. Details about the change of language – from topology to graph theory – can be found in [11].

Some additional terminology is required; in particular, certain distinguished vertices that play a basic role in the constructions shall be emphasised by suitable adjectives. Leaving details to the next sections, we anticipate that all the constructions will rely on particular pairs of vertices which are analogous to the extremes of a rectangular strip before the identification that leads to a Möbius strip. In our setting, any such pair will undergo a transformation which is similar to the topological identification of the extremes of the rectangular strip. The change from orientability to non-orientability, caused by the identification, is rephrased as the change from class 1 to class 2 as a consequence of the prescribed transformation.

Many standard definitions in this paper are in accordance with the textbook [3] by Bondy and Murty. As a further source, we mention the textbook [5] by Bryant. Edges like $\{u, v\}$ are simply denoted by uv. We use the term *t-colouring* if the colour set has size t. Given a vertex v of a graph G, the *palette of* v, in symbols $P_\gamma(v)$ or simply $P(v)$, is the set of colours that a colouring γ of G assigns to the edges containing v. In some cases, we will need to write γ_G so as to specify the graph we are colouring. The complementary set $\overline{P_\gamma(v)}$ or $\overline{P(v)}$ is the *complementary palette* of v with respect to the colour set of γ. If a colour is missing at a vertex v, we say that v *lacks* that colour. Finally, a vertex of degree h is an *h-vertex*.

For our purposes we also recall Vizing's Adjacency Lemma (VAL), concerning the structure of critical (simple) graphs, and the quite elementary, still very useful, Parity Lemma (PL):

Theorem 1.2 (VAL [13]). *If uv is an edge of a Δ-critical graph, then u is adjacent to at least $\Delta - \deg(v) + 1$ Δ-vertices (different from v).*

Lemma 1.3 (PL [1]). *For any colouring of a graph G, the number of vertices that lack a given colour has the same parity as $|V(G)|$.*

Although there exist several generalisations of VAL to multigraphs, for our purposes it suffices to consider the simple graph version (see the lines just above Remark 2.8).

2 Fertile pairs of vertices

As hinted in the Introduction, the constructions of critical graphs that follow can be thought of as identifications of special pairs of vertices which change the colouring class from 1 to 2. Accordingly, the first step in each construction is the choice of a suitable pair of vertices which we are going to define as *fertile pair*. There are three kinds of fertile pairs, but after a little thought all of them can be related to the same kind – as we will soon explain. Conversely, given a critical graph, we will show that it is obtained as a suitable identification of a fertile pair which collapses to a unique vertex. In this reconstruction process, it is important to note that the identification could be arbitrarily performed on every vertex, but the choice of a particular vertex is essential both for proving criticality in a comfortable way, and for generating new critical graphs using a pattern which is readily suggested by the fertile pair.

Definition 2.1. Let u, v be vertices of a graph G.

Assume that the following conditions hold:

(*) u is not adjacent to v, $\deg(u) + \deg(v) \leq \Delta$ and, for every Δ-colouring, $P(u) \cap P(v) \neq \emptyset$.

(**) For any edge e, $G - e$ admits a Δ-colouring such that $P(u) \cap P(v) = \emptyset$.

Then, u and v are said to be *conflicting*.

Assume, instead, the following:

(*) $\deg(u) = \deg(v) = \Delta - 1$ and, for every Δ-colouring, $P(u) = P(v)$.

(**) For any edge e which does not contain u nor v, $G - e$ admits a Δ-colouring such that $P(u) \neq P(v)$.

In this case, u and v are *same-lacking*.

Finally, assume the following:

(*) $\deg(u), \deg(v)$ are smaller than Δ and, for every Δ-colouring, $|P(u) \cup P(v)| = \Delta$.

(**) For any edge e, $G - e$ admits a Δ-colouring such that $|P(u) \cup P(v)| < \Delta$.

In this last case, u and v are said to be *saturating*.

In all of the three cases, we say that (u, v) is a *fertile* pair of vertices.

Remark 2.2. After the removal of e in the same-lacking case, we equivalently require that $|\overline{P(u)} \cup \overline{P(v)}| \geq 2$; this is trivial if e contains one or both vertices u, v. Furthermore, notice that in the saturating case condition $|P(u) \cup P(v)| = \Delta$ is equivalent to $\overline{P(u)} \cap \overline{P(v)} = \emptyset$.

The following lemma is the basic link between topology and graph theory in the present context, and should be considered the starting point for all the next constructions.

Lemma 2.3. *Let (u, v) be a fertile pair of a graph G having $\chi'(G) = \Delta \geq 2$. For each of the following cases, the corresponding operation yields a Δ-critical graph.*

(i) If u and v are non-adjacent and conflicting, identify u and v.

(ii) If u and v are same-lacking, add a new vertex w and edges uw, vw.

(iii) If u and v are saturating, add the edge uv.

Proof. If we identify a pair of conflicting vertices, we obtain a graph G' having maximum degree Δ and no proper Δ-coloring, since the palettes of two conflicting vertices share at least one color; hence G' is class 2. By definition 2.1, if we remove any edge e from G', we find at lest one Δ-coloring of $G' - e$ such that the two conflicting vertices have disjoint palettes with respect to it; therefore, G' is Δ-critical. The same-lacking and saturating cases can be managed analogously. □

Notice that adding two pendant edges uw, vw' when u and v are same-lacking yields conflicting 1-vertices w, w'. Similarly, adding one pendant edge uw when u and v are saturating yields conflicting vertices w, v. Therefore, the above operations can be regarded as identifications of conflicting vertices in all cases. These procedures could be rephrased in terms of atlases and orientability, as explained in [11]; the prototype of this analogy is given by the odd cycle C_{2n+1} of any fixed length. Such a graph is the result of the identification of two conflicting vertices, namely, the extremes of the path P_{2n+2} having the same number of edges. The path is "orientable" (i.e. 2-colourable) but the identification of conflicting vertices increases the chromatic index and compromises orientability. More precisely, the orientation of P_{2n+2} starts from a "local chart" (a colouring of the 2-star containing a non-extremal vertex v), and the local chart is subsequently extended so as to cover as many edges as possible. In the case of the path, we succeed in covering all the graph (so we have a "global atlas", that is, a global 2-colouring) whereas the cycle does not allow for a global 2-colouring because one edge must be excluded (the atlas cannot be extended to the whole graph). Notice that the hypothesis (**) for conflicting vertices is crucial to prove criticality.

Remark 2.4. The 4-critical graphs in Figure 1 can be obtained in the way described in Lemma 2.3, by considering the graphs G_{17}, G_{19} in Figure 8(b), 9(a), respectively, and identifying the vertices v, v'. Such vertices are conflicting, as we will show in Section 3.

Here follow some examples as a first step towards the main theorems.

Example 2.5. Let us show that the graph G_5 in Figure 2(a) has saturating vertices u_i, u_j, with $1 \leq i < j \leq 4$. For every 4-colouring the number of vertices that lack a fixed colour is odd, according to PL, whence every 3-vertex lacks a different colour; on the other hand, one can easily verify that the removal of any edge allows for a 4-colouring such that $|P(u_i) \cup P(u_j)| = 3$ for any pair of 3-vertices.

Example 2.6. The graphs G_7 and G_9 in Figure 2(c) – (d) have saturating vertices u_1, u_2, because PL implies that these vertices have disjoint palettes for any 4-colouring, and it remains to make routine checks after the removal of any arbitrary edge.

Example 2.7. The graph G_6 in Figure 2(b) has same-lacking vertices v_1, v_2, because PL forces the palettes to be equal and this is no longer true if we remove any edge not containing one or both vertices v_1, v_2.

Notice that graphs with same-lacking vertices can be replicated so as to form a chain along which a color is "transmitted". Such a transmission of colour is a fundamental concept in this paper and will be described more thoroughly in the next section.

In the following remark, we consider critical graphs having at least three vertices of maximum degree. VAL implies that this property holds for every simple graph, but in the

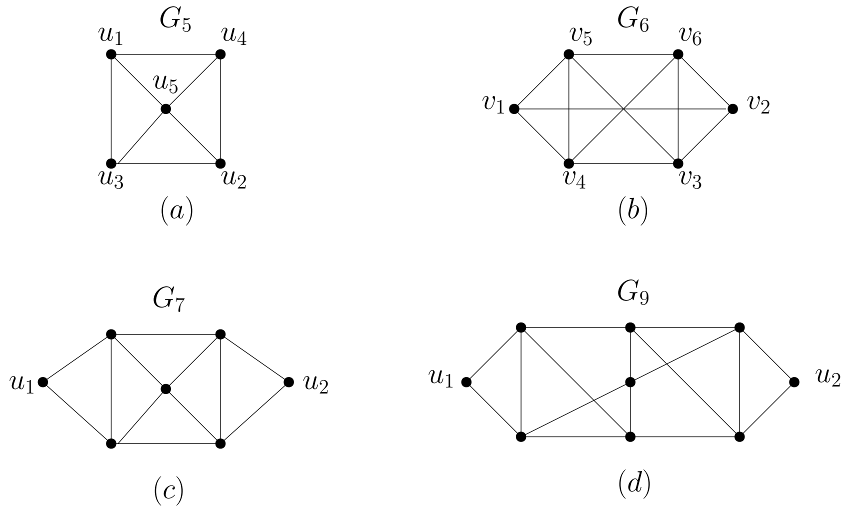

Figure 2: Fertile pairs of vertices: u_1 and u_2 are saturating, v_1 and v_2 are same-lacking.

presence of multiple edges the number of vertices of maximum degree might be smaller than 3. For instance, the complete graph K_3 with $\Delta - 1$ parallel edges connecting two fixed vertices is Δ-critical and has only two vertices of maximum degree.

Remark 2.8. Let G be a Δ-critical graph having at least three vertices of maximum degree. Let u, v be adjacent vertices that are connected by h parallel edges (possibly $h = 1$). After deleting one of the parallel edges, u and v become saturating and the degree remains equal to Δ.

According to the above remark, Chetwynd's 4-critical graph can also be obtained by inserting an additional edge between the saturating vertices u_1, u_2.

3 Construction of graphs with fertile pairs

Graphs with fertile pairs of vertices can be obtained in several ways from smaller graphs with the same property. The methods we present here will be applied to prove the main theorems.

Lemma 3.1. *Let H_1 and H_2 be vertex-disjoint graphs of degree $\Delta \geq 2$ and such that $\chi'(H_1) = \chi'(H_2) = \Delta$. Assume that v_1, v_2 are same-lacking in H_1 and u_1, u_2 are same-lacking (resp. saturating) in H_2. The graph H obtained from H_1 and H_2 by adding the edge u_2v_2 has again maximum degree Δ, chromatic index Δ, and has same-lacking (resp. saturating) vertices u_1, v_1.*

Proof. Let us analyse the same-lacking case. A colouring of H can be obtained by assuming that u_2 and v_2 lack the same colour in two given Δ-colourings of H_1 and H_2; by the hypothesis, u_1 and v_1 lack that colour. If we now remove any edge, say in H_1, u_2v_2 can be coloured with a colour which is present at u_1. Such a colour is instead missing at v_1. A similar argument applies to the saturating case. □

Example 3.2. We consider two copies of G_6 – see Figure 2(b) – as the graphs H_1 and H_2. We can actually iterate the gluing process m times, $m \geq 1$, so as to obtain a graph of order $6m$, of maximum degree 4, whose 3-vertices are still fertile (same-lacking). Let us denote this graph by G_6^m – see Figure 3. This graph will play a basic role in the proofs of Theorem 5.1 and 5.2.

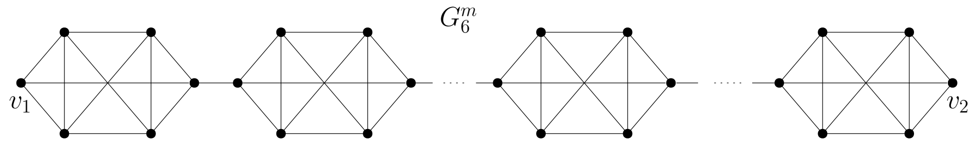

Figure 3: The graph G_6^m in Example 3.2 is a concatenation of graphs with same-lacking pairs.

The purpose of the next couple of definitions is twofold. On one hand, they allow to recover Chetwynd and Fiol's counterexamples in the light of our approach via transmission of colours along the edges of a graph. On the other hand, they play an important role in the construction of critical graphs of even order that will follow in the next pages. These definitions involve graphs with maximum degree 4, although they can be extended to graphs with $\Delta > 4$.

Before providing the definitions, some further observations are in order. What we refer to as *transmitting* vertices should be regarded as terminal nodes which lend themselves to being connected to other graphs so as to yield a global graph with conflicting vertices and, eventually, a critical graph. The fundamental property of 2- or 3-colour transmitting vertices concerns the complementary palettes, that is, the colours actually missing at each vertex. For, the missing colours can be seen as the admissible colours of any edge which is added to the graph and contains that vertex. In the two definitions, it is the interplay between the colours missing at each distinguished vertex to ensure that the connecting edges, when added, will transmit some prescribed colour across the whole graph, and will eventually increase the chromatic index. Indeed, the vertices we are going to introduce are the first step towards the construction of graphs with conflicting vertices (see Propositions 3.8 and 3.12).

Let $S \ominus T$ denote the symmetric difference between the sets S and T.

Definition 3.3. Let G be a graph having $\chi'(G) = \Delta = 4$, and u, v, u_1, u_2 be distinct vertices of G, where $\deg(u) = \deg(v) = 2$, $\deg(u_1) = \deg(u_2) = 3$. We say that G is 3-*colour transmitting with respect to* u, v, u_1, u_2 if the following conditions hold:

(1) there exists a 4-colouring such that u_1 and u_2 lack distinct colours A and B, exactly one colour is missing simultaneously in u, v and this colour is either A or B;

(2) for every 4-colouring such that u_1 and u_2 lack distinct colours A and B, $|\{A, B\} \cup (\overline{P(u)} \ominus \overline{P(v)})| \neq 3$ (in particular, in the colouring in (1) the two other colours missing at u and v are different from A and B);

(3) for every edge e there exists a 4-colouring of $G-e$ with colours A, B, C, D satisfying $A \in \overline{P(u_1)}$, $B \in \overline{P(u_2)}$, $C \in \overline{P(u)} \cap \overline{P(v)}$ and the set $\{A, D\}$ or $\{B, D\}$ is contained in $\overline{P(u)} \ominus \overline{P(v)}$.

If we slightly alter the above definition by setting $u_1 = u_2$ and $\deg(u_1) = 2$, the resulting graph is said 3-*colour transmitting with respect to* u, v, u_1. In this case, the first requirement in (1) and (2) clearly becomes "u_1 lacks colours A and B", in symbols $A, B \in \overline{P(u_1)}$.

Definition 3.4. Let G be a graph of maximum degree $\Delta = 4$ and $\chi'(G) = 4$. Let w, w_1, w_2 be distinct vertices of G, where $\deg(w) = 2$, $\deg(w_1) = \deg(w_2) = 3$. We say that G is 2-*colour transmitting with respect to* w, w_1, w_2, if the following conditions hold:

(1) for every 4-colouring of G the set $|\overline{P(w_1)} \cup \overline{P(w_2)}|$ contains exactly two colours and coincides with $\overline{P(w)}$;

(2) for every edge e there exists a 4-colouring of $G - e$ with colours A, B, C such that $A \in \overline{P(w_1)}$, $B \in \overline{P(w_2)}$ and $\overline{P(w)}$ contains $\{A, C\}$ or $\{B, C\}$.

Similarly as above, if the vertices w_1, w_2 coincide and $\deg(w_1) = 2$, we say that the graph is 2-*colour transmitting with respect to* w, w_1; the requirement in condition (2) becomes "w_1 lacks colours A and B".

Example 3.5. The graph G_{12} in Figure 4(a) is 3-colour transmitting with respect to u, v, u_1, u_2, as we are going to explain by testing the conditions of Definition 3.3. Condition (1) holds as shown in Figure 4(a). Condition (3) can be checked by setting: $P(u) \subseteq \{2, 3\}$, $P(v) \subseteq \{2, 4\}$, and $P(z_1) \subseteq \{1, 4\}$. In the graph $G_{12} - e$, the palettes of the vertices u_1, u_2 take the following values: $P(u_1) \subseteq \{1, 2, 3\}$ and $P(u_2) \subseteq \{1, 3, 4\}$; $P(u_1) \subseteq \{2, 3, 4\}$ and $P(u_2) \subseteq \{1, 2, 3\}$; $P(u_1) \subseteq \{2, 3, 4\}$ and $P(u_2) \subseteq \{1, 2, 4\}$. Notice that $P(u) \subseteq \{2, 3\}$, $P(v) \subseteq \{2, 4\}$ mean that $1 \in \overline{P(u)} \cap \overline{P(v)}$ and $\{3, 4\} \subseteq \overline{P(u)} \ominus \overline{P(v)}$, that is, colour 1 corresponds to colour C in Condition (3) and $\{3, 4\}$ corresponds to one of the sets $\{A, D\}$ or $\{B, D\}$, where $A \in \overline{P(u_1)}$, $B \in \overline{P(u_2)}$. Thus, for instance, if $P(u_1) \subseteq \{1, 2, 3\}$ and $P(u_2) \subseteq \{1, 3, 4\}$, then $A = 4$, $B = 2$ and $D = 3$.

It remains to prove Condition (2). By PL, the number of vertices that lack a given colour is even, and there are 6 vertices of degree smaller than 4. However, a color missing in all these vertices would make the two palettes of degree 3 equal, which is not allowed by assumption. Now let us partition the $2 \cdot 3 + 4 \cdot 2$ colours on the above 6 vertices either as $2 + 2 + 4 + 6$ or as $2 + 4 + 4 + 4$, where each part counts the occurrences of a fixed colour (0 is missing, by the above discussion). Up to permutations of colours there are two colourings of the first type and three of the second type (in the table, palettes of size 4 are not present and we assume that palettes of size 3 are the same in all cases):

$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$
$\{1,2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$
$\{1,2\}$	$\{1,3\}$	$\{1,2\}$	$\{1,4\}$	$\{1,3\}$
$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{1,4\}$	$\{1,4\}$
$\{1,3\}$	$\{1,3\}$	$\{2,4\}$	$\{2,3\}$	$\{3,4\}$
$\{1,4\}$	$\{1,4\}$	$\{3,4\}$	$\{2,4\}$	$\{3,4\}$

Whatever the assignments of palettes to the 2-vertices, column 2 and column 4 satisfy (2). For the colouring γ_1 in the 1st column, condition $|\overline{P(u_1)} \cup \overline{P(u_2)} \cup (\overline{P(u)} \ominus \overline{P(v)})| \neq 3$ is not satisfied if we choose $\{P(u), P(v)\} = \{\{1, 2\}, \{1, 3\}\}$ or $\{P(u), P(v)\} = \{\{1, 2\}, \{1, 4\}\}$. The permutation of colours 3 and 4 leaves γ_1 invariant and switches the sets $\{\{1, 2\}, \{1, 3\}\}, \{\{1, 2\}, \{1, 4\}\}$. Therefore, in order to show that Condition (2) is satisfied for the colouring γ_1, it suffices to show that the graph G_{12} cannot be coloured according to γ_1 by setting $\{P(u), P(v)\} = \{\{1,2\},\{1,3\}\}$.

Suppose, on the contrary, that G_{12} can be coloured according to γ_1 by setting $\{P(u), P(v)\} = \{\{1,2\},\{1,3\}\}$. The set of palettes of γ_1 shows that colour 1 induces a perfect matching of the graph G_{12}. As shown in Figure 5, there are exactly four perfect matchings of G_{12}. By the symmetry of the graph and by the fact that the sets $\{\{1,2\},\{1,3\}\}$, $\{\{1,2\},\{1,4\}\}$ can be obtained one from the other by a permutation of colours 3 and 4, we can consider the first two perfect matchings of Figure 5. The set of palettes of γ_1 also shows that colour 2 induces a matching of cardinality 5, where exactly one of the vertices u, v (respectively, z_1, z_2) is unmatched since we are supposing $\{P(u), P(v)\} = \{\{1,2\},\{1,3\}\}$ and $\{P(z_1), P(z_2)\} = \{\{1,2\},\{1,4\}\}$. Figure 6 shows how to colour the edges of G_{12} with 1 and 2. In each of the four cases represented in Figure 6, one can see that is not possible to colour to edges of G_{12} according to the colouring γ_1 by setting $\{P(u), P(v)\} = \{\{1,2\},\{1,3\}\}$. Therefore, if G_{12} can be coloured by γ_1, then γ_1 satisfies Condition (2). The same can be repeated for the remaining colourings in the 3rd and 5th column. It is thus proved that every 4-colouring of G_{12} with $|\overline{P(u_1)} \ominus \overline{P(u_2)}| = 2$ satisfies Condition (2).

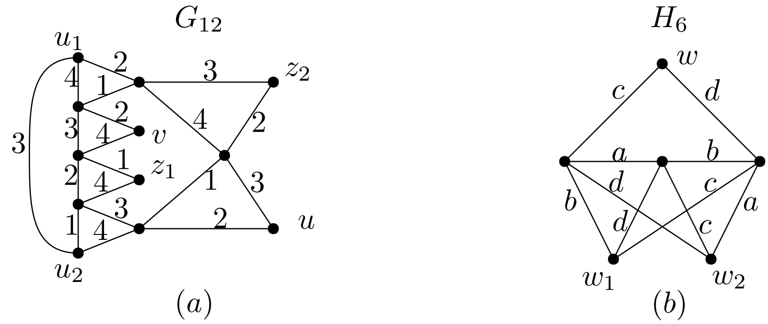

Figure 4: (a): A 4-colouring of the graph G_{12} in Example 3.5 that satisfies Conditions (1) and (2) of Definition 3.3. (b): A 4-colouring of the graph H_6 in Example 3.7.

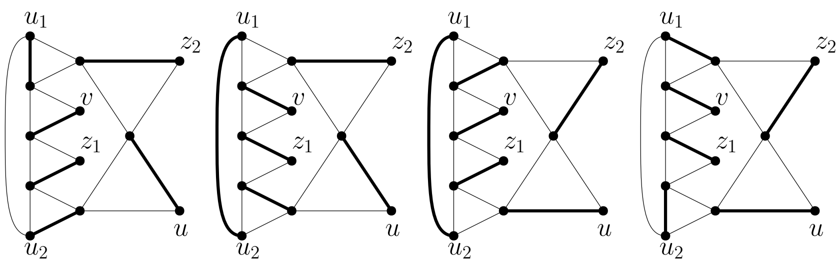

Figure 5: Perfect matchings of the graph G_{12} that are considered in Example 3.5.

There are several methods for obtaining a 3-colour transmitting graph starting from a smaller one. For instance, in the graph G_{12} of Figure 4(a), we can delete the edge u_1u_2 and connect the remaining graph to the graph G_6^m in Figure 3 by adding the edges u_1v_1, u_2v_2. The resulting graph is 3-colour transmitting with respect to u, v, u_1, u_2. In the next example, we show a more elaborate method for obtaining a 3-colour transmitting graph starting

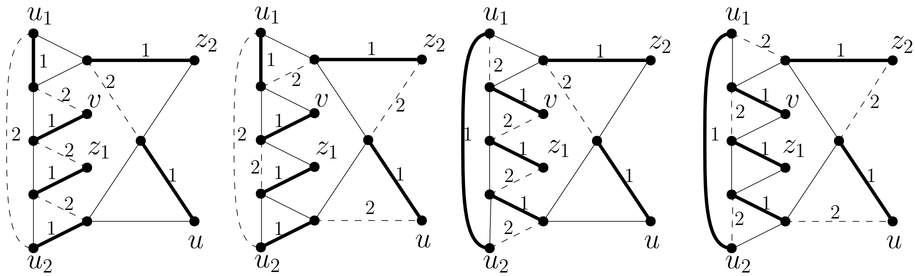

Figure 6: The edges of the graph G_{12} are coloured according to the palettes $\{1,2,3,4\},\{1,2,3\},\{1,2,4\},\{1,2\},\{1,2\},\{1,3\},\{1,4\}$ by setting $\{P(u),P(v)\} = \{\{1,2\},\{1,3\}\}$ and $\{P(z_1),P(z_2)\} = \{\{1,2\},\{1,4\}\}$; colour 1 induces a perfect matching, colour 2 induces a matching of cardinality 5, where exactly one of the vertices u,v (respectively, z_1,z_2) is unmatched (see Example 3.5).

from a smaller one. This method allows to find a graph that will be used to construct Fiol's 4-critical graph of order 18.

Example 3.6. Consider the graph N in Figure 7(a). Notice that $\overline{P(w)} = P(w_1) \ominus P(w_2)$ for every 4-colouring of the graph N, as a straightforward consequence of PL. We denote by L the graph obtained from G_{12} in Figure 4 by deleting the edge u_1u_2. Let G_{16} be the graph resulting from the identification of the vertices $w_1 \in V(N)$ with $u_1 \in V(L)$ and of $w_2 \in V(N)$ with $u_2 \in V(L)$. We have that $\chi'(L) = \Delta = 4$ (see the colouring in Figure 7(b)).

Let us show that G_{16} is 3-colour transmitting with respect to u,v,w by testing Definition 3.3 with $u_1 = u_2$. Condition (1) follows from the colouring in Figure 7(b).

Condition (2) is satisfied if every 4-coloring of G_{16} satisfies the relation $|\overline{P(w)} \cup (\overline{P(u)} \ominus \overline{P(v)})| \neq 3$. Suppose that there exists a 4-colouring γ of G_{16} such that $|\overline{P_\gamma(w)} \cup (\overline{P_\gamma(u)} \ominus \overline{P_\gamma(v)})| = 3$, that is, $\overline{P_\gamma(w)} = \{A,B\}$, $\overline{P_\gamma(u)} \ominus \overline{P_\gamma(v)} = \{A,C\}$ or $\{B,C\}$. The colouring γ induces a colouring γ' of G_{12} such that $\overline{P_{\gamma'}(u_1)} \ominus \overline{P_{\gamma'}(u_2)} = \{A,B\}$ and $\overline{P_{\gamma'}(u)} \ominus \overline{P_{\gamma'}(v)} = \{A,C\}$ or $\{B,C\}$, that is, γ' does not satisfies Condition (2) of Definition 3.3. That yields a contradiction, since G_{12} is 3-colour transmitting with respect to u,v,u_1,u_2.

Condition (3) holds if for every edge $e \in E(G_{16})$ there exists a 4-colouring of $G_{16} - e$ such that $\{A,B\} \subseteq \overline{P(w)}$, $C \in \overline{P(u)} \cap \overline{P(v)}$ and $\{A,D\} \subseteq \overline{P(u)} \ominus \overline{P(v)}$ where A,B,D are distinct. Assume $e \in E(G_{12})$. Since G_{12} is 3-colour transmitting with respect to u,v,u_1,u_2, there exists a suitable colouring which can be easily extended to the whole graph G_{16}.

If $e \in E(N)$, we colour the edges of G_{16} belonging to G_{12} by the 4-colouring in Figure 4(a), so that $P(u) = \{2,3\}$ and $P(v) = \{2,4\}$. One can verify that the edges of $N - e$ can be coloured in such a way that $P(w) \subseteq \{2,4\}$. Therefore, $\{1,3\} \subseteq \overline{P(w)}$, $1 \in \overline{P(u)} \cap \overline{P(v)}$ and $\{3,4\} \subseteq \overline{P(u)} \ominus \overline{P(v)}$, that is, Condition (3) is satisfied if $e \in E(N)$.

Example 3.7. The graph H_6 in Figure 4(b) is 2-colour transmitting with respect to w,w_1, w_2. The conditions of Definition 3.4 are satisfied: Condition (1) follows from Parity Lemma; Condition (2) can be verified by coluring the edges with A,B,C,D and setting $P(w_1) \subseteq \{B,C,D\}, P(w_2) \subseteq \{A,C,D\}, P(w) \subseteq \{A,D\}$.

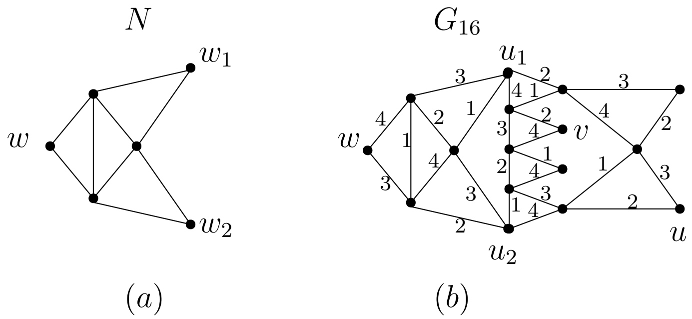

Figure 7: (a): The graph N. (b): A 4-colouring of the graph G_{16} that satisfies Conditions (1) and (2) of Definition 3.3; as proved in Example 3.6, the graph G_{16} is 3-colour transmitting with respect to u, v, w.

Definitions 3.3 and 3.4 are used to construct graphs having fertile vertices. The next result is a construction of graphs having fertile vertices and whose maximum degree Δ is 4. The construction can be extended to graphs whose maximum degree is larger than 4 and having multiple edges. In this context, we limit ourselves to consider $\Delta = 4$.

We recall that a *bowtie* is the graph obtained by identifying two vertices belonging to two distinct 3-cycles, thus obtaining a *centre* of degree 4 and four 2-vertices. If the 3-cycle are (x, y_1, y_2) and (x', y_1', y_2'), then we denote by $B(x, y_1, y_2, y_1', y_2')$ the bowtie resulting from the identification of the vertices x and x'.

Proposition 3.8. *Let $\mathbb{B} = B(x, u', v', w, y)$ be a bowtie with centre x and 2-vertices u', v', w, y. Let K and M be graphs of maximum degree 4 and $\chi'(K) = \chi'(M) = 4$, with the following features. The graph K is 3-colour transmitting with respect to u, v, u_1, u_2, where $\deg_K(u) = \deg_K(v) = 2, \deg_K(u_1) = \deg_K(u_2) = 3$; either M is 2-colour transmitting with respect to w, w_1, w_2, where $\deg_M(w) = 2, \deg_M(w_1) = \deg_M(w_2) = 3$, or M is 2-colour transmitting with respect to w, w_1, where $\deg_M(w) = \deg_M(w_1) = 2$.*

Let H be the graph obtained from $\mathbb{B}, K$ and M by identifying the vertices u' with u, w' with w and by adding the edges u_1w_1, u_2w_2 or u_1w_1, u_2w_1 according to whether M is 2-colour transmitting with respect to w, w_1, w_2 or with respect to w, w_1, respectively. The graph H has maximum degree 4, $\chi'(H) = 4$ and the vertices v, v' are conflicting.

Proof. We identify the edge u_2w_2 with the edge u_2w_1 if $w_1 = w_2$, that is, if M is 2-colour transmitting with respect to w, w_1. Since the identification of the vertices u, u' and w, w' does not increase the maximum degree of K, M and of the bowtie, the maximum degree of H is still 4. We show that $\chi'(H) = 4$. By Condition (1) of Definition 3.4, there exists a 4-colouring γ_M^* such that w_1, w_2 lack distinct colours A, B and these colours are missing in w (if $w_1 = w_2$, then w_1 lacks both colours A, B). By Condition (1) and (2) of Definition 3.3, there exists a 4-colouring γ_K^* such that u_1, u_2 lack distinct colours A, B and exactly one of these two colours, say A, is missing simultaneously in u and v; the other two missing colours are different from B, that is, $\overline{P_{\gamma_K^*}(u)} = \{A, C\}$ $\overline{P_{\gamma_K^*}(v)} = \{A, D\}$. We define a 4-colouring γ^* of H such that the restriction of γ^* to the edges of M (respectively, of K) coincides with γ_M^* (respectively, with γ_K^*); the edges of the bowtie and u_1w_1, u_2w_2 are coloured as follows: $\{\gamma^*(u_1w_1), \gamma^*(u_2w_2)\} = \{A, B\}$; $\gamma^*(wx) = A$; $\gamma^*(wy) = B$; $\gamma^*(ux) = C$; $\gamma^*(uv') = A$; $\gamma(v'x) = B$; and $\gamma^*(xy) = D$. In conclusion $\chi'(H) = 4$.

We prove that the vertices $v, v' \in V(H)$ are conflicting. Firstly, we show that for every 4-colouring of H, the palettes of v and v' share at least one colour. Suppose, on the contrary, that there exists a 4-colouring γ_1 of H such that v and v' have disjoint palettes. The restriction of γ_1 to the edges of K (respectively, of M) is a 4-colouring γ_K (respectively, γ_M). The following relations hold: $\overline{P_{\gamma_K}(u_1)} = \overline{P_{\gamma_M}(w_1)} = \gamma_1(u_1w_1) = A$; $\overline{P_{\gamma_K}(u_2)} = \overline{P_{\gamma_M}(w_2)} = \gamma_1(u_2w_2) = B$ (if $w_1 = w_2$ then $A \neq B$ and $\overline{P_{\gamma_M}(w_1)} = \{A, B\}$). Moreover, $\overline{P_{\gamma_K}(v)} = \overline{P_{\gamma_1}(v)} = P_{\gamma_1}(v') = \{\gamma_1(uv'), \gamma_1(v'x)\}$ since we are supposing that v and v' have disjoint palettes with respect to γ_1. Therefore $\overline{P_{\gamma_K}(u)} \ominus \overline{P_{\gamma_K}(v)} = \{\gamma_1(uv'), \gamma_1(ux)\} \ominus \{\gamma_1(uv'), \gamma_1(v'x)\} = \{\gamma_1(ux), \gamma_1(v'x)\}$. By Condition (1) of Definition 3.4, the colours A, B are distinct and $\overline{P_{\gamma_M}(w)} = \{A, B\}$. It follows that $\{\gamma_1(wx), \gamma_1(wy)\} = \{A, B\}$ and $\gamma_1(xy) \neq A, B, \gamma_1(ux), \gamma_1(v'x)$. Therefore, exactly one of the colours $\gamma_1(ux), \gamma_1(v'x)$ is in $\{A, B\}$. Consequently, the set $\overline{P_{\gamma_K}(u)} \ominus \overline{P_{\gamma_K}(v)} = \{\gamma_1(ux), \gamma_1(v'x)\}$ contains exactly one of the colours A, B. It follows that $|\overline{P_{\gamma_K}(u_1)} \cup \overline{P_{\gamma_K}(u_2)} \cup (\overline{P_{\gamma_K}(u)} \ominus \overline{P_{\gamma_K}(v)})| = 3$, a contradiction since K is 3-colour transmitting with respect to u, v, u_1, v_1. Hence, for every 4-colouring of H the palettes of the vertices v, v' share at least one colour.

We show that for every edge $e \in E(H)$ there exists a 4-colouring γ' of $H - e$ such that v and v' have disjoint palettes. We distinguish the cases: $e \in E(K)$; $e \in E(M)$; $e \in E(\mathbb{B})$; and $e \in \{u_1w_1, u_2w_2\}$.

Case $e \in E(K)$.

By Condition (3) of Definition 3.3, there exists a 4-colouring $\tilde{\gamma}$ of $K - e$ such that $A \in \overline{P_{\tilde{\gamma}}(u_1)}$, $B \in \overline{P_{\tilde{\gamma}}(u_2)}$, $C \in \overline{P_{\tilde{\gamma}}(u)} \cap \overline{P_{\tilde{\gamma}}(v)}$, and the set $\{A, D\}$ or $\{B, D\}$ is contained in $\overline{P_{\tilde{\gamma}}(u)} \ominus \overline{P_{\tilde{\gamma}}(v)}$, where A, B, D are distinct. Without loss of generality, we can assume $\{A, D\} \subseteq \overline{P_{\tilde{\gamma}}(u)} \ominus \overline{P_{\tilde{\gamma}}(v)}$. Now $\{A, D\}$ can be contained in exactly one of the complementary palettes $\overline{P_{\tilde{\gamma}}(u)}, \overline{P_{\tilde{\gamma}}(v)}$ or in neither of them. The first case occurs only if e contains exactly one of the vertices u, v, and in this case $\{\overline{P_{\tilde{\gamma}}(u)}, \overline{P_{\tilde{\gamma}}(v)}\} = \{\{A, D, C\}, \{B, C\}\}$. If, instead, e does not contain u, v, then $\{\overline{P_{\tilde{\gamma}}(u)}, \overline{P_{\tilde{\gamma}}(v)}\} = \{\{A, C\}, \{D, C\}\}$.

We colour the edges of M according to an arbitrary 4-colouring γ_M of the graph M. By a permutation of the colours and by Condition (1) of Definition 3.4, we can assume that the colours A, B are missing in w and w_1, w_2 lack A, B, respectively (if $w_1 = w_2$, then w_1 lacks both colours A, B). We define a 4-colouring γ' of $H - e$ such that the restriction of γ' to $K - e$ (respectively, to M) corresponds to the colouring $\tilde{\gamma}$ (respectively, γ_M) and $\gamma'(u_1w_1) = A$; $\gamma'(u_2w_2) = B$; $\gamma'(uv') = C$; $\gamma'(xy) = C$. The colouring of the edges ux, $v'x$, wx, wy depends on the set $\{\overline{P_{\gamma'}(u)}, \overline{P_{\gamma'}(v)}\}$. If $\{\overline{P_{\gamma'}(u)}, \overline{P_{\gamma'}(v)}\} = \{\{A, C\}, \{D, C\}\}$, then we set $\gamma'(wx) = B$, $\gamma'(wy) = A$ and the edges $ux, v'x$ are coloured by A, D or D, A, respectively, according to whether $\overline{P_{\gamma'}(u)} = \{A, C\}$ or $\overline{P_{\gamma'}(u)} = \{D, C\}$, respectively. If $\{\overline{P_{\gamma'}(u)}, \overline{P_{\gamma'}(v)}\} = \{\{A, D, C\}, \{B, C\}\}$, then we set $\gamma'(wx) = A$, $\gamma'(wy) = B$ and the edges $ux, v'x$ are coloured by D, B or B, D, respectively, according to whether $\overline{P_{\gamma'}(u)} = \{A, D, C\}$ or $\overline{P_{\gamma'}(u)} = \{B, C\}$, respectively. Notice that $P_{\gamma'}(v') \subseteq \overline{P_{\gamma'}(v)}$, hence v, v' have disjoint palettes with respect to γ'.

Case $e \in E(M)$.

We define a 4-colouring γ' of $H - e$ such that the edges of K are coloured according to the 4-colouring γ_K^* of K defined at the beginning of the proof. We have that $\overline{P_{\gamma'}(u)} = \overline{P_{\gamma_K^*}(u)} = \{A, C\}$, $\overline{P_{\gamma'}(v)} = \overline{P_{\gamma_K^*}(v)} = \{A, D\}$. Since u_1, u_2 lack distinct colours A, B

with respect to γ_K^*, we can assume that u_1 lacks A and u_2 lacks B.

By Condition (2) of Definition 3.4, we can colour the edges of $M - e$ according to the 4-colouring γ_M' of M such that the vertices w_1, w_2 lack distinct colours, say A, B, and the colours A, C are missing in w, where A, B, C are distinct (if $w_1 = w_2$, then w_1 lacks both colours A, B). The remaining edges of $H - e$ are coloured as follows: $\gamma'(u_1w_1) = A$; $\gamma'(u_2w_2) = B$; $\gamma'(uv') = A$; $\gamma'(ux) = C$; $\gamma'(v'x) = D$; $\gamma'(wx) = A$; $\gamma'(wy) = C$; and $\gamma'(xy) = B$. The vertices v, v' have disjoint palettes with respect to γ', since $P_{\gamma'}(v') = \overline{P_{\gamma'}(v)} = \{A, D\}$.

Case $e \in E(\mathbb{B})$.

We define a 4-colouring γ' of $H - e$ that corresponds to the 4-colouring γ^* of H defined at the beginning of the proof, except on the remaining edges of $\mathbb{B} - e$. The edges of $\mathbb{B} - e$ are coloured in such a way that $P_{\gamma'}(v') \subseteq \{A, D\}$, $P_{\gamma'}(u) \subseteq \{A, C\}$ and $\{\gamma'(wx), \gamma'(wy)\} \subseteq \{A, B\}$. The vertices v, v' have disjoint palettes with respect to γ', since $P_{\gamma'}(v') \subseteq \overline{P_{\gamma'}(v)} = \{A, D\}$.

Case $e \in \{u_1w_1, u_2w_2\}$.

We define a 4-colouring γ' of $H - e$ which coincides with γ_K^* on the subgraph K. So we have that $\overline{P_{\gamma'}(u)} = \overline{P_{\gamma_K^*}(u)} = \{A, C\}$, $\overline{P_{\gamma'}(v)} = \overline{P_{\gamma_K^*}(v)} = \{A, D\}$ and $\{\gamma_K^*(u_1w_1), \gamma_K^*(u_2w_2)\} = \{A, B\}$. Without loss of generality, we can assume that the edge e that has been removed is coloured with A. By Condition (1) of Definition 3.4, we can colour the edges of M in such a way that w_1, w_2 lack two distinct colurs, say B, C, and these two colours are missing in w. The edges of $\mathbb{B}$ are coloured as follows: $\gamma'(uv') = A$; $\gamma'(ux) = C$; $\gamma'(v'x) = D$; $\gamma'(wx) = B$; $\gamma'(wy) = C$; and $\gamma'(xy) = A$. The vertices v, v' have disjoint palettes with respect to γ', since $P_{\gamma'}(v') = \overline{P_{\gamma'}(v)} = \{A, D\}$. $\square$

Remark 3.9. The argument of the above proof is still valid if we assume that K is 3-colour transmitting with respect to u, v, u_1, where u, v, u_1 have degree 2 in K.

Example 3.10. We apply Proposition 3.8 to the graphs $K = G_{12}$ and $M = H_6$ in Figure 4. As remarked in Example 3.5, the graph G_{12} is 3-colour transmitting with respect to u, v, u_1, u_2. Similarly, in Example 3.7 we have seen that H_6 is 2-colour transmitting with respect to w, w_1, w_2. By Proposition 3.8, we obtain the graph G_{21} in Figure 8(a). The graph G_{21} has order 21, maximum degree 4, and $\chi'(G_{21}) = 4$. The vertices $v, v' \in V(G_{21})$ are conflicting. Following the proof of Proposition 3.8 we can colour the edges of G_{21} according to the 4-colourings γ_K^* and γ_M^* in Figure 4 by setting $a = 1$, $b = 2$, $c = 3$ and $d = 4$ (or $c = 4$ and $d = 3$). This graph will be used in the proof of Theorem 5.2.

Example 3.11 (Chetwynd's counterexample). We can apply Proposition 3.8 to the graph $K = G_{12}$ in Figure 4(a) and to the dipole $M = D_2$ with two parallel edges even thought the dipole D_2 is not 2-colour transmitting with respect to its vertices. More precisely, as remarked in Example 3.5, the graph G_{12} is 3-colour transmitting with respect to u, v, u_1, u_2. It is easy to see that every 4-colouring of the graph D_2 satisfies conditions (1) and (2) of Definition 3.4 with $w_1 = w_2$. Therefore, we can repeat the proof of Proposition 3.8 and obtain the graph G_{17} in Figure 8(b) having order 17, maximum degree 4 and $\chi'(G_{17}) = 4$. The vertices $v, v' \in V(G_{17})$ are conflicting. By Lemma 2.3, the identification of the vertices v, v' yields a 4-critical graph, namely, Chetwynd's 4-critical graph in Figure 1(b).

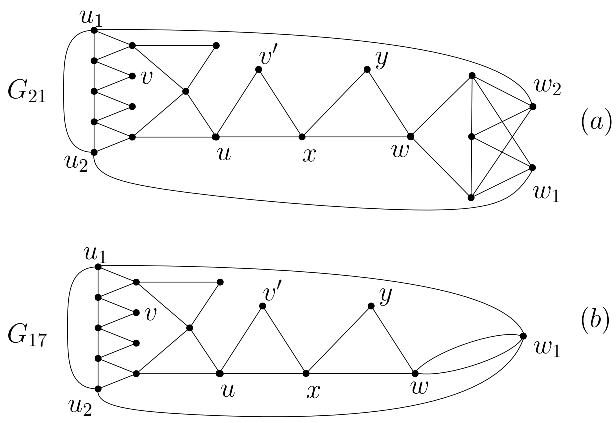

Figure 8: (a): The graph G_{21} constructed in Example 3.10. (b): The graph G_{17} constructed in Example 3.11.

Proposition 3.12. *Let $\mathbb{B} = B(x, u', v', w, y)$ be a bowtie with centre x and 2-vertices u', v', w, y. Let K and M be graphs of maximum degree 4 and $\chi'(K) = \chi'(M) = 4$ with the following features. The graph K is 3-colour transmitting with respect to u, v, u_1 where $\deg_K(u) = \deg_K(v) = \deg_K(u_1) = 2$. The 2-vertices $w, w_1 \in V(M)$ are saturating and for every $e \in E(M)$ not containing w nor w_1 there exists a 4-colouring of $M - e$ such that w, w_1 lack exactly one colour simultaneously.*

Let H be the graph obtained from $\mathbb{B}$, K and M by identifying the vertices u' with u; w' with w; and u_1 with w_1. The graph H has maximum degree 4, $\chi'(H) = 4$ and the vertices v, v' are conflicting.

Proof. The argument is the same as in the proof of Proposition 3.8. It is different in the case $e \in E(M)$. We show that if we remove an edge $e \in E(M)$, then there exists a 4-colouring γ' of $H - e$ such that v, v' have disjoint palettes with respect to it. As in the proof of Proposition 3.8, the restriction of γ' to the edges of K corresponds to a 4-colouring γ_K^* of K such that $\overline{P_{\gamma_K^*}(u_1)} = \{A, B\}$, $\overline{P_{\gamma_K^*}(u)} = \{A, C\}$, $\overline{P_{\gamma_K^*}(v)} = \{A, D\}$. We set $\gamma'(uv') = A$, $\gamma'(ux) = C$, $\gamma'(v'x) = D$. The restriction of γ' to the edges of $M - e$ corresponds to a 4-colouring γ'_M of $M - e$. Since u_1 and w_1 are identified, the palette of w_1 with respect to γ'_M is contained in $\{A, B\}$. We define γ'_M on the other edges of $M - e$ as follows.

If $e \in E(M)$ does not contain w nor w_1, then $P_{\gamma'_M}(w_1) = \{A, B\}$. By the assumptions, there exists a 4-colouring of $M - e$ such that w, w_1 lack exactly one colour simultaneously. By a permutation of the colours, we can set $P_{\gamma'_M}(w) = \{A, C\}$. We can colour the remaining edges of $H - e$ as follow: $\gamma'(wx) = B$, $\gamma'(wy) = D$, $\gamma'(xy) = A$. The colouring γ' of $H - e$ is thus defined and v, v' have disjoint palettes with respect to it, since $P_{\gamma'}(v') = \overline{P_{\gamma'}(v)} = \{A, D\}$. We can repeat similar arguments if the edge $e \in E(M)$ contains w but not w_1.

If $e \in E(M)$ contains w_1, then we can assume that $P_{\gamma'_M}(w_1) = \{A\}$. We can permute the colours in $M - e$ so that $P_{\gamma'_M}(w) \subseteq \{B, C\}$ or $P_{\gamma'_M}(w) \subseteq \{B, D\}$. The remaining edges of $H - e$ are coloured as follows: $\gamma'(wx) = A$, $\gamma'(xy) = B$ and $\gamma'(wy) = D$ or C according to whether $P_{\gamma'_M}(w) \subseteq \{B, C\}$ or $P_{\gamma'_M}(w) \subseteq \{B, D\}$, respectively. The colouring γ' of $H - e$ is thus defined and v, v' have disjoint palettes with respect to it, since $P_{\gamma'}(v') = \overline{P_{\gamma'}(v)} = \{A, D\}$. $\square$

Example 3.13. The graph G_{25} in Figure 9(b) has order 25, maximum degree 4 and $\chi'(G_{25}) = 4$. The vertices v, v' are conflicting. It is obtained by applying Proposition 3.12 to the graphs $K = G_{16}$ in Figure 7(b) and $M = G_7$ in Figure 2(c). The vertices u_1, w_1 are identified. As remarked in Example 3.6, the graph G_{16} is 3-colour transmitting with respect to u, v, u_1. As remarked in Example 2.6, the 2-vertices $w, w_1 \in V(G_{16})$ are saturating. Moreover, for every $e \in G_7$ not containing w nor w_1 there exists a colouring of $G_7 - e$ such that $P(w_1) \subseteq \{A, B\}$ and $P(w) \subseteq \{A, C\}$, that is, the assumption in Proposition 3.12 is satisfied. By Lemma 2.3, the identification of the conflicting vertices v, v' yields a 4-critical graph of order 24.

Example 3.14 (Fiol's counterexample). Proposition 3.12 is still true if we assume that M consists of exactly one vertex. For instance, consider the graph G_{19} in Figure 9(a) obtained from the graph G_{16} in Figure 7(b) and M consisting of exactly one vertex. The vertices u_1 and w_1 are identified. The vertices $v, v' \in V(G_{19})$ are conflicting (we can repeat the proof of Proposition 3.8 without considering the case $e \in E(M)$). By Lemma 2.3, the identification of the vertices v, v' yields a 4-critical graph, namely, Fiol's 4-critical graph in Figure 1(a).

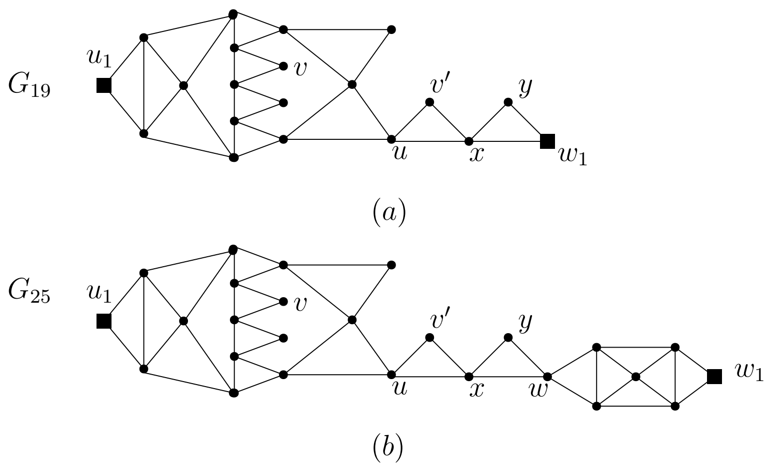

Figure 9: u_1 and w_1 should be identified in both graphs. (a): The graph G_{19} has order 19, maximum degree 4 and $\chi'(G_{19}) = 4$. (b): The graph G_{25} has order 25, maximum degree 4 and $\chi'(G_{25}) = 4$. As shown in Example 3.14, the vertices v, v' are conflicting.

4 Counterexamples to the Critical Graph Conjecture

In 1971, Jacobsen showed that there are no 3-critical graphs of order ≤ 10 and no 3-critical multigraphs of order ≤ 8. This led him to formulate the Critical Graph Conjecture. As we already mentioned, the first counterexamples to the conjecture were constructed by Goldberg [9], and afterwards by Chetwynd [6] and Fiol [7]. In this section we show that also Goldberg's counterexample can be obtained by a Möbius-type technique. Furthermore, combining our technique with Goldberg's construction we show that for every even value value of n, $n \geq 22$, there exists a 3-critical graph of order n.

Goldberg was the first to disprove the Critical Graph Conjecture by constructing an infinite family of 3-critical graphs of even order, the smallest of which has order 22 [9]. The graph of order 22 is represented in Figure 10(a). A 3-critical graph of the infinite family can be obtained from the 3-critical graph of order 22 in Figure 10(a) by adding in pairs the graph H_7 of order 7 in Figure 10(b). The result is the graph in Figure 11(a). A 3-critical graph of the infinite family has order $n \equiv 8 \pmod{16}$, $n \geq 24$.

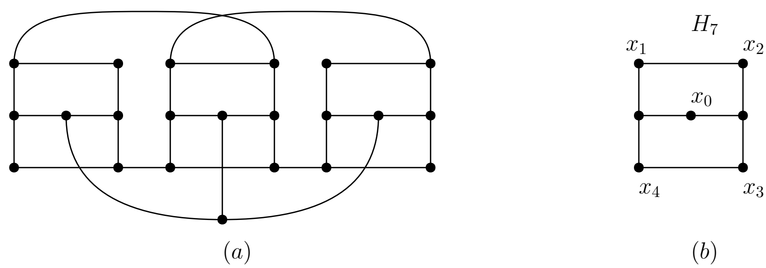

Figure 10: (a): The 3-critical graph of order 22 constructed by Goldberg. (b): The graph H_7 which is used to construct 3-critical graphs of order $n \equiv 8 \pmod{16}$, $n \geq 24$.

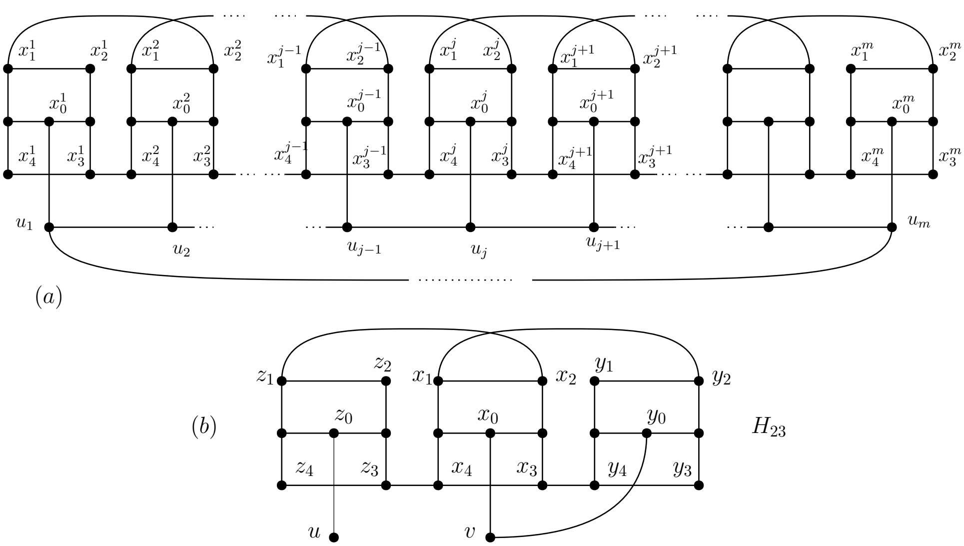

Figure 11: (a): The infinite family of 3-critical graphs of order $8m$, $m \geq 3$, m odd, constructed by Goldberg. (b): The graph H_{23} that yields the 3-critical graph of order 22 constructed by Goldberg by identifying the conflicting vertices u, v.

In what follows, we show that the 3-critical graphs constructed by Goldberg can be obtained by a Möbius type technique, namely, by identifying a pair of conflicting vertices in the case of the graph in Figure 10(a), or by connecting a pair of saturating vertices in the case of the graph in Figure 11(a). In Lemma 4.2, we will show that the vertices u, v of the graph H_{23} in Figure 11(b) are conflicting. We give a proof of the fact that u, v are conflicting showing that the structure of the graph H_7 forces to colour the edges of the graph in Figure 10(a) in a prescribed way, thus determining which vertex has to be split into two conflicting vertices. Analogously, for the proof of Lemma 4.3. The proofs of Lemmas 4.2 and 4.3 are based on the following result.

Lemma 4.1. *Every* 3*-colouring of the graph* H_7 *in Figure 10(b) satisfies the following condition:*

$$|\overline{P(x_0)} \cup \overline{P(x_i)} \cup \overline{P(x_{i+2})}| = 3 \quad \textit{and} \quad P(x_{i+1}) = P(x_{i+3}) = P(x_r)$$

where $i = 1$ *or* $i = 2$*,* $r \in \{0, i, i+2\}$ *and the subscripts are* $\pmod 4$*.*

Proof. Since the colour set has cardinality 3 and PL holds, exactly three vertices of H_7 lack the same colour A and the remaining 2-vertices of H_7 lack distinct colours B, C, both different from A. A direct inspection on the graph shows that the vertices lacking the same colours are x_{i+1}, x_{i+3} and x_r, where $i = 1$ or $i = 2$ and $r \in \{0, i, i+2\}$. □

Lemma 4.2. *The graph* H_{23} *in Figure 11(b) is class* 1 *and the vertices* $u, v \in V(H_{23})$ *are conflicting.*

The 3*-critical graph of order* 22 *in Figure 10(a) constructed by Goldberg can be obtained from the graph* H_{23} *by identifying the conflicting vertices* $u, v \in V(H_{23})$*.*

Proof. It is easy to see that H_{23} is class 1. We show that the vertices $u, v \in V(H_{23})$ are conflicting. Firstly, we prove that $P(u) \cap P(v) \neq \emptyset$ for every 3-colouring of the the graph H_{23}.

Let γ be a 3-colouring of H_{23}. Since γ induces a 3-colouring of the subgraphs of H_{23} that are isomorphic to H_7 and Lemma 4.1 holds, it is either $|\{\gamma(x_1y_2), \gamma(x_3y_4), \gamma(x_0v)\}| = 3$ or $|\{\gamma(x_2z_1), \gamma(x_4z_3), \gamma(x_0v)\}| = 3$. If $|\{\gamma(x_1y_2), \gamma(x_3y_4), \gamma(x_0v)\}| = 3$, then $\gamma(x_0v) = \gamma(y_0v)$, by virtue of Lemma 4.1 on the subgraph of H_{23} which is isomorphic to H_7 and contains the vertices y_i, $0 \le i \le 4$. That yields a contradiction, hence $|\{\gamma(x_2z_1), \gamma(x_4z_3), \gamma(x_0v)\}| = 3$. Since Lemma 4.1 holds on the subgraph of H_{23} which is isomorphic to H_7 and contains the vertices z_i, $0 \le i \le 4$, we have $\gamma(x_0v) = \gamma(z_0u)$. It is thus proved that $P(u) \cap P(v) \neq \emptyset$ for every 3-colouring of H_{23}.

It remains to prove that for every edge $e \in E(H_{23})$ there exists a 3-colouring γ' of $H_{23} - e$ such that the vertices u, v have disjoint palettes with respect to it. The existence is straightforward if e is incident to u, since u has degree 1. Let $\{1, 2, 3\}$ be the colour set of γ'. To define γ', it suffices to define γ' on the edges in $\{x_0v, y_0v, z_0u, x_iy_{i+1}, x_{i+1}z_i : i = 1, 3\}$ and colour the remaining edges according to Lemma 4.1. For instance, if e is incident to the vertices in $\{x_i, y_i : 0 \le i \le 4\}$, $e \notin \{x_0v, y_0v, z_0u, x_iy_{i+1}, x_{i+1}z_i : i = 1, 3\}$, then we set $\gamma'(x_1y_2) = \gamma'(z_0u) = 1$; $\gamma'(x_3y_4) = \gamma'(x_0v) = 2$; $\gamma'(y_0v) = 3$; $\gamma'(x_2z_1) = \gamma'(x_4z_3) = a \in \{1, 2\}$. The remaining cases can be managed in a similar way. It is thus proved that u, v are conflicting. Now the assertion follows from Lemma 2.3 by identifying the vertices u, v. □

Lemma 4.3. *Let* H_{8m}*,* $m \ge 3$*,* m *odd, be the graph obtained from the graph in Figure 11(a) by deleting the edge* u_1u_m*. The graph is class* 1 *and the vertices* u_1, u_m *are saturating. The* 3*-critical graphs of the infinite family constructed by Goldberg can be obtained by connecting a pair of saturating vertices.*

Proof. One can easily verify that the graph H_{8m} is class 1. We prove that u_1, u_m are saturating. Firstly, we show that $|P(u_1) \cup P(u_m)| = 3$ for every 3-colouring of the graph H_{8m}. For $1 \leq j \leq m$, let H_j be the subgraph of H_{8m} which is isomorphic to the graph H_7 in Figure 10(b) and contains the vertices x_i^j, $0 \leq i \leq 4$. Every 3-colouring γ of H_{8m} induces a 3-colouring γ' of the graph H_7, that is, Lemma 4.1 holds. By the symmetry of the graph, we can assume that $|\overline{P_{\gamma'}(x_0^1)} \cup \overline{P_{\gamma'}(x_2^1)} \cup \overline{P_{\gamma'}(x_4^1)}| = 3$ and $P_{\gamma'}(x_1^1) = P_{\gamma'}(x_3^1)$. Consequently, $\overline{P_{\gamma'}(x_2^2)} = \overline{P_{\gamma'}(x_4^2)}$ and $|\overline{P_{\gamma'}(x_0^2)} \cup \overline{P_{\gamma'}(x_1^2)} \cup \overline{P_{\gamma'}(x_3^2)}| = 3$. From this we deduce that $|\overline{P_{\gamma'}(x_0^j)} \cup \overline{P_{\gamma'}(x_2^j)} \cup \overline{P_{\gamma'}(x_4^j)}| = 3$ and $P_{\gamma'}(x_1^j) = P_{\gamma'}(x_3^j)$ if j is odd, $1 \leq j \leq m$; $|\overline{P_{\gamma'}(x_0^j)} \cup \overline{P_{\gamma'}(x_1^j)} \cup \overline{P_{\gamma'}(x_3^j)}| = 3$ and $P_{\gamma'}(x_2^j) = P_{\gamma'}(x_4^j)$ if j is even, $1 \leq j \leq m$. It follows that $\gamma(x_0^j u_j) = \gamma(x_0^{j+1} u_{j+1})$ for every $2 \leq j \leq m-1$, j even. We colour the edges of H_{8m} by $\{1, 2, 3\}$ and set $\gamma(x_0^2 u_2) = \gamma(x_0^3 u_3) = 3$. Without loss of generality we can set $\gamma(u_2 u_3) = 1$, whence $\gamma(u_1 u_2) = 2$. One can see that $\{\gamma(x_0^j u_j), \gamma(u_j u_{j+1})\} = \{\gamma(x_0^j u_{j+1}), \gamma(u_j u_{j+1})\} = \{1, 3\}$ for every $2 \leq j \leq m-1$, j even. As a consequence, $P(u_m) = \{1, 3\}$. It is thus proved that $|P(u_1) \cup P(u_m)| = 3$ for every 3-colouring of H_{8m}, since $2 \in P(u_1)$.

We omit the routine proof that for every $e \in E(H_{8m})$ there exists a colouring of H_{8m} such that $|P(u_1) \cup P(u_m)| < 3$. It is thus proved that u_1, u_m are saturating and the assertion follows from Lemma 2.3. □

It is known that the 3-critical graph of order 22 constructed by Goldberg is the smallest 3-critical graph [4]. Combining our construction with that one of Goldberg, we can prove the following result.

Theorem 4.4. *For every even value of n, $n \geq 22$, there exists a 3-critical graph of order n.*

Proof. A critical graph of the infinite family constructed by Goldberg has order $n \equiv 8 \pmod{16}$, $n \geq 24$. We construct a 3-critical graph of order $n \equiv 2 \pmod 4$, $n \geq 26$; and $n \equiv 0 \pmod 4$, $n \geq 28$. We define the auxiliary graphs H', K' and H'' that will be used in the construction. The graph H' is defined as follows. Consider $m \geq 1$ copies of the complete graph $K_4 - e$; the 2-vertices of $K_4 - e$ are same-lacking. For $1 \leq i \leq m-1$, connect the ith copy of $K_4 - e$ to the $(i+1)$th by adding exactly one edge joining a 2-vertex in the ith copy to a 2-vertex in the $(i+1)$th copy. The resulting graph H' has exactly two 2-vertices, say v_1, v_2. By Lemma 3.1, the graph H' has maximum degree 3, $\chi'(H') = 3$ and the vertices v_1, v_2 are same-lacking. Let K' be the graph of order 6 that can be obtained from the graph G_6 in Figure 2(b) by deleting the edges v_1v_2, v_3v_5, v_4v_6. The graph K' has maximum degree 3, $\chi'(K') = 3$ and the vertices v_1, v_2 are same-lacking. The graph H'' is obtained from the graphs H' and K' by connecting the vertex $v_2 \in V(K')$ to the vertex $v_1 \in V(H')$. By Lemma 3.1, the graph H'' has maximum degree 3, $\chi'(H'') = 3$ and the vertices v_1, v_2 are same-lacking. Let H be the graph obtained from the graph H_{23} in Figure 11(b) and the graph Γ, where $\Gamma \in \{H', K', H''\}$, by deleting the edge $z_0 u \in E(H_{23})$ and adding the edges $z_0 v_1, u v_2$. As remarked in Example 2.7, a graph with same-lacking vertices is able to transmit a color, therefore the graph H has maximum degree 3, $\chi'(H) = 3$ and the vertices $u, v \in V(H)$ are conflicting. Notice the following: $|V(H)| = 23 + 4m \geq 27$ if $\Gamma = H'$; $|V(H)| = 29$ if $\Gamma = K'$; $|V(H)| = 29 + 4m \geq 33$ if $\Gamma = H''$. By Lemma 2.3, the identification of the conflicting vertices $u, v \in V(H)$ yields a 3-critical graph of order $|V(H)| - 1$. Hence, the assertion follows. □

The 3-critical graphs of order $n \equiv 0 \pmod 4$, $n \geq 28$, that are constructed in the proof of Theorem 4.4, include the orders of Goldberg's infinte family but are not isomorphic to them. In fact, Goldberg's graphs have girth larger than 3; the 3-critical graphs in the proof of Theorem 4.4 have girth 3 as K' contains a 3-cycle.

5 From graphs with fertile vertices to 4-critical graphs

We show that it is possible to obtain 4-critical graphs of order n, for every $n \geq 5$, starting from the four graphs in Figure 2, the two graphs in Figure 1 and the graph G_{21} in Figure 8(a); these graphs have a pair of fertile vertices.

Theorem 5.1. *For every odd integer $n \geq 5$ there exists a 4-critical simple graph of order n.*

Proof. For every odd integer $n \geq 5$, we exhibit a graph H of maximum degree 4, $\chi'(H) = 4$ and order n having a pair of saturating vertices u_1, v_1. The assertion follows from Lemma 2.3 by adding the edge u_1v_1.

The graph H is obtained from Lemma 3.1 as follows. We take the graph G_6^m in Figure 3 as the graph H_1 in Lemma 3.1, where $m \geq 1$. As remarked in Example 3.2, it has order $6m \geq 6$, maximum degree 4 and the vertices $v_1, v_2 \in V(G_6^m)$ are same-lacking. We define the graph H_2 in Lemma 3.1 as follows: if $n \equiv 1 \pmod 6$, then H_2 is the graph G_7 in Figure 2(c); if $n \equiv 3 \pmod 6$, then H_2 is the graph G_9 in Figure 2(d); if $n \equiv 5 \pmod 6$, then H_2 is the graph G_5 in Figure 2(a). By the remarks in Examples 2.5 and 2.6, the vertices $u_1, u_2 \in V(H_2)$ are saturating. By Lemma 3.1, the graph H obtained from $H_1 = G_6^m$ and H_2 by adding the edge u_2v_2 has maximum degree 4, $\chi'(H) = 4$ and the vertices $u_1, v_1 \in V(H)$ are saturating. Notice that $|V(H)| = 6m + |V(H_2)| \geq 11$, where $m \geq 1$ and $|V(H_2)| \in \{5, 7, 9\}$. The graph G obtained from H by adding the edge u_1v_1 is 4-critical, since Lemma 2.3 holds. By construction, the graph G is simple. Since $|V(G)| = |V(H)|$, for every odd integer $n \geq 11$ there exists a 4-critical simple graph of order n. For $n = 5, 7, 9$, the assertion follows from Lemma 2.3 by setting $H = G_5, G_7, G_9$, respectively, and by adding the edge u_1u_2. □

Theorem 5.2. *For every even integer $n \geq 16$ there exists a 4-critical graph of order n. The graph is simple unless n is equal to 16.*

Proof. For $n = 16, 18$, we resort to the well known graphs in Figure 1. For $n = 20$ we consider the graph G_{21} in Figure 8(a). As remarked in Example 3.10, the vertices $v, v' \in G_{21}$ are conflicting. The existence of a 4-critical graph of order 20 follows from Lemma 2.3 by identifying the vertices v and v'. Notice that the graph is simple.

For every even integer $n \geq 22$, we exhibit a graph H of maximum degree 4, $\chi'(H) = 4$ and order n having a pair of saturating vertices u_1, v_1. The assertion follows from Lemma 2.3 by adding the edge u_1v_1. The graph H is obtained from Lemma 3.1 as follows. We take G_6^m in Figure 3 as the graph H_1 in Lemma 3.1, where $m \geq 1$. The graph H_2 in Lemma 3.1 has even order and its definition depends on the congruence class of n modulo 6.

Case $n \equiv 0 \pmod 6$, $n > 18$.

The graph H_2 is obtained from the 4-critical graph of order 18 in Figure 1(a) by the deletion of the edge u_1u_2. Alternatively, we can consider the 4-critical graph arising from the graph G_{25} in Figure 9(b) by identifying the vertices v, v' (see Example 3.13); H_2 can be obtained by deleting one of the two edges containing u_1.

Case $n \equiv 2 \pmod 6, n > 20$.

Consider the 4-critical graph G_{20} of order 20 obtained from the graph G_{21} in Figure 8(a) by identifying the vertices v, v'. Let H_2 be the graph obtained from G_{20} by deleting the edge u_1u_2.

Case $n \equiv 4 \pmod 6, n > 16$.

The graph H_2 is obtained from the 4-critical graph of order 16 in Figure 1(b) by the deletion of one parallel edge connecting the vertices u_1, u_2. For each congruence class of n, the vertices $u_1, u_2 \in V(H_2)$ are saturating, since Remark 2.8 holds. Moreover, H_2 is a simple graph of maximum degree 4, $\chi'(H_2) = 4$ and $|V(H_2)| = 18, 20, 16$ according to whether $n \equiv 0, 2, 4 \pmod 6$, respectively. By Lemma 3.1, the graph H obtained from $H_1 = G_6^m$ and H_2 by adding the edge u_2v_2 has maximum degree 4, $\chi'(H) = 4$ and the vertices $u_1, v_1 \in V(H)$ are saturating. Notice that $|V(H)| = 6m + |V(H_2)| \geq 22$, where $m \geq 1$ and $|V(H_2)| \in \{16, 18, 20\}$. By Lemma 2.3, the graph G obtained from H by adding the edge u_1v_1 is 4-critical. Since $|V(G)| = |V(H)|$, for every even integer $n \geq 22$ there exists a 4-critical graph of order n. Notice that these graphs are simple. Combining this result with the remarks on the existence of 4-critical graphs of order 16, 18 and 20, the assertion follows. □

There are alternative methods for constructing 4-critical graphs. For instance, consider the 4-critical graph G of order 20 obtained from the graph G_{21} in Figure 8(a) by identifying the vertices v, v'. Delete the edge $u_1u_2 \in E(G)$ and connect the remaining graph to the graph G_6^m in Figure 3. For every $m \geq 1$ we obtain a 4-critical graph of order $6m + 20$.

6 A concluding remark

We are confident that the present work will provide suggestions and tools for constructing infinite families of critical graphs even beyond degree 4. The next step should be inevitably the degree 5. The key definitions are compatible with the general case, and we believe that the method is versatile enough. With some effort and further investigation, new infinite families are expected to be found in the near future.

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On resolving sets in the point-line incidence graph of $\mathrm{PG}(n, q)$

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Abstract

Lower and upper bounds on the size of resolving sets and semi-resolving sets for the point-line incidence graph of the finite projective space $\mathrm{PG}(n, q)$ are presented. It is proved that if $n > 2$ is fixed, then the metric dimension of the graph is asymptotically $2q^{n-1}$.

Keywords: Point-line incidence graph, resolving sets, finite projective spaces.

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1 Introduction

For a simple, connected, finite graph $\Gamma = (V, E)$ and $x, y \in V$ let $d(x, y)$ denote the length of a shortest path joining x and y.

Definition 1.1. Let $\Gamma = (V, E)$ be a finite, connected, simple graph. Two vertices $v_1, v_2 \in V$ are *divided* by $S = \{s_1, s_2, \dots, s_r\} \subset V$ if there exists $s_i \in S$ so that $d(v_1, s_i) \neq d(v_2, s_i)$. A vertex $v \in V$ is *resolved* by S if the ordered sequence $(d(v, s_1), d(v, s_2), \dots, d(v, s_r))$ is unique. $\mathcal{S}$ is a *resolving set* in Γ if it resolves all the elements of V. The *metric dimension* of Γ, denoted by $\mu(\Gamma)$, is the size of the smallest example of resolving set in it.

The study of metric dimension is an interesting problem in its own right and it is also motivated by the connection with the base size of the corresponding graph. The *base size of a permutation group* is the smallest number of points whose stabilizer is the identity. The *base size* of Γ, denoted by $b(\Gamma)$, is the base size of its automorphism group $\mathrm{Aut}(\Gamma)$. The study of base size dates back more than 50 years, see [18]. A resolving set in Γ is obviously a base for $\mathrm{Aut}(\Gamma)$, so the metric dimension of a graph gives an upper bound on its base size. The difference $\mu(\Gamma) - b(\Gamma)$ is called the *dimension jump* of Γ. Distance-transitive graphs whose dimension jump is large with respect to the number of vertices are rare, and hence interesting objects. For more information about general results on metric dimension and base size we refer the reader to the survey paper of *Bailey* and *Cameron* [2].

Resolving sets for incidence graphs of some linear spaces were investigated by several authors [4, 9, 10, 12]. In these cases much better bounds than the general ones are known. Estimates on the size of blocking sets can be used to prove lower bounds on the metric dimension, and the knowledge of geometric properties is useful for constructions and upper bounds. It was shown by *Héger* and *Takáts* [12] that the metric dimension of the point-line incidence graph of a projective plane of order q is $4q - 4$ if $q \geq 23$. In a recent paper *Héger et al.* [11] extended this result for small values of q, too.

There are two natural generalizations of this planar result in higher dimensional spaces: one can consider either the point-hyperplane incidence graph, or the point-line incidence graph of $\mathrm{PG}(n, q)$. In the former case resolving sets are connected with lines in a higgledy-piggledy arrangement which were investigated by *Fancsali* and *Sziklai* [9]. Their results were recently improved by the authors of this paper [5]. The latter case is studied in the present paper. We assume that the reader is familiar with finite projective geometries. For a detailed description of these spaces we refer to [14, 16].

Let $\Gamma_{n,q}$ denote the point-line incidence graph of the finite projective space $\mathrm{PG}(n, q)$. The two sets of vertices of this bipartite graph correspond to points and lines of $\mathrm{PG}(n, q)$, respectively, and there is an edge between two vertices if and only if the corresponding point is incident with the corresponding line. In $\Gamma_{n,q}$ the distance of two different lines is 2 if they intersect each other and 4 if they are skew. The distance of a point P and a line ℓ is 1 if P is on ℓ, and it is 3 if P is not on ℓ. Finally, the distance of two different points is always 2. Hence, points cannot be resolved by other points. Considering these properties, the following definitions are natural.

Definition 1.2. A set $\mathcal{S}$ of points and lines of $\mathrm{PG}(n, q)$ is a *semi-resolving set* for points (lines) in $\Gamma_{n,q}$ if it resolves all the vertices of $\Gamma_{n,q}$ corresponding to points (lines).

Definition 1.3. Let $\mathcal{S}$ be a (semi-)resolving set in $\Gamma_{n,q}$. A point or a line is called *inner (outer)* if it is (not) in $\mathcal{S}$. An outer point is called *t-covered* if it is incident with exactly t

lines of $\mathcal{S}$. A point or a line is called *uncovered* if it has empty intersection with all elements of $\mathcal{S}$.

The paper is organized as follows. First, in Section 2 we provide a pure combinatorial proof of a lower bound for resolving sets in $\Gamma_{n,q}$. In the second part of the section interesting constructions are presented for $n \geq 3$ which yield examples asymptotically close to the lower bound. These resolving sets are related with regular spreads of lines in projective spaces. We prove that the metric dimension of $\Gamma_{n,q}$ is asymptotically $2q^{n-1}$ and its dimension jump is roughly $2\sqrt{v}$ where v denotes the number of its vertices. In Section 3 algebraic curves and blocking sets are applied. We consider a different type of line spread in $\mathrm{PG}(3,q)$ and we obtain examples of resolving sets in $\Gamma_{n,q}$ of smaller size when $q = p^h$, p prime and $h > 1$. Finally, in Section 4 computer aided results for small values of q are given.

2 General bounds

In this section we present lower and upper bounds on $\mu(\Gamma_{n,q})$ for all q and $n \geq 3$. Particular attention is paid to the case $n = 3$, since general upper bounds in any dimension depend on the 3-dimensional upper bound, see Proposition 2.15 and Theorem 2.11.

Theorem 2.1. *The size of any semi-resolving set for points in* $\Gamma_{n,q}$ *is at least*

$$2\frac{q^{n+1}-q}{q^2+q-2}.$$

Proof. Let $\mathcal{S}$ be a semi-resolving set for points in $\Gamma_{n,q}$ which consists of k lines and m points.

Count in two different ways the number of incident point-line pairs (P,ℓ) with $\ell \in \mathcal{S}$. On the one hand, this number is exactly $k(q+1)$. On the other hand, there is at most one uncovered point, the number of 1-covered points is at most k and any other outer point must be covered by at least two lines of $\mathcal{S}$. Hence

$$k(q+1) \geq k + 2\left(\frac{q^{n+1}-1}{q-1} - 1 - k - m\right),$$

so

$$k+m \geq \frac{2(q^{n+1}-q)}{(q-1)(q+2)} + \frac{qm}{q+2}.$$

This gives the required inequality at once. □

Corollary 2.2. *The size of any resolving set in* $\Gamma_{3,q}$ *is at least* $2\left(q^2-q+3-\frac{6}{q+2}\right)$.

Corollary 2.3. *The metric dimension of* $\Gamma_{3,q}$ *is at least* $2(q^2-q+3)$ *for* $q > 10$.

From now on we focus on upper bounds which will be given by constructions. For our examples we need the notion of spreads, in particular line spreads.

Definition 2.4. A k-spread $\mathcal{S}^k$ of $\mathrm{PG}(n,q)$ is a set of k-dimensional subspaces with the property that each point of $\mathrm{PG}(n,q)$ is incident with exactly one element of $\mathcal{S}^k$.

By definition, a k-spread of $\mathrm{PG}(n,q)$ consists of $\frac{q^{n+1}-1}{q^{k+1}-1}$ elements. The following theorem about the existence of spreads was proved independently by several authors, see [1, 7, 17].

Theorem 2.5. *The projective space* $\mathrm{PG}(n,q)$ *has a k-spread if and only if* $(k+1) \mid (n+1)$.

Hence there exists a line spread in any odd dimensional projective space. Two line spreads are said to be *disjoint*, if they do not share any common line. Our first construction for a semi-resolving set for points in $\Gamma_{n,q}$ is based on disjoint line spreads. We use the following theorem of *Etzion* [8] about the existence of disjoint line spreads.

Theorem 2.6 (Etzion). *If* $n \geq 3$ *is odd, then there exist at least two disjoint line spreads in* $\mathrm{PG}(n,q)$.

Theorem 2.7. *If* $n \geq 3$ *is odd, then there exists a semi-resolving set for points in* $\Gamma_{n,q}$ *of size*

$$r_{\mathcal{P}}(n,q) = 2q^2\frac{q^{n-1}-1}{q^2-1}. \tag{2.1}$$

Proof. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two disjoint line spreads in $\mathrm{PG}(n,q)$, and $\ell_i \in \mathcal{L}_i$ be arbitrary lines. We claim that $\mathcal{S} = \mathcal{L}_1 \cup \mathcal{L}_2 \setminus \{\ell_1,\ell_2\}$ is a semi-resolving set for points in $\Gamma_{n,q}$.

Each point not in ℓ_i is contained in a unique pair of lines $(r_1,r_2) \in \mathcal{L}_1 \times \mathcal{L}_2$. Each point of $\ell_1 \setminus \ell_2$ is contained in a unique line of $\mathcal{L}_1 \cup \mathcal{L}_2 \setminus \{\ell_1,\ell_2\}$ and each point of $\ell_2 \setminus \ell_1$ is contained in a unique line of $\mathcal{L}_1 \cup \mathcal{L}_2 \setminus \{\ell_1,\ell_2\}$. The (possible) unique point $\ell_1 \cap \ell_2$ is the only point of $\mathrm{PG}(n,q)$ not contained in any line of $\mathcal{L}_1 \cup \mathcal{L}_2 \setminus \{\ell_1,\ell_2\}$. The size of $\mathcal{S}$ is $2\frac{q^{n+1}-1}{q^2-1} - 2$, hence the statement follows. □

Proposition 2.8. *Let* Σ *be a hyperplane and* $\mathcal{L}$ *be a line spread in* $\mathrm{PG}(n,q)$, $n \geq 3$ *odd. Then* Σ *contains exactly* $\frac{q^{n-1}-1}{q^2-1}$ *elements of* $\mathcal{L}$.

Proof. Any element of $\mathcal{L}$ is either fully contained in Σ, or intersects it in exactly 1 point. The elements of $\mathcal{L}$ partition the set of points of Σ. Hence, if x denotes the number of fully contained lines, then

$$\frac{q^n-1}{q-1} = (q+1)x + \left(\frac{q^{n+1}-1}{q^2-1} - x\right).$$

The claim follows from this equation at once. □

Theorem 2.9. *Let* $\mathcal{L}_1$ *be a line spread in* $\mathrm{PG}(3,q)$. *Then there exists another line spread* $\mathcal{L}_2$ *in* $\mathrm{PG}(3,q)$ *such that* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *do not share any common line.*

Proof. Let $f(X,Y)$ be an irreducible homogeneous quadratic polynomial and $\mathcal{H}'_i$ denote the hyperbolic quadric in $\mathrm{PG}(3,q)$ with equation

$$f(X_0,X_1) + if(X_2,X_3) = 0$$

for $i = 1,2,\dots,q-1$. Apply a suitable linear transformation so that the images ℓ_1 and ℓ_2 of the lines $\ell'_1\colon X_0 = X_1 = 0$ and $\ell'_2\colon X_2 = X_3 = 0$ do not belong to $\mathcal{L}_1$. Let $\mathcal{H}_i$ denote the image of $\mathcal{H}'_i$, and let $\mathcal{E}_i$ and $\mathcal{F}_i$ denote the two reguli of lines on $\mathcal{H}_i$. Then for each i at most one of $\mathcal{E}_i$ and $\mathcal{F}_i$ contains some elements of $\mathcal{L}_1$, because any line of $\mathcal{E}_i$ intersects any line of $\mathcal{F}_i$ and no two elements of $\mathcal{L}_1$ intersect each other. Hence we can choose the notation so that $\mathcal{E}_i$ does not contain any element of $\mathcal{L}_1$ for all i. This implies that the spread

$$\mathcal{L}_2 = \bigcup_{i=1}^{q-1} \mathcal{E}_i \cup \{\ell_1, \ell_2\}$$

does not share any common line with $\mathcal{L}_1$. □

The next proposition gives a useful recursive construction method.

Proposition 2.10. *Let $\mathcal{S}$ be a semi-resolving set for points in $\Gamma_{d,q}$ of size k. Suppose that m elements of $\mathcal{S}$ are contained in a hyperplane Σ_{d-1} of $\mathrm{PG}(d, q)$, and Σ_{d-1} also contains the (at most one) uncovered point. Then $\Gamma_{d+1,q}$ has a semi-resolving set for points of size $(q + 1)k - qm$.*

Moreover, if $\mathcal{S}$ is a resolving set in $\Gamma_{d,q}$ and Σ_{d-1} also contains the (at most one) uncovered line, then $\Gamma_{d+1,q}$ admits a resolving set of size $(q + 1)k - qm + \frac{q^{d-1}-1}{q-1}$.

Proof. Embed $\Sigma_{d-1} \subset \mathrm{PG}(d, q)$ into $\mathrm{PG}(d+1, q)$, and consider in $\mathrm{PG}(d+1, q)$ the pencil of hyperplanes with carrier Σ_{d-1}. These hyperplanes, $\Sigma_d^1, \Sigma_d^2, \dots, \Sigma_d^{q+1}$, are isomorphic to $\mathrm{PG}(d, q)$. Take a copy of $\mathcal{S}$ in Σ_d^i and denote it by $\mathcal{S}^i$ for $i = 1, 2, \dots, q + 1$. Finally, let

$$\overline{\mathcal{S}} = \bigcup_{i=1}^{q+1} \mathcal{S}^i.$$

We claim that $\overline{\mathcal{S}}$ is a semi-resolving set for points in $\Gamma_{n+1,q}$. Inner points are resolved by definition. If two outer points, P_1 and P_2, are in the same Σ_d^i, then they are already divided by $\mathcal{S}^i$. If P_1 is in $\mathcal{S}^i$ and P_2 is in $\mathcal{S}^j$ with $i \neq j$, then, as none of P_1 and P_2 is uncovered and none of them is in Σ_{d-1}, there exist distinct lines $\ell^i \in \mathcal{S}^i$ through P_1 and $\ell^j \in \mathcal{S}^j$ through P_2. Hence ℓ^i does not contain P_2, so $d(P_1, \ell^i) \neq d(P_1, \ell^i)$. Since the size of $\overline{\mathcal{S}}$ is $m + (q + 1)(k - m)$, the first part of the statement is proved.

Now suppose that $\mathcal{S}$ is a resolving set in $\Gamma_{d,q}$. Then the elements of any point-line pair are obviously divided by $\overline{\mathcal{S}}$. Let ℓ_1 and ℓ_2 be two lines. If at least one of them is an element of $\overline{\mathcal{S}}$, then they are divided by definition. From now on we assume that none of the two lines is an element of $\overline{\mathcal{S}}$. We distinguish three main cases and some subcases.

1. If both of them are entirely contained in the same Σ_d^i, then they are divided by $\mathcal{S}^i$.
2. If there is no Σ_d^i that contains both ℓ_1 and ℓ_2, but each of the lines is entirely contained in some Σ_d^i, say $\ell_1 \subset \Sigma_d^{i_1}$ and $\ell_2 \subset \Sigma_d^{i_2}$, then none of the lines is in Σ_{d-1}. Let P_j denote the unique point $\ell_j \cap \Sigma_{d-1}$ for $j = 1, 2$.
   - If $P_1 = P_2$, then let $P_3 \neq P_1$ be a point on ℓ_1. Since $\mathcal{S}^{i_1}$ is a semi-resolving set for points in $\Sigma_d^{i_1}$ and P_3 is not an uncovered point, either $P_3 \in \mathcal{S}^{i_1}$ or there exists at least one line $\ell \in \mathcal{S}^{i_1}$ which contains P_3 but does not contain P_1. In the former case $d(\ell_1, P_3) = 1 \neq 3 = d(\ell_2, P_3)$. In the latter case $d(\ell_1, \ell) = 2 \neq 4 = d(\ell_2, \ell)$, so we are done.
   - If $P_1 \neq P_2$, then we may assume that P_1 is not an uncovered point, because there is at most one uncovered point. Again, either $P_1 \in \mathcal{S}^{i_1}$ or there exists at least one line $\ell \in \mathcal{S}^{i_1}$ which contains P_1 but does not contain P_2. In the former case $d(\ell_1, P_1) = 1 \neq 3 = d(\ell_2, P_1)$, while in the latter case $d(\ell_1, \ell) = 2 \neq 4 = d(\ell_2, \ell)$, so ℓ_1 and ℓ_2 are divided by $\mathcal{S}^{i_1}$.

3. If ℓ_1 is not contained in any Σ_d^i, then it cannot meet Σ_{d-1}, so there exists a unique point $P_1^i = \ell_1 \cap \Sigma_d^i$ for all $i = 1, 2, \ldots, q+1$.

   - If ℓ_2 is not contained in any Σ_d^i, then it cannot meet Σ_{d-1}, so there exists a unique point $P_2^i = \ell_2 \cap \Sigma_d^i$ for all $i = 1, 2, \ldots, q+1$. The two lines have at most one point of intersection, hence there exist at least q superscripts so that $P_1^i \neq P_2^i$. Since $\mathcal{S}^i$ is a semi-resolving set for points in Σ_d^i, there exists at least one element $s \in \mathcal{S}^i$ so that $d(P_1^i, s) \neq d(P_2^i, s)$. Hence

     $$d(\ell_1, s) = d(P_1^i, s) + 1 \neq d(P_2^i, s) + 1 = d(\ell_2, s),$$

     so the lines are divided by $\mathcal{S}^i$.
   - If ℓ_2 is contained in a unique Σ_d^i, then it is not contained in Σ_{d-1}, so there exists a unique point $P_2 = \ell_2 \cap \Sigma_{d-1}$. Let $j \neq i$ and consider Σ_d^j, which contains both P_1^j and P_2. Since $\mathcal{S}^j$ is a semi-resolving set for points in Σ_d^j, there exists at least one element $s \in \mathcal{S}^j$ so that $d(P_1^j, s) \neq d(P_2, s)$. Hence

     $$d(\ell_1, s) = d(P_1^j, s) + 1 \neq d(P_2, s) + 1 = d(\ell_2, s),$$

     the claim is proved.
   - Finally, suppose that ℓ_2 is contained in Σ_{d-1}. Then ℓ_1 and ℓ_2 are not necessarily divided by $\overline{\mathcal{S}}$. Suppose that $\overline{\mathcal{S}}$ consists of lines only. Then ℓ_2 and P_1^i are divided by $\mathcal{S}^i$, but it could happen that a line of $\mathcal{S}^i$ intersects ℓ_2 if and only if it contains P_1^i. If it holds for all i, then ℓ_1 and ℓ_2 have the same distance sequence with respect to $\overline{\mathcal{S}}$. We can handle this problem by extending $\overline{\mathcal{S}}$ with all the $\frac{q^{d-1}-1}{q-1}$ points of a hyperplane in Σ_{d-1}. Then ℓ_2 contains at least one of these points and ℓ_1 does not contain any of them. Hence the two lines are divided.

The size of the constructed resolving set is $(q+1)k - qm + \frac{q^{d-1}-1}{q-1}$, the statement is proved. $\square$

Theorem 2.11. *If $n \geq 4$ is even, then there exists a semi-resolving set for points in $\Gamma_{n,q}$ of size*

$$r_{\mathcal{P}}(n,q) = 2q^{n-1} + 2q^{n-2} + 2(q^{n-4} + q^{n-6} + \cdots + q^2). \tag{2.2}$$

Proof. We apply Proposition 2.10 for $d = n-1$. Let $\mathcal{S}$ be the semi-resolving set for points in $\Gamma_{n-1,q}$ which was constructed in Theorem 2.7. Its size is

$$k = 2\frac{q^n - q^2}{q^2 - 1}.$$

By Proposition 2.8, we can choose the hyperplane Σ_{n-2} so that it contains

$$m = 2\left(\frac{q^{n-2}-1}{q^2-1} - 1\right) = 2\frac{q^{n-2}-q^2}{q^2-1}$$

elements of $\mathcal{S}$. Thus we get from Proposition 2.10 that there exists a semi-resolving set for points in $\Gamma_{n,q}$ of size

$$\begin{aligned} r_{\mathcal{P}}(n,q) &= 2(q+1)\frac{q^n - q^2}{q^2-1} - 2q\frac{q^{n-2}-q^2}{q^2-1} \\ &= 2q^{n-1} + 2q^{n-2} + 2(q^{n-4} + q^{n-6} + \cdots + q^2). \end{aligned}$$ $\square$

Now we turn to semi-resolving sets for lines. Let us start with a simple, but very useful observation.

Lemma 2.12. *Let Σ be a hyperplane in $\mathrm{PG}(n, q)$, $\mathcal{S}$ be a semi-resolving set for points in Σ and ℓ_1 and ℓ_2 be two distinct lines in $\mathrm{PG}(n, q)$. Suppose that none of the lines is contained in Σ and the points $P_1 = \Sigma \cap \ell_1$ and $P_2 = \Sigma \cap \ell_2$ are distinct. Then the lines ℓ_1 and ℓ_2 are divided by $\mathcal{S}$ in $\Gamma_{n,q}$.*

Proof. Since $\mathcal{S}$ is a semi-resolving set for points in Σ, there exists at least one element $s \in \mathcal{S}$ so that $d(P_1^1, s) \neq d(P_2^1, s)$. Hence

$$d(\ell_1, s) = d(P_1^1, s) + 1 \neq d(P_2^1, s) + 1 = d(\ell_2, s),$$

the statement follows. □

Theorem 2.13. *For all $n > 3$ and $q \geq 2n - 1$ there exists a semi-resolving set for lines in $\Gamma_{n,q}$ of size $r_{\mathcal{L}}(n, q) = 2nr_{\mathcal{P}}(n - 1, q)$, where $r_{\mathcal{P}}$ follows* (2.1) *or* (2.2) *depending on the parity of $n - 1$.*

Proof. Let $\mathcal{H} = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_{2n}\}$ be a subset of $2n$ hyperplanes of the $(q + 1)$-element set formed by the dual hyperplanes of points on a normal rational curve. Then these hyperplanes are in general position, no $n + 1$ of them have a point in common. Let $\mathcal{S}^i$ be a semi-resolving set for points in Σ_i. We claim that $\mathcal{S} = \bigcup_{i=1}^{2n} \mathcal{S}^i$ is a semi-resolving set for lines in $\Gamma_{n,q}$.

Let ℓ_1 and ℓ_2 be two distinct lines in $\mathrm{PG}(n, q)$. We may assume that ℓ_j is contained in the intersection of m_j elements of $\mathcal{H}$ for $j = 1, 2$, and $m_1 \geq m_2$. The elements of $\mathcal{H}$ are in general position, so $n - 1 \geq m_j$, hence $2n - m_1 - m_2 \geq 2$. We may assume without loss of generality that ℓ_j intersects Σ_i in a single point, denoted by P_j^i, for $i = 1, 2, \ldots, 2n - m_1 - m_2$ and $j = 1, 2$. It could happen, that $P_1^{i_1} = P_1^{i_2} = \cdots = P_1^{i_k}$ for some indices, but $k \leq n - m_2$, otherwise the point would be a common point of at least $m_1 + (n - m_2 + 1) > n$ elements of $\mathcal{H}$. So we may assume that $P_1^1 \neq P_1^2$. As ℓ_2 contains at most one point of ℓ_1, we may also assume that P_1^1 is not on ℓ_2. Then, by Lemma 2.12, ℓ_1 and ℓ_2 are divided by $\mathcal{S}^1$.

By Theorems 2.7 and 2.11, the size of $\mathcal{S}$ is at most $2nr_{\mathcal{P}}(n - 1, q)$ for $n > 3$, thus the theorem is proved. □

The union of a semi-resolving set for points and a semi-resolving set for lines is a resolving set. Thus Theorems 2.7, 2.11 and 2.13 give our first general upper bound.

Corollary 2.14. *For all $n > 3$ and $q \geq 2n - 1$ there exists a resolving set in $\Gamma_{n,q}$ of size*

$$r(n, q) = 2q^{n-1} + (4n + 1 + (-1)^n)q^{n-2} + g_n(q),$$

where g_n is a polynomial of degree $n - 3$ whose coefficients depend only on n.

In this bound the coefficient of the second highest degree term depends on the dimension. In the next part, by a more sophisticated construction, we prove an upper bound in which the coefficient of the second highest degree term is a constant.

Proposition 2.15. *Let $q = p^h$, p prime. Suppose that there exists a resolving set $\mathcal{S}_3$ in $\Gamma_{3,q}$ of size $2q^2 + aq + g_3(p)$, where $a \in \mathbb{R}$, g_3 is a polynomial of degree $s \leq h-1$, and $\mathcal{S}_3$ contains the $2q^2+2$ elements of two disjoint line spreads. Then there exists a resolving set $\mathcal{S}_{4m+3}$ in $\Gamma_{4m+3,q}$ of size*

$$2q^{n-1} + aq^{n-2} + g_{4m+3}(p)$$

where g_{4m+3} is a polynomial of degree at most $(n-3)h + s$.

Proof. As $(4m+3)+1$ is divisible by $3+1$, there exists a 3-spread in $\mathrm{PG}(4m+3, q)$. This 3-spread contains $t = \frac{q^{4(m+1)}-1}{q^4-1}$ elements, say $\Sigma_3^1, \Sigma_3^2, \ldots, \Sigma_3^t$, each of them is isomorphic to $\mathrm{PG}(3, q)$. By the assumption of the theorem, in each Σ_3^i there exists a resolving set $\mathcal{S}_3^i$ of size $2q^2 + aq + g_3(p)$. We claim that

$$\mathcal{S} = \bigcup_{i=1}^{t} \mathcal{S}_3^i$$

is a resolving set in $\Gamma_{4m+3,q}$. The elements of any pair of points and any point-line pair are obviously divided by $\mathcal{S}$. Let ℓ_1 and ℓ_2 be two lines. If at least one of them is contained in a Σ_3^i, then they are divided by $\mathcal{S}_3^i$. If none of them is contained in any Σ_3^i, then we may assume without loss of generality that $\ell_1 \cap \Sigma_3^1$ is a point P which is not on ℓ_2. Let s_1 and s_2 be the two elements of the disjoint line spreads in $\mathcal{S}_3^1$ which are incident with P. Then $d(\ell_1, s_1) = d(\ell_1, s_2) = 2$. As ℓ_2 is not contained in $\mathcal{S}_3^1$, it cannot intersect both s_1 and s_2. Hence at least one of the distances $d(\ell_2, s_1)$ and $d(\ell_2, s_2)$ is 4. Thus ℓ_1 and ℓ_2 are divided by $\mathcal{S}_3^1 \subset \mathcal{S}$.

The size of $\mathcal{S}$ is

$$(2q^2 + aq + g_3(p))\frac{q^{n+1}-1}{q^4-1} = 2q^{n-1} + aq^{n-2} + g_{4m+3}(p),$$

where the degree of g_{4m+3} is $(n-3)h + \deg g_{4m+3} = (n-3)h + s \leq (n-2)h - 1$, so we are done. □

Theorem 2.16. *Let $q = p^h$, p prime. Suppose that there exists a resolving set in $\Gamma_{3,q}$ of size $2q^2 + aq + g_3^3(p)$ where g_3^3 is a polynomial of degree $s \leq h-1$. Then for $n \geq 3$ there exists a resolving set in $\Gamma_{n,q}$ of size*

$$r(n,q) = \begin{cases} 2q^{n-1} + (a+2)q^{n-2} + g_{n,0}(p), & \text{if } n \equiv 0 \pmod 4, \\ 2q^{n-1} + (a+2)q^{n-2} + g_{n,1}(p), & \text{if } n \equiv 1 \pmod 4, \\ 2q^{n-1} + (a+4)q^{n-2} + g_{n,2}(p), & \text{if } n \equiv 2 \pmod 4, \\ 2q^{n-1} + aq^{n-2} + g_{n,3}(p), & \text{if } n \equiv 3 \pmod 4, \end{cases}$$

where $g_{n,i}$ $(i = 0, 1, 2, 3)$ is a polynomial of degree $(n-3)h + s$ whose coefficients depend only on n.

Proof. We prove it by induction on the dimension modulo 4. For $n \equiv 3 \pmod 4$ the statement follows from Proposition 2.15.

If $n \equiv 0 \pmod 4$, then we apply Proposition 2.10 for $d = n-1$ with $k = r_{\mathcal{P}}(n-1, q)$ and $m = 0$. Therefore by the induction hypothesis

$$\begin{aligned} r_{\mathcal{P}}(n,q) &\leq (q+1)r_{\mathcal{P}}(n-1,q) + \frac{q^{n-2}-1}{q-1} \\ &= 2q^{n-1} + (a+2)q^{n-2} + (q+1)g_{n-1,3}(p) + (a+1)q^{n-3} + q^{n-4} + \cdots + 1. \end{aligned}$$

Thus $g_{n,0}(p) = (q+1)g_{n-1,3}(p) + (a+1)q^{n-3} + q^{n-4} + \cdots + 1$, hence its degree is $(n-3)h + s \leq (n-2)h - 1$.

If $n \equiv 1 \pmod 4$, then $n-2 \equiv 3 \pmod 4$, hence we can apply Proposition 2.10 for $d = n-1$ so that Σ_{d-1} contains a resolving set constructed in Proposition 2.15. Then $k = 2q^{n-2} + (a+2)q^{n-3} + g_{n-1,0}(p)$ and $m = 2q^{n-3} + aq^{n-4} + g_{n-2,3}(p)$. Hence

$$r_{\mathcal{P}}(n,q) = (q+1)k - qm + \frac{q^{n-2}-1}{q-1} = 2q^{n-1} + (a+2)q^{n-2} + g_{n,1}(p),$$

where

$$g_{n,1}(p) = (q+1)g_{n-1,0}(p) - qg_{n-2,3}(p) + 3q^{n-3} + q^{n-4} + \cdots + 1,$$

so its degree is $(n-3)h + s \leq (n-2)h - 1$.

Finally, if $n \equiv 2 \pmod 4$, then $n-3 \equiv 3 \pmod 4$. Hence we cannot do better than apply Proposition 2.10 for $d = n-1$ so that Σ_{d-1} contains entirely only elements of a $(d-2)$-dimensional resolving set constructed in Proposition 2.15. Now $k = 2q^{n-2} + (a+2)q^{n-3} + g_{n-1,1}(p)$ and $m = 2q^{n-4} + aq^{n-5} + g_{n-3,3}(p)$. This gives

$$r_{\mathcal{P}}(n,q) = (q+1)k - qm + \frac{q^{n-2}-1}{q-1} = 2q^{n-1} + (a+4)q^{n-2} + g_{n,2}(p),$$

where

$$g_{n,2}(p) = (q+1)g_{n-1,1}(p) - qg_{n-3,3}(p) + (a+1)q^{n-3} + q^{n-4} + \cdots + 1,$$

thus its degree is $(n-3)h + s \leq (n-2)h - 1$ again. The theorem is proved. □

Let us remark that the polynomials $g_{n,i}$ can be determined exactly. We omit the long, but straightforward calculations, because their coefficients do not play any role in the rest of the paper.

In the next part of the section semi-resolving sets for lines in $\Gamma_{n,q}$ are investigated. In their constructions double blocking sets and their duals play an important role. For the relevant definitions and estimates on their sizes we refer to the paper of *Ball* and *Blokhuis* [3].

Theorem 2.17. *For all $q > 3$ there exists a semi-resolving set for lines in $\Gamma_{3,q}$ of size*

$$r_{\mathcal{L}}(3,q) = \min\{12q - 22, 4\tau_2(q) - 10\},$$

where $\tau_2(q)$ denotes the size of the smallest minimal double blocking set in PG$(2,q)$.

Proof. First, we construct two sets of lines in PG$(2,q)$ which are semi-resolving sets for points.

1. Let E_1, E_2, and E_3 be the vertices of a triangle, ℓ_i denote the line E_jE_k and $\mathcal{P}_i$ be the pencil of lines with carrier E_i. Let

   $$\mathcal{S} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \setminus \{\ell_1, \ell_2, \ell_3, \ell\},$$

   where $\ell \in \mathcal{P}_1$, $\ell_2 \neq \ell \neq \ell_3$. Then $\mathcal{S}$ is a semi-resolving set for points in $\Gamma_{2,q}$, because $U = \ell \cap \ell_1$ is a unique uncovered point, every point in the set $\ell_1 \cup \ell_2 \cup \ell_3 \setminus \{E_1, E_2, E_3, U\}$ is 1-covered and all other points are at least 2-covered, hence resolved. The size of $\mathcal{S}$ is $3q - 4$.

2. Let $\mathcal{D}$ be a dual double blocking set in $\mathrm{PG}(2,q)$. Then, by definition, each point is incident with at least two lines of $\mathcal{D}$. Thus if we delete an arbitrary line ℓ from $\mathcal{D}$, then the set of lines $\mathcal{D} \setminus \{\ell\}$ is still a semi-resolving set for points and, by the Principle of Duality, its size is at most $\tau_2(q) - 1$.

Hence, for all $q > 3$ there is a set of lines in $\mathrm{PG}(2,q)$ of size $\min\{3q-4, \tau_2(q)-1\}$ which is a semi-resolving set for points. Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, and $\mathcal{H}_4$ be the faces of a tetrahedron $\mathcal{K}$ in $\mathrm{PG}(3,q)$. Let $\mathcal{T}^i$ be a semi-resolving set for points in $\mathcal{H}_i$ which consists of lines only. We can choose $\mathcal{T}^i$ so that each edge of $\mathcal{K}$ belongs to both corresponding semi-resolving sets, because the full collineation group of $\mathrm{PG}(2,q)$ acts transitively on triangles. We claim that $\mathcal{S} = \cup_{i=1}^{4} \mathcal{T}^i$ is a semi-resolving set for lines in $\Gamma_{3,q}$.

The edges of $\mathcal{K}$ belong to $\mathcal{S}$, thus they are resolved by definition. Let ℓ_1 and ℓ_2 be lines such that none of them is an edge of $\mathcal{K}$. Then each of them is contained in at most one face of $\mathcal{K}$, so we may assume without loss of generality that ℓ_1 intersects $\mathcal{H}_i$ in a single point, denoted by P_1^i, for $i = 1,2,3$. We distinguish two main cases.

1. If $P_1^1 = P_1^2 = P_1^3$, then this point is a vertex K of $\mathcal{K}$.
   - If ℓ_2 also contains K, then $\mathcal{H}_4 \cap \ell_1 \neq \mathcal{H}_4 \cap \ell_2$, hence, by Lemma 2.12, the two lines are divided by $\mathcal{T}^4$.
   - If ℓ_2 does not contain K, then we may assume that $\mathcal{H}_2 \cap \ell_2$ is a single point P_2^2. Since $P_2^2 \neq K$, by Lemma 2.12, the two lines are divided by $\mathcal{T}^2$.
2. If none of ℓ_1 and ℓ_2 contains any vertex, then we may assume that $P_1^1 \neq P_1^2$.
   - If ℓ_2 is not contained in neither $\mathcal{H}_1$ nor $\mathcal{H}_2$, then it intersects $\mathcal{H}_i$ in a single point, denoted by P_2^i, for $i = 1,2$. Since $\ell_1 \cap \ell_2$ contains at most one point, we may assume that $P_1^1 \neq P_2^1$. Then, by Lemma 2.12, the two lines are divided by $\mathcal{T}^1$.
   - Finally, if ℓ_2 is contained in one of $\mathcal{H}_1$ and $\mathcal{H}_2$, then we may assume that $\ell_2 \subset \mathcal{H}_1$ and $\ell_2 \cap \mathcal{H}_2$ in a single point P_2^2. Then P_2^2 is in $\mathcal{H}_1$, so $P_2^2 \neq P_1^2$, because otherwise $\ell_1 \subset \mathcal{H}_1$. Hence, by Lemma 2.12, the two lines are divided by $\mathcal{T}^2$.

Since $\mathcal{S}$ has the required size, we are done. □

Remark 2.18. Let us remark that if the double blocking set $\mathcal{D}$ in the proof of Theorem 2.17 is the disjoint union of two dual blocking sets, then not only one, but two lines can be deleted without violating the semi-resolving set property. We will consider this case in Section 3, Theorem 3.1.

Unfortunately, the exact value of $\tau_2(q)$ is not known in general. It is known that $\tau_2(q) = 2q + 2\sqrt{q} + 2$ for q is a square and $q > 16$ [3, Theorem 3.1], and for some small values of q. In the latter case for the known values $\tau_2(q) > 3q - 3$ always holds. Combining the semi-resolving set for points constructed in Corollary 2.7 and the semi-resolving set for lines of Theorem 2.17, we get the following upper bound on $\mu(\Gamma_{3,q})$.

Theorem 2.19. *The metric dimension of $\Gamma_{3,q}$ satisfies the inequality*

$$\mu(\Gamma_{3,q}) \leq 2q^2 + 12q - 24$$

for all $q > 3$.

Proof. For $q > 3$ let $\mathcal{S}_{\mathcal{L}}$ be a semi-resolving set for lines of size $12q - 22$ constructed in Theorem 2.17. Let $\mathcal{L}_1$ be a regular line spread of $\mathrm{PG}(3,q)$ which contains two skew (non-intersecting) elements of $\mathcal{S}_{\mathcal{L}}$. Such spread exists, because the collineation group of $\mathrm{PG}(3,q)$ acts transitively on the pairs of skew lines. Create a semi-resolving set for points $\mathcal{S}_{\mathcal{P}}$ which contains $\mathcal{L}_1$ as we did it in Corollary 2.7. Then $\mathcal{S} = \mathcal{S}_{\mathcal{L}} \cup \mathcal{S}_{\mathcal{P}}$ is a resolving set in $\Gamma_{3,q}$ and its size is $2q^2 + 12q - 24$. This proves the inequality. □

By combining Theorem 2.16, with $s = 0$, and Theorem 2.19, we get the following bounds.

Corollary 2.20. *Let $n \geq 3$ and $q > 3$. Then the metric dimension of $\Gamma_{n,q}$ satisfies the inequality*

$$\mu(\Gamma_{n,q}) \leq \begin{cases} 2q^{n-1} + 14q^{n-2} + h_{n,2}(q), & \text{if } n \equiv 0 \text{ or } n \equiv 1 \pmod 4, \\ 2q^{n-1} + 16q^{n-2} + h_{n,3}(q), & \text{if } n \equiv 2 \pmod 4, \\ 2q^{n-1} + 12q^{n-2} + h_{n,1}(q), & \text{if } n \equiv 3 \pmod 4, \end{cases}$$

where $h_{n,i}$ $(i = 1,2,3)$ is a polynomial of degree at most $n-3$ whose coefficients depend only on n.

The metric dimension of $\Gamma_{2,q}$ for $q \geq 23$ was determined by *Héger* and *Takáts* [12]. For higher dimensions we do not know the exact value, but Theorems 2.1, 2.19, and Corollary 2.20 imply the following result.

Corollary 2.21. *For all $n > 2$ and $q > 3$*

$$|\mu(\Gamma_{n,q}) - 2q^{n-1}| = O(q^{n-2}).$$

This means that $\mu(\Gamma_{n,q})$ is asymptotically $2q^{n-1}$. The number of vertices in $\Gamma_{n,q}$ is $v = \frac{q^{n+1}-1}{q-1} + \frac{(q^{n+1}-1)(q^n-1)}{(q+1)(q-1)^2}$, so its metric dimension is roughly $2\sqrt{v}$. The automorphism group of $\mathrm{PG}(n,q)$ is $\mathrm{P\Gamma L}(n+1,q)$ and it is well-known that its base size is $n+1$ if q is a prime, and it is $n+2$ if $q = p^h$ with $h > 1$. Hence the dimension jump of $\Gamma_{n,q}$ is roughly $2\sqrt{v}$.

3 Bounds for $q = p^h, h \geq 2$

In this section we consider the case $q = p^h$, $h > 1$. In the case h even, we will present a better bound on the size of a semi-resolving set for points in $\Gamma_{3,q}$ using small dual double blocking sets in $\mathrm{PG}(2,q)$. When $h > 2$, then we will show that a particular type of spread of lines in $\mathrm{PG}(3,q)$ can be used to resolve the lines. In fact, for a regular spread, there exist many pairs of lines of the spaces intersecting the same set of elements of the spread. We now investigate a different type of spread, called aregular, and we determine all the lines of the space intersecting the same set of elements of the spread; see Theorem 3.6. The main goal is to construct a set of lines of $\mathrm{PG}(3,q)$ which resolves all the lines of the spaces; see Theorem 3.7.

Theorem 3.1. *If q is a square, then the metric dimension of $\Gamma_{3,q}$ satisfies the inequality*

$$\mu(\Gamma_{3,q}) \leq 2q^2 + 8q + 8\sqrt{q} - 8.$$

Proof. The union of the sets of lines of two disjoint Baer subplanes is a dual double blocking set in $\mathrm{PG}(2,q)$ and its size is $2q+2\sqrt{q}+2$. This set is the disjoint union of two dual blocking sets. Hence, by a result of *Héger* and *Takáts* [12, Proposition 22], we can delete two of its lines so that the remaining set is still a semi-resolving set for points in $\mathrm{PG}(2,q)$; see also Remark 2.18.

Thus we can construct a semi-resolving set for lines $\mathcal{S}_{\mathcal{L}}$ of size $8q+8\sqrt{q}-6$ by the method applied in the proof of Theorem 2.17. Finally, we can extend it to a resolving set of size $2q^2+8q+8\sqrt{q}-8$ in the same way as we did in the proof of Theorem 2.19. $\square$

By combining Theorem 2.16, with $s=0$, and Theorem 2.19, we get the following bounds.

Corollary 3.2. *If q is a square and $n\geq 3$, then the metric dimension of $\Gamma_{n,q}$ satisfies the inequality*

$$\mu(\Gamma_{n,q}) \leq \begin{cases} 2q^{n-1}+10q^{n-2}+h_{n,2}(q), & \text{if } n\equiv 0 \text{ or } 1 \pmod 4,\\ 2q^{n-1}+12q^{n-2}+h_{n,3}(q), & \text{if } n\equiv 2 \pmod 4,\\ 2q^{n-1}+8q^{n-2}+h_{n,1}(q), & \text{if } n\equiv 3 \pmod 4,\end{cases}$$

where $h_{n,i}$ $(i=1,2,3)$ is a polynomial of degree at most $n-3$ whose coefficients depend only on n.

Theorem 3.3 ([13, Theorem 17.3.3])**.** *Let $q=p^h$, $h>1$, and choose $b,c\in\mathbb{F}_q^*$ such that the polynomial $t^{p+1}-tb+c$ has no roots in $\mathbb{F}_q$. Let*

$$\mathcal{A}_{b,c}=\{t_{\alpha,\beta}:\alpha,\beta\in\mathbb{F}_q\}\cup\{Z=T=0\},$$

where $t_{\alpha,\beta}$ is the line through the points $(\alpha:\beta:1:0)$ and $(c\beta^p:\alpha^p+b\beta^p:0:1)$. Then $\mathcal{A}_{b,c}$ is a spread, called the aregular spread.

In what follows, we will associate to each line r of the space an algebraic curve $\mathcal{C}_r\colon F_r(X,Y,T)=0$ such that $t_{\alpha,\beta}$ intersects r if and only if $F_r(\alpha,\beta,1)=0$. We distinguish four types of lines.

1. Lines $r_{x,y,\ell,m}$ through the points $(x:y:0:1)$ and $(\ell:m:1:0)$. Note that if $x=cm^p$ and $y=\ell^p+bm^p$ then $r_{x,y,\ell,m}$ coincides with $t_{\ell,m}\in\mathcal{A}_{b,c}$. From now on we consider $(x,y)\neq(cm^p,\ell^p+bm^p)$. A line $t_{\alpha,\beta}$ intersects $r=r_{x,y,\ell,m}$ if and only if for some $\lambda\in\mathbb{F}_q$ the points

$$(x+\lambda\ell:y+\lambda m:\lambda:1),\quad (\alpha:\beta:1:0),\quad (c\beta^p:\alpha^p+b\beta^p:0:1)$$

are collinear, that is

$$x+\lambda\ell-c\beta^p=\lambda\alpha,\qquad y+\lambda m-(\alpha^p+b\beta^p)=\lambda\beta.$$

This implies

$$\lambda(\ell-\alpha)=c\beta^p-x,\qquad \lambda(m-\beta)=\alpha^p+b\beta^p-y,$$

and therefore

$$(c\beta^p-x)(m-\beta)=(\alpha^p+b\beta^p-y)(\ell-\alpha).$$

In this case, $F_r(X,Y,T)=(cY^p-xT^p)(mT-Y)-(X^p+bY^p-yT^p)(\ell T-X)$.

2. Lines $s = s_{x,y,z,\ell}$ through the points $(x : y : z : 1)$ and $(\ell : 1 : 0 : 0)$. A line $t_{\alpha,\beta}$ intersects $s_{x,y,z,\ell}$ if and only if for some $\lambda \in \mathbb{F}_q$ the points

$$(x+\lambda\ell : y+\lambda : z : 1), \quad (\alpha : \beta : 1 : 0), \quad (c\beta^p : \alpha^p + b\beta^p : 0 : 1)$$

are collinear, that is

$$x+\lambda\ell - c\beta^p = z\alpha, \qquad y+\lambda-(\alpha^p+b\beta^p) = z\beta.$$

This implies

$$\lambda\ell = -x + c\beta^p + z\alpha, \qquad \lambda = -y + z\beta + \alpha^p + b\beta^p$$

and therefore

$$-x + c\beta^p + z\alpha = \ell(-y + z\beta + \alpha^p + b\beta^p).$$

So, $F_s(X,Y,T) = -\ell X^p + (c-\ell b)Y^p + zXT^{p-1} - \ell zYT^{p-1} + (\ell y - x)T^p$.

3. Lines $u = u_{x,y,z}$ through the points $(x : y : z : 1)$ and $(1 : 0 : 0 : 0)$. In this case, $F_u(X,Y,T) = X^p + bY^p + zYT^{p-1} - yT^p$.
4. Lines $v = v_{x,y,z}$ contained in the planes $T = 0$ and $xX + yY + zZ = 0$. Then, $F_v(X,Y,T) = xX + yY + zT$.

Such a curve is absolutely irreducible if $z \neq 0$, otherwise it collapses into a single line.

Proposition 3.4. *Consider the curves $\mathcal{C}_r$, $\mathcal{C}_s$, $\mathcal{C}_u$, and $\mathcal{C}_v$. Then*

1. *$\mathcal{C}_r$ is absolutely irreducible;*
2. *$\mathcal{C}_s$ is either absolutely irreducible or a line repeated p times;*
3. *$\mathcal{C}_u$ is either absolutely irreducible or a line repeated p times.*

Proof.

1. Now we prove that $\mathcal{C}_r$ is absolutely irreducible. Let $\varphi(X,Y,T) = (X + x_0T, Y + y_0T, T)$ with $x_0^p = y - bx/c$, $y_0^p = x/c$. Then

$$\begin{aligned}
&F_r(\varphi(X,Y,T))\\
&\quad = (c(Y+y_0T)^p - xT^p)(mT - Y - y_0T)\\
&\qquad - ((X+x_0T)^p + b(Y+y_0T)^p - yT^p)(\ell T - X - x_0T)\\
&\quad = (cY^p + cy_0^pT^p - xT^p)(mT - Y - y_0T)\\
&\qquad - (X^p + x_0^pT^p + bY^p + y_0^pT^p - yT^p)(\ell T - X - x_0T)\\
&\quad = (cY^p + xT^p - xT^p)(mT - Y - y_0T)\\
&\qquad - (X^p + yT^p - bx/cT^p + bY^p + bx/cT^p - yT^p)(\ell T - X - x_0T)\\
&\quad = cY^p(mT - Y - y_0T) - (X^p + bY^p)(\ell T - X - x_0T)\\
&\quad = G_r(X,Y,T).
\end{aligned}$$

Finally

$$G_r(X,1,Y) = c(mY - 1 - y_0Y) - (X^p + b)(\ell Y - X - x_0Y),$$

that is the curve $\mathcal{C}_r$ is $\mathbb{F}_q$-isomorphic to

$$\mathcal{C}'\colon Y = \frac{c - X(X^p + b)}{cm - cy_0 + (x_0 - \ell)(X^p + b)},$$

which is an irreducible rational curve with $q+1$ $\mathbb{F}_q$-rational points (note that $(x, y) \neq (cm^p, \ell^p + bm^p)$ yields $m \neq y_0$ or $\ell \neq x_0$). This means that the curve $\mathcal{C}_r$ is absolutely irreducible.

2. First, note that the homogeneous term $-\ell X^p + (c - \ell b)Y^p$ cannot vanish otherwise $c = 0$, a contradiction.

   - If $(\ell, z) = (0, 0)$, $\mathcal{C}_s$ is a line of type $b_0 Y + c_0 T = 0$ repeated p times.
   - If $\ell = 0$ and $z \neq 0$, then $F_s(X, Y, T)$ reads $cY^p + zXT^{p-1} - xT^p$ and $\mathcal{C}_s$ is absolutely irreducible.
   - If $\ell \neq 0$ and $z = 0$ then $\mathcal{C}_s$ is a (repeated) line $a_0 X + b_0 Y + c_0 T = 0$, where $a_0^p = -\ell$, $b_0^p = (c - \ell b)$, $c_0^p = (\ell y - x)$.
   - If $\ell \neq 0$ and $z \neq 0$ then consider $\varphi(X, Y, T) = (X + \sqrt[p]{(c - \ell b)/\ell}\, Y, Y, T)$ and so

   $$\begin{aligned} G_s(X, Y, T) &= F_s(\varphi(X, Y, T)) = \\ &- \ell X^p + zXT^{p-1} + z(\sqrt[p]{(c - \ell b)/\ell} - \ell)YT^{p-1} + (\ell y - x)T^p. \end{aligned}$$

   By our assumption of b, c, there is no $\ell \in \mathbb{F}_q$ such that $\sqrt[p]{(c - \ell b)/\ell} - \ell = 0$. The curve $G_s(X, Y, T) = 0$ is rational and irreducible and it is $\mathbb{F}_q$-isomorphic to $\mathcal{C}_s$.

3. Clear. □

Proposition 3.5. *Let $q = p^h$, $h > 2$. Two lines of the same type (r, s, u, v) do not intersect the same set of lines of the aregular spread $\mathcal{A}_{b,c}$.*

Proof. The assumption $h > 2$ implies $q + 1 > (p + 1)^2$. The curves $\mathcal{C}_r$, $\mathcal{C}_s$, $\mathcal{C}_u$, $\mathcal{C}_v$ have degree at most p and they have $q + 1$ $\mathbb{F}_q$-rational points (corresponding to the lines of the spread intersecting them). By Proposition 3.4, such curves are either absolutely irreducible or they consist of a repeated line. Thus, if two curves attached to the lines w_1 and w_2 of the same type share $q + 1$ $\mathbb{F}_q$-rational points, the corresponding polynomials must be proportional. By direct computations, this yields $w_1 = w_2$. □

Theorem 3.6. *Let $q = p^h$, $h > 2$. If two lines in* $\mathrm{PG}(3, q)$ *intersect the same set of lines of the spread $\mathcal{A}_{b,c}$ then one of them lies on the plane $Z = 0$ and the other on the plane $T = 0$.*

Proof. The reduced (absolutely irreducible) curves associated with the different types of lines (r, s, u, v) have degree $p + 1$, degree p or 1, degree p or 1, and degree 1, respectively. They can share $q + 1$ $\mathbb{F}_q$-rational points only in the following cases:

- both $\mathcal{C}_s$ and $\mathcal{C}_u$ have degree p;
- both $\mathcal{C}_s$ and $\mathcal{C}_u$ have degree 1;
- both $\mathcal{C}_s$ and $\mathcal{C}_v$ have degree 1;

- both $\mathcal{C}_u$ and $\mathcal{C}_v$ have degree 1.

The first case is not possible. The second case would imply $c = 0$, a contradiction. Recall that the lines $v = v_{x,y,z}$ are contained in the plane $T = 0$ of $\mathrm{PG}(3,q)$. The claim follows observing that if $\mathcal{C}_s$ or $\mathcal{C}_u$ have degree 1, then $s = s_{x,y,0,\ell}$ or $u = u_{x,y,0}$. So, both s and u are contained in the plane $Z = 0$. □

Theorem 3.7. *Let $q = p^h$, $h > 2$. Then there exists a set of $q^2 + 3$ lines resolving all the lines of $\mathrm{PG}(3,q)$.*

Proof. Consider the aregular spread $\mathcal{A}_{b,c}$ with $b, c \in \mathbb{F}_q^*$ and such that the polynomial $t^{p+1} - tb + c$ has no roots in $\mathbb{F}_q$. We already know by Theorem 3.6 that lines of $\mathrm{PG}(3,q)$ intersecting the same set of elements of $\mathcal{A}_{b,c}$ are contained in the planes $Z = 0$ or $T = 0$. Note that two lines in a fixed plane cannot intersect the same elements of $\mathcal{A}_{b,c}$. Consider two distinct extra lines w_1 and w_2 contained in $Z = 0$ intersecting the line $Z = T = 0$ at two distinct points. It is readily seen that $\mathcal{A}_{b,c} \cup \{w_1, w_2\}$ resolves all the lines of $\mathrm{PG}(3,q)$. □

Corollary 3.8. *If $q = p^h$, $h > 2$, then there exists a resolving set in $\Gamma_{3,q}$ of size $2q^2 + 2$.*

Proof. Consider the set of $q^2 + 3$ lines from Theorem 3.7 and use the argument of Theorem 2.9. One of the two extra lines could be an element of the other spread. Finally, delete one line from the modified regular spread. □

Finally, the following bounds are obtained combining again Theorem 2.16, with $s = 0$, and Theorem 2.19.

Corollary 3.9. *If $q = p^h$, $h > 2$, then the metric dimension of $\Gamma_{n,q}$ satisfies the inequality*

$$\mu(\Gamma_{n,q}) \leq \begin{cases} 2q^{n-1} + 2q^{n-2} + h_{n,2}(q), & \textit{if } n \equiv 0 \textit{ or } 1 \pmod 4, \\ 2q^{n-1} + 4q^{n-2} + h_{n,3}(q), & \textit{if } n \equiv 2 \pmod 4, \\ 2q^{n-1} + h_{n,1}(q), & \textit{if } n \equiv 3 \pmod 4, \end{cases}$$

where $h_{n,i}$ $(i = 1, 2, 3)$ is a polynomial of degree at most $n - 3$ whose coefficients depend only on n.

4 Resolving sets for small q

We performed a computer search to obtain sets of lines that are semi-resolving sets for lines in $\mathrm{PG}(3,q)$ for small q. We used MAGMA, a computer algebra system for symbolic computation developed at the University of Sydney; see [6]. We started classifying all set of lines of a certain size k. Then we extended the non-equivalent sets of size k using a backtracking algorithm.

In $\mathrm{PG}(3,2)$ there are 35 lines, so a semi-resolving set for lines must contain at least six elements. We found that there are 165 non-equivalent sets of lines of size six. Forty-eight of them are semi-resolving sets for lines in $\mathrm{PG}(3,2)$. An example is the following set of six lines:

$$\begin{gathered}\{\langle(1:0:0:0),(0:1:0:0)\rangle,\langle(0:1:0:0),(0:0:0:1)\rangle,\\ \langle(0:0:1:0),(0:0:0:1)\rangle,\langle(0:0:0:1),(1:1:0:0)\rangle,\\ \langle(0:1:0:0),(1:1:1:1)\rangle,\langle(1:1:0:0),(0:0:1:1)\rangle\}.\end{gathered}$$

In $\mathrm{PG}(3,3)$ there are 130 lines, so a semi-resolving set for lines must contain at least eight elements. We found that there are 10681 non-equivalent sets of lines of size seven. An exhaustive search by backtracking has proved that no set of lines of size eight or nine is a semi-resolving set for lines in $\mathrm{PG}(3,3)$. There exist semi-resolving sets for lines of size ten. An example is the following set of ten lines:

$$\begin{aligned}&\{\langle(1:0:0:0),(0:1:0:0)\rangle,\langle(1:0:0:0),(0:0:0:1)\rangle,\\&\ \langle(0:1:0:0),(0:0:1:0)\rangle,\langle(0:1:0:0),(1:0:0:1)\rangle,\\&\ \langle(1:0:1:2),(0:1:1:0)\rangle,\langle(1:0:0:2),(0:0:1:1)\rangle,\\&\ \langle(1:0:0:0),(0:1:1:2)\rangle,\langle(1:0:0:0),(0:1:1:0)\rangle,\\&\ \langle(0:1:1:1),(1:2:0:0)\rangle,\langle(1:1:1:0),(0:1:2:0)\rangle\}.\end{aligned}$$

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Building maximal green sequences via component preserving mutations*

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Abstract

We introduce a new method for producing both maximal green and reddening sequences of quivers. The method, called component preserving mutations, generalizes the notion of direct sums of quivers and can be used as a tool to both recover known reddening sequences as well as find reddening sequences that were previously unknown. We use the method to produce and recover maximal green sequences for many bipartite recurrent quivers that show up in the study of periodicity of T-systems and Y-systems. Additionally, we show how our method relates to the dominance phenomenon recently considered by Reading. Given a maximal green sequence produced by our method, this relation to dominance gives a maximal green sequence for infinitely many other quivers. Other applications of this new methodology are explored including computing of quantum dilogarithm identities and determining minimal length maximal green sequences.

Keywords: Cluster algebra, maximal green sequence, direct sum.

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1 Introduction

Quiver mutation is the fundamental combinatorial process which determines the generators and relations in Fomin and Zelevinsky's cluster algebras [15]. Cluster algebras have arisen in a variety of mathematical areas including Poisson geometry, Teichmüller theory, applications to mathematical physics, representation theory, and more. Quiver mutation is a local procedure that alters a quiver and produces a new quiver. Understanding how a quiver mutates is essential to understanding the corresponding cluster algebra. We will consider the problem of explicitly constructing sequences of mutations with some special properties.

1.1 Some history of the problem

A maximal green sequence, and more generally a reddening sequence, is a special sequence of quiver mutations related to quantum dilogarithm identities which was introduced by Keller [28, 29]. Such sequences of mutations do not exist for all quivers and determining their existence or nonexistence is an important problem. For a good introduction to the study of maximal green and reddening sequences see the work of Brüstle, Dupont, and Pérotin [3]. In addition to the role they play in quantum dilogrithm identities, these sequences of mutations are a key tool utilized in other cluster algebra areas. For example, the existence of a maximal green sequence allows one to categorify the associated cluster algebras following the work of Amiot [2]. Also the existence of a maximal green sequence is a condition which plays a role in the powerful results of Gross, Hacking, Keel, and Kontsevich [25] regarding canonical bases. These results use the notion of scattering diagrams to prove the positivity conjecture for a large class of cluster algebras. Additionally the existence of a reddening sequence is thought to be related to when a cluster algebra equals its upper cluster algebra [5, 34]. Maximal green sequences are also related to representation theory [3] and in the computation of BPS states in physics [1]. Our notion of a *component preserving* sequence of mutations, which will be defined in Section 3, is closely related to what has been called a *factorized* sequence of mutations [9, 10, 12] in the physics literature where particular attention has been paid to ADE Dynkin quivers. Our definition is more general which allows for use with both maximal green sequences and reddening sequences. Being able to work with reddening sequences is desirable since the existence of a reddening sequence is mutation invariant while the existence of a maximal green sequence is not [35]. Hence, the existence of a reddening sequences ends up being a invariant of the cluster algebra as opposed to just the quiver.

In general it can be a difficult problem to determine if a quiver admits a maximal green or reddening sequence. These sequences have been found or shown to not exist in the case of finite mutation type quivers by the work of a variety of authors [1, 4, 7, 33, 40] leaving the question of existence only to quivers that are not of finite mutation type. This makes finding these sequences particularly difficult as the exchange graph for such quivers can be very complicated. Additionally there are branches of the exchange graph, in which no amount of mutations can lead to a maximal green sequence; meaning random computer generated mutations are extremely unlikely to produce maximal green sequences for these quivers. In addition to finite mutation type quivers, headway has been made on specific families of quivers such as minimal mutation-infinite quivers [32] and quivers which are associated to reduced plabic graphs [17]. This gives us many quivers for which we know reddening or maximal green sequences for. This provides a foundation to produce reddening and maximal green sequences for quivers which are built out of these.

When a quiver does admit a maximal green or reddening sequence it is desirable to have an explicit and well understood construction of the sequence. Having the specific sequence of mutations and understanding the corresponding c-vectors gives us a product of quantum dilogarithms [28, 29] and an expression for the Donaldson-Thomas transformation of Kontsevich and Soibelman [31]. The method which we present in this paper allows one to explicitly produce the sequence so that it can be used to for the corresponding computation.

Work by Garver and Musiker [22], as inspired by [2] and [1], and later by Cao and Li [8] looked at using what has been called *direct sums* of quivers to produce maximal green and reddening sequences when the induced subquivers being summed exhibit the appropriate sequences. This heuristic approach of building large sequences of mutations from subquivers is essentially the direction we want to expand upon in this paper. Component preserving mutations are a way of taking known maximal green and reddening sequences for induced subquivers (which we will call components) and combining them together to obtain a maximal green or reddening sequence for the whole quiver. The direct sum procedure becomes a particular instance of the theory of component preserving mutations.

The methodology presented has an assortment of applications. It can be used to produce maximal green sequences for bipartite recurrent quivers, recover known results regarding admissible source mutation sequences for acyclic quivers, and show that the existence of a maximal green or reddening sequence is an example of a certain dominance phenomena in the sense of recent work by Reading [37].

1.2 Summary of the methodology

The goal of this paper is to develop a methodology which allows one to use reddening sequences of subquivers of a given quiver to build reddening sequences for larger quivers. Since mutation is a local procedure, only affecting neighboring vertices, this is a natural approach. Moreover, it is known that when a quiver has a maximal green or reddening sequence, then the same is true for any induced subquiver [35]. Hence, developing a method to produce a maximal green or reddening sequence from induced subquivers is a type of converse to this fact.

The method starts by breaking the quiver, Q, into subquivers which we call components; each of which has a known reddening sequence. The components will partition the vertices of the quiver, giving a *partitioned quiver* (Q, π), where $\pi := \pi_1/\pi_2/\cdots/\pi_\ell$ is a partition of the vertices of Q. We label the components Q_i. We start with the framed quiver, where we partition all of the frozen and mutable pairs into the same parts. We call this quiver the *framed partition quiver* $(\widehat{Q}, \widehat{\pi})$. We then try to *shuffle* the respective reddening sequences together to see if they form a reddening sequence for the entire quiver. It is not the case that one can always find a shuffle which works on the entire quiver. To guarantee that they do build a reddening sequence, we must check that at each mutation step the mutation vertex satisfies the component preserving condition which will be given in Definition 3.6. If this condition holds the main result of this paper shows that you have constructed a reddening sequence for the larger quiver.

Theorem 1.1 (Main Result). *Let $(\widehat{Q}, \widehat{\pi})$ be a framed partition quiver where for each $\widehat{Q_i}$ we have a reddening sequence σ_i. Let τ be a shuffle of the σ_i such that at every mutation step of the sequence τ we have that k is component preserving with respect to π. Then τ is a reddening sequence for $\widehat{Q}$.*

This main result is proven in Section 3 where is it restated in Theorem 3.11. This approach gives one a starting point as to where to search for reddening sequences given an arbitrary quiver. First break the quiver into subquivers you are comfortable constructing reddening sequences for; and then attempt to shuffle these sequences. This approach may initially seem overwhelming as you could consider any partition of the quiver into subquivers along with any shuffle of reddening sequences. However, as we explored utilizing this technique what we realized was that there are often very natural shuffles and partitions present in many commonly studied quivers. For instance, this concept generalizes the idea of direct sums of quivers where the shuffle takes the particular simple form of concatenation. Additionally, it can be used to give short and effective constructions of maximal green sequences for bipartite recurrent quivers, and many more examples where some *well behaved properties* of a specific quiver provides the recipe for how to shuffle and partition the vertices.

This article is structured in the following way. Section 2 will give some preliminaries for quiver mutation and the study of reddening sequences. In Section 3 we will present the main results of the paper outlining how the component preserving procedure can produce new maximal green and reddening sequences from induced subquivers. Within Section 3 we present a large amount of examples to try and illustrate how this procedure works. In the sections following this we look at some applications of this procedure to produce interesting and new results. Results related to dominance phenomena are in Section 4 and bipartite recurrent quivers are considered in Section 5. In Section 6 we consider the computation of Donaldson-Thomas invariants and minimal length maximal green sequences. We have added a large amount of examples to the article in an effort to try and give the reader an opportunity to become familiar with how one uses this method in a hands-on manner. This is intentional, as from exploring these methods it appears that many reddening sequences are built in this manner from small set of "basic reddening sequences." The intuition of the authors is that there may be a way to describe a list of "basic reddening sequences" from which any reddening sequence can be built. It is our hope that this paper is the first step in building the concrete theory behind this intuition.

2 Preliminaries

A *quiver* Q is a directed multigraph with vertex set $V(Q)$ and whose edge set $E(Q)$ contains no loops or 2-cycles. Elements of $E(Q)$ will typically be referred to as *arrows*. An *ice quiver* is a pair (Q, F) where Q is a quiver, $F \subseteq V(Q)$, and Q contains no arrows between elements of F. Vertices in F are called *frozen* while vertices in $V(Q) \setminus F$ are called *mutable*. The *framed quiver* associated to a quiver Q, denoted $\widehat{Q}$, is the ice quiver whose vertex set, edge set, and set of frozen vertices are the following:

$$\begin{aligned} V(\widehat{Q}) &\coloneqq V(Q) \sqcup \{i' \mid i \in V(Q)\}, \\ E(\widehat{Q}) &\coloneqq E(Q) \sqcup \{i \to i' \mid i \in V(Q)\}, \\ F &= \{i' \mid i \in V(Q)\}. \end{aligned}$$

The framed quiver corresponds to considering a cluster algebra with principal coefficients.

Given an ice quiver (Q, F) for any mutable vertex i, *mutation* at the vertex i produces a new quiver denoted by $(\mu_i(Q), F)$ obtained from Q by doing the following:

(1) For each pair of arrows $j \to i$, $i \to k$ such that not both i and j are frozen add an

arrow $j \to k$.

(2) Reverse all arrows incident on i.

(3) Delete a maximal collection of disjoint 2-cycles.

Mutation is not allowed at any frozen vertex. Since mutation does not change the set of frozen vertices we will often abbreviate an ice quiver (Q, F) by Q and $(\mu_i(Q), F)$ by $\mu_i(Q)$ where the set of frozen vertices is understood from context. We will be primarily focused on framed quivers and quivers which are obtained from a framed quiver by a sequence of mutations. In fact, whenever we have an ice quiver with a nonempty set of frozen vertices we will assume it is obtainable from a framed quiver by some sequence of mutations. So, the set of frozen vertices will be of a very particular form.

A mutable vertex is *green* if it there are no incident incoming arrows from frozen vertices. Similarly, a mutable vertex is *red* if there are no incident outgoing arrows to frozen vertices. If we start with an initial quiver Q and perform mutations at mutable vertices of the framed quiver $\widehat{Q}$, then any mutable vertex will always be either green or red. The result is known as *sign-coherence* and was established by Derksen, Weyman,and Zelevinsky [13]. For each vertex i in a quiver obtained from $\widehat{Q}$ by some sequence of mutations, the corresponding *c-vector* is defined by its jth entry being the number of arrows from i to j' (with arrows j' to i counting as negative). In these terms sign-coherence says a c-vector's entries are either nonnegative or nonpositive. Notice also that all vertices are initially green when starting with $\widehat{Q}$. Keller [28, 29] has introduced the following types of sequences of mutations which will be our main interest. A sequence mutations is called a *reddening sequence* if after preforming this sequence of mutations all mutable vertices are red. A *maximal green sequence* is a reddening sequence where each mutation occurs at a green vertex. When a sequence of mutations is a reddening sequence we may say it is a reddening sequence for either Q or $\widehat{Q}$. In terms of being a reddening sequence or not, the quiver Q and the framed quiver $\widehat{Q}$ are equivalent data.

We may write a maximal green or reddening sequence as either a sequence of vertices (read from left to right) or as a composition of mutations (read from right to left as is usual with composition of functions). For a quiver Q we will let $\mathrm{green}(Q)$ denote the set of maximal green sequences for Q. If we consider the quiver $Q = 1 \to 2$ there are exactly two maximal green sequences and we can record them either as

$$\mathrm{green}(Q) = \{(1, 2), (2, 1, 2)\}$$

in sequence of vertices notation or as

$$\mathrm{green}(Q) = \{\mu_2\mu_1, \mu_2\mu_1\mu_2\}$$

in composition notation.

We will need to modify and combine sequences of vertices when producing maximal green and reddening sequences. This is done by *shuffling* mutation sequences together.

Definition 2.1. A shuffle of two sequences $(a_1, a_2, \dots, a_k)$ and $(b_1, b_2, \dots, b_\ell)$ is any sequence whose entries are exactly the elements of $\{a_1, a_2, \dots, a_k\} \cup \{b_1, b_2, \dots, b_\ell\}$ (considered as a multiset) with the relative orders of $(a_1, a_2, \dots, a_k)$ and $(b_1, b_2, \dots, b_\ell)$ are preserved.

For example there are 6 shuffles of the sequences $(1,2)$ and (a,b). They are the sequences $(1,2,a,b)$, $(1,a,2,b)$, $(1,a,b,2)$, $(a,1,2,b)$, $(a,1,b,2)$, and $(a,b,1,2)$. In the next section we will define component preserving mutations and show how by checking for the component preserving property you can create shuffles of reddening sequences on induced subquivers whose result is a reddening sequence for a larger quiver.

3 Component preserving mutations

We start by establishing some basic definitions and notation of what we mean by a component of the quiver.

Definition 3.1. Let Q be an ice quiver with vertex set V. Then let $\pi = \pi_1/\pi_2/\cdots/\pi_\ell$ be a set partition of V. Then let Q_i be the induced subquiver of Q obtained by deleting every vertex $v \notin \pi_i$. We will call the Q_i the ***components*** of Q and the pair (Q,π) a ***partitioned quiver***.

Definition 3.2. When (Q,π) is a partitioned quiver with $\pi = \pi_1/\pi_2/\cdots/\pi_\ell$, we will define $\widehat{\pi}$ as the partition of $\widehat{V}$ where each $\widehat{\pi_i} = \{v, \widehat{v} \mid v \in \pi_i\}$. Then $(\widehat{Q},\widehat{\pi})$ will be called a ***partitioned ice quiver***.

Remark 3.3. In other words, for each mutable vertex v, the frozen copy of a vertex, $\widehat{v}$, lies in the same component as v. It is straight forward to see that $\widehat{(Q_i)} = (\widehat{Q})_i$.

Definition 3.4. Mutation of a partitioned ice quiver is defined as the following:

$$\mu_k((Q,\pi)) := (\mu_k(Q),\pi).$$

Definition 3.5. Let (Q,π) be a partitioned ice quiver. A ***bridging arrow*** $a \to b$ is any arrow in Q in which a and b are in different components.

Now we can talk about the definition that is crucial to all the results in the rest of the paper. This is the notion of component preserving vertices and component preserving mutations.

Definition 3.6. A vertex $k \in Q_i$ is ***component preserving*** with respect to π when one of the following occurs:

- If $\exists\, k \to j'$ for a frozen vertex j', then $\forall\, a \to k$ we have $a \in V(Q_i)$; or
- If $\exists\, j' \to k$ for a frozen vertex j', then $\forall\, k \to a$ we have $a \in V(Q_i)$.

Remark 3.7. Another way of thinking about component preserving mutations is in terms of sign-coherence. One can think of a component preserving vertex, k, as a vertex where *freezing* each mutable vertex outside of its component results in an ice quiver in which the extended exchange matrix is still sign-coherent with respect to this larger set of frozen vertices. In this way one can think of component preserving mutations as being a type of *locally sign-coherent* mutation.

Remark 3.8. Another observation to make is that whenever one starts from a framed quiver, mutation at component preserving vertices does not result in creating bridging arrows that involve frozen vertices. This means that any quiver which is the result of a sequence of component preserving mutations starting from a framed quiver has the ***support***

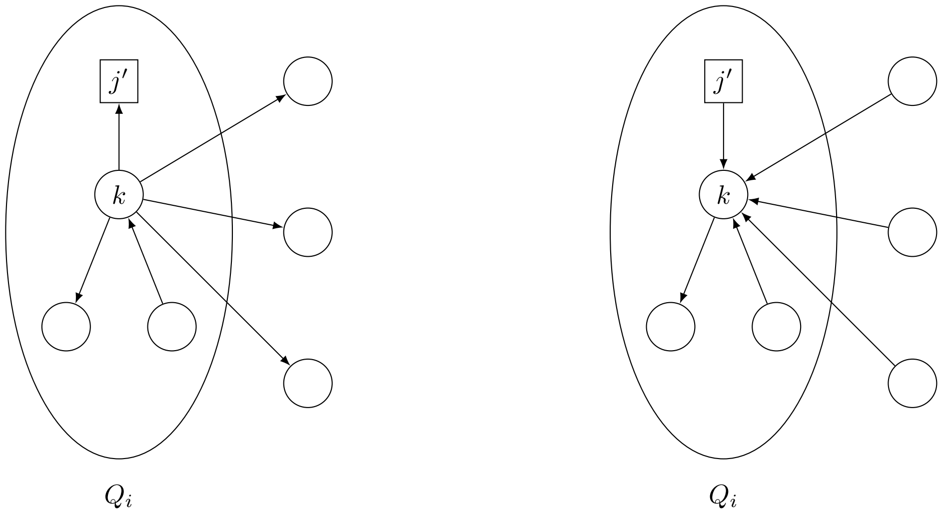

Figure 1: An illustration of a component preserving vertex $k \in Q_i$ on the left with arrow $k \to j'$ and on the right with arrow $j' \to k$.

of all of its c-vectors contained entirely within a component. In terms of the quiver, this means that the sequence of component preserving mutations results in a quiver in which all arrows involving frozen vertices are between mutable vertices and frozen vertices within the same component.

The choice of terminology is because performing mutation at a component preserving vertex, k, does not affect Q_i unless $k \in \pi_i$. We will prove this fact and then show how one can use this fact to shuffle maximal green sequences together if at every mutation step you mutated at a component preserving vertex.

3.1 Preservation proof

Now that we have the language to talk about components of the quiver, we want to set up a condition on a vertex, k, which forces μ_k to only affect the component which contains k and none of the other induced subquivers. This is exactly the property that component preserving vertices have.

Lemma 3.9. *Let (Q, π) be a partitioned ice quiver. If k is a component preserving vertex then $\mu_k(Q)_i = \mu_k(Q_i)\ \forall\ 1 \leq i \leq \ell$.*

Proof. First notice that these are in fact two ice quivers on the same set of vertices. To check that the lemma holds we need to see that each step of mutation has the same effect on the subquivers $\mu_k(Q)_i$ and $\mu_k(Q_i)$ for each i. The key step of mutation to check is where new arrows are created, which is step one in our definition of mutation. There are two cases to consider:

Case 1: $k \in \pi_i$. Let $a \to b$ be an arrow in $\mu_k(Q_i)$ created by mutation at vertex k. Then since Q_i is the quiver Q restricted to the component π_i we know that a, b along with k are elements of $V(Q_i)$. Therefore the arrows $a \to k$ and $k \to b$ are elements of $E(Q_i)$.

Therefore all of these arrows are present in Q and hence the arrow $a \to b$ is present in $\mu_k(Q)$. Since both endpoints of the arrow are in π_i the arrow $a \to b$ is also created in the mutation $\mu_k(Q)_i$.

We will now show this is a biconditional relationship. Assume $a \to b$ is an arrow in $\mu_k(Q)_i$ which is created from mutation. This occurs if and only if $a \to k \to b$ is present in Q and $a, b \in \pi_i$. Since we have assumed that $k \in \pi_i$ we know that $a, b, k \in \pi_i$ and the arrow $a \to b$ is also created in $\mu_k(Q_i)$.

Case 2: $k \notin \pi_i$. Since k is not a vertex in Q_i we will not be able to mutate the quiver Q_i in direction k. Therefore $\mu_k(Q_i) = Q_i$. Now what we must check is that *no* arrow $a \to b$ is created in $\mu_k(Q)_i$ by step (1) of mutation.

By way of contradiction, assume that $a \to b$ in $\mu_k(Q)_i$ is created by the composition of mutation and restriction. Then $a \to k \to b$ is present in Q and also $a, b \in \pi_i$. But since k is not in the same component as a and b, arrows $a \to k$ and $k \to b$ are bridging arrows in opposite directions. This is a contradiction since each component preserving vertex is incident to bridging arrows in at most one direction. □

3.2 Applications to reddening sequences and maximal green sequences

We have seen that if k is a component preserving vertex, then μ_k only affects arrows in Q_i and possibly bridging arrows. This can be extremely useful in the context of reddening sequences. The goal is to utilize reddening sequences on each component to create a reddening sequence for the larger quiver. This turns out to be possible if at each mutation step you are performing a component preserving mutation. The following is a useful consequence which follows directly from the sign-coherence of c-vectors as presented in [13] and Remark 3.8 on the support of c-vectors.

Lemma 3.10. *Let $(\widehat{Q}, \widehat{\pi})$ be a partitioned framed quiver. Let σ be any sequence of component preserving mutations. Also, let v be a vertex in the component π_i. Then the color of a vertex v in $\mu_\sigma(\widehat{Q})$ is the same as the color of the vertex v in $\mu_\sigma(\widehat{Q})_i$.*

Theorem 3.11. *Let $(\widehat{Q}, \widehat{\pi})$ be a framed partition quiver where for each $\widehat{Q_i}$ we have a reddening sequence σ_i. Then let τ be a shuffle of the σ_i such that at every mutation step of the sequence τ we have that k is component preserving with respect to π. Then τ is a reddening sequence for $\widehat{Q}$.*

Proof. Let $(\widehat{Q}, \widehat{\pi})$ be a framed partition quiver. Then since each mutation in τ is component preserving you have from the Lemma 3.9 that

$$\mu_\tau(\widehat{Q})_i = \mu_\tau(\widehat{Q_i}) = \mu_{\sigma_i}(\widehat{Q_i}).$$

Meaning that for each i any vertex $v \in \pi$ is red in $\mu_\tau(\widehat{Q})_i$ since it is the result of running a reddening sequence. It then follows from Lemma 3.10 that v is red in the larger quiver $\mu_\tau(\widehat{Q})$. □

Corollary 3.12. *Furthermore if additionally you have that each σ_i is a maximal green sequence for the component $\widehat{Q}_i$ then you have that τ is a maximal green sequence for $\widehat{Q}$.*

Proof. By Theorem 3.11 we know we have a reddening sequence. By Lemma 3.10 and Lemma 3.9 to decide if a mutation step occurred at a green vertex we only need to look at the component containing that vertex. Then we consider that each σ_i is a maximal green sequence and it follows from the same equation:

$$\mu_\tau(\widehat{Q})_i = \mu_\tau(\widehat{Q_i}) = \mu_{\sigma_i}(\widehat{Q_i}). \qquad \square$$

This can be quite useful. In practice what it tells you is that if you partition your quiver up into components, and you know a reddening (or maximal green) sequence for each component then you can try and shuffle the sequences together. If every mutation in the shuffle is component preserving, then you have successfully created a reddening (or maximal green) sequence for the larger quiver. In the sections that follow we will show some of the applications of using this approach to find maximal green and reddening sequences for a variety of quivers. Before showing new applications of the component preserving mutation method, we first provide some examples of previously known maximal green sequences that come from component preserving mutations. These known examples serve to show that our framework unifies many known maximal green sequences. Also the following examples aim to demonstrate that applications of Corollary 3.12 occur "in nature" and thus Definition 3.6 is not too restrictive as it includes many naturally occurring examples.

3.3 Example: Admissible source sequences

A sequence of vertices $(i_1, i_2, \ldots, i_n)$ of a quiver Q with n vertices is called an *admissible numbering* by sources if $\{i_1, i_2, \ldots, i_n\} = V(Q)$ and i_j is a source of $\mu_{i_{j-1}} \circ \cdots \circ \mu_{i_1}(Q)$. It is well known that any acyclic quiver Q admits an admissible numbering by sources and that any such admissible numbering by sources $(i_1, i_2, \ldots, i_n)$ is a maximal green sequence [3, Lemma 2.20]. In terms of component preserving mutations, $(i_1, i_2, \ldots, i_n)$ being an admissible numbering by sources means that $\tau = \mu_{i_n} \circ \mu_{i_{n-1}} \circ \cdots \circ \mu_{i_1}$ is a component preserving sequence of mutations with respect to the partition $\{i_1\}/\{i_2\}/\cdots/\{i_n\}$ of $V(Q)$ into singletons. Corollary 3.12 states $(i_1, i_2, \ldots, i_n)$ is a maximal green sequence in this special case. Figure 2 shows an example of an acyclic quiver where $(4, 1, 2, 3, 5)$ is a maximal green sequence from an admissible numbering by sources with the vertices as labeled in the figure.

$$\begin{tikzcd} & 3 & \\ 4 \arrow[r] & 1 \arrow[u] \arrow[r] \arrow[d] & 2 \\ & 5 & \end{tikzcd}$$

Figure 2: An acyclic quiver with maximal green sequence $(4, 1, 2, 3, 5)$.

3.4 Example: Direct sum

A *direct sum* of quivers A and B is any quiver Q with

$$V(Q) = V(A) \sqcup V(B)$$
$$E(Q) = E(A) \sqcup E(B) \sqcup E$$

where E is any set of arrows such which has for any $i \to j \in E$ implies $i \in V(A)$ and $j \in V(B)$. In other words, a direct sum of quivers simply takes the disjoint union of the two quivers then adds additional arrows between the quivers with the condition that all arrows are directed from one quiver to the other. We can take the partition $V(A)/V(B)$ of $V(Q)$ and the consider the concatenation $\tau = \tau_B\tau_A$ for any reddening sequence τ_A of A and τ_B of B. Then τ will be component preserving and hence a reddening sequence by Theorem 3.11

An example of a direct sum of quivers A and B where $V(A) = \{1, 2\}$ and $V(B) = \{4, 5, 6\}$ is given in Figure 3. We can take the maximal green sequences $(2, 1, 2)$ and $(4, 6, 5)$ on the components and obtain maximal green sequence $(2, 1, 2, 4, 6, 5)$ on the direct sum. We will not prove that such sequences of mutations are component preserving since proofs for maximal green sequences and reddening sequences of direct sums are already in the literature [22, Theorem 3.12], [8, Theorem 4.5].

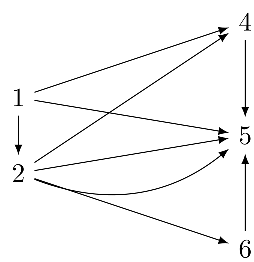

Figure 3: A direct sum of quivers with maximal green sequence $(2, 1, 2, 4, 6, 5)$.

3.5 Example: Square products

The *square product* of two Dynkin quivers is considered by Keller in his work on periodicity [30]. For two type A quivers the square product is a grid with all square faces oriented in a directed cycle. In Figure 4 we show a square product of type (A_2, A_n). Consider the partition $\pi = B/B'$ of the quiver in Figure 4 where B is the set of vertices in the top row and B' is the set of vertices in the bottom row. Then the quiver restricted to either B or B' is an alternating path which has a maximal green sequence of repeatedly applying sink mutations. A component preserving shuffle for these quivers can be found by alternating between mutations in B and B' until you have completed both maximal green sequences. This example generalizes to many other quivers in a family called *bipartite recurrent quivers*. Maximal green sequences for bipartite recurrent quivers will be investigated in more depth in Section 5.

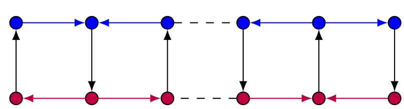

Figure 4: An arbitrary length square product of type (A_2, A_n).

3.6 Example: Dreaded torus

Let Q be the quiver shown in Figure 5 which comes from a triangulation of the torus with one boundary component and a single marked point on the boundary. With vertices as labeled in the figure we can take the partition $\{1,4\}/\{2,3\}$ and the maximal green sequences $(1,4,1)$ and $(3,2,3)$ on the two components. The sequence $(1,3,4,2,1,3)$ is component preserving and hence a maximal green sequence by Corollary 3.12. The quiver Q is an example of a quiver which admits a maximal green sequence, and hence a reddening sequence, but is not a member of the class $\mathcal{P}$ of Kontsevich and Soibelman [31]. So, Q should be included in a solution to a question posed by the first two authors which seeks to identify a collection of quivers which generate all quivers with reddening sequences by using quiver mutation and the direct sum construction [5, Question 3.6].

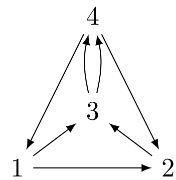

Figure 5: The quiver for the torus with one boundary component and one marked point. A maximal green sequence for this quiver is $(1,3,4,2,1,3)$.

3.7 Example: Cremmer-Gervais

In the Gekhtman, Shapiro, and Vainshtein approach to cluster algebras with Poisson geometry there is an exotic cluster structure on SL_n known as the Cremmer-Gervais cluster structure [23, 24]. The mutable part of the quiver defining this cluster structure for the case $n = 3$ is shown in Figure 6. The cluster algebra has the interesting property that whether or not it agrees with its upper cluster algebra is ground ring dependent [6, Proposition 4.1]. A maximal green sequence for the quiver in Figure 6 is $(2,3,4,1,5,1,6,3)$ which can be obtained by considering the partition $\{1,2,5\}/\{3,6\}/\{4\}$ along with maximal green sequences $(2,1,5,1)$, $(3,6,3)$, and (4). The authors believe it would be interesting to try the technique of component preserving maximal green sequences on quivers for the Cremmer-Gervais cluster structure for larger values for n.

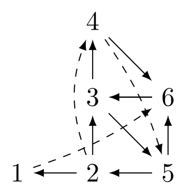

Figure 6: The mutable part of the quiver defining the Cremmer-Gervais cluster structure.

4 Applications to quiver dominance

One natural question that arises when discussing any algebraic object is to ask questions about what information can be extracted from considering the smaller sub-objects inside your larger object. The methods we have presented thus far give a way of producing reddening sequences on larger quivers by considering reddening sequences on quivers with fewer vertices. In this section we will give a way of producing reddening sequences on larger quivers by considering reddening sequences on quivers with fewer arrows but the same number of vertices.

Component preserving mutations give rise to a *dominance phenomenon* of quivers. In terms of matrices dominance is given by the following definition. One obtains a definition of dominance in quivers by considering its skew-symmetric exchange matrix.

Definition 4.1. Given $n \times n$ exchange matrices $B = [b_{ij}]$ and $A = [a_{ij}]$, we say B dominates A if for each i and j, we have $b_{ij}a_{ij} \geq 0$ and $|b_{ij}| \geq |a_{ij}|$.

An initiation of a systematic study of dominance for exchange matrices was put forth by Reading [37]. Dominance had previously been considered by Huang, Li, and Yang [26] as part of their definition of a *seed homomorphism*. One instance of the dominance phenomenon observed by Reading is the following observation about scattering fans.

Phenomenon 4.2 ([37, Phenomenon III])**.** Suppose that B and B' are exchange matrices such that B dominates B'. In many cases, the scattering fan of B refines the scattering fan of B'.

Remark 4.3. Following [25] to any quiver one can associate a cluster scattering diagram inside some ambient vector space. Reddening sequences and maximal green sequences then correspond to paths in the ambient vector space subject to certain restrictions coming from the scattering diagram. A cluster scattering diagram partitions the ambient vector into a complete fan called the scattering fan [38]. Hence, the phenomenon that the scattering fan of B often refines the scattering fan of B' when B dominates B' means that it should be more difficult to find a reddening sequence for B since the scattering diagram of B has additional walls imposing more constraints. However, we will find certain conditions for when a reddening sequence for B' will still work as a reddening sequence for B.

In this section we will apply the results of Section 3 to show that the existence of a reddening (maximal green) sequence passes through the dominance relationship in many cases. The interesting aspect of this result is it appears to go in the wrong direction; the property is passed from the dominated quiver to the dominating quiver. Let B dominate A. If A has a reddening (maximal green) sequence then, we wish to produce a reddening (maximal green) sequence for B. This is not a true statement in general, but if we put some restrictions on *how* B dominates A and *extra conditions* on the reddening or maximal green sequence this turns out to be true. Going forward we will consider dominance in terms of the quivers instead of exchange matrices. A reformulation of dominance is the following.

Definition 4.4. Given quivers B and A on the same vertex set we say that B dominates A if:

- for every pair of vertices (i, j) any arrows between i and j in A are in the same direction as any arrow between i and j in B; and

- for every pair of vertices (i, j) the number of arrows in B involving vertices i and j is greater than or equal to the number of arrows in A involving i and j.

For an example of quiver dominance see Figure 7 where multiplicity of an arrow greater than 1 is denoted by the number next to the arrow. We now need to establish the notion of π-*dominance*. This is a restrictive form of dominance, where we the quivers A and B have the same component subquivers with respect to a partition π but have the multiplicity of the bridging arrows altered in a consistent way.

Figure 7: An example where the quiver on the right dominates the quiver on the left.

Definition 4.5. Let (A, π) and (B, π) be two partitioned ice quivers with the same vertex set and same set partition π. We say that B π-dominates[1] A if:

- the component quivers $A_i = B_i$ for each i;
- for all $u \in B_i$ and $v \in B_j$ with $i \neq j$ we have the $\#(u \to v \text{ in } B)$ is equal to $d_{ij} \times \#(u \to v \text{ in } A)$, where d_{ij} is a positive integer that is the same for the entire i-th and j-th components.

The d_{ij} are called the *dominance constants* associated to (B, π) and (A, π). As usual in Definition 4.5 arrows in the opposite direction are counted as negative. A practical way of thinking about π-dominance is that B is obtained from the A by scaling up the multiplicity of the bridging arrows between components by the appropriate dominance constant. Notice that the dominance constants are always positive, and hence bridging arrows are always in the same direction after scaling by the dominance constants. An example of π-dominance can be seen in Figure 8. This example has the type (A_2, A_4) square product on the left side and the Q-system quiver of type A_4 on the right side.

Figure 8: This is π-dominance where the components are the horizontal rows of the quiver. The right hand quiver π-dominates the left hand quiver and $d_{12} = 2$.

Theorem 4.6. *Let k be a component preserving vertex in (A, π) and (B, π) be an ice quiver which π-dominates A with dominance constants d_{ij}. Then $\mu_k(B)$ dominates $\mu_k(A)$ with dominance constants d_{ij}.*

Proof. Since k is a component preserving vertex in (A, π) we know that k is also a component preserving vertex in (B, π) since the direction of the bridging arrows is unchanged by scaling by the multiple d_{ij}. Also as k is component preserving in both A and B we know

[1]This is a more restrictive version of the dominance phenomena presented by Reading. In general, not all quivers B which dominate a quiver A will π-dominate the quiver.

by Lemma 3.9 that $\mu_k(A)_i = \mu_k(A_i) = \mu_k(B_i) = \mu_k(B)_i$. Therefore we only need to consider the bridging arrows between components.

The bridging arrows incident to k are only affected by the step of mutation which reverses arrows incident to k. Therefore dominance is preserved for these arrows because they are reversed by mutation at k in both A and B.

Now we must check the number of bridging arrows created during mutation for both $\mu_k(B)$ and $\mu_k(A)$. For some nonnegative integer α, we will use the notation $i \xrightarrow{\alpha} j$ to denote that there are α arrows from i to j in a quiver.

Assume $s \xrightarrow{\alpha} k \xrightarrow{\beta} t$ is present in A with $\alpha, \beta \geq 0$. Then mutation will create arrows from $s \to t$ with multiplicity $\alpha\beta$. Since we need only consider bridging arrows we will assume the $\alpha\beta$ many arrows from s to t created are bridging arrows. In the case that k is green we know that s must be in the same component as k because k is component preserving. Assume $k, s \in V(A_i)$ and $t \in V(A_j)$ for $i \neq j$. We now will show that $\mu_k(B)$ creates $d_{ij}\alpha\beta$ arrows from s to t. The presence of $s \xrightarrow{\alpha} k \xrightarrow{\beta} t$ in A implies that there is $s \xrightarrow{\alpha} k \xrightarrow{d_{ij}\beta} t$ in B. Therefore mutation at k in B creates $d_{ij}\alpha\beta$ arrows $s \to t$. Now we can consider the multiplicity of bridging arrows resulting from cancellation of 2-cycles mutation. In $\mu_k(A)$ the multiplicity of the arrows from s to t is $\alpha\beta + \gamma$, where γ is the number of arrows from s to t in A (here we allow γ to be negative if there are arrows from t to s). In $\mu_k(B)$ the multiplicity of arrows from s to t is $d_{ij}\alpha\beta + d_{ij}\gamma$ since there are $d_{ij}\gamma$ arrows from s to t in B by the assumption that B π-dominates A. Therefore there are exactly $d_{ij}(\alpha\beta + \gamma)$ arrows from s to t in $\mu_k(B)$ which is exactly the condition needed to say that $\mu_k(B)$ π-dominates $\mu_k(A)$.

The case where k is red is very similar. In this case t must be in the same component as k because k is component preserving. The presence of $s \xrightarrow{\alpha} k \xrightarrow{\beta} t$ in A now implies that there is $s \xrightarrow{d_{ij}\alpha} k \xrightarrow{\beta} t$ in B. Again mutation at k in B creates $d_{ij}\alpha\beta$ arrows $s \to t$ and the rest of the argument follows the case where k was green. □

We can now state our main result regarding dominance, that certain reddening sequences can be passed from a quiver A to a π-dominating quiver B.

Corollary 4.7. *Let (A, π) be a partitioned quiver, with $\pi = \pi_1/\pi_2/\cdots/\pi_\ell$. Let $\sigma_1, \sigma_2, \ldots, \sigma_\ell$ be reddening sequences for $A_1, A_2, \ldots, A_\ell$ respectively. If A admits a reddening sequence, τ, which is a component preserving shuffle of $\sigma_1, \sigma_2, \ldots, \sigma_\ell$ and B π-dominates A, then τ is also a reddening sequence for B. Moreover, if τ is a maximal green sequence for A, then τ is a maximal green sequence for B.*

Proof. Theorem 4.6 shows that each component preserving mutation in A is also a component preserving mutation in B. Therefore the mutation sequence τ is a component preserving sequence for B since it is a component preserving sequence for A. The definition of π-dominance tells us that $A_1 = B_1, A_2 = B_2, \ldots, A_\ell = B_\ell$. Therefore since $\sigma_1, \sigma_2, \ldots \sigma_\ell$ are reddening sequences for $A_1, A_2, \ldots, A_\ell$, they are also reddening sequences for $B_1, B_2, \ldots, B_\ell$. Then by Theorem 3.11 and Corollary 3.12 we have that they are in fact reddening sequences and additionally maximal green in the case where each σ_i is a maximal green sequence. □

Now we are equipped to use π-dominance to produce reddening and maximal green sequences for the dominating quivers by having *well behaved* sequences on the dominated quiver. We conclude this section with a few examples each providing a family of applications of Corollary 4.7.

4.1 Examples of applying Corollary 4.7

Corollary 4.7 applies to any case where one can produce a maximal green or reddening seqeunce using component preserving mutations. Thus, this result can be applied in many cases to produce infinite families of examples. In this section we highlight a few examples.

Example 4.8 (Dreaded torus)**.** Previously much attention has been paid to maximal green sequences for finite mutation type quivers (see [33]). In Section 3.6 we saw one example of a maximal green sequence for a finite mutation type quiver using component preserving mutations. Now we revisit this example, except we can scale the bridging arrows between the components and leave the case of finite mutation type. By Corollary 4.7 we know that the original maximal green sequence for the dreaded torus will also be a maximal green sequence for all π-dominating quivers. Therefore $(1, 3, 4, 2, 1, 3)$ is a maximal green sequence for all of the quivers in Figure 9, where a is a positive integer. This is an example of a quiver where the shuffle is not one that can be obtained from direct sum results as the partition does not form a direct sum of either the original quiver or the π-dominating quivers.

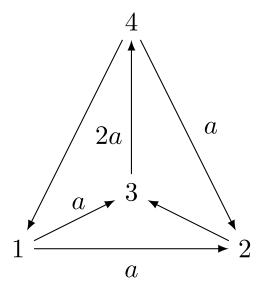

Figure 9: For each positive integer a, Corollary 4.7 produces a maximal green sequence for the quiver, which was the maximal green sequence from the dreaded torus. The maximal green sequence is $(1, 3, 4, 2, 1, 3)$.

Example 4.9 (The cycle)**.** Another example of finite mutation type quiver is the directed cycle quiver with vertex set $\{1, 2, \ldots, n\}$ and arrow set $\{i \to (i+1) : 1 \leq i < n\} \cup \{n \to 1\}$. In [4, Lemma 4.2] it is shown this quiver has the maximal green sequence

$$(1, 2, \ldots, n-2, n-1, n, n-2, n-3, \ldots, 2, 1)$$

which can be seen to be component preserving with respect to the partition $\{1, 2, \ldots, n-3, n-2, n\}/\{n-1\}$. By applying Corollary 4.7 we then obtain maximal green sequences for many quivers of infinite mutation type. The case $n = 6$ is shown in Figure 10.

Example 4.10 (Q-systems)**.** Consider Figure 11 when $\alpha = 2$ in which we can produce a maximal green sequence for the Q-system quiver of type A_4 by utilizing the maximal green sequence from the square product quiver of type (A_2, A_4). This technique also produces

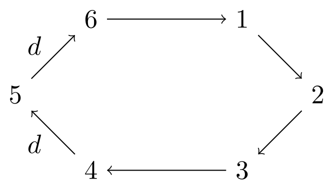

Figure 10: A quiver dominating the cycle which has the maximal green sequence $(1, 2, 3, 4, 5, 6, 4, 3, 2, 1)$.

maximal green sequences for other Q-system quivers (see [14, 27]) which are dominating quivers of square products. The next section will focus on producing maximal green sequences for a variety of bipartite recurrent quivers.

Figure 11: This is π-dominance where the components are the horizontal rows of the quiver. The square product quiver on the left has a maximal green sequence compatible with a π component preserving shuffle of $(2, 3, 6, 7, 1, 4, 5, 8, 2, 3, 6, 7, 1, 4, 5, 8, 2, 3, 6, 7)$. Corollary 4.7 shows that the quiver on the left where α is any positive integer admits the same maximal green sequence.

5 Bipartite recurrent quivers

In this section we consider certain quivers arising in the setting of T-systems and Y-systems. An early application of cluster algebras was Fomin and Zelevinsky's proof of periodicity for Y-systems associated to root systems [16] which was conjectured by Zamolodchikov [42]. This has lead to many more applications of cluster algebra theory in periodicity for T-systems and Y-systems. We will focus on work of Galashin and Pylyavskyy on bipartite recurrent quivers [18, 19, 20]. For certain bipartite recurrent quivers we will produce maximal green sequences in Theorem 5.3. An important ingredient in our constructions of maximal green sequences will be an extension of Stembridge's bigraphs [41]. The pattern for the maximal green sequences produced in this section was originally observed by Keller in the case of square products [30]. For a quantum field theory perspective on the results in this section we refer the reader to [10] where some of the same mutation sequences we construct are also considered. The main contribution of this section is to demonstrate how component preserving mutation neatly establishes the existence of a maximal green sequence for all quivers in Galashin and Pylyavskyy's classification of Zamolodchikov periodic quivers [18] as well as for some additional bipartite recurrent quivers.

We call a quiver Q *bipartite* if there exists a map $\epsilon\colon V(Q) \to \{0, 1\}$ such that $\epsilon(i) \neq \epsilon(j)$ for every arrow $i \to j$ of Q. The choice of such a map ϵ when it exists for a quiver Q is called a *bipartition*. Given a bipartition ϵ for Q a vertex $i \in V(Q)$ will be called *white* if $\epsilon(i) = 0$ and *black* if $\epsilon(i) = 1$. Let $i_1, i_2, \ldots, i_\ell$ denote the white vertices and Q and

$j_1, j_2, \ldots, j_m$ denote the black vertices. We then let

$$\mu_\circ = \mu_{i_1} \circ \mu_{i_2} \circ \cdots \circ \mu_{i_\ell}$$

and

$$\mu_\bullet = \mu_{j_1} \circ \mu_{j_2} \circ \cdots \circ \mu_{j_m}$$

denote the mutations at all white vertices or black vertices respectively. Since the quiver is bipartite no white vertex is adjacent to any other white vertex and so the order of mutation among the white vertices in $\mu_\circ$ does not matter. Similarly the order among the black vertices in $\mu_\bullet$ does not matter. A bipartite quiver Q is *recurrent* if both $\mu_\circ(Q) = Q^{op}$ and $\mu_\bullet(Q) = Q^{op}$ where Q^{op} denotes the quiver obtained from Q by reserving the direction of all arrows. Thus for a bipartite recurrent quiver we have $\mu_\bullet(\mu_\circ(Q)) = Q$ and $\mu_\circ(\mu_\bullet(Q)) = Q$.

A *bigraph* is a pair (Γ, Δ) of undirected graphs on the same underlying vertex set with no edges in common. Let A_Γ and A_Δ denote the adjacency matrices of Γ and Δ respectively. Given any bipartite quiver Q with bipartition ϵ we obtain a bigraph $(\Gamma(Q), \Delta(Q))$ on vertex set $V(Q)$ where $\Gamma(Q)$ has an edge $\{i, j\}$ for each arrow $i \to j$ in Q with $\epsilon(i) = 0$ and $\Delta(Q)$ has an edge $\{i, j\}$ for each arrow $i \to j$ of Q with $\epsilon(i) = 1$. By abuse of notation we may also think of $\Gamma(Q)$ and $\Delta(Q)$ as directed graphs with the direction of edge inherited from the quiver. Galashin and Pylyavskyy have shown that a bipartite quiver Q is recurrent if and only if $A_{\Gamma(Q)}$ and $A_{\Delta(Q)}$ commute [18, Corollary 2.3]. A bigraph (Γ, Δ) is called an *admissible ADE bigraph* if every component of both Γ and Δ is an ADE Dynkin diagram and the adjacency matrices of Γ and Δ commute. In the case of an admissible ADE bigraph, each connected component of Γ, and similarly of Δ, will be an ADE Dynkin diagram will the same Coxter number [41, Corollary 4.4]. More generally, we wish to also consider what we will refer to as *half-finite* bigraphs where for at least one of Γ or Δ each connected component is a ADE Dynkin diagram. Note the half-finite case includes both the admissible ADE bigraph case (which are exactly those quivers which are Zamolodchikov periodic [18]) as well as the *affine* $\boxtimes$ *finite* case in the classification of Galashin and Pylyavskyy [20]. An example of a bipartite recurrent quiver is shown in Figure 12. Let Q denote the bipartite recurrent quiver in Figure 12. The edges of $\Gamma(Q)$ correspond to the thick red arrows while the edges of $\Delta(Q)$ correspond to the thin blue arrows.

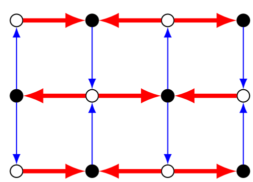

Figure 12: An example of a bipartite recurrent quiver.

For an ADE Dynkin diagram Λ we denote its Coxeter number by $h(\Lambda)$ and its number of positive roots by $|\Phi_+(\Lambda)|$. These quantities will be important in the maximal green sequences we construct. Table 1 shows the values for $h(\Lambda)$ and $|\Phi_+(\Lambda)|$ for each ADE Dynkin diagram Λ. We now present a result due to Galashin and Pylyavskyy generalizing the result for admissible ADE bigraphs.

Table 1: Coxeter numbers and number of positive roots for ADE types.

Λ	A_n	D_n	E_6	E_7	E_8
$h(\Lambda)$	$n+1$	$2n-2$	12	18	30
$\lvert\Phi_+(\Lambda)\rvert$	$\binom{n+1}{2}$	n^2-n	36	63	120

Lemma 5.1 ([20, Corollary 1.1.9]). *If (Γ,Δ) is a half-finite bigraph so that each component of Γ is an ADE Dynkin diagram, then the Coxeter number of each component of Γ will be the same.*

If Q is an orientation of an ADE Dynkin diagram Γ, then the length of the longest possible maximal green sequence is $|\Phi_+(\Lambda)|$ which has been shown in [3, Theorem 4.4] and [36, Proposition 7.3]. A quiver Q is an *alternating orientation* of an ADE Dynkin diagram Λ if it is an orientation of Λ so that every vertex is either a source or sink. In the case we have an alternating orientation, we will be interested in a certain maximal green sequence of length $|\Phi_+(\Lambda)|$ coming from bipartite dynamics. We may assume we have a bipartition of Q such that all sinks are the white vertices and all sources are the black vertices. The maximal green sequence in the following lemma was first observed by Keller [29].

Lemma 5.2 ([29]). *Let Q be an alternating orientation of an ADE Dynkin diagram with Coxeter number h. If $h=2k$, then $(\mu_\bullet\mu_\circ)^k$ is a maximal green sequence. If $h=2k+1$, then $\mu_\circ(\mu_\bullet\mu_\circ)^k$ is a maximal green sequence.*

We are ready to state and prove our theorem which gives a maximal green sequence for any half-finite bipartite recurrent quiver. Notice the assumption that $\Gamma(Q)$ consists of connected components which are all ADE Dynkin diagrams can easily be exchanged for the assumption that $\Delta(Q)$ consists of connected components which are all ADE Dynkin diagrams. Also the assumption on white vertices is only to allow us to explicitly state the maximal green sequences. An easy modification gives the correct statement of the theorem with the roles of black and white vertices reversed.

Theorem 5.3. *Let Q be a half-finite bipartite recurrent quiver. Assume that $\Gamma(Q)$ consists of connected components which are all ADE Dynkin diagrams. Further assume that with the orientation induced by Q the white vertices are sinks in $\Gamma(Q)$ and sources is $\Delta(Q)$. Let h be the Coxeter number of some component of $\Gamma(Q)$. If $h=2k$ is even, then $(\mu_\bullet\mu_\circ)^k$ is a maximal green sequence of Q. If $h=2k+1$ is odd, then $\mu_\circ(\mu_\bullet\mu_\circ)^k$ is a maximal green sequence of Q.*

Proof. We will construct a maximal green sequence for Q via component preserving mutations where components are given by the connected components of $\Gamma(Q)$. By construction within each component every vertex will be either a source or sink. Under our assumptions white vertices are initially sinks while black vertices are initially sources within each component. Since Q is a bipartite recurrent quiver $\mu_\circ(Q)=Q^{op}$ and $\mu_\bullet(\mu_\circ(Q))=Q$. Initially, mutation at any white vertex will be component preserving as each white vertex is a sink within its component and thus all arrows to other components will be outgoing. Mutation at a given white vertex will not change the fact another white vertex is component preserving. For the same reason mutation at any black vertex is component preserving in Q^{op}. It follows that $(\mu_\bullet\mu_\circ)^m$ and $\mu_\circ(\mu_\bullet\mu_\circ)^m$ are component preserving sequences of mutations for any m. By Lemma 5.1 each component has the same Coxeter number. Lemma 5.2 says that we do indeed have maximal green sequences on each component and therefore the theorem is proven by appealing to Corollary 3.12. □

6 Other applications

In this section we provide a variety of uses of the technique of component preserving mutations.

6.1 Quantum dilogarithms

We will review Keller's [28] association of a product of quantum dilogarithms with a sequence of mutations. We will then consider properties of such products of quantum dilogarithms which come from component preserving mutations. Let $q^{\frac{1}{2}}$ be an indeterminant. We define the *quantum dilogarithm* as

$$\mathbb{E}(y) = 1 + \frac{q^{\frac{1}{2}}y}{q-1} + \cdots + \frac{q^{\frac{n^2}{2}}y^n}{(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})} + \cdots$$

which is consider as an element of the power series ring $\mathbb{Q}(q^{\frac{1}{2}})[[y]]$. Keller has shown how reddening sequences give identities of quantum dilogarithms in a certain quantum algebra determined by a quiver.

Given a quiver Q with vertex set V and skew-symmetric adjacency matrix $B = (b_{uv})$ we obtain a lattice $\Lambda = \mathbb{Z}^V$ with basis $\{e_v\}_{v\in V}$. There is a skew-symmetric bilinear form $\lambda\colon \Lambda \times \Lambda \to \mathbb{Z}$ defined by

$$\lambda(e_u, e_v) := b_{uv}.$$

The *completed quantum algebra* of the quiver Q, denoted by $\widehat{\mathbb{A}}_Q$, is then the noncommutative power series ring modulo relations defined as

$$\widehat{\mathbb{A}}_Q := \mathbb{Q}(q^{\frac{1}{2}})\langle\langle y^\alpha, \alpha \in \Lambda : y^\alpha y^\beta = q^{\frac{1}{2}\lambda(\alpha,\beta)}y^{\alpha+\beta}\rangle\rangle.$$

For any sequence $\sigma = (i_1, i_2, \ldots, i_N)$ of vertices in Q we define

$$Q_{\sigma,t} := \mu_{i_t} \circ \mu_{i_{t-1}} \circ \cdots \circ \mu_{i_1}(Q)$$

for $0 \le t \le N$ where $Q_{\sigma,0} = Q$. We then define the product $\mathbb{E}_{Q,\sigma} \in \widehat{\mathbb{A}}_Q$ as

$$\mathbb{E}_{Q,\sigma} := \mathbb{E}(y^{\epsilon_1\beta_1})^{\epsilon_1}\mathbb{E}(y^{\epsilon_2\beta_2})^{\epsilon_2}\cdots\mathbb{E}(y^{\epsilon_N\beta_N})^{\epsilon_N}$$

where β_t is the c-vector corresponding to vertex i_t in $Q_{\sigma,t-1}$ and $\epsilon_t \in \{\pm 1\}$ is the common sign on the entries of β_t. If σ is a reddening sequence, then $\mathbb{E}_{Q,\sigma}$ is known as the *combinatorial Donaldson-Thomas invariant* of the quiver Q. If σ and σ' are two reddening sequences, then we have the quantum dilogarithm identity $\mathbb{E}_{Q,\sigma} = \mathbb{E}_{Q,\sigma'}$ [29, Theorem 6.5].

In the case that $\alpha = \sum_{i\in I} e_i$ where $I = \{i_1, i_2, \ldots, i_\ell\}$ we may write $y_{i_1 i_2\cdots i_\ell}$ in place of y^α. Using this abbreviated notation, the well known *pentagon identity* is

$$\mathbb{E}(y_1)\mathbb{E}(y_2) = \mathbb{E}(y_2)\mathbb{E}(y_{12})\mathbb{E}(y_1) \tag{6.1}$$

and can be seen by looking at the two maximal green sequences for the quiver $Q = (1 \to 2)$. Now consider the quiver in Figure 13 which is an alternating orientation of the Dynkin diagram A_3. The two maximal green sequences

$$(2, 1, 3)$$

and

$$(1, 3, 2, 1, 3, 2)$$

give the quantum dilogarithm identity

$$\mathbb{E}(y_2)\mathbb{E}(y_1)\mathbb{E}(y_3) = \mathbb{E}(y_1)\mathbb{E}(y_3)\mathbb{E}(y_{123})\mathbb{E}(y_{23})\mathbb{E}(y_{12})\mathbb{E}(y_2). \tag{6.2}$$

Reineke [39] has given quantum dilogarithm identities associated to any alternating orientation of an ADE Dynkin diagram which generalize Equations (6.1) and (6.2). Using cluster algebra theory, Keller [29] has further generalized these identities to square products associated to pairs of ADE Dynkin diagrams. Even more general identities follow from Theorem 5.3 since we have now produced two maximal green sequences for any Zamolodchikov periodic quiver.

$$1 \longleftarrow 2 \longrightarrow 3$$

Figure 13: An alternating orientation of the Dynkin diagram A_3.

Let us give a few properties of quantum dilogarithm products coming from component preserving mutations. For $\alpha = \sum_i a_i e_i \in \Lambda$ we define its *support* to be $\mathrm{Supp}(\alpha) := \{i : a_i \neq 0\}$. Consider a quiver Q, a subset of vertices $C \subseteq V(Q)$, and a sequence of vertices $\sigma = (i_1, i_2, \ldots, i_N)$. Define $\sigma|_C$ to be the restriction of σ to C (i.e. σ where all vertices not in C have been deleted). Again write

$$\mathbb{E}_{Q,\sigma} = \mathbb{E}(y^{\epsilon_1\beta_1})^{\epsilon_1}\mathbb{E}(y^{\epsilon_2\beta_2})^{\epsilon_2}\cdots\mathbb{E}(y^{\epsilon_N\beta_N})^{\epsilon_N}$$

and define $(\mathbb{E}_{Q,\sigma})|_C$ to be the product $\mathbb{E}_{Q,\sigma}$ (taken in the same order) with the terms $\mathbb{E}(y^{\epsilon_t\beta_t})^{\epsilon_t}$ removed whenever $i_t \notin C$. We now provide a proposition which tells us that when a reddening sequence of component preserving mutations is performed, there is a restriction on the support of the c-vectors occurring in the combinatorial Donaldson-Thomas invariant. The proposition follows readily from the definitions and Remark 3.8. When π is a set partition of a set X and $x \in X$ is an element of that set, we will use $\pi(x)$ to denote the block of the set partition π which contains x.

Proposition 6.1. *Let (Q, π) be a partitioned quiver so that $\sigma = (i_1, i_2, \ldots, i_N)$ is a component preserving sequence of vertices. If $C = Q_j$ is some component, then $\mathbb{E}_{Q,\sigma|_C} = (\mathbb{E}_{Q,\sigma})|_C$. Moreover, we have that $\mathrm{Supp}(\beta_t) \subseteq \pi(i_t)$ for each $1 \leq t \leq N$.*

When Q is such that $(\Gamma(Q), \Delta(Q))$ is an admissible ADE bigraph we can obtain a second maximal green sequence from Theorem 5.3 by exchanging the roles of $\Gamma(Q)$ and $\Delta(Q)$. A square product of two ADE Dynkin diagrams produces a quiver Q such that $(\Gamma(Q), \Delta(Q))$ is an admissible ADE bigraph. For square products of ADE Dynkin diagrams Keller [29] has previously produced the maximal green sequences in Theorem 5.3. The square product of A_3 and A_4 is shown in Figure 12. Stembridge's classification [41] of admissible ADE bigraphs includes more than just those bigraphs encoding square products of ADE Dynkin diagrams. Thus, Theorem 5.3 provides new quantum dilogarithm identites which can be thought of as generalizations of the pentagon identity. An infinite family examples of quivers which are not square products are the *twists* of an ADE Dynkin diagrams [41, Example 1.4]. The quiver Q which is the twist of A_3 is shown in Figure 14. On the left of Figure 14 the quiver is pictured to indicated the bigraph $(\Gamma(Q), \Delta(Q))$, and on the right we show the quiver with vertex labels. The two expressions of the combinatorial Donaldson-Thomas invariant of Q obtain from the maximal green sequences constructed in Theorem 5.3 are

$$\mathbb{E}(y_1)\mathbb{E}(y_3)\mathbb{E}(y_4)\mathbb{E}(y_6)\mathbb{E}(y_{123})\mathbb{E}(y_{456})\mathbb{E}(y_{23})\mathbb{E}(y_{12})\mathbb{E}(y_{56})\mathbb{E}(y_{45})\mathbb{E}(y_2)\mathbb{E}(y_4) \quad (6.3)$$

and

$$\mathbb{E}(y_2)\mathbb{E}(y_5)\mathbb{E}(y_{15})\mathbb{E}(y_{35})\mathbb{E}(y_{24})\mathbb{E}(y_{26})\mathbb{E}(y_{246})\mathbb{E}(y_{135})\mathbb{E}(y_1)\mathbb{E}(y_3)\mathbb{E}(y_4)\mathbb{E}(y_6). \quad (6.4)$$

These expressions are equal and give one example of the quantum dilogarithm identities obtained from Theorem 5.3. Looking at supports we can verify Proposition 6.1 in this example. Expression (6.3) comes from considering $\{1, 2, 3\}$ and $\{4, 5, 6\}$ as components while Expression (6.4) comes from considering $\{1, 3, 5\}$ and $\{2, 4, 6\}$ as components. The maximal green sequences corresponding to the products of quantum dilogarithms in Equations (6.3) and (6.4) are

$$(1, 3, 4, 6, 2, 5, 1, 3, 4, 6, 2, 5)$$

and

$$(2, 5, 1, 3, 4, 6, 2, 5, 1, 3, 4, 6)$$

respectively.

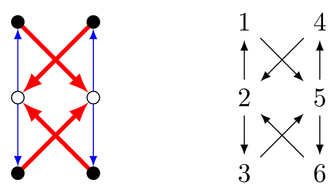

Figure 14: The quiver obtained from the twist of A_3.

6.2 Minimal length maximal green sequences

There has been recent interest in finding maximal green sequences of minimal possible length for a given quiver [11, 21]. We will now show how minimal length maximal green sequences can be constructed with component preserving mutations. In additional to being a natural question to ask about maximal green sequences, it has been observed by Garver, McConville, and Serhiyenko that the minimal possible length of a maximal green sequence may be related to derived equivalence of cluster tilted algebras (see [21, Question 10.1]). The following result is a component preserving generalization of [21, Proposition 4.4] which considers the direct sum case.

Lemma 6.2. *Let (Q, π) be a partitioned quiver with $\pi = \pi_1/\pi_2/\cdots/\pi_\ell$. Also let σ_i be a minimal length maximal green sequence for Q_i for each $1 \leq i \leq \ell$. If τ is a component preserving shuffle of $\sigma_1, \sigma_2, \ldots, \sigma_n$, then τ is a minimal length maximal green sequence for Q.*

Proof. Let L_i be the length of a minimal length maximal green sequence of Q_i for each $1 \leq i \leq \ell$ and let $L = L_1 + L_2 + \cdots + L_\ell$. By Corollary 3.12 we know that τ is a maximal green sequence and will have length L. So, we now need to show that there are no shorter maximal green sequences. Consider any maximal green sequence τ' for Q. By [21, Theorem 3.3] it follows that for each $1 \leq i \leq \ell$ there is a subsequence of mutations in τ' at vertices in Q_i which is a maximal green sequence of Q_i. This means τ' must mutate at vertices of Q_i at least L_i times for each $1 \leq i \leq \ell$. Since π is a partition, Q_i and Q_j share no vertices when $i \neq j$. It follows that τ' has length at least $L = L_1 + L_2 + \cdots + L_\ell$. $\square$

To illustrate a use of Lemma 6.2, let Q be the quiver[2] in Figure 15. We will take the set partition $\{v_1, v_2, v_3, v_4, v_5\}/\{u_1, u_2, u_3, u_4\}$. A minimal length maximal green sequence for Q is then

$$(u_1, u_2, u_3, v_1, v_2, v_3, v_4, v_5, v_3, v_2, v_1, u_4)$$

which is a shuffle of $(v_1, v_2, v_3, v_4, v_5, v_3, v_2, v_1)$ and (u_1, u_2, u_3, u_4). The first is a maximal green sequence for the cycle by [4, Lemma 4.2] and is of minimal length by [21, Theorem 6.1]. The second is a maximal green sequence coming from an admissible numbering by sources.

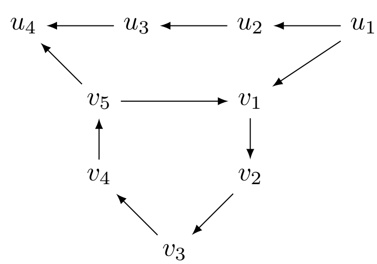

Figure 15: A quiver where a minimal length maximal green sequence can be found by component preserving mutations.

6.3 Exponentially many maximal green sequences for Dynkin quivers

In [3, Remark 4.2 (3)] the authors observe that the number of maximal green sequences of the lineary oriented Dynkin quiver of type A_n seems to grow exponentially with n. The main result of this section will affirm this observation. A *Dynkin quiver of type A_n* is any orientation of the Dynkin diagram of type A_n. The *linearly oriented Dynkin quiver of type A_n* has vertex set $\{i : 1 \leq i \leq n\}$ and arrow set $\{i \to i+1 : 1 \leq i < n\}$. Figure 16 shows the linearly oriented Dynkin quiver of type A_5. We will show that the number of maximal green sequences of arbitrarily oriented Dynkin quiver of type A_n is at least expontential. We give a simple and explicit proof of an exponential lower bound to $|\operatorname{green}(Q)|$ where Q

[2]The use of Lemma 6.2 readily generalizes to quivers similar to Q with longer cycle or longer path.

is any Dynkin quiver of type A_n. After we will provide an improved bound in the case Q is a linearly oriented Dynkin quiver of type A_n.

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$$

Figure 16: The linearly oriented Dynkin quiver A_5.

Recall the Fibonacci numbers are defined by the recurrence $F_1 = 1$, $F_2 = 2$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. A closed form expression for F_n is

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$$

where

$$\phi = \frac{1+\sqrt{5}}{2}\psi = \frac{1-\sqrt{5}}{2}.$$

Proposition 6.3. *If Q is a Dynkin quiver of type A_n for any $n \geq 1$, then* $|\operatorname{green}(Q)| \geq F_{n+1}$.

Proof. It can be easily checked that $|\operatorname{green}(Q)| = 1 = F_2$ for $n = 1$ and $|\operatorname{green}(Q)| = 2 = F_3$ for $n = 2$. For $n \geq 3$ assume inductively that $|\operatorname{green}(Q)| \geq F_{m+1}$ for all $1 \leq m < n$. We first consider components of Q coming from the set partition C/C' where $C = \{i : 1 \leq i \leq n-1\}$ and $C' = \{n\}$. Here Q is isomorphic to a direct sum of a Dynkin quiver of type A_{n-1} and a Dynkin quiver of type A_1. Hence, Q has at least $|\operatorname{green}(Q|_C)|$ maximal green sequences by considering any maximal green sequence on $Q|_C$ with (n) either appended or prepened depending of whether $(n-1) \to n \in Q$ or $n \to (n-1) \in Q$.

Next consider components of Q coming from the set partition D/D' where $D = \{i : 1 \leq i \leq n-2\}$ and $D' = \{n-1, n\}$. Now Q is isomorphic to a direct sum of Dynkin quiver of type A_{n-2} and a Dynkin quiver of type A_2. Thus, Q has at least $|\operatorname{green}(Q|_D)|$ maximal green sequences by considering any maximal green sequence on D with:

- $(n, n-1, n)$ appended if $(n-2) \to (n-1), (n-1) \to n \in Q$.
- $(n, n-1, n)$ prepended if $(n-1) \to (n-2), (n-1) \to n \in Q$.
- $(n-1, n, n-1)$ appended if $(n-2) \to (n-1), n \to (n-1) \in Q$.
- $(n-1, n, n-1)$ prepended if $(n-1) \to (n-2), n \to (n-1) \in Q$.

We see that the set of maximal green sequences for Q coming from $\operatorname{green}(Q|_C)$ are disjoint from those coming from $\operatorname{green}(Q|_D)$. In the former n is mutated at only once and is either mutated first or last in the sequence. In the latter n is either mutated at twice or otherwise is neither the first nor the last mutation. It follows that

$$|\operatorname{green}(Q)| \geq |\operatorname{green}(Q|_C)| + |\operatorname{green}(Q|_D)| \geq F_n + F_{n-1} = F_{n+1}$$

and the proposition is proven. □

For a linearly oriented Dynkin quiver Q of type A_n, we have the maximal green sequence

$$(n, n-1, \ldots, 1, n, n-1, \ldots, 2, \ldots, n, n-1, n)$$

which we will call the *long sequence*.[3] As an example in the case $n = 4$ the long seqeunce is

$$(4, 3, 2, 1, 4, 3, 2, 4, 3, 4).$$

The long sequence is a maximal green sequence coming from a reduced factorization of the longest element in the corresponding Coxeter group.

Proposition 6.4. *If Q is the linearly oriented Dynkin quiver of type A_n for any $n \geq 1$, then* $|\operatorname{green}(Q)| \geq 2^{n-1}$.

Proof. For $n = 1$ we have $|\operatorname{green}(Q)| = 1$ and for $n = 2$ and $|\operatorname{green}(Q)| = 2$. Given $n \geq 3$, assume inductively that $|\operatorname{green}(Q)| \geq 2^{m-1}$ for all $1 \leq m < n$. Consider components from the set partition $C^{(k)}/D^{(k)}$ where $C^{(k)} = \{1, 2, \ldots, k\}$ and $D^{(k)} = \{k+1, k+2, \ldots, n\}$ for $0 \leq k < n$. For each k, our quiver Q has at least $|\operatorname{green}(Q|_{C^{(k)}})|$ many maximal green sequences by appending the long sequence of $Q|_{D^{(k)}}$ to any maximal green sequence of $Q|_{C^{(k)}}$. Here we count one maximal green sequence, the long sequence for Q, when $k = 0$. In the long sequence for $Q|_{D^{(k)}}$ vertex n is mutated at $n - k$ times, and thus the maximal green sequences coming from $\operatorname{green}(Q|_{C^{(k_1)}})$ and $\operatorname{green}(Q|_{C^{(k_2)}})$ are disjoint for $k_1 \neq k_2$. So,

$$|\operatorname{green}(Q)| \geq \sum_{k=0}^{n-1} |\operatorname{green}(Q|_{C^{(k)}})| \geq 1 + \sum_{k=1}^{n-1} 2^{k-1} = 2^{n-1}$$

and the proposition follows. □

Let $\operatorname{green}(A_n)$ denote the set of maximal green sequences of a linearly oriented type A_n quiver. Proposition 6.4 is constructive starting from knowing $\operatorname{green}(A_1) = \{(1)\}$ and $\operatorname{green}(A_2) = \{(1, 2), (2, 1, 2)\}$. The method in the proof of Proposition 6.4 produces

$$\{(1, 2, 3), (2, 1, 2, 3), (1, 3, 2, 3), (3, 2, 1, 3, 2, 3)\} \subseteq \operatorname{green}(A_3),$$

and we show in Table 2 the 8 maximal green sequences in $\operatorname{green}(A_4)$ constructed by applying the proof of Proposition 6.4 one more time. The maximal green sequences in Table 2 are arranged according to the set partition $C^{(k)}/D^{(k)}$.

Table 2: Maximal green sequences in $\operatorname{green}(A_4)$ constructed in proof of Proposition 6.4 according to set partition $C^{(k)}/D^{(k)}$.

k	Maximal green sequences
0	$(4, 3, 2, 1, 4, 3, 2, 4, 3, 4)$
1	$(1, 4, 3, 2, 4, 3, 4)$
2	$(1, 2, 4, 3, 4), (2, 1, 2, 4, 3, 4)$
3	$(1, 2, 3, 4), (2, 1, 2, 3, 4), (1, 3, 2, 3, 4), (3, 2, 1, 3, 2, 3, 4)$

[3]There are many possible maximal green sequences of this maximal length. So, we should perhaps say *a* long sequence instead of *the* long sequence. However, we wish to emphasize that in this section we will be using only this particular sequence of mutations.

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The Cayley isomorphism property for the group $C_2^5 \times C_p$

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Abstract

A finite group G is called a DCI-group if two Cayley digraphs over G are isomorphic if and only if their connection sets are conjugate by a group automorphism. We prove that the group $C_2^5 \times C_p$, where p is a prime, is a DCI-group if and only if $p \neq 2$. Together with the previously obtained results, this implies that a group G of order $32p$, where p is a prime, is a DCI-group if and only if $p \neq 2$ and $G \cong C_2^5 \times C_p$.

Keywords: Isomorphisms, DCI-*groups, Schur rings.*

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1 Introduction

Let G be a finite group and $S \subseteq G$. The *Cayley digraph* $\mathrm{Cay}(G, S)$ over G with connection set S is defined to be the digraph with vertex set G and arc set $\{(g, sg) : g \in G, s \in S\}$. Two Cayley digraphs over G are called *Cayley isomorphic* if there exists an isomorphism between them which is also an automorphism of G. Clearly, two Cayley isomorphic Cayley digraphs are isomorphic. The converse statement is not true in general (see [3, 10]). A subset $S \subseteq G$ is called a CI-*subset* if for each $T \subseteq G$ the Cayley digraphs $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, T)$ are isomorphic if and only if they are Cayley isomorphic. A finite group G is called a DCI-*group* (CI-*group*, respectively) if each subset of G (each inverse-closed subset of G, respectively) is a CI-subset.

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The investigation of DCI-groups was initiated by Ádám [1] who conjectured, in our terms, that every cyclic group is a DCI-group. This conjecture was disproved by Elspas and Turner in [10]. The problem of determining of finite DCI- and CI-groups was suggested by Babai and Frankl in [5]. For more information on DCI- and CI-groups we refer the readers to the survey paper [21].

In this paper we are interested in abelian DCI-groups. The cyclic group of order n is denoted by C_n. Elspas and Turner [10] and independently Djoković [8] proved that every cyclic group of prime order is a DCI-group. The fact that C_{pq} is a DCI-group for distinct primes p and q was proved by Alspach and Parsons in [3] and independently by Klin and Pöschel in [17]. The complete classification of cyclic DCI-groups was obtained by Muzychuk in [23, 24]. He proved that a cyclic group of order n is a DCI-group if and only if $n = k$ or $n = 2k$, where k is square-free.

Denote the class of all finite abelian groups where every Sylow subgroup is elementary abelian by $\mathcal{E}$. From [18, Theorem 1.1] it follows that every DCI-group is the coprime product (i.e. the direct product of groups of coprime orders) of groups from the following list:

$$C_p^k,\ C_4,\ Q_8,\ A_4,\ H \rtimes \langle z\rangle,$$

where p is a prime, H is a group of odd order from $\mathcal{E}$, $|z| \in \{2, 4\}$, and $h^z = h^{-1}$ for every $h \in H$. One can check that the class of DCI-groups is closed under taking subgroups. So one of the crucial steps towards the classification of all DCI-groups is to determine which groups from $\mathcal{E}$ are DCI.

The following non-cyclic groups from $\mathcal{E}$ are DCI-groups (p and q are assumed to be distinct primes): C_p^2 [2, 14]; C_p^3 [2, 9]; C_2^4, C_2^5 [7]; C_p^4, where p is odd [15] (a proof for C_p^4 with no condition on p was given in [22]); C_p^5, where p is odd [13]; $C_p^2 \times C_q$ [18]; $C_p^3 \times C_q$ [27]; $C_p^4 \times C_q$ [20]. The smallest example of a non-DCI-group from $\mathcal{E}$ was found by Nowitz [28]. He proved that C_2^6 is non-DCI. This implies that C_2^n is non-DCI for every $n \geq 6$. Also C_3^n is non-DCI for every $n \geq 8$ [33] and C_p^n is non-DCI for every prime p and $n \geq 2p + 3$ [32].

In this paper we find a new infinite family of DCI-groups from $\mathcal{E}$ which are close to the smallest non-DCI-group from $\mathcal{E}$. The main result of the paper can be formulated as follows.

Theorem 1.1. *Let p be a prime. Then the group $C_2^5 \times C_p$ is a* DCI*-group if and only if $p \neq 2$.*

Theorem 1.1 extends the results obtained in [18, 20, 27] which imply that the group $C_p^k \times C_q$ is a DCI-group whenever p and q are distinct primes and $k \leq 4$. Note that the "only if" part of Theorem 1.1, in fact, was proved by Nowitz in [28]. The next corollary immediately follows from [18, Theorem 1.1] and Theorem 1.1.

Corollary 1.2. *Let p be a prime. Then a group G of order $32p$ is a* DCI*-group if and only if $p \neq 2$ and $G \cong C_2^5 \times C_p$.*

To prove Theorem 1.1, we use the S-ring approach. An *S-ring* over a group G is a subring of the group ring $\mathbb{Z}G$ which is a free $\mathbb{Z}$-module spanned by a special partition of G. If every S-ring from a certain family of S-rings over G is a CI-S-ring then G is a DCI-group (see Section 4). The definition of an S-ring goes back to Schur [31] and Wielandt [34]. The usage of S-rings in the investigation of DCI-groups was proposed by

Klin and Pöschel [17]. Most recent results on DCI-groups were obtained using S-rings (see [15, 18, 19, 20, 27]).

The text of the paper is organized in the following way. In Section 2 we provide definitions and basic facts concerned with S-rings. Section 3 contains a necessary information on isomorphisms of S-rings. In Section 4 we discuss CI-S-rings and their relation with DCI-groups. We also prove in this section a sufficient condition of CI-property for S-rings (Lemma 4.4). Section 5 is devoted to the generalized wreath and star products of S-rings. Here we deduce from previously obtained results two sufficient conditions for the generalized wreath product of S-rings to be a CI-S-ring (Lemma 5.5 and Lemma 5.8). Section 6 and 7 are concerned with p-S-rings and S-rings over a group of order pk, where p is a prime and $\mathrm{GCD}(p,k)=1$, (so-called non-powerful order) respectively. In Section 8 we provide properties of S-rings over the groups C_2^n, $n \leq 5$, and prove that all S-rings over these groups are CI. The material of this section is based on computational results obtained with the help of the GAP package COCO2P [16]. Finally, in Section 9 we prove Theorem 1.1.

Notation. Let G be a finite group and $X \subseteq G$. The element $\sum_{x\in X} x$ of the group ring $\mathbb{Z}G$ is denoted by $\underline{X}$.

The set $\{x^{-1} : x \in X\}$ is denoted by X^{-1}.

The subgroup of G generated by X is denoted by $\langle X\rangle$; we also set $\mathrm{rad}(X) = \{g \in G : gX = Xg = X\}$.

Given a set $X \subseteq G$ the set $\{(g, xg) : x \in X, g \in G\}$ of arcs of the Cayley digraph $\mathrm{Cay}(G, X)$ is denoted by $A(X)$.

The group of all permutations of G is denoted by $\mathrm{Sym}(G)$.

The subgroup of $\mathrm{Sym}(G)$ consisting of all right translations of G is denoted by G_{right}.

The set $\{K \leq \mathrm{Sym}(G) : K \geq G_{\text{right}}\}$ is denoted by $\mathrm{Sup}(G_{\text{right}})$.

For a set $\Delta \subseteq \mathrm{Sym}(G)$ and a section $S = U/L$ of G we set $\Delta^S = \{f^S : f \in \Delta,\ S^f = S\}$, where $S^f = S$ means that f permutes the L-cosets in U and f^S denotes the bijection of S induced by f.

If $K \leq \mathrm{Sym}(\Omega)$ and $\alpha \in \Omega$ then the stabilizer of α in K and the set of all orbits of K on Ω are denoted by K_α and $\mathrm{Orb}(K, \Omega)$ respectively.

If $H \leq G$ then the normalizer of H in G is denoted by $N_G(H)$.

The cyclic group of order n is denoted by C_n.

The class of all finite abelian groups where every Sylow subgroup is elementary abelian is denoted by $\mathcal{E}$.

2 S-rings

In this section we give a background of S-rings. In general, we follow [20], where the most part of the material is contained. For more information on S-rings we refer the readers to [6, 25].

Let G be a finite group and $\mathbb{Z}G$ the integer group ring. Denote the identity element of G by e. A subring $\mathcal{A} \subseteq \mathbb{Z}G$ is called an *S-ring (a Schur ring)* over G if there exists a partition $\mathcal{S}(\mathcal{A})$ of G such that:

(1) $\{e\} \in \mathcal{S}(\mathcal{A})$,

(2) if $X \in \mathcal{S}(\mathcal{A})$ then $X^{-1} \in \mathcal{S}(\mathcal{A})$,

(3) $\mathcal{A} = \mathrm{Span}_{\mathbb{Z}}\{\underline{X} :\ X \in \mathcal{S}(\mathcal{A})\}$.

The elements of $\mathcal{S}(\mathcal{A})$ are called the *basic sets* of $\mathcal{A}$ and the number $\mathrm{rk}(\mathcal{A}) = |\mathcal{S}(\mathcal{A})|$ is called the *rank* of $\mathcal{A}$. If $X, Y \in \mathcal{S}(\mathcal{A})$ then $XY \in \mathcal{S}(\mathcal{A})$ whenever $|X| = 1$ or $|Y| = 1$.

Let $\mathcal{A}$ be an S-ring over a group G. A set $X \subseteq G$ is called an $\mathcal{A}$-*set* if $\underline{X} \in \mathcal{A}$. A subgroup $H \leq G$ is called an $\mathcal{A}$-*subgroup* if H is an $\mathcal{A}$-set. From the definition it follows that the intersection of $\mathcal{A}$-subgroups is also an $\mathcal{A}$-subgroup. One can check that for each $\mathcal{A}$-set X the groups $\langle X\rangle$ and $\mathrm{rad}(X)$ are $\mathcal{A}$-subgroups. By the *thin radical* of $\mathcal{A}$ we mean the set defined as

$$\mathbf{O}_\theta(\mathcal{A}) = \{x \in G : \{x\} \in \mathcal{S}(\mathcal{A})\}.$$

It is easy to see that $\mathbf{O}_\theta(\mathcal{A})$ is an $\mathcal{A}$-subgroup.

Lemma 2.1 ([11, Lemma 2.1])**.** *Let $\mathcal{A}$ be an S-ring over a group G, H an $\mathcal{A}$-subgroup of G, and $X \in \mathcal{S}(\mathcal{A})$. Then the number $|X \cap Hx|$ does not depend on $x \in X$.*

Let $L \trianglelefteq U \leq G$. A section U/L is called an $\mathcal{A}$-*section* if U and L are $\mathcal{A}$-subgroups. If $S = U/L$ is an $\mathcal{A}$-section then the module

$$\mathcal{A}_S = \mathrm{Span}_{\mathbb{Z}}\left\{\underline{X}^\pi : X \in \mathcal{S}(\mathcal{A}),\ X \subseteq U\right\},$$

where $\pi\colon U \to U/L$ is the canonical epimorphism, is an S-ring over S.

3 Isomorphisms and schurity

Let $\mathcal{A}$ and $\mathcal{A}'$ be S-rings over groups G and G' respectively. A bijection $f\colon G \to G'$ is called an *isomorphism* from $\mathcal{A}$ to $\mathcal{A}'$ if

$$\{A(X)^f : X \in \mathcal{S}(\mathcal{A})\} = \{A(X') : X' \in \mathcal{S}(\mathcal{A}')\},$$

where $A(X)^f = \{(g^f,\ h^f) : (g,\ h) \in A(X)\}$. If there exists an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ then we say that $\mathcal{A}$ and $\mathcal{A}'$ are *isomorphic* and write $\mathcal{A} \cong \mathcal{A}'$.

The group of all isomorphisms from $\mathcal{A}$ onto itself contains a normal subgroup

$$\{f \in \mathrm{Sym}(G) : A(X)^f = A(X) \text{ for every } X \in \mathcal{S}(\mathcal{A})\}$$

called the *automorphism group* of $\mathcal{A}$ and denoted by $\mathrm{Aut}(\mathcal{A})$. The definition implies that $G_{\mathrm{right}} \leq \mathrm{Aut}(\mathcal{A})$. The S-ring $\mathcal{A}$ is called *normal* if G_{right} is normal in $\mathrm{Aut}(\mathcal{A})$. One can verify that if S is an $\mathcal{A}$-section then $\mathrm{Aut}(\mathcal{A})^S \leq \mathrm{Aut}(\mathcal{A}_S)$. Denote the group $\mathrm{Aut}(\mathcal{A}) \cap \mathrm{Aut}(G)$ by $\mathrm{Aut}_G(\mathcal{A})$. It easy to check that if S is an $\mathcal{A}$-section then $\mathrm{Aut}_G(\mathcal{A})^S \leq \mathrm{Aut}_S(\mathcal{A}_S)$. One can verify that

$$\mathrm{Aut}_G(\mathcal{A}) = (N_{\mathrm{Aut}(\mathcal{A})}(G_{\mathrm{right}}))_e.$$

Let $K \in \mathrm{Sup}(G_{\mathrm{right}})$. Schur proved in [31] that the $\mathbb{Z}$-submodule

$$V(K, G) = \mathrm{Span}_{\mathbb{Z}}\{\underline{X} : X \in \mathrm{Orb}(K_e,\ G)\},$$

is an S-ring over G. An S-ring $\mathcal{A}$ over G is called *schurian* if $\mathcal{A} = V(K, G)$ for some $K \in \mathrm{Sup}(G_{\mathrm{right}})$. One can verify that given $K_1, K_2 \in \mathrm{Sup}(G_{\mathrm{right}})$,

$$\text{if } K_1 \leq K_2 \text{ then } V(K_1, G) \geq V(K_2, G). \tag{3.1}$$

If $\mathcal{A} = V(K, G)$ for some $K \in \mathrm{Sup}(G_{\mathrm{right}})$ and S is an $\mathcal{A}$-section then $\mathcal{A}_S = V(K^S, S)$. So if $\mathcal{A}$ is schurian then $\mathcal{A}_S$ is also schurian for every $\mathcal{A}$-section S. It can be checked that

$$V(\mathrm{Aut}(\mathcal{A}), G) \geq \mathcal{A} \tag{3.2}$$

and the equality is attained if and only if $\mathcal{A}$ is schurian.

An S-ring $\mathcal{A}$ over a group G is defined to be *cyclotomic* if there exists $K \leq \mathrm{Aut}(G)$ such that $\mathcal{S}(\mathcal{A}) = \mathrm{Orb}(K, G)$. In this case we write $\mathcal{A} = \mathrm{Cyc}(K, G)$. Obviously, $\mathcal{A} = V(G_{\mathrm{right}}K, G)$. So every cyclotomic S-ring is schurian. If $\mathcal{A} = \mathrm{Cyc}(K, G)$ for some $K \leq \mathrm{Aut}(G)$ and S is an $\mathcal{A}$-section then $\mathcal{A}_S = \mathrm{Cyc}(K^S, S)$. Therefore if $\mathcal{A}$ is cyclotomic then $\mathcal{A}_S$ is also cyclotomic for every $\mathcal{A}$-section S.

Two permutation groups K_1 and K_2 on a set Ω are called 2-*equivalent* if $\mathrm{Orb}(K_1, \Omega^2) = \mathrm{Orb}(K_2, \Omega^2)$ (here we assume that K_1 and K_2 act on Ω^2 componentwise). In this case we write $K_1 \approx_2 K_2$. The relation $\approx_2$ is an equivalence relation on the set of all subgroups of $\mathrm{Sym}(\Omega)$. Every equivalence class has a unique maximal element with respect to inclusion. Given $K \leq \mathrm{Sym}(\Omega)$, this element is called the 2-*closure* of K and denoted by $K^{(2)}$. If $\mathcal{A} = V(K, G)$ for some $K \in \mathrm{Sup}(G_{\mathrm{right}})$ then $K^{(2)} = \mathrm{Aut}(\mathcal{A})$. An S-ring $\mathcal{A}$ over G is called 2-*minimal* if

$$\{K \in \mathrm{Sup}(G_{\mathrm{right}}) : K \approx_2 \mathrm{Aut}(\mathcal{A})\} = \{\mathrm{Aut}(\mathcal{A})\}.$$

Two groups $K_1, K_2 \leq \mathrm{Aut}(G)$ are said to be *Cayley equivalent* if $\mathrm{Orb}(K_1, G) = \mathrm{Orb}(K_2, G)$. In this case we write $K_1 \approx_{\mathrm{Cay}} K_2$. If $\mathcal{A} = \mathrm{Cyc}(K, G)$ for some $K \leq \mathrm{Aut}(G)$ then $\mathrm{Aut}_G(\mathcal{A})$ is the largest group which is Cayley equivalent to K. A cyclotomic S-ring $\mathcal{A}$ over G is called *Cayley minimal* if

$$\{K \leq \mathrm{Aut}(G) : K \approx_{\mathrm{Cay}} \mathrm{Aut}_G(\mathcal{A})\} = \{\mathrm{Aut}_G(\mathcal{A})\}.$$

It is easy to see that $\mathbb{Z}G$ is 2-minimal and Cayley minimal.

4 CI-S-rings

Let $\mathcal{A}$ be an S-ring over a group G. Put

$$\mathrm{Iso}(\mathcal{A}) = \{f \in \mathrm{Sym}(G) : f \text{ is an isomorphism from } \mathcal{A} \text{ onto an } S\text{-ring over } G\}.$$

One can see that $\mathrm{Aut}(\mathcal{A})\,\mathrm{Aut}(G) \subseteq \mathrm{Iso}(\mathcal{A})$. However, the converse statement does not hold in general. The S-ring $\mathcal{A}$ is defined to be a CI-*S-ring* if $\mathrm{Aut}(\mathcal{A})\,\mathrm{Aut}(G) = \mathrm{Iso}(\mathcal{A})$. It is easy to check that $\mathbb{Z}G$ and the S-ring of rank 2 over G are CI-S-rings.

Put

$$\mathrm{Sup}_2(G_{\mathrm{right}}) = \{K \in \mathrm{Sup}(G_{\mathrm{right}}) : K^{(2)} = K\}.$$

The group $M \leq \mathrm{Sym}(G)$ is said to be *G-regular* if M is regular and isomorphic to G. Following [15], we say that a group $K \in \mathrm{Sup}(G_{\mathrm{right}})$ is *G-transjugate* if every G-regular subgroup of K is K-conjugate to G_{right}. Babai proved in [4] the statement which can be formulated in our terms as follows: a set $S \subseteq G$ is a CI-subset if and only if the group $\mathrm{Aut}(\mathrm{Cay}(G, S))$ is G-transjugate. The next lemma provides a similar criterion for a schurian S-ring to be CI.

Lemma 4.1. *Let $K \in \mathrm{Sup}_2(G_{\mathrm{right}})$ and $\mathcal{A} = V(K, G)$. Then $\mathcal{A}$ is a* CI-*S-ring if and only if K is G-transjugate.*

Proof. The statement of the lemma follows from [15, Theorem 2.6]. □

Let $K_1, K_2 \in \mathrm{Sup}(G_{\mathrm{right}})$ such that $K_1 \leq K_2$. Then K_1 is called a *G-complete subgroup* of K_2 if every G-regular subgroup of K_2 is K_2-conjugate to some G-regular subgroup of K_1 (see [15, Definition 2]). In this case we write $K_1 \preceq_G K_2$. The relation $\preceq_G$ is a partial order on $\mathrm{Sup}(G_{\mathrm{right}})$. The set of the minimal elements of $\mathrm{Sup}_2(G_{\mathrm{right}})$ with respect to $\preceq_G$ is denoted by $\mathrm{Sup}_2^{\min}(G_{\mathrm{right}})$.

Lemma 4.2 ([20, Lemma 3.3])**.** *Let G be a finite group. If $V(K,G)$ is a* CI-*S-ring for every $K \in \mathrm{Sup}_2^{\min}(G_{\mathrm{right}})$ then G is a* DCI-*group.*

Remark 4.3. The condition that $V(K,G)$ is a CI-S-ring for every $K \in \mathrm{Sup}_2^{\min}(G_{\mathrm{right}})$ is equivalent to, say, that every schurian S-ring over G is a CI-S-ring. The latter condition means that every 2-closed overgroup of G_{right} is G-transjugate. However, 2-closed overgroup of G_{right} may not be the automorphism group of a Cayley digraph over G. So the condition that the automorphism group of every Cayley digraph over G is G-transjugate or, equivalently, that G is a DCI-group, seems weaker than the condition of Lemma 4.2. It is a natural question whether there exists a DCI-group for which the condition of Lemma 4.2 does not hold.

We finish the section with the lemma that gives a sufficient condition for an S-ring to be a CI-S-ring. In order to formulate this condition, we need to introduce some further notations. Let $\mathcal{A}$ be a schurian S-ring over an abelian group G and L a normal $\mathcal{A}$-subgroup of G. Then the partition of G into the L-cosets is $\mathrm{Aut}(\mathcal{A})$-invariant. The kernel of the action of $\mathrm{Aut}(\mathcal{A})$ on the latter cosets is denoted by $\mathrm{Aut}(\mathcal{A})_{G/L}$. Since $\mathrm{Aut}(\mathcal{A})_{G/L}$ is a normal subgroup of $\mathrm{Aut}(\mathcal{A})$, we can form the group $K = \mathrm{Aut}(\mathcal{A})_{G/L}G_{\mathrm{right}}$. Clearly, $K \leq \mathrm{Aut}(\mathcal{A})$. From [15, Proposition 2.1] it follows that $K = K^{(2)}$.

Lemma 4.4. *Let $\mathcal{A}$ be a schurian S-ring over an abelian group G, L an $\mathcal{A}$-subgroup of G, and $K = \mathrm{Aut}(\mathcal{A})_{G/L}G_{\mathrm{right}}$. Suppose that both $\mathcal{A}_{G/L}$ and $V(K,G)$ are* CI-*S-rings and $\mathcal{A}_{G/L}$ is normal. Then $\mathcal{A}$ is a* CI-*S-ring.*

Proof. Firstly we prove that the group $\mathrm{Aut}(\mathcal{A})^{G/L}$ is G/L-transjugate. Suppose that F is a G/L-regular subgroup of $\mathrm{Aut}(\mathcal{A})^{G/L}$. The S-ring $\mathcal{A}_{G/L}$ is a CI-S-ring by the assumption of the lemma. So Lemma 4.1 implies that the group $\mathrm{Aut}(\mathcal{A}_{G/L})$ is G/L-transjugate. Since $F \leq \mathrm{Aut}(\mathcal{A})^{G/L} \leq \mathrm{Aut}(\mathcal{A}_{G/L})$, we conclude that F and $(G/L)_{\mathrm{right}}$ are $\mathrm{Aut}(\mathcal{A}_{G/L})$-conjugate. However, $\mathcal{A}_{G/L}$ is normal and hence $F = (G/L)_{\mathrm{right}}$. Therefore $\mathrm{Aut}(\mathcal{A})^{G/L}$ is G/L-transjugate.

Now let us show that $K \preceq_G \mathrm{Aut}(\mathcal{A})$. Let H be a G-regular subgroup of $\mathrm{Aut}(\mathcal{A})$. Then $H^{G/L}$ is abelian transitive subgroup of $\mathrm{Aut}(\mathcal{A})^{G/L}$ and hence $H^{G/L}$ is regular on G/L. Therefore $H^{G/L} \cong (G/L)_{\mathrm{right}} = (G_{\mathrm{right}})^{G/L}$. There exists $\gamma \in \mathrm{Aut}(\mathcal{A})$ such that $(H^{G/L})^{\gamma^{G/L}} = (G/L)_{\mathrm{right}} = (G_{\mathrm{right}})^{G/L}$ because $\mathrm{Aut}(\mathcal{A})^{G/L}$ is G/L-transjugate. This yields that $H^\gamma \leq K$. Thus, $K \preceq_G \mathrm{Aut}(\mathcal{A})$.

Finally, let us prove that $\mathrm{Aut}(\mathcal{A})$ is G-transjugate. Again, let H be a G-regular subgroup of $\mathrm{Aut}(\mathcal{A})$. Since $K \preceq_G \mathrm{Aut}(\mathcal{A})$, there exists $\gamma \in \mathrm{Aut}(\mathcal{A})$ such that $H^\gamma \leq K$. The S-ring $V(K,G)$ is a CI-S-ring by the assumption of the lemma. So K is G-transjugate by Lemma 4.1. Therefore H^γ and G_{right} are K-conjugate and hence H and G_{right} are $\mathrm{Aut}(\mathcal{A})$-conjugate. Thus, $\mathrm{Aut}(\mathcal{A})$ is G-transjugate and $\mathcal{A}$ is a CI-S-ring by Lemma 4.1. □

It should be mentioned that the proof of Lemma 4.4 is similar to the proof of [20, Lemma 3.6].

5 Generalized wreath and star products

Let $\mathcal{A}$ be an S-ring over a group G and $S = U/L$ an $\mathcal{A}$-section of G. An S-ring $\mathcal{A}$ is called the *S-wreath product* or the *generalized wreath product* of $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ if $L \trianglelefteq G$ and $L \leq \mathrm{rad}(X)$ for each basic set X outside U. In this case we write $\mathcal{A} = \mathcal{A}_U \wr_S \mathcal{A}_{G/L}$ and omit S when $U = L$. The construction of the generalized wreath product of S-rings was introduced in [12].

The S-wreath product is called *nontrivial* or *proper* if $L \neq \{e\}$ and $U \neq G$. An S-ring $\mathcal{A}$ is said to be *decomposable* if $\mathcal{A}$ is the nontrivial S-wreath product for some $\mathcal{A}$-section S of G; otherwise $\mathcal{A}$ is said to be indecomposable. We say that an $\mathcal{A}$-subgroup $U < G$ has a *gwr-complement* with respect to $\mathcal{A}$ if there exists a nontrivial normal $\mathcal{A}$-subgroup L of G such that $L \leq U$ and $\mathcal{A}$ is the S-wreath product, where $S = U/L$.

Lemma 5.1 ([19, Theorem 1.1])**.** *Let $G \in \mathcal{E}$, $\mathcal{A}$ an S-ring over G, and $S = U/L$ an $\mathcal{A}$-section of G. Suppose that $\mathcal{A}$ is the nontrivial S-wreath product and the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are* CI-*S-rings. Then $\mathcal{A}$ is a* CI-*S-ring whenever*

$$\mathrm{Aut}_S(\mathcal{A}_S) = \mathrm{Aut}_U(\mathcal{A}_U)^S \, \mathrm{Aut}_{G/L}(\mathcal{A}_{G/L})^S.$$

In particular, $\mathcal{A}$ is a CI-*S-ring if*

$$\mathrm{Aut}_S(\mathcal{A}_S) = \mathrm{Aut}_U(\mathcal{A}_U)^S \quad \text{or} \quad \mathrm{Aut}_S(\mathcal{A}_S) = \mathrm{Aut}_{G/L}(\mathcal{A}_{G/L})^S.$$

Lemma 5.2 ([19, Proposition 4.1])**.** *In the conditions of Lemma 5.1, suppose that $\mathcal{A}_S = \mathbb{Z}S$. Then $\mathcal{A}$ is a* CI-*S-ring. In particular, if $U = L$ then $\mathcal{A}$ is a* CI-*S-ring.*

Lemma 5.3 ([20, Lemma 4.2])**.** *In the conditions of Lemma 5.1, suppose that at least one of the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ is cyclotomic and $\mathcal{A}_S$ is Cayley minimal. Then $\mathcal{A}$ is a* CI-*S-ring.*

Lemma 5.4. *Let $\mathcal{A}$ be an S-ring over an abelian group G. Suppose that $\mathcal{A}$ is the nontrivial $S = U/L$-wreath product for some $\mathcal{A}$-section $S = U/L$ and L_1 is an $\mathcal{A}$-subgroup containing L. Then $\mathcal{B} = V(K, G)$, where $K = \mathrm{Aut}(\mathcal{A})_{G/L_1} G_{\mathrm{right}}$, is also the S-wreath product.*

Proof. Since $K \leq \mathrm{Aut}(\mathcal{A})$, from Equations (3.1) and (3.2) it follows that

$$\mathcal{B} = V(K, G) \geq V(\mathrm{Aut}(\mathcal{A}), G) \geq \mathcal{A}.$$

So U and L are also $\mathcal{B}$-subgroups.

Let $\mathcal{C} = \mathbb{Z}U \wr_S \mathbb{Z}(G/L)$. The S-rings $\mathcal{C}_U$ and $\mathcal{C}_{G/L}$ are schurian and $\mathcal{C}_S$ is 2-minimal because $\mathcal{C}_S = \mathbb{Z}S$. So $\mathcal{C}$ is schurian by [26, Corollary 10.3]. This implies that

$$\mathcal{C} = V(\mathrm{Aut}(\mathcal{C}), G). \tag{5.1}$$

Every element from $\mathrm{Aut}(\mathcal{C})_e$ fixes every basic set of $\mathcal{C}$ and hence it fixes every L-coset. Since $L_1 \geq L$, every element from $\mathrm{Aut}(\mathcal{C})_e$ fixes every L_1-coset. We conclude that

$\mathrm{Aut}(\mathcal{C})_e \le \mathrm{Aut}(\mathcal{A})_{G/L_1}$ and hence $\mathrm{Aut}(\mathcal{C}) \le K$. Now from Equations (3.1) and (5.1) it follows that

$$\mathcal{C} = V(\mathrm{Aut}(\mathcal{C}), G) \ge V(K, G) = \mathcal{B}. \tag{5.2}$$

The group U is a $\mathcal{B}$- and a $\mathcal{C}$-subgroup. Due to Equation (5.2), every basic set of $\mathcal{B}$ which lies outside U is a union of some basic sets of $\mathcal{C}$ which lie outside U. So $L \le \mathrm{rad}(X)$ for every $X \in \mathcal{S}(\mathcal{B})$ outside U. Thus, $\mathcal{B}$ is the S-wreath product. □

Lemma 5.5. *In the conditions of Lemma 5.1, suppose that: (1) every S-ring over U is a* CI-*S-ring; (2) $\mathcal{A}_{G/L}$ is 2-minimal or normal. Then $\mathcal{A}$ is a* CI-*S-ring.*

Proof. Let $\mathcal{B} = V(K, G)$, where $K = \mathrm{Aut}(\mathcal{A})_{G/L} G_{\mathrm{right}}$. From Lemma 5.4 it follows that $\mathcal{B}$ is the S-wreath product. Since $L_1 = L$, the definition of $\mathcal{B}$ implies that $\mathcal{B}_{G/L} = \mathbb{Z}(G/L)$ and hence $\mathcal{B}_S = \mathbb{Z}S$. Clearly, $\mathcal{B}_{G/L}$ is a CI-S-ring. The S-ring $\mathcal{B}_U$ is a CI-S-ring by the assumption of the lemma. Therefore $\mathcal{B}$ is a CI-S-ring by Lemma 5.2. The S-ring $\mathcal{A}_{G/L}$ is a CI-S-ring by the assumption of the lemma. Thus, $\mathcal{A}$ is a CI-S-ring by [20, Lemma 3.6] whenever $\mathcal{A}_{G/L}$ is 2-minimal and by Lemma 4.4 whenever $\mathcal{A}_{G/L}$ is normal. □

Let V and W be $\mathcal{A}$-subgroups. The S-ring $\mathcal{A}$ is called the *star product* of $\mathcal{A}_V$ and $\mathcal{A}_W$ if the following conditions hold:

(1) $V \cap W \trianglelefteq W$;

(2) each $T \in \mathcal{S}(\mathcal{A})$ with $T \subseteq (W \setminus V)$ is a union of some $V \cap W$-cosets;

(3) for each $T \in \mathcal{S}(\mathcal{A})$ with $T \subseteq G \setminus (V \cup W)$ there exist $R \in \mathcal{S}(\mathcal{A}_V)$ and $S \in \mathcal{S}(\mathcal{A}_W)$ such that $T = RS$.

In this case we write $\mathcal{A} = \mathcal{A}_V \star \mathcal{A}_W$. The construction of the star product of S-rings was introduced in [15]. The star product is called *nontrivial* if $V \ne \{e\}$ and $V \ne G$. If $V \cap W = \{e\}$ then the star product is the usual *tensor product* of $\mathcal{A}_V$ and $\mathcal{A}_W$ (see [11, p. 5]). In this case we write $\mathcal{A} = \mathcal{A}_V \otimes \mathcal{A}_W$. One can check that if $\mathcal{A} = \mathcal{A}_V \otimes \mathcal{A}_W$ then $\mathrm{Aut}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A}_V) \times \mathrm{Aut}(\mathcal{A}_W)$. If $V \cap W \ne \{e\}$ then $\mathcal{A}$ is the nontrivial $V/(V \cap W)$-wreath product. Indeed, let $T \in \mathcal{S}(\mathcal{A})$ such that $T \nsubseteq V$. If $T \subseteq W \setminus V$ then $V \cap W \le \mathrm{rad}(T)$ by Condition (2) of the definition. If $T \subseteq G \setminus (V \cup W)$ then $T = RS$ for some $R \in \mathcal{S}(\mathcal{A}_V)$ and some $S \in \mathcal{S}(\mathcal{A}_W)$ such that $S \subseteq W \setminus V$ by Condition (3) of the definition. Since $V \cap W \le \mathrm{rad}(S)$, we obtain $V \cap W \le \mathrm{rad}(T)$.

Lemma 5.6. *Let $G \in \mathcal{E}$ and $\mathcal{A}$ a schurian S-ring over G. Suppose that $\mathcal{A} = \mathcal{A}_V \star \mathcal{A}_W$ for some $\mathcal{A}$-subgroups V and W of G and the S-rings $\mathcal{A}_V$ and $\mathcal{A}_{W/(V\cap W)}$ are* CI-*S-rings. Then $\mathcal{A}$ is a* CI-*S-ring.*

Proof. The statement of the lemma follows from [18, Proposition 3.2, Theorem 4.1]. □

Lemma 5.7 ([13, Lemma 2.8])**.** *Let $\mathcal{A}$ be an S-ring over an abelian group $G = G_1 \times G_2$. Assume that G_1 and G_2 are $\mathcal{A}$-groups. Then $\mathcal{A} = \mathcal{A}_{G_1} \otimes \mathcal{A}_{G_2}$ whenever $\mathcal{A}_{G_1}$ or $\mathcal{A}_{G_2}$ is the group ring.*

Lemma 5.8. *In the conditions of Lemma 5.1, suppose that $|G : U|$ is a prime and there exists $X \in \mathcal{S}(\mathcal{A}_{G/L})$ outside S with $|X| = 1$. Then $\mathcal{A}$ is a* CI-*S-ring.*

Proof. Let $X = \{x\}$ for some $x \in G/L$. Due to $G \in \mathcal{E}$, we conclude that $|\langle x\rangle|$ is prime. So $|\langle x\rangle \cap S| = 1$ because x lies outside S. Since $|G : U|$ is a prime, $G/L = \langle x\rangle \times S$. Note that $\mathcal{A}_{\langle x\rangle} = \mathbb{Z}\langle x\rangle$. Therefore

$$\mathcal{A}_{G/L} = \mathbb{Z}\langle x\rangle \otimes \mathcal{A}_S$$

by Lemma 5.7.

Let $\varphi \in \mathrm{Aut}_S(\mathcal{A}_S)$. Define $\psi \in \mathrm{Aut}(G/L)$ in the following way:

$$\psi^S = \varphi,\ x^\psi = x.$$

Then $\psi \in \mathrm{Aut}_{G/L}(\mathcal{A}_{G/L})$ because $\mathcal{A}_{G/L} = \mathbb{Z}\langle x\rangle \otimes \mathcal{A}_S$. We obtain that

$$\mathrm{Aut}_{G/L}(\mathcal{A}_{G/L})^S \geq \mathrm{Aut}_S(\mathcal{A}_S),$$

and hence $\mathrm{Aut}_{G/L}(\mathcal{A}_{G/L})^S = \mathrm{Aut}_S(\mathcal{A}_S)$. Thus, $\mathcal{A}$ is a CI-S-ring by Lemma 5.1. □

6 p-S-rings

Let p be a prime. An S-ring $\mathcal{A}$ over a p-group G is called a *p-S-ring* if every basic set of $\mathcal{A}$ has a p-power size. Clearly, if $|G| = p$ then $\mathcal{A} = \mathbb{Z}G$. In the next three lemmas G is a p-group and $\mathcal{A}$ is a p-S-ring over G.

Lemma 6.1. *If $\mathcal{B} \geq \mathcal{A}$ then $\mathcal{B}$ is a p-S-ring.*

Proof. The statement of the lemma follows from [29, Theorem 1.1]. □

Lemma 6.2. *Let $S = U/L$ be an $\mathcal{A}$-section of G. Then $\mathcal{A}_S$ is a p-S-ring.*

Proof. From Lemma 2.1 it follows that for every $X \in \mathcal{S}(\mathcal{A})$ the number $\lambda = |X \cap Lx|$ does not depend on $x \in X$. So λ divides $|X|$ and hence λ is a p-power. Let $\pi\colon G \to G/L$ be the canonical epimorphism. Note that $|\pi(X)| = |X|/\lambda$ and hence $|\pi(X)|$ is a p-power. Therefore every basic set of $\mathcal{A}_S$ has a p-power size. Thus, $\mathcal{A}_S$ is a p-S-ring. □

Lemma 6.3 ([13, Proposition 2.13])**.** *The following statements hold:*

(1) $|\mathbf{O}_\theta(\mathcal{A})| > 1$;

(2) there exists a chain of $\mathcal{A}$-subgroups $\{e\} = G_0 < G_1 < \cdots < G_s = G$ such that $|G_{i+1} : G_i| = p$ for every $i \in \{0, \ldots, s-1\}$.

Lemma 6.4. *Let G be an abelian group, $K \in \mathrm{Sup}_2^{\min}(G_{\mathrm{right}})$, and $\mathcal{A} = V(K, G)$. Suppose that H is an $\mathcal{A}$-subgroup of G such that G/H is a p-group for some prime p. Then $\mathcal{A}_{G/H}$ is a p-S-ring.*

Proof. The statement of the lemma follows from [18, Lemma 5.2]. □

7 S-rings over an abelian group of non-powerful order

A number n is called *powerful* if p^2 divides n for every prime divisor p of n. From now throughout this section $G = H \times P$, where H is an abelian group and $P \cong C_p$, where p is a prime coprime to $|H|$. Clearly, $|G|$ is non-powerful. Let $\mathcal{A}$ be an S-ring over G, H_1 a maximal $\mathcal{A}$-subgroup contained in H, and P_1 the least $\mathcal{A}$-subgroup containing P. Note that H_1P_1 is an $\mathcal{A}$-subgroup.

Lemma 7.1 ([20, Lemma 6.3])**.** *In the above notations, if $H_1 \neq (H_1P_1)_{p'}$, the Hall p'-subgroup of H_1P_1, then $\mathcal{A}_{H_1P_1} = \mathcal{A}_{H_1} \star \mathcal{A}_{P_1}$.*

Lemma 7.2 ([27, Proposition 15])**.** *In the above notations, if $\mathcal{A}_{H_1P_1/H_1} \cong \mathbb{Z}C_p$ then $\mathcal{A}_{H_1P_1} = \mathcal{A}_{H_1} \star \mathcal{A}_{P_1}$.*

Lemma 7.3 ([11, Lemma 6.2])**.** *In the above notations, suppose that $H_1 < H$. Then one of the following statements holds:*

(1) $\mathcal{A} = \mathcal{A}_{H_1} \wr \mathcal{A}_{G/H_1}$ with $\mathrm{rk}(\mathcal{A}_{G/H_1}) = 2$;

(2) $\mathcal{A} = \mathcal{A}_{H_1P_1} \wr_S \mathcal{A}_{G/P_1}$, where $S = H_1P_1/P_1$ and $P_1 < G$.

8 S-rings over C_2^n, $n \leq 5$

All S-rings over the groups C_2^n, where $n \leq 5$, were enumerated with the help of the GAP package COCO2P [16]. The list of all S-rings over these groups is available on the web-page [30] (see also [35]). The next lemma is an immediate consequence of the above computational results (see also [11, Theorem 1.2]).

Lemma 8.1. *Every S-ring over C_2^n, where $n \leq 5$, is schurian.*

To prove Theorem 1.1, we will show that every schurian S-ring over $C_2^5 \times C_p$ is CI. Since the most of schurian S-rings over $C_2^5 \times C_p$ are generalized wreath or star products of S-rings over its proper subgroups, we need to check that all schurian S-rings over proper subgroups of $C_2^5 \times C_p$ are CI. In this section we will do it for $G \cong C_2^n$, where $n \leq 5$. Note that G is a DCI-group by [2, 7] but this does not imply that every S-ring over G is CI (see Remark 4.3). We will describe 2-S-rings over G using computational results and check that all S-rings over G are CI. Until the end of the section G is an elementary abelian 2-group of rank n and $\mathcal{A}$ is a 2-S-ring over G.

Lemma 8.2. *Let $n \leq 3$. Then $\mathcal{A}$ is cyclotomic. Moreover, $\mathcal{A}$ is Cayley minimal except for the case when $n = 3$ and $\mathcal{A} \cong \mathbb{Z}C_2 \wr \mathbb{Z}C_2 \wr \mathbb{Z}C_2$.*

Proof. The first part of the lemma follows from [20, Lemma 5.2]; the second part follows from [20, Lemma 5.3]. □

Analyzing the lists of all S-rings over C_2^4 and C_2^5 available on the web-page [30], we conclude that up to isomorphism there are exactly nineteen 2-S-rings over G if $n = 4$ and there are exactly one hundred 2-S-rings over G if $n = 5$. It can can be established by inspecting the above 2-S-rings one after the other that there are exactly fifteen decomposable and four indecomposable 2-S-rings over G if $n = 4$ and there are exactly ninety six decomposable and four indecomposable 2-S-rings over G if $n = 5$.

Lemma 8.3. *Let $n \in \{4, 5\}$ and $\mathcal{A}$ indecomposable. Then $\mathcal{A}$ is normal. If in addition $n = 5$ then $\mathcal{A} \cong \mathbb{Z}C_2 \otimes \mathcal{A}'$, where $\mathcal{A}'$ is indecomposable 2-S-ring over C_2^4.*

Proof. Let $n = 4$. One can compute $|\operatorname{Aut}(\mathcal{A})|$ and $|N_{\operatorname{Aut}(\mathcal{A})}(G_{\mathrm{right}})|$ using the GAP package COCO2P [16]. It turns out that for each of the four indecomposable 2-S-rings over G the equality

$$|\operatorname{Aut}(\mathcal{A})| = |N_{\operatorname{Aut}(\mathcal{A})}(G_{\mathrm{right}})|$$

is attained. So every indecomposable 2-S-ring over G is normal whenever $n = 4$.

Let $n = 5$. The straightforward check for each of the four indecomposable 2-S-rings over G yields that $\mathcal{A} = \mathcal{A}_H \otimes \mathbb{Z}L$, where $H \cong C_2^4$, $L \cong C_2$, and $\mathcal{A}_H$ is indecomposable 2-S-ring. Clearly, $\mathbb{Z}L$ is normal. By the above paragraph, $\mathcal{A}_H$ is normal. Since $\mathrm{Aut}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A}_H) \times \mathrm{Aut}(\mathcal{A}_L)$, we obtain that $\mathcal{A}$ is normal. □

Note that if $p > 2$ then Lemma 8.3 does not hold. In fact, if $p > 2$ then there exists an indecomposable p-S-ring over C_p^5 which is not normal (see [13, Lemma 6.4]).

Lemma 8.4. *Let $n \leq 5$. Then $\mathcal{A}$ is normal whenever one of the following statements holds:*

(1) $\mathcal{A}$ is indecomposable;

(2) $|G : \mathbf{O}_\theta(\mathcal{A})| = 2$;

(3) $n = 4$ and $\mathcal{A} \cong (\mathbb{Z}C_2 \wr \mathbb{Z}C_2) \otimes (\mathbb{Z}C_2 \wr \mathbb{Z}C_2)$.

Proof. If $n \leq 3$ and $\mathcal{A}$ is indecomposable then $\mathcal{A} = \mathbb{Z}G$ by [20, Lemma 5.2]. Clearly, in this case $\mathcal{A}$ is normal. If $n \in \{4, 5\}$ and $\mathcal{A}$ is indecomposable then $\mathcal{A}$ is normal by Lemma 8.3. There are exactly $n - 1$ 2-S-rings over G for which Statement (2) of the lemma holds. For every $\mathcal{A}$ isomorphic to one of these 2-S-rings and for $\mathcal{A} \cong (\mathbb{Z}C_2 \wr \mathbb{Z}C_2) \otimes (\mathbb{Z}C_2 \wr \mathbb{Z}C_2)$ one can compute $|\mathrm{Aut}(\mathcal{A})|$ and $|N_{\mathrm{Aut}(\mathcal{A})}(G_{\mathrm{right}})|$ using the GAP package COCO2P [16]. It turns out that in each case the equality $|\mathrm{Aut}(\mathcal{A})| = |N_{\mathrm{Aut}(\mathcal{A})}(G_{\mathrm{right}})|$ holds and hence $\mathcal{A}$ is normal. □

Lemma 8.5. *Let $n = 4$. Then $\mathcal{A}$ is cyclotomic.*

Proof. If $\mathcal{A}$ is decomposable then $\mathcal{A}$ is cyclotomic by [20, Lemma 5.6]. If $\mathcal{A}$ is indecomposable then $\mathcal{A}$ is normal by Lemma 8.3. This implies that

$$\mathrm{Aut}(\mathcal{A})_e = (N_{\mathrm{Aut}(\mathcal{A})}(G_{\mathrm{right}}))_e \leq \mathrm{Aut}(G).$$

The S-ring $\mathcal{A}$ is schurian by Lemma 8.1. So from Equation (3.2) it follows that $\mathcal{A} = V(\mathrm{Aut}(\mathcal{A}), G)$ and hence $\mathcal{A} = \mathrm{Cyc}(\mathrm{Aut}(\mathcal{A})_e, G)$. □

Lemma 8.6. *Let $n = 5$. Suppose that $\mathcal{A}$ is decomposable and $|\mathbf{O}_\theta(\mathcal{A})| = 8$. Then $\mathcal{A}$ is cyclotomic.*

Proof. Let $\mathcal{A}$ be the nontrivial S-wreath product for some $\mathcal{A}$-section $S = U/L$. Note that $|U| \leq 16$, $|G/L| \leq 16$, and $|S| \leq 8$. The S-rings $\mathcal{A}_U$, $\mathcal{A}_{G/L}$, and $\mathcal{A}_S$ are 2-S-rings by Lemma 6.2. So each of these S-rings is cyclotomic by Lemma 8.2 whenever the order of the corresponding group is at most 8 and by Lemma 8.5 otherwise. Since $|\mathbf{O}_\theta(\mathcal{A})| = 8$, we conclude that $|S| \leq 4$ or $|S| = 8$ and $|\mathbf{O}_\theta(\mathcal{A}_S)| \geq 4$. In both cases $\mathcal{A}_S$ is Cayley minimal by Lemma 8.2. This implies that

$$\mathrm{Aut}_U(\mathcal{A}_U)^S = \mathrm{Aut}_{G/L}(\mathcal{A}_{G/L})^S = \mathrm{Aut}_S(\mathcal{A}_S).$$

Now from [20, Lemma 4.3] it follows that $\mathcal{A}$ is cyclotomic. □

In the next two lemmas we establish some properties of decomposable 2-S-rings over $G \cong C_2^5$ whose thin radical is of size 2 or 4. These properties will be used in the proof of Theorem 1.1. The statements of Lemma 8.7 and Lemma 8.8 can be verified by analysis of computational results obtained with the help of the GAP package COCO2P [16]. For every decomposable 2-S-ring $\mathcal{A}$ with $|\mathbf{O}_\theta(\mathcal{A})| \in \{2,4\}$ over G (see the list [30]), we compute all $\mathcal{A}$-subgroups, automorphism groups, and Cayley automorphism groups of some restrictions and quotients.

Lemma 8.7. *Let $n = 5$. Suppose that $\mathcal{A}$ is decomposable and $|\mathbf{O}_\theta(\mathcal{A})| = 4$. Then one of the following statements holds:*

(1) there exists an $\mathcal{A}$-subgroup $L \leq \mathbf{O}_\theta(\mathcal{A})$ of order 2 such that $\mathcal{A} = \mathbb{Z}\mathbf{O}_\theta(\mathcal{A}) \wr_S \mathcal{A}_{G/L}$, where $S = \mathbf{O}_\theta(\mathcal{A})/L$;

(2) $|\operatorname{Aut}_G(\mathcal{A})| \geq |\operatorname{Aut}_U(\mathcal{A}_U)|$ for every $\mathcal{A}$-subgroup U with $|U| = 16$ and $U \geq \mathbf{O}_\theta(\mathcal{A})$;

(3) $\mathcal{A}$ is normal;

(4) there exist an $\mathcal{A}$-subgroup $L \leq \mathbf{O}_\theta(\mathcal{A})$ and $X \in \mathcal{S}(\mathcal{A})$ such that $|L| = |X| = 2$, $L \neq \operatorname{rad}(X)$, and $\mathcal{A}_{G/L}$ is normal.

Lemma 8.8. *Let $n = 5$. Suppose that $\mathcal{A}$ is decomposable, $|\mathbf{O}_\theta(\mathcal{A})| = 2$, and there exists $X \in \mathcal{S}(\mathcal{A})$ with $|X| > 1$ and $|\operatorname{rad}(X)| = 1$. Then $|X| = 4$ and one of the following statements holds:*

(1) $\mathcal{A} \cong \mathcal{B} \wr \mathbb{Z}C_2$, where $\mathcal{B}$ is a 2-S-ring over C_2^4;

(2) $|\operatorname{Aut}_G(\mathcal{A})| \geq |\operatorname{Aut}_U(\mathcal{A}_U)|$ for every $\mathcal{A}$-subgroup U with $|U| = 16$;

(3) there exists an $\mathcal{A}$-subgroup L such that $|L| \in \{2,4\}$ and $\mathcal{A}_{G/L}$ is normal.

Lemma 8.9. *Let $D \in \mathcal{E}$ such that every S-ring over a proper section of D is CI, $\mathcal{D}$ an S-ring over D, and $S = U/L$ a $\mathcal{D}$-section. Suppose that $\mathcal{D}$ is the nontrivial S-wreath product. Then $\mathcal{D}$ is a CI-S-ring whenever $D/L \cong C_2^k$ for some $k \leq 4$ and $\mathcal{D}_{D/L}$ is a 2-S-ring.*

Proof. The S-ring $\mathcal{D}_{D/L}$ is cyclotomic by Lemma 8.2 whenever $|D/L| \leq 8$ and by Lemma 8.5 whenever $|D/L| = 16$. The S-ring $\mathcal{D}_S$ is a 2-S-ring by Lemma 6.2. If $\mathcal{D}_S \ncong \mathbb{Z}C_2 \wr \mathbb{Z}C_2 \wr \mathbb{Z}C_2$ then $\mathcal{D}_S$ is Cayley minimal by Lemma 8.2. The S-rings $\mathcal{D}_U$ and $\mathcal{D}_{D/L}$ are CI-S-rings by the assumption of the lemma. So $\mathcal{D}$ is a CI-S-ring by Lemma 5.3.

Assume that

$$\mathcal{D}_S \cong \mathbb{Z}C_2 \wr \mathbb{Z}C_2 \wr \mathbb{Z}C_2.$$

In this case $|D/L| = 16$, $|S| = 8$, and there exists the least $\mathcal{D}_S$-subgroup A of S of order 2. Every basic set of $\mathcal{D}_{D/L}$ outside S is contained in an S-coset because $\mathcal{D}_{(D/L)/S} \cong \mathbb{Z}C_2$. So $\operatorname{rad}(X)$ is a $\mathcal{D}_S$-subgroup for every $X \in \mathcal{S}(\mathcal{D}_{D/L})$ outside S. If $|\operatorname{rad}(X)| > 1$ for every $X \in \mathcal{S}(\mathcal{D}_{D/L})$ outside S then $\mathcal{D}_{D/L}$ is the S/A-wreath product because A is the least $\mathcal{D}_S$-subgroup. This implies that $\mathcal{D}$ is the $U/\pi^{-1}(A)$-wreath product, where $\pi\colon D \to D/L$ is the canonical epimorphism. One can see that $|D/\pi^{-1}(A)| \leq 8$ and $|U/\pi^{-1}(A)| \leq 4$. The S-rings $\mathcal{D}_{D/\pi^{-1}(A)}$ and $\mathcal{D}_{U/\pi^{-1}(A)}$ are 2-S-rings by Lemma 6.2. The S-ring $\mathcal{D}_{D/\pi^{-1}(A)}$ is cyclotomic by Lemma 8.2 and the S-ring $\mathcal{D}_{U/\pi^{-1}(A)}$ is Cayley minimal by

Lemma 8.2. The S-rings $\mathcal{D}_U$ and $\mathcal{D}_{D/\pi^{-1}(A)}$ are CI-S-rings by the assumption of the lemma. Thus, $\mathcal{D}$ is a CI-S-ring by Lemma 5.3.

Suppose that there exists a basic set X of $\mathcal{D}_{D/L}$ outside S with $|\operatorname{rad}(X)| = 1$. If $\mathcal{D}_{D/L}$ is decomposable then

$$\operatorname{Aut}_{D/L}(\mathcal{D}_{D/L})^S = \operatorname{Aut}_S(\mathcal{D}_S)$$

by [20, Lemma 5.8]. Therefore $\mathcal{D}$ is a CI-S-ring by Lemma 5.1.

If $\mathcal{D}_{D/L}$ is indecomposable then $\mathcal{D}_{D/L}$ is normal by Lemma 8.3. So all conditions of Lemma 5.5 hold for $\mathcal{D}$. Thus, $\mathcal{D}$ is a CI-S-ring. □

Lemma 8.10. *Let $n \leq 5$. Then every S-ring over G is a* CI-*S-ring.*

Proof. Every S-ring over G is schurian by Lemma 8.1. So to prove the lemma, it is sufficient to prove that $\mathcal{B} = V(K, G)$ is a CI-S-ring for every $K \in \operatorname{Sup}_2^{\min}(G_{\text{right}})$ (see Remark 4.3). The S-ring $\mathcal{B}$ is a 2-S-ring by Lemma 6.4. If $n \leq 4$ then $\mathcal{B}$ is CI by [20, Lemma 5.7]. Thus, if $n = 4$ then the statement of the lemma holds.

Let $n = 5$. Suppose that $\mathcal{B}$ is indecomposable. Then the second part of Lemma 8.3 implies $\mathcal{B} \cong \mathbb{Z}C_2 \otimes \mathcal{B}'$, where $\mathcal{B}'$ is indecomposable 2-S-ring over C_2^4. Since $\mathcal{B}$ is schurian by Lemma 8.1 and every S-ring over an elementary abelian group of rank at most 4 is CI by the above paragraph, we conclude that $\mathcal{B}$ is a CI-S-ring by Lemma 5.6.

Now suppose that $\mathcal{B}$ is decomposable, i.e. $\mathcal{B}$ is the nontrivial $S = U/L$-wreath product for some $\mathcal{B}$-section $S = U/L$. Clearly, $|G/L| \leq 16$. The S-ring $\mathcal{B}_{G/L}$ is a 2-S-ring by Lemma 6.2. Since every S-ring over an elementary abelian group of rank at most 4 is CI, $\mathcal{B}$ is a CI-S-ring by Lemma 8.9. □

9 Proof of Theorem 1.1

Let $G = H \times P$, where $H \cong C_2^5$ and $P \cong C_p$, where p is a prime. These notations are valid until the end of the paper. If $p = 2$ then G is not a DCI-group by [28]. So in view of Lemma 4.2, to prove Theorem 1.1, it is sufficient to prove the following theorem.

Theorem 9.1. *Let p be an odd prime and $K \in \operatorname{Sup}_2^{\min}(G_{\text{right}})$. Then $\mathcal{A} = V(K, G)$ is a* CI-*S-ring.*

The proof of Proposition 9.1 will be given at the end of the section. We start with the next lemma concerned with proper sections of G.

Lemma 9.2. *Let S be a section of G such that $S \neq G$. Then every schurian S-ring over S is a* CI-*S-ring.*

Proof. If $S \cong C_2^n$ for some $n \leq 5$ then we are done by Lemma 8.10. Suppose that $S \cong C_2^n \times C_p$ for some $n \leq 4$. Then the statement of the lemma follows from [20, Remark 3.4] whenever $n \leq 3$ and from [20, Remark 3.4, Theorem 7.1] whenever $n = 4$. □

A key step towards the proof of Theorem 9.1 is the following lemma.

Lemma 9.3. *Let $\mathcal{A}$ be an S-ring over G and U an $\mathcal{A}$-subgroup with $U \geq P$. Suppose that P is an $\mathcal{A}$-subgroup, $\mathcal{A}$ is the nontrivial S-wreath product, where $S = U/P$, $|S| = 16$, and $\mathcal{A}_{G/P}$ is a 2-S-ring. Then $\mathcal{A}$ is a* CI-*S-ring.*

Proof. Firstly we prove two lemmas concerned with some special cases of Lemma 9.3.

Lemma 9.4. *Suppose that S has a gwr-complement with respect to $\mathcal{A}_{G/P}$. Then $\mathcal{A}$ is a* CI-*S-ring.*

Proof. The condition of the lemma implies that there exists an $\mathcal{A}_{G/P}$-subgroup A such that $\mathcal{A}_{G/P}$ is the nontrivial S/A-wreath product. This means that $\mathcal{A}$ is the nontrivial $U/\pi^{-1}(A)$-wreath product, where $\pi\colon G \to G/P$ is the canonical epimorphism. Note that $|G/\pi^{-1}(A)| \leq 16$ and $\mathcal{A}_{G/\pi^{-1}(A)} \cong \mathcal{A}_{(G/P)/A}$ is a 2-S-ring by Lemma 6.2. Therefore $\mathcal{A}$ is a CI-S-ring by Lemma 9.2 and Lemma 8.9. □

Lemma 9.5. *Suppose that S does not have a gwr-complement with respect to $\mathcal{A}_{G/P}$. Then*

$$|\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})^S| = |\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})|.$$

Proof. To prove the lemma it is sufficient to prove that the group

$$(\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P}))_S = \{\varphi \in \operatorname{Aut}_{G/P}(\mathcal{A}_{G/P}) :\ \varphi^S = \mathrm{id}_S\}$$

is trivial. Let $\varphi \in (\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P}))_S$. Put $\mathcal{C} = \operatorname{Cyc}(\langle\varphi\rangle, G/P)$. Clearly, $\langle\varphi\rangle \leq \operatorname{Aut}(\mathcal{A}_{G/P})$. So from Equations (3.1) and (3.2) it follows that $\mathcal{C} \geq \mathcal{A}_{G/P}$. Lemma 6.1 yields that $\mathcal{C}$ is a 2-S-ring. Since $\varphi^S = \mathrm{id}_S$, we conclude that $\mathbf{O}_\theta(\mathcal{C}) \geq S$.

If $\mathcal{C} \neq \mathbb{Z}(G/P)$ then $\mathbf{O}_\theta(\mathcal{C}) = S$. Therefore $\mathcal{C} = \mathbb{Z}S \wr_{S/A} \mathbb{Z}((G/P)/A)$ for some $\mathcal{C}$-subgroup A by Statement (i) of [19, Proposition 4.3]. This implies that $\mathcal{A}_{G/P} = \mathcal{A}_S \wr_{S/A} \mathcal{A}_{((G/P)/A)}$ because $\mathcal{C} \geq \mathcal{A}_{G/P}$ and S is both $\mathcal{A}_{G/P}$, $\mathcal{C}$-subgroup. We obtain a contradiction with the assumption of the lemma. Thus, $\mathcal{C} = \mathbb{Z}(G/P)$ and hence φ is trivial. So the group $(\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P}))_S$ is trivial. □

If $\mathcal{A}_{G/P}$ is indecomposable then $\mathcal{A}_{G/P}$ is normal by Lemma 8.3. So $\mathcal{A}$ is a CI-S-ring by Lemma 9.2 and Lemma 5.5. Further we assume that $\mathcal{A}_{G/P}$ is decomposable. Due to Lemma 9.4, we may assume also that

$$S \text{ does not have a gwr-complement with respect to } \mathcal{A}_{G/P}. \tag{9.1}$$

If there exists $X \in \mathcal{S}(\mathcal{A}_{G/P})$ outside S with $|X| = 1$ then $\mathcal{A}$ is a CI-S-ring by Lemma 9.2 and Lemma 5.8. So we may assume that

$$\mathbf{O}_\theta(\mathcal{A}_{G/P}) \leq S. \tag{9.2}$$

Note that $|\mathbf{O}_\theta(\mathcal{A}_{G/P})| > 1$ by Statement (1) of Lemma 6.3 and $|\mathbf{O}_\theta(\mathcal{A}_{G/P})| \leq 16$ by Equation (9.2). So $|\mathbf{O}_\theta(\mathcal{A}_{G/P})| \in \{2, 4, 8, 16\}$. We divide the rest of the proof into four cases depending on $|\mathbf{O}_\theta(\mathcal{A}_{G/P})|$.

Case 1: $|\mathbf{O}_\theta(\mathcal{A}_{G/P})| = 16$.

Due to Equation (9.2), we conclude that $\mathcal{A}_S = \mathbb{Z}S$. So $\mathcal{A}$ is a CI-S-ring by Lemma 9.2 and Lemma 5.2.

Case 2: $|\mathbf{O}_\theta(\mathcal{A}_{G/P})| = 8$.

Since $\mathcal{A}_{G/P}$ is decomposable, Lemma 8.6 implies that $\mathcal{A}_{G/P}$ is cyclotomic. The S-ring $\mathcal{A}_S$ is a 2-S-ring by Lemma 6.2. In view of Equation (9.2), we obtain that $|\mathbf{O}_\theta(\mathcal{A}_S)| = 8$. So Statement (ii) of [19, Proposition 4.3] yields that the S-ring $\mathcal{A}_S$ is Cayley minimal. Thus, $\mathcal{A}$ is a CI-S-ring by Lemma 9.2 and Lemma 5.3.

Case 3: $|\mathbf{O}_\theta(\mathcal{A}_{G/P})| = 4$.

In this case one of the statements of Lemma 8.7 holds for $\mathcal{A}_{G/P}$. If Statement (1) of Lemma 8.7 holds for $\mathcal{A}_{G/P}$ then we obtain a contradiction with Equation (9.1).

If Statement (2) of Lemma 8.7 holds for $\mathcal{A}_{G/P}$ then $|\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})| \geq |\operatorname{Aut}_S(\mathcal{A}_S)|$. From Lemma 9.5 it follows that $|\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})^S| = |\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})|$ and hence

$$|\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})^S| \geq |\operatorname{Aut}_S(\mathcal{A}_S)|.$$

Since $\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})^S \leq \operatorname{Aut}_S(\mathcal{A}_S)$, we conclude that $\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})^S = \operatorname{Aut}_S(\mathcal{A}_S)$. Thus, $\mathcal{A}$ is a CI-S-ring by Lemma 9.2 and Lemma 5.1.

If Statement (3) of Lemma 8.7 holds for $\mathcal{A}_{G/P}$ then $\mathcal{A}_{G/P}$ is normal. In this case $\mathcal{A}$ is a CI-S-ring by Lemma 9.2 and Lemma 5.5.

Suppose that Statement (4) of Lemma 8.7 holds for $\mathcal{A}_{G/P}$, i.e. there exists an $\mathcal{A}_{G/P}$-subgroup $A \leq \mathbf{O}_\theta(\mathcal{A}_{G/P})$ of order 2 and $X = \{x_1, x_2\} \in \mathcal{S}(\mathcal{A}_{G/P})$ such that $\mathcal{A}_{(G/P)/A}$ is normal and $A \neq \operatorname{rad}(X)$. Let $L = \pi^{-1}(A)$, where $\pi\colon G \to G/P$ is the canonical epimorphism, and $\mathcal{B} = V(N, G)$, where $N = \operatorname{Aut}(\mathcal{A})_{G/L} G_{\text{right}}$.

Prove that $\mathcal{B}$ is a CI-S-ring. Lemma 5.4 implies that $\mathcal{B}$ is the S-wreath product. From Equations (3.1) and (3.2) it follows that $\mathcal{B} \geq \mathcal{A}$. So $\mathcal{B}_{G/P} \geq \mathcal{A}_{G/P}$ and hence $\mathcal{B}_{G/P}$ is a 2-S-ring by Lemma 6.1. We obtain that $\mathcal{B}$ and U satisfy the conditions of Lemma 9.3.

One can see that X is a $\mathcal{B}_{G/P}$-set and

$$\mathbf{O}_\theta(\mathcal{B}_{G/P}) \geq \mathbf{O}_\theta(\mathcal{A}_{G/P}) \tag{9.3}$$

because $\mathcal{B}_{G/P} \geq \mathcal{A}_{G/P}$. The definition of $\mathcal{B}$ yields that every basic set of $\mathcal{B}$ is contained in an L-coset and hence every basic set of $\mathcal{B}_{G/P}$ is contained in an A-coset. Therefore

$$\{x_1\}, \{x_2\} \in \mathcal{S}(\mathcal{B}_{G/P}) \tag{9.4}$$

because X is a $\mathcal{B}_{G/P}$-set and $A \neq \operatorname{rad}(X)$. Now from Equations (9.3) and (9.4) it follows that

$$|\mathbf{O}_\theta(\mathcal{B}_{G/P})| \geq 8. \tag{9.5}$$

If $\mathcal{B}_{G/P}$ is indecomposable then $\mathcal{B}_{G/P}$ is normal by Lemma 8.3 and hence $\mathcal{B}$ is CI by Lemma 9.2 and Lemma 5.5. If S has a gwr-complement with respect to $\mathcal{B}_{G/P}$ then $\mathcal{B}$ is CI by Lemma 9.4. If $\mathbf{O}_\theta(\mathcal{B}_{G/P}) \not\leq S$ then $\mathcal{B}$ is CI by Lemma 9.2 and Lemma 5.8. Suppose that none of the above conditions does not hold for $\mathcal{B}$. Then, in view of Equation (9.5), $\mathcal{B}$ satisfies all conditions from one of the Cases 1 or 2. Therefore, $\mathcal{B}$ is CI.

Clearly, $\mathcal{A}_{G/L} \cong \mathcal{A}_{(G/P)/A}$ and hence $\mathcal{A}_{G/L}$ is normal. Also $\mathcal{A}_{G/L}$ is CI by Lemma 9.2. The S-ring $\mathcal{B}$ is CI by the above paragraph. Thus, $\mathcal{A}$ is CI by Lemma 4.4.

Case 4: $|\mathbf{O}_\theta(\mathcal{A}_{G/P})| = 2$.

Let $A = \mathbf{O}_\theta(\mathcal{A}_{G/P})$. Clearly, A is the least $\mathcal{A}_{G/P}$-subgroup. If $|\operatorname{rad}(X)| > 1$ for every $X \in \mathcal{S}(\mathcal{A}_{G/P})$ outside S then $A \leq \operatorname{rad}(X)$ for every $X \in \mathcal{S}(\mathcal{A}_{G/P})$ outside S and we obtain a contradiction with Equation (9.1). So there exists $X \in \mathcal{S}(\mathcal{A}_{G/P})$ outside S with $|\operatorname{rad}(X)| = 1$. From Equation (9.2) it follows that $|X| > 1$. Lemma 8.8 implies that $|X| = 4$. The number $\lambda = |X \cap Ax|$ does not depend on $x \in X$ by Lemma 2.1. If $\lambda = 2$ then $A \leq \operatorname{rad}(X)$, a contradiction. Therefore

$$\lambda = 1. \tag{9.6}$$

One of the statements of Lemma 8.8 holds for $\mathcal{A}_{G/P}$. If Statement (1) of Lemma 8.8 holds for $\mathcal{A}_{G/P}$ then there exists $Y \in \mathcal{S}(\mathcal{A}_{G/P})$ with $|Y| = 16$ and $|\operatorname{rad}(Y)| = 16$. Since $|S| = 16$, we conclude that Y lies outside S and hence $Y = (G/P) \setminus S$. This means that S is a gwr-complement to S with respect to $\mathcal{A}_{G/P}$. However, this contradicts Equation (9.1).

If Statement (2) of Lemma 8.8 holds for $\mathcal{A}_{G/P}$ then $|\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})| \geq |\operatorname{Aut}_S(\mathcal{A}_S)|$. So Lemma 9.5 implies that $\operatorname{Aut}_{G/P}(\mathcal{A}_{G/P})^S = \operatorname{Aut}_S(\mathcal{A}_S)$. Therefore, $\mathcal{A}$ is CI by Lemma 9.2 and Lemma 5.1

Suppose that Statement (3) of Lemma 8.8 holds for $\mathcal{A}_{G/P}$, i.e. there exists an $\mathcal{A}_{G/P}$-subgroup B such that $|B| \in \{2, 4\}$ and $\mathcal{A}_{(G/P)/B}$ is normal. Let $L = \pi^{-1}(B)$, where $\pi\colon G \to G/P$ is the canonical epimorphism, and $\mathcal{B} = V(N, G)$, where $N = \operatorname{Aut}(\mathcal{A})_{G/L} G_{\text{right}}$.

We prove that $\mathcal{B}$ is a CI-S-ring. As in Case 3, $\mathcal{B}$ is the S-wreath product by Lemma 5.4 and $\mathcal{B} \geq \mathcal{A}$ by Equations (3.1) and (3.2). So $\mathcal{B}_{G/P} \geq \mathcal{A}_{G/P}$ and hence $\mathcal{B}_{G/P}$ is a 2-S-ring by Lemma 6.1. Therefore $\mathcal{B}$ and U satisfy the conditions of Lemma 9.3.

Note that X is a $\mathcal{B}_{G/P}$-set and Equation (9.3) holds because $\mathcal{B}_{G/P} \geq \mathcal{A}_{G/P}$. By the definition of $\mathcal{B}$, every basic set of $\mathcal{B}$ is contained in an L-coset and hence every basic set of $\mathcal{B}_{G/P}$ is contained in a B-coset. The set X is a $\mathcal{B}_{G/P}$-set with $|X| = 4$ and $|\operatorname{rad}(X)| = 1$. So there exists $X_1 \in \mathcal{S}(\mathcal{B}_{G/P})$ such that

$$X_1 \subset X \text{ and } |X_1| \in \{1, 2\}.$$

If $|X_1| = 1$ then $X_1 \subseteq \mathbf{O}_\theta(\mathcal{B}_{G/P})$. If $|X_1| = 2$ then X_1 is a coset by a $\mathcal{B}_{G/P}$-subgroup A_1 of order 2. Clearly, $A_1 \subseteq \mathbf{O}_\theta(\mathcal{B}_{G/P})$. In view of Equation (9.6), we have $A_1 \neq A$. Thus, in both cases $\mathbf{O}_\theta(\mathcal{B}_{G/P}) \nleq A$. Together with Equation (9.3) this implies that

$$|\mathbf{O}_\theta(\mathcal{B}_{G/P})| \geq 4. \tag{9.7}$$

If $\mathcal{B}_{G/P}$ is indecomposable then $\mathcal{B}_{G/P}$ is normal by Lemma 8.3 and hence $\mathcal{B}$ is CI by Lemma 9.2 and Lemma 5.5. If S has a gwr-complement with respect to $\mathcal{B}_{G/P}$ then $\mathcal{B}$ is CI by Lemma 9.4. If $\mathbf{O}_\theta(\mathcal{B}_{G/P}) \nleq S$ then $\mathcal{B}$ is CI by Lemma 9.2 and Lemma 5.8. Suppose that none of the above conditions does not hold for $\mathcal{B}$. Then, in view of Equation (9.7), $\mathcal{B}$ satisfies all conditions from one of the Cases 1, 2 or 3. Therefore, $\mathcal{B}$ is CI.

The S-ring $\mathcal{A}_{G/L}$ is normal because it is isomorphic to $\mathcal{A}_{(G/P)/B}$. The S-rings $\mathcal{A}_{G/L}$ and $\mathcal{B}$ are CI by Lemma 9.2 and the above paragraph respectively. Thus, $\mathcal{A}$ is CI by Lemma 4.4.

All cases were considered. □

Proof of Theorem 9.1. Let H_1 be a maximal $\mathcal{A}$-subgroup contained in H and P_1 the least $\mathcal{A}$-subgroup containing P.

Lemma 9.6. *If $H_1 = H$ then $\mathcal{A}$ is a* CI-*S-ring.*

Proof. The S-ring $\mathcal{A}_{G/H}$ is a p-S-ring over $G/H \cong C_p$ by Lemma 6.4. So $\mathcal{A}_{G/H} \cong \mathbb{Z}C_p$. Clearly, $G = HP_1$. Therefore $\mathcal{A} = \mathcal{A}_H \star \mathcal{A}_{P_1}$ by Lemma 7.2. Since H and $P_1/(H \cap P_1)$ are proper sections of G, the S-rings $\mathcal{A}_H$ and $\mathcal{A}_{P_1/(H\cap P_1)}$ are CI by Lemma 9.2. Thus, $\mathcal{A}$ is CI by Lemma 5.6. □

Lemma 9.7. *If $H_1 < H$ and $H_1 P_1 = G$ then $\mathcal{A}$ is a* CI-*S-ring.*

Proof. Since $H_1 \neq (H_1P_1)_{p'} = H$, Lemma 7.1 implies that $\mathcal{A} = \mathcal{A}_{H_1} \star \mathcal{A}_{P_1}$. The S-rings $\mathcal{A}_{H_1}$ and $\mathcal{A}_{P_1/(H \cap P_1)}$ are CI by Lemma 9.2 because H_1 and $P_1/(H_1 \cap P_1)$ are proper sections of G. Therefore $\mathcal{A}$ is CI by Lemma 5.6. □

In view of Lemma 9.6, we may assume that $H_1 < H$. Then one of the statements of Lemma 7.3 holds for $\mathcal{A}$. If Statement (1) of Lemma 7.3 holds for $\mathcal{A}$ then

$$\mathcal{A} = \mathcal{A}_{H_1} \wr \mathcal{A}_{G/H_1},$$

where $\mathrm{rk}(\mathcal{A}_{G/H_1}) = 2$. If H_1 is trivial then $\mathrm{rk}(\mathcal{A}) = 2$. Obviously, $\mathcal{A}$ is CI in this case. If H_1 is nontrivial then $\mathcal{A}$ is CI by Lemma 9.2 and Lemma 5.2.

Assume that Statement (2) of Lemma 7.3 holds for $\mathcal{A}$, i.e.

$$\mathcal{A} = \mathcal{A}_U \wr_S \mathcal{A}_{G/P_1},$$

where $U = H_1P_1$, $S = U/P_1$, and $P_1 < G$. In view of Lemma 9.7, we may assume that $H_1P_1 < G$, i.e. $\mathcal{A}$ is the nontrivial S-wreath product. The group G/P_1 is a 2-group of order at most 32 because $P_1 \geq P$. Lemma 6.4 implies that $\mathcal{A}_{G/P_1}$ is a 2-S-ring. If $|G/P_1| \leq 16$ then $\mathcal{A}$ is CI by Lemma 9.2 and Lemma 8.9. So we may assume that $|G/P_1| = 32$. Clearly, in this case

$$P_1 = P.$$

In view of Statement (2) of Lemma 6.3, we may assume that

$$|S| = 16.$$

Indeed, if $|S| < 16$ then S is contained in an $\mathcal{A}_{G/P}$-subgroup S' of order 16 by Statement (2) of Lemma 6.3. Clearly, $\mathcal{A} = \mathcal{A}_{U'} \wr_{S'} \mathcal{A}_{G/P}$, where $U' = \pi^{-1}(S')$ and $\pi\colon G \to G/P$ is the canonical epimorphism. Replacing S by S', we obtain the required.

Now all conditions of Lemma 9.3 hold for $\mathcal{A}$ and U. Thus, $\mathcal{A}$ is CI by Lemma 9.3. □

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On the divisibility of binomial coefficients

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Abstract

Shareshian and Woodroofe asked if for every positive integer n there exist primes p and q such that, for all integers k with $1 \leq k \leq n-1$, the binomial coefficient $\binom{n}{k}$ is divisible by at least one of p or q. We give conditions under which a number n has this property and discuss a variant of this problem involving more than two primes. We prove that every positive integer n has infinitely many multiples with this property.

Keywords: Binomial coefficients, divisibility, primorials.

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1 Introduction

Binomial coefficients display interesting divisibility properties. Conditions under which a prime power p^a divides a binomial coefficient $\binom{n}{k}$ are given by Kummer's Theorem [10] and also by a generalized form of Lucas' Theorem [5, 13].

Still, there are problems involving divisibility of binomial coefficients that remain unsolved. In this article we investigate the following question, which was asked by Shareshian and Woodroofe in [16].

Question 1.1. Is it true that for every positive integer n there exist primes p and q such that, for all integers k with $1 \leq k \leq n-1$, the binomial coefficient $\binom{n}{k}$ is divisible by p or q?

As in [16], we say that n *satisfies Condition* 1 if such primes p and q exist for n. In this article we discuss sufficient conditions under which an integer n satisfies Condition 1. In Sections 2 and 3 we prove a variation of the Sieve Lemma from [16] and use it to show that

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n satisfies Condition 1 if certain inequalities hold. In Section 5 we infer that every positive integer has infinitely many multiples for which Condition 1 is satisfied.

The collection of numbers for which Condition 1 is not known to hold has asymptotic density 0 assuming the truth of Cramér's conjecture (as first shown in [16]) and includes most *primorials* $p_1p_2\cdots p_i$, where $p_1,\dots,p_i$ are the first i primes, namely those primorials such that $(p_1p_2\cdots p_i)-1$ is not a prime.

In addition, we introduce the following variant of Condition 1:

Definition 1.2. A positive integer n satisfies the N*-variation* of Condition 1 if there exist N different primes $p_1,\dots,p_N$ such that if $1\le k\le n-1$ then $\binom{n}{k}$ is divisible by at least one of $p_1,\dots,p_N$.

For example, it follows from Kummer's Theorem or from Lucas' Theorem that a positive integer n satisfies the 1-variation of Condition 1 if and only if n is a prime power, and every integer n satisfies the m-variation of Condition 1 if $n = p_1^{a_1}\cdots p_m^{a_m}$ where $p_1,\dots,p_m$ are distinct primes. In Section 4 we discuss upper bounds on N so that a given n satisfies the N-variation of Condition 1.

2 An extended Sieve Lemma

Our results in this section will be based on Lucas' Theorem:

Theorem 2.1 (Lucas [13])**.** *Let p be a prime and let*

$$n = n_rp^r + n_{r-1}p^{r-1} + \cdots + n_1p + n_0$$
$$k = k_rp^r + k_{r-1}p^{r-1} + \cdots + k_1p + k_0$$

be base p expansions of two positive integers, where $0\le n_i<p$ and $0\le k_i<p$ for all i, and $n_r\neq 0$. Then

$$\binom{n}{k}\equiv\prod_{i=0}^{r}\binom{n_i}{k_i}\pmod p.$$

By convention, a binomial coefficient $\binom{n_i}{k_i}$ is zero if $n_i<k_i$. Hence, if any of the digits of the base p expansion of n is 0 whereas the corresponding digit in the base p expansion of k is nonzero, then $\binom{n}{k}$ is divisible by p. As a particular case, if a prime power p^a with $a>0$ divides n and does not divide k, then $\binom{n}{k}$ is divisible by p.

Observe that, if n satisfies Condition 1 with two primes p and q, then at least one of these primes has to be a divisor of n, because otherwise $\binom{n}{1}$ would not be divisible by any of them. The next two results are elementary consequences of Lucas' Theorem.

Proposition 2.2. *If $n=p^a+1$ with p a prime and $a>0$, then n satisfies Condition 1 with p and any prime dividing n.*

Proof. If $n-1$ is a prime power then the two summands in the left-hand term of the equality

$$\binom{n-1}{k-1}+\binom{n-1}{k}=\binom{n}{k}$$

are divisible by p by Lucas' Theorem if $2\le k\le n-2$, and hence $\binom{n}{k}$ is also divisible by p. If $k=1$ or $k=n-1$, then $\binom{n}{k}=n$, so any prime factor of n divides $\binom{n}{k}$. □

Proposition 2.3. *If a positive integer n is equal to the product of two prime powers p_1^a and p_2^b with $a > 0$, $b > 0$, and $p_1 \neq p_2$, then n satisfies Condition 1 with p_1 and p_2.*

Proof. The base p_1 expansion of n ends with a zeroes and the base p_2 expansion of n ends with b zeroes. Because a positive integer k smaller than n cannot be divisible by both p_1^a and p_2^b, it is not possible that k ends with a zeroes in base p_1 and b zeroes in base p_2. Consequently, we can apply Lucas' Theorem modulo p_1 if p_1^a does not divide k or modulo p_2 if p_2^b does not divide k. □

Proposition 2.3 generalizes as follows.

Proposition 2.4. *If $p_1, \ldots, p_m$ are distinct primes and $n = p_1^{a_1} \cdots p_m^{a_m}$ with $a_i > 0$ for all i, then n satisfies the m-variation of Condition 1 with $p_1 \ldots, p_m$.*

Proof. If $1 \leq k \leq n-1$, then the base p_i expansion of k ends with less zeroes than the base p_i expansion of n for at least one prime factor p_i of n. □

The following result extends [16, Lemma 4.3]. It is the starting point of our discussion of Question 1.1 in the next sections. By symmetry, we only need to consider those values of k with $k \leq n/2$. Moreover, we may restrict our study further to those values of k that are multiples of p^a, since otherwise $\binom{n}{k}$ is divisible by p.

Theorem 2.5. *Let n be a positive integer and suppose that p^a divides n where p is a prime and $a > 0$. Suppose that there is a prime q with $n/(d+1) < q < n/d$, where $d \geq 1$, and let $k \leq n/2$. Then $\binom{n}{k}$ is divisible by p or q except possibly when k is a multiple of p^a belonging to one of the intervals $[cq, cq + \beta]$ with $\beta = n - dq$ and $0 \leq c < (d+1)/2$.*

Proof. Since $q < n/d$, the number $\beta = n - dq$ is positive. If $k \leq \beta$ then k is in the interval $[0, \beta]$, which is the case $c = 0$ in the statement of the theorem.

The assumption that $n/(d+1) < q$ is equivalent to assuming the inequality $n - dq < q$, which implies that the last digit in the base q expansion of n is equal to β. Hence, if $\beta < k < q$ then we may infer from Lucas' Theorem that $\binom{n}{k}$ is divisible by q.

The remaining range of values of k to be considered is $q \leq k \leq n/2$. In this case we look at the last digit of the base q expansion of k. If this last digit is bigger than β, then $\binom{n}{k}$ is again divisible by q. Thus the undecided cases are those in which the residue of k modulo q is smaller than or equal to β. This happens when $cq \leq k \leq cq + \beta$ for some positive integer c, and if $cq \leq k \leq n/2$ then $c \leq n/(2q) < (d+1)/2$. □

By the Bertrand-Chebyshev Theorem [2], for every integer $n > 2$ there exists a prime q such that $n/2 < q < n$. This yields the following particular instance of Theorem 2.5, which is also a special case of [16, Lemma 4.3].

Corollary 2.6. *For a positive integer n, suppose that p^a divides n where p is a prime and $a > 0$. If q is a prime such that $n/2 < q < n$ and $n - q < p^a$, then n satisfies Condition 1 with p and q.*

Proof. Pick $d = 1$ in Theorem 2.5. □

Note that, under the assumptions of Corollary 2.6, the equality $n - q = p^a$ cannot hold, since p divides n and $p \neq q$ because q does not divide n. Hence there remains to study the case when $n - q > p^a$ and q is the largest prime smaller than n while p^a is the largest prime power dividing n. In other words, Condition 1 holds for n whenever there is a prime between $n - p^a$ and n.

The sequence of integers n for which there is no prime between $n - p^a$ and n can be found in the On-Line Encyclopedia of Integer Sequences (OEIS) [17] with the reference A290203 [3]. Its first terms are the following:

$$126, 210, 330, 630, 1144, 1360, 2520, 2574, 2992, 3432, 3960, 4199, \ldots \tag{2.1}$$

Banderier's conjecture [1] claims that if $p_n\#$ denotes the n-th *primorial*, that is,

$$p_n\# = p_1 p_2 \cdots p_n$$

where $p_1, \ldots, p_n$ are the first n primes, and q is the largest prime below $p_n\#$, then either $p_n\# - q = 1$ or $p_n\# - q$ is a prime.

Proposition 2.7. *If Banderier's conjecture is true, then the sequence* (2.1) *contains all primorials $p_n\#$ such that $p_n\# - 1$ is not a prime.*

Proof. If $p_n\# - 1$ is not a prime, then $p_n\# - q$ is a prime according to Banderier's conjecture. Since $p_n\# - q$ does not divide $p_n\#$, we infer that $p_n\# - q$ is bigger than p_n, which is the largest prime power dividing $p_n\#$. □

The first primorials $p_n\#$ such that $p_n\# - 1$ is not a prime are

$$p_4\# = 210, \quad p_7\# = 510510, \quad p_8\# = 9699690, \quad p_9\# = 223092870.$$

Inspecting this list could be a strategy to seek for a counterexample for Question 1.1. The complementary list of primorials can be found in OEIS with reference A057704 [11].

For any fixed value of d, the number β in Theorem 2.5 is smallest when q is as close as possible to n/d. For this reason, we focus our attention on the largest prime q_d below n/d for various values of d. This motivates the next definition.

Definition 2.8. For positive integers n and $1 \leq d < n/2$, let q_d be the largest prime smaller than n/d and let $\beta_d = n - dq_d$. For each integer c with $0 \leq c < (d+1)/2$, we call $[cq_d,\ cq_d + \beta_d]$ a *dangerous interval.*

By Theorem 2.5, if we attempt to prove that Condition 1 holds with p and q_d assuming that $q_d > n/(d+1)$ —that is, assuming that the dangerous intervals are disjoint— we only need to care about values of k that lie in a dangerous interval and are multiples of the largest power of p dividing n.

In the case $d = 1$, the only dangerous interval below $n/2$ is $[0, n - q_1]$. When $d = 2$, we have that $[0, n - 2q_2]$ and $[q_2, n - q_2]$ are dangerous intervals. Since $n - q_2 > n/2$, the second interval may be replaced by $[q_2, n/2]$ to carry our study further, as we do in the next section.

Example 2.9. The largest prime below $n = p_7\# = 510510$ is $q_1 = 510481$ and the largest prime dividing n is $p = 17$. Here $n - q_1 = 29$ and therefore $\binom{n}{k}$ is divisible by 17 or 510481 for all k except for $k = 17$.

On the other hand, the largest prime below $n/2 = 255255$ is $q_2 = 255253$. Thus $\beta_2 = n - 2q_2 = 4$ and therefore $[0, 4]$ and $[255253,\ 255257]$ are dangerous intervals. The second interval contains a multiple of 17, namely $n/2$. However, since

$$
\begin{aligned}
510510 &= 6 \cdot 17^4 + 1 \cdot 17^3 + 15 \cdot 17^2 + 8 \cdot 17, \\
255255 &= 3 \cdot 17^4 + 0 \cdot 17^3 + 16 \cdot 17^2 + 4 \cdot 17,
\end{aligned}
$$

we infer from Lucas' Theorem that $\binom{510510}{255255}$ is divisible by 17. Consequently, $\binom{n}{k}$ is divisible by 17 or 255253 for all k.

3 Using the nearest prime below $n/2$

Nagura showed in [14] that, if $m \geq 25$, then there is a prime between m and $(1 + 1/5)m$. Therefore, there is a prime q such that $5n/6 < q < n$ when $n \geq 30$. This implies that, if $n \geq 30$ and the largest prime-power divisor p^a of n satisfies $p^a \geq n/6$, then there is a prime q between $n - p^a$ and n and hence Condition 1 holds for n with p and q.

The following result is sharper.

Proposition 3.1. *If $n \geq 2010882$ and the largest prime-power divisor p^a of n satisfies $p^a \geq n/16598$, then n satisfies Condition 1 with p and the nearest prime q below n.*

Proof. Schoenfeld proved in [15] that for $m \geq 2010760$ there is a prime between m and $(1 + 1/16597)m$. Hence, if $n \geq 2010882$ and the largest prime-power divisor p^a of n satisfies $p^a \geq n/16598$ then there is a prime between $n - p^a$ and n, and therefore Condition 1 holds for n by Corollary 2.6. □

The following are consequences of Nagura's and Schoenfeld's bounds.

Lemma 3.2. *Let q_d be the largest prime below n/d for positive integers n and d.*

(a) If $n \geq 120$ and $d < 5$, then $n/(d+1) < q_d$.

(b) If $n \geq 3.34 \cdot 10^{10}$ and $d < 16597$, then $n/(d+1) < q_d$.

Proof. By Nagura's bound [14], if $n/d \geq 30$, then $5n/6d < q_d < n/d$. Therefore, $n - dq_d < n/6$. If $d < 5$, then $6d < 5(d+1)$ and hence

$$
n < \frac{5n(d+1)}{6d} < q_d(d+1),
$$

as claimed. The proof of part (b) is analogous using Schoenfeld's bound [15]. □

In order to apply Theorem 2.5 with $d = 2$ for a given n, we need that there is a prime q such that $n/3 < q < n/2$. If q_2 denotes the nearest prime below $n/2$, then the inequality $n/3 < q_2$ holds if $n \geq 120$ by Lemma 3.2. Since by (2.1) we have that $n - q_1 < p^a$ if $n < 126$, we may assume that $n/3 < q_2$ without any loss of generality.

Note that the inequality $n/3 < q$ is equivalent to $n - 2q < q$, so the intervals $[0, n-2q]$ and $[q, n-q]$ are disjoint.

Theorem 3.3. *For an odd positive integer n and a prime power p^a dividing n, suppose that there is a prime q with $n/3 < q < n/2$ and $n - 2q < p^a$. Then n satisfies Condition 1 with p and q.*

Proof. By Theorem 2.5, in order to infer that $\binom{n}{k}$ is divisible by p or q, the only cases that we need to discuss are those values of k that are multiples of p^a with $k \in [0, n-2q]$ or $k \in [q, n-q]$. By assumption, there are no multiples of p^a in $[0, n-2q]$. Since $n-q > n/2$, we may focus on the interval $[q, n/2]$. Since n is odd, $n/2$ is not an integer; hence we are only left to prove that there is no multiple k of p^a with $q \leq k < n/2$. We will prove this by contradiction.

Thus suppose that $q \leq \lambda p^a < n/2$ for some integer λ. The assumption that $n-2q < p^a$ implies that $n - p^a < 2q$ and hence

$$n/2 - p^a/2 < q \leq \lambda p^a.$$

Consequently, $\lambda p^a < n/2 < (\lambda + 1/2)p^a$. If we now write $n = mp^a$, we obtain that $2\lambda < m < 2\lambda + 1$, which is impossible for an integer m. □

The rest of this section is devoted to the case when n is even.

Lemma 3.4. *Suppose that n is even and there is a prime q with $q < n/2$ and $n - 2q < p^a$, where p^a is the largest power of p dividing n. If there is a multiple k of p^a in the interval $[q, n/2]$, then p is odd and $k = n/2$.*

Proof. Suppose first that p is odd. Then the integer $n/2$ is a multiple of p^a, so we may write $n/2 = \lambda p^a$ for some integer λ. If there is another multiple of p^a in the interval $[q, n/2]$, then $q \leq (\lambda - 1)p^a < n/2$, and this implies that

$$n/2 - p^a = \lambda p^a - p^a = (\lambda - 1)p^a \geq q.$$

Hence $n - 2q \geq 2p^a$, which is incompatible with our assumption that $n - 2q < p^a$.

In the case $p = 2$ (so that 2^a is the largest power of 2 dividing n), we have that $n/2$ is divisible by 2^{a-1}, and we may write $n/2 = \lambda 2^{a-1}$ with λ odd. If there is a multiple of 2^a in the interval $[q, n/2)$, then $q \leq \mu 2^a < n/2$, so $\mu < \lambda/2$ and $\mu \leq (\lambda - 1)/2$ because λ is odd. Therefore

$$n/2 - 2^{a-1} = (\lambda - 1)2^{a-1} \geq \mu 2^a \geq q.$$

Hence, as above, $n - 2q \geq 2^a$, which contradicts that $n - 2q < 2^a$. □

Theorem 3.5. *For an even positive integer n, suppose that there is a prime q such that $n/3 < q < n/2$ and $n - 2q < p^a$, where p^a is the largest power of p dividing n.*

(a) If $p = 2$, then n satisfies Condition 1 with 2 and q.

(b) If $p \neq 2$, then n satisfies Condition 1 with p and q if and only if $\binom{n}{n/2}$ is divisible by p.

Proof. By Theorem 2.5 and Lemma 3.4, the only case left is $k = n/2$ for p odd. Consequently, if $\binom{n}{n/2}$ is divisible by p, then n satisfies Condition 1 with p and q. Moreover, $\binom{n}{n/2}$ is not divisible by q, since the base q expansions of n and $n/2$ are, respectively, $2 \cdot q + (n - 2q)$ and $1 \cdot q + (n/2 - q)$. Hence the assumption that $\binom{n}{n/2}$ be divisible by p is necessary. □

Our last remarks in this section correspond to the case when n is even, and they are only relevant if $p \neq 2$, by Theorem 3.5. Next we give sufficient conditions to infer that a prime p divides $\binom{n}{n/2}$. The greatest integer less than or equal to a real number x is denoted by $\lfloor x \rfloor$, and we write $v_p(n) = a$ if p^a is the maximum power of p such that p^a divides n.

Recall from [12] that

$$v_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor = \frac{n - s_p(n)}{p-1}, \tag{3.1}$$

where $s_p(n)$ denotes the sum of all the digits in the base p expansion of n.

Proposition 3.6. *Suppose that n is even. A prime p divides $\binom{n}{n/2}$ if and only if at least one of the numbers $\lfloor n/p^r \rfloor$ with $r \geq 1$ is odd.*

Proof. By comparing $v_p(n!)$ and $v_p((n/2)!)$ we see that, for each r,

$$\left\lfloor \frac{n}{p^r} \right\rfloor = 2 \left\lfloor \frac{n/2}{p^r} \right\rfloor$$

if $\lfloor n/p^r \rfloor$ is even. If $\lfloor n/p^r \rfloor$ is even for all r, we conclude that $v_p(n!) = 2v_p((n/2)!)$, and hence p does not divide $\binom{n}{n/2}$. However, if $\lfloor n/p^r \rfloor$ is odd, then

$$\left\lfloor \frac{n}{p^r} \right\rfloor = 2 \left\lfloor \frac{n/2}{p^r} \right\rfloor + 1$$

and consequently $v_p(n!)$ is greater than $2v_p((n/2)!)$. □

Corollary 3.7. *If n is even and $(n - s_p(n))/(p-1)$ is odd, then p divides $\binom{n}{n/2}$.*

Proof. This follows from Proposition 3.6 and Legendre's formula (3.1). □

Corollary 3.8. *Suppose that n is even.*

(a) If any of the digits in the base p expansion of $n/2$ is larger than $\lfloor p/2 \rfloor$, then p divides $\binom{n}{n/2}$.

(b) If one of the digits in the base p expansion of n is odd, then p divides $\binom{n}{n/2}$.

Proof. If a digit of $n/2$ in base p is larger than $\lfloor p/2 \rfloor$, then when we add $n/2$ to itself in base p to obtain n there is at least one carry. Similarly, if n has an odd digit in base p, then there is a carry when adding $n/2$ and $n/2$ in base p. Hence, by Kummer's Theorem [10] with $k = n/2$, if there is at least one carry when adding $n/2$ to itself in base p, then p divides $\binom{n}{n/2}$. □

Corollary 3.9. *Let n be an even positive integer. Suppose that there is a prime q such that $n/3 < q < n/2$ and $n - 2q < p^a$, where p^a denotes the largest power of p dividing n. If $p^{\lfloor \log n / \log p \rfloor} > n/2$, then p divides $\binom{n}{n/2}$ and therefore n satisfies Condition 1 with p and q.*

Proof. The largest value of r such that $p^r < n < p^{r+1}$ is $\lfloor \log n / \log p \rfloor$. Therefore, in Proposition 3.6, the exponent r is bounded by $\lfloor \log n / \log p \rfloor$. Also note that $r \geq a$, where a is the largest exponent of p such that p^a divides n. If $p^{\lfloor \log n / \log p \rfloor} > n/2$, then $\lfloor n/p^r \rfloor = 1$. Because this is odd, p divides $\binom{n}{n/2}$ by Proposition 3.6. □

In those cases when the inequalities $n - q_1 < p^a$ and $n - 2q_2 < p^a$ both fail for the largest prime power p^a dividing n, a possible strategy would be to analyze the inequality $n - dq_d < p^a$ for bigger values of d, where q_d is the largest prime below n/d.

Up to 1,000,000 there are 88 integers that do not satisfy $n - 2q_2 < p^a$, where p^a is the largest prime power dividing n. The On-Line Encyclopedia of Integer Sequences has published these numbers with the reference A290290 [4]. Among these, there are 25 that do not satisfy the inequality $n - 3q_3 < p^a$; there are 7 that do not satisfy the inequality $n - 4q_4 < p^a$ either; there are 5 for which the inequality $n - 5q_5 < p^a$ also fails, and there is only one integer for which the inequality $n - 6q_6 < p^a$ still fails (namely, $n = 875160$). However, the value of $n - dq_d$ need not decrease as d grows, and the number of dangerous intervals that one needs to inspect when $n - dq_d < p^a$ increases linearly with d. Therefore this strategy is not conclusive, although it often works in practice.

Example 3.10. The largest prime power dividing $n = p_{14}\# = 13082761331670030$ is $p = 43$. In this case, $n - q_1 = 89$ and $n - 2q_2 = 268$. Thus, Condition 1 fails for p and q_1 and it also fails for p and q_2. Nevertheless, $n - 3q_3 = 27$ works, as the dangerous interval $[q_3, n - 2q_3]$ contains one multiple of 43, namely $n/3$, and $\binom{n}{n/3}$ is divisible by 43. Therefore Condition 1 holds for $p = 43$ and $q_3 = 4360920443890001$.

Example 3.11. For $n = 210$, the inequality $n - q_1 < 7$ fails while $n - 2q_2 < 7$ is true. However, $\binom{210}{105}$ is not divisible by 7. Hence we look for greater values of d and find that $n - 5q_5 < 7$ with $q_5 = 41$. Now $42 \in [41, 46]$ and $84 \in [82, 87]$, yet $\binom{210}{42}$ and $\binom{210}{84}$ are both divisible by 7. Hence Condition 1 is satisfied with $p = 7$ and $q_5 = 41$.

Example 3.12. For $n = 875160$, the inequality $n - dq_d < 17$ is satisfied with $d = 11$ but not with any smaller value of d. There are 6 dangerous intervals of length $n - 11q_{11} = 11$. Each of these intervals (except the first) contains one multiple of 17, and in each case the corresponding binomial coefficient $\binom{n}{k}$ happens to be divisible by 17. Therefore Condition 1 is satisfied with $p = 17$ and $q_{11} = 79559$.

4 On the N-variation of Condition 1

Recall from Definition 1.2 that n satisfies the *N-variation* of Condition 1 if there are N primes $p_1, \dots, p_N$ such that if $1 \le k \le n - 1$ then $\binom{n}{k}$ is divisible by at least one of $p_1, \dots, p_N$.

Theorem 4.1. *If an even positive integer n satisfies $n - 2q < p^a$ for a prime q with $n/3 < q < n/2$, where p^a is the largest power of p dividing n and $p \ne 2$, then n satisfies the 3-variation of Condition 1 with p, q and any prime that divides $\binom{n}{n/2}$.*

Proof. According to the statement of part (b) of Theorem 3.5, the only binomial coefficient $\binom{n}{k}$ with $1 \le k \le n - 1$ that might fail to be divisible by p or q is $\binom{n}{n/2}$. Hence it suffices to add an extra prime with this purpose. □

Proposition 4.2. *For a positive integer n, let q_1 be the largest prime smaller than n, let $p_1^{a_1}$ be the largest prime-power divisor of n and let $p_2^{a_2}$ be the second largest prime-power divisor of n. If $p_1^{a_1} p_2^{a_2} > n - q_1$, then n satisfies the 3-variation of Condition 1 with p_1, p_2 and q_1.*

Proof. By Lucas' Theorem, for any k such that $1 \le k < p_1^{a_1}$, the binomial coefficient $\binom{n}{k}$ is divisible by p_1, and for any k such that $n - q_1 < k \le n/2$ the binomial coefficient $\binom{n}{k}$ is divisible by q_1. Thus we need to add a prime that divides at least the binomial coefficients $\binom{n}{k}$ with $p_1^{a_1} \le k \le n - q_1$ in which k is a multiple of $p_1^{a_1}$. For this, we pick p_2 and therefore we only need to consider those values of k that are, in addition, multiples of $p_2^{a_2}$. The least k that is a multiple of both prime powers is $p_1^{a_1} p_2^{a_2}$. Therefore, if $p_1^{a_1} p_2^{a_2} > n - q_1$, then all values of k lying in the interval $p_1^{a_1} \le k \le n - q_1$ are such that $\binom{n}{k}$ is divisible by p_1 or p_2. □

In the statement of Proposition 4.2, the condition that $p_1^{a_1} p_2^{a_2} > n - q_1$ holds by Nagura's bound [14] if we impose instead that $p_1^{a_1} p_2^{a_2} > n/6$.

For each n, we are interested in the minimum number N of primes such that n satisfies the N-variation of Condition 1. We next discuss upper bounds for N.

Proposition 4.3. *For positive integers n and d, suppose that there is a prime q such that $n/(d+1) < q < n/d$ and a prime-power divisor p^a of n such that $n - dq < p^a$. Then n satisfies the N-variation of Condition* 1 *with $N = 2 + \lfloor d/2 \rfloor$.*

Proof. By Theorem 2.5, the binomial coefficients $\binom{n}{k}$ are divisible by q except possibly if k lies in a dangerous interval. In the dangerous intervals we only need to consider those integers that are multiples of p^a, since otherwise $\binom{n}{k}$ is divisible by p. Since we are assuming that $n - dq < p^a$, we know that in each dangerous interval there is at most one multiple of p^a. This means that the worst case is the one in which there is a multiple of p^a in every dangerous interval $[cq, cq + \beta]$ with $1 \le c \le \lfloor d/2 \rfloor$. Hence we pick one extra prime for each such interval. □

Corollary 4.4. *If $1 < d < 5$ and $p^a > q_d + \beta_d$ where p^a divides n and q_d is the largest prime below n/d, and $\beta_d = n - dq_d$, then n satisfies Condition* 1 *with p and q_d.*

Proof. By Lemma 3.2, we may assume that $n/(d+1) < q_d$. If $1 < d < 5$, then $\lfloor d/2 \rfloor$ equals 1 or 2. If $\lfloor d/2 \rfloor = 1$, then the assumption that $p^a > q_d + \beta_d$ implies that no multiple of p^a falls into any dangerous interval until $n/2$. If $\lfloor d/2 \rfloor = 2$, then we need to check that $2p^a > 2q_d + \beta_d$ in order to ensure that $2p^a$ does not fall into the third dangerous interval. The minimum value of p^a such that our assumption $p^a > q_d + \beta_d$ holds is $q_d + \beta_d + 1$. The next multiple of $q_d + \beta_d + 1$ is $2q_d + 2\beta_d + 2$, which is greater than $2q_d + \beta_d$ and therefore $2p^a$ does not fall into the third dangerous interval. □

In order to refine the conclusion of Proposition 4.3, we consider the Diophantine equation

$$p^a x - q_d y = \delta, \tag{4.1}$$

for $0 \le \delta \le \beta_d = n - dq_d$, where p^a is a prime-power divisor of a given number n and q_d is the largest prime below n/d with $d \ge 1$. We keep assuming, as above, that $q_d > n/(d+1)$. We will also assume that $p \ne q_d$, which guarantees that (4.1) has infinitely many solutions for each value of δ. Specifically, if (x_1, y_1) is a particular solution for some value of δ, then the general solution for this δ is

$$x = x_1 + rq_d, \qquad y = y_1 + rp^a,$$

where r is any integer. In the next theorem we denote by $N(\delta)$ the number of solutions (x, y) of (4.1) with $x > 0$ and $0 \le y \le \lfloor d/2 \rfloor$ for each value of δ with $0 \le \delta \le \beta_d$. Thus $N(\delta) = 0$ precisely when (4.1) has no solution (x, y) subject to these conditions.

Theorem 4.5. *For positive integers n and d, suppose that the largest prime q_d below n/d satisfies $q_d > n/(d+1)$, and let $\beta_d = n - dq_d$. Let p^a be a prime power dividing n with $p \neq q_d$. Then n satisfies the N-variation of Condition 1 with*

$$N = 2 + \sum_{\delta=0}^{\beta_d} N(\delta),$$

where $N(\delta)$ is the number of solutions (x, y) of $p^a x - q_d y = \delta$ with $x > 0$ and $0 \leq y \leq \lfloor d/2 \rfloor$ for each value of δ with $0 \leq \delta \leq \beta_d$.

Proof. The number $N(\delta)$ counts how many times a multiple of p^a falls into a dangerous interval $[cq_d, cq_d + \beta_d]$ at a distance δ from the origin of that interval. Thus we pick an extra prime for each such case, and add two to the sum in order to account for p and q_d. □

Example 4.6. The largest prime-power divisor of $n = 96135$ is $p = 29$. For $d = 4$ we find that $q_4 = 24029$ and $\beta_4 = 19$. Since $24029 \equiv 17 \pmod{29}$, the only solution (x, y) of the Diophantine equation $29x - 24029y = \delta$ with $x > 0$ and $0 \leq y \leq 2$ is $(829, 1)$ for $\delta = 12$. Thus, $N(12) = 1$ and $N = 3$ for $d = 4$. In other words, the only occurrence of a multiple of 29 in a dangerous interval for $d = 4$ is $24041 \in [24029,\ 24048]$. This example shows that the bound $2 + \lfloor d/2 \rfloor$ given in Proposition 4.3 can be lowered.

The number N given by Theorem 4.5 is not a sharp bound. For those multiples $p^a x$ of p^a falling into a dangerous interval $[cq_d, cq_d + \beta_d]$, it often happens that the corresponding binomial coefficient $\binom{n}{p^a x}$ is divisible by p, as in Example 4.6 or in other examples given in the previous sections. It could also be divisible by q_d if $d \geq q_d$. When $d < q_d$, we have that n satisfies Condition 1 with p and q_d if and only if the binomial coefficient $\binom{n}{p^a x}$ is divisible by p for every solution (x, y) of (4.1) with $x > 0$ and $0 \leq y \leq \lfloor d/2 \rfloor$, since $n = dq_d + \beta_d$ and $p^a x = yq_d + \delta$ with $\delta \leq \beta_d < q_d$ and $y \leq \lfloor d/2 \rfloor < d$, so $\binom{n}{p^a x}$ is not divisible by q_d by Lucas' Theorem. Note also, for practical purposes, that $\binom{n}{p^a x} \equiv \binom{n/p^a}{x} \pmod{p}$.

5 Every number has multiples for which Condition 1 holds

We next prove that every positive integer n has infinitely many multiples for which Condition 1 holds. We are indebted to R. Woodroofe for simplifying and improving our earlier statement of this result, which was based on prime gap conjectures.

It follows from the Prime Number Theorem [7] that, given any real number $\varepsilon > 0$, there is a prime between m and $m(1+\varepsilon)$ for sufficiently large m. This fact can be used to prove the following:

Theorem 5.1. *For every positive integer n and every prime p, the number np^k satisfies Condition 1 with p and another prime, for all sufficiently large values of k.*

Proof. For any prime p and any $k > 0$, let $m = np^k - p^k = p^k(n-1)$. Then

$$np^k = m + p^k = m\left(1 + \frac{1}{n-1}\right).$$

Therefore, by the Prime Number Theorem, there is a prime between m and np^k for all sufficiently large values of k. Choose the largest prime q with this property. Thus,

$$np^k - p^k < q < np^k,$$

so $np^k - q < p^k$, from which it follows, according to Corollary 2.6, that np^k satisfies Condition 1 with p and q. $\square$

Theorem 5.2. *For every positive integer n there is a number M such that if p is any prime with $p > M$ then np satisfies Condition 1 with p and another prime.*

Proof. Given n, let $\varepsilon = 1/(n-1)$. Choose m_0 such that there is a prime between m and $m(1+\varepsilon)$ for all $m \geq m_0$, and let $M = \varepsilon m_0$. If p is any prime such that $p > M$, then for $m = p(n-1)$ we have

$$np = m + p = m\left(1 + \frac{p}{m}\right) = m\left(1 + \frac{1}{n-1}\right) = m(1+\varepsilon).$$

Therefore, by our choice of m_0, there is a prime between m and np. If q is the largest prime with this property, then $np - p < q < np$, and consequently np satisfies Condition 1 with p and q. $\square$

Prime gap conjectures provide information relevant to our problem. For example, if p_i denotes the i-th prime, then Cramér's conjecture [6] claims that there exist constants M and N such that if $p_i \geq N$ then

$$p_{i+1} - p_i \leq M(\log p_i)^2.$$

Proposition 5.3. *Let m be the number of distinct prime factors of n. If Cramér's conjecture is true and n grows sufficiently large keeping m fixed, then n satisfies Condition 1.*

Proof. If n has m distinct prime factors, then $\sqrt[m]{n} \leq p^a$, where p^a is the largest prime-power divisor of n. Let M and N be the constants given by Cramér's conjecture. Pick n_0 such that if $n \geq n_0$ then $M(\log n)^2 < \sqrt[m]{n}$. For every n such that $n \geq n_0$ and $N \leq p_i < n \leq p_{i+1}$ (where p_i denotes the i-th prime), we have

$$n - p_i \leq p_{i+1} - p_i \leq M(\log p_i)^2 < M(\log n)^2 < \sqrt[m]{n} \leq p^a,$$

from which it follows that n satisfies Condition 1 with p and p_i. $\square$

We note that the argument used in the proof of Proposition 5.3 yields an alternative proof of the fact that Condition 1 holds for a set of integers of asymptotic density 1 if Cramér's conjecture holds, a result first found in [16, § 5]:

Theorem 5.4 ([16]). *If Cramér's conjecture is true, then the set of numbers in the sequence* (2.1) *has asymptotic density zero.*

Proof. Suppose that Cramér's conjecture holds with constants M and N, and denote by $\omega(n)$ the number of distinct prime divisors of n. Thus $n^{1/\omega(n)} \leq p^a$, where p^a is the largest prime-power divisor of n. According to [8, § 3.2], for every $\varepsilon > 0$ the inequality

$$\omega(n) < (1+\varepsilon)\log\log n \tag{5.1}$$

holds for all n except those of a set of asymptotic density zero. Since

$$\lim_{n\to\infty} \frac{n^{1/\log\log n}}{(\log n)^k} = \infty$$

for all k, there is an n_0 such that $n^{1/\omega(n)} > M(\log n)^2$ if $n \geq n_0$. Now, if n is bigger than n_0 and satisfies $N \leq p_i < n \leq p_{i+1}$, and moreover n is not in the set of integers for which (5.1) fails, then

$$n - p_i \leq p_{i+1} - p_i \leq M(\log p_i)^2 < M(\log n)^2 < n^{1/w(n)} \leq p^a.$$

Therefore, n satisfies Condition 1 with p and p_i. □

6 Multinomials

We also consider a generalization of Condition 1 to multinomials. We say that a positive integer n satisfies *Condition* 1 *for multinomials of order* m if there are primes p and q such that the multinomial coefficient

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}$$

is divisible by either p or q whenever $k_1 + \cdots + k_m = n$ with $1 \leq k_i \leq n - 1$ for all i.

Proposition 6.1. *If n satisfies Condition 1 with two primes p and q, then n satisfies Condition 1 for multinomials of any order $m \leq n$ with p and q.*

Proof. This follows from the equality

$$\binom{n}{k_1, k_2, \ldots, k_m} = \binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}\cdots\binom{k_m}{k_m},$$

and the fact that $\binom{n}{k_1}$ is divisible by p or q by assumption. □

Therefore, if Condition 1 is proven for binomial coefficients, then it automatically holds for multinomial coefficients.

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Distance-balanced graphs and travelling salesman problems*

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Abstract

For every probability $p \in [0,1]$ we define a distance-based graph property, the pTS-distance-balancedness, that in the case $p = 0$ coincides with the standard property of distance-balancedness, and in the case $p = 1$ is related to the Hamiltonian-connectedness. In analogy with the classical case, where the distance-balancedness of a graph is equivalent to the property of being self-median, we characterize the class of pTS-distance-balanced graphs in terms of their equity with respect to certain probabilistic centrality measures, inspired by the Travelling Salesman Problem. We prove that it is possible to detect this property looking at the classical distance-balancedness (and therefore looking at the classical centrality problems) of a suitable graph composition, namely the wreath product of graphs. More precisely, we characterize the distance-balancedness of a wreath product of two graphs in terms of the pTS-distance-balancedness of the factors.

Keywords: Distance-balanced graph, pTS-distance-balanced graph, total distance, wreath product of graphs.

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1 Introduction

The investigation of *distance-balanced graphs* began in [13], though an explicit definition was provided later in [15, 18]. Such graphs generated a certain degree of interest also by virtue of their connection with centrality measures [2, 8] and with some well known

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and largely studied distance-based invariants, such as the Wiener and the Szeged index [2, 14, 15, 16]. For instance, it was proven in [14] that, in the bipartite case, distance-balanced graphs maximize the Szeged index.

Throughout the paper we will denote by $G = (V_G, E_G)$ a simple connected finite graph G with vertex set V_G and edge set E_G. We say that such a graph has order n if $|V_G| = n$. For a pair of adjacent vertices $u, v \in V_G$ (we say $u \sim v$ in G) we define

$$W_{uv}^G = \{z \in V_G : d_G(z, u) < d_G(z, v)\}, \tag{1.1}$$

where $d_G(u, v)$ denotes the geodesic distance in G. In other words, W_{uv}^G is the set of vertices of G which are closer to u than to v.

Definition 1.1. A graph $G = (V_G, E_G)$ is *distance-balanced* if $|W_{uv}^G| = |W_{vu}^G|$, for every pair of adjacent vertices $u, v \in V_G$.

Cyclic graphs and complete graphs are simple examples of distance-balanced graphs. More generally, it is known that the class of distance-balanced graphs contains vertex-transitive graphs [18], which are graphs $G = (V_G, E_G)$ whose group of automorphisms $\mathrm{Aut}(G)$ acts transitively on the vertex set. On the other hand, the Handa graph H_{24}, introduced in [13], is an example of a non-vertex-transitive distance-balanced graph.

Recall that semisymmetric graphs are regular graphs which are edge-transitive (the group $\mathrm{Aut}(G)$ acts transitively on the edge set) but not vertex-transitive graphs. In particular, a semisymmetric graph is bipartite, and the two sets of the bipartition coincide with the orbits of $\mathrm{Aut}(G)$. As for such graphs there exists no automorphism switching two adjacent vertices, they appear as good candidates to be not distance-balanced. However, in [18] it is explicitly proven that there exist infinitely many semisymmetric graphs which are not distance-balanced, as well as infinitely many semisymmetric graphs which are distance-balanced.

In [15], the behaviour of the four classical graph compositions with respect to the distance-balanced property is investigated. More precisely, it is shown that the Cartesian product $G \,\square\, H$ of two connected graphs is distance-balanced if and only if both G and H are distance-balanced; the lexicographic product $G \circ H$ of two connected graphs is distance-balanced if and only if G is distance-balanced and H is regular; on the other hand, it is shown there, by explicit counterexamples, that the direct product $G \times H$ and the strong product $G \boxtimes H$ do not preserve the property of being distance-balanced. A generalization of the distance-balancedness, called ℓ-distance-balancedness, is studied in [19]. In [12], Cartesian and lexicographic graph products which are 2-distance-balanced are characterized.

In [3], in order to construct an algorithm that recognizes whether a given graph is distance-balanced or not, the authors establish a connection with some *graph centrality measures*; more precisely, they characterize the distance-balancedness of a graph in terms of its median vertices, and therefore in terms of their total distance (also known as transmission in the literature).

We denote the *normalized total distance* of a vertex $u \in V_G$ as

$$d_G(u) := \frac{1}{|V_G|} \sum_{v \in V_G} d_G(u, v),$$

which is the average of the distances of u from each vertex of G. A vertex $u \in V_G$ is said to be *median* if $d_G(u) = \min_{v \in V_G} d_G(v)$. The graph G is said to be *self-median* if every vertex $u \in V_G$ is median.

Theorem 1.2 ([3, Theorem 3.1])**.** *A graph $G = (V_G, E_G)$ is distance-balanced if and only if it is self-median.*

Using this characterization, Cabello and Lukšič studied in [6] the complexity of the problem of finding the minimum number of edges that can be added to a given graph to obtain a distance-balanced graph.

According to Theorem 1.2, distance-balanced graphs are graphs where all vertices have the same relevance in some sense, but they are not necessarily indistinguishable (notice that there are even examples of distance-balanced graphs with a trivial automorphism group [17]). Therefore, distance-balanced graphs are of special interest in the study of social networks, as all people in such graphs are *equal* with respect to the total distance. A related measure of this equality is given by the *opportunity index*, which is defined as follows. Given a graph $G = (V_G, E_G)$ with $|V_G| = 2n$, and two subsets V_1 and V_2 of V_G such that $|V_1| = |V_2| = n$ and $V_1 \cup V_2 = V_G$ (called a half-partition of G), the opportunity index of G is defined as $opp(G) = \max\{|W_{V_1}(G) - W_{V_2}(G)| : \{V_1, V_2\}$ is a half-partition of $G\}$ where, for a given $U \subseteq V_G$, $W_U(G)$ denotes the sum of the distances between all pairs of vertices in U. In particular, distance-balanced graphs are characterized as those graphs whose opportunity index is zero [2].

In the present paper, aimed at generalizing distance-balancedness in a probabilistic direction, we start exactly from this point of view, and we interpret the set of median vertices of a graph, and the whole class of distance-balanced graphs itself, as solutions of particular *facility location* problems, very typical in graph centrality investigations. In order to deeper understand this correspondence, let us suppose that $G = (V_G, E_G)$ represents a city; its vertex set is the set of the buildings/locations, the edges are connections between the buildings and then, for any $u, v \in V_G$, the geodesic distance $d_G(u, v)$ represents the distance between buildings u and v, or the *cost* of reaching the vertex v from the vertex u. In these terms, the quantity $d_G(u)$ is the average distance of the location u from all locations, and the median vertices are those vertices solving the following problem.

Problem 1.3. Find the location for a facility in order to minimize its average distance from all the buildings of the city.

Consequently, distance-balanced graphs are those graphs whose vertices are all equal with respect to Problem 1.3. That is, they solve this second problem.

Problem 1.4. Find a city where Problem 1.3 is solved by any location.

From another point of view, that we will develop in the last part of the paper, our work can be interpreted as the investigation of the distance-balancedness in a *wreath product of graphs*. In this sense, it is the natural continuation of [8]. The wreath product of graphs represents the graph analogue of the classical wreath product of groups, as it is true that the wreath product of the Cayley graphs of two finite groups is the Cayley graph of the wreath product of the groups. In [10], this correspondence is proven in the more general context of generalized wreath products of graphs, inspired by the construction introduced in [1] for permutation groups. Also, observe that in [9] the wreath product of matrices has been defined, in order to describe the adjacency matrix of the wreath product of two graphs: spectral computations using this matrix representation have been developed for some infinite families of wreath products in [5, 4, 11].

The paper is organized as follows. In Section 2, we consider two optimization problems, namely Problem 2.5 and Problem 2.6, that are the analogue, respectively, of Problem 1.3 and Problem 1.4, where the centrality measure at the vertex u is not yet the normalized total distance, but the quantity $d_G^p(u)$, that is, the expectation of the length of a shortest path starting from u that satisfies some random requirements depending on the probability p. In particular, these new problems collapse to the classical ones in the case $p = 0$.

Problems 2.5 and 2.6 are of some interest on their own, given their connection with the Travelling Salesman Problem, which is among the most studied optimization problems, largely investigated in literature also in its several probabilistic versions.

Then we extend the classical definition of distance-balanced graph by introducing the notion of *pTS-distance-balanced* graph in Definition 2.9, and we prove in Theorem 2.10 a pTS analogue of Theorem 1.2: pTS-distance-balanced graphs are exactly the graphs that solve Problem 2.6 (that is, the TS-version of Problem 1.4). We present examples and non-examples of pTS-distance-balanced graphs.

In Section 3, we recall the definition of the wreath product $G \wr H$ of two graphs G and H (Definition 3.1). It turns out that, when the order of H is m, the uniform probability distribution on the vertices of $G \wr H$ is compatible, in a precise sense explained in Lemma 3.3, with the model introduced in Section 2 for G, when $p = \frac{m-1}{m}$.

It follows that the TS-problems considered on the graph G are equivalent to the classical problems on the wreath product $G \wr H$, for a suitable choice of the graph H. More precisely we characterize, in Theorem 3.4, the distance-balancedness of a wreath product in terms of pTS-distance-balancedness of its factors. Finally, we investigate the class of graphs that are pTS-distance-balanced for every $p \in [0, 1]$, giving several equivalent characterizations in Theorem 3.11. We conclude the paper by asking if this class actually coincides with the class of vertex-transitive graphs (Question 4.1).

2 pTS centrality

As a natural generalization of Problem 1.3, suppose that every day each building (vertex) of the city (graph) $G = (V_G, E_G)$, independently, with the same probability $p \in [0, 1]$, requires a visit from the facility and with probability $1 - p$ does not. An example could be a postoffice with a postman delivering parcels. We want to find a location for the postoffice in order to minimize the expectation of the length of a shortest walk starting from the postoffice, visiting at least once each building waiting for a parcel, and finally arriving at the postman's house, that can be on each vertex with the same probability $\frac{1}{n}$ (observe that the postoffice and the postman's house locations may coincide). This set-up is justified if, for example, we have to decide the postoffice location prior to be aware of the location of the postman's house, or for example if every day the postman can be different. We are going to formalize this model in what follows.

Definition 2.1. Let $G = (V_G, E_G)$ be a graph and let $A \subseteq V_G$. We define a map ρ_A on $V_G \times V_G$ such that, for any pair of vertices u and v in V_G, the number $\rho_A(u, v)$ is the length of a shortest walk joining u and v, visiting at least once all vertices in A.

Remark 2.2. Let $G = (V_G, E_G)$ be a graph of order n, and let $u, v, z \in V_G$ and $A \subseteq V_G$. We list some properties of the map ρ_A; see [7] for more details.

- $\rho_\emptyset(u, v) = d_G(u, v)$.
- $\rho_A(u, v) = \rho_A(v, u)$. (Symmetry)

- $\rho_{A\cup B}(u,v) \leq \rho_A(u,z) + \rho_B(z,v)$. (Triangle inequality)
- $B \subseteq A \implies \rho_B(u,v) \leq \rho_A(u,v)$. (Monotonicity)
- $\rho_A(u,v) < n^2$.

Combining the first with the third property we have

$$|\rho_A(u,z) - \rho_A(v,z)| \leq d_G(u,v). \tag{2.1}$$

Let $G = (V_G, E_G)$ be a graph of order n, and let $u, v \in V_G$. A *Hamiltonian path* from u to v in G is a path from u to v visiting each vertex of G exactly once. A *Hamiltonian cycle* is a Hamiltonian path between adjacent vertices u and v. A graph is *Hamiltonian* if it admits a Hamiltonian cycle, that is equivalent to say that $\rho_{V_G}(u,u) = n$ for some, or equivalently, for every $u \in V_G$. A graph G is *Hamilton-connected* if, for every pair $u, v \in V_G$, there exists a Hamiltonian path from u to v. It is easy to observe that

$$\forall u, v \in V_G,\ \rho_{V_G}(u,v) = \begin{cases} n-1 & \text{if } u \neq v \\ n & \text{if } u = v \end{cases} \iff G \text{ is Hamilton-connected.} \tag{2.2}$$

The computation of ρ_{V_G} for Hamilton-connected graphs is rather easy; however, to determine ρ_A is in general very hard. This is not the case for the easiest example of Hamilton-connected graph, that is, the complete graph K_n.

Example 2.3. Let $K_n = (V_{K_n}, E_{K_n})$ be the *complete graph on n vertices*. For every nonempty $A \subseteq V_{K_n}$ and every $u, v \in V_{K_n}$ we have

$$\rho_A(u,v) = \begin{cases} |A|+1 & \text{if } u, v \notin A \\ |A| & \text{if } u \notin A, v \in A \text{ or viceversa} \\ |A|-1 & \text{if } u, v \in A, u \neq v \\ |A| & \text{if } u = v \in A, |A| > 1 \\ 0 & \text{if } u = v \in A, |A| = 1. \end{cases}$$

The hypothesis that each vertex independently with probability p requires a visit implies that the probability for a given subset $A \subseteq V_G$ to be the random subset waiting for the parcels is

$$p_A := p^{|A|}(1-p)^{n-|A|}. \tag{2.3}$$

Then we define the quantity $d_G^p(u)$, that is, the expected length of a walk from u, visiting the random set A and arriving to the random vertex v (uniformly distributed on V_G), as follows:

$$d_G^p(u) := \frac{1}{n} \sum_{v \in V_G} \sum_{A \subseteq V_G} p_A\, \rho_A(u,v). \tag{2.4}$$

Remark 2.4.

- If $p = 0$ we have $p_A = \begin{cases} 1 & \text{if } A = \emptyset \\ 0 & \text{otherwise} \end{cases}$ and $d_G^p(u) = d_G(u)$.
- If $p = \frac{1}{2}$ we have $p_A = \frac{1}{2^n}$ and $d_G^p(u) = \frac{1}{2^n n} \sum_{v \in V_G} \sum_{A \subseteq V_G} \rho_A(u,v)$.

- If $p = 1$ we have $p_A = \begin{cases} 1 & \text{if } A = V_G \\ 0 & \text{otherwise} \end{cases}$ and $d_G^p(u) = \frac{1}{n}\sum_{v\in V_G} \rho_{V_G}(u, v)$.

We are now in position to formulate the pTS versions of Problem 1.3 and Problem 1.4, respectively.

Problem 2.5. Find a vertex $u \in V_G$ such that $d_G^p(u) = \min_{v\in V_G} d_G^p(v)$.

Problem 2.6. Find a graph such that Problem 2.5 is solved by any vertex.

This leads us to introduce a notion of *medianity* in this setting, as a solution of the above mentioned problems.

Definition 2.7. In a graph $G = (V_G, E_G)$ a vertex $u \in V_G$ is *pTS-median* if it solves Problem 2.5. The graph G is *self-pTS-median* if it solves Problem 2.6.

Remark 2.8. Notice that, as a consequence of Remark 2.4, when $p = 0$ the Problem 1.3 and Problem 1.4 and their pTS versions, Problem 2.5 and Problem 2.6 respectively, are equivalent.

In analogy with Equation (1.1), for any subset $A \subseteq V_G$ and any pair of adjacent vertices $u, v \in V_G$, we define the vertex subsets

$$W_{uv}^A := \{z \in V_G : \rho_A(z, u) < \rho_A(z, v)\},$$

and the expectation of their cardinality is

$$w_{uv}^p := \sum_{A\subseteq V_G} p_A |W_{uv}^A|. \tag{2.5}$$

A natural generalization of the distance-balancedness is given in the following definition.

Definition 2.9. A graph $G = (V_G, E_G)$ is *pTS-distance-balanced* if $w_{uv}^p = w_{vu}^p$, for every pair of adjacent vertices $u, v \in V_G$. A graph G is *TS-distance-balanced* if it is pTS-distance-balanced for each $p \in [0, 1]$.

The following is the pTS-version of Theorem 1.2.

Theorem 2.10. *A graph $G = (V_G, E_G)$ is pTS-distance-balanced if and only if it is self-pTS-median.*

Proof. Observe that, as the graph $G = (V_G, E_G)$ is connected, the statement is proved if we show that, for every pair of adjacent vertices $u, v \in V_G$, one has:

$$d_G^p(u) - d_G^p(v) = \frac{1}{n}(w_{vu}^p - w_{uv}^p). \tag{2.6}$$

Now, by the definition of $d_G^p(u)$ in Equation (2.4), we get

$$\begin{aligned} d_G^p(u) - d_G^p(v) &= \frac{1}{n} \sum_{z\in V_G} \sum_{A\subseteq V_G} p_A(\rho_A(u, z) - \rho_A(v, z)) \\ &= \frac{1}{n} \sum_{A\subseteq V_G} p_A \sum_{z\in V_G} (\rho_A(u, z) - \rho_A(v, z)) = \frac{1}{n} \sum_{A\subseteq V_G} p_A(|W_{vu}^A| - |W_{uv}^A|) \end{aligned}$$

since, being u and v adjacent, by virtue of Equation (2.1), we have

$$\rho_A(u,z) - \rho_A(v,z) = \begin{cases} 1 & \text{if } z \in W_{vu}^A \\ -1 & \text{if } z \in W_{uv}^A \\ 0 & \text{otherwise.} \end{cases}$$

Finally, by Equation (2.5), we have

$$d_G^p(u) - d_G^p(v) = \frac{1}{n} \sum_{A \subseteq V_G} p_A(|W_{vu}^A| - |W_{uv}^A|) = \frac{1}{n}(w_{vu}^p - w_{uv}^p),$$

that proves Equation (2.6). □

Remark 2.11. As we have already observed,

$$G \text{ is distance-balanced} \iff G \text{ is } {}_0\text{TS-distance-balanced.}$$

Moreover, if, for two given vertices $u, v \in V_G$, there exists $\varphi \in \mathrm{Aut}(G)$ such that $\varphi(u) = v$, one has $d_G^p(u) = d_G^p(v)$, for every $p \in [0,1]$. This implies

$$G \text{ is vertex-transitive} \implies G \text{ is TS-distance-balanced.}$$

On the other hand, when $p = 1$, Hamilton-connected graphs satisfy $d_G^1(u) = n - 1 + \frac{1}{n}$ by Equation (2.2) and Remark 2.4 for every vertex $u \in V_G$, and then:

$$G \text{ is Hamilton-connected} \implies G \text{ is } {}_1\text{TS-distance-balanced.}$$

Notice that the converse of the last implication is not true, since there exist graphs which are vertex-transitive but not Hamilton-connected (e.g., cyclic graphs).

Example 2.12. The graph W_7 depicted in Figure 1 is the *Wheel graph* on 7 vertices. Being Hamilton-connected, the graph W_7 is ${}_1$TS-distance-balanced. Clearly, it is not distance-balanced and then it is not ${}_0$TS-distance-balanced.

By an explicit computation (brute-force) we computed $d_{W_7}^{1/2}(u) = \frac{1}{7 \cdot 2^7} \cdot 3842$, whether u is the central vertex or it belongs to the external cycle. As a consequence, the graph W_7 is ${}_{\frac{1}{2}}$TS-distance-balanced. We found quite surprising that this graph, that has a so recognizable central vertex, presents such an equality property.

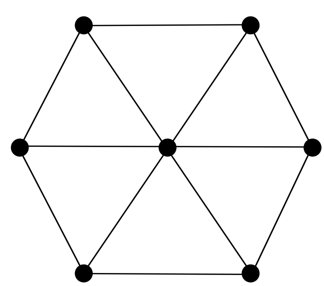

Figure 1: The Wheel graph W_7 on 7 vertices.

Example 2.13. Let H_9 be the graph on 9 vertices depicted in Figure 2. This graph has been introduced in [14] as the smallest example of a non-regular distance-balanced graph. In particular, it is 0TS-distance-balanced, but an explicit computation gives $d_{H_9}^{1/2}(v_1) = \frac{1}{9 \cdot 2^9} \cdot 26688$ and $d_{H_9}^{1/2}(v_2) = \frac{1}{9 \cdot 2^9} \cdot 26656$. By virtue of Theorem 2.10, it is not $\frac{1}{2}$TS-distance-balanced.

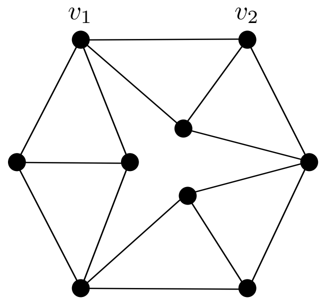

Figure 2: The graph H_9.

3 pTS-distance-balancedness and wreath product of graphs

We start this section by recalling the definition of the wreath product of two graphs.

Definition 3.1. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs. Let us fix an enumeration of the vertices of G so that $V_G = \{x_1, \dots, x_n\}$. The *wreath product* $G \wr H$ is the graph with vertex set $V_H^{V_G} \times V_G = \{(y_1, \dots, y_n)x_i : x_i \in V_G, y_j \in V_H, j \in \{1, \dots, n\}\}$, where two vertices $u = (y_1, \dots, y_n)x_i$ and $v = (y'_1, \dots, y'_n)x_k$ are connected by an edge if:

- (edges of type I) either $i = k$ and $y_j = y'_j$ for every $j \neq i$, and $y_i \sim y'_i$ in H;
- (edges of type II) or $y_j = y'_j$, for every $j \in \{1, \dots, n\}$, and $x_i \sim x_k$ in G.

It follows that if $|V_G| = n$ and $|V_H| = m$, the graph $G \wr H$ has nm^n vertices. It is proved that $G \wr H$ is connected, bipartite or vertex-transitive, if and only if both G and H are, respectively, connected, bipartite or vertex-transitive [7]. Moreover, if G is a regular graph of degree r_G and H is a regular graph of degree r_H, then the graph $G \wr H$ is an $(r_G + r_H)$-regular graph.

There exists a practical and clarifying interpretation of the graph wreath product, given by the *Lamplighter random walk* [20]. Suppose that a lamplighter moves along G, so that each vertex of G represents a possible *position* of the lamplighter: at each vertex of G, there is a *lamp*. The vertices of the graph H represent the possible *colors* of each lamp. Therefore, a vertex $(y_1, \dots, y_n)x_i$ in $G \wr H$ can be regarded as a *configuration* of colors $(y_1, \dots, y_n)$ (each y_j is from V_H) together with a particular position x_i (from V_G) of the lamplighter: the lamp at the position $x_j \in V_G$ has the color $y_j \in V_H$. Two vertices of $G \wr H$ are adjacent if either they share the same configuration of colors and have adjacent positions for the lamplighter in G (such an edge of type II corresponds to the situation of the lamplighter moving to a neighbor vertex in G but leaving all lamps unchanged); or they share the same position but the configuration of colors differ, by two adjacent colors, exactly for the lamp associated at that position (such an edge of type I corresponds to the situation of the lamplighter changing the color of the lamp, to an adjacent color in H, in the position where he stays). For this reason, the wreath product $G \wr H$ is also called the Lamplighter graph, with base graph G and color graph H.

The Lamplighter interpretation allows us to highlight the relationship between the wreath product of graphs and Problems 2.5 and 2.6. We explicit this connection in the two following lemmas, where the distance and the normalized total distance in $G \wr H$ are expressed in terms of distance and normalized total distance in H, and of their TS-version in G.

Lemma 3.2 ([7]). *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs of order n and m, respectively. For any vertices $u = (y_1, \dots, y_n)x$, $v = (y'_1, \dots, y'_n)x' \in V_{G\wr H}$, we have:*

$$d_{G\wr H}(u, v) = \rho_{\delta(u,v)}(x, x') + \sum_{i=1}^{n} d_H(y_i, y'_i),$$

where $\delta(u, v) := \{x_i \in V_G : y_i \neq y'_i\}$.

Observe that the sum $\sum_{i=1}^{n} d_H(y_i, y'_i)$ can be interpreted as the geodesic distance in the n-th iterated Cartesian product of H with itself. Moreover, in the Lamplighter interpretation, the subset $\delta(u, v)$ consists of those vertices x_i of the base graph G where the color of the associated lamps y_i and y'_i does not coincide. In other words, $\delta(u, v)$ is the set of the vertices of G that the lamplighter has to visit in order to move from the lamp configuration of u to that of v.

Notice that fixing a vertex u in the graph $G \wr H$ and considering a random vertex $v = (y'_1, \dots, y'_n)x'$, with uniform probability $\frac{1}{nm^n}$ in $V_{G\wr H}$, induces a random subset $\delta(u, v)$ of V_G and a random vertex x' in V_G. More precisely, for a given subset $A \subseteq V_G$, the probability that the random set $\delta(u, v)$ is equal to A is exactly $n(m-1)^{|A|}/nm^n$, because the Lamplighter may be in any position, and there are $|A|$ vertices where the lamps may take $m-1$ distinct colors, whereas the colors of the lamps at the remaining $n-|A|$ vertices are fixed. This probability is exactly p_A of Equation (2.3) for $p = \frac{m-1}{m}$. This means that the model considered in Section 2 can be simply derived by assigning a uniform probability distribution to the vertices of the wreath product. This is formally proved in Lemma 3.3.

Lemma 3.3. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs of order n and m, respectively. Let $u = (y_1, \dots y_n)x \in V_{G\wr H}$ and $p = \frac{m-1}{m}$. Then:*

$$d_{G\wr H}(u) = d^p_G(x) + \sum_{i=1}^{n} d_H(y_i).$$

Proof. From Theorem 17 in [8], where the focus is on the non-normalized total distance, we have:

$$nm^n d_{G\wr H}(u) = nm^{n-1} \sum_{i=1}^{n} m d_H(y_i) + \sum_{A\subseteq V_G} (m-1)^{|A|} \sum_{x'\in V_G} \rho_A(x, x').$$

Taking into account that $p = \frac{m-1}{m}$, we obtain:

$$\begin{aligned} d_{G\wr H}(u) &= \sum_{i=1}^{n} d_H(y_i) + \frac{1}{n} \sum_{A \subseteq V_G} \sum_{x' \in V_G} \frac{(m-1)^{|A|}}{m^n} \rho_A(x, x') \\ &= \sum_{i=1}^{n} d_H(y_i) + \frac{1}{n} \sum_{A \subseteq V_G} \sum_{x' \in V_G} p^{|A|}(1-p)^{n-|A|} \rho_A(x, x'). \end{aligned}$$

The claim follows by using Equations (2.3) and (2.4). □

In the following theorem we give necessary and sufficient conditions for a vertex of a wreath product to be median and for the wreath product itself to be distance-balanced.

Theorem 3.4. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs with $|V_G| = n$, $|V_H| = m$. Let $u = (y_1, \ldots, y_n)x \in V_{G\wr H}$, and $p = \frac{m-1}{m}$. Then u is median in $G \wr H$ if and only if $y_1, \ldots, y_n$ are median in H and x is pTS-median in G. In particular:*

$$\begin{matrix} G \text{ is } p\text{TS-distance-balanced} \\ H \text{ is distance-balanced} \end{matrix} \iff G \wr H \text{ is distance-balanced .}$$

Proof. Suppose that y_{i^*} is not median in H, so that there exists $\bar{y} \in V_H$ such that $d_H(\bar{y}) < d_H(y_{i^*})$. Denoting by u' the vertex $(y_1, \ldots, y_{i^*-1}, \bar{y}, y_{i^*+1}, \ldots, y_n)x \in V_{G\wr H}$, by Lemma 3.3 we have $d_{G\wr H}(u') < d_{G\wr H}(u)$, and then u is not median in $G \wr H$.

Similarly, suppose that x is not pTS-median in G, so that there exists $\bar{x} \in V_G$ such that $d_G^p(\bar{x}) < d_G^p(x)$. Denoting by u'' the vertex $(y_1, \ldots, y_n)\bar{x} \in V_{G\wr H}$, by Lemma 3.3 again we have $d_{G\wr H}(u'') < d_{G\wr H}(u)$, and then u is not median in $G \wr H$.

Viceversa, suppose that u is not median in $G \wr H$ and then $d_{G\wr H}(u)$ is not minimal, then one among $\{d_H(y_i)\}_{i=1,\ldots,n}$ or $d_G^p(x)$ cannot be minimal, and the statement follows. □

Corollary 3.5. *If H and H' are two distance-balanced graphs of the same order, then $G \wr H$ is distance-balanced if and only if $G \wr H'$ is distance-balanced.*

Remark 3.6. Another consequence of Theorem 3.4 is the equivalence of the TS-problems for G with the classical problems for $G \wr H$, where H is any distance-balanced graph. More precisely, let $H = (V_H, E_H)$ be a distance-balanced graph of order m, and $p = \frac{m-1}{m}$. Then we have:

$$\begin{matrix} x \in V_G \\ \text{is solution of Problem 2.5} \end{matrix} \iff \begin{matrix} (y_1, \ldots, y_n)x \in V_{G\wr H} \\ \text{is solution of Problem 1.3} \end{matrix}$$

$$\begin{matrix} G \\ \text{is solution of Problem 2.6} \end{matrix} \iff \begin{matrix} G \wr H \\ \text{is solution of Problem 1.4.} \end{matrix}$$

Example 3.7. We know from Example 2.12 that the graph W_7 is $\frac{1}{2}$TS-distance-balanced. By virtue of Theorem 3.4, the graph $W_7 \wr K_2$ is distance-balanced. Moreover, the graph $W_7 \wr K_2$ has order 896, it is non-regular (since W_7 is non-regular), and it is not bipartite (since W_7 is not bipartite).

Example 3.8. We know from Example 2.13 that the distance-balanced graph H_9 is not $\frac{1}{2}$TS-distance-balanced. As a consequence, the graph $H_9 \wr K_2$ is not distance-balanced.

In the light of Example 2.12 and Example 3.7, we deduce that the distance-balancedness of the wreath product $G \wr H$ does not imply the distance-balancedness of the first factor G. Moreover, in the light of Example 2.13 and Example 3.8, we deduce that the distance-balancedness of the graphs G and H does not imply the distance-balancedness of their wreath product.

We conclude this section by investigating the class of graphs G such that $G \wr H$ is distance-balanced whenever H is distance-balanced. By virtue of Theorem 3.4, this class must contain the class of TS-distance-balanced graphs. The two classes actually coincide, as we will prove in Theorem 3.11. We need a preliminary definition and lemma.

Definition 3.9. The *total distance vector of* the vertex $u \in V_G$ is the $(n+1)$-vector

$$W_\rho(u,G) = (W_{\rho_0}(u,G), W_{\rho_1}(u,G), \dots, W_{\rho_n}(u,G)),$$

where, for each $k \in \{0,1,\dots,n\}$, we set

$$W_{\rho_k}(u,G) := \sum_{A \subseteq V_G,\, |A|=k} \sum_{v \in V_G} \rho_A(u,v).$$

In particular, observe that $W_{\rho_0}(u,G)$ is the non-normalized total distance of the vertex u in G.

Lemma 3.10. *For every $u \in V_G$ and for every $p \in [0,1]$, we have:*

$$d_G^p(u) = \frac{1}{n}\sum_{k=0}^{n} p^k (1-p)^{n-k} W_{\rho_k}(u,G). \tag{3.1}$$

Proof. The claim follows by combining Equation (2.4) with Definition 3.9, since the expression of p_A in Equation (2.3) only depends on the cardinality of A. □

Theorem 3.11. *Let $G = (V_G, E_G)$ be a graph of order n. The following are equivalent.*

(1) G is TS-distance-balanced;

(2) $G \wr H$ is distance-balanced for every distance-balanced graph H;

(3) $G \wr K_{n^3 2^n}$ is distance-balanced;

(4) G is pTS-distance-balanced for more than n distinct values of $p \in [0,1]$;

(5) the total distance vector $W_\rho(u,G)$ does not depend on the particular vertex $u \in V_G$.

Proof. (1) $\Longrightarrow$ (2): It is a consequence of Theorem 3.4.

(2) $\Longrightarrow$ (4): If $G \wr H$ is distance-balanced for every distance-balanced graph H, in particular $G \wr K_m$ is distance-balanced for $m = 2, \dots, n+2$, and then, by Theorem 3.4, the graph G is pTS-distance-balanced for each $p \in \left\{\frac{1}{2}, \frac{2}{3}, \dots, \frac{n+1}{n+2}\right\}$.

(4) $\Longrightarrow$ (5): For a given vertex $u \in V_G$, we define the following polynomial of degree n in the variable x

$$P_u(x) := \sum_{k=0}^{n} x^k (1-x)^{n-k} W_{\rho_k}(u,G).$$

By Lemma 3.10, we have $P_u(p) = nd_G^p(u)$. Combining with hypothesis (4), for any pair $u, v \in V_G$, the polynomials P_u and P_v share more than n evaluations, and so $P_u = P_v$. It is an easy exercise to prove that this implies that $W_{\rho_k}(u, G) = W_{\rho_k}(v, G)$, for each $k \in \{0, 1, \ldots, n\}$, and so $W_\rho(u, G) = W_\rho(v, G)$.

(5) $\implies$ (1): It is a consequence of Lemma 3.10 and Theorem 2.10.

(2) $\implies$ (3): It is true since the graph $K_{n^3 2^n}$ is distance-balanced.

(3) $\implies$ (5): As we already observed in Remark 2.2, for every $A \subseteq V_G$, for every $u, v \in V_G$ we have $\rho_A(u, v) < n^2$. Since the number of subsets of V_G having cardinality k is clearly less than 2^n, this implies

$$0 < W_{\rho_k}(u, G) = \sum_{A \subseteq V_G,\, |A|=k} \sum_{v \in V_G} \rho_A(u, v) < 2^n n^3. \tag{3.2}$$

We set $m := 2^n n^3$ and $p := \frac{m-1}{m}$. Since, by hypothesis, $G \wr K_m$ is distance-balanced, it follows that, for every $u, v \in V_G$:

$$d_G^p(u) n m^n = d_G^p(v) n m^n,$$

and then, by Lemma 3.10:

$$\sum_{k=0}^{n} (m-1)^k W_{\rho_k}(u, G) = \sum_{k=0}^{n} (m-1)^k W_{\rho_k}(v, G). \tag{3.3}$$

By virtue of Equation (3.2) we can regard the quantities $W_{\rho_k}(u, G)$ (resp. $W_{\rho_k}(v, G)$) as the digits of $d_G^p(u) n m^n$ (resp. $d_G^p(v) n m^n$) in base $(m-1)$; therefore, Equation (3.3) implies that $W_{\rho_k}(u, G) = W_{\rho_k}(v, G)$, for each $k \in \{0, 1, \ldots, n\}$, and so $W_\rho(u, G) = W_\rho(v, G)$. □

Example 3.12. Lemma 3.10 and the characterization (5) in Theorem 3.11 make us able to investigate distance-balancedness (at least in those cases for which the total distance vector is known) simply by studying roots of polynomials. For the graph H_9 from Example 2.13, we first computed the total distance vectors, which are given by

$$W_\rho(v_1, H_9) = (14, 252, 1345, 3711, 6279, 6941, 5065, 2363, 641, 77)$$
$$W_\rho(v_2, H_9) = (14, 252, 1360, 3762, 6333, 6933, 5001, 2307, 620, 74).$$

By using Equation (3.1) we were able to determine $d_G^p(v_1)$ and $d_G^p(v_2)$ for a general p. In particular, the graph H_9 is pTS-distance-balanced exactly for all values of $p \in [0, 1]$ satisfying the equation $d_G^p(v_1) = d_G^p(v_2)$. It turns out that these values are $p = 0$ and $p \approx 0.48219$, which is the unique real root of the polynomial $2x^5 - 13x^4 + 38x^3 - 63x^2 + 54x - 15$.

As we already observed in Remark 2.11, vertex-transitive graphs satisfy the equivalent properties of Theorem 3.11. Moreover, we recall that regularity, vertex-transitivity and bipartiteness are all properties preserved by the wreath product. This yields the following infinite families of examples.

Example 3.13. Let H_9 be the non-regular non-bipartite distance-balanced graph from Example 2.13. Let H_{24} be the Handa graph which is non-regular, bipartite and distance-balanced [13]. Consider the *Generalized Petersen Graph* $P(7, 3)$, that is regular, distance-balanced but not vertex-transitive [15]. Then:

- $\{K_n \wr H_9\}_{n\in\mathbb{N}}$ is a family of non-regular, non-bipartite, distance-balanced graphs;
- $\{K_n \wr P(7,3)\}_{n\in\mathbb{N}}$ is a family of regular, non-vertex transitive, distance-balanced graphs;
- $\{K_{n,n} \wr H_{24}\}_{n\in\mathbb{N}}$ is a family of non-regular, bipartite, distance-balanced graphs.

It is clear that, in order to obtain other infinite families with the same properties, one can replace $K_{n,n}$ or K_n with any (bipartite or not) vertex-transitive graph, and the second factor with any distance-balanced graph sharing the same properties of regularity, vertex-transitivity, bipartiteness.

4 Conclusions

Vertex-transitive graphs are TS-distance-balanced. More generally, if u and v are vertices of a graph $G = (V_G, E_G)$ for which there exists $\varphi \in \mathrm{Aut}(G)$ such that $\varphi(u) = v$, one has $W_\rho(u, G) = W_\rho(v, G)$. In other words, the total distance vector is constant on the orbits of V_G under the action of $\mathrm{Aut}(G)$. This suggests that it is possible to use it as an *invariant* in order to distinguish vertices: it is a finer invariant than the standard total distance (see Example 3.12). Therefore, it is natural to ask whether this invariant is *complete* on the orbit partition of vertices. The following question is a total-distance-analogue of Question 1 in [7] about *the Wiener vector* and the *isomorphism problem*.

Question 4.1. Does there exist a graph $G = (V_G, E_G)$ with two vertices $u, v \in V_G$ for which there exists no automorphism φ such that $\varphi(u) = v$, but $W_\rho(u, G) = W_\rho(v, G)$?

A negative answer would imply that the equivalent conditions of Theorem 3.11 are also equivalent to the vertex-transitivity property.

A last remark is that, regardless of the answer to our Question 4.1, the wreath product construction produces new infinite families of distance-balanced graphs, which cannot be obtained via the classical graph products. Moreover, the graphs in these families possibly inherit good properties from their factors (see Example 3.13). We believe that this new approach may provide different examples and counterexamples in the field of distance-balancedness and its generalizations, and for this reason it deserves to be further investigated and exploited.

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On generalized truncations of complete graphs*

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Abstract

For a k-regular graph Γ and a graph Υ of order k, a generalized truncation of Γ by Υ is constructed by replacing each vertex of Γ with a copy of Υ. E. Eiben, R. Jajcay and P. Šparl introduced a method for constructing vertex-transitive generalized truncations. For convenience, we call a graph obtained by using Eiben *et al.*'s method a *special generalized truncation*. In their paper, Eiben *et al.* proposed a problem to classify special generalized truncations of a complete graph $\mathbf{K}_n$ by a cycle of length $n-1$. In this paper, we completely solve this problem by demonstrating that with the exception of $n = 6$, every special generalized truncation of a complete graph $\mathbf{K}_n$ by a cycle of length $n-1$ is a Cayley graph of $\mathrm{AGL}(1, n)$ where n is a prime power. Moreover, the full automorphism groups of all these graphs and the isomorphisms among them are determined.

Keywords: Truncation, vertex-transitive, Cayley graph, automorphism group.

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1 Introduction

In [6], the symmetry properties of graphs constructed by using the generalized truncations was investigated. In particular, a method for constructing vertex-transitive generalized truncations was proposed (see [6, Construction 4.1 and Theorem 5.1]), and this method was used to construct vertex-transitive generalized truncations of a complete graph $\mathbf{K}_n$ by a cycle of length $n-1$ for some small values of n. The vertex-transitive generalized truncations of a complete graph $\mathbf{K}_n$ by a graph Υ in context of [6, Theorem 5.1] can be defined as follows.

Let $\mathbf{K}_n$ be a complete graph of order n with $n \geq 4$, and let $V(\mathbf{K}_n) = \{v_1, v_2, \ldots, v_n\}$. Let G be an arc-transitive group of automorphisms of $\mathbf{K}_n$. Then G acts 2-transitively

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on $V(\mathbf{K}_n)$. Let $v = v_1$, and let $\mathcal{O}_v$ be a union of orbits of the stabilizer G_v acting on $\{\{x, y\} \mid x \neq y, x, y \in V(\mathbf{K}_n) \setminus \{v\}\}$. Let Υ be the graph with vertex set $\{v_2, v_3, \ldots, v_n\}$ and edge set $\mathcal{O}_v$. For each $u \in V(\mathbf{K}_n)$, let $V_u = \{(u, w) \mid w \in V(\mathbf{K}_n) \setminus \{u\}\}$. The *special generalized truncation of* $\mathbf{K}_n$ *by* Υ, denoted by $T(\mathbf{K}_n, G, \Upsilon)$, is the graph with the vertex set $\bigcup_{u \in V(\mathbf{K}_n)} V_u$, and the adjacency relation in which a vertex (u, w) is adjacent to the vertex (w, u) and to all the vertices (u, w') for which there exists a $g \in G$ with the property $u^g = v$ and $\{w, w'\}^g \in \mathcal{O}_v$.

Based on the analysis of special generalized truncations of a complete graph $\mathbf{K}_n$ by a cycle of length $n-1$ for some small values of n, the authors of [6] proposed the following problem.

Problem 1.1 ([6, Problem 5.4])**.** Classify the special generalized truncations of $\mathbf{K}_n$ $(n \geq 4)$ by a cycle of length $n-1$.

The main purpose of this paper is to give a solution of this problem. Before stating the main result of this paper, we first set some notation. For a positive integer n, we denote by $\mathbb{Z}_n$ the cyclic group of order n, and by D_{2n} the dihedral group of order $2n$. Let $\mathbb{Z}_n^*$ be the multiplicative group of units mod n ($\mathbb{Z}_n^*$ consists of all positive integers less than n and coprime to n). Also we use A_n and S_n respectively to denote the alternating and symmetric groups of degree n. For two groups M and N, $N \rtimes M$ denotes a semidirect product of N by M. For a group G, the automorphism group of G and the socle of G will be represented by $\mathrm{Aut}(G)$ and $\mathrm{soc}(G)$, respectively. For a graph Γ we denote by $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ the vertex set, edge set, arc set and full automorphism group of Γ, respectively. A graph Γ is said to be *vertex-transitive* (resp. *arc-transitive* (or *symmetric*)) if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$ (resp. $A(\Gamma)$). Cayley graphs form an important class of vertex-transitive graphs. Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is the graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. Finally, we use $\mathbf{K}_n$ and C_n respectively to denote the complete graph and cycle with n vertices.

Let p be a prime and e a positive integer. Let $\mathrm{GF}(p^e)$ be the Galois field of order p^e and let x be a primitive root of $\mathrm{GF}(p^e)$. Then

$$\mathrm{AGL}(1, p^e) = \{\alpha_{x^i, z'} \colon z \mapsto zx^i + z', \forall z \in \mathrm{GF}(p^e) \mid i \in \mathbb{Z}_{p^e-1}, z' \in \mathrm{GF}(p^e)\},$$

and $\mathrm{AGL}(1, p^e)$ is a 2-transitive permutation group on $\mathrm{GF}(p^e)$. Let

$$H = \{\alpha_{1, z'} \colon z \mapsto z + z', \forall z \in \mathrm{GF}(p^e) \mid z' \in \mathrm{GF}(p^e)\},$$
$$K = \{\alpha_{x^i, 0} \colon z \mapsto zx^i, \forall z \in \mathrm{GF}(p^e) \mid i \in \mathbb{Z}_{p^e-1}\}.$$

Then H is regular on $\mathrm{GF}(p^e)$ and the point stabilizer $\mathrm{AGL}(1, p^e)_0$ of the zero element 0 of $\mathrm{GF}(p^e)$ is K. So $\mathrm{AGL}(1, p^e) = H \rtimes K$.

Construction 1.2. *Let* z' *be a non-zero element of* $\mathrm{GF}(p^e)$*. For each* $i \in \mathbb{Z}_{p^e-1}^*$ *with* $i < \frac{p^e-1}{2}$*, let*

$$\mathbf{K}_{p^e}^i = \mathrm{Cay}(\mathrm{AGL}(1, p^e), \{\alpha_{-1, z'}, \alpha_{x^i, 0}, \alpha_{x^{-i}, 0}\}) \quad (p > 2),$$
$$\mathbf{K}_{2^e}^i = \mathrm{Cay}(\mathrm{AGL}(1, 2^e), \{\alpha_{1, z'}, \alpha_{x^i, 0}, \alpha_{x^{-i}, 0}\}) \quad (p = 2).$$

Remark 1.3. Let z', z'' be two non-zero elements of $\mathrm{GF}(p^e)$. There exists $x^j \in \mathrm{GF}(p^e) \setminus \{0\}$ such that $z'x^j = z''$. So

$$\{\alpha_{-1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}^{\alpha_{x^j,0}} = \{\alpha_{-1,z''}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\} \quad (p > 2),$$
$$\{\alpha_{1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}^{\alpha_{x^j,0}} = \{\alpha_{1,z''}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\} \quad (p = 2).$$

It follows that

$$\begin{aligned}&\mathrm{Cay}(\mathrm{AGL}(1,p^e), \{\alpha_{-1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \cong \\ &\qquad \mathrm{Cay}(\mathrm{AGL}(1,p^e), \{\alpha_{-1,z''}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \quad (p > 2),\\ &\mathrm{Cay}(\mathrm{AGL}(1,2^e), \{\alpha_{1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \cong \\ &\qquad \mathrm{Cay}(\mathrm{AGL}(1,2^e), \{\alpha_{1,z''}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \quad (p = 2).\end{aligned}$$

In view of this fact, up to graph isomorphism, $\mathbf{K}^i_{p^e}$ is independent of the choice of z'.

The following is the main result of this paper.

Theorem 1.4. *Let $\widetilde{\mathbf{K}}_n$ be a special generalized truncation of $\mathbf{K}_n$ $(n \geq 4)$ by C_{n-1}. Then $\widetilde{\mathbf{K}}_n$ is isomorphic to either $T(\mathbf{K}_6, A_5, C_5)$ (see Figure 1), or one of the graphs $\mathbf{K}^i_{p^e}$ ($i \in \mathbb{Z}^*_{p^e-1}$, $i < \frac{p^e-1}{2}$). Conversely, each of the above graphs is indeed a special generalized truncation of $\mathbf{K}_n$ $(n \geq 4)$ by a cycle of length $n-1$, where $n = 6$ or a prime power.*

Furthermore, for any distinct $i, i' \in \mathbb{Z}^*_{p^e-1}$ with $i, i' < \frac{p^e-1}{2}$, $\mathbf{K}^i_{p^e} \cong \mathbf{K}^{i'}_{p^e}$ if and only if $i' \equiv ip^j$ or $-ip^j \pmod{p^e-1}$ for some $1 \leq j \leq e$. Moreover, the following hold:

- *(i)* $\mathrm{Aut}(T(\mathbf{K}_6, A_5, C_5)) \cong A_5$;
- *(ii)* $\mathrm{Aut}(\mathbf{K}^1_4) \cong S_4$;
- *(iii)* $\mathrm{Aut}(\mathbf{K}^1_7) \cong D_{42} \rtimes \mathbb{Z}_3$;
- *(iv)* $\mathrm{Aut}(\mathbf{K}^3_{11}) \cong \mathrm{PGL}_2(11)$;
- *(v)* $\mathrm{Aut}(\mathbf{K}^7_{23}) \cong \mathrm{PGL}_2(23)$;
- *(vi) if $\mathbf{K}^i_{p^e}$ is not isomorphic to one of the graphs: $\mathbf{K}^1_4$, $\mathbf{K}^1_7$, $\mathbf{K}^3_{11}$ and $\mathbf{K}^7_{23}$, then* $\mathrm{Aut}(\mathbf{K}^i_{p^e}) \cong \mathrm{AGL}(1,p^e)$.

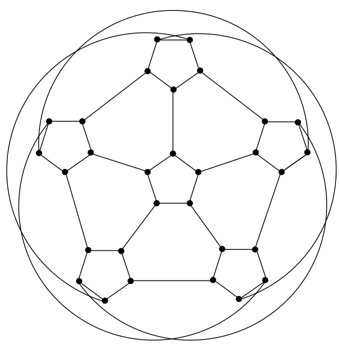

Figure 1: The graph $T(\mathbf{K}_6, A_5, C_5)$.

2 Preliminaries

All groups considered in this paper are finite and all graphs are finite, connected, simple and undirected. For the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [3, 12].

Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph on a group G relative to a subset S of G. It is easy to prove that Γ is connected if and only if S is a generating subset of G. For any $g \in G$, $R(g)$ is the permutation of G defined by $R(g)\colon x \mapsto xg$ for $x \in G$. Set $R(G) = \{R(g) \mid g \in G\}$. It is well-known that $R(G)$ is a subgroup of $\mathrm{Aut}(\Gamma)$. For briefness, we shall identify $R(G)$ with G in the following. In 1981, Godsil [7] proved that the normalizer of G in $\mathrm{Aut}(\Gamma)$ is $G \rtimes \mathrm{Aut}(G, S)$, where $\mathrm{Aut}(G, S)$ is the group of automorphisms of G fixing the set S set-wise. Clearly, $\mathrm{Aut}(G, S)$ is a subgroup of the stabilizer $\mathrm{Aut}(\Gamma)_1$ of the identity 1 of G in $\mathrm{Aut}(\Gamma)$. We say that the Cayley graph $\mathrm{Cay}(G, S)$ is *normal* if G is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$ (see [13]). If $\Gamma = \mathrm{Cay}(G, S)$ is a normal Cayley graph on G, then we have $\mathrm{Aut}(G, S) = \mathrm{Aut}(\Gamma)_1$, and if, in addition, Γ is also arc-transitive, then $\mathrm{Aut}(G, S)$ is transitive on S. From this we can easily obtain the following lemma.

Lemma 2.1. *There does not exist an arc-transitive normal Cayley graph of odd valency at least three on a cyclic group.*

A Cayley graph $\mathrm{Cay}(G, S)$ on a group G relative to a subset S of G is called a *CI-graph* of G, if for any Cayley graph $\mathrm{Cay}(G, T)$, whenever $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$ we have $T = S^\alpha$ for some $\alpha \in \mathrm{Aut}(G)$. The following proposition is a criterion for a Cayley graph to be a CI-graph.

Proposition 2.2 ([1, Lemma 3.1])**.** *Let Γ be a Cayley graph on a finite group G. Then Γ is a CI-graph of G if and only if all regular subgroups of $\mathrm{Aut}(\Gamma)$ isomorphic to G are conjugate.*

Let Γ be a connected vertex-transitive graph, and let $G \leq \mathrm{Aut}(\Gamma)$ be vertex-transitive on Γ. For a G-invariant partition $\mathcal{B}$ of $V(\Gamma)$, the *quotient graph* $\Gamma_{\mathcal{B}}$ is defined as the graph with vertex set $\mathcal{B}$ such that, for any two different vertices $B, C \in \mathcal{B}$, B is adjacent to C if and only if there exist $u \in B$ and $v \in C$ which are adjacent in Γ. Let N be a normal subgroup of G. Then the set $\mathcal{B}$ of orbits of N in $V(\Gamma)$ is a G-invariant partition of $V(\Gamma)$. In this case, the symbol $\Gamma_{\mathcal{B}}$ will be replaced by Γ_N.

In view of [11, Theorem 9], we have the following proposition.

Proposition 2.3. *Suppose that Γ is a connected trivalent graph with an arc-transitive group G of automorphisms. If $N \trianglelefteq G$ has more than two orbits in $V(\Gamma)$, then N is semiregular on $V(\Gamma)$, and Γ_N is a trivalent symmetric graph with G/N as an arc-transitive group of automorphisms.*

3 Proof of Theorem 1.4

3.1 Special generalized truncations of K_n by C_{n-1}

In this subsection, we shall prove the first part of Theorem 1.4 by determining all special generalized truncations of $\mathbf{K}_n$ ($n \geq 4$) by C_{n-1}. Throughout this subsection, we shall use the following assumptions and notations.

Assumption 3.1.

(1) Let $\mathbf{K}_n$ be a complete graph of order n with $n \geq 4$, and let $V(\mathbf{K}_n) = \{v_1, v_2, \ldots, v_n\}$.

(2) Let $G \leq \mathrm{Aut}(\mathbf{K}_n)$ be an arc-transitive group of automorphisms.

(3) Let $v = v_1$, and let $\mathcal{O}_v$ be a union of orbits of the stabilizer G_v acting on $\{\{x, y\} \mid x \neq y,\ x, y \in V(\mathbf{K}_n) \setminus \{v\}\}$. Let Υ be the graph with vertex set $\{v_2, v_3, \ldots, v_n\}$ and edge set $\mathcal{O}_v$.

(4) For each $u \in V(\mathbf{K}_n)$, let $V_u = \{(u, w) \mid w \in V(\mathbf{K}_n) \setminus \{u\}\}$.

(5) Let $\widetilde{\mathbf{K}}_n = T(\mathbf{K}_n, G, \Upsilon)$ be the graph with the vertex set $\bigcup_{u \in V(\mathbf{K}_n)} V_u$, and the adjacency relation in which a vertex (u, w) is adjacent to the vertex (w, u) and to all the vertices (u, w') for which there exists a $g \in G$ with the property $u^g = v$ and $\{w, w'\}^g \in \mathcal{O}_v$.

In view of [6, Theorem 5.1], we have the following proposition.

Proposition 3.2. *Use the notations in Assumption 3.1. Then $\mathrm{Aut}(\widetilde{\mathbf{K}}_n)$ has a vertex-transitive subgroup $\widetilde{G}$ such that $\mathcal{P} = \{V_u \mid u \in V(\mathbf{K}_n)\}$ is an imprimitivity block system for $\widetilde{G}$. Furthermore, the following hold.*

(1) The quotient graph of $\widetilde{\mathbf{K}}_n$ relative to $\mathcal{P}$ is isomorphic to $\mathbf{K}_n$.

(2) $\widetilde{G} \cong G$.

(3) $\widetilde{G}$ acts faithfully on $\mathcal{P}$.

For the two groups $\widetilde{G}, G$ in the above proposition, we shall follow [6] to say that $\widetilde{G}$ is the *lift* of G. The next lemma shows that if $\Upsilon \cong C_{n-1}$ then $\widetilde{G}$ is a 2-transitive permutation group on $\mathcal{P}$ and the point stabilizer $\widetilde{G}_{V_u}$ is either cyclic or dihedral.

Lemma 3.3. *Use the notations in Assumption 3.1. Let $\Upsilon \cong C_{n-1}$ and let $\widetilde{G}$ be the lift of G. Then for each $u \in V(\mathbf{K}_n)$, the subgraph of $\widetilde{\mathbf{K}}_n$ induced by V_u is a cycle of length $n-1$, and the subgroup $\widetilde{G}_{V_u}$ of $\widetilde{G}$ fixing V_u set-wise acts faithfully and transitively on V_u. In particular, $\widetilde{G}$ acts faithfully and 2-transitively on $\mathcal{P}$, and $\widetilde{G}_{V_u} \cong \mathbb{Z}_{n-1}$, or D_{n-1} (if n is odd), or $D_{2(n-1)}$.*

Proof. By Assumption 3.1 (3) and (5), the subgraph of $\widetilde{\mathbf{K}}_n$ induced by V_v is isomorphic to Υ. By Proposition 3.2, $\mathcal{P} = \{V_u \mid u \in V(\mathbf{K}_n)\}$ is an imprimitivity block system for $\widetilde{G}$, and so for each $u \in V(\mathbf{K}_n)$, the subgraph of $\widetilde{\mathbf{K}}_n$ induced by V_u is a cycle of length $n-1$.

For any two vertices u, w of $\mathbf{K}_n$, by Assumption 3.1 (5), $\{(u, w), (w, u)\}$ is the unique edge of $\widetilde{\mathbf{K}}_n$ connecting V_u and V_w. This implies that the subgroup K of $\widetilde{G}_{V_u}$ fixing V_u point-wise will fix every block in $\mathcal{P}$. It then follows from Proposition 3.2 (3) that $K = 1$, and so $\widetilde{G}_{V_u}$ acts faithfully on V_u. Since $\widetilde{G}$ is transitive on $V(\widetilde{\mathbf{K}}_n)$, $\widetilde{G}_{V_u}$ is transitive on V_u. Since the subgraph of $\widetilde{\mathbf{K}}_n$ induced by V_u is a cycle of length $n-1$, one has $\widetilde{G}_{V_u} \cong \mathbb{Z}_{n-1}$, or D_{n-1} (if n is odd), or $D_{2(n-1)}$.

Again since $\{(u, w), (w, u)\}$ is the unique edge of $\widetilde{\mathbf{K}}_n$ connecting V_u and V_w, it follows that $\widetilde{G}_{V_u}$ also acts transitively on $\mathcal{P} \setminus \{V_u\}$. This implies that $\widetilde{G}$ acts 2-transitively on $\mathcal{P}$. By Proposition 3.2 (3), $\widetilde{G}$ acts faithfully on $\mathcal{P}$. □

The above lemma enables us to determine the structure of $\widetilde{G}$ in the case when $\Upsilon \cong C_{n-1}$.

Lemma 3.4. *Use the notations in Assumption 3.1. Let $\Upsilon \cong C_{n-1}$ and let $\widetilde{G}$ be the lift of G. Then one of the following holds:*

(1) $n = 6$ and $\mathrm{soc}(\widetilde{G}) = A_5$;

(2) $n = 4$ and $\widetilde{G} \cong \mathrm{AGL}(1, 2^2)$ or $\mathrm{A\Gamma L}(1, 2^2)$;

(3) $n = p^e \neq 4$ and $\widetilde{G} \cong \mathrm{AGL}(1, p^e)$, where p is a prime and e is a positive integer.

Proof. By Lemma 3.3, $\widetilde{G}$ can be viewed as a 2-transitive permutation group on $\mathcal{P}$ with point stabilizer isomorphic to $\mathbb{Z}_{n-1}$, or D_{n-1} (if n is odd), or $D_{2(n-1)}$. By [5, Propositon 5.2], $\mathrm{soc}(\widetilde{G})$ is either elementary abelian or non-abelian simple, and furthermore, if $\mathrm{soc}(\widetilde{G})$ is non-abelian simple, then by checking the list of the simple groups which can occur as socles of 2-transitive groups in [5, p. 8], we have $\mathrm{soc}(\widetilde{G}) = A_5$. In order to complete the proof of this lemma, it remains to deal with the case when $\mathrm{soc}(\widetilde{G})$ is elementary abelian.

In what follows, assume that $\mathrm{soc}(\widetilde{G}) \cong \mathbb{Z}_p^e$ for some prime p and positive integer e. View $\mathrm{soc}(\widetilde{G})$ as an e-dimensional vector space over a field of order p, and let 0 denote the zero vector of $\mathrm{soc}(\widetilde{G})$. Recall that $\widetilde{G}_0 \cong \mathbb{Z}_{p^e-1}$, D_{p^e-1} (p odd), or $D_{2(p^e-1)}$. By checking Hering's theorem on classification of 2-transitive affine permutation groups [8] (see also [10, Appendix 1]), we have $\widetilde{G} \leq \mathrm{A\Gamma L}(1, p^e)$ with point-stabilizer $\widetilde{G}_0 \leq \Gamma\mathrm{L}(1, p^e)$. As $\widetilde{G} = \mathrm{soc}(\widetilde{G}) \rtimes \widetilde{G}_0$, to determine $\widetilde{G}$, we only need to determine all possible subgroups of $\Gamma\mathrm{L}(1, p^e)$ which are isomorphic to $\mathbb{Z}_{p^e-1}$, D_{p^e-1} (p odd), or $D_{2(p^e-1)}$, and transitive on $\mathrm{soc}(\widetilde{G}) \setminus \{0\}$.

Note that $\Gamma\mathrm{L}(1, p^e)$ can be constructed in the following way. Let $\mathrm{GF}(p^e)$ be the Galois field of order p^e, and view $\mathrm{soc}(\widetilde{G})$ as the additive group of $\mathrm{GF}(p^e)$. It is well-known that the multiplicative group $\mathrm{GF}(p^e)^*$ of $\mathrm{GF}(p^e)$ is cyclic, and let x be a generator of $\mathrm{GF}(p^e)^*$. Then $\mathrm{GL}(1, p^e) = \langle x \rangle$. Let y be the Frobenius automorphism of $\mathrm{GF}(p^e)$ such that y maps every $g \in \mathrm{GF}(p^e)$ to g^p. Then we have

$$\Gamma\mathrm{L}(1, p^e) = \langle x, y \mid x^{p^e-1} = y^e = 1, y^{-1}xy = x^p \rangle.$$

In the following, we shall first determine all possible cyclic subgroups of $\Gamma\mathrm{L}(1, p^e)$ of order either $p^e - 1$ or $\frac{p^e-1}{2}$ (p odd) (Claim 1), and then this is used to determine all possible subgroups of $\Gamma\mathrm{L}(1, p^e)$ which are isomorphic to $\mathbb{Z}_{p^e-1}$, D_{p^e-1} (p odd), or $D_{2(p^e-1)}$, and transitive on $\mathrm{soc}(\widetilde{G}) \setminus \{0\}$.

Claim 1. *Let T be a cyclic subgroup of $\Gamma\mathrm{L}(1, p^e)$ of order $\frac{p^e-1}{r}$ with either $r = 1$ or $r = 2$ and p is odd. Then either $T = \langle x^r \rangle$, or $p^e = 3^2$, $T \cong \mathbb{Z}_{\frac{p^e-1}{2}}$ and $T = \langle xy \rangle$ or $\langle x^3 y \rangle$.*

Proof of Claim 1. Let $\ell = p^e - 1$ or $\frac{p^e-1}{2}$ (p odd). Since T is a cyclic subgroup of $\Gamma\mathrm{L}(1, p^e)$ of order ℓ, we may let $T = \langle x^j y^k \rangle$ with $0 \leq j \leq p^e - 2$ and $0 \leq k \leq e - 1$. If $k = 0$, then $T \leq \langle x \rangle$ and so $T = \langle x^r \rangle$ with either $r = 1$ or $r = 2$ and p is odd.

Assume now that $0 < k \leq e - 1$. Then $y^k \neq 1$. Since $y^{-1}xy = x^p$, one has $yx^py^{-1} = x$, and hence $(yxy^{-1})^p = x$. Clearly, $p^e \equiv 1 \pmod{p^e - 1}$, so $yxy^{-1} = x^{p^{e-1}}$.

It follows that $y^k x^j y^{-k} = x^{jp^{k(e-1)}}$, and so $y^k x^j = x^{jp^{k(e-1)}} y^k$. By this equality, we have for any positive integer m,

$$(x^j y^k)^m = x^{j(1+p^{k(e-1)}+p^{2k(e-1)}+\cdots+p^{(m-1)k(e-1)})} y^{mk} = x^{j\frac{p^{mk(e-1)}-1}{p^{k(e-1)}-1}} y^{mk}. \tag{3.1}$$

From Equation (3.1) it follows that $(x^j y^k)^e = x^{j\frac{p^{ek(e-1)}-1}{p^{k(e-1)}-1}}$. Since $p^e - 1 \mid p^{ke(e-1)} - 1$, one has

$$(x^j y^k)^{e(p^{k(e-1)}-1)} = x^{j(p^{ek(e-1)}-1)} = 1.$$

This implies that the order of $x^j y^k$ divides $e(p^{k(e-1)} - 1)$, namely, $\ell \mid e(p^{k(e-1)} - 1)$. Since $\ell = p^e - 1$ or $\frac{p^e-1}{2}$ (p odd), we have $p^e - 1 \mid 2e(p^{k(e-1)} - 1)$.

Suppose that $e \geq 3$. If $(p, e) = (2, 6)$, then $\ell = p^e - 1 = 63$. However, it is easy to check that $63 \nmid 6(2^{5k} - 1)$ for any $k \leq 5$, contrary to $\ell \mid e(p^{k(e-1)} - 1)$. Thus, $(p, e) \neq (2, 6)$. Then by a result of Zsigmondy [14], there exists at least one prime q such that q divides $p^e - 1$ but does not divide $p^t - 1$ for any positive integer $t < e$. Clearly, $p \neq q$, so p is an element of $\mathbb{Z}_q^* \cong \mathbb{Z}_{q-1}$ of order e. In particular, we have $q > e$. Since $q \mid p^e - 1$ and $p^e - 1 \mid 2e(p^{k(e-1)} - 1)$, we have $q \mid p^{k(e-1)} - 1$, implying $k(e-1) > e$. Since $k \leq e - 1$, we may let $k(e-1) = me + t$ for some positive integers m and $t < e$, and since $p^{me}(p^t - 1) = (p^{k(e-1)} - 1) - (p^{me} - 1)$, we have $q \mid p^t - 1$. However, this is impossible because it is assumed that $q \nmid p^t - 1$ for any $t < e$.

Thus, $e < 3$. Since $0 < k \leq e-1$, one has $e = 2$ and $k = 1$, and then $p^2 - 1 \mid 4(p-1)$. It follows that $p + 1 \mid 4$ and hence $p = 3$. Then $(x^j y)^2 = x^{4j}$ has order at most 2 since $\langle x \rangle \cong \mathbb{Z}_8$, and then $x^j y$ has order dividing 4. This implies that $\ell = \frac{p^e-1}{2} = 4$ and $T = \langle xy \rangle$ or $\langle x^3 y \rangle$. This completes the proof of Claim 1. $\square$

By now, we have shown that Claim 1 is true. Recall that $\widetilde{G}_0 \leq \Gamma\mathrm{L}(1, p^e)$, $\widetilde{G}_0 \cong \mathbb{Z}_{p^e-1}$, D_{p^e-1} (p odd), or $D_{2(p^e-1)}$ and $\widetilde{G}_0$ is transitive on $\mathrm{soc}(\widetilde{G}) \setminus \{0\}$. We shall finish the proof by considering the following three cases.

Case 1. $\widetilde{G}_0 \cong \mathbb{Z}_{p^e-1}$.

In this case, by Claim 1, we must have $\widetilde{G}_0 = \langle x \rangle = \mathrm{GL}(1, p^e)$ and so $\widetilde{G} \cong \mathrm{AGL}(1, p^e)$.

Case 2. $\widetilde{G}_0 \cong D_{p^e-1}$ (p odd).

In this case, by Claim 1, either $x^2 \in \widetilde{G}_0$, or $p^e = 9$ and $\widetilde{G}_0$ contains xy or $x^3 y$. For the former, we have $\widetilde{G}_0 = \langle x^2, f \rangle$, where f is an involution of $\Gamma\mathrm{L}(1, p^e)$ such that $f x^2 f = x^{-2}$ and $f \notin \langle x \rangle$. Note that $\widetilde{G}_0$ is transitive on $\mathrm{soc}(\widetilde{G}) \setminus \{0\}$. We may let $f = xy^k$ and $0 < k \leq e - 1$. By Equation (3.1), $f^2 = (xy^k)^2 = 1$ implies that e is even and $k = \frac{e}{2}$, and furthermore, $x^{p^{\frac{e(e-1)}{2}}+1} = 1$. It follows that $p^e - 1 \mid p^{\frac{e(e-1)}{2}} + 1$. However, since $p^{e\frac{(e-2)}{2}}(p^{\frac{e}{2}} + 1) = (p^{\frac{e(e-1)}{2}} + 1) + (p^{e\frac{(e-2)}{2}} - 1)$, we would have $p^e - 1 \mid p^{\frac{e}{2}} + 1$, forcing that $p = 2$, a contradiction.

For the latter, we have $\widetilde{G}_0 \cong D_8$. However, it is easy to check that in $\Gamma\mathrm{L}(1, 9) = \langle x, y \mid x^8 = y^2 = 1, y^{-1} xy = x^3 \rangle$ there does not exist an involution inverting xy or $x^3 y$, a contradiction.

Case 3. $\widetilde{G}_0 \cong D_{2(p^e-1)}$.

By Claim 1, we must have $\widetilde{G}_0 = \langle x \rangle \rtimes \langle y^{\frac{e}{2}} \rangle$ with $y^{\frac{e}{2}} x y^{\frac{e}{2}} = x^{-1}$. On the other hand, since $y^{-1} xy = x^p$, we have $y^{\frac{e}{2}} x y^{\frac{e}{2}} = x^{p^{\frac{e}{2}}}$ and hence $x^{p^{\frac{e}{2}}} = x^{-1}$. It follows that

$p^{\frac{e}{2}} \equiv -1 \pmod{p^e - 1}$ and hence $p^e - 1$ divides $p^{\frac{e}{2}} + 1$. Consequently, we have $p^e = 4$, $\widetilde{G}_0 = \langle x, y\rangle = \Gamma\mathrm{L}(1,4)$, and $\widetilde{G} \cong \mathrm{A\Gamma L}(1,4) \cong S_4$. $\square$

Now we are ready to determine all possible special generalized truncations of $\mathbf{K}_n$ by C_{n-1}.

Lemma 3.5. *Use the notations in Assumption 3.1. Let $\Upsilon \cong C_{n-1}$ and let $\widetilde{G}$ be the lift of G. Then $\widetilde{\mathbf{K}}_n = T(\mathbf{K}_n, G, \Upsilon)$ is isomorphic to either $T(\mathbf{K}_6, A_5, C_5)$ (see Figure 1), or one of the graphs $\mathbf{K}^i_{p^e}$ ($i \in \mathbb{Z}^*_{p^e-1}$, $i < \frac{p^e-1}{2}$) (see Construction 1.2 for the definition of these graphs).*

Proof. If $\mathrm{soc}(\widetilde{G}) \cong A_5$, then by [6, Example 5.3], we have $\widetilde{G} \cong A_5$ and up to graph isomorphism, there exists a unique graph, and so we may denote this graph by $T(\mathbf{K}_6, A_5, C_5)$ (see Figure 1).

In what follows, we assume that $\mathrm{soc}(\widetilde{G}) \ncong A_5$. Then from Lemma 3.4 we see that $\widetilde{G}$ has a subgroup, say $\widetilde{T}$ such that $\widetilde{T} \cong \mathrm{AGL}(1,p^e)$ and $\widetilde{T}$ acts regularly on $V(\widetilde{\mathbf{K}}_n)$, where p is a prime and e is a positive integer such that $p^e \geq 4$. It follows that $\widetilde{\mathbf{K}}_n$ is a Cayley graph on $\widetilde{T}$ ($\cong$ $\mathrm{AGL}(1,p^e)$) and $n = p^e$. For each $u \in V(\mathbf{K}_n)$, by Lemma 3.3, the subgraph of $\widetilde{\mathbf{K}}_n$ induced by V_u is a cycle of length $n-1$, and the subgroup $\widetilde{G}_{V_u}$ of $\widetilde{G}$ fixing V_u set-wise acts faithfully and transitively on V_u. Furthermore, $\widetilde{G}$ acts faithfully and 2-transitively on $\mathcal{P}$. For convenience, we may identify $\mathcal{P}$ with $\mathrm{GF}(p^e)$, identify V_u with the zero element 0 of $\mathrm{GF}(p^e)$ and identify $\widetilde{T}$ with $\mathrm{AGL}(1,p^e)$. We shall use the notations for $\widetilde{T} = \mathrm{AGL}(1,p^e)$ as well as its elements and subgroups H and K introduced in the paragraph before Construction 1.2. Then $\widetilde{T}_{V_u} = K \cong \mathbb{Z}_{p^e-1}$.

Take $(u,w) \in V_u$, and assume that (u,w_1) and (u,w_2) are two vertices in V_u adjacent to (u,w). Since $\widetilde{T}_{V_u} = K \cong \mathbb{Z}_{p^e-1}$ is transitive on V_u, there exists a unique $\alpha_{x^i,0} \in \widetilde{T}_{V_u}$ such that $(u,w)^{\alpha_{x^i,0}} = (u,w_1)$ and $(u,w)^{\alpha_{x^{-i},0}} = (u,w_2)$, and since the subgraph of $\widetilde{\mathbf{K}}_n$ induced by V_u is a cycle of length $n-1$, i is coprime to p^e-1 ($n = p^e$). So we may let

$$\widetilde{\mathbf{K}}_n = \mathrm{Cay}(\mathrm{AGL}(1,p^e), \{\alpha_{x^i,0}, \alpha_{x^{-i},0}, \alpha_{x^j,z'}\}),$$

where $\alpha_{x^j,z'}$ is an involution. Since $\widetilde{\mathbf{K}}_n$ is connected, if p is odd, then we have $\alpha_{x^j,z'} = \alpha_{x^{\frac{p^e-1}{2}},z'} = \alpha_{-1,z'}$ and $z' \neq 0$, and if $p = 2$, then we have $\alpha_{x^j,z'} = \alpha_{1,z'}$ and $z' \neq 0$, and correspondingly, we obtain the two graphs $\mathbf{K}^j_{p^e}$ ($p > 2$) and $\mathbf{K}^j_{2^e}$ (see Construction 1.2). $\square$

From Figure 1 it is easy to see that $T(\mathbf{K}_6, A_5, C_5)$ is a special generalized truncation of $\mathbf{K}_6$ by a cycle of length 5. The following lemma shows that each of the Cayley graphs $\mathbf{K}^i_{p^e}$ ($i \in \mathbb{Z}^*_{p^e-1}$, $i < \frac{p^e-1}{2}$) is also indeed a special generalized truncation of $\mathbf{K}_{p^e}$ by a cycle of length $p^e - 1$.

Lemma 3.6. *Each of the graphs $\mathbf{K}^i_{p^e}$ ($i \in \mathbb{Z}^*_{p^e-1}$, $i < \frac{p^e-1}{2}$) (see Construction 1.2) is a special generalized truncation of $\mathbf{K}_{p^e}$ by a cycle of length $p^e - 1$.*

Proof. Recall that each $\mathbf{K}^i_{p^e}$ ($i \in \mathbb{Z}^*_{p^e-1}$, $i < \frac{p^e-1}{2}$) is a trivalent Cayley graph on $\mathrm{AGL}(1,p^e)$ defined as follows:

$$\mathbf{K}^i_{p^e} = \mathrm{Cay}(\mathrm{AGL}(1,p^e), \{\alpha_{-1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \quad (z' \neq 0, p > 2),$$
$$\mathbf{K}^i_{2^e} = \mathrm{Cay}(\mathrm{AGL}(1,2^e), \{\alpha_{1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \quad (z' \neq 0).$$

(Keep in mind we use the notations for $\mathrm{AGL}(1,p^e)$ as well as its elements and subgroups H and K introduced in the paragraph before Construction 1.2.) Note that $\mathrm{AGL}(1,p^e) = H \rtimes K$, where

$$H = \{\alpha_{1,z''}\colon z \mapsto z + z'', \forall z \in GF(p^e) \mid z'' \in GF(p^e)\},$$
$$K = \{\alpha_{x^j,0}\colon z \mapsto zx^j, \forall z \in GF(p^e) \mid j \in \mathbb{Z}_{p^e-1}\}.$$

Moreover, K is maximal in $\mathrm{AGL}(1,p^e)$ since $\mathrm{AGL}(1,p^e)$ is 2-transitive on $GF(p^e)$. As $i \in \mathbb{Z}^*_{p^e-1}$, one has $K = \langle \alpha_{x^i,0}\rangle$ and then the maximality of K implies that $\langle \alpha_{-1,z'}, \alpha_{x^i,0}\rangle = \mathrm{AGL}(1,p^e)$ for $p > 2$ and $\langle \alpha_{1,z'}, \alpha_{x^i,0}\rangle = \mathrm{AGL}(1,2^e)$. Thus, every $\mathbf{K}^i_{p^e}$ ($i \in \mathbb{Z}^*_{p^e-1}, i < \frac{p^e-1}{2}$) is connected.

It is easy to see that $\mathrm{Cay}(K, \{\alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \cong C_{p^e-1}$ is a subgraph of $\mathbf{K}^i_{p^e}$ ($i \in \mathbb{Z}^*_{p^e-1}$, $i < \frac{p^e-1}{2}$). Since $\mathrm{AGL}(1,p^e)$ acts on $V(\mathbf{K}^i_{p^e})$ by right multiplication, the subgraph of $\mathbf{K}^i_{p^e}$ induced by Kg for any $g \in \mathrm{AGL}(1,p^e)$ is a cycle of length $p^e - 1$. As $\mathrm{AGL}(1,p^e)$ acts 2-transitively on $\mathcal{B} = \{Kg \mid g \in \mathrm{AGL}(1,p^e)\}$, the quotient graph of $\mathbf{K}^i_{p^e}$ relative to $\mathcal{B}$ is a complete graph $\mathbf{K}_{p^e}$. So we have $\mathbf{K}^i_{p^e} \cong T(\mathbf{K}_{p^e}, \mathrm{AGL}(1,p^e), \Upsilon_i)$, where Υ_i is the subgraph with vertex set $\mathcal{B} - \{K\}$ and edge set $\{\{K\gamma g, K\gamma\alpha_{x^i,0}g\} \mid g \in K\}$ where $\gamma = \alpha_{-1,z'}$ for $p > 2$ and $\gamma = \alpha_{1,z'}$ for $p = 2$. □

3.2 Automorphisms and isomorphisms

In this subsection, we shall determine the automorphism groups and isomorphisms of special generalized truncations of $\mathbf{K}_n$ by C_{n-1}, and thus prove the second part of Theorem 1.4. By checking [6, Table 1], we have the following lemma.

Lemma 3.7. $\mathrm{Aut}(T(\mathbf{K}_6, A_5, C_5)) \cong A_5$.

In the following two lemmas, we shall determine the automorphisms and isomorphisms of the graphs $\mathbf{K}^i_{p^e}$ ($i \in \mathbb{Z}^*_{p^e-1}$, $i < \frac{p^e-1}{2}$). We keep using the notations for $\mathrm{AGL}(1,p^e)$ as well as its elements and subgroups H and K introduced in the paragraph before Construction 1.2.

Lemma 3.8. *Let Γ be one of the graphs $\mathbf{K}^i_{p^e}$ ($i \in \mathbb{Z}^*_{p^e-1}$, $i < \frac{p^e-1}{2}$) (see Construction 1.2). Then Theorem 1.4 (ii)–(vi) hold.*

Proof. Recall that Γ is a connected trivalent Cayley graph on $X = \mathrm{AGL}(1,p^e)$. Let $A = \mathrm{Aut}(\Gamma)$. For convenience of the statement, we view X as a regular subgroup of A.

Suppose first that Γ is arc-transitive. Let $N = \bigcap_{g\in A} X^g$. If $N = 1$, then by [9, Theorem 1.1], we have $\mathrm{Aut}(\Gamma) \cong \mathrm{PGL}_2(p^e)$ with $p^e = 11$ or 23. If $p^e = 11$, then since $i \in \mathbb{Z}^*_{10}$ and $i < 5$, we have $i = 3$ and hence $\Gamma = \mathbf{K}^3_{11}$. If $p^e = 23$, then $i = 3, 5, 7$ or 9 as $i \in \mathbb{Z}^*_{22}$ and $i < 11$, and by MAGMA [4], $\mathrm{Aut}(\mathbf{K}^i_{23}) \cong \mathrm{PGL}_2(23)$ if and only if $i = 7$, and hence $\Gamma = \mathbf{K}^7_{23}$. If $N > 1$, then $N \trianglelefteq A$, and in particular, $N \trianglelefteq X$. Since $\mathrm{soc}(X) \cong \mathbb{Z}^e_p$ is the unique minimal normal subgroup of $X = \mathrm{AGL}(1,p^e)$, one has $\mathrm{soc}(X) \le N$. Clearly, $\mathrm{soc}(X)$ is a Sylow p-subgroup of N since $N \le X$. So $\mathrm{soc}(X)$ is characteristic in N and hence normal in A. Consider the quotient graph Σ of Γ relative to $\mathrm{soc}(X)$. Clearly, Σ has $p^e - 1$ vertices. Since $p^e - 1 > 2$, by Proposition 2.3, Σ would be a trivalent arc-transitive Cayley graph on $X/\mathrm{soc}(X) \cong \mathbb{Z}_{p^e-1}$. Furthermore, by [2, Corollary 1.3], either $\Sigma \cong \mathbf{K}_{3,3}$, or Σ is a trivalent normal arc-transitive Cayley graph on $X/\mathrm{soc}(X) \cong \mathbb{Z}_{p^e-1}$. However, the latter case cannot happen by Lemma 2.1. For the former, we have $p^e - 1 = 6$

and so $p = 7$ and $e = 1$. In this case, we have $i = 1$ and $\Gamma = \mathbf{K}_7^1$. By MAGMA [4], we have $\mathrm{Aut}(\mathbf{K}_7^1) \cong D_{42} \rtimes \mathbb{Z}_3$.

Suppose now that Γ is not arc-transitive. If $A > X$, then the vertex-stabilizer A_a is a 2-group for any $a \in V(\Gamma)$. Then A_a fixes one and only one neighbor of a. Assume that the neighbor of a fixed by A_a is b. Then $B = \{\{a, b\}^g \mid g \in A\}$ is a system of blocks of imprimitivity of A on $V(\Gamma)$. It follows that $\Gamma - B$ is a union of several cycles with equal lengths, and the set of vertex-sets of these cycles forms an A-invariant partition of $V(\Gamma)$. Let C be the cycle of Γ containing the identity 1 of X. Since Γ is a Cayley graph on X, X acts on $V(\Gamma) = X$ by right multiplication, and since $V(C)$ is a block of imprimitivity of A acting on $V(\Gamma)$, C is actually a subgroup of X. From the definition of $\Gamma = \mathbf{K}_{p^e}^i$, one may see that $V(C) = K = \{\alpha_{x^i,0} \colon z \mapsto zx^i, \forall z \in GF(p^e) \mid i \in \mathbb{Z}_{p^e-1}\}$, and the vertex set of every cycle of $\Gamma - B$ is just a right coset of K. Let $\mathcal{B} = \{Kg \mid g \in X\}$. Then $\mathcal{B}$ is an A-invariant partition of Γ. Clearly, X acts 2-transitively and faithfully on $\mathcal{B}$, so the quotient graph of Γ relative $\mathcal{B}$ is $\mathbf{K}_{p^e}$. Now it is easy to see that $\Gamma \cong T(\mathbf{K}_{p^e}, A, \Upsilon_i)$, where Υ_i is the subgraph with vertex set $\mathcal{B} - K$ and edge set $\{\{K\gamma g, K\gamma\alpha_{x^i,0}g\} \mid g \in A_K\}$ where $\gamma = \alpha_{-1,z'}$ for $p > 2$ and $\gamma = \alpha_{1,z'}$ for $p = 2$. Clearly, $\Upsilon_i \cong C_{p^e-1}$. From Lemma 3.4 it follows that either $p^e = 4$ and $A = \mathrm{A\Gamma L}(1, 4) \cong S_4$, or $A = X = \mathrm{AGL}(1, p^e)$. □

Lemma 3.9. *For any distinct* $i, i' \in \mathbb{Z}_{p^e-1}^*$ *with* $i, i' < \frac{p^e-1}{2}$, $\mathbf{K}_{p^e}^i \cong \mathbf{K}_{p^e}^{i'}$ *if and only if there exists* $1 \le j \le e$ *such that* $i' \equiv ip^j$ *or* $-ip^j \pmod{p^e - 1}$.

Proof. If $p^e = 4$ or 7, then we must have $i = 1$, and so we have only one graph for each of these two cases. If $p^e = 11$ or 23, then by MAGMA [4], for any distinct $i, i' \in \mathbb{Z}_{p^e-1}^*$ with $i, i' < \frac{p^e-1}{2}$, one may check that $\mathbf{K}_{p^e}^i \cong \mathbf{K}_{p^e}^{i'}$ if and only if $i' \equiv ip^j$ or $-ip^j \pmod{p^e - 1}$.

Suppose that $\mathbf{K}_{p^e}^i$ is not isomorphic to one of the graphs: $\mathbf{K}_4^1, \mathbf{K}_7^1, \mathbf{K}_{11}^3$ and $\mathbf{K}_{23}^7$. By Lemma 3.8, $\mathrm{Aut}(\mathbf{K}_{p^e}^i) \cong \mathrm{AGL}(1, p^e)$ and by Proposition 2.2, $\mathbf{K}_{p^e}^i$ is a CI-graph. Recall that

$$\mathbf{K}_{p^e}^i = \mathrm{Cay}(\mathrm{AGL}(1, p^e), \{\alpha_{-1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \quad (p > 2),$$
$$\mathbf{K}_{2^e}^i = \mathrm{Cay}(\mathrm{AGL}(1, 2^e), \{\alpha_{1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \quad (p = 2).$$

Since $\mathbf{K}_{p^e}^i$ is a CI-graph, for any distinct $i, i' \in \mathbb{Z}_{p^e-1}^*$ with $i, i' < \frac{p^e-1}{2}$, $\mathbf{K}_{p^e}^i \cong \mathbf{K}_{p^e}^{i'}$ if and only if there exists $\gamma \in \mathrm{Aut}(\mathrm{AGL}(1, p^e))$ such that $\{\alpha_{x^i,0}, \alpha_{x^{-i},0}\}^\gamma = \{\alpha_{x^{i'},0}, \alpha_{x^{-i'},0}\}$ and either $\alpha_{-1,z'}^\gamma = \alpha_{-1,z'}$ for $p > 2$ or $\alpha_{1,z'}^\gamma = \alpha_{1,z'}$ for $p = 2$.

Note that $\mathrm{Aut}(\mathrm{AGL}(1, p^e)) = \mathrm{A\Gamma L}(1, p^e) = \mathrm{AGL}(1, p^e) \rtimes \langle\eta\rangle$, where η is induced by the Frobenius automorphism of $\mathrm{GF}(p^e)$ such that $\alpha_{a,b}^\eta = \alpha_{a^p,b^p}$ for any $\alpha_{a,b} \in \mathrm{AGL}(1, p^e)$. Suppose first that $i' \equiv ip^j$ or $-ip^j \pmod{p^e - 1}$ for some $1 \le j \le e$. Then one may check that $\alpha_{\pm 1,z'}^{\eta^j \alpha_{(z')^{-p^j} z',0}} = \alpha_{\pm 1,z'}$ and $\{\alpha_{x^i,0}, \alpha_{x^{-i},0}\}^{\eta^j \alpha_{(z')^{-p^j} z',0}} = \{\alpha_{x^{i'},0}, \alpha_{x^{-i'},0}\}$. So $\mathbf{K}_{p^e}^i \cong \mathbf{K}_{p^e}^{i'}$. Conversely, if $\mathbf{K}_{p^e}^i \cong \mathbf{K}_{p^e}^{i'}$, then there exists $\gamma \in \mathrm{Aut}(\mathrm{AGL}(1, p^e))$ such that $\{\alpha_{x^i,0}, \alpha_{x^{-i},0}\}^\gamma = \{\alpha_{x^{i'},0}, \alpha_{x^{-i'},0}\}$ and either $\alpha_{-1,z'}^\gamma = \alpha_{-1,z'}$ for $p > 2$ or $\alpha_{1,z'}^\gamma = \alpha_{1,z'}$ for $p = 2$. Since $K = \langle\alpha_{x^i,0}\rangle$, γ normalizes K, and since $N_{\mathrm{A\Gamma L}(1,p^e)}(K) = K \rtimes \langle\eta\rangle$, one has $\gamma = \alpha_{x^k,0}\eta^j$, for some $k \in \mathbb{Z}_{p^e-1}^*$ and $1 \le j \le e$. Then

$$\alpha_{x^i,0}^\gamma = \alpha_{x^i,0}^{\alpha_{x^k,0}\eta^j} = \alpha_{x^i,0}^{\eta^j} = \alpha_{x^{ip^j},0} \in \{\alpha_{x^{i'},0}, \alpha_{x^{-i'},0}\}.$$

It follows that $i' \equiv ip^j$ or $-ip^j \pmod{p^e - 1}$. □

3.3 Proof of Theorem 1.4

From Lemmas 3.5 and 3.6 we can obtain the proof of the first part of Theorem 1.4, and from Lemmas 3.8 and 3.9, we obtain the proof of the second part of Theorem 1.4.

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Properties of double Roman domination on cardinal products of graphs*

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Abstract

Double Roman domination is a stronger version of Roman domination that doubles the protection. The areas now have 0, 1, 2 or 3 legions. Every attacked area needs 2 legions for its defence, either their own, or borrowed from 1 or 2 neighbouring areas, which still have to keep at least 1 legion to themselves. The minimal number of legions in all areas together is equal to the double Roman domination number.

In this paper we determine an upper bound and a lower bound for double Roman domination numbers on cardinal product of any two graphs. Also we determine the exact values of double Roman domination numbers on $P_2 \times G$ (for many types of graph G). Also, the double Roman domination number is found for $P_2 \times P_n$, $P_3 \times P_n$, $P_4 \times P_n$, while upper and lower bounds are given for $P_5 \times P_n$ and $P_6 \times P_n$.

Finally, we will give a case study to determine the efficiency of double protection. We will compare double Roman domination versus Roman domination by running a simulation of a battle.

Keywords: Roman domination, double Roman domination, cardinal products of graphs, paths, cycles.

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1 Introduction

In the 4th century AD Constantine I (274 – 337 AD) ruled the Roman Empire. To defend the Empire against barbarians, he had to arrange Roman legions in a way that all strategically important places were protected with as low costs as possible.

If at least one Roman legion was stationed at a certain location, that location was considered to be secured. Unsecured locations, on the other hand, had no legions, but they had to be adjacent to at least one secured location. If an unsecured location was under attack, one could send a legion from some neighbouring secured location. But to avoid making that secured location unsecure, it had to have at least two legions itself. Maintaining of an army was expensive, so Constantine had to secure the Empire with as few legions as possible.

This historical background motivated Ian Stewart (1999) to suggest the new variant of graph domination known as Roman domination (RD). If we represent locations of the Empire as graph vertices and roads of the Empire as graph edges, the problem of defending the Roman Empire becomes a problem of graph domination. Double Roman domination (DRD) is stronger version in which we double protection.

There are many works dealing with Roman domination [8, 9, 13, 14], but only few about double Roman dominations. Foundations of DRD are set in [4]. In [3, 15, 16] we can find bounds on the DRD and the most recent work is [2]. For more details on Roman domination and double Roman domination and their variants see [5, 6, 7].

In this paper we determine exact values or upper and lower bounds for double Roman domination numbers on cardinal products of some graphs.

Apart from this introduction, the work is organized in the following way. In Section 2 we define dominating function on a graph G, Roman dominating function on G, double Roman dominating function on G and on a cardinal product of graphs. Domination number, Roman domination number and double Roman domination number are defined and some basic relations among them are given.

In Section 3 we determine one upper and one lower bound for double Roman domination numbers on cardinal product of any two graphs. Then we determined the exact values of double Roman domination numbers of $P_2 \times G$ for many types of graph G. Finally, the double Roman domination number is found for $P_2 \times P_n$, $P_3 \times P_n$, $P_4 \times P_n$, while upper and lower bounds are given for $P_5 \times P_n$ and $P_6 \times P_n$.

In Section 4 we give a case study to determine the efficiency of double protection. We will simulate a battle between Romans and barbarians in the cases of double Roman domination and standard Roman domination.

2 Definitions

Dominating function (DF) on $G = (V, E)$ is a function $f \colon V \to \{0, 1\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 1$. Depending on values of f, we get the ordered partition (V_0, V_1) of V where each vertex in V_0 is adjacent to at least one vertex in V_1. The set V_1 is called *a dominating set*.

We have bijection between the set of all functions $f \colon V \to \{0, 1\}$ and the set of all ordered partitions (V_0, V_1). Thus we are allowed to write $f = (V_0, V_1)$. *The weight of* f equals $w(f) = \sum_{v \in V} f(v) = 0 \cdot |V_0| + 1 \cdot |V_1| = |V_1|$. Of course, we will look for dominating functions with the minimum weight. This weight $\gamma(G)$ is called *the domination* *number* of G.

Further, *Roman dominating function* (RDF) on $G = (V, E)$ is a function $f\colon V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. Since this function also induces the ordered partition of V with $V_i = \{v \in V : f(v) = i\}$, $i \in \{0, 1, 2\}$, we are allowed to write $f = (V_0, V_1, V_2)$. The set $V_1 \cup V_2$ is called *a Roman dominating set*. The weight of an RDF f equals $w(f) = \sum_{v \in V} f(v) = 0 \cdot |V_0| + 1 \cdot |V_1| + 2 \cdot |V_2| = |V_1| + 2|V_2|$. The minimum such weight $\gamma_R(G)$ is called *the Roman domination number* on G.

In Roman domination, one legion is required to defend any attacked area. Double Roman domination is a stronger version of Roman domination that doubles the protection by ensuring that any attack can be defended by at least two legions.

Finally, *double Roman dominating function* (DRDF) on $G = (V, E)$ is a function $f\colon V \to \{0, 1, 2, 3\}$ if it satisfies the following conditions:

1. If $f(v) = 0$, then the vertex v has at least two neighbours in V_2 or one neighbour in V_3.

2. If $f(v) = 1$, then the vertex v has at least one neighbour in $V_2 \cup V_3$,

where by V_i we denote the set of vertices assigned with i by the function f. The set $V_1 \cup V_2 \cup V_3$ is called *a double Roman dominating set*. The weight of a DRDF equals $w(f) = \sum_{v \in V} f(v) = |V_1| + 2|V_2| + 3|V_3|$.

Double Roman domination number $\gamma_{dR}(G)$ equals the minimum weight among all double Roman dominating functions on G. A double Roman dominating function on G with weight $\gamma_{dR}(G)$ is called a γ_{dR}*-function* of G.

In Roman domination at most two Roman legions are deployed at any location. But as we will see in what follows, the ability to deploy three legions at a given location provides a level of defense that is both stronger and more flexible. Also, the additional security we get is usually greater than the additional costs.

Here we can see a real benefit of double Roman domination. In the example of the star graph $K_{1,n-1}$ (see Figure 1), it is obvious that $\gamma_{dR}(K_{1,n-1}) = 3$. Note that this is only one more than $\gamma_R(K_{1,n-1}) = 2$. So we doubled the defense of a graph (at least two legions against each attack) with an added cost of no more than 50% of the cost of defending against each attack with one legion.

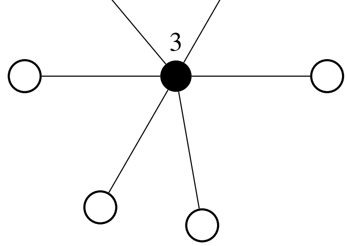

Figure 1: Double Roman domination on star graph.

In [4], it is observed that $\gamma_{dR}(G) \leq 2|V_1| + 3|V_2|$ for any RDF $f = (V_0, V_1, V_2)$. It is also proved a relation between domination and double Roman domination number of any graph G, i.e.

$$2\gamma(G) \leq \gamma_{dR}(G) \leq 3\gamma(G),$$

and a relation between Roman domination and double Roman domination number of any nontrivial connected graph G

$$\gamma_R(G) \leq \gamma_{dR}(G) \leq 2\gamma_R(G).$$

Graphs where $\gamma_{dR}(G) = 3\gamma(G)$ are called *double Roman graphs*. There is an open problem to characterize such graphs. Double Roman trees are characterized in [1]. For more domination parameters and for the terminology see [10, 11, 12].

In this paper we will consider double Roman domination number of cardinal product of graphs. For arbitrary graphs G and H, the *cardinal product of* G *and* H is the graph $G \times H$ which satisfies the following:

1. Its vertex set is $V(G \times H) = V(G) \times V(H)$.
2. Two vertices $(g, h), (g', h') \in V(G \times H)$ are adjacent if and only if g is adjacent to g' in G and h is adjacent to h' in H.

If $H \subset V(G)$ then $G[H]$ is the subgraph induced with H. The cardinal product of two paths $P_m \times P_n$ has two connected components. If the vertices of P_m and P_n are denoted by $\{1, 2, 3, \ldots, m\}$ and $\{1, 2, 3, \ldots, n\}$ respectively, then the component of $P_m \times P_n$ containing the vertex $(1, 1)$ will be denoted by K_1 and the other component by K_2. If at least one of the parameters m or n is even, the components K_1 and K_2 are isomorphic (see Figure 2). Otherwise, the component K_1 has one vertex more than the component K_2.

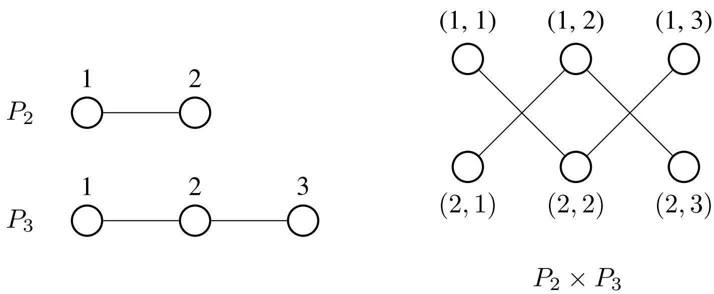

Figure 2: $K_1 = P_2 \times P_3\,[\{(1, 1), (2, 2), (1, 3)\}]$ and $K_2 = P_2 \times P_3\,[\{(2, 1), (1, 2), (2, 3)\}]$.

3 Specific values of double Roman domination numbers for cardinal products of graphs

As for introduction, we will show here some basic results and bounds for double Roman domination.

Remark 3.1. In [2] it is proved that

$$\gamma_{dR}(P_n) = \begin{cases} 3\lceil \frac{n}{3} \rceil, & n \equiv 0,2 \pmod 3 \\ 3\lceil \frac{n}{3} \rceil - 1, & n \equiv 1 \pmod 3 \end{cases}$$
$$= \begin{cases} n, & n \equiv 0 \pmod 3 \\ n+1, & n \equiv 1,2 \pmod 3 \end{cases}$$

and

$$\gamma_{dR}(C_n) = \begin{cases} n, & n \equiv 0,2,3,4 \pmod 6 \\ n+1, & n \equiv 1,5 \pmod 6). \end{cases}$$

Observartion 3.2. *For any graphs G and H of order n and m*

$$\gamma_{dR}(G \times H) \geq \left\lceil \frac{3mn}{\Delta(G)\Delta(H)+1} \right\rceil,$$
$$\gamma_{dR}(G \times H) \leq 2mn \frac{2 + ln((1+\delta(G)\delta(H))/2)}{\delta(G)\delta(H)+1},$$

where by $\Delta(G)$ ($\delta(G)$) we denote the maximum (minimum) degree of all vertices on G.

Proof. In [16] it is proved that for any graph G of order n with maximum degree $\Delta(G) \geq 1$

$$\gamma_{dR}(G) \geq \left\lceil \frac{3n}{\Delta(G)+1} \right\rceil.$$

Further, it holds

$$\Delta(G \times H) = \Delta(G) \cdot \Delta(H).$$

Combining two previous statements we get the lower bound. Next, in [14] it is proved that for cardinal product any graphs G and H of order n and m

$$\gamma_R(G \times H) \leq mn \frac{2 + ln((1+\delta(G)\delta(H))/2)}{\delta(G)\delta(H)+1}.$$

Then the statement follows from $\gamma_{dR}(G) \leq 2\gamma_R(G)$. □

Now we will calculate the exact values of double Roman domination numbers for cardinal products of some graphs.

Theorem 3.3. *For any tree T and any graph G without cycles of odd length we have*

$$\gamma_{dR}(P_2 \times T) = 2\gamma_{dR}(T) < \gamma_{dR}(P_2) \cdot \gamma_{dR}(T),$$
$$\gamma_{dR}(P_2 \times G) = 2\gamma_{dR}(G) < \gamma_{dR}(P_2) \cdot \gamma_{dR}(G).$$

Proof. The proof is trivial, since $P_2 \times T$ and $P_2 \times G$ consist of two disjoint copies of T and G, respectively and $\gamma_{dR}(P_2) = 3$. □

Theorem 3.4. *For the path P_2 and any odd cycle $C_{2n+1}, n \geq 1$,*

$$\gamma_{dR}(P_2 \times C_{2n+1}) = 4n + 2.$$

Proof. Note that the cardinal product of P_2 and C_{2n+1} is a cycle C_{4n+2}. Namely, if we denote the vertices of P_2 with a and b, and the vertices of C_{2n+1} with $1, 2, \dots, 2n+1$, then the vertices of the product $P_2 \times C_{2n+1}$ are adjacent in this order: $(a, 1), (b, 2), (a, 3), (b, 4), \dots,$ $(a, 2n+1), (b, 1), (a, 2), \dots, (b, 2n+1)$ and the last vertex $(b, 2n+1)$ is adjacent to $(a, 1)$, which makes a cycle of length $2(2n + 1) = 4n + 2$. Remark 3.1 implies that

$$\gamma_{dR}(C_{4n+2}) = 4n + 2. \qquad \square$$

Definition 3.5. For a fixed m, $1 \le m \le n$, the set $(P_k)_m = \{(i, m) : i = 1, \dots, k\}$ is called a column of $P_k \times P_n$. Similary, for a fixed l, $1 \le l \le k$, the set $(P_n)_l = \{(l, j) : j = 1, \dots, n\}$ is called a row of $P_k \times P_n$.

Theorem 3.6. *Let $n \ge 2$. Then*

$$\gamma_{dR}(P_3 \times P_n) = \begin{cases} 7, & n = 3 \\ 2n + 2, & \textit{otherwise.} \end{cases}$$

Proof. It is easy to see that $\gamma_{dR}(P_3 \times P_3) = 7$. Hence we assume $n \ge 4$. First we show that $\gamma_{dR}(P_3 \times P_n) \le 2n + 2$. Define $f\colon V(P_3 \times P_n) \to \{0, 1, 2, 3\}$ by $f((2, 2)) = f((2, n-1)) = 3$, $f((2, j)) = 2$ for $j \in \{1, \dots, n\} - \{2, n-1\}$ and $f(x) = 0$ otherwise. Clearly f is a double Roman dominating function on $P_3 \times P_n$ of weight $2n + 2$ and so $\gamma_{dR}(P_3 \times P_n) \le 2n+2$. To prove inverse inequality, let $f = (V_0, \emptyset, V_2, V_3)$ be a $\gamma_{dR}(P_3 \times P_n)$-function. Since the vertices $(2, 2)$ and $(2, n-1)$ are strong support vertices, we have $(2, 2), (2, n-1) \in V_3$. On the other hand, since $V_2 \cup V_3$ is a dominating set of $P_3 \times P_n$, we have $|V_2 \cup V_3| \ge \gamma(P_3 \times P_n) = n$ (see [11]). Thus we have $\gamma_{dR}(P_3 \times P_n) = 2|V_2| + 3|V_3| = 2(|V_2| + |V_3|) + |V_3| \ge 2n + 2$. Thus $\gamma_{dR}(P_3 \times P_n) = 2n + 2$ for $n \ge 4$ and the proof is complete. $\square$

Theorem 3.7. *Let $n \ge 2$. Then*

$$\gamma_{dR}(P_4 \times P_n) = \begin{cases} 3n, & n \equiv 0 \pmod 4 \\ 3n + 3, & n \equiv 1 \pmod 4 \\ 3n + 2, & n \equiv 2 \pmod 4 \\ 3n + 1, & n \equiv 3 \pmod 4. \end{cases}$$

Proof. First we show that

$$\gamma_{dR}(P_4 \times P_n) \le \begin{cases} 3n, & n \equiv 0 \pmod 4 \\ 3n + 3, & n \equiv 1 \pmod 4 \\ 3n + 2, & n \equiv 2 \pmod 4 \\ 3n + 1, & n \equiv 3 \pmod 4. \end{cases}$$

Since $P_4 \times P_n$ consists of two isomorphic components, we consider only K_1 and we multiply the result by 2.

Case 1: $n \equiv 0 \pmod 4$. Define $f\colon V(K_1) \to \{0, 1, 2, 3\}$ by $f((2, 4j + 2)) = f((3, 4j + 3)) = 3, j = 0, 1, \dots, \lfloor \frac{n}{4} \rfloor - 1$, and $f(x) = 0$ otherwise. Clearly f is a double Roman dominating function of weight $\frac{3n}{2}$ on K_1 and so $\gamma_{dR}(P_4 \times P_n) \le 3n$, for $n \equiv 0 \pmod 4$.

Case 2: $n \equiv 1 \pmod 4$. Define $f\colon V(K_1) \to \{0,1,2,3\}$ by $f((2,4j+2)) = f((3,4j+3)) = 3, j = 0,1,\dots,\lfloor\frac{n}{4}\rfloor - 1$, $f((2,n-1)) = 3$ and $f(x) = 0$ otherwise. It can easily be seen that f is a double Roman dominating function of weight $6\left(\frac{n-1}{4}\right)+3 = \frac{3n+3}{2}$ on K_1 and so $\gamma_{dR}(P_4 \times P_n) \le 3n+3$, for $n \equiv 1 \pmod 4$.

Case 3: $n \equiv 2 \pmod 4$. Define $f\colon V(K_1) \to \{0,1,2,3\}$ by $f((2,4j+2)) = 3$, $j = 0,1,\dots,\lfloor\frac{n}{4}\rfloor - 1$, $f((3,4j+3)) = 3, j = 0,1,\dots,\lfloor\frac{n}{4}\rfloor - 2$, $f((3,n-1)) = 3$, $f((1,n-1)) = f((4,n-4)) = 2$ and $f(x) = 0$ otherwise. Hence f is a double Roman dominating function of weight $6\left(\frac{n-2}{4}\right)+4 = \frac{3n+2}{2}$ on K_1 and so $\gamma_{dR}(P_4 \times P_n) \le 3n+2$, for $n \equiv 2 \pmod 4$.

Case 4: $n \equiv 3 \pmod 4$. Define $f\colon V(K_1) \to \{0,1,2,3\}$ by $f((2,4j+2)) = f((3,4j+3)) = 3, j = 0,1,\dots,\lfloor\frac{n}{4}\rfloor - 1$, $f((2,n-1)) = 3$, $f((4,n-1)) = 2$ and $f(x) = 0$ otherwise. Therefore f is a double Roman dominating function of weight $6\left(\frac{n-3}{4}\right)+5 = \frac{3n+1}{2}$ on K_1 and so $\gamma_{dR}(P_4 \times P_n) \le 3n+1$, for $n \equiv 3 \pmod 4$.

Proof of the minimality: In [16] is proved that if G is a graph of order n with maximum degree $\Delta(G) \ge 1$

$$\gamma_{dR}(G) \ge \left\lceil \frac{3n}{\Delta(G)+1} \right\rceil.$$

The order of $P_4 \times P_n$ is $4n$ and $\Delta(P_4 \times P_n) = 4$. Therefore

$$\gamma_{dR}(P_4 \times P_n) \ge \left\lceil \frac{12n}{5} \right\rceil = 3n. \tag{3.1}$$

Let $n \equiv 0 \pmod 4$. From the fact that $\gamma_{dR}(P_4 \times P_n) \le 3n$ and (3.1), it follows

$$\gamma_{dR}(P_4 \times P_n) = 3n, \quad n \equiv 0 \pmod 4.$$

In more details, for this case each vertex from V_0 is double Roman dominated by only one vertex from V_3. Next, $V_2 = \emptyset$, and V_3 is dominating set (see [11]). Also, on the last n-th column from $P_4 \times P_n$ all vertices are from V_0 (see Figure 3).

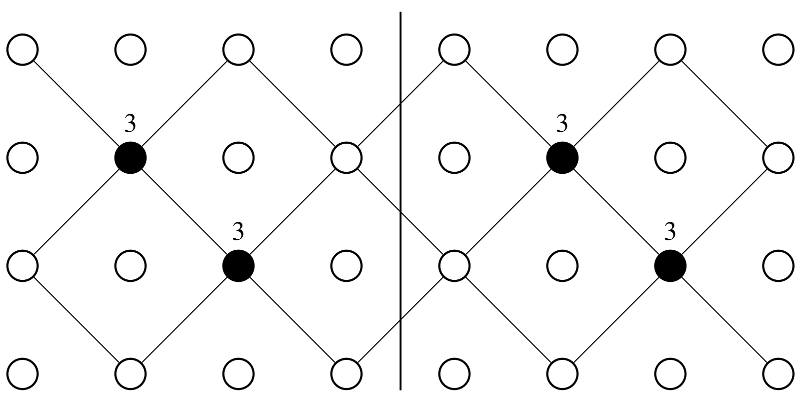

Figure 3: The function $f(V(K_1))$ on $P_4 \times P_8$.

Let $n \equiv 1 \pmod 4$. Then from (3.1) on the first $n-1$ columns on $P_4 \times P_n$ double Roman function f has a weight at least $3(n-1)$. Further, if the function f has the exactly weight $3n-3$, then the vertex $(2,n-1) \in V_0$ (on K_1). But $(2,n-1)$ is strong suport vertex, so must be in V_3. The same situation is on K_2. It follows that $\gamma_{dR}(P_4 \times P_n) \geq 3n-3+6 = 3n+3$. Hence,

$$\gamma_{dR}(P_4 \times P_n) = 3n+3, \ \ n \equiv 1 \pmod 4.$$

Let $n \equiv 2 \pmod 4$. It is easy to see that $\gamma_{dR}(P_4 \times P_6) = 20$ (on each component 10) and that the last 4×4 block and n^{th} and $(n-1)^{\text{th}}$ columns make a 4×6 block. It follows that on the last 6 columns we need at least weight 20, and on the first $n-6$ columns $3(n-6)$. Then $\gamma_{dR}(P_4 \times P_n) \geq 3n-18+20 = 3n+2$. So,

$$\gamma_{dR}(P_4 \times P_n) = 3n+2, \ \ n \equiv 2 \pmod 4.$$

Let $n \equiv 3 \pmod 4$. Then on the first $n-3$ columns on $P_4 \times P_n$ double Roman function f has a weight at least $3(n-3)$. On the last 3 columns we need at least weight 5 on one, or 10 on both components giving $\gamma_{dR}(P_4 \times P_3) = 10$. It follows that $\gamma_{dR}(P_4 \times P_n) \geq 3n-9+10 = 3n+1$. Therefore,

$$\gamma_{dR}(P_4 \times P_n) = 3n+1, \ \ n \equiv 3 \pmod 4. \qquad \square$$

For $P_5 \times P_n$ and $P_6 \times P_n$ from the formula

$$2\gamma(G) \leq \gamma_{dR}(G) \leq 3\gamma(G),$$

and [11] we have the following bounds:

$$2\begin{cases} n+2, & n = 2,3,4 \\ 11, & n = 7 \\ \frac{4n+6}{3}, & n \equiv 0,3 \pmod 6 \\ \frac{4n+4}{3}, & n \equiv 2,5 \pmod 6 \\ \frac{4n+8}{3}, & n \equiv 1,4 \pmod 6,\ n > 7 \end{cases} \leq \gamma_{dR}(P_5 \times P_n),$$

$$\gamma_{dR}(P_5 \times P_n) \leq 3\begin{cases} n+2, & n = 2,3,4 \\ 11, & n = 7 \\ \frac{4n+6}{3}, & n \equiv 0,3 \pmod 6 \\ \frac{4n+4}{3}, & n \equiv 2,5 \pmod 6 \\ \frac{4n+8}{3}. & n \equiv 1,4 \pmod 6,\ n > 7, \end{cases}$$

$$4\Big(n - \Big\lfloor \frac{n}{5} \Big\rfloor\Big) \leq \gamma_{dR}(P_6 \times P_n) \leq 6\Big(n - \Big\lfloor \frac{n}{5} \Big\rfloor\Big).$$

4 A case study

In this section we simulate a battle between Romans and barbarians to test efficiency of the double protection versus the ordinary protection (standard Roman domination). Cardinal product $P_4 \times P_n$ is used to model the battlefield, more precisely component K_1. We could use any other cardinal product of graphs, but we use $P_4 \times P_n$ because of its convenience: it is large enough to have multiple outcomes, but not too large for visualization.

Instead of Romans and barbarians, we could have ambulances and patients or firefighters and fires. Ambulances would respond to medical emergencies and firefighters would extinguish fires in their local area. Position of hospitals and fire stations would correspond base vertices, respectively. We are still speaking about Romans and barbarians in order to conform with the usual terminology dealing with dominations. But as we can see, the whole situation has also some modern interpretations, which are more practical and more useful.

First, we will give some basic rules and restrictions to avoid exceptions. The following rules could be easily adapted to ambulances and patients, and to firefighters and fires.

- The simulation is organized in turns. The first turn is played by the barbarians.
- The simulation stops if all cities are destroyed by the barbarians or if all barbarians are defeated by legions and legions are returned to their base cities.
- The legions and the barbarian groups move only by one edge in each turn.
- The barbarian groups destroy an unprotected city if Romans do not send enough help in the next turn.
- If the barbarian group attacks a city with a legion, they fight immediately (no waiting for the next turn).
- The destroyed city stays destroyed, but both the legions and barbarian groups can pass through it.
- If the legions are outnumbered, they all die and no barbarian group dies. An analogous rule holds if the barbarians are outnumbered.
- If there is an equal number of legions and barbarian groups, Romans always win.
- Base cities defend only their direct neighbours.
- A base city does not send help to a neighbour if it cannot send enough help.
- At least one legion must stay in its base city.
- If a direct neighbour is secured, the legion returns to its base city.
- If a city is destroyed, the barbarian group moves to the closest undestroyed city. If there is more then one, then it moves randomly.
- Different barbarian groups move independently.

In the case of double Roman dominations, the initial number of Roman legions and their positions on the graph will be defined as for minimum double Roman domination sets in Theorem 3.7. In the case of standard Roman dominations, the layout of Roman legions will be defined as for minimal Roman domination sets [14], i.e. for $P_4 \times P_n$, $n \equiv 0 \pmod 6$ and K_1 the minimal Roman domination set is:

$$V_1 = \Big\{(1, 6j+5), (4, 6j+2) : j = 0, 1, \ldots, \Big\lfloor \frac{n}{6} \Big\rfloor - 1\Big\} \quad \text{and}$$
$$V_2 = \Big\{(2, 6j+2), (3, 6j+5) : j = 0, 1, \ldots, \Big\lfloor \frac{n}{6} \Big\rfloor - 1\Big\}.$$

Vertices with initial legions are called the base vertices or base cities. The initial number and placement of barbarian groups is arbitrary, but we will put at most 2 barbarian groups into one city. We do not want to destroy all cities in the very beginning.

We consider the placement of the barbarian groups as the first move done by barbarians. The next turn is on the legions. In each turn we have to check the state of each city i.e. the number of barbarian groups and legions and determine their next move. The number of legions and barbarian groups is fixed (it can only decay by turns).

Because on $P_4 \times P_n$ we have two symmetrical components we will consider only K_1. On this component from Theorem 3.7, in the case of double Roman dominations, on the graph $P_4 \times P_{12}$ we have 6 base cities and 3 legions in each of them. So 18 legions defend 24 cities. In case of standard Roman dominations, we have 8 base cities with total of 12 legions.

We have noticed that for $P_4 \times P_{12}$ there exists a second layout for base vertices. It has also total sum of 12 legions, but they are placed differently while still satisfying Roman domination.

In Figure 4, Figure 5 and Figure 6, we see an initial placement for the standard and double protections.

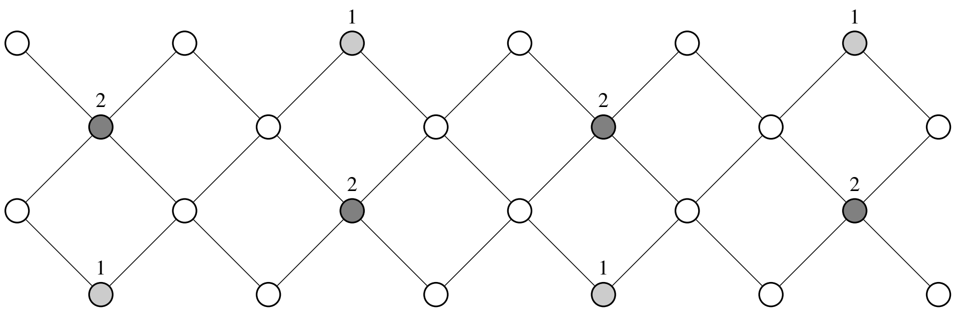

Figure 4: First version of initial placement of legions for $P_4 \times P_{12}$ according to Roman dominating set with the lowest weight [14].

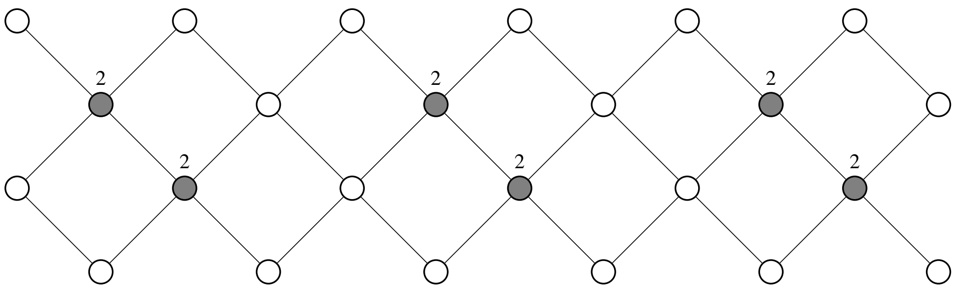

Figure 5: Second version of initial placement of legions for $P_4 \times P_{12}$ according to Roman dominating set with the lowest weight.

Now we will test the both cases simultaneously. For the standard case we will take both layouts into consideration. Further, for a fixed number of barbarians, we will reproduce 30 random possibilities of attack for each case and measure number of destroyed cities and legions. Numbers of destroyed cities and legions will be represented with their arithmetic means.

First, we compare Roman dominating set with the first layout and double Roman dominating set. As shown in Table 1, Roman dominating set of 12 legions can survive at most

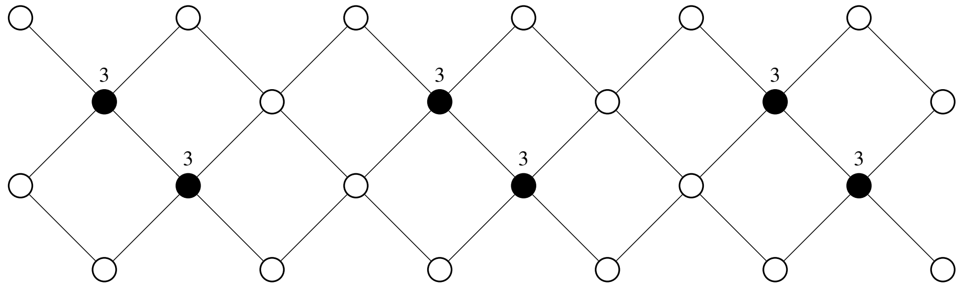

Figure 6: Initial placement of legions for $P_4 \times P_{12}$ according to double Roman dominating set with the lowest weight.

25 barbarian groups according to our simulation, while double Roman dominating set of 18 legions can survive maximum 45 barbarion groups. So 50% more legions can survive 80% more barbarian groups on $P_4 \times P_{12}$ which is efficiency increase of 30%. Also, when all cities and legions are destroyed in case of Roman dominations, only 27% of cities and 9% of legions are destroyed for double Roman dominations.

Table 1: Average number of destroyed cities and legions at the end of the simulations.

Barb. legions	**RD 1. layout** Destroyed cities	**RD 1. layout** Destroyed legions	**RD 2. layout** Destroyed cities	**RD 2. layout** Destroyed legions	**DRD** Destroyed cities	**DRD** Destroyed legions
10	6.67	1.46	3.70	0.50	0.90	0.10
20	20.66	9.13	15.87	6.13	4.53	0.73
25	23.63	11.63	20.97	9.33	6.56	1.66
29	24	12	23.53	11.53	9.80	3.67
30	24	12	24	12	10.53	4.40
40	24	12	24	12	21.53	15.2
45	24	12	24	12	23.83	17.76
46	24	12	24	12	24	18

Second, we compare Roman dominating set with the second layout and double Roman dominating set. Now Roman dominating set of 12 legions can survive at most 29 barbarian groups. So 50% more legions can survive 55% more barbarion groups. The increase in efficiency is considerably less than for the first layout.

What is common to the second layout of Roman dominating set and double Roman dominating set is that base cities are closer and bigger. It means that it is better to have few base cities with higher number of legions than more base cities with smaller number.

5 Conclusion

In this paper bounds for double Roman domination numbers for the cardinal product of any two graphs are given. Also, the exact values are given for the cardinal product of P_2 with any graph, for $P_3 \times P_n$ and for $P_4 \times P_n$. Furthermore, upper and lower bounds for double Roman domination numbers of $P_5 \times P_n$ and $P_6 \times P_n$ are given.

We have also created a case study in wich we have compared Roman domination and double Roman domination on a cardinal product of graphs. The case study has confirmed that double Roman domination is more efficient because for small cost we can multiple protection.

Double Roman domination can be useful even today, not only in military sense. For example, in unsecure parts of a town, where calls for police are common, there should be at least three teams ready to go out after a call. So, when two teams are gone, the thmining team can react to some new call. Such services already exist in emergency medical stations.

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On the Smith normal form of the Varchenko matrix*

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Abstract

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{A}$ isomorphic to $\mathbb{R}^n$. Let V_q be the q-Varchenko matrix for the arrangement $\mathcal{A}$ with all hyperplane parameters equal to q. In this paper, we consider three interesting cases of q-Varchenko matrices associated to hyperplane arrangements. We show that they have a Smith normal form over $\mathbb{Z}[q]$.

Keywords: Hyperplane arrangement, Smith normal form, Varchenko matrix.

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1 Introduction

Let M be an $n \times n$ matrix over a commutative unital ring R. We say that M has a Smith normal form (SNF for short) over R if there are matrices $P, Q \in R^{n\times n}$ such that $\det(P)$ and $\det(Q)$ are units in R and PMQ is a diagonal matrix $\operatorname{diag}(d_1, d_2, \ldots, d_n)$ where d_i divides d_j in R for all $i < j$.

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Recently, there is an interest in SNF in combinatorics. A survey of this topic was given by Stanley in [11]. The SNF of a matrix of a differential operator was considered by Stanley and the first author in [2], where they proved a special case of a conjecture given by Miller and Reiner [7]. In [13], interesting results concerning the SNF of random integer matrix were found.

It is well known that M has an SNF if R is a principal ideal domain (PID), but not much is known for general rings. In this paper we are interested in the integer polynomial ring $\mathbb{Z}[q]$. Some matrices in $\mathbb{Z}[q]^{n\times n}$ do not have an SNF over R. For example, it is not hard to show that $\left[\begin{smallmatrix} 2 & 0 \\ 0 & q \end{smallmatrix}\right]$ does not have an SNF over $\mathbb{Z}[q]$. However, lots of matrices in $\mathbb{Z}[q]^{n\times n}$ do have SNF over $\mathbb{Z}[q]$. For example, it is asked whether every matrix of the form $A = (q^{a_{ij}})$, where a_{ij} are nonnegative integers, has an SNF over $\mathbb{Z}[q]$. There is not a general solution to this question. But we could give a positive answer which arises from some special cases of geometrical structures. The matrices we are interested in are called Varchenko matrices (see [12]). These matrices are associated to a hyperplane arrangement (see Definition 1.2). The Varchenko matrix was studied in the papers of Varchenko [12], Schechtman and Varchenko [8], and Brylawski and Varchenko [1]. These matrices describe the analogue of Serre's relations for quantum Kac-Moody Lie algebras and are relevant to the study of hypergeometric functions and the representation theory of quantum groups [6]. Entries appearing in the diagonal of a Smith normal form of a matrix are called invariant factors. Applications of invariant factors of a q-matrix can be found in [3, 4, 9]. We are going to prove that Varchenko matrices associated to some hyperplane arrangements do have an SNF.

We use the notation and terminology on hyperplane arrangements in [10]. A finite (real) hyperplane arrangement $\mathcal{A}$ is a finite set of affine hyperplanes in some affine space $\mathbb{A}$ isomorphic to $\mathbb{R}^n$.

For a hyperplane H in $\mathcal{A}$, let

$$\mathcal{A}^H = \{H \cap H' : H' \in \mathcal{A} \text{ such that } H' \cap H \neq \emptyset \text{ and } H' \neq H\}.$$

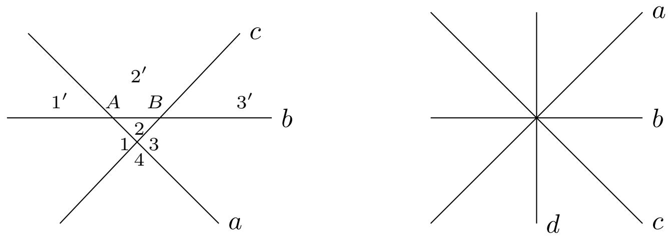
This is a hyperplane arrangement in the affine space H. We also write $\mathcal{A} - \{H\}$ for the arrangement from $\mathcal{A}$ with H removed.

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{A}$. Then $\mathbb{A}$ is divided into some regions by these hyperplanes. Explicitly, a region is a connected component of $\mathbb{A} - \bigcup_{H\in\mathcal{A}} H$. We let $\mathcal{R}(\mathcal{A})$ denote the set of regions of $\mathcal{A}$.

Example 1.1. In the following picture, arrangement $\mathcal{A}_{\mathrm{p}}$ is an example of the so-called peelable arrangement, which is treated in Section 2. Here we see straight lines a, b, c form a hyperplane arrangement in the plane $\mathbb{R}^2$. There are 7 regions of $\mathcal{A}_{\mathrm{p}}$ which we denote by $1', 2', 3', 1, 2, 3, 4$. (We write it in this way for the example in Section 2.) The hyperplane arrangement $\mathcal{A}_{\mathrm{p}}^b$ contains two affine hyperplanes $A = b \cap a, B = b \cap c$ (two points in b).

Arrangement $\mathcal{A}_{\mathrm{p}}$ is also an example of the regular n-gon arrangement $\mathcal{G}_n$, which is treated in Section 4. It is a regular triangle arrangement. (Although in Figure 1 the central triangle is not so much like a equilateral triangle. This does not matter, because the Varchenko matrix that we are concerned with is a topological invariant.) As another example for the n-gon arrangement, a picture of the pentagon arrangement $\mathcal{G}_5$ is given in Section 4.

Arrangement $\mathcal{C}_4$ in Figure 1 is an example of arrangement $\mathcal{C}_n$, which is treated in Section 3.



(a) Arrangement $\mathcal{A}_{\mathrm{p}}$ (b) Arrangement $\mathcal{C}_4$

Figure 1: Arrangements $\mathcal{A}_{\mathrm{p}}$ and $\mathcal{C}_4$.

Definition 1.2. Let $\mathcal{A}$ be a finite hyperplane arrangement and $\mathcal{R}(\mathcal{A})$ its set of regions, and let a_H for $H \in \mathcal{A}$ be indeterminates. The *Varchenko matrix* $V = V(\mathcal{A})$ is indexed by $\mathcal{R}(\mathcal{A})$ with the entries given by

$$V_{RR'} = \prod_{H \in \mathrm{Sep}_{\mathcal{A}}(R,R')} a_H, \tag{1.1}$$

where $\mathrm{Sep}_{\mathcal{A}}(R, R')$ is the set of hyperplanes in $\mathcal{A}$ which separate R and R'. We write $V_q = V_q(\mathcal{A})$ for $V(\mathcal{A})$ when we set each $a_H = q$, an indeterminate, and call V_q the *q-Varchenko matrix* of $\mathcal{A}$.

Thus $(V_q)_{RR'} = q^{\#\mathrm{Sep}(R,R')}$. Also note that $V(\mathcal{A})$ and $V_q(\mathcal{A})$ are symmetric matrices with 1's on the main diagonal.

We are interested mostly in the q-Varchenko matrix V_q. We are going to prove that $V_q(\mathcal{A})$ has an SNF over the ring $\mathbb{Z}[q]$ for the peelable arrangements (in Section 2), arrangement $\mathcal{C}_n$ (in Section 3) and regular n-gon arrangement $\mathcal{G}_n$ (in Section 4). (Since this ring is not a PID, an SNF does not a priori exist.) In Section 5, we compute the SNF of the Varchenko matrices for two arrangements which are not included in the previous sections.

2 Peelable hyperplane arrangements

Example 2.1. Let us look at the arrangement $\mathcal{A}_{\mathrm{p}}$ in Example 1.1. Its Varchenko matrix $V_q = V_q(\mathcal{A}_{\mathrm{p}})$ is

$$V_q = \left[\begin{array}{ccc|ccc|c} 1 & q & q^2 & q & q^2 & q^3 & q^2 \\ q & 1 & q & q^2 & q & q^2 & q^3 \\ q^2 & q & 1 & q^3 & q^2 & q & q^2 \\ \hline q & q^2 & q^3 & 1 & q & q^2 & q \\ q^2 & q & q^2 & q & 1 & q & q^2 \\ q^3 & q^2 & q & q^2 & q & 1 & q \\ \hline q^2 & q^3 & q^2 & q & q^2 & q & 1 \end{array}\right],$$

where the columns are indexed by the regions in the order $1', 2', 3', 1, 2, 3, 4$, and so are the rows. We will briefly show that this matrix has an SNF. We write V_q as a block matrix the way it is partitioned:

$$V_q = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}.$$

Notice that $(B_1, C_1) = q(B_2, C_2)$ and $A_2 = qA_1$. (This is not a coincidence. We see that $[B_1, C_1]$ is the submatrix indexed by $1', 2', 3'$ (rows) and $1, 2, 3, 4$ (columns), while $[B_2, C_2]$ is the submatrix indexed by $1, 2, 3$ (rows) and $1, 2, 3, 4$ (columns). There is one more line, line b, to separate regions i' and j' than regions i and j.) We can multiply by the following matrix on the left to cancel B_1:

$$P = \begin{bmatrix} I_3 & -qI_3 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have

$$PV_q = \begin{bmatrix} A_1 - qA_2 & 0 & 0 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}.$$

As V_q is a symmetric matrix, so is PVP^{t}. We thus have

$$PV_qP^{\mathrm{t}} = \begin{bmatrix} A_1 - qA_2 & 0 & 0 \\ 0 & B_2 & C_2 \\ 0 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} (1-q^2)A_1 & 0 \\ 0 & M_1 \end{bmatrix},$$

where we write

$$M_1 = \begin{pmatrix} B_2 & C_2 \\ B_3 & C_3 \end{pmatrix},$$

and we use that $A_2 = qA_1$. The matrix A_1 is the q-Varchenko matrix of $\mathcal{A}_{\mathrm{p}}^b$. (See Example 1.1 for the notation $\mathcal{A}_{\mathrm{p}}^b$). The matrix M_1 is the q-Varchenko matrix of $\mathcal{A}_{\mathrm{p}} - \{b\}$. We can use induction to transform PV_qP^{t} into an SNF.

This example motivates us to define a peelable hyperplane in an arrangement.

Definition 2.2. Let $\mathcal{A}$ be a finite hyperplane arrangement and H be a hyperplane in $\mathcal{A}$. We say that H is *peelable* (from $\mathcal{A}$) if there is one side H_f of H such that if R is a region of $\mathcal{A}$ and R is in H_f, then $\bar{R} \cap H$ is the closure of a region of $\mathcal{A}^H$.

For example, the hyperplane b is peelable from $\mathcal{A}_{\mathrm{p}}$ in Example 1.1. Let us see why this is. On the side above b there are three regions $1'$, $2'$ and $3'$. For each one of these regions, the intersection of its closure with b is actually a closure of a region of $\mathcal{A}_{\mathrm{p}}^b$. For instance, the closure of region $2'$ intersects b at a line section AB, and this line section is actually a closure of a region of $\mathcal{A}_{\mathrm{p}}^b$. (In fact $\mathcal{A}_{\mathrm{p}}^b$ has 3 regions: the part to the left of A, the part between A and B, and the part to the right of B.)

Theorem 2.3. *Assume that H is peelable from $\mathcal{A}$. Then there is a matrix P with entries in $\mathbb{Z}[q]$ such that $\det(P) = 1$ and*

$$PV_q(\mathcal{A})P^t = \begin{bmatrix} (1-q^2)V_q(\mathcal{A}^H) & 0 \\ 0 & V_q(\mathcal{A} - \{H\}) \end{bmatrix}.$$

Remark 2.4. Under the same assumption, a similar result can be given for the Varchenko matrix $V(\mathcal{A})$, and the proof is almost the same. Using this result, we can prove that the Varchenko matrix $V(\mathcal{A})$ associated to a peelable hyperplane arrangement (as defined below) has a "diagonal form" in $\mathbb{Z}[a_H : H \in \mathcal{A}]$, that is, we can find matrices P, Q whose determinants are units and $PV(\mathcal{A})Q$ is a diagonal matrix. Let us mention that, subsequent to our work, Gao and Zhang [5] gave a necessary and sufficient condition on an arrangment $\mathcal{A}$ for $V(\mathcal{A})$ to have a diagonal form.

The main idea of the proof of this theorem is in the previous example. We will give a rigorous proof in a while, in order to make sure there is no gap that might have occurred when we move from the more visualizable two-dimensional example.

Iteratively using this result, the Varchenko matrices of a special type of hyperplane arrangement can be shown to have an SNF.

Definition 2.5. Let $\mathcal{A} = \{H_1, H_2, \dots, H_m\}$ be a finite hyperplane arrangement. We inductively define $\mathcal{A}$ to be *peelable* as follows.

1. If $m = 1$ then $\mathcal{A} = \{H_1\}$ is peelable.
2. If there is one peelable hyperplane H in $\mathcal{A}$ such that both $\mathcal{A} - \{H\}$ and $\mathcal{A}^H$ are peelable, then we say that $\mathcal{A}$ is peelable.

Now it is easy to see that we have the following result.

Corollary 2.6. *The q-Varchenko matrix $V_q(\mathcal{A})$ of a peelable hyperplane arrangement $\mathcal{A}$ has an SNF over $\mathbb{Z}[q]$. Moreover, its SNF is of the form*

$$\operatorname{diag}((1-q^2)^{n_1}, (1-q^2)^{n_2}, \dots, (1-q^2)^{n_r}),$$

where $0 \le n_1 \le n_2 \le \cdots \le n_r$ is a sequence of nonnegative integers and r is the number of regions of $\mathcal{A}$.

We will need the following two results, which are not hard to prove.

Lemma 2.7. *Let H be a hyperplane in $\mathcal{A}$. Assume that R is a region such that $\bar{R} \cap H$ contains a point which is an interior point of some region R_1 in $\mathcal{A}_H$. Then $\bar{R}_1 = \bar{R} \cap H$.*

Lemma 2.8. *Let H be a hyperplane in $\mathcal{A}$. Assume that R is a region such that $\bar{R} \cap H$ is the closure of some region of $\mathcal{A}_H$. Then there is a unique region R' on the other side of H such that $\bar{R}' \cap H = \bar{R} \cap H$.*

To simplify the wording of the proof of Theorem 2.3, we introduce a new notation.

Definition 2.9. Let $\mathcal{A}$ be a hyperplane arrangement. Let $\mathcal{R}_1, \mathcal{R}_2$ be two subsets of $\mathcal{R}(\mathcal{A})$. We denote by $V_q(\mathcal{R}_1, \mathcal{R}_2)$ the submatrix of $V_q(\mathcal{A})$ with rows indexed by $\mathcal{R}_1$ and column indexed by $\mathcal{R}_2$.

Now let us prove Theorem 2.3. Assume that H is peelable from $\mathcal{A}$ and H_f is a side of H with the properties as in Definition 2.2. Let $R_1, R_2, \dots, R_s$ be the set of the regions in H_f. Let H, $\mathcal{A}$, H_f be as in the Definition 2.2. Let $\mathcal{R}_1' = \{1', 2', \dots, r'\}$ denote the set of regions in H_f, and let $\mathcal{R}_1 = \{1, 2, \dots, r\}$ denote the corresponding regions on the other side of H_f as given by the previous lemma. Let $\mathcal{R}_2 = \{r+1, \dots, r+s\}$ be the set of other regions. Let $\mathcal{R}' = \{1, 2, \ldots, r+t\}$, i.e., $\mathcal{R}'$ is the union of $\mathcal{R}_1$ and $\mathcal{R}_2$. It is not difficult to prove the following facts:

$$\begin{aligned} V_q(\mathcal{R}'_1, \mathcal{R}'_1) &= V_q(\mathcal{A}^H) \\ V_q(\mathcal{R}', \mathcal{R}') &= V_q(\mathcal{A} - \{H\}) \\ V_q(\mathcal{R}'_1, \mathcal{R}') &= qV_q(\mathcal{R}_1, \mathcal{R}') \\ V_q(\mathcal{R}'_1, \mathcal{R}'_1) &= qV_q(\mathcal{R}_1, \mathcal{R}'_1). \end{aligned}$$

The q-Varchenko matrix $V = V(\mathcal{A})$ has the following block matrix form:

$$V_q(\mathcal{A}) = \begin{bmatrix} V_q(\mathcal{R}'_1, \mathcal{R}'_1) & V_q(\mathcal{R}'_1, \mathcal{R}_1) & V_q(\mathcal{R}'_1, \mathcal{R}_2) \\ V_q(\mathcal{R}_1, \mathcal{R}'_1) & V_q(\mathcal{R}_1, \mathcal{R}_1) & V_q(\mathcal{R}_1, \mathcal{R}_2) \\ V_q(\mathcal{R}_2, \mathcal{R}'_1) & V_q(\mathcal{R}_2, \mathcal{R}_1) & V_q(\mathcal{R}_2, \mathcal{R}_2) \end{bmatrix}.$$

Now an argument similar to Example 2.1 can be applied to prove the theorem.

3 The case that all lines go through the same point

From now on, we consider hyperplane arrangements in $\mathbb{R}^2$. Define $\mathcal{C}_n$ to be the arrangement consisting of n lines intersecting in a common point in $\mathbb{R}^2$. We prove that the q-Varchenko matrix $V(n)$ associated to $\mathcal{C}_n$ has a Smith normal form (over $\mathbb{Z}[q]$, as usual). This matrix has the form

$$V(n) = \begin{bmatrix} 1 & q & q^2 & q^3 & \cdots & q^n & q^{n-1} & \cdots & q \\ q & 1 & q & q^2 & \cdots & q^{n-1} & q^n & \cdots & q^2 \\ & & & & \vdots & & & \vdots & \\ q & q^2 & q^3 & q^4 & \cdots & q^{n-1} & q^{n-2} & \cdots & 1 \end{bmatrix}.$$

Remark 3.1. This matrix is an example of circulant matrices $C(c_1, c_2, \ldots, c_n)$ which is defined by

$$C(c_1, c_2, \ldots, c_n) = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_n \\ c_n & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_n & c_1 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_n & c_1 \end{bmatrix}. \tag{3.1}$$

We see that $V(n)$ is circulant because the regions of $\mathcal{C}_n$ are in a circular mode. Similar but more complicated situations occur in the regular n-gon arrangement, which is considered in the next section.

Proposition 3.2. *Let n be a positive integer. Then the Varchenko matrix $V(n)$ has the following Smith normal form over $\mathbb{Z}[q]$:*

$$\mathrm{diag}(1, \underbrace{1-q^2, \ldots, 1-q^2}_{n}, (1-q^2)^2, \underbrace{(1-q^2)(1-q^{2n}), \ldots, (1-q^2)(1-q^{2n})}_{n-2}). \tag{3.2}$$

Proof. First successively apply the row operations $r_i - qr_{i-1}$ $(i = n, n-1, \ldots, 2)$, $r_{n+i} - qr_{n+i+1}$ $(i = 1, 2, \ldots, n-1)$, $r_{2n} - qr_1$. This transforms $V(n)$ into the block matrix

$$\begin{bmatrix} 1 & \alpha & q \\ O & M & O \\ 0 & \beta & 1-q^2 \end{bmatrix},$$

where M is a $2(n-1) \times 2(n-1)$ matrix, α, β are row vectors, O is a zero column vector and β's components are all multiples of $1-q^2$. It's easy to see that we only need to find the Smith normal form of M. Factoring $1-q^2$ out of M, one finds that

$$M = (1-q^2)\begin{bmatrix} A & B \\ B^{\mathrm{t}} & A^{\mathrm{t}} \end{bmatrix},$$

where

$$A = \sum_{k=0}^{n-2} q^k T^k, \quad B = \sum_{k=0}^{n-2} q^{n-1-k} (T^{\mathrm{t}})^k$$

and $T = (t_{ij})$ with $t_{i,j} = \delta_{i+1,j}$. Note that A is a unitriangular matrix; in particular, it is invertible in $\mathbb{Z}[q]$. Multiplying M on the left by

$$P = \begin{bmatrix} I & O \\ -B^{\mathrm{t}} A^{-1} & I \end{bmatrix},$$

we transform M into

$$(1-q^2)\begin{bmatrix} A & B \\ O & A^{\mathrm{t}} - B^{\mathrm{t}} A^{-1} B \end{bmatrix}.$$

We see that we only need to find the Smith normal form of $A^{\mathrm{t}} - B^{\mathrm{t}} A^{-1} B$, but it can be seen from the following lemma that its SNF is

$$\mathrm{diag}(1-q^2, \underbrace{1-q^{2n}, \ldots, 1-q^{2n}}_{n-2}). \tag{3.3}$$

Now the SNF of $V(n)$ follows. $\square$

Lemma 3.3. *Let $m \times m$ matrix $T = (t_{ij})$ with $t_{i,j} = \delta_{i+1,j}$. Let*

$$A = \sum_{k=0}^{m-1} q^k T^k, \quad B = \sum_{k=0}^{m-1} q^{m-k} (T^t)^k.$$

Then the matrix

$$C = (I_m - qT^t)(A^t - B^t A^{-1} B)$$

is equal to a matrix with first row

$$(1-q^2, q^3 - q^{2m+1}, q^4 - q^{2m}, q^5 - q^{2m-1}, \ldots, q^{m+1} - q^{m+3}),$$

the other diagonal entries all equal to $1 - q^{2m+2}$, and all other entries zero.

Proof. First $A^{-1} = I_m - qT$, so

$$B^{\mathrm{t}} A^{-1} = q^m I_m + \sum_{k=1}^{m-1} (q^{m-k} - q^{m+2-k}) T^k.$$

Then one computes $B^{\mathrm{t}}A^{-1}B$ and finds it is equal to

$$M = \begin{bmatrix} q^2 & q^3 - q^{2m+1} & q^4 - q^{2m} & \dots & q^{m+1} - q^{m+3} \\ q^3 & q^4 & q^5 - q^{2m+1} & \dots & q^{m+2} - q^{m+4} \\ q^4 & q^5 & q^6 & \dots & q^{m+3} - q^{m+5} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q^m & q^{m+1} & q^{m+2} & \dots & q^{2m-1} - q^{2m+1} \\ q^{m+1} & q^{m+2} & q^{m+3} & \dots & q^{2m} \end{bmatrix}.$$

Now let $N = (I_m - qT^{\mathrm{t}})M$. We find that the first row of N is the same as that of M, the other diagonal entries of N are all equal to q^{2m+2} and all other entries are zero. Now we see that C is as claimed in the lemma since

$$C = (I_m - qT^{\mathrm{t}})A^{\mathrm{t}} - (I_m - qT^{\mathrm{t}})M = I_m - (I_m - qT^{\mathrm{t}})M = I_m - N. \qquad \square$$

4 The case of regular n-gon arrangement $\mathcal{G}_n$

Let $\mathcal{G}_n$ be the arrangement in $\mathbb{R}^2$ obtained by extending the sides of a regular n-gon. Let $V_q(\mathcal{G}_n)$ be the Varchenko matrix associated to $\mathcal{G}_n$. We are going to prove the following

Theorem 4.1. *Let $V_q(\mathcal{G}_n)$ be the Varchenko matrix associated to the regular n-gon arrangement $\mathcal{G}_n$. A Smith normal form of $V_q(\mathcal{G}_n)$ over $\mathbb{Z}[q]$ is*

$$\mathrm{diag}(1, \underbrace{1 - q^2, \dots, 1 - q^2}_{n}, \underbrace{(1 - q^2)^2, \dots, (1 - q^2)^2}_{(p-1)n}), \tag{4.1}$$

where p is the integer part of $(n+1)/2$.

The above result can be proved by using some results and tools in [3, 12]. But we want to prove it directly. First, it is easy to calculate the number of regions of the arrangement $\mathcal{G}_n$. For instance, one uses the formula that the number of regions is one more than the sum of the number of the lines and the number of intersection points.

Lemma 4.2. *The number of the regions associated to the regular n-gon arrangement P_n is $np + 1$, where p is the integer part of $(n+1)/2$.*

The main idea of the proof of Theorem 4.1 is to group the regions by their shapes. We then write the Varchenko matrix as a block matrix. The columns of each block are labeled by regions of a same shape and so are the rows of a block. For regions of the same shape, we order them clockwise. The key property of this treatment is that each block is a circulant matrix. Once we write the block matrix down, it will be relatively easy to do cancelations and turn it into an SNF, although it takes some space to write the process down. To show how to write the block matrix, we consider the example of $\mathcal{G}_5$. Then in the proof we write the block matrix for general n and then do the cancelation.

Example 4.3. We mark the regions of $\mathcal{G}_5$ (see Figure 2) as in the following.

They are regions $\Delta_i^{(j)}$ ($i = 1, 2, \dots, 5$; $j = 1, 2, 3$) together with a unmarked central region. Note that we mark the regions according to their shape. Precisely, for each j, the shape of the $\Delta_i^{(j)}$ for $i = 1, 2, \dots, 5$ are the same. Let us call them the regions of type

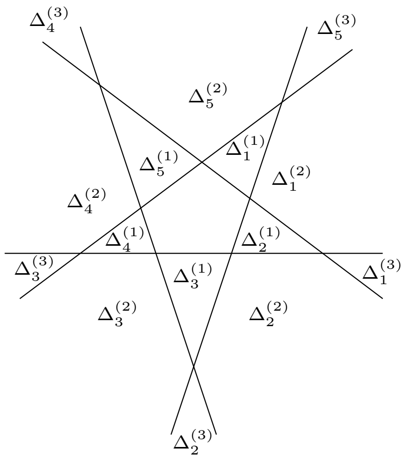


Figure 2: Arrangement $\mathcal{G}_5$.

j. For regions of the same type, we label them clockwise as $\Delta_1^{(j)}, \Delta_2^{(j)}, \dots, \Delta_5^{(j)}$. We call $\Delta_1^{(j)}$ the leading region of the type j regions. The union of the three leading regions $\Delta_1^{(1)}, \Delta_1^{(2)}, \Delta_1^{(3)}$ is the region inside an exterior angle of the pentagon. (So is the union of three region $\Delta_i^{(1)}, \Delta_i^{(2)}, \Delta_i^{(3)}$.) We obtain the Varchenko matrix

$$V_q(\mathcal{G}_5) = \begin{bmatrix} 1 & Q_1 & Q_2 & Q_3 \\ Q_1^{\mathrm{t}} & E_{11} & E_{12} & E_{13} \\ Q_2^{\mathrm{t}} & E_{21} & E_{22} & E_{23} \\ Q_3^{\mathrm{t}} & E_{31} & E_{32} & E_{33} \end{bmatrix},$$

where the first (block) column is indexed by the central non-marked region. For $j = 2, 3, 4$, the jth block column is indexed by the type j regions. The block rows are indexed in the same way. For example, the rows of the matrix E_{12} are indexed by the type 1 regions and the columns of it are indexed by type 2 regions. Because regions of the same type are ordered in a circular mode, the blocks E_{ij} should all be circulant matrices (see Remark 3.1). In fact, it can be checked that the blocks are as follows

$$\begin{aligned} &Q_k = (q^k, q^k, q^k, q^k, q^k) \quad \text{for } k = 1, 2, 3, \\ &E_{11} = C(1, q^2, q^2, q^2, q^2) \quad E_{12} = C(q, q^3, q^3, q^3, q) \quad E_{13} = C(q^2, q^4, q^4, q^2, q^2), \\ &E_{22} = C(1, q^2, q^4, q^4, q^2) \quad E_{23} = C(q, q^3, q^5, q^3, q) \quad E_{33} = C(1, q^2, q^4, q^4, q^2). \end{aligned}$$

We then use Gaussian elimination (in blocks) to turn the matrix into an SNF. For instance, at the beginning, we subtract the q times of the third block row $(Q_2^{\mathrm{t}}\ E_{21}\ E_{22}\ E_{23})$ from the fourth block row $(Q_3^{\mathrm{t}}\ E_{31}\ E_{32}\ E_{33})$.

Proof. We write the Varchenko matrix of $\mathcal{G}_n$ in the following form of block matrix:

$$V_q(\mathcal{G}_n) = \begin{bmatrix} 1 & Q_1 & Q_2 & \dots & Q_p \\ Q_1^{\mathrm{t}} & E_{11} & E_{12} & \dots & E_{1p} \\ Q_2^{\mathrm{t}} & E_{21} & E_{22} & \dots & E_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Q_p^{\mathrm{t}} & E_{p1} & E_{p2} & \dots & E_{pp} \end{bmatrix}$$

where $E_{kl} = E_{lk}^{\mathrm{t}}$, Q_k is the row vector

$$Q_k = \underbrace{(q^k, q^k, \dots, q^k)}_{n} \tag{4.2}$$

and E_{ij} $(i \leq j)$ is a circulant matrix:

$$\begin{aligned} E_{ij} = C\Big(&q^{j-i}, q^{j-i+2}, q^{j-i+4}, \dots, q^{j-i+2(i-1)}, \underbrace{q^{i+j}, \dots, q^{i+j}}_{n+1-i-j}, \\ &q^{j+i-2}, q^{j+i-4}, \dots, q^{j+i-2(i-1)}, \underbrace{q^{j-i}, \dots, q^{j-i}}_{j-i}\Big). \end{aligned}$$

Now we apply Gaussian elimination to $V_q(\mathcal{G}_n)$ and transform it into the desired diagonal form. We do this in blocks and we will use the multiplication of elementary block matrices to realize the elimination. We proceed in four steps.

Step 1: We first apply some row eliminations. Let

$$R_1 = \begin{bmatrix} 1 & & & & \\ -qI_{n\times 1} & I_n & & & \\ & & I_n & & \\ & & & \ddots & \\ & & & & I_n \end{bmatrix} \quad \text{and} \quad R_k = \begin{bmatrix} 1 & & & & & \\ & I_n & & & & \\ & & \ddots & & & \\ & & -qI_n & I_n & & \\ & & & & \ddots & \\ & & & & & I_n \end{bmatrix}$$

for $k \geq 2$, where $I_{n\times 1}$ is a column of n 1's and R_k comes from the (block) identity matrix by adding the $-q$ times of its $(k-1)$th block row to it's kth block row. Now compute the matrix $M_1 = R_1 R_2 \cdots R_p V_q(P_n)$.

Step 2: We apply some column eliminations. Let

$$S_1 = \begin{bmatrix} 1 & -qI_{1\times n} & & & \\ & I_n & & & \\ & & I_n & & \\ & & & \ddots & \\ & & & & I_n \end{bmatrix} \quad \text{and} \quad S_k = \begin{bmatrix} 1 & & & & & \\ & I_n & & & & \\ & & \ddots & & & \\ & & & I_n & -qI_n & \\ & & & & \ddots & \\ & & & & & I_n \end{bmatrix}$$

for $k \geq 2$, where $I_{1\times n}$ is a row of n 1's and S_k comes from the (block) identity matrix by adding the $-q$ times of its $(k-1)$th block column to its kth block column. (So $S_k = T_k^{\mathrm{t}}$.) Now compute the matrix $M_2 = M_1 S_p \cdots S_2 S_1$.

Step 3: We apply some more row eliminations. Let

$$T_k = \begin{bmatrix} {}^{1}I_n & & & \\ & \ddots & & \\ & -qJ & I_n & \\ & & \ddots & \\ & & & I_n \end{bmatrix} \quad \text{with} \quad J = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where T_k come from the (block) identify matrix by adding the $-qJ$ times the kth block row to the $(k+1)$th block row. Now compute $M_3 = T_1 \cdots T_{p-1} M_2$. We find the Varchenko matrix is now transformed to

$$M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & D & N_{12} & N_{13} & \dots & N_{1p-1} & N_{1p} \\ 0 & 0 & D' & N_{23} & \dots & N_{2p-1} & N_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & D' & N_{p-1p} \\ 0 & 0 & 0 & 0 & \dots & 0 & D' \end{bmatrix},$$

where $D = (1-q^2)I_n, D' = (1-q^2)^2 I_n$ and all non-diagonal entries are the multiple of the diagonal entry on the same row. This ensures that we can do the following:

Step 4: We apply more column eliminations to cancel the non-diagonal entries. This does not change the diagonal entries of M_3. We finish the proof as the diagonal of M_3 is the same as that of (4.1). □

5 Two more examples

We now simply say that a hyperplane arrangement $\mathcal{A}$ has SNF if its Varchenko matrix $V_q(\mathcal{A})$ has an SNF over $\mathbb{Z}[q]$. We can use Theorem 2.3 to give more examples of hyperplane arrangements who have SNF. For example, starting from an arrangement which has SNF, for instance C_n, we can keep adding straight lines to it. As long as every time the line added does not separate the set of intersection points of the previous arrangement, the new arrangement will have SNF. This helps us to construct lots of examples of hyperplane arrangements having SNF. We now give two examples which can not be constructed this way. We found that they both have SNF.

1. The *Shi arrangement* $\mathcal{S}_3$ with hyperplanes $x_i - x_j = 0, 1$ for $1 \le i < j \le 3$. We write the multiplicity of a diagonal element in brackets following that entry. For instance, $1 - q^2$ [3] indicates that $1 - q^2$ occurs three times as a diagonal element of the SNF. The diagonal elements of the SNF of $V_q(\mathcal{S}_3)$ are 1 [1], $1 - q^2$ [6], $(1-q^2)^2$ [6], and $(1-q^2)(1-q^6)$ [3].
2. Define a hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^3$ by the equations $x = 0$, $y = 0$, $z = 0$, $x - y - z = 0$. We verified that its q-Varchenko matrix has an SNF over $\mathbb{Z}[q]$, with diagonal entries 1 [1], $1-q^2$ [4], $(1-q^2)^2$ [6], $(1-q^2)^3$ [2], and $(1-q^2)^2(1-q^8)$ [1].

Based on the previous examples, it is natural to consider the following problem.

Problem 5.1. Do all hyperplane arrangements have SNF?

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Frobenius groups which are the automorphism groups of orientably-regular maps*

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Abstract

In this paper, we prove that a Frobenius group (except for those which are dihedral groups) can only be the automorphism group of an orientably-regular chiral map. The necessary and sufficient conditions for a Frobenius group to be the automorphism group of an orientably-regular chiral map are given. Furthermore, these orientably-regular chiral maps with Frobenius automorphisms are proved to be normal Cayley maps. Frobenius groups conforming to these conditions are also constructed.

Keywords: Frobenius group, (orientably) regular map, automorphism group.

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1 Introduction

Maps are 2-cell embeddings of graphs in compact, connected surfaces. A flag of a map is a topological triangle whose corners are a vertex, the midpoint of an edge incident with the vertex, and the midpoint of a face incident to both the vertex and the edge. Thus, the supporting surface of any map can be decomposed into flags (considered as closed discs bounded by the triangles).

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It is well known that the automorphism group of a map acts semi-regularly on its flags. If the automorphism group of a map is regular on the flags, then the map is called regular. Regular maps have the largest automorphism groups, acting regularly on flags of the map. Similarly, orientably-regular maps are maps in orientable surfaces that have the largest orientation preserving automorphism groups acting regularly on darts (edges with direction).

Regular and orientably-regular maps constitute the most meaningful generalization of the Platonic solids. Early recognition of the importance of regular maps in modern mathematics goes back to Kepler [12]; more recent development of the theory of maps was closely related to the theory of map colorings, with the topic of highly symmetric maps always at the center of interest. The study of regular maps is nowadays considered one of the 'classical' areas of mathematics (e.g., Heffter [7], Klein [13], Dyck [5], or Burnside [3]).

A group G acting on a set X is said to act *regularly*, if for any pair of elements $x, y \in X$ there exists a unique element $g \in G$ mapping x to y, $x^g = y$. In such a case, X can be identified with the elements of G, and consequently, any mathematical structure with an automorphism group acting regularly on its base set can be identified with the group itself, the building blocks of the structure being identified with cosets of stabilizers of some blocks. This identification has been used in the theory of regular and orientably-regular maps as well and we just sum up the basics, referring for details to [11] and [2] for the theory of maps on orientable and on general surfaces, and to [16] for a recent survey of the theory of regular and orientably-regular maps. In all the forthcoming group presentations we will assume that the listed exponents are the *true orders* of the corresponding elements.

A finite regular map $\mathcal{M}$ can in this way be identified with a (partial) three-generator presentation of a finite group G, isomorphic to the automorphism group $\mathrm{Aut}(\mathcal{M})$ of $\mathcal{M}$, of the form

$$G = \langle x, y, z \mid x^2, y^2, z^2, (xy)^2, (yz)^\ell, (zx)^m, \ldots \rangle \tag{1.1}$$

where the dots indicate possible presence of additional relators (at least one if the carrier surface of the map is not simply connected). In particular, all vertices of $\mathcal{M}$ have degree ℓ and all the face boundary walks in $\mathcal{M}$ have length m; we will often refer just to *face length* m. The pair (ℓ, m) is the *type* of the regular map $\mathcal{M}$. In such a representation of $\mathcal{M}$, its flags are elements of G, the darts are (say) right cosets of the subgroup $\langle x \rangle$, while edges, vertices and faces are right cosets of the dihedral subgroups $\langle x, y \rangle$, $\langle y, z \rangle$ and $\langle z, x \rangle$ of order 4, 2ℓ and $2m$, respectively. The three generators x, y, z correspond to involutory automorphisms of $\mathcal{M}$ taking a fixed flag onto its three neighboring flags, and the three dihedral subgroups correspond to the edge-, vertex- and face-stabilizers of $\mathcal{M}$.

We will write $\mathcal{M} = \mathrm{Map}(G; x, y, z)$ to formally identify a regular map $\mathcal{M}$ with a group presentation as in (1.1). The algebraic situation with finite orientably-regular maps is similar. Each such map $\mathcal{M}$ can be identified with a partial two-generator presentation of a group H, isomorphic to the group $\mathrm{Aut}^+(\mathcal{M})$ of orientation-preserving automorphisms of $\mathcal{M}$, of the form

$$H = \langle \rho, \lambda \mid \rho^\ell, \lambda^2, (\rho\lambda)^m, \ldots \rangle \, . \tag{1.2}$$

Here, elements of H represent darts of $\mathcal{M}$; right cosets of the cyclic groups $\langle \lambda \rangle$, $\langle \rho \rangle$ and $\langle \rho\lambda \rangle$ represent edges, vertices and faces of $\mathcal{M}$. The generators λ and ρ, stabilizing an edge e and a vertex v incident to e, represent a half-turn of $\mathcal{M}$ about the center of e and a $2\pi/\ell$ turn of $\mathcal{M}$ about v in accord with a chosen orientation of the carrier surface of the map. Again, the pair (ℓ, m) is the type of the map, and we will use the notation $\mathcal{M} = \mathrm{Map}(H; \rho, \lambda)$ in this case.

If a regular map $\mathcal{M} = \mathrm{Map}(G; x, y, z)$ is orientable (meaning that its carrier surface is orientable), $\mathcal{M}$ is also orientably-regular, with $\mathrm{Aut}^+(\mathcal{M}) = \langle \rho, \lambda \rangle$ for $\lambda = xy$ and $\rho = yz$. In fact, a regular map $\mathrm{Map}(G; x, y, z)$ is orientable if and only if the subgroup $\langle xy, yz \rangle$ has index 2 in G. Reversing this line of thought, an orientably-regular map $\mathcal{M} = \mathrm{Map}(H; \rho, \lambda)$ may also be regular. It happens if and only if the map admits an orientation-reversing automorphism, which (see e.g. [16]) is equivalent to the existence of an automorphism of H that fixes λ and inverts ρ. In such a case we call the orientably-regular map $\mathcal{M}$ *reflexible*; otherwise, that is, when $H \cong \mathrm{Aut}^+(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})$, the map is called *chiral*.

A *Cayley graph* $\mathrm{Cay}(G, X)$ is a graph whose vertex set can be identified with the elements of a group G generated by a set X closed under taking inverses and not containing the identity 1_G, with the pairs of adjacent vertices consisting of all pairs g, gx with $g \in G$ and $x \in X$. A graph Γ is isomorphic to a Cayley graph $\mathrm{Cay}(G, X)$ if and only if $\mathrm{Aut}\,\Gamma$ contains a subgroup G acting regularly on the vertices of Γ [15]. A *Cayley map* is an orientable map $\mathcal{M}$ that admits a group of orientation preserving automorphisms G acting regularly on its set of vertices. Therefore, the underlying graphs of Cayley maps are Cayley graphs. It turns out that many of the orientably-regular maps obtained in the forthcoming sections fall in the class of Cayley maps the theory of which (without regularity assumptions) was initiated in [14] and further developed e.g. in [8] and [4].

An *orientably-regular Cayley map* can therefore be distinguished by $\mathcal{M} = \mathrm{Map}(H; \rho, \lambda)$, where $H = J\langle \rho \rangle$ for some subgroup $J \le H$ such that $J \cap \langle \rho \rangle = 1$, vertices of $\mathcal{M}$ are right cosets of $\langle \rho \rangle$ in H, and the underlying graph of $\mathcal{M}$ is a Cayley graph $\mathrm{Cay}(J, S)$ for some unit-free inverse-closed generating set S of J. In the even more special instance when J is normal in H, i.e., when H is a semi-direct product $J \rtimes \langle \rho \rangle$, we speak about a *normal* (orientably-regular) Cayley map. In this case, conjugation by ρ induces an automorphism $\hat{\rho}$ of J and its restriction $\pi = \pi_{\hat{\rho}}$ to S is a cyclic permutation of S. It turns out that either all elements in S are involutions, or none of them is and then $s^{-1} = s\hat{\rho}^{\ell/2} = \rho^{-\ell/2} s \rho^{\ell/2}$ for every $s \in S$, where ℓ is the order of ρ (necessarily even in this case). Moreover, since we also know that $J\langle \rho \rangle = \langle \lambda, \rho \rangle$, the involution λ can be taken to be equal to an arbitrary element of S in the all-involutions case, or to $s\rho^{\ell/2}$ for an arbitrary $s \in S$ if no element in S is involutory.

In our paper we address the natural question whether for a given finite Frobenius group G there exists some orientably-regular or even regular map whose automorphism group is G. In Section 2, we list some properties of Frobenius groups which we will refer to in Section 3. In Section 3, necessary and sufficient conditions (Theorems 3.3, 3.5 and 3.6) for a Frobenius group to be the automorphism group of an orientably-regular chiral map are given. The Frobenius groups conforming to these conditions are also constructed.

2 Frobenius groups

A Frobenius group is a transitive permutation group G on a set Ω which is not regular on Ω, but has the property that the only element of G which fixes more than one point is the identity element. It was shown by Thompson [17, 18] that a finite Frobenius group G has a nilpotent normal subgroup K, called the Frobenius kernel, which acts regularly on Ω. Thus, K is the direct product of its Sylow subgroups and G is the semidirect product $K \rtimes H$, where H is the stabilizer of a point of Ω. Because of the vertex transitivity of the action, any two point stabilizers are conjugate. As a result, every point stabilizer has the form $(hk)^{-1}H(hk) = k^{-1}Hk = H^k$ for some $h \in H$ and $k \in K$. Each point stabilizer is called a Frobenius complement of K in G, so the choice of Frobenius complement is not unique. Because of the regularity of K acting on Ω, one may identify Ω with K such that K acts on itself by multiplication. Moreover, Gorenstein [6, pp. 38, 339] showed that every element of $H \setminus \{1\}$ induces an automorphism of K by conjugation which fixes only the identity element of K. Combining all these results we give a lemma to express the relation between a Frobenius group and its Frobenius kernel as well as its Frobenius complements.

Lemma 2.1. *Let $G = K \rtimes H$ be a Frobenius group, where K is the Frobenius kernel and H is a Frobenius complement. Then, G can be divided in the following two ways.*

(1) $G = \cup_{k \in K} Hk$, *where* $Hk_1 \cap Hk_2 = \emptyset$ *for any two different elements* $k_1, k_2 \in K$*;*

(2) $G = (\cup_{k \in K} H^k) \cup K$, *where* $H^k = k^{-1}Hk$ *denotes the conjugation of H by k, and* $H^{k_1} \cap H^{k_2} = H^k \cap K = \{1\}$ *for any elements k_1, k_2, k in K and $k_1 \neq k_2$.*

Given a(several) Frobenius group(s), one can get new Frobenius groups. In the following Lemmas 2.3 and 2.4, we give two methods to get new Frobenius groups from original ones.

Lemma 2.2 ([19, Lemma 3.8, p. 13])**.** *Assume A, B are two groups and B acts on A. If A has a subgroup P which is invariant under the action of B, $(|B|, |P|) = 1$ and $(Pa)^b = Pa$ for some $a \in A$ and each $b \in B$, then there is an element $x \in Pa$ such that $x^b = x$ for every $b \in B$.*

Lemma 2.3. *Assume $G = K \rtimes H$, $1 < N < K$ and $N \trianglelefteq G$. Then G is a Frobenius group with H as a Frobenius complement if and only if both $N \rtimes H$ and $K/N \rtimes H$ are Frobenius groups with H as a Frobenius complement.*

Proof. Assume $G = K \rtimes H$ is a Frobenius group with H as a Frobenius complement. It is obvious that $N \rtimes H$ is a Frobenius group with H a Frobenius complement. So we only need to show that $K/N \rtimes H$ is a Frobenius group. If not, then there is an $h \in H$ such that h fixes some non-identity element of K/N. That is, there is an element $k \in K$ but $k \notin N$ such that Nk is fixed by h. Consider $K \rtimes \langle h \rangle$. From Lemma 2.2, there is an element $x \in Nk$ which is fixed by h. It is obvious that $x \neq 1$, and so h fixes at least two elements in K. This contradicts the assumption of G being a Frobenius group.

Conversely, assume both $N \rtimes H$ and $K/N \rtimes H$ are Frobenius groups with H as a Frobenius complement. If G is not a Frobenius group, then there exists $1 \neq k \in K$ and $1 \neq h \in H$ such that $k^h = k$. Since $N \rtimes H$ is a Frobenius group, $k \notin N$. Thus $Nk \neq \bar{1}$ in K/N. Clearly, $(Nk)^h = Nk^h = Nk$. This contradicts the assumption of $K/N \rtimes H$ being a Frobenius group. □

Lemma 2.4. *Let $K_1 \rtimes H$ and $K_2 \rtimes H$ be two Frobenius groups. Then, $(K_1 \times K_2) \rtimes H$ is a Frobenius group, where H acts on $K_1 \times K_2$ by $(k_1k_2)^h = k_1^h k_2^h$, for any elements $k_1 \in K_1, k_2 \in K_2$ and $h \in H$.*

Proof. Note that each non-identity element $h \in H$ fixes exactly the identity element of $K_1 \times K_2$. □

Lemma 2.5. *Let $G = K \rtimes H$ be a Frobenius group. For each $g \in G \setminus K$, it satisfies the following two relations:*

(1) $\langle g \rangle \cap K = \{1\}$*;*

(2) *As an element in the quotient group G/K, Kg has order $o(g)$, where $o(g)$ denotes the order of g in group G.*

Proof. According to Lemma 2.1, there is an element $k \in K$ such that $g \in H^k$. So $\langle g\rangle \cap K \leq H^k \cap K = \{1\}$. As a result, $o(Kg) = |K\langle g\rangle/K| = |\langle g\rangle|/\langle g\rangle \cap K| = |\langle g\rangle| = o(g)$. □

Corollary 2.6. *Let $G = K \rtimes H$ be a Frobenius group. For each $h \in H, h \neq 1$ and for each $k \in K$, the orders of h, kh and hk are equal, that is $o(h) = o(kh) = o(hk)$.*

Proof. Note that $Kh = K(kh) = K(hk)$, so $o(h) = o(kh) = o(hk)$ according to Lemma 2.5(2). □

Remark 2.7. If a group $G = N \rtimes P$ is not a Frobenius group, then it may not satisfy the results in Lemma 2.5. For example, take $G = \mathrm{SL}_2(3) \cong Q_8 \rtimes \mathbb{Z}_3$. Let $x = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. There $x \in G \setminus Q_8$, $o(x) = 6$. But $\langle x\rangle \cap Q_8 = \langle x^3\rangle \neq 1$ and $o(Q_8x) = 3 \neq o(x)$.

3 Maps having Frobenius groups as automorphism groups

The following Lemma 3.1 will be referred to several times in this paper. The result is known and one can prove it very quickly. But for easy reference, we give a short proof.

Lemma 3.1. *Let G be a finite group. If there is an involution $\tau \in \mathrm{Aut}(G)$ such that τ only fixes the identity element of G, then τ maps each element in G to its inverse and G is an abelian group of odd order.*

Proof. According to the property of τ, one can check that $G = \{g^{-1}g^\tau \mid g \in G\}$. Clearly $(g^{-1}g^\tau)^\tau = (g^{-1}g^\tau)^{-1}$. It follows that τ maps each element in G to its inverse. So, for any two elements $a, b \in G$, one can get $(ab)^\tau = b^{-1}a^{-1} = a^\tau b^\tau = a^{-1}b^{-1}$. That is to say, G is an abelian group. Since τ only fixes the identity element of G, the group G does not have involutions. Thus, G is of odd order. □

If $a \in L$, then we use $\langle a\rangle^L$ to denote the group generated by the elements $x^{-1}ax$ for $x \in L$.

Theorem 3.2. *Other than dihedral groups of order $2n$ for any odd integer n, Frobenius groups cannot be the automorphism groups of regular maps.*

Proof. Let $G = K \rtimes H$ be a Frobenius group. If G can be the automorphism group of a regular map, then G has the following generating relations:

$$G = \langle x, y, z \mid x^2, y^2, z^2, (xy)^2, (yz)^k, (zx)^m, \ldots\rangle,$$

where $2, k, m$ are the true orders of xy, yz and zx, respectively. As $G/K = \langle Kx, Ky, Kz\rangle \cong H$, $|H|$ is even. By Lemma 3.1, K is an abelian group of odd order and H has a unique involution. Consequently, $H \cong \mathbb{Z}_2$. Moreover, H is a Sylow-2 subgroup of G.

It is easy to see that $\langle x, y\rangle$ is a 2-group. So $|\langle x, y\rangle| \leq |H| = 2$. Thus $\langle x, y\rangle = \langle xy\rangle$. It follows that $x = 1$ or $y = 1$. In either case, G is a dihedral group of order $2n$ for some odd integer n. In this situation, the map is an embedding of a semi-star of valency n in the sphere or the disc, or the dual of the latter, an embedding of a circuit of length n in the boundary of the disc. It is obvious that these two infinite families of maps are reflexible with their full automorphism groups being the dihedral groups of order $2n$. Apart from these two infinite families of maps, the only other possibilities for Frobenius automorphism groups are orientably-regular chiral maps. □

According to Theorem 3.2, we only need to concentrate on Frobenius groups which can be automorphism groups of orientably-regular chiral maps. There are several well-known infinite families of examples of these, such as the embeddings of complete graphs K_n [9], Paley graphs, and generalized Paley graphs [10]. In the following Theorem 3.3, we will give the necessary conditions that a Frobenius group $G = K \rtimes H$ should satisfy to be the automorphism group of an orientably-regular chiral map.

Theorem 3.3. *Let $G = K \rtimes H$ be a Frobenius group. If $G = \langle \rho, \lambda \mid \rho^k, \lambda^2, (\rho\lambda)^m, \ldots \rangle$, $k, m \geq 3$, is the automorphism group of an orientably-regular chiral map $\mathcal{M} = \mathrm{Map}(G; \rho, \lambda)$, then one of the following two cases will happen.*

(1) H is a cyclic group of even order and K is an abelian group of odd order. There are two subcases corresponding to the parity of k.

(1.1) If k is even, then $H \cong \mathbb{Z}_k$ and

$$m = \begin{cases} \frac{k}{2}, & \text{if } k \equiv 2 \pmod 4, \\ k, & \text{if } k \equiv 0 \pmod 4. \end{cases}$$

Moreover, the map $\mathcal{M}$ is an orientably-regular normal Cayley map of K. When $k \equiv 2 \pmod 4$, $\mathcal{M}$ has $\frac{|G|}{k}$ vertices, $\frac{|G|}{2}$ edges, $\frac{2|G|}{k}$ faces and the genus of the corresponding orientable surface is $1 - \frac{|G|(6-k)}{4k}$; when $k \equiv 0 \pmod 4$, $\mathcal{M}$ has $\frac{|G|}{k}$ vertices, $\frac{|G|}{2}$ edges, $\frac{|G|}{k}$ faces and the genus of the corresponding orientable surface is $1 - \frac{|G|(4-k)}{4k}$.

(1.2) If k is odd, then $H \cong \mathbb{Z}_{2k}$ and $m = 2k$. The map $\mathcal{M}$ is an orientably-regular normal Cayley map of a group isomorphic to $K \rtimes \mathbb{Z}_2$. In this situation, $\mathcal{M}$ has $\frac{|G|}{k}$ vertices, $\frac{|G|}{2}$ edges, $\frac{|G|}{2k}$ faces, so the genus of the corresponding orientable surface is $1 - \frac{|G|(3-k)}{4k}$.

(2) H is a cyclic group of odd order and $H \cong \mathbb{Z}_k$, K is a 2-group and $m = k$. In this situation, $\mathcal{M}$ is an orientably-regular normal Cayley map of K. The map $\mathcal{M}$ has $\frac{|G|}{k}$ vertices, $\frac{|G|}{2}$ edges, $\frac{|G|}{k}$ faces and the genus of the corresponding orientable surface is $1 - \frac{|G|(4-k)}{4k}$.

Proof. (1): If $|H|$ is even, then there is an involution in $\mathrm{Aut}(K)$ which only fixes the identity element. So, K is an abelian group of odd order by Lemma 3.1. In this case, $\lambda \notin K$ and so $o(K\lambda) = 2$ in the quotient group G/K. Note that $\rho \notin K$. Otherwise, by Corollary 2.6 one can get $o(\rho\lambda) = o(\lambda) = 2$, that is $m = 2$. So, $o(K\rho) = o(\rho) = k$ by Lemma 2.5. According to Lemma 3.1, there is only one involution in $H \cong G/K = \langle K\rho, K\lambda \rangle$, so $K\lambda$ belongs to the center of G/K which is therefore abelian.

(1.1): If k is even, then $K\lambda \in \langle K\rho \rangle$. So, $\langle K\rho, K\lambda \rangle = \langle K\rho \rangle$ and $H \cong \mathbb{Z}_k$. According to Lemma 2.1, one can assume $H = \langle \rho \rangle$ and $\lambda = a\rho^{\frac{k}{2}}$ for some non-identity element $a \in K$ without loss of generality. The vertices of $\mathcal{M}$ can be looked as the cosets of H.

Therefore, K acts regularly on the vertices of $\mathcal{M}$ which implies that $\mathcal{M}$ is an orientably-regular Cayley map of K. Now, we know that $\mathrm{Aut}(\mathcal{M}) = K \rtimes \langle\rho\rangle$. So, $\mathcal{M}$ is normal and from the construction method of $\mathcal{M}$ from G, one can get the corresponding Cayley subset $\{a, a^{\rho}, a^{\rho^2}, \ldots, a^{\rho^{k-1}}\}$. In this case, $K = \langle a\rangle^H$.

Since $K\lambda = K\rho^{\frac{k}{2}}$, $K\rho K\lambda = K\rho^{\frac{k}{2}+1}$. If $k \equiv 2 \pmod 4$, then $m = o(\rho\lambda) = o(K\rho K\lambda) = o(K\rho^{\frac{k}{2}+1}) = \frac{k}{2}$. The type of $\mathcal{M}$ is $(k, \frac{k}{2})$. Moreover, $\mathcal{M}$ has $\frac{|G|}{k}$ vertices, $\frac{|G|}{2}$ edges, $\frac{2|G|}{k}$ faces and the genus of the corresponding orientable surface is $1 - \frac{|G|(6-k)}{4k}$. If $k \equiv 0 \pmod 4$, then $m = k$ and consequently the type of $\mathcal{M}$ is (k, k). And $\mathcal{M}$ has $\frac{|G|}{k}$ vertices, $\frac{|G|}{2}$ edges, $\frac{|G|}{k}$ faces and the genus of the corresponding orientable surface is $1 - \frac{|G|(4-k)}{4k}$.

(1.2): If k is odd, then $K\lambda$ belongs to the center of G/K which is therefore abelian. So, $\langle K\rho, K\lambda\rangle = \langle K\rho K\lambda\rangle$ and as a result $H \cong \mathbb{Z}_{2k}$. Because $\rho \notin K$, according to Lemma 2.1, we may assume $\rho \in H$ and $H = \langle\tilde{\rho}\rangle$ with $\rho = \tilde{\rho}^2$. As a result, $\lambda = a\tilde{\rho}^k$ for some non-identity element $a \in K$. Because $\lambda\rho = a\tilde{\rho}^{k+2}$, it follows that $m = o(\lambda\rho) = o(\tilde{\rho}^{k+2}) = 2k$ according to Lemma 2.6. The type of $\mathcal{M}$ in this subcase is therefore $(k, 2k)$.

Let $\tilde{H} = \langle\rho\rangle$ be the index two subgroup of H and $\tilde{K} = K \rtimes \langle\tilde{\rho}^k\rangle \cong K \rtimes \mathbb{Z}_2$. It is clear that $G = \tilde{K} \rtimes \tilde{H}$. Now, $\lambda \in \tilde{K}$ and so we have the relations $G = \langle\rho, \lambda\rangle \leq \langle\lambda\rangle^{\langle\rho\rangle}\langle\rho\rangle \leq \tilde{K}\tilde{H} = G$. Therefore, $\tilde{K} = \langle\lambda\rangle^{\tilde{H}}$.

The vertices of $\mathcal{M}$ can be looked as the cosets of $\tilde{H}$. Therefore, $\tilde{K}$ acts regularly on the vertices of $\mathcal{M}$ which implies that $\mathcal{M}$ is an orientably-regular normal Cayley map of $\tilde{K}$ with corresponding Cayley subset $\{\lambda, \lambda^{\rho}, \lambda^{\rho^2}, \ldots, \lambda^{\rho^{k-1}}\}$. In this case, $\mathcal{M}$ has $\frac{|G|}{k}$ vertices, $\frac{|G|}{2}$ edges, $\frac{|G|}{2k}$ faces, so the genus of the corresponding orientable surface is $1 - \frac{|G|(3-k)}{4k}$.

(2): If $|H|$ is odd, then $\lambda \in K$ and so $H \cong G/K = \langle K\rho\rangle$ is cyclic. Similar to (1.1), we can assume $H = \langle\rho\rangle$ and $\mathcal{M}$ is an orientalby-regular normal Cayley map of K with the corresponding Cayley subset $\{\lambda, \lambda^{\rho}, \ldots, \lambda^{\rho^{k-1}}\}$. Also in this case $K = \langle\lambda\rangle^H$. It is known that K is nilpotent, so the Sylow-2 subgroup P of K is a characteristic subgroup of G. Note that $G = \langle\lambda, \rho\rangle = \langle\lambda\rangle^G\langle\rho\rangle \leq P \rtimes \langle\rho\rangle \leq K \rtimes \langle\rho\rangle$. So, $K = \langle\lambda\rangle^G = P$ is a 2-group. According to Corollary 2.6, $o(\rho\lambda) = o(\rho) = k$ and so the type of $\mathcal{M}$ is (k, k). The map $\mathcal{M}$ has $\frac{|G|}{k}$ vertices, $\frac{|G|}{2}$ edges, $\frac{|G|}{k}$ faces and the genus of the corresponding orientable surface is $1 - \frac{|G|(4-k)}{4k}$. □

In the proof of Theorem 3.3, for a Frobenius group $G = K \rtimes H$ that can be the automorphism group of an orientably-regular chiral map, we have described the relations between K and H. To be more clear, we rewrite these relations in Corollary 3.4.

Corollary 3.4. *Let $G = K \rtimes H$ be a Frobenius group. If $G = \langle\rho, \lambda \mid \rho^k, \lambda^2, (\rho\lambda)^m, \ldots\rangle$ is the automorphism group of an orientably-regular chiral map* $\mathrm{Map}(G; \rho, \lambda)$*, then one of the following three cases will happen:*

(1) k is even, $H = \langle\rho\rangle \cong \mathbb{Z}_k$, K is an abelian group and $K = \langle\lambda\rho^{\frac{k}{2}}\rangle^H$;

(2) k is odd, $H \cong \mathbb{Z}_{2k}$, K is abelian and $G = \tilde{K} \rtimes \tilde{H}$, where $\tilde{K} \cong K \rtimes \mathbb{Z}_2$, $\tilde{H} = \langle\rho\rangle$ is the index two subgroup of H and $\tilde{K} = \langle\lambda\rangle^{\tilde{H}}$;

(3) k is odd, $H = \langle\rho\rangle \cong \mathbb{Z}_k$ and $K = \langle\lambda\rangle^H$ is a 2-group.

In the following Theorems 3.5 and 3.6, we will show that a Frobenius group whose Frobenius kernel and Frobenius complement conforming to the conditions in Corollary 3.4

can be the automorphism group of an orientably-regular normal Cayley map which implies that the conditions are also sufficient.

Theorem 3.5. *Let $G = K \rtimes H$ be a Frobenius group, where K is abelian and $K = \langle x\rangle^H$ for some $x \in K$, $H = \langle y\rangle$ is cyclic of order $2n, n \geq 2$. Then, there is an orientably-regular normal Cayley map $\mathcal{M}$ such that $G = \mathrm{Aut}(\mathcal{M})$ and the type of $\mathcal{M}$ is*

$$(k,m) = \begin{cases} (2n,n) \text{ or } (n,2n), & \text{if } n \text{ is odd,} \\ (2n,2n), & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $\rho = y, \lambda = xy^n$. Then $G = \langle \rho, \lambda\rangle$. It is clear that $o(\rho) = 2n, o(\lambda) = 2$, $o(\rho\lambda) = n$ if n is odd and $o(\rho\lambda) = 2n$ if n is even. So, G is the automorphism group of an orientably-regular map $\mathcal{M}$ of type $(2n,n)$ or $(2n,2n)$ depending on whether n is odd or even. Because $H = \langle y\rangle = \langle\rho\rangle$, it follows that the vertex set consists of the cosets of H in G. So, K acts regularly on the vertex set of $\mathcal{M}$ and as a result $\mathcal{M}$ is an orientably-regular normal Cayley map of K.

When n is odd, if we set $\rho = y^2, \lambda = xy^n$, then $o(\rho) = n, o(\lambda) = 2$ and $o(\rho\lambda) = 2n$. We claim that $G = \langle\rho,\lambda\rangle$. Set $Q = \langle\rho,\lambda\rangle$, then $Q = \langle y^2, xy\rangle$ because n is odd. From the requirement of $n \geq 2$, we have $y^2 \neq 1$ and so $C_K(y^2) = 1$ in the Frobenius group G. A calculation shows that the commutator $[y^2, x^{-1}] = (xy)^{y^2}(xy)^{-1} \in Q$. Also, $[y^2, x^{-1}]$ belongs to K. Note that K is abelian. We have

$$\begin{aligned} Q \geq [y^2, x^{-1}]^{\langle xy\rangle} &= [y^2, x^{-1}]^{K\langle xy\rangle} = [y^2, x^{-1}]^{K\langle y\rangle} \\ &= [y^2, x^{-1}]^{\langle y\rangle} = \langle [y^2, (x^{-1})^g] \mid g \in \langle y\rangle\rangle. \end{aligned}$$

Define a function $\sigma\colon K \to K$ such that $b^\sigma = [y^2, b]$ for each $b \in K$. Now,

$$(b_1b_2)^\sigma = [y^2, b_1b_2] = [y^2, b_2][y^2, b_1]^{b_2} = [y^2, b_1][y^2, b_2] = b_1^\sigma b_2^\sigma.$$

From $C_K(y^2) = 1$, one can get $\sigma \in \mathrm{Aut}(K)$. Therefore,

$$\langle [y^2, (x^{-1})^g] \mid g \in \langle y\rangle\rangle = \langle ((x^{-1})^g)^\sigma \mid g \in \langle y\rangle\rangle = (\langle x^{-1}\rangle^{\langle y\rangle})^\sigma = K^\sigma = K.$$

So, $K \leq Q$ and $\langle xy\rangle K \leq Q$. Consequently, $Q = G$. Let $\tilde{K} = K \rtimes \langle y^n\rangle$ and $\tilde{H} = \langle\rho\rangle$. Then, $\tilde{K} \cong K \rtimes \mathbb{Z}_2$, $\tilde{H}$ is the index two subgroup of H and $G = \tilde{K} \rtimes \tilde{H}$. Therefore, G is the automorphism group of an orientably-regular normal Cayley map of $\tilde{K}$ of type $(n, 2n)$. □

Theorem 3.6. *Let $G = K \rtimes H$ be a Frobenius group, where K is a 2-group and $K = \langle x\rangle^H$ for some involution $x \in K$, $H = \langle y\rangle$ is cyclic of order n for some odd integer n. Then, there is an orientably-regular normal Cayley map $\mathcal{M}$ such that $G = \mathrm{Aut}(\mathcal{M})$ and the type of $\mathcal{M}$ is (n,n).*

Proof. Let $\rho = y, \lambda = x$. Then $G = \langle\rho,\lambda\rangle$. It is clear that $o(\rho) = n, o(\lambda) = 2, o(\rho\lambda) = n$. So, G is the automorphism group of an orientably-regular map $\mathcal{M}$ of type (n,n). Because $H = \langle y\rangle = \langle\rho\rangle$, it follows that the vertex set consists of the cosets of H in G. So, K acts regularly on the vertex set of $\mathcal{M}$ and as a result $\mathcal{M}$ is an orientably-regular normal Cayley map of K. □

Corollary 3.7. *Let $K_1 \rtimes H$ and $K_2 \rtimes H$ be Frobenius groups, where $K_1 = \langle x_1 \rangle^H$, $K_2 = \langle x_2 \rangle^H$ are both abelian groups whose orders are coprime with each other, $H \cong \langle y \rangle$ and $o(y) = 2n$ for some integer $n \geq 2$. Then, the following two results follow from Lemma 2.4 and Theorem 3.5.*

- *$(K_1 \times K_2) \rtimes H$ is a Frobenius group, $K_1 \times K_2 = \langle x_1 x_2 \rangle^H$ and for each $a_1 \in K_1, a_2 \in K_2, b \in H$ the element b acts on $a_1 a_2$ in the way $(a_1 a_2)^b = a_1^b a_2^b$,*
- *$(K_1 \times K_2) \rtimes H$ is the automorphism group of an orientably-regular normal Cayley map.*

According to Theorem 3.5 and Corollary 3.7, one may concentrate on Frobenius groups whose Frobenius kernels are p-groups and satisfy the conditions in Theorem 3.5. Now, we want to give an example of Frobenius groups satisfying the conditions in Theorem 3.5.

In a finite group G, for each element $g \in G$ we use $C_G(g)$ to denote the centralizer of g in G, that is $C_G(g) = \{h \in G \mid hg = gh\}$.

Example 3.8. Let $K = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_d \rangle$, where $o(a_i) = p^{e_i}$, p is an odd prime number and these positive integers $e_i, 1 \leq i \leq d$, satisfy $e_1 \geq e_2 \geq \cdots \geq e_d$. Let $H \cong \mathbb{Z}_k = \langle b \rangle$ for some positive even integer k satisfying $k \mid p-1$. Assume $d \leq \phi(k)$, where ϕ is the Euler's totient function, t_i is a positive integer such that $t_i + p^{e_i}\mathbb{Z}$ is an element in $\mathbb{Z}_{p^{e_i}}^*$ of order k and $t_i + p\mathbb{Z} \neq t_j + p\mathbb{Z}$ for any $1 \leq i \neq j \leq d$. Set $G = K \rtimes H$, where $a_i^b = a_i^{t_i}$, then G is a Frobenius group. Take $a = \prod_{i=1}^d a_i$, then

$$K = \langle a \rangle^H = \langle a, a^b, \ldots, a^{b^{d-1}} \rangle.$$

Proof. To show that G is a Frobenius group, we only need to show that for each element $y \in H \setminus \{1\}$, the equality $C_K(y) = 1$ holds. Suppose $x = \prod_{i=1}^d x_i \in C_K(y)$, where $x_i \in \langle a_i \rangle$. It is obvious that $\prod_{i=1}^d x_i = \prod_{i=1}^d x_i^y$. From the defining relation $a_i^b = a_i^{t_i}$, then $\langle a_i \rangle$ is an H-invariant subgroup, and so $x_i = x_i^y$ for each i. That is $x_i \in C_{\langle a_i \rangle}(y)$. While y is a power of b and the action of b on $\langle a_i \rangle$ has only one fixed point, that is the identity of $\langle a_i \rangle$, so $C_{\langle a_i \rangle}(y) = 1$.

For each $1 \leq \ell \leq d-1$, $a^{b^\ell} = \prod_{i=1}^d a_i^{t_i^\ell}$. If we look at the determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_d \\ \vdots & \vdots & \vdots & \vdots \\ t_1^{d-1} & t_2^{d-1} & \cdots & t_d^{d-1} \end{vmatrix}$$

in the finite field F_p, then from the choices of t_i this is a non-zero Vandermonde determinant. As a result,

$$a + \Phi(K), a^b + \Phi(K), \ldots, a^{b^{d-1}} + \Phi(K)$$

is a basis of the linear space $K/\Phi(K)$, where $\Phi(K)$ is the Frattini subgroup of K. From the Burnside basis theorem, the result $K = \langle a, a^b, \ldots, a^{b^{d-1}} \rangle$ follows. □

Corollary 3.9. *Let K and H be groups in Example 3.8. Then, the Frobenius group $G = K \rtimes H$ is the automorphism group of an orientably-regular normal Cayley map described in Theorem 3.5.*

Lemma 3.10. *Let A be a group, B be a subgroup of A of index 3, and each $a \in A \backslash B$, $a^3 = 1$. Then, $[b, b^a] = 1$ for any $b \in B, a \in A \setminus B$.*

Proof. Note that if $b \in B$ and $a \in A \setminus B$, then $ba \in A \setminus B$. So, $(ba)^3 = 1$ and $bab = a^{-1}b^{-1}a^{-1}$. The commutator $[b, b^a] = b^{-1}a^{-1}b^{-1}aba^{-1}ba = b^{-1}(a^{-1}b^{-1}a^{-1}) \cdot (a^{-1}ba^{-1})ba = b^{-1}(bab)(b^{-1}ab^{-1})ba = a^3 = 1$. □

Corollary 3.11. *Let $G = K \rtimes H$ be a Frobenius group. If G satisfies the following two conditions:*

(1) G can be generated by two elements,

(2) $H \cong \mathbb{Z}_3$,

then K is abelian. Moreover, if G is the automorphism group of an orientably-regular map, then K is isomorphic to the Klein group K_4 and G is isomorphic to the alternating group A_4.

Proof. Assume $G = \langle a, b\rangle$ and $a \notin K$. By Lemma 2.1 and Corollary 2.6, $a^3 = 1$ and so $G = K \cup Ka \cup Ka^2$. As a result, one of the three elements b, ba^{-1}, ba^{-2} must belong to K. Suppose $b \in K$, then $G = \langle a, b\rangle = \langle a\rangle\langle b\rangle^G \leq \langle a\rangle K = G$. Because $\langle a\rangle \cap K = 1$, it follows that $K = \langle b\rangle^G$. While $\langle b\rangle^G = \langle b, b^a, b^{a^2}\rangle$, so K is abelian according to Lemma 3.10.

If G is the automorphism group of an orientably-regular map, then without loss of generality we can assume $H \cong \langle a\rangle$. So, $K = \langle b, b^a, b^{a^2}\rangle$ is a 2-group according to Theorem 3.3. The fact of K being abelian implies that the rank $d(K)$ of K satisfies $d(K) \leq 3$. Therefore, $K \cong \mathbb{Z}_2^{d(K)}$. Moreover, from $3 \mid |K| - 1$, one can get $d(K) = 2$ and K is isomorphic to K_4 and $G \cong A_4$. □

Remark 3.12. In Corollary 3.11, the condition $K = \langle b\rangle^H$ for some element $b \in K$ is necessary. In fact, one may check the list of small groups to find $\mathrm{SmallGroup}(192, 1023)$ in MAGMA [1] to get a Frobenius group satisfying $K = \langle a, b\rangle^H$ for two different elements a and b of K, $H \cong \mathbb{Z}_3$, but K is not abelian.

According to Theorem 3.3, if the Frobenius group $G = K \rtimes H$ is the automorphism group of an orientably-regular map and $|H|$ is odd, then K is a 2-group. By Corollary 3.11, in order to find a non-abelian 2-group as the Frobenius kernel, the smallest order of the Frobenius complement is 5.

Theorem 3.13. *Let $G = K \rtimes H$ be a Frobenius group. If K is a non-abelian 2-group, H is a cyclic group of odd order and G is the automorphism group of an orientably-regular map, then the group G of the smallest order is* $\mathrm{SmallGroup}(1280, 1116310)$ *in* MAGMA.

Proof. Since K is a non-abelian 2-group, its commutator subgroup K' is non-trivial and is a proper subgroup of K. Set $|H| = n$. Because $K/K' \rtimes H$ and $K' \rtimes H$ are both Frobenius groups, $n \mid (|K/K'| - 1) = 2^{n_1} - 1$ and $n \mid (|K'| - 1) = 2^{n_2} - 1$ for some integers n_1 and n_2. According to Corollary 3.11, n is an odd integer but $n \neq 3$.

If $n = 5$, then the smallest choices of n_1, n_2 are 4 and in this case $|G| = 2^8 \times 5$. The Frobenius group satisfying these conditions really exists. It is $\mathrm{SmallGroup}(1280, 1116310)$ in the list of groups in MAGMA.

We claim that no Frobenius groups of order less than $2^8 \times 5$ with non-abelian 2-groups as Frobenius kernels, cyclic groups of odd orders as Frobenius complements, exist that can be automorphism groups of orientably-regular maps. Otherwise, suppose a group $G = \langle \rho, \lambda \mid \rho^n, \lambda^2, \ldots \rangle$ satisfies these conditions. Then, $n = 7$, $n_1 = n_2 = 3$ and $|G| = 2^6 \times 7 = 448$. It is $\mathrm{SmallGroup}(448, 1394)$ in the list of groups of MAGMA. But, its Frobenius kernel is abelian which is a contradiction. □

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