On constructing expander families of G-graphs[*]

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Abstract

Like Cayley graphs, G-graphs are graphs that are constructed from groups. A method for constructing expander families of G-graphs is presented and is used to construct new expander families of irregular graphs. This technique depends on a relation between some known expander families of Cayley graphs and certain expander families of G-graphs. Several other properties of expander families of G-graphs are presented.

Keywords: Cayley graph, diameter of a graph, abelian group, G-graph, expander family.

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1 Introduction

Expander graphs are sparse graphs that have strong connectivity properties. Expander constructions have found extensive applications in computer science [13, 16], in constructing of algorithms, error correcting codes [12], random walks [23], and sorting networks [1]. If one chooses at random a family of d-regular graphs, it is almost certain to be an expander graph [10]. Nevertheless, constructing expander families is not an easy task. Most constructions use deep algebraic and combinatorial techniques; mainly through Cayley graphs and the Zig-Zag product (see for example [15, 19]).

Like Cayley graphs, G-graphs are defined from groups, but they correspond to an alternative construction. These graphs, introduced in [6], have highly regular properties. In particular, because the algorithm for constructing G-graphs is simple, it appears to be a useful tool to construct new symmetric and semi-symmetric graphs [7]. Several extensively studied problems in graph theory such as the hamiltonicity of Cayley graphs (see e.g. [3, 18] for the latest development on this problem) may as well be approached using these objects. For instance, G-graphs are used to characterize new classes of Hamiltonian Cayley graphs [4], and to improve some upper bounds in the cage graphs problem [6]. Recently in [9], the authors studied some robustness properties of G-graphs such as edge/vertex-connectivity and vertex/edge-transitivity. It turns out, that several families of G-graphs are optimally connected where an optimally connected graph can be thought of as a graph whose vertex-connectivity is equal to its minimum degree. Because of their nice properties, it is natural to consider the problem of constructing an expander family of G-graphs.

One of the chief tools for constructing a family of expander graphs is the concept of Cayley graphs. The main advantage for using such graphs is that at first it enables us when fixing the size of the generating set, to construct a large family of sparse graphs in an effective and concise way. Additionally, the underlying properties of a group G and its generating set S can give us an insightful gaze on the expansion properties of its corresponding Cayley graph $\mathrm{Cay}(G, S)$. Generally speaking, it is hard to prove that a certain family of Cayley graphs is an expander family. Concerning this, a huge amount of research in the last few decades has been devoted to dealing with the following question: which sequence of groups corresponds to an expander family of Cayley graphs? Using some algebraic techniques that depend mainly on Kazhdan constant, many partial results were obtained. In fact, most of these results gave negative answers to this question for certain groups (see [14] and [17], see also Example 3.2 below). The purpose of this article is to present a technique for constructing such families. Our construction is based on a relation between some known expander families of Cayley graphs and certain expander families of G-graphs. More precisely, for a group G and a subset S of G with $S^* = \bigcup_{s \in S} \langle s \rangle \setminus \{e\}$ (i.e. if $S = \{s_1, \ldots, s_k\}$, then $S^* = \{s_1, \ldots, s_1^{o(s_1)-1}, \ldots, s_k, \ldots, s_k^{o(s_k)-1}\}$, where $o(s_i)$ denotes the order of s_i), we prove the following main result (see below for terminology).

Theorem 1.1. *If $\{\mathrm{Cay}(G_n, S_n^*), n \in \mathbb{N}^+\}$ is an expander family, then $\{\tilde{\Phi}(G_n, S_n), n \in \mathbb{N}^+\}$ is also an expander family.*

The rest of the paper is organized as follows. In Section 2, we give a review of some basic facts concerning groups, multigraphs, G-graphs and expander graphs that are needed for our purposes. In Section 3, we shall prove the preceding theorem. In addition, just like in the case of Cayley graphs, we prove that abelian groups can not yield an expander family of G-graphs. In Sections 4 and 5, we first identify a new method for generating

an infinite regular family of Cayley graphs from another one by switching specific edges. This leads to a new infinite expander family of Cayley graphs on the projective special linear group $\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z})$. Consequently, we construct several new infinite families of expander G-graphs on the special linear group $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ and projective special linear group $\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z})$. These families are formed of irregular graphs, in particular semi-regular, which are of the very few ones.

2 Preliminaries

This section has been designed to give a general introduction to some of the basic facts needed from the theory of groups, multigraphs, expanders and G-graphs. This introduction is given here to provide a convenient repository for all readers. We discuss briefly the material we shall require from these theories and for more details on any of these subjects, see for example [2, 11, 14, 17, 20].

2.1 Groups

Throughout this paper, all groups are assumed to be finite. Let $(G, ., e)$ be a group, where e denotes the identity element of G and "." denotes the group operation (multiplicative notation). For every g in G we define the order of g, denoted by $o(g)$, as the smallest integer l such that $g^l = e$. Let $S = \{s_1, \ldots, s_k\}$ be a non-empty subset of G, and let $O_{\max}(S)$ and $O_{\min}(S)$ be respectively the maximum and the minimum $o(s_i)$ for all $i \in \{1, \ldots, k\}$. A subset S of G is said to be *symmetric* if every element in S has its inverse in S. We define $\langle S \rangle$ to be the smallest subgroup of G which contains S. If $\langle S \rangle = G$, then set S is said to be a *generating set* of G, or G is *generated* by S. If H is a subgroup of G then the set Hx is called right coset of H in G, and we denote by G/H to be the set of all right cosets of H in G. A subset T_H of G is said to be a *right transversal* for H if T_H contains exactly one element from each right coset of H in G. Let A and B be subsets of a set U, then we denote $B \setminus A = \{x \in B \text{ and } x \notin A\}$ and $\overline{A} = U \setminus A$. The special linear group $\mathrm{SL}(2, \mathbb{Z}/q\mathbb{Z})$ is defined as follows:

$$\mathrm{SL}(2, \mathbb{Z}/q\mathbb{Z}) = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}; \ a_1, a_2, a_3, a_4 \in \mathbb{Z}/q\mathbb{Z} \text{ and } a_1 a_4 - a_2 a_3 = 1 \right\}.$$

The projective special linear group $\mathrm{PSL}(2, \mathbb{Z}/q\mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z}/q\mathbb{Z})/\{\pm I_2\}$, where I_2 is the 2×2 identity matrix.

2.2 Multigraphs

All multigraphs considered in this paper are undirected and finite. Generally, we define an *undirected multigraph* Γ as the triple $(V(\Gamma), E(\Gamma), \xi_\Gamma)$, where $V(\Gamma)$ is the *set of vertices*, $E(\Gamma)$ is the *set of edges*, and ξ_Γ is an incidence function that associates with each edge of Γ an unordered pair of vertices of Γ. In addition, we denote by $\{u, v\}$ the multi-edge that links vertices u and v. The multiplicity of the multi-edge $\{u, v\}$ is the cardinality of the set of edges that links u and v. A multi-edge with identical end-points is called a *loop*. A multigraph is a *simple graph* if it has neither loops nor multi-edges with multiplicity greater than or equal to 2.

The *neighborhood* of vertex u denoted by $N(u)$ is the set of all vertices that are adjacent to u. The *degree of a vertex* v in a multigraph Γ, denoted by $d(v)$ is the number of edges

of Γ incident to v where each loop counts as two edges. The *maximum* and *minimum degree* of a multigraph Γ are denoted by $\Delta(\Gamma)$ and $\delta(\Gamma)$ respectively. A multigraph Γ is d-regular if $d(u) = d$ for all $u \in V(\Gamma)$. A family of d-regular multigraphs is formed of regular multigraphs where each has degree d, while a family of regular multigraphs is formed of regular multigraphs with possibly varying degrees. The *distance* $d(u, v)$ between two vertices u and v is the number of edges in a shortest path that connects u and v. The *diameter* $\operatorname{diam}(\Gamma)$ of a multigraph Γ is defined by:

$$\operatorname{diam}(\Gamma) = \max\{d(u, v); u, v \in V(\Gamma)\}.$$

Let $\Gamma_1 = (V_1, E_1, \xi_1)$ and $\Gamma_2 = (V_2, E_2, \xi_2)$ be two multigraphs, a homomorphism from Γ_1 to Γ_2 is a couple $(f, f^\#)$ where $f \colon V_1 \to V_2$ and $f^\# \colon E_1 \to E_2$ such that if $\xi_1(a) = \{u, v\}$ then $\xi_2(f^\#(a)) = \{f(u), f(v)\}$. A graph isomorphism is the couple $(f, f^\#)$ where f and $f^\#$ are bijective. We say that Γ_1 is isomorphic to Γ_2 if there exists an isomorphism between Γ_1 and Γ_2. In such a case, we write $\Gamma \simeq \Gamma'$.

A multigraph $\Gamma = (V, E, \xi_\Gamma)$ is k-*partite* if there is a partition of V into k parts such that each part is a stable set. We will write $\Gamma = (\bigsqcup_{i \in I} V_i; E)$ where $I = \{1, \ldots, k\}$. A multigraph is *minimum k-partite* $(k \geq 1)$ if it is k-partite and not $(k-1)$-partite. It is easy to verify that for any multigraph Γ, there exists k such that Γ is minimum k-partite. If a multigraph Γ is k-partite, then we will say that $(V_i)_{i \in \{1,2,\ldots,k\}}$ is a k-representation of Γ.

Cayley graphs offer a combinatorial depiction of groups and their generators. More precisely, the Cayley graph $\operatorname{Cay}(G, S)$ is the multigraph with vertex set G and two elements x and y of G are adjacent if and only if $y = s.x$ for some $s \in S$. It is well-known that $\operatorname{Cay}(G, S)$ is connected if and only if $G = \langle S \rangle$ (see for example [14]).

2.3 G-graphs

Definition 2.1. Let $(G, ., e)$ be a finite group. Let S be a nonempty subset of G. For $s \in S$, consider the cycles $(s)x = (x, sx, \ldots, s^{o(s)-1}x)$ of permutation $g_s \colon x \mapsto sx$. Note that $\langle s \rangle x$ is the set $\{x, sx, \ldots, s^{o(s)-1}x\}$. We define the G-graph $\Phi(G, S)$ in the following way:

1. The vertex set of $\Phi(G, S)$ is $V = \bigsqcup_{s \in S} V_s$ where $V_s = \{(s)x, x \in T_{\langle s \rangle}\}$ where $T_{\langle s \rangle}$ is a right transversal for the subgroup $\langle s \rangle$.

2. For each $(s)x, (t)y \in V$, there exists edge between $(s)x$ and $(t)y$ labeled g for each $g \in \langle s \rangle x \cap \langle t \rangle y$, such an edge will be denoted by $(\{(s)x, (t)y\}, g)$. If $\operatorname{card}(\langle s \rangle x \cap \langle t \rangle y) = p$, $p \geq 1$ then there exists p labeled edges between $(s)x$ and $(t)y$, or $\{(s)x, (t)y\}$ is a multi-edge with multiplicity p.

Since the cosets of $\langle s \rangle$ form a partition of G, $(V_s)_{s \in S}$ is a $|S|$-representation of $\Phi(G, S)$. Every vertex $(s)x$ has $o(s)$ loops. We denote by $\tilde{\Phi}(G, S)$ the multigraph $\Phi(G, S)$ without loops. The multigraph $\tilde{\Phi}(G, S)$ is the simple graph underlying $\Phi(G, S)$, that is, the vertices $(s)x$ and $(t)y$ in $V(\tilde{\Phi}(G, S))$ are connected by a single edge if $\langle s \rangle x \cap \langle t \rangle y$ is non-empty. If $S = \{s_1, \ldots, s_k\}$ then the *level* of any s_i, noted V_{s_i} (or simply V_i), is the stable set of $\tilde{\Phi}(G, S)$ which comprises all the vertices of the form $(s_i)x$ where $x \in G$. Note that each level V_s contains $\frac{|G|}{o(s)}$ vertices, therefore we have the following relation:

$$|V(\tilde{\Phi}(G, S))| = |G| \sum_{s \in S} \frac{1}{o(s)}.$$

The *principal clique*[1] of $x \in G$, denoted by C_x, is the subgraph of $\tilde{\Phi}(G, S)$ induced by the set of vertices which contain x. In $\tilde{\Phi}(G, S)$ there are $|G|$ principal cliques; each contains $|S|$ vertices.

Example 2.2. Let G be the cyclic group of order 6, i.e. $G = \{e, a, a^2, a^3, a^4, a^5\}$. Clearly G can be generated by an element of order 3 and another of order 2. Let S be $\{a^2, a^3\}$. Then the vertices of the corresponding G-graph without loops $\tilde{\Phi}(G, S)$ are

$$(a^2)e = (e, a^2e, a^4e) = (e, a^2, a^4), \quad (a^2)a = (a, a^2a, a^4a) = (a, a^3, a^5)$$

which are the 3-cycles and

$$(a^3)e = (e, a^3), \quad (a^3)a = (a, a^3a) = (a, a^4), \quad (a^3)a^2 = (a^2, a^3a^2) = (a^2, a^5)$$

which are the 2-cycles. Obviously, in this case the multigraph $\tilde{\Phi}(G, S)$ is isomorphic to $K_{2,3}$ (Figure 1). The levels V_{a^2} and V_{a^3} are respectively $\{(a^2)e, (a^2)a\}$ and $\{(a^3)e, (a^3)a, (a^3)a^2\}$. There are 6 principal cliques each of size $|S| = 2$. For instance, the principal cliques C_e and C_a are the induced subgraphs of $\tilde{\Phi}(G, S)$ with vertex set $\{(a^2)e, (a^3)e\}$ and $\{(a^2)a, (a^3)a\}$ respectively.
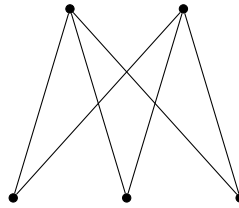

Figure 1: The bipartite multigraph $K_{2,3}$.

The next 3 propositions can be found in [5].

Proposition 2.3 ([5])**.** $\Phi(G, S)$ *and* $\tilde{\Phi}(G, S)$ *are minimum* $|S|$*-partite graphs.*

Proposition 2.4 ([5])**.** $\tilde{\Phi}(G, S)$ *is connected if and only if* S *is a generating set of* G.

Proposition 2.5 ([5])**.** *Let* $\tilde{\Phi}(G, S) = (\bigsqcup_{s \in S} V_s; E)$ *be a* G*-graph with* $|G| = n$ *and* $|S| = k$*. Then the following holds.*

$$d((s)x) = o(s)(k - 1), \quad \text{for all } (s)x \in V_s,$$

$$\sum_{(s)x \in V_s} d((s)x) = n(k - 1), \quad \text{for all } s \in S,$$

$$|E(\tilde{\Phi}(G, S))| = \frac{nk(k - 1)}{2}.$$

2.3.1 New results on G-graphs

Proposition 2.6. *Let* $\tilde{\Phi}(G, S)$ *be any* G*-graph such that* $|S| = \{s_1, \ldots, s_k\}$*. Then the following are equivalent:*

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[1]This definition is due to [4].

    i. $\tilde{\Phi}(G,S)$ *is d-regular,*

    ii. $o(s_i) = \frac{d}{k-1}$, *for all* $i \in \{1, \ldots, k\}$,

    iii. $|V_{s_i}| = |V_{s_j}|$, *for all* $i, j \in \{1, \ldots, k\}$.

Proof. Let $(s)x \in V_s$, where $s \in S$. From Proposition 2.5, we have

$$d((s)x) = o(s)(k-1) \quad \text{or} \quad o(s) = \frac{d((s)x)}{k-1},$$

and then

$$|V_s| = \frac{|G|}{o(s)} = \frac{|G|(k-1)}{d((s)x)}.$$

Therefore $o(s_i) = o(s_j)$ if and only if $|V_{s_i}| = |V_{s_j}|$, for all $i, j \in \{1, \ldots, k\}$. $\qquad \square$

Remark 2.7. When $\tilde{\Phi}(G,S)$ is a regular multigraph, we use the notation O instead of $o(s)$ for any $s \in S$.

    The following lemma can be found in [22].

Lemma 2.8 ([22]). *Let* $\Phi(G,S)$ *be a G-graph with* $S = \{s_1, \ldots, s_k\}$ *a generating set of* G, *then all the multi-edges between levels* V_{s_i} *and* V_{s_j} *have the same multiplicity* $|\langle s_i \rangle \cap \langle s_j \rangle|$.

    As a result, we have the following corollary.

Corollary 2.9. *Let* $\tilde{\Phi}(G,S)$ *be a G-graph with* $S = \{s_1, \ldots, s_k\}$. *Then* $\tilde{\Phi}(G,S)$ *is a simple graph if and only if* $\langle s_i \rangle \cap \langle s_j \rangle = \{e\}$ *for all* $i, j \in \{1, \ldots, k\}$ *with* $i \neq j$.

2.4 Expanders

Before we define expander graphs, we need to define some expansion parameters. Let $\Gamma = (V, E, \xi_\Gamma)$ be a non-oriented multigraph with $|V| \geq 2$ and V' be a subset of V. The *edge boundary* of V' in Γ denoted by $\partial V'(\Gamma)$ (or simply $\partial V'$ when no ambiguity occurs) is defined as follows:

$$\partial V'(\Gamma) = \{\alpha \in E; \; \xi_\Gamma(\alpha) \in V' \times \bar{V}'\}.$$

In other words, this is the set of edges emanating from the set V' to its complement. The *rate of expansion* of Γ is then defined as follows:

$$h(\Gamma) = \min_{0 < |V'| \leq \frac{|V|}{2}} \frac{|\partial V'|}{|V'|}.$$

Definition 2.10. For $\epsilon \in \mathbb{R}_+^*$, a multigraph Γ is said to be an ϵ-*expander* if

$$\epsilon \leq h(\Gamma).$$

Definition 2.11. If a family of multigraphs $\{\Gamma_i = (V_i, E_i, \xi_i), \; i \in \mathbb{N}^+\}$ satisfies the following three conditions:

    i. $|V_i| \to \infty$ as $i \to \infty$,

ii. There exists $r \in \mathbb{N}^+$ such that $\Delta(\Gamma_i) \leq r$ for all $i \in \mathbb{N}^+$. That is $\{\Gamma_i, \ i \in \mathbb{N}^+\}$ is a sequence of bounded degree multigraphs,

iii. There exists $\epsilon \in \mathbb{R}_+^*$ such that Γ_i is an ϵ-expander for all $i \in \mathbb{N}^+$,

then this family is called an *expander family* and an element of this family is an *expander graph*.

If Γ is a d-regular multigraph, then in [14] it is proved that $\log_d |V(\Gamma)| \leq \mathrm{diam}(\Gamma)$. The next proposition is a simple generalization of this result.

Proposition 2.12. *Let Γ be a connected multigraph such that $\Delta(\Gamma) \leq r \in \mathbb{N}^+$. Then*

$$\log_r |V(\Gamma)| \leq \mathrm{diam}(\Gamma).$$

Proof. Consider $v \in V(\Gamma)$ and define $B_l(v) = \{u \in V(\Gamma); \ d(v, u) \leq l\}$. We show by induction that $|B_l(v)| \leq r^l$. The result is trivial for $l = 0$. Suppose it is true up to $l - 1$ and let's prove it for l. Since every vertex in $B_{l-1}(v)$ has at most $r - 1$ neighbors in $\overline{B_{l-1}(v)}$, then $|B_l(v)| \leq (r-1)|B_{l-1}(v)| + |B_{l-1}(v)| = r|B_{l-1}(v)| \leq rr^{l-1} = r^l$. If $l = \mathrm{diam}(\Gamma)$, then $B_l(v) = V(\Gamma)$ and therefore $|V(\Gamma)| \leq r^{\mathrm{diam}(\Gamma)}$. □

3 $\mathbb{C}\mathrm{ay}$-expanders and $\mathbb{G}$-expanders

In this section, we are mainly concerned with proving Theorem 1.1. First, we need to introduce more auxiliary materials. We start with the following definition which is virtually an interpretation of Definition 2.11 for the G-graph and Cayley graph cases.

Definition 3.1. Let $\{G_i, \ i \in \mathbb{N}^+\}$ be a family of finite groups. We say that $\{G_i, \ i \in \mathbb{N}^+\}$ is a $\mathbb{G}$-*expander family*, if for every $i \in \mathbb{N}^+$ there exists a generating subset S_i of G_i such that $\{\tilde{\Phi}(G_i, S_i), \ i \in \mathbb{N}^+\}$ is an expander family. More precisely, $\{G_i, \ i \in \mathbb{N}^+\}$ is a $\mathbb{G}$-*expander family* if the following 3 conditions are satisfied:

i. $|V(\tilde{\Phi}(G_i, S_i))| = |G_i| \sum_{s \in S_i} \frac{1}{o(s)} \to \infty$ as $i \to \infty$.

ii. There exists a positive integer r such that $\Delta(\tilde{\Phi}(G_i, S_i)) \leq r$ for all $i \in \mathbb{N}^+$ which by Proposition 2.5 means that for every $(s)x \in V_s$ we have $d((s)x) = (|S_i| - 1)o(s) \leq \Delta(\tilde{\Phi}(G_i, S_i)) \leq r \in \mathbb{N}^+$ for all $i \in \mathbb{N}^+$. This in turn means that there exists $r_1, r_2 \in \mathbb{N}^+$ such that $2 \leq |S_i| \leq r_1$ and $o(s) \leq r_2$ for all $s \in S_i$ and for all $i \in \mathbb{N}^+$. In addition, since $\Delta(\tilde{\Phi}(G_i, S_i)) \leq r$ for all $i \in \mathbb{N}^+$, then clearly Condition i. is equivalent to saying that $|G_i| \to \infty$ as $i \to \infty$.

iii. There exists an $\epsilon \in \mathbb{R}_+^*$ such that $\epsilon \leq h(\tilde{\Phi}(G_i, S_i))$ for all $i \in \mathbb{N}^+$.

Note that $2 \leq |S_i|$ since otherwise $\tilde{\Phi}(G_i, S_i)$ will be a disconnected multigraph so that $h(\tilde{\Phi}(G_i, S_i)) = 0$, and so it is clear that $\max\{r_1, r_2\} \leq r$.

On the other hand, we say that $\{G_i, \ i \in \mathbb{N}^+\}$ is a $\mathbb{C}\mathrm{ay}$-*expander family*, if for every $i \in \mathbb{N}^+$ there exists a symmetric generating subset S_i of G_i such that $\{|S_i| : i \in \mathbb{N}^+\}$ is uniformly bounded and provided that $\{\mathrm{Cay}(G_i, S_i), \ i \in \mathbb{N}^+\}$ is an expander family. More explicitly, $\{G_i, i \in \mathbb{N}^+\}$ is a $\mathbb{C}\mathrm{ay}$-*expander family* if the following 2 conditions are satisfied:

i. $|V(\mathrm{Cay}(G_i, S_i))| = |G_i| \to \infty$ as $i \to \infty$,

ii. There exists an $\epsilon \in \mathbb{R}_+^*$ such that $\epsilon \le h(\mathrm{Cay}(G_i, S_i))$ for all $i \in \mathbb{N}^+$.

Example 3.2. For every $i \in \mathbb{N}^+$, let D_{2i} be the dihedral group:

$$D_{2i} = \left\langle s, f \mid s^2 = f^i = e, sf = f^{-1}s \right\rangle.$$

In 2002, Rosenhouse [21] showed that $h(\mathrm{Cay}(D_{2i}, \{f, f^{-1}, s\})) = \frac{4}{i}$. Hence, $h(\mathrm{Cay}(D_{2i}, \{f, f^{-1}, s\})) \to 0$ as $i \to \infty$. Thus $\{\mathrm{Cay}(D_{2i}, \{f, f^{-1}, s\}), i \in \mathbb{N}^+\}$ is not an expander family. In fact, it was shown later (see [14]) that for any set of generator S_i of D_{2i}, $\{\mathrm{Cay}(D_{2i}, S_i), i \in \mathbb{N}^+\}$ is not an expander family. Thus $\{D_{2i}, i \in \mathbb{N}^+\}$ is not a $\mathbb{C}\mathrm{ay}$-expander family.

It is well-known that no family of abelian groups is a $\mathbb{C}\mathrm{ay}$-expander [14]. Before we prove the same result for the $\mathbb{G}$-expander case, we need the following lemma.

Lemma 3.3. *Let G be an abelian group generated by $S = \{s_1, \ldots, s_k\}$ and let $\tilde{\Phi}(G, S)$ be the corresponding G-graph, then*

$$\mathrm{diam}(\tilde{\Phi}(G, S)) \le |S|.$$

Proof. Let $(s_p)x, (s_q)y \in V(\tilde{\Phi}(G, S))$, where $x, y \in G$ and $1 \le p, q \le |S| = k$. Since $G = \langle S \rangle$ is an abelian group, then

$$x = s_1^{i_1} \ldots s_p^{i_p} \ldots s_q^{i_q} \ldots s_k^{i_k} y = s_1^{i_1} \ldots s_p^{i_p} \ldots s_{q-1}^{i_{q-1}} s_{q+1}^{i_{q+1}} \ldots s_k^{i_k} s_q^{i_q} y,$$

where $1 \le i_l \le o(s_l)$ for all $1 \le l \le k$. It is easy to see that $(s_p)x$ is adjacent to $(s_1)s_2^{i_2} \ldots s_k^{i_k} y$ which is in turn connected to $(s_2)s_3^{i_3} \ldots s_k^{i_k} y$ and so on up to $(s_k)s_q^{i_q} y$ which is connected to $(s_q)y$. Thus $d((s_p)x, (s_q)y) \le |S|$. $\square$

Corollary 3.4. *No family of abelian groups is a $\mathbb{G}$-expander.*

Proof. Suppose that $\{G_i, i \in \mathbb{N}^+\}$ is a family of finite abelian groups and that $\{\tilde{\Phi}(G_i, S_i), i \in \mathbb{N}^+\}$ is an expander family. Then there exists $r \in \mathbb{N}^+$ such that $|S_i| \le r$ for all $i \in \mathbb{N}^+$. But then by the preceding lemma $\mathrm{diam}(\tilde{\Phi}(G_i, S_i)) \le |S_i| \le r \in \mathbb{N}^+$, and that contradicts Proposition 2.12. $\square$

Now we are ready to prove the main result of this paper which is Theorem 1.1.

Proof of Theorem 1.1. Since $\mathrm{Cay}(G_n, S_n^*)$ is an expander family, then $|G_n| \to \infty$ as $n \to \infty$ and there is an $r \in \mathbb{N}^+$ such that $|S_n^*| \le r$ for all $n \in \mathbb{N}^+$. Hence $|S_n| \le r$ and $O_{\max}(S_n) \le r$ for every $n \in \mathbb{N}^+$. Then $|V(\tilde{\Phi}(G_n, S_n))| \to \infty$ as $n \to \infty$ and $\Delta(\tilde{\Phi}(G_n, S_n)) < r^2$ for all $n \in \mathbb{N}^+$.

Suppose that $H \subset V(\tilde{\Phi}(G_n, S_n))$ where $0 < |H| < \frac{|V(\tilde{\Phi}(G_n, S_n))|}{2}$, and $H_i = H \cap V_i$ for every $1 \le i \le |S_n|$. Then, clearly we have

$$H = \bigsqcup_i H_i.$$

Let $W = \bigcap_i \bigcup_{(s)x \in H_i} \langle s \rangle x \subset G$. Since $|H| \le \frac{|V(\tilde{\Phi}(G_n, S_n))|}{2}$, we have $|W| \le \frac{|G|}{2}$. Now let $X_i = \{(s_i)x \in H_i \mid \langle s_i \rangle x \subset W\}$, then $|X_i| \le |W|$. Denote by X and Y the following sets of vertices,

$$X = \bigsqcup_{i=1}^{|S_n|} X_i, \quad \text{and} \quad Y = H \setminus X.$$

If $(s)x \in Y$, there is an edge between $(s)x$ and a vertex in $V(\tilde{\Phi}(G_n, S_n)) \setminus H$. Hence

$$|\partial H| \geq |Y|.$$

In $\mathrm{Cay}(G_n, S_n^*)$, we have $|\partial W| \geq \epsilon|W|$. Let $f \colon \partial W \to \partial H, \{x, y\} \mapsto (\{(s_i)x, (s_j)y\}, y)$, where $x \in W, y \notin W, i$ and j are chosen so that $xy^{-1} \in \langle s_i \rangle$ and $y \notin \bigsqcup_{(s)x \in H_j} \langle s \rangle x$ (note here that there may be several possible choices for i and j). Now observe that if $f(x, y) = f(x', y')$, then $xx'^{-1} \in \langle s_i \rangle$ and $y = y'$. So for all $\alpha \in \partial H$, $|f^{-1}(\alpha)| \leq O_{\max}(S_n)$. Hence,

$$|\partial H| \geq \frac{|\partial W|}{O_{\max}(S_n)} \geq \frac{\epsilon|W|}{O_{\max}(S_n)} \geq \frac{\epsilon \max_i |X_i|}{O_{\max}(S_n)} \geq \frac{\epsilon|X|}{O_{\max}(S_n)|S_n|}.$$

Using $|\partial H| \geq |Y|$ and $|H| = |X| + |Y|$, we obtain

$$|\partial H| \geq \frac{1}{2} \min\left\{\frac{\epsilon}{O_{\max}(S_n)|S_n|}, 1\right\} |H| \geq \frac{1}{2} \min\left\{\frac{\epsilon}{r^2}, 1\right\} |H|.$$

This completes the proof. $\qquad\square$

The following results are obvious consequences of Theorem 1.1.

Corollary 3.5. *If $\{G_n, \ n \in \mathbb{N}^+\}$ is a $\mathbb{Cay}$-expander family, then it is also a $\mathbb{G}$-expander family.*

Corollary 3.6. *If $\{\mathrm{Cay}(G_i, S_i^*), \ i \in \mathbb{N}^+\}$ is an expander family, then $\{\tilde{\tilde{\Phi}}(G_i, S_i), \ i \in \mathbb{N}^+\}$ is also an expander family.*

Proof. By Theorem 1.1, $\{\tilde{\Phi}(G_i, S_i), \ i \in \mathbb{N}^+\}$ is an expander family. By Definition 3.1, there exists $r \in \mathbb{N}^+$ such that $o(s_j) \leq r$, for every $s_j \in S_i$. Then $|\langle s_{j_1} \rangle \cap \langle s_{j_2} \rangle| \leq r$ for all $s_{j_1}, s_{j_2} \in S_i$. Thus $\frac{h(\tilde{\Phi}(G_i, S_i))}{r} \leq h(\tilde{\tilde{\Phi}}(G_i, S_i))$. $\qquad\square$

Remark 3.7.

1. Unlike most constructed expander families which are d-regular, our construction produces expander families that may be d-regular, regular, or irregular. More specifically, by Proposition 2.6, if the order of all elements in the generating set S_i is the same, then the constructed family is either a d-regular or regular family depending on whether there exist $s_i \in S_i$ and $s_j \in S_j$ such that $o(s_i) \neq o(s_j)$. Otherwise, it will be an irregular family.

2. By Corollary 2.9, if $\langle s_{j_1} \rangle \cap \langle s_{j_2} \rangle = \{e\}$ for all $s_{j_1} \in S_i, s_{j_2} \in S_i \setminus s_{j_1}$, and for every $i \in \mathbb{N}^+$, then the constructed expander family $\{\tilde{\Phi}(G_i, S_i), \ i \in \mathbb{N}^+\}$ is formed of simple graphs. Note that $\{\tilde{\tilde{\Phi}}(G_i, S_i), \ i \in \mathbb{N}^+\}$ is always an expander family of simple graphs.

3. In Table 1, we compare some graph invariants for the Cayley graph $\mathrm{Cay}(G, S^*)$ and the G-graph $\tilde{\Phi}(G, S)$.

   It is worthy to note that $|S^*| = \sum_{s \in S} o(s) - |S|$ and every vertex in level V_s of $\tilde{\Phi}(G, S)$ has degree equal to $o(s)(|S| - 1)$ where $|V_s| = \frac{|G|}{o(s)}$. Thus, the degree

Table 1: Some graph invariants of $\mathrm{Cay}(G, S^*)$ and $\tilde{\Phi}(G, S)$.

	$\mathrm{Cay}(G, S^*)$	$\tilde{\Phi}(G, S)$
Order	$\lvert G \rvert$	$\sum_{s \in S} \frac{\lvert G \rvert}{o(s)}$
Degree	$\lvert S^* \rvert$-regular multigraph	$d(u) = o(s)(\lvert S \rvert - 1)$ for all $u \in V_s$ and $s \in S$
Size	$\frac{1}{2}\lvert G \rvert \lvert S^* \rvert = \frac{1}{2}\lvert G \rvert (\sum_{s \in S} o(s) - \lvert S \rvert)$	$\frac{1}{2}\lvert G \rvert \lvert S \rvert (\lvert S \rvert - 1)$

of most vertices of $\tilde{\Phi}(G, S)$ is smaller than $\lvert S^* \rvert$ (see also the remark after Theorem 5.10). In other words, this means that G-graphs enable us to construct sparser multigraphs than those which can be constructed using the family $\mathrm{Cay}(G, S^*)$, and in some cases even sparser than the ones constructed from the family $\mathrm{Cay}(G, S)$, with possibly smaller expansion ratios (see the proof of Theorem 1.1).

4   Applications

In this section, we present some direct results of Theorem 1.1. But first we start with some auxiliary materials.

Proposition 4.1. *Let $x_i \in G_i \setminus S_i$. If $\{\mathrm{Cay}(G_i, S_i),\ i \in \mathbb{N}^+\}$ is an expander family, then $\{\mathrm{Cay}(G_i, S_i \cup x_i^{\pm 1}),\ i \in \mathbb{N}^+\}$ is also an expander family.*

Proof. Since $\{\mathrm{Cay}(G_i, S_i), i \in \mathbb{N}^+\}$ is an expander family, then there exists $r \in \mathbb{N}^+$ such that $\lvert S_i \rvert \leq r$, for all $i \in \mathbb{N}^+$. Thus $\lvert S_i \cup x_i^{\pm 1} \rvert \leq r + 2$ for all $i \in \mathbb{N}^+$, so the second condition of Definition 2.11 is satisfied. Note that $\mathrm{Cay}(G_i, S_i)$ is a spanning subgraph of $\mathrm{Cay}(G_i, S_i \cup x_i^{\pm 1})$, hence

$$0 < \epsilon \leq h(\mathrm{Cay}(G_i, S_i)) \leq h(\mathrm{Cay}(G_i, S_i \cup x_i^{\pm 1})). \qquad \square$$

A direct consequence of the preceding proposition is the following.

Corollary 4.2. *Let $\{\mathrm{Cay}(G_i, S_i),\ i \in \mathbb{N}^+\}$ be an expander family. If there exists $l \in \mathbb{N}^+$ such that $\lvert S_i^* \rvert \leq l$ for all $i \in \mathbb{N}^+$, then $\{\mathrm{Cay}(G_i, S_i^*),\ i \in \mathbb{N}^+\}$ is also an expander family.*

The following theorem was proved by Breuillard and Gamburd in [8].

Theorem 4.3 ([8]). *There exists $\epsilon \in \mathbb{R}_+^*$ and an infinite set of prime numbers $\mathbb{P}'$ such that for every $p \in \mathbb{P}'$ and every generating set $\{x, y\}$ of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$, the family*

$$\mathrm{Cay}(\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z}), \{x^{\pm 1}, y^{\pm 1}\})$$

is an ϵ-expander.

Let

$$S_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad S_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is well-known that $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z}) = \langle S_1, S_3 \rangle = \langle S_2, S_3 \rangle$. The order of S_1, S_2 is 4, while the order of S_3 in $\mathbb{Z}/p\mathbb{Z}$ is p. Thus $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ is also generated by one of the following sets: $\{S_1, S_1 S_3\}, \{S_1, S_3 S_1\}, \{S_2, S_2 S_3\}, \{S_2, S_3 S_2\}$. Where

$$S_1 S_3 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \qquad\qquad S_3 S_1 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

$$S_2 S_3 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \text{ and} \qquad\qquad S_3 S_2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that the orders of $S_1 S_3, S_3 S_1, S_2 S_3$, and $S_3 S_2$ are respectively 6, 6, 3 and 3. With the above notation, we have the following conclusion.

Corollary 4.4. *Let*

$$A_1 = \{S_1^{\pm 1}, S_1 S_3, S_3^{-1} S_1^{-1}\}, \qquad\qquad A_2 = \{S_1^{\pm 1}, S_3 S_1, S_1^{-1} S_3^{-1}\},$$
$$A_3 = \{S_2^{\pm 1}, S_2 S_3, S_3^{-1} S_2^{-1}\}, \text{ and} \qquad A_4 = \{S_2^{\pm 1}, S_3 S_2, S_2^{-1} S_3^{-1}\}.$$

There exist sets $\mathbb{P}_i^a$ of prime numbers such that $\{\mathrm{Cay}(\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z}), A_i); \ p \in \mathbb{P}_i^a\}$ is an expander family for all $1 \leq i \leq 4$.

Let $B_1 = \{S_1, S_1 S_3\}$, by Corollaries 4.2 and 4.4 we directly deduce that there exists a set $\mathbb{P}'$ of prime numbers such that $\{\mathrm{Cay}(\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z}), B_1^*); \ p \in \mathbb{P}'\}$ is an expander family. Using Theorem 1.1, we can easily deduce that $\{\tilde{\Phi}(\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z}), B_1); \ p \in \mathbb{P}'\}$ is an expander family. By the same analogy, we obtain the following.

Corollary 4.5. *Let*

$$B_1 = \{S_1, S_1 S_3\}, \qquad\qquad B_2 = \{S_1, S_3 S_1\},$$
$$B_3 = \{S_2, S_2 S_3\}, \text{ and} \qquad\qquad B_4 = \{S_2, S_3 S_2\}.$$

There exist sets $\mathbb{P}_i^b$ of prime numbers such that $\{\tilde{\Phi}(\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z}), B_i); \ p \in \mathbb{P}_i^b\}$ is an expander family for all $1 \leq i \leq 4$.

In a similar fashion, many other G-graph families on the special linear group $\mathrm{SL}(2, \mathbb{Z}/q\mathbb{Z})$ can be constructed.

5   New expander families of G-graphs

In this section, we present a method for constructing a family of Cayley graphs from another given family by rearrangement of edges in such a way to almost maintain the same expansion ratio. Consequently, we prove that if the family of Cayley graphs $\{\mathrm{Cay}(G_i, \{s_1^{\pm 1}, s_2^{\pm 1}\}); \ i \in \mathbb{N}^+\}$ is an expander, then so is the family of Cayley graphs $\{\mathrm{Cay}(G_i, \{s_1^{\pm 1}, s_1 s_2, s_2^{-1} s_1^{-1}\}); \ i \in \mathbb{N}^+\}$. Then using Theorem 1.1, several expander families of G-graphs are constructed. But first we need to introduce more notation.

Remark 5.1. Let $\mathrm{Cay}(G, S)$ be a Cayley graph and let $H' \subseteq H \subseteq G$. Let $s \in S$, we denote by $N_s(H)$ and $N_s(H)(H')$ the set of vertices of $\mathrm{Cay}(G, S)$ that are defined in the following way:

    i. $N_s(H) = sH \cap \bar{H}$,

ii. $N_s(H)(H') = sH' \cap \bar{H}$.

Next, we start by the following simple lemma.

Lemma 5.2. *Let* $\mathrm{Cay}(G, S)$ *be a Cayley graph, where* $S = \{s_1^{\pm 1}, \ldots, s_k^{\pm 1}\}$. *Let* $H \subseteq G$, *then*

$$|\partial H(\mathrm{Cay}(G, S))| = 2 \sum_{i|o(s_i)>2} |N_{s_i}(H)| + \sum_{i|o(s_i)=2} |N_{s_i}(H)| = \sum_{1 \le i \le k} |N_{s_i^{\pm 1}}(H)|.$$

Proof. Let $x, y \in H$ such that $y = s_i x$ for some $s_i \in S$, then $x = s_i^{-1} y$. Thus the number of edges in the subgraph H of G that corresponds to s_i is equal to that of s_i^{-1} and $|N_{s_i}| = |N_{s_i^{-1}}|$. It is easy to see that: $|\partial H(\mathrm{Cay}(G, S))| = \sum_{1 \le i \le k} |N_{s_i^{\pm 1}}(H)|$ and the proof is complete. $\square$

Example 5.3. Let $(\mathbb{Z}/n\mathbb{Z}, +, 0)$, $n \ge 10$ and $S = \{\pm 1, \pm 2\}$. Then $\mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, S)$ is 4-regular multigraph on n vertices. Let H be a subgraph of $\mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, S)$ such that $V(H) = \{1, 2, 3, 7\}$. Let $s_1 = +1$ and $s_2 = +2$. Then $N_{s_1}(H) = \{4, 8\}$, $N_{s_1^{-1}}(H) = \{0, 6\}$, $N_{s_2}(H) = \{4, 5, 9\}$, and $N_{s_2^{-1}}(H) = \{0, 5, n-1\}$. Thus

$$|\partial H(\mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, S))| = 2(|N_{s_1}(H)| + |N_{s_2}(H)|) = 10.$$

Next, we shall show that it is possible to construct an expander family of Cayley graphs from another one by switching some of its edges.

Corollary 5.4. *Let* $\{\mathrm{Cay}(G_i, \{s_1^{\pm 1}, s_2^{\pm 1}\}); \ i \in \mathbb{N}^+\}$ *be an expander family. If* $o(s_1)$, $o(s_2)$, *and* $o(s_1 s_2) > 2$, *then* $\{\mathrm{Cay}(G_i, \{s_1^{\pm 1}, s_1 s_2, s_2^{-1} s_1^{-1}\}); \ i \in \mathbb{N}^+\}$ *is also an expander family.*

Proof. Let $V(H) = \{x_1, \ldots, x_t\} \in G$. Define $\partial' H$, $\partial'' H$ to be the sets of emanating edges from $V(H)$ in the multigraphs $\mathrm{Cay}(G_i, \{s_1^{\pm 1}, s_2^{\pm 1}\})$ and $\mathrm{Cay}(G_i, \{s_1^{\pm 1}, s_1 s_2, s_2^{-1} s_1^{-1}\})$ respectively. By Lemma 5.2, we have:

$$|\partial' H| = 2|N_{s_1}(H)| + 2|N_{s_2}(H)|, \text{ and} \tag{1}$$
$$|\partial'' H| = 2|N_{s_1}(H)| + 2|N_{s_1 s_2}(H)|. \tag{2}$$

Let $y \in N_{s_2}(H), y = s_2 x$ for some $x \in H$.

    i. If $s_1 y \notin H$, then $s_1 s_2 x \notin H$ and $s_1 s_2 x \in N_{s_1 s_2}(H)$.

    ii. And if $s_1 y \in H$, then $s_1 s_2 x \in H$.

Let H_1 and H_2 be the set of vertices of H defined as follows:

$$H_1 = \{x \in H/s_2 x \notin H \text{ and } s_1 s_2 x \notin H\},$$
$$H_2 = \{x \in H/s_2 x \notin H \text{ and } s_1 s_2 x \in H\}.$$

From equalities (1) and (2), we have

$$2|N_{s_1}(H)| + 2|N_{s_2}(H)(H_1)| + 2|N_{s_2}(H)(H_2)| = |\partial' H|,$$
$$2|N_{s_1}(H)| + 2|N_{s_1 s_2}(H)(H_1)| + 2|N_{s_1 s_2}(H)(H_2)| \le |\partial'' H|.$$

From the definition of H_2, we have $|N_{s_1 s_2}(H)(H_2)| = 0$, then

$$2|N_{s_1}(H)| + 2|N_{s_1 s_2}(H)(H_1)| \leq |\partial'' H|.$$

Therefore, it holds that

$$2|N_{s_1}(H)| + 4|N_{s_1 s_2}(H)(H_1)| - 2|N_{s_2}(H)(H_1)| - 2|N_{s_2}(H)(H_2)| \leq 2|\partial'' H| - |\partial' H|.$$

From the definition of H_1, we have $|N_{s_1 s_2}(H)(H_1)| = |N_{s_2}(H)(H_1)|$ and similarly from the definition of H_2, we have $|N_{s_2}(H)(H_2)| = |N_{s_1^{-1}}(H) \cap N_{s_2}(H)|$. Thus,

$$2|N_{s_1}(H)| + 2|N_{s_2}(H)(H_1)| - 2|N_{s_1^{-1}}(H) \cap N_{s_2}(H)| \leq 2|\partial'' H| - |\partial' H|.$$

Noticing that

$$|N_{s_1^{-1}}(H) \cap N_{s_2}(H)| \leq |N_{s_1^{-1}}(H)| = |N_{s_1}(H)|,$$

then

$$2|N_{s_2}(H)(H_1)| \leq 2|\partial'' H| - |\partial' H|.$$

Finally, we obtain

$$0 < \epsilon \leq \frac{|\partial' H|}{2|H|} \leq \frac{|\partial'' H|}{|H|}. \qquad \square$$

Remark 5.5. Note that in general $\{\mathrm{Cay}(G_i, \{s_1^{\pm 1}, s_2^{\pm 1}\}); \ i \in \mathbb{N}^+\}$ and $\{\mathrm{Cay}(G_i, \{s_1^{\pm 1}, s_1 s_2, s_2^{-1} s_1^{-1}\}); \ i \in \mathbb{N}^+\}$ may be not isomorphic. An example of this situation is given by the dihedral group D_{2i} which is defined earlier as follows:

$$D_{2i} = \left\langle s, f \mid s^2 = f^i = e, sf = f^{-1}s \right\rangle.$$

Let $s_1 = s$ and $s_2 = f$, then $S = \{s_1^{\pm 1}, s_2^{\pm 1}\} = \{s, f^{\pm 1}\}$ and $L = \{s_1^{\pm 1}, s_1 s_2, s_2^{-1} s_1^{-1}\} = \{s, sf\}$. Clearly, the 3-regular $\mathrm{Cay}(D_{2i}, \{s_1^{\pm 1}, s_2^{\pm 1}\})$ is not isomorphic to the 2-regular multigraph $\mathrm{Cay}(D_{2i}, \{s_1^{\pm 1}, s_1 s_2, s_2^{-1} s_1^{-1}\})$.

The next theorem is Theorem 4.4.2 in [16].

Theorem 5.6 ([16])**.** *Let $\mathbb{P}$ be the set of all prime numbers, then $\{\mathrm{Cay}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2^{\pm 1}, S_3^{\pm 1}\}); \ p \in \mathbb{P}\}$ is an expander family.*

As a consequence, we have the following.

Corollary 5.7. *Let $\mathbb{P}$ be the set of all prime numbers, then $\{\mathrm{Cay}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}); \ \{S_2^{\pm 1}, S_2 S_3, S_3^{-1} S_2^{-1}\}); \ p \in \mathbb{P}\}$ is an expander family.*

Corollary 5.8. *Let $\mathbb{P}$ be the set of all prime numbers, then $\{\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}); \ p \in \mathbb{P}\}$ is a $\mathbb{Cay}$-expander family.*

Remark 5.9. The order of S_2 and $S_2 S_3$ are 4 and 3 respectively. Let $L = \{S_2, S_2 S_3\}$ and $W = \{S_2, S_2^2, S_2 S_3\}$, then we see that $\max\{|L^*|, |W^*|\} \leq 7$. Using Corollaries 4.2 and 5.7, we deduce that $\{\mathrm{Cay}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), L^*); \ p \in \mathbb{P}\}$ and $\{\mathrm{Cay}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), W^*); \ p \in \mathbb{P}\}$ are all expander families. Now by Theorem 1.1, we are able to directly construct several expander families of G-graphs.

Thus, we can conclude the following.

Theorem 5.10. *Let $\mathbb{P}$ be the set of all prime numbers. Then the G-graphs families given by*

1. $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2, S_2 S_3\}); \ p \in \mathbb{P}\}$,

2. $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2, S_2^2, S_2 S_3\}); \ p \in \mathbb{P}\}$.

are expanders.

Remark 5.11.

1. Using Corollary 2.9, it is easy to check that the first expander family given in Theorem 5.10 is formed of simple graphs, while the second one is not. By Proposition 2.6, we also deduce that the multigraphs in both families are semiregular; in otherwords the above two expander families are irregular.

2. Each $\{\mathrm{Cay}(G_i, S_i^*); \ i \in \mathbb{N}^+\}$ expander family enables us to construct several expander families of G-graphs depending on the choice of S_i in S_i^* with the possibility that some of these families may be isomorphic. For example, the following expander families $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2, S_3^{-1} S_2^{-1}\}); \ p \in \mathbb{P}\}$, $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2^{-1}, S_2 S_3\}); \ p \in \mathbb{P}\}$, and $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2^{-1}, S_3^{-1} S_2^{-1}\}); \ p \in \mathbb{P}\}$ are all isomorphic to $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2, S_2 S_3\}); \ p \in \mathbb{P}\}$. Similarly, the expander families $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2, S_2^2, S_3^{-1} S_2^{-1}\}); \ p \in \mathbb{P}\}$, $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2^{-1}, S_2^2, S_2 S_3\}); \ p \in \mathbb{P}\}$, and $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2^{-1}, S_2^2, S_3^{-1} S_2^{-1}\}); \ p \in \mathbb{P}\}$ are all isomorphic to $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2, S_2^2, S_2 S_3\}); \ p \in \mathbb{P}\}$.

Table 2: Comparison of some graph invariants between $\mathrm{Cay}(G, S^*)$ and $\tilde{\Phi}(G, S)$ for $S = L$ and $S = W$.

	$\mathrm{Cay}(G, L^*)$	$\tilde{\Phi}(G, L)$
Order	$|G|$	$\sum_{s \in S} \frac{|G|}{o(s)} = \frac{7}{12}|G|$
Degree	5-regular multigraph	$d(u) = 4$ for all $u \in V_{S_2}$ $d(v) = 3$ for all $v \in V_{S_2 S_3}$
Size	$\frac{5}{2}|G|$	$|G|$

	$\mathrm{Cay}(G, W^*)$	$\tilde{\Phi}(G, W)$
Order	$|G|$	$\frac{13}{12}|G|$
Degree	6-regular multigraph	$d(u) = 8$ for all $u \in V_{S_2}$ $d(v) = 6$ for all $v \in V_{S_2 S_3}$ $d(w) = 4$ for all $w \in V_{S_2^2}$
Size	$3|G|$	$3|G|$

3. Let G be the projective special linear group, that is $G = \mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z})$. In Table 2, we compare the order, the degree, and the size of the following expander family of

Cayley graphs $\{\mathrm{Cay}(G, L^*), p \in \mathbb{P}\}$ (resp. $\{\mathrm{Cay}(G, W^*), p \in \mathbb{P}\}$) with their corresponding ones in the G-graphs family $\{\tilde{\Phi}(G, L); \ p \in \mathbb{P}\}$ (resp. $\{\tilde{\Phi}(G, W); \ p \in \mathbb{P}\}$) (see Theorem 5.10). Form the preceding table, it is easy to see that the infinite expander family of G-graphs $\tilde{\Phi}(G, L)$ is sparser than the original expander family of the 4-regular graphs $\mathrm{Cay}(G, L^{\pm 1})$ and the 5-regular graphs $\mathrm{Cay}(G, L^*)$. The same can be said concerning the infinite expander family of G-graphs $\tilde{\Phi}(G, W)$ and the Cayley graph one $\mathrm{Cay}(G, W^*)$.

We close this section by the following corollary which can be easily obtained by using Theorem 5.10 and Corollary 3.6.

Corollary 5.12. *Let $\mathbb{P}$ be the set of all prime numbers. Then the family of G-graphs given by*

$$\{\tilde{\tilde{\Phi}}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2, S_2^2, S_2 S_3\}); \ p \in \mathbb{P}\}$$

is an expander family.

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