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HAMILTONICITY OF CERTAIN CARTESIAN PRODUCTS OF GRAPHS

HAMILTONSKOST KARTEZIČNEGA PRODUKTA GRAFOV

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Abstract

A graph is Hamiltonian if it contains a spanning cycle. In this paper, we examine the hamiltonicity of the Cartesian product of a tree with a path. We offer sufficient conditions for the Cartesian product of a tree with a path to be Hamiltonian.

Povzetek

Graf je Hamiltonov, če vsebuje cikel, ki gre skozi vsako vozlišče natanko enkrat. V tem članku preučujemo hamiltonskost kartezičnega produkta drevesa in poti. Podamo zadostne pogoje, da bo kartezični produkt drevesa in poti Hamiltonov.

1 INTRODUCTION

A *Hamiltonian path or traceable path* is a path that visits each vertex of the graph exactly once. If there exists a Hamiltonian path in *G*, then *G* is referred to as *traceable*, and a graph is *Hamiltonian* if it contains a spanning cycle. In this article, we consider the hamiltonicity of the Cartesian product of two graphs. Our goal is to investigate the necessary and sufficient conditions for the Cartesian product to be Hamiltonian. We summarise some previous results and provide new ones. Certain results are related to those obtained in *[2, 4]*.

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Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and the edge set $E(G)$. The number of vertices in $V(G)$ is the *order* of G. The degree of a vertex v is denoted by $degG(v)$. The maximum degree in G is denoted by $\Delta(G)$. The number of isolated vertices of G is denoted by $i(G)$. Let Pn denote a path of order n and Cn the cycle of order n. For convenience, we write $V(Pn) = \{1, 2,\ldots,n\}$ and $E(P_n) = \{i(i + 1) | i = 1, 2,\ldots, n - 1\}$. An *end-vertex* of G is a vertex of degree 1 in G. A *path factor* of a graph G is a spanning subgraph of G such that each component of the spanning subgraph is a nontrivial path. A graph has a $\{P2, P3\}$-factor if it has a spanning subgraph such that each component is isomorfic to $P2$ or $P3$.

Lemma 1.1 *([4])* A *graph G has a path factor if and only if G has a* $\{P2, P3\}$*-factor.*

If each component in a $\{P2, P_3\}$-factor is isomorfic to P_2, the path factor is called *perfect matching.* The number of components of a graph G is denoted by $c(G)$. A graph G is *t-tough* ($t \in \mathbb{R}$) if $|S| > t\cdot c(G \setminus S)$ for every subset $S \subseteq V(G)$ with $c(G \setminus S) > 1$.

Let $G = (V(G),E(G))$ and $H = (V(H), E(H))$ be graphs. The *Cartesian product* of G and H is the graph $G\Box H$ defined by $V(G\Box H) = V(G) \times V(H)$, where $(x_1, y_1)(x_2,y_2)$ is an edge in $G\Box H$ if $x_1 = x_2$ and $y_1y_2 \in E(H)$, or $x_1x_2 \in E(G)$, and $y_1 = y_2$. The graphs G and H are termed *factors* of the product. For an $x \in V(G)$, the *H-layer* H_x is the set $H_x = \{(x, y) \mid y \in V(H)\}$.

2 CARTESIAN PRODUCT OF A TREE WITH A PATH

In this section we deal with Cartesian products of a tree with a path, i.e., we consider $T\Box Pn$, for $n \geq 4$ even.

Proposition 2.1 *([3]) Let G and H be both of odd order. If both G and H are bipartite, then G□H is not Hamiltonian.*

Notice that when the order of T and n is both odd, the $T\Box Pn$ is not Hamiltonian by Proposition 2.1, so we will focus on even paths. The lemma below is from *[1].*

Theorem 2.2 *([1]) Let T be a tree with* $\Delta(T) \geq 2$ *and Cn a cycle of order n. Then T□Cn is Hamiltonian if and only if* $\Delta(T) \leq n$.

In *[4]*, the authors showed that in the above theorem, $T\Box Cn$ cannot be replaced by $T\Box Pn$. They give an example of a tree such that $n = \Delta(T) + 1$ and $T\Box Pn$ is not Hamiltonian, proving that for a tree T_1 with the vertex set $V(T_1) = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and the edge set $E(T_1) = \{12, 23, 34, 45, 26, 37, 48\}$, the graph $T_1\Box P_4$ is not Hamiltonian.

From the figure below we can see that $T_1\Box P_6$ is Hamiltonian. Therefore, we are interested in other examples of when this is possible.

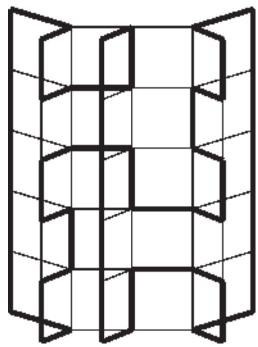

Figure 1: *The Hamiltonian cycle in $T_1 \square P_6$*

In *[4]*, the following result is proven.

Proposition 2.3 *([4]) Let H be a connected bipartite graph. Let n be an even integer and $n \geq 4\,\Delta(H) - 2$. The following three statements are equivalent:* (*i*) *$H \square P_n$ is Hamiltonian;* (*ii*) *$H \square P_n$ is 1-tough;* (*iii*) *H has a path factor.*

Motivated by the example above (Figure 1), we will be interested in examples of such trees *T*, for which the condition $n \geq 4\Delta(H) - 2$ in proposition 2.3 is not fulfilled, yet $T \square P_n$ is Hamiltonian.

Proposition 2.4 *([4]) Let T be a tree with perfect matching and n be a positive integer. The following three statements are equivalent:* (*i*) *$T \square P_n$ is Hamiltonian;* (*ii*) *$T \square P_n$ is 1-tough;* (*iii*) $n \geq \Delta(T)$.

Let *T* be a tree with $\{P_2, P_3\}$-factor *F*. We define the *type* of a vertex *v* with respect to *F* as follows:

- *v* has type *EPL* if *v* is the left endpoint of a P_3 in *F*,
- *v* has type *EPR* if *v* is the right endpoint of a P_3 in *F*,
- *v* has type *M* if *v* is the middle vertex of a P_3 in *F*,
- *v* has type EP2 if *v* is a vertex of P_2 in *F*.

Theorem 2.5 *Suppose that T has a $\{P_2, P_3\}$-factor F and n is an even integer. If $deg_T(x) \leq (n+2)/2$ for every x of type M in F, $deg_T(x) \leq n/2$ for every x of type EP2 in F and $deg_T(x) + deg_T(y) \leq (n+2)/2$ for every x, y of type EPL and EPR on every component in F isomorfic to P_3, then $T \square P_n$ contains a Hamiltonian cycle.*

Proof. Let *F* be a $\{P_2, P_3\}$-factor which satisfies the conditions in the theorem. If each component in *F* is isomorfic to P_2, then $T \square P_n$ by proposition 2.4 contains a Hamiltonian cycle, since every vertex *x* in *T* has type *EP2* and therefore $degT(x) \leq \Delta(T) \leq n/2 \leq n$.

So, we can assume that there exist a component isomorfic to P_3.

First, we define the standard Hamiltonian cycle for $P_3 \square P_n$ and for $P_2 \square P_n$.

For $\{x, y, z\} \in V(P_3)$, $\{xy, yz\} \in E(P_3)$ and an even *n*, we define the set $\{(x, 1)(y, 1)\} \cup \{(y, 2i-1)(z, 2i-1), (z, 2i-1)(z, 2i), (z, 2i)(y, 2i) \mid 1 \leq i \leq n/2\} \cup \{(y, 2i)(y, 2i+1) \mid 1 \leq i \leq (n-2)/2\} \cup \{(y,n)(x,n)\} \cup \{(x,i)(x,i+1) \mid 1 \leq i < n\}$ of edges in $P_3 \square P_n$ as the standard Hamiltonian cycle for $P_3 \square P_n$ (see Figure 2 (left)).

For $\{u, v\} \in V(P_2)$, we define the set $\{(u, 1)(v, 1)\} \cup \{(v, i)(v, i+1) \mid 1 \leq i \leq n\} \cup \{(u, i)(u, i+1) \mid 1 \leq i < n\} \cup \{(u, n)(v, n)\}$ of edges in $P_2 \square P_n$ as the standard Hamiltonian cycle for $P_2 \square P_n$ (see Figure 2 (right)).

Notice that there are (n-2)/2 vertical edges on every P_n-layer that correspond to a vertex $y \in F$ of type *M* on the standard Hamiltonian cycle *P3□Pn* and that there are n/2 vertical edges on every P_n-layer that correspond to a vertex $y \in F$ of type *EPR* on the standard Hamiltonian cycle *P3□Pn*.

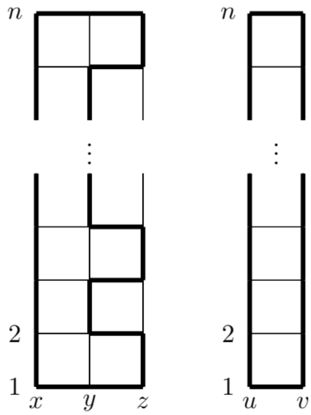

Figure 2: *The standard Hamiltonian cycle for P3□Pn and for P2□Pn*

We now use a recursive construction to reach a Hamiltonian cycle in *T□Pn*. We start with the standard Hamiltonian cycle for *C'* = *C1□Pn* of initially chosen component C_1 in *F*. Let C_2 be a component in *F* such that there is a vertex $y \in C_2$ adjacent with a vertex $x \in C_1$ (note that $xy \in$ E(*T*)) and let *C"* = *C2□Pn* be a standard Hamiltonian cycle as described above. We can join such two standard cycles *C'* and *C"* with cycle *C'''* with vertex set *V* (*C'''*) = *V* (*C'*) U *V* (*C''*) and edge set *E*(*C'''*) as described below.

We distinguish several cases:

(i) C_1 and C_2 are isomorfic to P_2

We can join cycles *C'* and *C"* with cycle *C'"* with edge set *E*(*C'"*) = ((*E*(*C'*) ∪ *E*(*C"*)) \ {(*x*, *i*)(*x*, *i* +1), (*y*, *i*)(*y*, *i* + 1)}) ∪ {(*x*, *i*)(*y*, *i*), (*x*, *i* + 1)(*y*, *i* + 1)} for every *i* = 1, 2, …, n–1 (see Figure 3 (a)).

(ii) C_1 is isomorfic to P_2 and C_2 is isomorfic to *P3* (or vice-versa).

If *y* has type *M*, we can join such cycles *C'* and *C"* with cycle *C'"* with edge set *E*(*C'"*) = ((*E*(*C'*) ∪ *E*(*C"*)) \ {(*x*, 2*i*)(*x*, 2*i* + 1), (*y*, 2*i*)(*y*, 2*i* + 1)}) ∪ {(*x*, 2*i*)(*y*, 2*i*), (*x*, 2*i* + 1)(*y*, 2*i* + 1)} for every *i* = 1,2,…, (*n*-2)/2 (see Figure 3 (b)).

If *y* has type *EPR* (or *EPL*), we can join cycles *C'* and *C"* with cycle *C'"* with edge set *E*(*C'"*) = ((*E*(*C'*) ∪ *E*(*C"*)) \ {(*x*, 2*i*–1)(*x*, 2*i*), (*y*, 2*i*–1)(*y*, 2*i*)}) ∪ {(*x*, 2*i* -1)(*y*, 2*i*–1), (*x*, 2*i*)(*y*, 2*i*)} for every *i* = 1, 2,…, *n*/2 (see Figure 3 (c)).

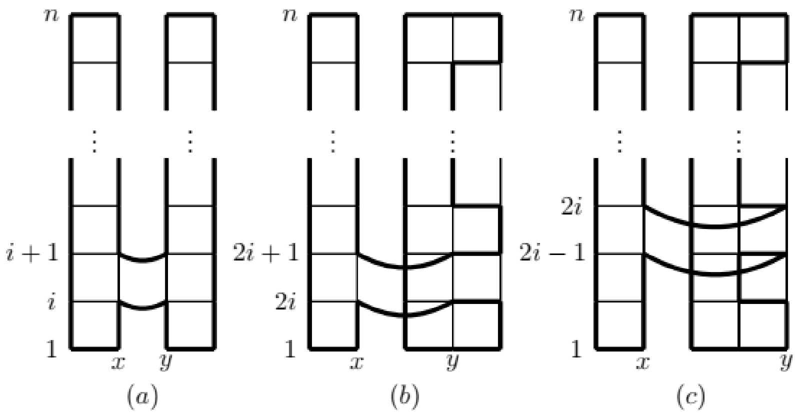

Figure 3: *Joining standard cycles C′=C1□Pn and C′′=C2□Pn for P2□Pn where C1 is isomorfic to P2*

(iii) C_1 and C_2 are isomorfic to P_3.

If *x* and *y* have type *M*, we can join cycles *C′* and *C″* with cycle *C′″* with edge set *E*(*C′″*) = ((*E*(*C′*) ∪ *E*(*C″*)) \ {(*x*, 2*i*)(*x*, 2*i* + 1), (*y*, 2*i*)(*y*, 2*i* + 1)}) ∪ {(*x*, 2*i*)(*y*, 2*i*), (*x*, 2*i* + 1)(*y*, 2*i* + 1)} for every *i* = 1, 2,…, (*n*-2)/2 (see Figure 4 (a)).

If *x* and *y* have type *EPR* (or *EPL*), we can join cycles *C′* and *C″* with cycle *C′″* with edge set *E*(*C′″*) = ((*E*(*C′*) ∪ *E*(*C″*)) \ {(*x*, 2*i*–1)(*x*, 2*i*), (*y*, 2*i*–1)(*y*, 2*i*)}) ∪ {(*x*, 2*i*–1)(*y*, 2*i*–1), (*x*, 2*i*)(*y*, 2*i*)} for every *i* = 1, 2,…, *n*/2 (see Figure 4 (b)).

If *x* has type *M* and *y* has type *EPL*, we can join such two standard cycles *C′* and *C″* with cycle *C′″* with edge set ((*E*(*C′*) ∪ *E*(*C″*)) \ {(*x*, 2*i*)(*x*, 2*i* + 1), (*y*, 2*i*)(*y*, 2*i*+1)})∪{(*x*, 2*i*)(*y*, 2*i*), (*x*, 2*i*+1)(*y*, 2*i*+1)} for every *i* = 1, 2,…, (*n*-2)/2 (see Figure 4 (c)).

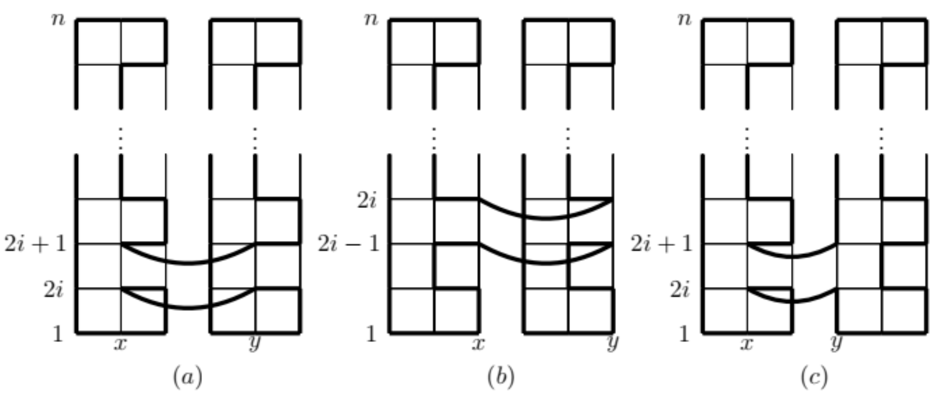

Figure 4: *Joining standard cycles C′=C1□Pn and C′′=C2□Pn for P2□Pn where C1 is isomorfic to P3*

If *x* has type *M* and *y* has type *EPR,* we reshape the standard Hamiltonian cycle *C″*= P_3□*Pn* into C^R. Denote $y = y_3$ and $\{y_1,y_2,y_3\} \in V(P_3)$ where $\{y_1y_2, y_2y_3\} \in E(P_3)$. Define, C^R = (*C′*\{(y_1, 2*i*)(y_1, 2*i*+1), (y_2,2*i*)(y_3,2*i*), (y_2,2*i*+1)(y_3,2*i* + 1)})∪{(*y***1**, 2*i*)(y_2, 2*i*), (y_1, 2*i* + 1)(y_2,2*i* + 1), (y_3,2*i*)(y_3,2*i* + 1)} for some

i = 1, 2,…, (n-2)/2 (see Figure 5 (a)). Now we can join two of such cycles *C'* and C^R with cycle *C'''* with edge set ((E(C') U *E(CR))* \ {(x, 2i)(x, 2i + 1), (y_3, 2i)(y_3, 2i + 1)}) U {(x,2i)(y_3,2i),(x,2i+1)(y_3,2i+1)} (see Figure 5 (b)).

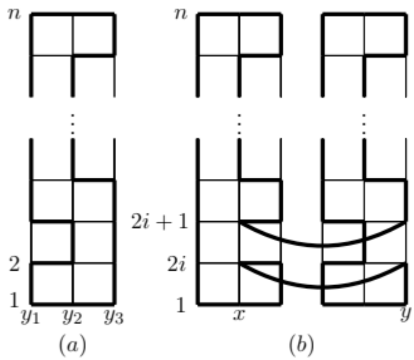

Figure 5: *The redesigned standard Hamiltonian cycle CR for P3□Pn (a) and joining standard cycles C' = P3□Pn and CR (b)*

For t = 2, 3,.. we repeat the following until we reach a Hamiltonian cycle for *T□Pn*. Let C_t be a component of $T \setminus C_{t-1}$ such that there is a vertex $x \in C_t$ incident with the vertex on C_{t-1}. We join standard Hamiltonian cycle C_t□*Pn* with the cycle C_{t-1}□*Pn* as described above. The construction is correct since it consists of the repeated joining of cycles at incident vertices in *T,* and there are enough free edges to join all standard Hamiltonian cycles, namely:

- for every $x \in C_{t-1}$ of type *EP2*, we have at most deg_T (x)–1 ≤ n/2–1 = (n-2)/2 component C_j adjacent with x, so there are enough free vertical edges on P_n-layer P_{nx} to join cycle C' = C_{t-1}□P_n with all cycles C" = C_j□P_n as described above;
- for every $x \in C_{t-1}$ of type *M*, we have at most deg_T(x)–2 ≤ (n+2)/2–2 = (n-2)/2 component C_j adjacent with x, so there are enough free vertical edges on P_n-layer P_{nx} to join cycle C' = C_{t-1}□P_n with all cycles C" = C_j□P_n as described above;
- for every $x,y \in C_{t-1}$ of type *EPL* and *EPR*, we have at most deg_T(x)+deg_T($_y$)–2 ≤ (n+2)/2–2 = (n-2)/2 component C_j adjacent with x and y, so there are enough free vertical edges on P_n-layer P_{nx} or P_{ny} to join cycle C' = C_{t-1}□P_n with all cycles C" = C_j□P_n as described above.□

References

[1] **V. Batagelj, T. Pisanski**: *Hamiltonian cycle in the cartesian product of a tree and a cycle*, Discrete Math. 38, 311 – 312, 1982

[2] **R. Čada, E. Flandrin, H. Li**: *Hamiltonicity and pancyclicity of cartesian products of graphs*, Discrete Math. 309, 6337 – 6343, 1987

[3] **V. Dimakopoulos, L. Palios, A. S. Poulakidas***: On the Hamiltonicity of the Cartesian product,* Inform. Process. Lett. 96, 49 – 53, 2005

[4] **L. Kao, C. Weng:** *The Relation Between Hamiltonian and 1-Tough Properties of the Cartesian Product Graphs*, Graphs Comb., 2020

[5] **M. Rosenfeld, D. Barnette:** *Hamiltonian circuits in certain prisms,* Discrete Math. 5, 389394, 1973