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9 Why Nature Made a Choice of Clifford and not Grassmann Coordinates? *

N.S. Mankoč Borštnik[1] and H.B.F. Nielsen[2]

[1]Department of Physics, University of Ljubljana,
SI-1000 Ljubljana, Slovenia
[2]Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17,
Copenhagen Ø, Denmark

Abstract. This is a discussion on fermion fields, the internal degrees of freedom of which are described by either the Grassmann or the Clifford anticommuting "coordinates". We prove that both fields can be second quantized so that their creation and annihilation operators fulfill the requirements of the commutation relations for fermion fields. However, while the internal spins determined by the generators of the Lorentz group of the Clifford objects S^{ab} and $\tilde{S}^{ab}$ (in the *spin-charge-family* theory S^{ab} determine the spin degrees of freedom and $\tilde{S}^{ab}$ the family degrees of freedom) are half integer, the internal spin determined by $\mathbf{S}^{ab}$ (expressible with $S^{ab} + \tilde{S}^{ab}$) is integer. *Nature "made"* obviously the choice of the Clifford algebra, at least in the so far observed part of our universe. We discuss here the quantization — first and second — of the fields, the internal degrees of freedom of which are functions of the Grassmann coordinates θ^a and their conjugate momenta, as well as of the fields, the internal degrees of freedom of which are functions of the Clifford γ^a. Inspiration comes from the *spin-charge-family* theory ([1,2,9,3], and the references therein), in which the action for fermions in d-dimensional space isequal to $\int\, d^dx\; E\; \frac{1}{2}\,(\bar{\psi}\,\gamma^a p_{0a}\psi) + h.c.$, with $p_{0a} = f^{\alpha}{}_a p_{0\alpha} + \frac{1}{2E}\,\{p_\alpha, Ef^{\alpha}{}_a\}_-$, $p_{0\alpha} = p_\alpha - \frac{1}{2}S^{ab}\omega_{ab\alpha} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{ab\alpha}$. We write the basic states as products of those either Grassmann or Clifford objects, which allow second quantization for fermion fields, and look for the action and solutions for free fields also in the Grassmann case in order to understand why the Clifford algebra "wins in the competition" for the physical (observable) degrees of freedom.

Povzetek. Avtorja obravnavata razliko med fermionskimi polji, katerih interne prostostne stopnje opišemo bodisi z Grassmannovimi bodisi s Cliffordovimi antikomutirajočimi "koordinatami". Dokažeta, da lahko v obeh primerih poiščemo kreacijske in anihilacijske operatorje, ki zadoščajo komutacijskim relacijam za fermionska polja v drugi kvantizaciji. Obe vrsti opisa fermionskih polj se vseeno bistveno razlikujeta: notranji spini, določeni z generatorji Lorenztove grupe Cliffordovih objektov S^{ab} in $\tilde{S}^{ab}$ (v teoriji *spinov-nabojev-družin* določajo S^{ab} spinsko kvantno število ter s tem spine in naboje kvarkov in leptonov, $\tilde{S}^{ab}$ pa določajo družinska kvantna števila), imajo polštevilčen spin, medtem ko je notanji spin, ki ga določajo $\mathbf{S}^{ab}$ (izrazljivi z $S^{ab} + \tilde{S}^{ab}$), celoštevilčen. *Narava* je očitno "*izbrala*" Cliffordovo algebro (vsaj v opazljivem delu vesolja). Avtorja obravnavata prvo in drugo kvantizacijo polj, katerih notranje prostostne stopnje opišeta s funkcijami Grassmannovih

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koordinat θ^a in ustreznih konjugiranih momentov, pa tudi polja, katerih notranje prostostne stopnje so opisane s funkcijami Cliffordovih koordinat γ^a. Uporabo za opis fermionov v Grassmannovem prostoru je navdihnila teorija *spinov-nabojev-družin* ([1,2,9,3], in reference v njih), v kateri akcijo v d-razsežnem prostoru opiše eden od avtorjev (N.S.M.B.) z $\int d^d x\, E\, \frac{1}{2}\, (\bar{\psi}\, \gamma^a p_{0a} \psi) + \text{h.c.}$, s kovariantnim odvodom $p_{0a} = f^{\alpha}{}_a p_{0\alpha} + \frac{1}{2E}\, \{p_\alpha, E f^{\alpha}{}_a\}_-$, $p_{0\alpha} = p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}$. Bazna stanja iščeta kot produkt bodisi Grassmannovih bodisi Cliffordovih "koordinat", ki dopuščajo drugo kvantizacijo, ponudita akcijo za prosta polja tudi v primeru Grassmannovih koordinat, da bi bolje razumela, zakaj je v tekmi za fizikalne prostostne stopnje "zmagala" Cliffordova algebra.

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9.1 Introduction

This paper is to look for the answers to the questions: Why our universe "uses" the Clifford rather than the Grassmann coordinates, although both lead in the second quantization procedure to the anti-commutation relations required for fermion degrees of freedom? Is the answer that the Clifford degrees of freedom offer the appearance of families, the half integer spin and the charges as observed so far for fermions, while the Grassmann coordinates offer the groups of (isolated) integer spin states with the charges in the adjoint representations and no families? Can the choice of the Clifford degrees of freedom explain why the simple starting action of the *spin-charge-family* theory of one of us (N.S.M.B.) [9,3,5,8,4,6,7] is doing so far extremely well in manifesting the observed properties of the fermion and boson fields in the observed low energy regime?

The questions are too demanding that this paper could offer the answers. We are trying only to make first steps towards understanding them.

Our working hypothesis is that "*nature knows all the mathematics*", accordingly therefore also both — the Grassmann and the Clifford "coordinates". In a trial to understand why Grassmann space "*was not the choice of nature*" to describe the internal degrees of freedom of fermions, we see that γ^a's and $\tilde{\gamma}^a$'s of the *spin-charge-family* theory enable to describe not only the spin and charges of fermions, but also the existence of families of fermions (in the first and second quantized theory of fields).

This work is a part of the project of both authors, which includes the *fermionization* procedure of boson fields or the *bosonization* procedure of fermion fields, discussed in Refs. [11,12,14] for any dimension d (by the authors of this contribution, while one of them, H.B.F.N. [13], has succeeded with another author to do the *fermionization* for $d = (1+1)$), and which would hopefully help to better understand the content and dynamics of our universe.

In the *spin-charge-family* theory [9,3,5,8,4,6,7] — which offers explanations for all the assumptions of the *standard model*, with the appearance of families, the scalar higgs and the Yukawa couplings included, offering also the explanation for the matter-antimatter asymmetry in our universe and for the appearance of the dark matter — a very simple starting action for massless fermions and bosons in $d = (1+13)$ is assumed, in which massless fermions interact with only gravity, the vielbeins $f^{\alpha}{}_{a}$ (the gauge fields of moments p_a) and the two kinds of the spin connections ($\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, the gauge fields of the two kinds of the Clifford algebra objects γ^a and $\tilde{\gamma}^a$, respectively).

$$\begin{aligned}\mathcal{A} = & \int d^d x \; E \; \frac{1}{2} \, (\bar{\psi} \, \gamma^a p_{0a} \psi) + \text{h.c.} + \\ & \int d^d x \; E \; (\alpha \, R + \tilde{\alpha} \, \tilde{R}) \, ,\end{aligned} \tag{9.1}$$

with $p_{0a} = f^{\alpha}{}_{a} p_{0\alpha} + \frac{1}{2E} \{p_\alpha, E f^{\alpha}{}_{a}\}_-$, $p_{0\alpha} = p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}$ and $R = \frac{1}{2} \{f^{\alpha[a} f^{\beta b]} \; (\omega_{ab\alpha,\beta} - \omega_{ca\alpha} \, \omega^{c}{}_{b\beta})\} + \text{h.c.}$, $\tilde{R} = \frac{1}{2} \{f^{\alpha[a} f^{\beta b]} \; (\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha} \, \tilde{\omega}^{c}{}_{b\beta})\} + \text{h.c.}$. The two kinds of the Clifford algebra objects, γ^a and $\tilde{\gamma}^a$,

$$\begin{aligned}\{\gamma^a, \gamma^b\}_+ &= 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ \, , \\ \{\gamma^a, \tilde{\gamma}^b\}_+ &= 0 \, .\end{aligned} \tag{9.2}$$

anticommute (γ^a and $\tilde{\gamma}^b$ are connected with the left and the right multiplication of the Clifford objects, there is no third kind of the Clifford operators). One kind of the objects, the generators $S^{ab} = \frac{i}{4} (\gamma^a \, \gamma^b - \gamma^b \, \gamma^a)$, determines spins and charges of spinors of any family, another kind, $\tilde{S}^{ab} = \frac{i}{4} (\tilde{\gamma}^a \, \tilde{\gamma}^b - \tilde{\gamma}^b \, \tilde{\gamma}^a)$, determines the family quantum numbers. Here [1] $f^{\alpha[a} f^{\beta b]} = f^{\alpha a} f^{\beta b} - f^{\alpha b} f^{\beta a}$. There are correspondingly two kinds of infinitesimal generators of the Lorentz transformations in the internal degrees of freedom — S^{ab} for $SO(13,1)$ and $\tilde{S}^{ab}$ for $\widetilde{SO}(13,1)$ — arranging states into representations.

The scalar curvatures R and $\tilde{R}$ determine dynamics of the gauge fields — the spin connections and the vielbeins, which manifest in $d = (3+1)$ all the known vector gauge fields as well as the scalar fields [5] which explain the appearance of higgs and the Yukawa couplings, provided that the symmetry breaks from the starting one $SO(13,1)$ to $SO(3,1) \times SU(3) \times U(1)$.

The infinitesimal generators of the Lorentz transformations for the gauge fields — the two kinds of the Clifford operators and the Grassmann operators — operate as follows, Eq. (9.25)

$$\begin{aligned}\{S^{ab}, \, \gamma^e\}_- &= -i \, (\eta^{ae} \, \gamma^b - \eta^{be} \, \gamma^a) \, , \\ \{\tilde{S}^{ab}, \, \tilde{\gamma}^e\}_- &= -i \, (\eta^{ae} \, \tilde{\gamma}^b - \eta^{be} \, \tilde{\gamma}^a) \, , \\ \{\mathbf{S}^{ab}, \, \theta^e\}_- &= -i \, (\eta^{ae} \, \theta^b - \eta^{be} \, \theta^a) \, , \\ \{\mathbf{M}^{ab}, \, A^{d \dots e \dots g}\}_- &= -i \, (\eta^{ae} \, A^{d \dots b \dots g} - \eta^{be} \, A^{d \dots a \dots g}) \, ,\end{aligned} \tag{9.3}$$

[1] $f^{\alpha}{}_{a}$ are inverted vielbeins to $e^{a}{}_{\alpha}$ with the properties $e^{a}{}_{\alpha} f^{\alpha}{}_{b} = \delta^{a}{}_{b}$, $e^{a}{}_{\alpha} f^{\beta}{}_{a} = \delta^{\beta}_{\alpha}$, $E = \det(e^{a}{}_{\alpha})$. Latin indices $a, b, .., m, n, .., s, t, ..$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, .., \mu, \nu, .. \sigma, \tau, ..$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index ($a, b, c, ..$ and $\alpha, \beta, \gamma, ..$), from the middle of both the alphabets the observed dimensions $0, 1, 2, 3$ ($m, n, ..$ and $\mu, \nu, ..$), indexes from the bottom of the alphabets indicate the compactified dimensions ($s, t, ..$ and $\sigma, \tau, ..$). We assume the signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.

where $\mathbf{M}^{ab}$ are defined by a sum of L^{ab} plus either S^{ab} or $\tilde{S}^{ab}$, in the Grassmann case $\mathbf{M}^{ab}$ is $L^{ab} + \mathbf{S}^{ab}$, which appear to be $\mathbf{M}^{ab} = L^{ab} + S^{ab} + \tilde{S}^{ab}$, as presented later in Eq. (9.26).

We discuss in what follows the first and the second quantization of the fields, the internal degrees of freedom of which are determined by the Grassmann coordinates θ^a, as well as of the fields, the internal degrees of freedom of which are determined by the Clifford coordinates γ^a (or $\tilde{\gamma}^a$) in order to understand why *"nature has made a choice"* of fermions of spins and charges (describable in the *spin-charge-family* theory by subgroups of the Lorentz group expressible with the generators S^{ab}) in the fundamental representations of the groups (which interact in the *spin-charge-family* theory through the boson gauge fields — the vielbeins and the spin connections of two kinds), rather than of fermions with the integer spins and charges. We choose correspondingly either θ^a's or γ^a's (or $\tilde{\gamma}^a$'s, either γ^a's or $\tilde{\gamma}^a$'s [6,7,9]) to describe the internal degrees of freedom of fields.

In all these cases we treat free massless fields; masses of the fields in $d = (3+1)$ are in the *spin-charge-family* theory due to their interactions with the gravitational fields in $d > 4$, described by the scalar vielbeins or spin connection fields [[1,2,9,3,5,8,4,6,7], and the references therein].

9.2 Observations helping to understand why Clifford algebra manifests in the observable $d = (3+1)$

We present in this section properties of fields with the integer spin in d-dimensional space, expressed in terms of the Grassmann algebra objects, and the spinor fields with the half integer spin, expressed in terms of the Clifford algebra objects. Since the Clifford algebra objects are expressible with the Grassmann algebra objects (Eqs. (9.17, 9.18)), the norms of both are determined by the integral in Grassmann space, Eqs. (9.28, 9.31) [2].

a. *Fields with the integer spin in Grassmann space*

A point in d-dimensional Grassmann space of real anticommuting coordinates θ^a, $(a = 0, 1, 2, 3, 5, \dots, d)$, is determined by a vector

$$\{\theta^a\} = (\theta^0, \theta^1, \theta^2, \theta^3, \theta^5, \dots, \theta^d).$$

A linear vector space over the coordinate Grassmann space has correspondingly the dimension 2^d, due to the fact that $(\theta^{a_i})^2 = 0$ for any $a_i \in (0, 1, 2, 3, 5, \dots, d)$.

Correspondingly are fields in Grassmann space expressed in terms of the Grassmann algebra objects

$$\mathbf{B} = \sum_{k=0}^{d} a_{a_1 a_2 \dots a_k}\, \theta^{a_1} \theta^{a_2} \dots \theta^{a_k} |\phi_{og}>, \quad a_i \le a_{i+1}\,, \tag{9.4}$$

[2] These observations might help also when fermionizing boson fields or bosonizing fermion fields.

where $|\phi_{og}>$ is the vacuum state, here assumed to be $|\phi_{og}>=|1>$, so that $\frac{\partial}{\partial\theta^a}|\phi_{og}>=0$ for any θ^a. The *Kalb-Ramond* boson fields $a_{a_1 a_2 \dots a_k}$ are antisymmetric with respect to the permutation of indexes, since the Grassmann coordinates anticommute

$$\{\theta^a, \theta^b\}_+ = 0 \,. \tag{9.5}$$

The left derivative $\frac{\partial}{\partial\theta_a}$ on vectors of the space of monomials $\mathbf{B}(\theta)$ is defined as follows

$$\begin{aligned} \frac{\partial}{\partial\theta_a}\mathbf{B}(\theta) &= \frac{\partial\mathbf{B}(\theta)}{\partial\theta_a}\,, \\ \left\{\frac{\partial}{\partial\theta_a}, \frac{\partial}{\partial\theta_b}\right\}_+ \mathbf{B} &= 0\,, \text{ for all } \mathbf{B}\,. \end{aligned} \tag{9.6}$$

Defining $p^{\theta a} = i\frac{\partial}{\partial\theta_a}$ it correspondingly follows

$$\{p^{\theta a}, p^{\theta b}\}_+ = 0\,, \quad \{p^{\theta a}, \theta^b\}_+ = i\eta^{ab}\,, \tag{9.7}$$

The metric tensor η^{ab} ($= diag(1, -1, -1, \dots, -1)$) lowers the indexes of a vector $\{\theta^a\}$: $\theta_a = \eta_{ab}\,\theta^b$, the same metric tensor lowers the indexes of the ordinary vector x^a of commuting coordinates.

Defining [3]

$$(\theta^a)^\dagger = \frac{\partial}{\partial\theta_a}\eta^{aa} = -i\,p^{\theta a}\eta^{aa}\,, \tag{9.8}$$

it follows

$$(\frac{\partial}{\partial\theta_a})^\dagger = \eta^{aa}\,\theta^a\,, \quad (p^{\theta a})^\dagger = -i\eta^{aa}\theta^a\,. \tag{9.9}$$

Making a choice for the complex properties of θ^a, and correspondingly of $\frac{\partial}{\partial\theta_a}$, as follows

$$\begin{aligned} \{\theta^a\}^* &= (\theta^0, \theta^1, -\theta^2, \theta^3, -\theta^5, \theta^6, \dots, -\theta^{d-1}, \theta^d)\,, \\ \{\frac{\partial}{\partial\theta_a}\}^* &= (\frac{\partial}{\partial\theta_0}, \frac{\partial}{\partial\theta_1}, -\frac{\partial}{\partial\theta_2}, \frac{\partial}{\partial\theta_3}, -\frac{\partial}{\partial\theta_5}, \frac{\partial}{\partial\theta_6}, \dots, -\frac{\partial}{\partial\theta_{d-1}}, \frac{\partial}{\partial\theta_d})\,, \end{aligned} \tag{9.10}$$

it follows for the two Clifford algebra objects $\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$, and $\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial\theta_a})$, Eqs. (9.17, 9.18), that γ^a is real if θ^a is real, and imaginary if θ^a is imaginary, while $\tilde{\gamma}^a$ is imaginary when θ^a is real and real if θ^a is imaginary, just as it is required in Eq. (9.23).

We define here the commuting object γ^a_G, which will be useful to find the action for Grassmann fermions, Eq. (9.37), and the appropriate discrete symmetry operators for this purpose — ($\mathcal{C}_\mathcal{G}$, $\mathcal{T}_\mathcal{G}$, $\mathcal{P}_\mathcal{G}$) in $((d-1)+1)$-dimensional space-time

[3] In Ref. [2] the definition of $\theta^{a\dagger}$ was differently chosen. Correspondingly also the scalar product needed a (slightly) different weight function in Eq. (9.28).

and $(\mathcal{C}_\mathcal{N}, \mathcal{T}_\mathcal{N}, \mathcal{P}_\mathcal{N})$ in $(3+1)$ space-time — while following the definitions of the discrete symmetry operators in the Clifford algebra case [21]

$$\begin{aligned}\gamma^a_G &= (1 - 2\theta^a \eta^{aa} \frac{\partial}{\partial \theta_a}) \\ &= -i\eta^{aa}\gamma^a \tilde{\gamma}^a\,, \\ \{\gamma^a_G, \gamma^b_G\}_- &= 0\,.\end{aligned} \tag{9.11}$$

Index ^{a} is not the Lorentz index in the usual sense. γ^a_G are commuting operators — $\{\gamma^a_G, \gamma^b_G\}_- = 0$ for all (a, b) — as expected. They are real and Hermitian.

$$\gamma^{a\dagger}_G = \gamma^a_G\,, \quad (\gamma^a_G)^* = \gamma^a_G\,. \tag{9.12}$$

Correspondingly it follows: $\gamma^{a\dagger}_G \gamma^a_G = I$, $\gamma^a_G \gamma^a_G = I$. I represents the unit operator.

By introducing [2] the generators of the infinitesimal Lorentz transformations in Grassmann space as

$$\mathbf{S}^{ab} = (\theta^a p^{\theta b} - \theta^b p^{\theta a})\,, \tag{9.13}$$

one finds

$$\begin{aligned}\{\mathbf{S}^{ab}, \mathbf{S}^{cd}\}_- &= i\{\mathbf{S}^{ad}\eta^{bc} + \mathbf{S}^{bc}\eta^{ad} - \mathbf{S}^{ac}\eta^{bd} - \mathbf{S}^{bd}\eta^{ac}\}\,, \\ \mathbf{S}^{ab\dagger} &= \eta^{aa}\eta^{bb}\mathbf{S}^{ab}\,.\end{aligned} \tag{9.14}$$

The basic states in Grassmann space can be arranged into representations with respect to the Cartan subalgebra of the Lorentz algebra as presented in Ref. [2,15]. The state in d-dimensional space, for example, with all the eigenvalues of the Cartan subalgebra of the Lorentz group of Eq. (9.84) equal to either i or 1 is: $(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)|\phi_{og}>$, with $|\phi_{og}>= |1>$. All the states of the representation, which start with this state, follow by the application of those $\mathbf{S}^{ab}$, which do not belong to the Cartan subalgebra of the Lorentz algebra. $\mathbf{S}^{01}$, for example, transforms $(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)|\phi_{og}>$ into $(\theta^0\theta^3 + i\theta^1 i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)|\phi_{og}>$, while $\mathbf{S}^{01} - i\mathbf{S}^{02}$ transforms this state into $(\theta^0 + \theta^3)(\theta^1 - i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)|\phi_{og}>$.

b. *Fermion fields with the half integer spin and the Clifford objects*

Let us present as well the properties of the fermion fields with the half integer spin, expressed by the Clifford algebra objects

$$\mathbf{F} = \sum_{k=0}^{d} a_{a_1 a_2 \ldots a_k}\, \gamma^{a_1}\gamma^{a_2}\ldots\gamma^{a_k}|\psi_{oc}>\,, \quad a_i \le a_{i+1}\,, \tag{9.15}$$

where $|\psi_{oc}>$ is the vacuum state. The *Kalb-Ramond* fields $a_{a_1 a_2 \ldots a_k}$ are again in general boson fields, which are antisymmetric with respect to the permutation of indexes, since the Clifford objects have the anticommutation relations, Eq. (9.2),

$$\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}\,. \tag{9.16}$$

A linear vector space over the Clifford coordinate space has again the dimension 2^d, due to the fact that $(\gamma^{a_i})^2 = \eta^{a_i a_i}$ for any $a_i \in (0, 1, 2, 3, 5, \ldots, d)$.

One can see that γ^a are expressible in terms of the Grassmann coordinates and their conjugate momenta as

$$\gamma^a = (\theta^a - i p^{\theta a}) \,. \tag{9.17}$$

We also find $\tilde{\gamma}^a$

$$\tilde{\gamma}^a = i (\theta^a + i p^{\theta a}) \,, \tag{9.18}$$

with the anticommutation relation of Eq. (9.16) for either γ^a and $\tilde{\gamma}^a$

$$\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2\eta^{ab} \,, \quad \{\gamma^a, \tilde{\gamma}^b\}_+ = 0 \,. \tag{9.19}$$

Taking into account Eqs. (9.8, 9.17, 9.18) one finds

$$\begin{aligned}
&(\gamma^a)^\dagger = \gamma^a \eta^{aa} \,, \quad (\tilde{\gamma}^a)^\dagger = \tilde{\gamma}^a \eta^{aa} \,, \\
&\gamma^a \gamma^a = \eta^{aa} \,, \quad \gamma^a (\gamma^a)^\dagger = I \,, \quad \tilde{\gamma}^a \tilde{\gamma}^a = \eta^{aa} \,, \quad \tilde{\gamma}^a (\tilde{\gamma}^a)^\dagger = I \,,
\end{aligned} \tag{9.20}$$

where I represents the unit operator. Making a choice for the θ^a properties as presented in Eq. (9.10), it follows for the Clifford objects

$$\begin{aligned}
&\{\gamma^a\}^* = (\gamma^0, \gamma^1, -\gamma^2, \gamma^3, -\gamma^5, \gamma^6, ..., -\gamma^{d-1}, \gamma^d) \,, \\
&\{\tilde{\gamma}^a\}^* = (-\tilde{\gamma}^0, -\tilde{\gamma}^1, \tilde{\gamma}^2, -\tilde{\gamma}^3, \tilde{\gamma}^5, -\tilde{\gamma}^6, ..., \tilde{\gamma}^{d-1}, -\tilde{\gamma}^d) \,,
\end{aligned} \tag{9.21}$$

All three choices for the linear vector space — spanned over either the coordinate Grassmann space, or over the vector space of γ^a, as well as over the vector space of $\tilde{\gamma}^a$ — have the dimension 2^d.

We can express Grassmann coordinates θ^a and momenta $p^{\theta a}$ in terms of γ^a and $\tilde{\gamma}^a$ as well [4]

$$\begin{aligned}
\theta^a &= \frac{1}{2}(\gamma^a - i\tilde{\gamma}^a) \,, \\
\frac{\partial}{\partial \theta_a} &= \frac{1}{2} (\gamma^a + i\tilde{\gamma}^a) \,.
\end{aligned} \tag{9.22}$$

It then follows $\frac{\partial}{\partial \theta_b} \theta^a |1> = \eta^{ab} |1>$.

Correspondingly we can use either γ^a or $\tilde{\gamma}^a$ instead of θ^a to span the vector space. In this case we change the vacuum from the one with the property $\frac{\partial}{\partial \theta^a} |\phi_{og}> = 0$ to $|\psi_{oc}>$ with the property [2,7,9]

$$\begin{aligned}
&< \psi_{oc}|\gamma^a|\psi_{oc}> = 0 \,, \quad \tilde{\gamma}^a|\psi_{oc}> = i\gamma^a|\psi_{oc}> \,, \quad \tilde{\gamma}^a\gamma^b|\psi_{oc}> = -i\gamma^b\gamma^a|\psi_{oc}> \,, \\
&\tilde{\gamma}^a\tilde{\gamma}^b|\psi_{oc}>|_{a\neq b} = -\gamma^a\gamma^b|\psi_{oc}> \,, \quad \tilde{\gamma}^a\tilde{\gamma}^b|\psi_{oc}>|_{a=b} = \eta^{ab}|\psi_{oc}> \,.
\end{aligned} \tag{9.23}$$

[4] In Ref. [28] the author suggested in Eq. (47) a choice of superposition of γ^a and $\tilde{\gamma}^a$, which resembles the choice of one of the authors (N.S.M.B.) in Ref. [2] and both authors in Ref. [16,17] and in present article.

This is in agreement with the requirement

$$\begin{aligned}
&\gamma^a\, \mathbf{F}(\gamma)\, |\psi_{oc}>:= \\
&(\, a_0\, \gamma^a + a_{a_1}\, \gamma^a \gamma^{a_1} + a_{a_1 a_2}\, \gamma^a \gamma^{a_1}\gamma^{a_2} + \cdots + a_{a_1 \cdots a_d}\, \gamma^a \gamma^{a_1} \cdots \gamma^{a_d}\,)\, |\psi_{oc}>, \\
&\tilde{\gamma}^a\, \mathbf{F}(\gamma)\, |\psi_{oc}>:= (\, i\, a_0 \gamma^a - i\, a_{a_1} \gamma^{a_1}\, \gamma^a + i\, a_{a_1 a_2} \gamma^{a_1} \gamma^{a_2}\, \gamma^a + \cdots + \\
&\qquad i\, (-1)^d\, a_{a_1 \cdots a_d} \gamma^{a_1} \cdots \gamma^{a_d}\, \gamma^a\,)\, |\psi_{oc}> . \qquad (9.24)
\end{aligned}$$

We find the infinitesimal generators of the Lorentz transformations in Clifford space

$$\begin{aligned}
S^{ab} &= \frac{i}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a)\,, \quad S^{ab\dagger} = \eta^{aa}\eta^{bb} S^{ab}\,, \\
\tilde{S}^{ab} &= \frac{i}{4}(\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a)\,, \quad \tilde{S}^{ab\dagger} = \eta^{aa}\eta^{bb} \tilde{S}^{ab}\,,
\end{aligned} \qquad (9.25)$$

with the commutation relations for either S^{ab} or $\tilde{S}^{ab}$ of Eq. (9.14), if $\mathbf{S}^{ab}$ is replaced by either S^{ab} or $\tilde{S}^{ab}$, respectively, while

$$\begin{aligned}
\mathbf{S}^{ab} &= S^{ab} + \tilde{S}^{ab}\,, \\
\{S^{ab}, \tilde{S}^{cd}\}_- &= 0\,.
\end{aligned} \qquad (9.26)$$

The basic states in Clifford space can be arranged in representations, in which any state is the eigenstate of the Cartan subalgebra operators of Eq. (9.84). The state, for example, in d-dimensional space with the eigenvalues of either S^{03}, S^{12}, $S^{56}, \ldots, S^{d-1\, d}$ or $\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \ldots, \tilde{S}^{d-1\, d}$ equal to $\frac{1}{2}(i, 1, 1, \ldots, 1)$ is $(\gamma^0 - \gamma^3)(\gamma^1 + i\gamma^2)(\gamma^5 + i\gamma^6) \cdots (\gamma^{d-1} + i\gamma^d)$, where the states are expressed in terms of γ^a. The states of one representation follow from the starting state by the application of S^{ab}, which do not belong to the Cartan subalgebra operators, while $\tilde{S}^{ab}$, which operate on family quantum numbers, cause jumps from the starting family to the new one.

9.2.1 Norms of vectors in Grassmann and Clifford space

Let us look for the norm of vectors in Grassmann space

$$\mathbf{B} = \sum_{k=0}^{d} a_{a_1 a_2 \ldots a_k}\, \theta^{a_1} \theta^{a_2} \ldots \theta^{a_k} |\phi_{og}>$$

and in Clifford space

$$\mathbf{F} = \sum_{k=0}^{d} a_{a_1 a_2 \ldots a_k}\, \gamma^{a_1} \gamma^{a_2} \ldots \gamma^{a_k} |\psi_{oc}>,$$

where $|\phi_{og}>$ and $|\phi_{oc}>$ are the vacuum states in the Grassmann and Clifford case, respectively. In what follows we refer to Ref. [2].

a. *Norms of the Grassmann vectors*

Let us define the integral over the Grassmann space [2] of two functions of the Grassmann coordinates $< \mathbf{B}|\mathbf{C} >$, $< \mathbf{B}|\theta >=< \theta|\mathbf{B} >^{\dagger}$, by requiring

$$
\begin{aligned}
&\{d\theta^a, \theta^b\}_+ = 0, \quad \int d\theta^a = 0\,, \quad \int d\theta^a \theta^a = 1\,,\\
&\int d^d\theta \; \theta^0\theta^1 \cdots \theta^d = 1\,,\\
&d^d\theta = d\theta^d \ldots d\theta^0\,,\\
&\omega = \Pi^d_{k=0}(\frac{\partial}{\partial\theta_k} + \theta^k)\,, \qquad (9.27)
\end{aligned}
$$

with $\frac{\partial}{\partial\theta_a}\theta^c = \eta^{ac}$. We shall use the weight function $\omega = \Pi^d_{k=0}(\frac{\partial}{\partial\theta_k} + \theta^k)$ to define the scalar product $< \mathbf{B}|\mathbf{C} >$

$$
< \mathbf{B}|\mathbf{C} > = \int d^{d-1}x d^d\theta^a \;\, \omega < \mathbf{B}|\theta >< \theta|\mathbf{C} >= \sum_{k=0}^{d} \int d^{d-1}x \, b^*_{b_1\ldots b_k} c_{b_1\ldots b_k}\,, \qquad (9.28)
$$

where, according to Eq. (9.8), follows:

$$
< \mathbf{B}|\theta >=< \phi_{og}| \sum_{p=0}^{d} (-i)^p \; a^*_{a_1\ldots a_p} p^{\theta a_p} \eta^{a_p a_p} \cdots p^{\theta a_1} \eta^{a_1 a_1}\,.
$$

The vacuum state is chosen to be $|\phi_{og} >= |1 >$, as taken in Eq. (9.4).

The norm $< \mathbf{B}|\mathbf{B} >$ is correspondingly always nonnegative.

b. *Norms of the Clifford vectors*

Let us look for the norm of vectors, expressed with the Clifford objects $\mathbf{F} = \sum_k^d a_{a_1 a_2 \ldots a_k} \gamma^{a_1}\gamma^{a_2} \ldots \gamma^{a_k}|\psi_{oc} >$, where $|\phi_{og} >$ and $|\psi_{oc} >$ are the two vacuum states when the Grassmann and the Clifford objects are concerned, respectively. By taking into account Eq. (9.20) it follows that

$$
(\gamma^{a_1}\gamma^{a_2} \ldots \gamma^{a_k})^{\dagger} = \gamma^{a_k}\eta^{a_k a_k} \ldots \gamma^{a_2}\eta^{a_2 a_2}\gamma^{a_1}\eta^{a_1 a_1}\,, \qquad (9.29)
$$

since $\gamma^a \gamma^a = \eta^{aa}$.

We can use Eqs. (9.27, 9.28) to evaluate the scalar product of two Clifford algebra objects $< \gamma|\mathbf{F} >=< (\theta^a - ip^{\theta a})|\mathbf{F} >$ and equivalently for $< (\theta^a - ip^{\theta a})|\mathbf{G} >$. These expressions follow from Eqs. (9.17, 9.18, 9.20)). We must then choose for the vacuum state the one from the Grassmann case — $|\psi_{oc} >= |\phi_{og} >= |1 >$. It follows

$$
< \mathbf{F}|\mathbf{G} > = \int d^{d-1}x d^d\theta^a \;\, \omega < \mathbf{F}|\gamma >< \gamma|\mathbf{G} >= \sum_{k=0}^{d} \int d^{d-1}x \, a^*_{a_1\ldots a_k} b_{b_1\ldots b_k}\,. \qquad (9.30)
$$

{Similarly we obtain, if we express $\tilde{\mathbf{F}} = \sum_{k=0}^d a_{a_1 a_2 \ldots a_k} \tilde{\gamma}^{a_1}\tilde{\gamma}^{a_2} \ldots \tilde{\gamma}^{a_k}|\phi_{oc} >$ and $\tilde{\mathbf{G}} = \sum_{k=0}^d b_{b_1 b_2 \ldots b_k} \tilde{\gamma}^{b_1}\tilde{\gamma}^{b_2} \ldots \tilde{\gamma}^{b_k}|\phi_{oc} >$ and take $|\psi_{oc} >= |\phi_{og} >= |1 >$, the scalar product

$$< \tilde{\mathbf{F}}|\tilde{\mathbf{G}} > = \int d^{d-1}x d^d\theta^a \; \omega < \tilde{\mathbf{F}}|\tilde{\gamma} >< \tilde{\gamma}|\tilde{\mathbf{G}} >= \sum_{k=0}^{d} \int d^{d-1}x \, a^*_{a_1\dots a_k} a_{b_1\dots b_k} .\} \tag{9.31}$$

Correspondingly we can write

$$\int d^d\theta^a \; \omega(a_{a_1 a_2 \dots a_k} \gamma^{a_1}\gamma^{a_2} \dots \gamma^{a_k})^\dagger (a_{a_1 a_2 \dots a_k} \gamma^{a_1}\gamma^{a_2} \dots \gamma^{a_k}) = a^*_{a_1 a_2 \dots a_k} a_{a_1 a_2 \dots a_k} . \tag{9.32}$$

The norm of each scalar term in the sum of **F** is nonnegative.

c. We have learned that in both spaces — Grassmann and Clifford — norms of basic states can be defined so that the states, which are eigenvectors of the Cartan subalgebra, are orthogonal and normalized using the same integral.

Studying the second quantization procedure in Subsect. 9.2.3 we learn that not all 2^d states can be represented as creation and annihilation operators, either in the Grassmann or in the Clifford case, since they must — in both cases — fulfill the requirements for the second quantized operators, either for states with integer spins in Grassmann space or for states with half integer spin in Clifford space.

9.2.2 Actions in Grassmann and Clifford space

Let us construct an action for free massless particles in which the internal degrees of freedom will be described: **i.** by states in Grassmann space, **ii.** by states in Clifford space. In the first case the internal degrees of freedom manifest the integer spin, in the second case the internal degrees of freedom manifest the half integer spin.

While the action in Clifford space is well known since long [22], the action in Grassmann space must be found. We shall represent it here. In both cases we look for actions for free massless states in $((d-1)+1)$ space [5]. States in Grassmann space as well as states in Clifford space will be organized to be — within each of the two spaces — orthogonal and normalized with respect to Eq. (9.27). We choose the states in each of two spaces to be the eigenstates of the Cartan subalgebra — with respect to $\mathbf{S}^{ab}$ in Grassmann space and with respect to S^{ab} and $\tilde{S}^{ab}$ in Clifford space, Eq. (9.84).

In both spaces the requirement that states are obtained by the application of creation operators on the vacuum states — $\hat{b}^{\theta}_i$ obeying the commutation relations of Eq. (9.48) on the vacuum state $|\phi_{og} >= |1>$ in Grassmann space, and $\hat{b}^{\alpha}_i$ obeying the commutation relation of Eq. (9.60) on the vacuum states $|\psi_{oc}>$, Eq. (9.67), in Clifford space — reduces the number of states, in Clifford space more than in Grassmann space. But while in Clifford space all physically applicable states are reachable by either S^{ab} (defining family members quantum numbers)

[5] In $(3+1)$ space the mass is due to the interaction of particles with the scalar fields, with which the particles interact in $((d-1)+1)$ space.

or by $\tilde{S}^{ab}$ (defining family quantum numbers), the states in Grassmann space, belonging to different representations with respect to the Lorentz generators, seem not to be connected.

a. *Action in Clifford space*

In Clifford space the action for a free massless object must be Lorentz invariant

$$\mathcal{A} = \int d^d x \, \frac{1}{2} \, (\psi^{\dagger} \gamma^0 \, \gamma^a p_a \psi) + h.c. \,, \tag{9.33}$$

$p_a = i \, \frac{\partial}{\partial x^a}$, leading to the equations of motion

$$\gamma^a p_a |\psi^\alpha> = 0 \,, \tag{9.34}$$

which fulfill also the Klein-Gordon equation

$$\gamma^a p_a \gamma^b p_b |\psi^\alpha_i> = p^a p_a |\psi^\alpha_i> = 0 \,, \tag{9.35}$$

for each of the basic states $|\psi^\alpha_i>$. Correspondingly γ^0 appears in the action since we pay attention that

$$\begin{aligned} S^{ab\dagger} \gamma^0 &= \gamma^0 \, S^{ab} \,, \\ S^{\dagger} \gamma^0 &= \gamma^0 S^{-1} \,, \\ S &= e^{-\frac{i}{2} \omega_{ab} (S^{ab} + L^{ab})} \,. \end{aligned} \tag{9.36}$$

We choose the basic states to be the eigenstates of all the members of the Cartan subalgebra, Eq. (9.84). Correspondingly all the states, belonging to different values of the Cartan subalgebra — they differ at least in one value of either the set of S^{ab} or the set of $\tilde{S}^{ab}$, Eq. (9.84) — are orthogonal with respect to the scalar product defined as the integral over the Grassmann coordinates, Eq. (9.27), for a chosen vacuum state. Correspondingly the states generated by the creation operators, Eq. (9.65), on the vacuum state, Eq. (9.67), are orthogonal as well (both last equations will appear later).

b. *Action in Grassmann space*

We define here the action in Grassmann space, for which we require — similarly as in the Clifford case — that the action for a free massless object

$$\mathcal{A} = \frac{1}{2} \{ \int d^d x \, d^d \theta \, \omega \, (\phi^{\dagger} (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) \, \frac{1}{2} \, (\theta^a p_a + \eta^{aa} \theta^{a\dagger} p_a) \phi \} \,, \tag{9.37}$$

is Lorentz invariant. We use the integral also over θ^a coordinates, with the weight function ω from Eq. (9.27). Requiring the Lorentz invariance we add after $\phi^\dagger$ the operator γ^0_G ($\gamma^a_G = (1 - 2\theta^a \frac{\partial}{\partial \theta^a})$), which takes care of the Lorentz invariance. Namely

$$\begin{aligned} \mathbf{S}^{ab\dagger} \, (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) &= (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) \, \mathbf{S}^{ab} \,, \\ \mathbf{S}^{\dagger} \, (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) &= (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) \, \mathbf{S}^{-1} \,, \\ \mathbf{S} &= e^{-\frac{i}{2} \omega_{ab} (L^{ab} + \mathbf{S}^{ab})} \,, \end{aligned} \tag{9.38}$$

while θ^a, $\frac{\partial}{\partial \theta_a}$ and p^a transform as Lorentz vectors. The equation of motion follow from the action, Eq. (9.37),

$$\frac{1}{2}[(1-2\theta^0\frac{\partial}{\partial\theta^0})\,\theta^a + ((1-2\theta^0\frac{\partial}{\partial\theta^0})\,\theta^a)^\dagger]\;p_a\,|\phi^\theta_i> \;=0\,, \tag{9.39}$$

as well as the Klein-Gordon equation

$$\{(1-2\theta^0\frac{\partial}{\partial\theta^0})\theta^a p_a\}^\dagger\,\theta^b p_b|\phi^\theta_i> = p^a p_a|\phi^\theta_i>=0\,, \tag{9.40}$$

for each of the basic states $|\psi^\alpha_i>$.

c. *We learned*:

In both spaces — in Clifford and in Grassmann space — there exists the action, which leads to the equations of motion and to the corresponding Klein-Gordon equation for free massless particles. In both cases we use the operator, which does not change the Clifford or Grassmann character of states.

We shall see that, if one identifies the creation operators in both spaces with the products of odd numbers of either θ^a (in the Grassmann case) or γ^a (in the Clifford case) and the annihilation operators with their Hermitian conjugate operators, the creation and annihilation operators fulfill the anticommutation relations, required for fermions. The internal parts of states are then defined by the application of the creation operators on the vacuum state. But while the Clifford algebra defines spinors with the half integer eigenvalues of the Cartan subalgebra operators of the Lorentz algebra, the Grassmann algebra defines states with the integer eigenvalues of the Cartan subalgebra.

9.2.3 Second quantization of Grassmann vectors and Clifford vectors

States in Grassmann space as well as states in Clifford space are organized to be — within each of the two spaces — orthogonal and normalized with respect to Eq. (9.27). All the states in each of spaces are chosen to be eigenstates of the Cartan subalgebra — with respect to $\mathbf{S}^{ab}$ in Grassmann space, and with respect to S^{ab} and $\tilde{S}^{ab}$ in Clifford space, Eq. (9.84).

In both spaces the requirement that states are obtained by the application of creation operators on vacuum states — $\hat{b}^\theta_i$ obeying the commutation relations of Eqs. (9.42, 9.48) on the vacuum state $|\phi_{og}>=|1>$ for Grassmann space, and $\hat{b}^\alpha_i$ obeying the commutation relation of Eq. (9.60) on the vacuum states $|\psi_{oc}>$, Eq. (9.67), for Clifford space — reduces the number of states arranged into the representations of the Lorentz group. The reduction of degrees of freedom depends on whether $d=2(2n+1)$ or $d=4n$, n is a positive integer. The second quantization procedure with creation operators expressed by the product of Grassmann or Clifford objects requires that the product has an odd number of objects.

We shall pay attention in this paper almost only to spaces with $d=2(2n+1)$ [6].

[6] The main reason that we treat here mostly $d=2(2n+1)$ spaces is that one Weyl representation, expressed by the product of the Clifford algebra objects, manifests in $d=(1+3)$ all the observed properties of quarks and leptons, if $d\geq 2(2n+1), n=3$.

We define in Grassmann space creation operators by an odd number of factors of superposition of θ^a's and annihilation operators by Hermitian conjugation of the corresponding creation operators. In Clifford space we define creation operators by an odd number of factors of superposition of γ^a's and the annihilation operators by Hermitian conjugate creation operators. Each basic state is a product of factors chosen to be eigenstates of the Cartan subalgebra of the Lorentz algebra.

But while in Clifford space all physically applicable states are reachable either by S^{ab} or by $\tilde{S}^{ab}$, the states, belonging to different groups with respect to the Lorentz generators, in Grassmann space two different representations of the Lorentz group are not connected by the Lorentz operators.

Let us construct creation and annihilation operators for the cases that we use **a.** Grassmann vector space, **b.** Clifford vector space. We shall see that from 2^d states in either of these two spaces there are reduced number of states generated by the creation operators, which fulfill the requirements for the creation and their Hermitian conjugate annihilation operators.

a. Quantization in Grassmann space

There are 2^d states in Grassmann space, orthogonal to each other with respect to Eq. (9.27). To any coordinate there exists the conjugate momentum. We pay attention in what follows mostly to spaces with $d = 2(2n+1)$, although also spaces with $d = 4n$ will be treated. In $d = 2(2n+1)$ spaces there are $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ states, Eq. (9.51), divided into two separated groups of states, all states of one group reachable from a starting state by $\mathbf{S}^{ab}$. These states are Grassmann odd products of eigenstates of the Cartan subalgebra. We use these products to define the creation operators and their Hermitian conjugate operators as the annihilation operators, fulfilling requirements of Eq. (9.41, 9.42). Let us see how it goes.

If $\hat{b}^{\theta\dagger}_i$ is a creation operator, which creates a state in the Grassmann space when operating on a vacuum state $|\psi_{og}>$ and $\hat{b}^{\theta}_i = (\hat{b}^{\theta\dagger}_i)^\dagger$ is the corresponding annihilation operator, then for a set of creation operators $\hat{b}^{\theta\dagger}_i$ and the corresponding annihilation operators $\hat{b}^{\theta}_i$ it must be

$$\begin{aligned} \hat{b}^{\theta}_i|\phi_{og}> &= 0\,, \\ \hat{b}^{\theta\dagger}_i|\phi_{og}> &\neq 0\,. \end{aligned} \tag{9.41}$$

We first pay attention on only the internal degrees of freedom — the spin.

Choosing $\hat{b}^{\theta}_a = \frac{\partial}{\partial\theta^a}$ it follows

$$\begin{aligned} \hat{b}^{\theta\dagger}_a &= \theta^a\,, \\ \hat{b}^{\theta}_a &= \frac{\partial}{\partial\theta^a}\,, \\ \{\hat{b}^{\theta}_a, \hat{b}^{\theta\dagger}_b\}_+|\phi_{og}> &= \delta_{ab}|\phi_{og}>\,, \\ \{\hat{b}^{\theta}_a, \hat{b}^{\theta}_b\}_+|\phi_{og}> &= 0\,, \\ \{\hat{b}^{\theta\dagger}_a, \hat{b}^{\theta\dagger}_b\}_+|\phi_{og}> &= 0\,, \\ \hat{b}^{\dagger\theta}_a\;|\phi_{og}> &= \theta^a|\phi_{og}>\,, \\ \hat{b}^{\theta}_a\;|\phi_{og}> &= 0\,. \end{aligned} \tag{9.42}$$

The vacuum state $|\phi_{og}>$ is in this case $|1>$.

The identity I ($I^\dagger = I$) can not be taken as a creation operator, since its annihilation partner does not fulfill Eq. (9.41).

We can use the products of superposition of θ^a's as creation and products of superposition of $\frac{\partial}{\partial\theta_a}$'s as annihilation operators provided that they fulfill the requirements for the creation and annihilation operators, Eq. (9.48), with the vacuum state $|\phi_{og}>=|1>$. In general they would not. Only an odd number of θ^a in any product would have the required anticommutation properties.

It is convenient to take products of superposition of vectors θ^a and θ^b to construct creation operators so that each factor is the eigenstate of one of the Cartan subalgebra member of the Lorentz algebra (9.84). We can start with the creation operators as products of $\frac{d}{2}$ states $\hat{b}^{\theta\dagger}_{a_i b_i} = \frac{1}{\sqrt{2}}(\theta^{a_i} \pm \epsilon\theta^{b_i})$. Then the corresponding annihilation operators have $\frac{d}{2}$ factors of $\hat{b}^{\theta}_{a_i b_i} = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial\theta^{a_i}} \pm \epsilon^* \frac{\partial}{\partial\theta_{b_i}})$, $\epsilon = i$, if $\eta^{a_i a_i} = \eta^{b_i b_i}$ and $\epsilon = -1$, if $\eta^{a_i a_i} \neq \eta^{b_i b_i}$.

In $d = 2(2n+1)$, n is a positive integer, we can start with the state

$$|\phi^{\theta 1}_1> = (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)|1> \,. \tag{9.43}$$

The rest of states, belonging to the same Lorentz representation, follows from the starting state by the application of the operators $\mathbf{S}^{cf}$, which do not belong to the Cartan subalgebra operators.

Let us add that in $d = 4n$ we should start with the state

$$|\phi^{\theta 1}_1>|_{4n} = \\ (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1} (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-3} + i\theta^{d-2})\theta^{d-1}\theta^d|1> \,. \tag{9.44}$$

Again the rest of states, belonging to the same Lorentz representation, follow from the starting state by the application of the operators $\mathbf{S}^{cf}$, which do not belong to the Cartan subalgebra operators.

i. Taking into account Eqs. (9.8, 9.9, 9.43) one can propose the following starting creation operator and the corresponding annihilation operator

$$\begin{aligned}
\hat{b}^{\theta 1\dagger}_i &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)\,,\\
\hat{b}^{\theta 1}_i &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^d})\cdots(\frac{\partial}{\partial\theta^0} - \frac{\partial}{\partial\theta^3})\,,\\
&\quad \text{for } d = 2(2n+1)\,,\\
\hat{b}^{\theta 1\dagger}_i &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1} (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-3} + i\theta^{d-2})\theta^{d-1}\theta^d\,,\\
\hat{b}^{\theta 1}_i &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1} \frac{\partial}{\partial\theta^d}\frac{\partial}{\partial\theta^{d-1}}(\frac{\partial}{\partial\theta^{d-3}} - i\frac{\partial}{\partial\theta^{d-2}})\cdots(\frac{\partial}{\partial\theta^0} - \frac{\partial}{\partial\theta^3})\,,\\
&\quad \text{for } d = 4n\,.
\end{aligned} \tag{9.45}$$

The rest of the creation operators belonging to this group in either $d = 2(2n+1)$ or in $d = 4n$ follows by the application of all the operators $\mathbf{S}^{ef}$, which do not belong to the Cartan subalgebra operators. The corresponding annihilation operators follow by the Hermitian conjugation of a particular creation operator. One finds, for example for $d = 2(2n+1)$,

$$\begin{aligned}
\hat{b}^{\theta 1 \dagger}_{j} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\, (\theta^0\theta^3 + i\theta^1\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^{d})\,,\\
\hat{b}^{\theta 1}_{j} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\, (\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^{d}})\cdots(\frac{\partial}{\partial\theta^{3}}\,\frac{\partial}{\partial\theta^{0}} - i\frac{\partial}{\partial\theta^{2}}\,\frac{\partial}{\partial\theta^{1}})\,.\\
&\cdots
\end{aligned} \tag{9.46}$$

For $d = 4n$ one finds equivalently

$$\begin{aligned}
\hat{b}^{\theta 1 \dagger}_{j} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-2}\, (\theta^0\theta^3 + i\theta^1\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-3} + i\theta^{d-2})\,\theta^{d-1}\theta^{d}\,,\\
\hat{b}^{\theta 1}_{j} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-2}\, \frac{\partial}{\partial\theta^{d}}\frac{\partial}{\partial\theta^{d-1}}\,(\frac{\partial}{\partial\theta^{d-3}} - i\frac{\partial}{\partial\theta^{d-2}})\cdots(\frac{\partial}{\partial\theta^{3}}\,\frac{\partial}{\partial\theta^{0}} - i\frac{\partial}{\partial\theta^{2}}\,\frac{\partial}{\partial\theta^{1}})\,.\\
&\cdots
\end{aligned} \tag{9.47}$$

It was taken into account in the above two equations that $\mathbf{S}^{01}$ transforms $(\frac{1}{\sqrt{2}})^2(\theta^0 - \theta^3)(\theta^1 + i\theta^2)$ into $\frac{1}{\sqrt{2}}(\theta^0\theta^3 + i\theta^1\theta^2)$ and that any $\mathbf{S}^{ac}$ ($a \neq c$), which does not belong to Cartan subalgebra, Eq.(9.82), transforms $(\frac{1}{\sqrt{2}})^2(\theta^a + i\theta^b)(\theta^c + i\theta^d)$ ($a \neq c$ and $a \neq d$, $b \neq c$ and $b \neq d$, $\eta^{aa} = \eta^{bb}$) into $\frac{1}{\sqrt{2}}(\theta^a\theta^b + \theta^c\theta^d)$. The states are normalized and the simplest phases are chosen.

One finds that $\mathbf{S}^{ab}(\theta^a \pm \epsilon\theta^b) = \mp i\,\frac{\eta^{aa}}{\epsilon}(\theta^a \pm \epsilon\theta^b)$, $\epsilon = 1$ for $\eta^{aa} = 1$ and $\epsilon = i$ for $\eta^{aa} = -1$, while either $\mathbf{S}^{ab}$ or $\mathbf{S}^{cd}$, applied on $(\theta^a\theta^b \pm \epsilon\theta^c\theta^d)$, gives zero.

Although all the states, generated by creation operators, which include one $(I \pm \epsilon\theta^a\theta^b)$ or several $(I \pm \epsilon\theta^{a_1}\theta^{b_1})\cdots(I \pm \epsilon\theta^{a_k}\theta^{a_k})$, are orthogonal with respect to the scalar product, Eq.(9.28), their Hermitian conjugate values include $I^\dagger$, which, when applying on the vacuum state $|\phi_{og}>=|1>$, does not give zero. Correspondingly such creation operators do not have appropriate annihilation partners, which would fulfill Eqs. (9.41, 9.42).

However, creation operators which are products of several θ's, let say n with $n = 2, 4 \ldots \frac{d}{2}-1$ — always of an even number of θ's, since $\mathbf{S}^{ab}$ is a Grassmann even operator, $\theta^{a_1}\cdots\theta^{a_n}$ (factors $\theta^a\theta^b$ can be "eigenstates" of the Cartan subalgebra operators provided that $\mathbf{S}^{ab}$ belong to the Cartan subalgebra: $\mathbf{S}^{ab}\theta^a\theta^b|1>=0$) — can appear in the expression for a creation operator, provided that the rest of expression has an odd number of factors ($\frac{d}{2} - n$ (with "eigenvalues" either ($+1$ or -1) or ($+i$ or $-i$), as can be seen in the states of Eqs. (9.45, 9.46, 9.47)). Then such creation and annihilation operators fulfill the relations, we skip the index 1 in $\hat{b}^{\theta 1}_{i}$

and in $\hat{b}^{\theta 1\dagger}_i$

$$
\begin{aligned}
\{\hat{b}^{\theta}_i, \hat{b}^{\theta\dagger}_j\}_+ |\phi_{og}> &= \delta_{ij}\, |\phi_{og}>\,,\\
\{\hat{b}^{\theta}_i, \hat{b}^{\theta}_j\}_+ |\phi_{og}> &= 0\, |\phi_{og}>\,,\\
\{\hat{b}^{\theta\dagger}_i, \hat{b}^{\theta\dagger}_j\}_+ |\phi_{og}> &= 0\, |\phi_{og}>\,,\\
\hat{b}^{\theta\dagger}_j |\phi_{og}> &= |\phi_j>\\
\hat{b}^{\theta}_j |\phi_{og}> &= 0\, |\phi_{og}> \,.
\end{aligned}
\tag{9.48}
$$

It is not difficult to see that states included into a representation, which started with $\hat{b}^{\theta\dagger}_i$ as presented in Eq. (9.45) for $d = (2n+1)2$ and $4n$ spaces, *have the properties, required by* Eq. (9.48):

i.a. In any d-dimensional space the product $\frac{\partial}{\partial\theta^{a_1}}\cdots\frac{\partial}{\partial\theta^{a_k}}$, with all different a_i (also if all or some of them are equal, since $(\frac{\partial}{\partial\theta^{a}})^2 = 0$), if applied on the vacuum $|1>$, is equal to zero. Correspondingly the second equation and the last equation of Eq. (9.48) are fulfilled.

i.b. In any d space the product of different θ^as — $\theta^{a_1}\theta^{a_2}\cdots\theta^{a_k}$ with all different θ^a's ($a_i \neq a_j$) for all a_i and a_j — applied on the vacuum $|1>$ is different from zero. Since all the θ's, appearing in Eqs. (9.45, 9.46, 9.47) are different, forming normalized states, the fourth equation of Eq. (9.48) is fulfilled.

i.c. The third equation of Eq. (9.48) is fulfilled provided that there is an odd number of θ^s in the expression for a creation operator. Then, when in the anticommutation relation different θ^a's appear (like in the case of $d = 6$ $\{\theta^0\theta^3\theta^5, \theta^1\theta^2\theta^6\}_+$), such a contribution gives zero. When two or several equal θ's appear in the anticommutation relation, the contribution is zero (since $(\theta^a)^2 = 0$).

i.d. Also for the first equation in Eq. (9.48) it is not difficult to show that it is fulfilled only for a particular creation operator and its Hermitian conjugate: Let us show this for $d = 1{+}3$ and the creation operator $\frac{1}{\sqrt{2}}(\theta^0{-}\theta^3)\,\theta^1\theta^2$ and its Hermitian conjugate (annihilation) operator: $\frac{1}{\sqrt{2}}\{\frac{\partial}{\partial\theta^2}\,\frac{\partial}{\partial\theta^1}\,(\frac{\partial}{\partial\theta^0}-\frac{\partial}{\partial\theta^3}), \frac{1}{\sqrt{2}}(\theta^0-\theta^3)\,\theta^1\theta^2\}_+$. Applying $(\frac{\partial}{\partial\theta^0}-\frac{\partial}{\partial\theta^3})$ on $(\theta^0-\theta^3)$ gives two, while $\frac{\partial}{\partial\theta^2}\,\frac{\partial}{\partial\theta^1}$ applied on $\theta^1\theta^2$ gives one.

ii. There is additional group of creation and annihilation operators which follows from the starting state

$$
\begin{aligned}
&|\phi^{\theta 2}_1>|_{2(2n+1)} =\\
&(\frac{1}{\sqrt{2}})^{\frac{1}{2}}(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})(\theta^{d-1}+i\theta^{d})\,,\\
&\text{for } d = 2(2n+1)\,,\\
&|\phi^{\theta 2}_1>|_{4n} =\\
&(\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})\,\theta^{d-1}\theta^{d}\,,\\
&\text{for } d = 4n\,.
\end{aligned}
\tag{9.49}
$$

These two states can not be obtained from the previous group of states, presented in Eqs. (9.43, 9.44) by the application of $\mathbf{S}^{ef}$, since each $\mathbf{S}^{ef}$ changes an even number of factors, never an odd one. Correspondingly both starting states form a new group of states, the first in $d = 2(2n+1)$, the second in $d = 4n$. All the rest states of this new group of states in either $d = 2(2n+1)$ or in $d = 4n$ follow from the starting one by the application of $\mathbf{S}^{ef}$. The corresponding creation and annihilation operators are

$$\begin{aligned}
\hat{b}^{\theta 2\dagger}_{01} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-1}+i\theta^d)\,,\\
\hat{b}^{\theta 2}_{01} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\frac{\partial}{\partial\theta^{d-1}}-i\frac{\partial}{\partial\theta^{d}})\cdots(\frac{\partial}{\partial\theta^{0}}+\frac{\partial}{\partial\theta^{3}})\,,\\
&\quad \text{for } d=2(2n+1)\,,\\
\hat{b}^{\theta 2\dagger}_{01} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})\theta^{d-1}\theta^d\,,\\
\hat{b}^{\theta 2}_{01} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,\frac{\partial}{\partial\theta^{d}}\frac{\partial}{\partial\theta^{d-1}}(\frac{\partial}{\partial\theta^{d-3}}-i\frac{\partial}{\partial\theta^{d-2}})\cdots(\frac{\partial}{\partial\theta^{0}}+\frac{\partial}{\partial\theta^{3}})\,,\\
&\quad \text{for } d=4n\,.
\end{aligned}\tag{9.50}$$

As in the first case all the rest of creation operators can be obtained from the starting one, in each of the two kinds of spaces, by the application of $\mathbf{S}^{ac}$, and the annihilation operators by the Hermitian conjugation of the creation operators. Also all these creation and annihilation operators fulfill the requirements for the creation and annihilation operators, presented in Eq. (9.48).

One can choose as the starting creation operator of the second group of operators by changing sign instead of in the factor $(\theta^0-\theta^3)$ in the starting creation operator of the first group in any of the rest of factors in the product. In each case the same group will follow.

Let us count the number of states with the odd Grassmann character in $d = 2(2n+1)$.

There are in $(d = 2)$ two creation $((\theta^0 \mp \theta^1$, for $\eta^{ab} = diag(1,-1))$ and correspondingly two annihilation operators $(\frac{\partial}{\partial\theta^0} \mp \frac{\partial}{\partial\theta^1})$, each belonging to its own group with respect to the Lorentz transformation operators, both fulfill Eq. (9.48).

It is not difficult to see that the number of all creation operators of an odd Grassmann character in $d = 2(2n+1)$-dimensional space is equal to $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$.

We namely ask: In how many ways can one put on $\frac{d}{2}$ places d different θ^a's. And the answer is — the central binomial coefficient for $x^{\frac{d}{2}}\,1^{\frac{d}{2}}$ — with all x different. This is just $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$. But we have counted all the states with an odd Grassmann character, while we know that these states belong to two different groups of representations with respect to the Lorentz group.

Correspondingly one concludes:*There are two groups of states in* $d = 2(2n+1)$ *with an odd Grassmann character, each of these two groups has*

$$\frac{1}{2}\,\frac{d!}{\frac{d}{2}!\frac{d}{2}!}\tag{9.51}$$

members.

In $d=2$ we have two groups with one state, which have an odd Grassmann character, in $d=6$ we have two groups of 10 states, in $d=10$ we have two groups of 126 states with an odd Grassmann characters. And so on.

Correspondingly we have in $d=2(2n+1)$-dimensional spaces two groups of creation operators with $\frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ members each, creating states with an odd Grassmann character and the same number of annihilation operators. Creation and annihilation operators fulfill anticommutation relations presented in Eq. (9.48).

The rest of creation operators [and the corresponding annihilation operators] have rather opposite Grassmann character than the ones studied so far — like $\theta^0\theta^1$ $[\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^0}]$ in $d=(1+1)$ $(\theta^0\mp\theta^3)(\theta^1\pm i\theta^2)$ $[(\frac{\partial}{\partial\theta^1}\mp i\frac{\partial}{\partial\theta^2})(\frac{\partial}{\partial\theta^0}\mp\frac{\partial}{\partial\theta^3}]$, $\theta^0\theta^3\theta^1\theta^2$ $[\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^3}\frac{\partial}{\partial\theta^0}]$ in $d=(3+1)$.

All the states $|\phi^\theta_i>$, generated by the creation operators, Eq. (9.48), on the vacuum state $|\phi_{og}>$ $(=|1>)$ are the eigenstates of the Cartan subalgebra operators and are orthogonal and normalized with respect to the norm of Eq. (9.27)

$$< \phi^\theta_i|\phi^\theta_j> = \delta_{ij}\,. \tag{9.52}$$

If we now extend the creation and annihilation operators to the ordinary coordinate space, the relations among creation and annihilation operators at one time read

$$\begin{aligned}
\{\hat{b}^\theta_i(\vec{x}),\hat{b}^{\theta\dagger}_j(\vec{x}')\}_+|\phi_{og}> &= \delta^i_j\,\delta(\vec{x}-\vec{x}')|\phi_{og}>\,,\\
\{\hat{b}^\theta_i(\vec{x}),\hat{b}^{\theta}_j(\vec{x}')\}_+|\phi_{og}> &= 0\,|\phi_{og}>\,,\\
\{\hat{b}^{\theta\dagger}_i(\vec{x}),\hat{b}^{\theta\dagger}_j(\vec{x}')\}_+|\phi_{og}> &= 0\,|\phi_{og}>\,,\\
\hat{b}^\theta_j(\vec{x})|\phi_{og}> &= 0\,|\phi_{og}>\\
|\phi_{og}> &= |1>\,.
\end{aligned} \tag{9.53}$$

Again the index 1 or 2 in $(\hat{b}^{\theta 1}_i, \hat{b}^{\theta\dagger 1}_i)$ or in $(\hat{b}^{\theta 2}_i, \hat{b}^{\theta\dagger 2}_i)$ is kept.

b. Quantization in Clifford space

In Grassmann space the requirement that products of eigenstates of the Cartan subalgebra operators represent the creation and annihilation operators, obeying the relations of Eq. (9.48), reduces the number of states from 2^d (allowed in the first quantization procedure) to two isolated groups of $\frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ (There is no operator that determines the family quantum number and would connect both isolated groups of states.)

Let us study what happens, when, let say, γ^a's are used to create the basis and correspondingly also to create the creation and annihilation operators.

Let us point out that γ^a is expressible with θ^a and its derivative ($\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$), Eq. (9.17), and that we again require that creation (annihilation) operators create (annihilate) states, which are eigenstates of the Cartan subalgebra, Eq. (9.84). We could as well make a choice of $\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial\theta_a})$ instead of γ^a's to create the basic states [7]. We shall follow here to some extent Ref. [19].

[7] In the case that we would choose $\tilde{\gamma}^a$'s instead of γ^a's, Eq.(9.17), the role of $\tilde{\gamma}^a$ and γ^a should be then correspondingly exchanged in Eq. (9.92).

Making a choice of the Cartan subalgebra eigenstates of S^{ab}, Eq. (9.84),

$$\stackrel{ab}{(k)}:=\frac{1}{2}(\gamma^a+\frac{\eta^{aa}}{ik}\gamma^b)\,,\qquad \stackrel{ab}{[k]}:=\frac{1}{2}(1+\frac{i}{k}\gamma^a\gamma^b)\,, \tag{9.54}$$

where $k^2=\eta^{aa}\eta^{bb}$, recognizing that the Hermitian conjugate values of $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$ are

$$\stackrel{ab}{(k)}^\dagger=\eta^{aa}\stackrel{ab}{(-k)},\quad \stackrel{ab}{[k]}^\dagger=\stackrel{ab}{[k]}\,, \tag{9.55}$$

while the corresponding eigenvalues of S^{ab}, Eq. (9.56), and $\tilde{S}^{ab}$, Eq. (9.101), are

$$\begin{aligned} S^{ab}\stackrel{ab}{(k)}&=\frac{1}{2}k\stackrel{ab}{(k)}\,,\quad S^{ab}\stackrel{ab}{[k]}=\frac{1}{2}k\stackrel{ab}{[k]}\\ \tilde{S}^{ab}\stackrel{ab}{(k)}&=\;\frac{k}{2}\stackrel{ab}{(k)}\,,\quad \tilde{S}^{ab}\stackrel{ab}{[k]}=-\frac{k}{2}\stackrel{ab}{[k]}\,, \end{aligned} \tag{9.56}$$

we find in $d=2(2n+1)$ that from the starting state with products of odd number of only nilpotents

$$|\psi^1_1>|_{2(2n+1)}=\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{35}{(+)}\cdots\stackrel{d-3\;d-2}{(+)}\stackrel{d-1\;d}{(+)}|\psi_{oc}>, \tag{9.57}$$

having correspondingly an odd Clifford character [8], all the other states of the same Lorentz representation, there are $2^{\frac{d}{2}-1}$ members, follow by the application of S^{cd} (which do not belong to the Cartan subalgebra) on the starting state [9], Eq. (9.84): $S^{cd}\,|\psi^1_1>|_{2(2n+1)}=|\psi^1_i>|_{2(2n+1)}$.

The operators $\tilde{S}^{cd}$, which do not belong to the Cartan subalgebra of Eq. (9.84), generate states with different eigenstates of the Cartan subalgebra ($\tilde{S}^{03}$, $\tilde{S}^{12}$, $\tilde{S}^{56}$, $\cdots$, $\tilde{S}^{d-1\,d}$), we call the eigenvalues of their eigenstates the "family" quantum numbers. There are $2^{\frac{d}{2}-1}$ families. From the starting new member with a different "family" quantum number the whole Lorentz representation with this "family" quantum number follows by the application of S^{ef}: $S^{ef}\;\tilde{S}^{cd}|\psi^1_1>|_{2(2n+1)}=|\psi^j_i>|_{2(2n+1)}$. All the states of one Lorentz representation of any particular "family" quantum number have an odd Clifford character, since neither S^{cd} nor $\tilde{S}^{cd}$, both with an even Clifford character, can change this character.

We are interested only in states with an odd Clifford character, in order that the corresponding creation operators defining these states when being applied on an appropriate vacuum state, and their annihilation operators, will fulfill anticommutation relations required for spinors with half integer spin. We shall discuss the number of states with an odd Clifford character after defining the creation and annihilation operators.

---

[8] We call the starting state in $d=2(2n+1)$ $|\psi^1_1>|_{2(2n+1)}$, and the starting state in $d=4n$ $|\psi^1_1>|_{4n}$.

[9] The smallest number of all the generators S^{ac}, which do not belong to the Cartan subalgebra, needed to create from the starting state all the other members, is $2^{\frac{d}{2}-1}-1$. This is true for both even dimensional spaces – $2(2n+1)$ and $4n$.

For $d = 4n$ the starting state must be the product of one projector and $4n - 1$ nilpotents applied on an appropriate vacuum state, since we again require that the corresponding creation and annihilation operators fulfill the anticommutation relations.

Let us start with the state

$$|\psi^1_1 > |_{4n} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{35}{(+)} \cdots \stackrel{d-3\; d-2}{(+)} \stackrel{d-1\; d}{[+]} \; |\psi_{oc} >, \tag{9.58}$$

All the other states belonging to the same Lorentz representation follow again by the application of S^{cd} on this state $|\psi^1_1 > |_{4n}$, while a new family starts by the application of $\tilde{S}^{cd}|\psi^1_1 > |_{4n}$ and from this state all the other members with the same "family" quantum number can be generated by $S^{ef}\tilde{S}^{cd}$ on $|\psi^1_1 > |_{4n}$: $S^{ef}\tilde{S}^{cd}$ $|\psi^1_1 > |_{4n} = |\psi^j_i > |_{4n}$.

All these states in either $d = 2(2n + 1)$ space or $d = 4n$ space are orthogonal with respect to Eq. (9.27).

However, let us point out that $(\gamma^a)^\dagger = \gamma^a \eta^{aa}$. Correspondingly it follows, Eq. (9.55), that $\stackrel{ab}{(k)}^\dagger = \eta^{aa} \stackrel{ab}{(-k)}$, and $\stackrel{ab}{[k]}^\dagger = \stackrel{ab}{[k]}$.

Since any projector is Hermitian conjugate to itself, while to any nilpotent $\stackrel{ab}{(k)}$ the Hermitian conjugated one has an opposite k, it is obvious that Hermitian conjugated product to a product of nilpotents and projectors can not be accepted as a new state [10].

The vacuum state $|\psi_{oc} >$ ought to be chosen so that $< \psi_{oc}|\psi_{oc} >= 1$, while all the states belonging to the physically acceptable states, like $\stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\stackrel{78}{[-]}$ $\cdots \stackrel{d-3\; d-2}{(+)} \stackrel{d-1\; d}{(+)} \; |\psi_{oc} >$ in $d = 2(2n + 1)$, must not give zero for either $d = 2(2n + 1)$ or for $d = 4n$. We also want that the states, obtained by the application of ether S^{cd} or $\tilde{S}^{cd}$ or both, are orthogonal. To make a choice of the vacuum it is needed to know the relations of Eq. (9.88). It must be

$$\begin{aligned}
< \psi_{oc}| \cdots \stackrel{ab}{(k)}^\dagger \cdots | \cdots \stackrel{ab}{(k')} \cdots |\psi_{oc} > &= \delta_{kk'}\,,\\
< \psi_{oc}| \cdots \stackrel{ab}{[k]}^\dagger \cdots | \cdots \stackrel{ab}{[k']} \cdots |\psi_{oc} > &= \delta_{kk'}\,,\\
< \psi_{oc}| \cdots \stackrel{ab}{[k]}^\dagger \cdots | \cdots \stackrel{ab}{(k')} \cdots |\psi_{oc} > &= 0\,.
\end{aligned} \tag{9.59}$$

Our experiences in the case, when states with the integer values of the Cartan subalgebra operators were expressed by Grassmann coordinates, teach us that the requirements, that creation and annihilation operators must fulfill, influence the choice of the number of states, as well as of the vacuum state.

---

[10] We could as well start with the state $|\psi^1_1 > |_{2(2n+1)} = \stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{35}{(-)} \cdots \stackrel{d-3\; d-2}{(-)} \stackrel{d-1\; d}{(-)} \; |\psi_{oc} >$ for $d = 2(2n + 1)$ and with $|\psi^1_1 > |_{4n} = \stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{35}{(-)} \cdots \stackrel{d-3\; d-2}{(-)} \stackrel{d-1\; d}{[-]} \; |\psi_{oc} >$ in the case of $d = 4n$. Then creation and annihilation operators will exchange their roles and also the vacuum state will be correspondingly changed.

Let us first repeat therefore the requirements which the creation and annihilation operators must fulfill

$$
\begin{aligned}
\{\hat{b}_i^{\alpha\gamma}, \hat{b}_k^{\beta\gamma\dagger}\}_+ |\psi_{oc}> &= \delta^\alpha_\beta\, \delta^i_k |\psi_{oc}>\,,\\
\{\hat{b}_i^{\alpha\gamma}, \hat{b}_k^{\beta\gamma}\}_+ |\psi_{oc}> &= 0 |\psi_{oc}>\,,\\
\{\hat{b}_i^{\alpha\gamma\dagger}, \hat{b}_k^{\beta\gamma\dagger}\}_+ |\psi_{oc}> &= 0 |\psi_{oc}>\,,\\
\hat{b}_i^{\alpha\gamma} |\psi_{oc}> &= 0 |\psi_{oc}>\,,\\
\hat{b}_i^{\alpha\gamma\dagger} |\psi_{oc}> &= |\psi_i^{\alpha\gamma}>\,,
\end{aligned} \tag{9.60}
$$

paying attention at this stage only at the internal degrees of freedom of the states, that is on their spins. Here $(\alpha, \beta, \dots)$ represent the family quantum number determined by $\tilde{S}^{ac}$ and $(i, j, \dots)$ the quantum number of one representation, determined by S^{ac} and index γ is to point out that these creation operators represent Clifford rather than Grassmann objects. In what follows we shall skip the index γ, since either states or creation and annihilation operators carry two indexes, while in Grassmann case there is no family quantum number.

From Eqs. (9.57, 9.58) is not difficult to extract the creation operator which, when applied on the vacuum state for either $d = 2(2n+1)$ or $d = 4n$, generates the starting state .

i. *One Weyl representation*

We define the creation $\hat{b}_1^{1\dagger}$ — and the corresponding annihilation operator $\hat{b}_1^1 = (\hat{b}_1^{1\dagger})^\dagger$ — which when applied on the vacuum state $|\psi_{oc}>$ create a vector of one of the two equations (9.57, 9.58), as follows

$$
\begin{aligned}
\hat{b}_1^{1\dagger} &:= \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} \cdots \overset{d-1\, d}{(+)}\,,\\
\hat{b}_1^{1} &:= \overset{d-1\, d}{(-)} \cdots \overset{56}{(-)}\overset{12}{(-)}\overset{03}{(-i)}\,,\\
&\quad \text{for } d = 2(2n+1)\,,\\
\\
\hat{b}_1^{1\dagger} &:= \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} \cdots \overset{d-3\, d-2}{(+)}\; \overset{d-1\, d}{[+]}\,,\\
\hat{b}_1^{1} &:= \overset{d-1, d}{[+]}\; \overset{d-2\; d-3}{(-)} \cdots \overset{56}{(-)}\overset{12}{(-)}\overset{03}{(-i)}\,,\\
&\quad \text{for } d = 4n\,.
\end{aligned} \tag{9.61}
$$

We shall call the $\hat{b}_1^{1\dagger}|\psi_{oc}>$, when operating on the vacuum state, the starting vector of the starting "family".

Now we can make a choice of the vacuum state for this particular "family" taking into account Eq. (9.88)

$$
\begin{aligned}
|\psi_{oc}> &= \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[-]} \cdots \overset{d-1\, d}{[-]}\; |0>\,,\\
&\quad \text{for } d = 2(2n+1)\,,\\
|\psi_{oc}> &= \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[-]} \cdots \overset{d-3\, d-2}{[-]}\; \overset{d-1\, d}{[+]}\; |0>\,,\\
&\quad \text{for } d = 4n\,,
\end{aligned} \tag{9.62}
$$

n is a positive integer, so that the requirements of Eq. (9.60) are fulfilled. We see: The creation and annihilation operators of Eq. (9.61) (both are nilpotents, $(\hat{b}^{1\dagger}_1)^2 = 0$ and $(\hat{b}^1_1)^2 = 0$), $\hat{b}^{1\dagger}_1$ (generating the vector $|\psi^1_1>$ when operating on the vacuum state) gives $\hat{b}^{1\dagger}_1|\psi_{oc}>\neq 0$, while the annihilation operator annihilates the vacuum state $\hat{b}^1_1|\psi_0>= 0$, giving $\{\hat{b}^1_1, \hat{b}^{1\dagger}_1\}_+|\psi_{oc}>= |\psi_{oc}>$, since we choose the appropriate normalization, Eq. (9.54).

All the other creation and annihilation operators, belonging to the same Lorentz representation with the same family quantum number, follow from the starting ones by the application of particular S^{ac}, which do not belong to the Cartan subalgebra (9.82).

We call $\hat{b}^{1\dagger}_2$ the one obtained from $\hat{b}^{1\dagger}_1$ by the application of one of the four generators ($S^{01}, S^{02}, S^{31}, S^{32}$). This creation operator is for $d = 2(2n + 1)$ equal to $\hat{b}^{1\dagger}_2 = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{35}{(+)}\cdots\stackrel{d-1\,d}{(+)}$, while it is for $d = 4n$ equal to $\hat{b}^{1\dagger}_2 = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}\cdots\stackrel{d-1\,d}{[+]}$. All the other family members follow from the starting one by the application of different S^{ef}, or by the product of several S^{gh}.

We accordingly have

$$
\begin{aligned}
\hat{b}^{1\dagger}_i &\propto S^{ab}..S^{ef}\hat{b}^{1\dagger}_1\,,\\
\hat{b}^1_i &\propto \hat{b}^1_1 S^{ef}..S^{ab}\,,
\end{aligned}
\tag{9.63}
$$

with $S^{ab\dagger} = \eta^{aa}\eta^{bb}S^{ab}$. We shall make a choice of the proportionality factors so that the corresponding states $|\psi^1_1>= \hat{b}^{1\dagger}_i|\psi_{oc}>$ will be normalized.
We recognize that [19]:

i.a. $(\hat{b}^{1\dagger}_i)^2 = 0$ and $(\hat{b}^1_i)^2 = 0$, for all i.

To see this one must recognize that S^{ac} (or S^{bc}, S^{ad}, S^{bd}) transforms $\stackrel{ab}{(+)}\stackrel{cd}{(+)}$ to $\stackrel{ab}{[-]}\stackrel{cd}{[-]}$, that is an even number of nilpotents $(+)$ in the starting state is transformed into projectors $[-]$ in the case of $d = 2(2n + 1)$. For $d = 4n$, S^{ac} (or S^{bc}, S^{ad}, S^{bd}) transforms $\stackrel{ab}{(+)}\stackrel{cd}{[+]}$ into $\stackrel{ab}{[-]}\stackrel{cd}{(-)}$. Therefore for either $d = 2(2n + 1)$ or $d = 4n$ at least one of factors, defining a particular creation operator, will be a nilpotent. For $d = 2(2n + 1)$ there is an odd number of nilpotents, at least one, leading from the starting factor $\stackrel{dg}{(+)}$ in the creator. For $d = 4n$ a nilpotent factor can also be $\stackrel{d-1\,d}{(-)}$ (since $\stackrel{d-1\,d}{[+]}$ can be transformed by $S^{e\,d-1}$, for example into $\stackrel{d-1\,d}{(-)}$). A square of at least one nilpotent factor (we started with an odd number of nilpotents, and oddness can not be changed by S^{ab}), is enough to guarantee that the square of the corresponding $(\hat{b}^{1\dagger}_i)^2$ is zero. Since $\hat{b}^1_i = (\hat{b}^{1\dagger}_i)^\dagger$, the proof is valid also for annihilation operators.

i.b. $\hat{b}^{1\dagger}_i|\psi_{oc}>\neq 0$ and $\hat{b}^1_i|\psi_{oc}>= 0$, for all i.

To see this in the case $d = 2(2n+1)$ one must recognize that $\hat{b}^{1\dagger}_i$ distinguishes from $\hat{b}^{1\dagger}_1$ in (an even number of) those nilpotents $(+)$, which have been transformed into $[-]$. When $\stackrel{ab}{[-]}$ from $\hat{b}^{1\dagger}_i$ meets $\stackrel{ab}{[-]}$ from $|\psi_{oc}>$, the product gives $\stackrel{ab}{[-]}$ back, and correspondingly a nonzero contribution. For $d = 4n$ also the factor $\stackrel{d-1\,d}{[+]}$ can be transformed. It is transformed into $\stackrel{d-1\,d}{(-)}$ which, when applied to a vacuum state, gives again a nonzero contribution ($\stackrel{d-1\,d}{(-)}\stackrel{d-1\,d}{[+]} = \stackrel{d-1\,d}{(-)}$, Eq. (9.88)).

In the case of $\hat{b}^1_i$ we recognize that in $\hat{b}^{1\dagger}_i$ at least one factor is nilpotent; that of the same type as in the starting $\hat{b}^\dagger_1$ — $(+)$ — or in the case of $d = 4n$ it can be also $\stackrel{d-1\,d}{(-)}$. Performing the Hermitian conjugation $(\hat{b}^{1\dagger}_i)^\dagger$, $(+)$ transforms into $(-)$, while $\stackrel{d-1\,d}{(-)}$ transforms into $\stackrel{d-1\,d}{(+)}$ in $\hat{b}^1_i$. Since $(-)[-]$ gives zero and $\stackrel{d-1\,d}{(+)}\stackrel{d-1\,d}{[+]}$ also gives zero, $\hat{b}^1_i|\psi_{oc}>= 0$.

i.c. $\{\hat{b}^{1\dagger}_i, \hat{b}^{1\dagger}_j\}_+ = 0$, for each pair (i, j).

There are several possibilities to be discussed. A trivial one is, if both $\hat{b}^{1\dagger}_i$ and $\hat{b}^{1\dagger}_j$ have a nilpotent factor (or more than one) for the same pair of indexes, say $\stackrel{kl}{(+)}$. Then the product of such two $\stackrel{kl}{(+)}\stackrel{kl}{(+)}$ gives zero. It also happens, that $\hat{b}^{1\dagger}_i$ has a nilpotent at the place (kl) ($\stackrel{03}{[-]}\cdots\stackrel{kl}{(+)}\cdots\stackrel{mn}{[-]}\cdots$) while $\hat{b}^{1\dagger}_j$ has a nilpotent at the place (mn) ($\stackrel{03}{[-]}\cdots\stackrel{kl}{[-]}\cdots\stackrel{mn}{(+)}\cdots$). Then in the term $\hat{b}^{1\dagger}_i\hat{b}^{1\dagger}_j$ the product $\stackrel{mn}{[-]}\stackrel{mn}{(+)}$ makes the term equal to zero, while in the term $\hat{b}^{1\dagger}_j\hat{b}^{1\dagger}_i$ the product $\stackrel{kl}{[-]}\stackrel{kl}{(+)}$ makes the term equal to zero. There is no other possibility in $d = 2(2n+1)$. In the case that $d = 4n$, it might appear also that $\hat{b}^{1\dagger}_i = \stackrel{03}{[-]}\cdots\stackrel{ij}{(+)}\cdots\stackrel{d-1\,d}{[+]}$ and $\hat{b}^{1\dagger}_j = \stackrel{03}{[-]}\cdots\stackrel{ij}{[-]}\cdots\stackrel{d-1\,d}{(-)}$. Then in the term $\hat{b}^{1\dagger}_i\hat{b}^{1\dagger}_j$ the factor $\stackrel{d-1\,d}{[+]}\stackrel{d-1\,d}{(-)}$ makes it zero, while in $\hat{b}^{1\dagger}_j\hat{b}^{1\dagger}_i$ the factor $\stackrel{ij}{[-]}\stackrel{ij}{(+)}$ makes it zero. Since there are no further possibilities, the proof is complete.

i.d. $\{\hat{b}^1_i, \hat{b}^1_j\}_+ = 0$, for each pair (i, j).

The proof goes similarly as in the case with creation operators. Again we treat several possibilities. $\hat{b}^1_i$ and $\hat{b}^1_j$ have a nilpotent factor (or more than one) with the same indexes, say $\stackrel{kl}{(-)}$. Then the product of such two $\stackrel{kl}{(-)}\stackrel{kl}{(-)}$ gives zero. It also happens, that $\hat{b}^1_i$ has a nilpotent at the place (kl) ($\cdots\stackrel{mn}{[-]}\cdots\stackrel{kl}{(-)}\cdots\stackrel{03}{[-]}$) while $\hat{b}^1_j$ has a nilpotent at the place (mn) ($\cdots\stackrel{mn}{(-)}\cdots\stackrel{kl}{[-]}\cdots\stackrel{03}{[-]}$). Then in the term $\hat{b}^1_i\hat{b}^1_j$ the product $\stackrel{kl}{(-)}\stackrel{kl}{[-]}$ makes the term equal to zero, while in the term $\hat{b}^1_j\hat{b}^1_i$ the product $\stackrel{mn}{(-)}\stackrel{mn}{[-]}$ makes the term equal to zero. In the case that $d = 4n$, it appears also that $\hat{b}^1_i = \stackrel{d-1\,d}{[+]}\cdots\stackrel{ij}{(-)}\cdots\stackrel{03}{[-]}$ and $\hat{b}^1_j = \stackrel{d-1\,d}{(+)}\cdots\stackrel{ij}{[-]}\cdots\stackrel{03}{[-]}$. Then in the term $\hat{b}^1_i\hat{b}^1_j$ the factor $\stackrel{ij}{(-)}\stackrel{ij}{[-]}$ makes it zero, while in $\hat{b}^1_j\hat{b}^1_i$ the factor $\stackrel{d-1\,d}{(+)}\stackrel{d-1\,d}{[+]}$ makes it zero.

i.e. $\{\hat{b}^1_i, \hat{b}^{1\dagger}_j\}_+|\psi_{oc}>= \delta_{ij}|\psi_{oc}>$.

To prove this we must recognize that $\hat{b}^1_i = \hat{b}_1 S^{ef}..S^{ab}$ and $\hat{b}^{1\dagger}_i = S^{ab}..S^{ef}\hat{b}_1$. Since any $\hat{b}^1_i|\psi_{oc}>= 0$, we only have to treat the term $\hat{b}^1_i\hat{b}^{1\dagger}_j$. We find $\hat{b}^1_i\hat{b}^{1\dagger}_j \propto \cdots\stackrel{lm}{(-)}\cdots\stackrel{03}{(-)}S^{ef}\cdots S^{ab}S^{lm}\cdots S^{pr}\stackrel{03}{(+)}\cdots\stackrel{lm}{(+)}\cdots$. If we treat the term $\hat{b}^1_i\hat{b}^{1\dagger}_i$, generators $S^{ef}\cdots S^{ab}S^{lm}\cdots S^{pr}$ are proportional to a number and we normalize

$< \psi_{oc}|\hat{b}^1_i \hat{b}^{1\dagger}_i|\psi_{oc} >$ to one. When $S^{ef}\cdots S^{ab}S^{lm}\cdots S^{pr}$ are proportional to several products of S^{cd}, these generators change $\hat{b}^{1\dagger}_1$ into $\stackrel{03}{(+)}\cdots\stackrel{kl}{[-]}\cdots\stackrel{np}{[-]}\cdots$, making the product $\hat{b}^1_i\,\hat{b}^{1\dagger}_j$ equal to zero, due to factors of the type $\stackrel{kl}{(-)}\stackrel{kl}{[-]}$. In the case of $d = 4n$ also a factor $\stackrel{d-1\,d}{[+]}\;\stackrel{d-1\,d}{(-)}$ might occur, which also gives zero.

We saw and proved that for the definition of the creation and annihilation operators, Eq. (9.61), for states in Eqs. (9.57, 9.58) and further for all the rest of creation and annihilation operators, Eq. (9.63), and for the choice of the vacuum states, Eq. (9.62), *all the requirements of* Eq. (9.60) *are fulfilled, provided that creation and correspondingly also the annihilation operators have an odd Clifford character, that is that the number of nilpotents in the product is odd.*

For an even number of factors of the nilpotent type in the starting state and accordingly in the starting $\hat{b}^{1\dagger}_1$, an annihilation operator $\hat{b}^1_i$ would appear with all factors of the type $[-]$, which on the vacuum state (Eq.(9.62)) would not give zero.

ii. *Families of Weyl representations*

Let $\hat{b}^{\alpha\dagger}_i$ be a creation operator, fulfilling Eq. (9.60), which creates one of the $(2^{d/2-1})$ Weyl basic states of an $\alpha-$th "family", when operating on a vacuum state $|\psi_{oc}>$ and let $\hat{b}^{\alpha}_i = (\hat{b}^{\alpha\dagger}_i)^\dagger$ be the corresponding annihilation operator. We shall now proceed to define $\hat{b}^{\alpha\dagger}_i$ and $\hat{b}^{\alpha}_i$ from a chosen starting state (9.57, 9.58), which $\hat{b}^{1\dagger}_1$ creates on the vacuum state $|\psi_{oc}>$.

When treating more than one Weyl representation, that is, more than one "family", we must take into account that: **i.** The vacuum state chosen to fulfill requirements for second quantization of the starting family might not and it will not be the correct one when all the families are taken into account. **ii.** The products of $\tilde{S}^{ab}$, which do not belong to the Cartan subalgebra set of the generators $\tilde{S}^{ab}$, when being applied on the starting family ψ^1_1, generate the starting member ψ^{α}_1 of each of the remaining families. There is correspondingly the same number of "families" as the number of vectors of one Weyl representation, namely $2^{d/2-1}$. Then the whole Weyl representation of a particular family ψ^{α}_1 follows again with the application of S^{ef}, which do not belong to the Cartan subalgebra of S^{ab} on this starting α family state.

Any vector $|\psi^{\alpha}_i>$ follows from the starting vector, Eqs. (9.57, 9.58), by the application of either $\tilde{S}^{ef}$, which change the family quantum number, or S^{gh}, which change the member of a particular family (as it can be seen from Eqs. (9.90, 9.102)) or with the corresponding product of S^{ef} and $\tilde{S}^{ef}$

$$|\psi^{\alpha}_i > \propto \tilde{S}^{ab}\cdots\tilde{S}^{ef}|\psi^1_i > \propto \tilde{S}^{ab}\cdots\tilde{S}^{ef}S^{mn}\cdots S^{pr}|\psi^1_1 > . \tag{9.64}$$

Correspondingly we define $\hat{b}^{\alpha\dagger}_i$ (up to a constant) to be

$$\begin{aligned}\hat{b}^{\alpha\dagger}_i &\propto \tilde{S}^{ab}\cdots\tilde{S}^{ef}S^{mn}\cdots S^{pr}\hat{b}^{1\dagger}_1\\ &\propto S^{mn}\cdots S^{pr}\hat{b}^{1\dagger}_1 S^{ab}\cdots S^{ef} .\end{aligned} \tag{9.65}$$

This last expression follows due to the property of the Clifford object $\tilde{\gamma}^a$ and correspondingly of $\tilde{S}^{ab}$, presented in Eqs. (9.92, 9.93).

For $\hat{b}_i^{\alpha} = (\hat{b}_i^{\alpha\dagger})^{\dagger}$ we accordingly have

$$\hat{b}_i^{\alpha} = (\hat{b}_i^{\alpha\dagger})^{\dagger} \propto S^{ef} \cdots S^{ab} \hat{b}_1^{1} S^{pr} \cdots S^{mn}\,. \tag{9.66}$$

The proportionality factor will be chosen so that the corresponding states $|\psi_i^{\alpha}> = \hat{b}_i^{\alpha\dagger}|\psi_{oc}>$ will be normalized.

We ought to generalize the vacuum state from Eq. (9.62) so that $\hat{b}_i^{\alpha\dagger}\,|\psi_{oc}> \neq 0$ and $\hat{b}_i^{\alpha}\,|\psi_{oc}> = 0$ for all the members i of any family α. Since any $\tilde{S}^{eg}$ changes $\overset{ef}{(+)}\overset{gh}{(+)}$ into $\overset{ef}{[+]}\overset{gh}{[+]}$ and $\overset{ab}{[+]}{}^{\dagger} = \overset{ab}{[+]}$, while $\overset{ab}{(+)}{}^{\dagger}\ \overset{ab}{(+)} = \overset{ab}{[-]}$, the vacuum state $|\psi_{oc}>$ from Eq. (9.62) must be replaced by

$$|\psi_{oc}> =$$
$$\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[-]}\cdots\overset{d-1\ d}{[-]} + \overset{03}{[+i]}\overset{12}{[+]}\overset{56}{[-]}\cdots\overset{d-1\ d}{[-]} + \overset{03}{[+i]}\overset{12}{[-]}\overset{56}{[+]}\cdots\overset{d-1\ d}{[-]} + \cdots|0>,$$
for $d = 2(2n+1)$,

$$|\psi_{oc}> =$$
$$\overset{03}{[-i]}\overset{12}{[-]}\overset{35}{[-]}\cdots\overset{d-3\ d-2}{[-]}\overset{d-1\ d}{[+]} + \overset{03}{[+i]}\overset{12}{[+]}\overset{56}{[-]}\cdots\overset{d-3\ d-2}{[-]}\overset{d-1\ d}{[+]} + \cdots|0>,$$
for $d = 4n$, (9.67)

n is a positive integer. There are $2^{\frac{d}{2}-1}$ summands, since we step by step replace all possible pairs of $\overset{ab}{[-]}\cdots\overset{ef}{[-]}$ in the starting part $\overset{03}{[-i]}\overset{12}{[-]}\overset{35}{[-]}\cdots\overset{d-1\ d}{[-]}$ (or $\overset{03}{[-i]}\overset{12}{[-]}\overset{35}{[-]}\cdots\overset{d-3\ d-2}{[-]}\overset{d-1\ d}{[+]}$) into $\overset{ab}{[+]}\cdots\overset{ef}{[+]}$ and include new terms into the vacuum state so that the last $2n+1$ summands have for $d = 2(2n+1)$ case, n is a positive integer, only one factor $[-]$ and all the rest $[+]$, each $[-]$ at different position. For $d = 4n$ also the factor $\overset{d-1\ d}{[+]}$ in the starting term $\overset{03}{[-i]}\overset{12}{[-]}\overset{35}{[-]}\cdots\overset{d-3\ d-2}{[-]}\overset{d-1\ d}{[+]}$ changes to $\overset{d-1\ d}{[-]}$. The vacuum state has then the normalization factor $1/\sqrt{2^{d/2-1}}$.

There is therefore

$$2^{\frac{d}{2}-1}\,2^{\frac{d}{2}-1} \tag{9.68}$$

number of creation operators, defining the orthonormalized states when applying on the vacuum state of Eqs. (9.67) and the same number of annihilation operators, which are defined by the creation operators on the vacuum state of Eqs. (9.67). $\tilde{S}^{ab}$ connect members of different families, S^{ab} generates all the members of one family.

We recognize that:

ii.a. The above creation and annihilation operators are nilpotent — $(\hat{b}_i^{a\dagger})^2 = 0 = (\hat{b}_i^{a})^2$ — since the "starting" creation operator $\hat{b}_1^{1\dagger}$ and annihilation operator $\hat{b}_i^{a}$ are both made of the product of an odd number of nilpotents, while products of either S^{ab} or $\tilde{S}^{ab}$ can change an even number of nilpotents into projectors. Any $\hat{b}_i^{a\dagger}$ is correspondingly a factor of an odd number of nilpotents (at least one) (and an even number of projectors) and its square is zero. The same is true for $\hat{b}_i^{a}$.

ii.b. All the creation operators operating on the vacuum state of Eq. (9.67) give a non zero vector — $\hat{b}_i^{a\dagger}\,|\psi_{oc}> \neq 0$ — while all the annihilation operators annihilate this vacuum state — $\hat{b}_i^{a}\,|\psi_0> = 0$ for any α and any i.

It is not difficult to see that $\hat{b}^{a}_{i}\,|\psi_{oc}>= 0$, for any α and any i. First we recognize that whatever the set of factors $S^{mn}\cdots S^{pr}$ appear on the right hand side of the annihilation operator $\hat{b}^{1}_{1}$ in Eq. (9.66), it leaves at least one factor $[-]$ unchanged. Since $\hat{b}^{1}_{1}$ is the product of only nilpotents $(-)$ and since $\stackrel{ab}{(-)}\stackrel{ab}{[-]}= 0$, this part of the proof is complete.

Let us prove now that $\hat{b}^{\alpha\dagger}_{i}\,|\psi_{oc}>\neq 0$ for any α and any i. According to Eq. (9.65) the operation S^{mn} on the left hand side of $\hat{b}^{1\dagger}_{1}$, with $(m,n,..)$, which does not belong to the Cartan subalgebra set of indices, transforms the term $\stackrel{03}{[-i]}\stackrel{12}{[-]}\cdots\stackrel{lm}{[-]}\cdots\stackrel{nk}{[-]}\cdots\stackrel{d-1\,d}{[-]}$ (or the term $\stackrel{03}{[-i]}\stackrel{12}{[-]}\cdots\stackrel{lm}{[-]}\cdots\stackrel{nk}{[-]}\cdots\cdots\stackrel{d-1\,d}{[+]}$) into the term $\stackrel{03}{[-i]}\stackrel{12}{[-]}\cdots\stackrel{lm}{(+)}\cdots\stackrel{nk}{(+)}\cdots\stackrel{d-1\,d}{[-]}$ (or into the term $\stackrel{03}{[-i]}\stackrel{12}{[-]}\cdots\stackrel{lm}{(+)}\cdots\stackrel{nk}{(+)}\cdots\cdots\stackrel{d-1\,d}{[+]}$) and $\hat{b}^{1\dagger}_{1}$ on such a term gives zero, since $\stackrel{lm}{(+)}\stackrel{lm}{(+)}= 0$ and $\stackrel{nk}{(+)}\stackrel{nk}{(+)}= 0$. Let us first assume that S^{mn} is the only term on the right hand side of $\hat{b}^{1\dagger}_{1}$ and that none of the operators from the left hand side of $\hat{b}^{1\dagger}_{1}$ in Eq. (9.65) has the indices m,n. It is only one term among all the summands in the vacuum state (Eq. (9.67)), which gives non zero contribution in this particular case, namely the term $\stackrel{03}{[-i]}\stackrel{12}{[-]}\cdots\stackrel{lm}{[+]}\cdots\stackrel{nk}{[+]}\cdots\stackrel{d-1\,d}{[-]}$ (or the term $\stackrel{03}{[-i]}\stackrel{12}{[-]}\cdots\stackrel{lm}{[+]}\cdots\stackrel{nk}{[+]}\cdots\cdots\stackrel{d-1\,d}{[+]}$). S^{mn} transforms the part $\cdots\stackrel{lm}{[+]}\cdots\stackrel{nk}{[+]}\cdots$ into $\cdots\stackrel{lm}{(-)}\cdots\stackrel{nk}{(-)}\cdots$ and since $\stackrel{lm}{(+)}\stackrel{lm}{(-)}$ gives $\eta^{ll}\stackrel{lm}{[+]}$, while for the rest of factors it was already proven that such a factor on $\hat{b}^{1\dagger}_{1}$ forms a $b^{1\dagger}_{i}$ giving non zero contribution on the vacuum, Eq. (9.62), the proof is complete.

It is also proved that what ever other S^{ab} but S^{mn} operate on the left hand side of $\hat{b}^{1\dagger}_{1}$ the contribution of this particular part of the vacuum state is nonzero. If the operators on the left hand side have the indexes m or n or both, the contribution on this term of the vacuum will still be nonzero, since then such a S^{mp} will transform the factor $\stackrel{lm}{(+)}$ in $\hat{b}^{1\dagger}_{1}$ into $\stackrel{lm}{[-]}$ and $\stackrel{lm}{[-]}\stackrel{lm}{(-)}$ is nonzero, Eq. (9.88).

It was proven that $\hat{b}^{\alpha\dagger}_{i}$ operating on the vacuum $|\psi_{oc}>$ of Eq. (9.67) gives a nonzero contribution. The vacuum state has namely a term which guarantees a non zero contribution for any possible set of $S^{mn}\cdots S^{pr}$ operating from the right hand side of $\hat{b}^{1\dagger}_{1}$ (that is for each family) (what we achieved just by the transformation of all possible pairs of $\stackrel{cd}{[-]},\stackrel{gh}{[-]}$ in the vacuum into $\stackrel{cd}{[+]},\stackrel{gh}{[+]}$). (When we speak about $[-]$ also $\stackrel{03}{[-i]}$ is understood.) It is not difficult to see that for each "family" of $2^{\frac{d}{2}-1}$ families it is only one term among all the summands in the vacuum state $|\psi_{oc}>$ of Eq. (9.67), which gives a nonzero contribution, since whenever $[+]$ appears on a wrong position, that is on the position, so that the product of $\stackrel{ab}{(+)}$ from $\hat{b}^{1\dagger}$ and $\stackrel{ab}{[+]}$ from the vacuum summand "meet", the contribution is zero.

ii.c. Any two creation operators anticommute: $\{\hat{b}^{\alpha\dagger}_{i},\hat{b}^{\beta\dagger}_{j}\}_{+}=0$. According to Eq. (9.65) we can rewrite $\{\hat{b}^{\alpha\dagger}_{i},\hat{b}^{\beta\dagger}_{j}\}_{+}$, up to a factor, as $\{S^{mn}\cdots S^{pr}\hat{b}^{1\dagger}_{1}S^{ab}\cdots S^{ef}, S^{m'n'}\cdots S^{p'r'}\hat{b}^{1\dagger}_{1}S^{a'b'}\cdots S^{e'f'}\}_{+}$. Whatever the product $S^{ab}\cdots S^{ef}S^{m'n'}\cdots S^{p'r'}$ (or $S^{a'b'}\cdots S^{e'f'}S^{mn}\cdots S^{pr}$) is, it always transforms an even number of $(+)$ in $\hat{b}^{1\dagger}_{1}$ into $[-]$. Since an odd number of nilpotents $(+)$ (at least one) remains unchanged in this right $\hat{b}^{1\dagger}_1$ after the application of all the S^{ab} in the product in front of it, or $\overset{d-1\;d}{[+]}$ transforms into $\overset{d-1\;d}{(-)}$, and since the left $\hat{b}^{1\dagger}_1$ is a product of only nilpotents $(+)$ in $d = 2(2n+1)$, or an odd number of nilpotents and $[+]$ for $d = 4n$, while $\overset{d-1\;d}{[+]}\overset{d-1\;d}{(-)} = 0$, the anticommutator for any two creation operators is zero.

ii.d. Any two annihilation operators anticommute: $\{\hat{b}^{\alpha}_i, \hat{b}^{\beta}_j\}_+ = 0$. According to Eq. (9.66) we can rewrite $\{\hat{b}^{\alpha}_i, \hat{b}^{\beta}_j\}_+$, up to a factor, as $\{S^{ab} \cdots S^{ef}\hat{b}^1_1 S^{mn} \cdots S^{pr}, S^{a'b'} \cdots S^{e'f'}\hat{b}^1_1 S^{m'n'} \cdots S^{p'r'}\}_+$. Whatever the product $S^{mn} \cdots S^{pr} S^{a'b'} \cdots S^{e'f'}$ (or $S^{m'n'} \cdots S^{p'r'} S^{ab} \cdots S^{ef}$) is, it always transforms an even number of $(-)$ in $\hat{b}^1_1$ into $[+]$. Since an odd number of nilpotents $(-)$ (at least one) remains unchanged in this $\hat{b}^1_1$ after the application of all the S^{ab} in the product in front of it or $\overset{d-1\;d}{[+]}$ is transformed into $\overset{d-1\;d}{(-)}$, and since $\hat{b}^1_1$ on the left hand side is a product of only nilpotents $(-)$ for $d = 2(2n+1)$ (or an odd number of nilpotents and $[+]$ for $d = 4n$), while $\overset{ab}{(-)}\overset{ab}{(-)} = 0$ and $\overset{ab}{[+]}\overset{ab}{[-]} = 0$, the anticommutator of any two annihilation operators is zero.

ii.e. For any creation and any annihilation operator it follows: $\{\hat{b}^{\alpha}_i, \hat{b}^{\beta\dagger}_j\}_+ |\psi_{oc}> = \delta^{\alpha\beta}\delta_{ij}|\psi_{oc}>$. Let us prove this. According to Eqs. (9.65, 9.66) we may rewrite $\{\hat{b}^{\alpha}_i, \hat{b}^{\beta\dagger}_j\}_+$ up to a factor as

$$\{S^{ab} \cdots S^{ef}\hat{b}^1_1 S^{mn} \cdots S^{pr}, S^{m'n'} \cdots S^{p'r'}\hat{b}^{1\dagger}_1 S^{a'b'} \cdots S^{e'f'}\}_+ .$$

We distinguish between two cases. It can be that both $S^{mn} \cdots S^{pr} S^{m'n'} \cdots S^{p'r'}$ and $S^{a'b'} \cdots S^{e'f'} S^{ab} \cdots S^{ef}$ are numbers. This happens when $\alpha = \beta$ and $i = j$. Then we follow **i.b.**. We normalize the states so that $<\psi^{\alpha}_i|\psi^{\alpha}_i> = 1$.

The second case is that at least one of products $S^{mn} \cdots S^{pr} S^{m'n'} \cdots S^{p'r'}$ and $S^{a'b'} \cdots S^{e'f'} S^{ab} \cdots S^{ef}$ is not a number. Then the factors like $\overset{ab}{(-)}\overset{ab}{[-]}$ or $\overset{ab}{[+]}\overset{ab}{(-)}$ or $\overset{ab}{(+)}\overset{ab}{[+]}$ make the anticommutator equal to zero. And the proof is completed.

Let us extend the creation and annihilation operators to the ordinary coordinate space

$$\begin{aligned}
\{\hat{b}^{\alpha}_i(\vec{x}), \hat{b}^{\beta\dagger}_j(\vec{x}')\}_+|\phi_{oc}> &= \delta^{\alpha}_{\beta}\,\delta^i_j\;\delta(\vec{x}-\vec{x}')|\phi_{oc}>\,,\\
\{\hat{b}^{\alpha}_i(\vec{x}), \hat{b}^{\beta}_j(\vec{x}')\}_+|\phi_{oc}> &= 0\;|\phi_{oc}>\,,\\
\{\hat{b}^{\alpha\dagger}_i(\vec{x}), \hat{b}^{\beta\dagger}_j(\vec{x}')\}_+|\phi_{oc}> &= 0\;|\phi_{oc}>\,,\\
\hat{b}^{\alpha}_j(\vec{x})|\phi_{oc}> &= 0\;|\phi_{oc}>\,,\\
\hat{b}^{\alpha\dagger}_j(\vec{x})|\phi_{oc}> &= |\psi^{\alpha}_i(\vec{x})>\,,
\end{aligned} \tag{9.69}$$

with the vacuum state $|\phi_{oc}>$ defined in Eq. (9.67).

c. Discrete symmetries in Grassmann space and in Clifford space in d **and in** $d = (3+1)$ **space**

Let $\underline{\Psi}^{\dagger}_p[\Psi_p]$ be the creation operator creating a fermion in the state Ψ_p (which is a function of $\vec{x}$) and let $\Psi_p(\vec{x})$ be the second quantized field creating a fermion at position $\vec{x}$ either in the Grassmann or in the Clifford case. Then

$$\underline{\Psi}^{\dagger}_{p}[\Psi_p] = \int \Psi^{\dagger}_{p}(\vec{x})\,\Psi_p(\vec{x})d^{(d-1)}x\,, \tag{9.70}$$

describes on a vacuum state a single particle in the state Ψ

$$\{\underline{\Psi}^{\dagger}_{p}[\Psi_p] = \int \Psi^{\dagger}_{p}(\vec{x})\,\Psi_p(\vec{x})d^{(d-1)}x\}|vac>$$

so that the anti-particle state becomes

$$\{\underline{\mathbb{C}\Psi}^{\dagger}_{p}[\Psi^{pos}_{p}] = \int \Psi_p(\vec{x})\,(\mathcal{C}\,\Psi^{pos}_{p}(\vec{x}))d^{(d-1)}x\}|vac> \;.$$

We distinguish in d-dimensional space two kinds of dicsrete operators $\mathcal{C}, \mathcal{P}$ and $\mathcal{T}$ operators with respect to the internal space which we use.

In the Clifford case we have [21]

$$\begin{aligned}
\mathcal{C}_{\mathcal{H}} &= \prod_{\gamma^a \in \Im} \gamma^a\; K\,,\\
\mathcal{T}_{\mathcal{H}} &= \gamma^0 \prod_{\gamma^a \in \Re} \gamma^a\; K\, I_{x^0}\,,\\
\mathcal{P}^{(d-1)}_{\mathcal{H}} &= \gamma^0\, I_{\vec{x}}\,,\\
I_x x^a &= -x^a\,, \quad I_{x^0} x^a = (-x^0, \vec{x})\,, \quad I_{\vec{x}}\vec{x} = -\vec{x}\,,\\
I_{\vec{x}_3} x^a &= (x^0, -x^1, -x^2, -x^3, x^5, x^6, \ldots, x^d)\,.
\end{aligned} \tag{9.71}$$

The product $\prod \gamma^a$ is meant in the ascending order in γ^a.

In the Grassmann case we correspondingly define

$$\begin{aligned}
\mathcal{C}_{G} &= \prod_{\gamma^a_G \in \Im\gamma^a} \gamma^a_G\, K\,,\\
\mathcal{T}_{G} &= \gamma^0_G \prod_{\gamma^a_G \in \Re\gamma^a} \gamma^a_G\, K\, I_{x^0}\,,\\
\mathcal{P}^{(d-1)}_{G} &= \gamma^0_G\, I_{\vec{x}}\,,
\end{aligned} \tag{9.72}$$

γ^a_G is defined in Eq. (9.11) as

$$\gamma^a_G = (1 - 2\theta^a \eta^{aa} \frac{\partial}{\partial\theta_a})\,, \tag{9.73}$$

while $I_x x^a = -x^a$, $I_{x^0} x^a = (-x^0, \vec{x})$, $I_{\vec{x}}\vec{x} = -\vec{x}$,

$$I_{\vec{x}_3} x^a = (x^0, -x^1, -x^2, -x^3, x^5, x^6, \ldots, x^d)\,.$$

Let be noticed, that since γ^a_G $(= -i\eta^{aa}\gamma^a\tilde{\gamma}^a)$ is always real as there is $\gamma^a i\tilde{\gamma}^a$, while γ^a is either real or imaginary, we use in Eq. (9.72) γ^a to make a choice of appropriate γ^a_G. In what follows we shall use the notation as in Eq. (9.72).

Let us define in the Clifford case and in the Grassmann case the operator "emptying" [7,9] (arxiv:1312.1541) the Dirac sea, so that operation of "emptying$_N$" after the charge conjugation $\mathcal{C}_{\mathcal{H}}$ in the Clifford case and "emptying$_G$" after the charge conjugation $\mathcal{C}_G$ in the Grassmann case (both transform the state put on the top of either the Clifford or the Grassmann Dirac sea into the corresponding negative energy state) creates the anti-particle state to the starting particle state, both put on the top of the Dirac sea and both solving the Weyl equation, either in the Clifford case, Eq. (9.34), or in the Grassmann case, Eq. (9.39), for free massless fermions

$$\begin{aligned}
\text{"emptying}_N\text{"} &= \prod_{\Re\gamma^a} \gamma^a\, K \quad \text{in Clifford space}\,, \\
\text{"emptying}_G\text{"} &= \prod_{\Re\gamma^a} \gamma^a_G\, K \quad \text{in Grassmann space}\,, \qquad (9.74)
\end{aligned}$$

although we must keep in mind that indeed the anti-particle state is a hole in the Dirac sea from the Fock space point of view. The operator "emptying" is bringing the single particle operator $\mathcal{C}_{\mathcal{H}}$ in the Clifford case and $\mathcal{C}_G$ in the Grassmann case into the operator on the Fock space in each of the two cases. Then the anti-particle state creation operator — $\underline{\Psi}^{\dagger}_a[\Psi_p]$ — to the corresponding particle state creation operator — can be obtained also as follows

$$\begin{aligned}
\underline{\Psi}^{\dagger}_a[\Psi_p]\,|\nu ac> &= \mathbb{\underline{C}}_{\mathcal{H}}\,\underline{\Psi}^{\dagger}_p[\Psi_p]\,|\nu ac> = \int \Psi^{\dagger}_a(\vec{x})\,(\mathbb{C}_{\mathcal{H}}\,\Psi_p(\vec{x}))\,d^{(d-1)}x\,|\nu ac>\,, \\
\mathbb{C}_{\mathcal{H}} &= \text{"emptying}_N\text{"}\cdot\mathcal{C}_{\mathcal{H}} \qquad (9.75)
\end{aligned}$$

in both cases.

The operators $\mathbb{C}_{\mathcal{H}}$ and $\mathbb{C}_G$

$$\mathbb{C}_{\mathcal{H}} = \text{"emptying}_N\text{"}\cdot\mathcal{C}_{\mathcal{H}}\,, \qquad \mathbb{C}_G = \text{"emptying}_{NG}\text{"}\cdot\mathcal{C}_G\,, \qquad (9.76)$$

operating on $\Psi_p(\vec{x})$ transforms the positive energy spinor state (which solves the corresponding Weyl equation for a massless free fermion) put on the top of the Dirac sea into the positive energy anti-fermion state, which again solves the corresponding Weyl equation for a massless free anti-fermion put on the top of the Dirac sea. Let us point out that either the operator "emptying$_N$" or the operator "emptying$_{NG}$" transforms the single particle operator either $\mathcal{C}_{\mathcal{H}}$ or $\mathcal{C}_G$ into the operator operating in the Fock space.

We use the Grassmann even, Hermitian and real operators γ^a_G, Eq. (9.11), to define discrete symmetry in Grassmann space, first in $((d+1)-1)$ space and then in $(3+1)$ space, as we did in [21] in the Clifford case. In the Grassmann case we do this in analogy with the operators in the Clifford case [21]

$$
\begin{aligned}
\mathcal{C}_{NG} &= \prod_{\gamma^m_G \in \Re\gamma^m} \gamma^m_G \, K \, I_{x^6 x^8 \ldots x^d} \,, \\
\mathcal{T}_{NG} &= \gamma^0_G \prod_{\gamma^m_G \in \Im\gamma^m} K \, I_{x^0} I_{x^5 x^7 \ldots x^{d-1}} \,, \\
\mathcal{P}^{(d-1)}_{NG} &= \gamma^0_G \prod_{s=5}^{d} \gamma^s_G I_{\vec{x}} \,, \\
\mathbb{C}_{NG} &= \prod_{\gamma^s_G \in \Re\gamma^s} \gamma^s_G \,, I_{x^6 x^8 \ldots x^d} \,, \\
\mathbb{C}_{NG}\mathcal{P}^{(d-1)}_{NG} &= \gamma^0_G \prod_{\gamma^s_G \in \Im\gamma^s, s=5}^{d} \gamma^s_G \, I_{\vec{x}_3} \, I_{x^6 x^8 \ldots x^d} \,, \\
\mathbb{C}_{NG}\mathcal{T}_{NG}\mathcal{P}^{(d-1)}_{NG} &= \prod_{\gamma^s_G \in \Im\gamma^a} \gamma^a_G \, I_x K \,.
\end{aligned} \tag{9.77}
$$

Let us try to understand the Grassmann fermions in the case $d = 5 + 1$, before the break, as well as after the break of $d = 5 + 1$ into $d = 3 + 1$, when the fifth and the sixth dimension determine the charge in $d = 3 + 1$. There are two decuplets in this case [15], both of an odd Grassmann character, which can be second quantized. The two triplets in the first decuplet— (ψ^I_1, ψ^I_2, ψ^I_3) and (ψ^I_4, ψ^I_5, ψ^I_6) — both solving the Eq. (9.39) for massless free fermions in Grassmann space with the space function $e^{-ip_a x^a}$. The Grassmann even opoerator operator $\mathbb{C}_{NG}\mathcal{P}^{(d-1)}_{NG}$ transforms ψ^I_1 with $p^a = (|p^0|, 0, 0, |p^3|, 0, 0)$ into the antiparticle state ψ^I_6, with the positive energy $|p^0|$ and with $-|p^3|$, for example. Correspondingly transforms $\mathbb{C}_{NG}\mathcal{P}^{(d-1)}_{NG}$ the particle state ψ^I_3 with the positive energy and into the antiparticle state ψ^I_4 with the positive energy, and the particle ψ^I_3 into the positive energy antiparticle state ψ^I_4. All belong to the same representation.

Applying the Grassmann even operators on one of the states of one the decuplets — $\mathcal{C}_G (= \gamma^2_G \gamma^5_G$, Eq. (9.72)), $\mathcal{C}_{NG}\mathcal{P}^{(d-1)}_{NG}$ $(= \gamma^1_G \gamma^3_G \gamma^5_G \gamma^6_G \, I_{x^6} I_{\vec{x}_3} K$, Eq. (9.72)) — one remains within the same decuplet. To get the positive energy antiparticle states the operator empting_N in $(d-1)+1$ and empting_{NG} in $d = (3+1)$ are needed, Eqs. (9.74, 9.76). The reader can find more discussions in Refs. [15,21].

d. What do we learn in the second quantization procedure in Grassmann and in Clifford space

We proved that basic states in both spaces can be written by creation operators operating on an appropriate vacuum state. The creation and annihilation operators fulfill in both spaces anticommutation relations as required for fermions, Eqs (9.48, 9.60).

In both spaces the creation operators are chosen to create states that are eigenstates of the corresponding Cartan subalgebra of the Lorentz algebra, the generators of which are $\mathbf{S}^{ab}$, Eq. (9.13), for the Grassmann case and $(S^{ab}, \tilde{S}^{ab})$, first generating spins and the second families, Eq. (9.25), for the Clifford case.

I		decuplet	$\mathbf{S}^{03}$	$\mathbf{S}^{12}$	$\mathbf{S}^{56}$
	1	$(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$	i	1	1
	2	$(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5+i\theta^6)$	0	0	1
	3	$(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$	$-i$	-1	1
	4	$(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$	i	-1	-1
	5	$(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5-i\theta^6)$	0	0	-1
	6	$(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$	$-i$	1	-1
	7	$(\theta^0-\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$	i	0	0
	8	$(\theta^0+\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$	$-i$	0	0
	9	$(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1+i\theta^2)$	0	1	0
	10	$(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1-i\theta^2)$	0	-1	0
II		decuplet	$\mathbf{S}^{03}$	$\mathbf{S}^{12}$	$\mathbf{S}^{56}$
	1	$(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$	$-i$	1	1
	2	$(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5+i\theta^6)$	0	0	1
	3	$(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$	i	-1	1
	4	$(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$	$-i$	-1	-1
	5	$(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5-i\theta^6)$	0	0	-1
	6	$(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$	i	1	-1
	7	$(\theta^0+\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$	$-i$	0	0
	8	$(\theta^0-\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$	i	0	0
	9	$(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1+i\theta^2)$	0	1	0
	10	$(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1-i\theta^2)$	0	-1	0

Table 9.1. The creation operators of the decuplet and the antidecouplet of the orthogonal group $SO(5,1)$ in Grassmann space are presented. Applying on the vacuum state $|\phi_0>=|1>$ the creation operators form eigenstates of the Cartan subalgebra, Eq. (9.84), $(\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56})$. The states within each decouplet are reachable from any member by $\mathbf{S}^{ab}$. The product of the discrete operators $\mathbb{C}_{NG}$ $(=\prod_{\Re\gamma^s}\gamma^s_G\, I_{x^6x^8\dots x^d})$ $\mathcal{P}^{(d-1)}_{NG}$ $(=\gamma^0_G\prod^d_{s=5}\gamma^s_G I_{\vec{x}_3})$ transforms, for example, ψ^I_1 into ψ^I_6, ψ^I_2 into ψ^I_5 and ψ^I_3 into ψ^I_4. Solutions of the Weyl equation, Eq. (9.39), with the negative energies belong to the "Grassmann sea", with the positive energy to the particles and antiparticles.

While in the Grassmann case the vacuum state is simple, $|\phi_{og}>=|1>$, in the Clifford case the vacuum state is a sum of products of $2^{\frac{d}{2}-1}$ projectors, Eq. (9.67).

In $2(2n+1)$-dimensional spaces there are in the Clifford case $2^{\frac{d}{2}-1}$ states in one representation reachable from (any) starting state by S^{ab}, while $\tilde{S}^{ab}$ transform each of these states changing its family quantum number. There are correspondingly $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ states reachable with either S^{ab} or $\tilde{S}^{ab}$. Each state is obtained by the corresponding creation operator on the vacuum state and is annihilated by its Hermitian conjugate operator.

In $2(2n+1)$-dimensional spaces there are in the Grassmann case two decoupled groups with $\frac{1}{2}\,\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ states in each representation. Each of states can be obtained by the corresponding creation operator and is annihilated by its Hermitian conjugated operator. While all of $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ states in Clifford space are reachable by even Clifford objects, either S^{ab} or $\tilde{S}^{ab}$, in Grassmann space the two groups of representations can not be reached by an even number of Grassmann objects.

9.3 Conclusions

We have learned in the present study that one can use either Grassmann or Clifford space to express the internal degrees of freedom of fermions in any even dimensional space, either for $d = 2(2n+1)$ or $d = 4n$. In both spaces the creation operators and their Hermitian conjugated annihilation operators fulfill the anticommutation relation requirements, needed for fermions, provided that they are expressed as odd products of either Grassmann (θ^a, $(\theta^a)^\dagger = \frac{\partial}{\partial\theta_a}\eta^{aa}$, Eq. (9.8)) or Clifford objects (either $\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$, Eq. (9.17) and correspondingly $\gamma^{a\dagger} = \gamma^a\eta^{aa}$, or $\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial\theta_a})$, Eq. (9.18), and correspondingly $\tilde{\gamma}^{a\dagger} = \tilde{\gamma}^a\eta^{aa}$). But while in the Clifford case states appear in the fundamental representations of the Lorentz group, carrying half integer spins, the states in the Grassmann case are in adjoint representations of the Lorentz group. The Clifford case, offering two kinds of the Clifford objects (γ^a and $\tilde{\gamma}^a$), enables to describe besides the spin degrees of freedom of fermion fields also their family degrees of freedom. The Grassmann case offers only one kind of objects. Assuming that "nature has both choices" for describing the internal degrees of freedom of fermion fields, the question arises why Grassmann choice is not chosen, or better, why the Clifford choice is chosen.

In the case that spin degrees in $d \geq 5$ manifest as charges in $d = (3+1)$, fermions in the Grassmann case manifest charges in the adjoint representations. On the other hand in the Clifford case — this is used in the *spin-charge-family* theory, which takes the Lorentz group $SO(13,1)$ — the spin and charges appear in the fundamental representations of the corresponding groups, offering also the family degrees of freedom.

We present in this paper the action describing free massless particles with the internal degrees of freedom describable in Grassmann space, Eqs. (9.37, 9.38). The action leads to the equation of motion analogous to the Weyl equation in Clifford space, fulfilling the Klein-Gordon equation.

Since the Clifford objects γ^a and $\tilde{\gamma}^a$ are expressible with the Grassmann coordinates θ^a and their conjugate moments $\frac{\partial}{\partial\theta^a}$, either basic states in Grassmann space, Eq. (9.4), or basic states in Clifford space, Eq. (9.15), can be normalized with the same integral, Eq. (9.27, 9.28, 9.30).

To understand better the difference in the description of the fermion internal degrees of freedom with either Clifford or Grassmann space, let us replace in the starting action of the *spin-charge-family* theory, Eq. (9.1), using the Clifford algebra to describe fermion degrees of freedom, the covariant momentum $p_{0a} = f^\alpha{}_a\, p_{0\alpha}$, $p_{0\alpha} = p_\alpha - \frac{1}{2}S^{ab}\omega_{ab\alpha} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{ab\alpha}$, with $p_{0\alpha} = p_\alpha - \frac{1}{2}\,\mathbf{S}^{ab}\Omega_{ab\alpha}$, where $\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}$, Eq. (9.26), and $\Omega_{ab\alpha}$ are the spin connection gauge fields of $\mathbf{S}^{ab}$ (which are the generators of the Lorentz transformations in Grassmann space!), while $f^\alpha{}_a\, p_{0\alpha}$ replaces the ordinary momentum when massless objects start to interact with the gravitational field through the vielbeins and the spin connections. Let us add that varying the action with respect to either $\omega_{ab\alpha}$ or $\tilde{\omega}_{ab\alpha}$ when no fermions are present, one learns that both spin connections are uniquely determined by the vielbeins ([9,3,5] and references therein) and correspondingly in this particular case $\omega_{ab\alpha} = \tilde{\omega}_{ab\alpha}$.

Let us use instead of p_a in the action for free massless fields using Grassmann space to describe the internal degrees of freedom, Eq. (9.37), the above covariant momentum $p_{0a} = f^{\alpha}{}_a\, (p_\alpha - \frac{1}{2}\, \mathbf{S}^{ab}\Omega_{ab\alpha})$. One finds in this case that the representations of the Lorentz group in $d = 2(2n+1) = 13+1$ and their subgroups $SO(7,1), SU(3)$ and $U(1)$ are all in the adjoint representations of the groups.

The *spin-charge-family* theory (using Clifford objects) offers the explanation for all the assumptions of the *standard model* of elementary fields, fermions and bosons, vector and scalar gauge fields, with the appearance of families included, explaining also the phenomena like the existence of the dark matter [10], of the matter-antimatter asymmetry [4], offering correspondingly the next step beyond both standard models — cosmological one and the one of the elementary fields.

We do notice, however, that the *Grassmann degrees of freedom do not offer the appearance of families at all*.

We also notice that the second quantization procedure allows in $d = 2(2n+1)$-dimensional space for each member of a Weyl representation in Clifford space (for each of $2^{\frac{d}{2}-1}$ "family member") $2^{\frac{d}{2}-1}$ "families", all together therefore $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ basic states which can be second quantized, according to this paper. From 2^d Clifford objects, only those of an odd Clifford character contribute to the second quantization — half of them as creation and half of them as annihilation operators, $2^{\frac{d}{2}-1}$ projectors from the rest of objects form the vacuum state.

We notice that in case of Grassmann space and $d = 2(2n+1)$ only twice two isolated groups of $\frac{1}{2}\, \frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ states of an odd Grassmann character can be second quantized.

To come to the low energy regime the symmetry must break, first from $SO(13,1)$ to $SO(7,1)\times SU(3)\times U(1)$ and then further to $SO(3,1)\times SU(3)\times U(1)$, in both spaces, in Grassmann and in Clifford. In Clifford case there are two kinds of generators and correspondingly two kinds of symmetries. We learned in Refs. [23–25] that when breaking symmetries only some of families stay massless and correspondingly observable in $d = (3+1)$.

This study is indeed to learn more about possibilities that "nature has". One of the authors (N.S.M.B.) wants to learn: **a.** Why is the simple starting action of the *spin-charge-family* theory doing so well in manifesting the observed properties of the fermion and boson fields? **b.** Under which condition can more general action lead to the starting action of Eq. (9.1)? **c.** What would more general action, if leading to the same low energy physics, mean for the history of our Universe? **d.** Could the fermionization procedure of boson fields or the bosonization procedure of fermion fields, discussed in Ref. [12] for any even dimension d (by the authors of this contribution, while one of them (H.B.F.N. [13]) has succeeded with another author to do the fermionization for $d = (1+1)$) tell more about the "decisions" of the universe in the history?

Although we have not yet learned enough to be able to answer these questions, yet we have learned at least that the description of the fermion internal degrees of freedom in Grassmann space would not offer families, and would not be in agreement with the spin and charges and other observations so far. We also learned that if there are no fermion present only one kind of dynamical fields manifests, since either $\omega_{ab\alpha}$ or $\tilde{\omega}_{ab\alpha}$ are uniquely expressed by vielbeins ([9] Eq. (C9) and references therein), which could mean that the appearance of the two kinds of the spin connection fields might be due to the break of symmetries.

9.4 Appenix: Lorentz algebra and representations in Grassmann and Clifford space

The Lorentz transformations of vector components θ^a, γ^a, or $\tilde{\gamma}^a$, which all could be used to describe internal degrees of freedom of fields with the anticommutation relations of fermions, and of vector components x^a, which are real (ordinary) commuting coordinates:

$\theta'^a = \Lambda^a{}_b\,\theta^b$, $\gamma'^a = \Lambda^a{}_b\,\gamma^b$, $\tilde{\gamma}'^a = \Lambda^a{}_b\,\tilde{\gamma}^b$ and $x^a = \Lambda^a{}_b\,x^b$, leave forms $a_{a_1 a_2 \dots a_i}\ \theta^{a_1}\theta^{a_2}\dots\theta^{a_i}$, $a_{a_1 a_2 \dots a_i}\ \gamma^{a_1}\gamma^{a_2}\dots\gamma^{a_i}$, $a_{a_1 a_2 \dots a_i}\ \tilde{\gamma}^{a_1}\tilde{\gamma}^{a_2}\dots\tilde{\gamma}^{a_i}$ and $b_{a_1 a_2 \dots a_i}\, x^{a_1}x^{a_2}\dots x^{a_i}$, $i = (1, \dots, d)$, invariant.

While $b_{a_1 a_2 \dots a_i}$ $(= \eta_{a_1 b_1}\eta_{a_2 b_2}\cdots\eta_{a_i b_i}\, b^{b_1 b_2 \dots b_i})$ is a symmetric tensor field, $a_{a_1 a_2 \dots a_i}$ $(= \eta_{a_1 b_1}\eta_{a_2 b_2}\cdots\eta_{a_i b_i}\, a^{b_1 b_2 \dots b_i})$ are antisymmetric tensor *Kalb-Ramond* fields.

The requirements: $x'^a\, x'^b \eta_{ab} = x^c\, x^d \eta_{cd}$, $\theta'^a\theta'^b\varepsilon_{ab} = \theta^c\theta^d\varepsilon_{cd}$, $\gamma'^a\gamma'^b\varepsilon_{ab} = \gamma^c\gamma^d\varepsilon_{cd}$ and $\tilde{\gamma}'^a\tilde{\gamma}'^b\varepsilon_{ab} = \tilde{\gamma}^c\tilde{\gamma}^d\varepsilon_{cd}$ lead to $\Lambda^a{}_b\,\Lambda^c{}_d\,\eta_{ac} = \eta_{bd}$. Here η^{ab} (in our case $\eta^{ab} = diag(1, -1, -1, \dots, -1)$) is the metric tensor lowering the indexes of vectors ($\{x^a\} = \eta^{ab}x_b$, $\{\theta^a\} = \eta^{ab}\,\theta_b$, $\{\gamma^a\} = \eta^{ab}\,\gamma_b$ and $\{\tilde{\gamma}^a\} = \eta^{ab}\,\tilde{\gamma}_b$) and ε_{ab} is the antisymmetric tensor. An infinitesimal Lorentz transformation for the case with $\det\Lambda = 1, \Lambda^0{}_0 \geq 0$ can be written as $\Lambda^a{}_b = \delta^a_b + \omega^a{}_b$, where $\omega^a{}_b + \omega_b{}^a = 0$.

According to Eqs. (9.17, 9.18, 9.25) one finds, Eq. (9.3),

$$\begin{aligned}
&\{\gamma^a, \tilde{S}^{cd}\}_- = 0 = \{\tilde{\gamma}^a, S^{cd}\}_-\,,\\
&\{\gamma^a, \mathbf{S}^{cd}\}_- = \{\gamma^a, S^{cd}\}_- = i\,(\eta^{ac}\gamma^d - \eta^{ad}\gamma^c)\,,\\
&\{\tilde{\gamma}^a, \mathbf{S}^{cd}\}_- = \{\tilde{\gamma}^a, \tilde{S}^{cd}\}_- = i\,(\eta^{ac}\tilde{\gamma}^d - \eta^{ad}\tilde{\gamma}^c)\,.
\end{aligned} \tag{9.78}$$

Comments: In cases with either the basis θ^a or with the basis of γ^a or $\tilde{\gamma}^a$ the scalar products — the norms $< \mathbf{B}|\mathbf{B} >$ and $< \mathbf{F}|\mathbf{F} >$ (where $< \theta|B >$, Eq. (9.4), and $< \gamma|F >$, Eq. (9.15), are vectors in Grassmann and Clifford space, respectively) — are non negative and equal to $\sum_{k=0}^{d} \int d^{d-1}x\, b^*_{b_1 \dots b_k} b_{b_1 \dots b_k}$.

9.4.1 Lorentz properties of basic vectors

What follows is taken from Ref. [2] and Ref. [9], Appendix B.

Let us first repeat some properties of the anticommuting Grassmann coordinates.

An infinitesimal Lorentz transformation of the proper ortochronous Lorentz group is then

$$
\begin{aligned}
\delta\theta^c &= -\frac{i}{2}\omega_{ab}\mathbf{S}^{ab}\theta^c = \omega^c{}_a\theta^a\,,\\
\delta\gamma^c &= -\frac{i}{2}\omega_{ab}S^{ab}\gamma^c = \omega^c{}_a\gamma^a\,,\\
\delta\tilde{\gamma}^c &= -\frac{i}{2}\omega_{ab}\tilde{S}^{ab}\tilde{\gamma}^c = \omega^c{}_a\tilde{\gamma}^a\,,\\
\delta x^c &= -\frac{i}{2}\omega_{ab}L^{ab}x^c = \omega^c{}_a x^a\,,
\end{aligned}
\tag{9.79}
$$

where ω_{ab} are parameters of a transformation and γ^a and $\tilde{\gamma}^a$ are expressed by θ^a and $\frac{\partial}{\partial\theta_a}$ in Eqs. (9.17, 9.18).

Let us write the operator of finite Lorentz transformations as follows

$$
\mathbf{S} = e^{-\frac{i}{2}\omega_{ab}(\mathbf{S}^{ab}+L^{ab})}\,. \tag{9.80}
$$

We see that the Grassmann θ^a and the ordinary x^a coordinates and the Clifford objects γ^a and $\tilde{\gamma}^a$ transform as vectors Eq. (9.80)

$$
\begin{aligned}
\theta'^c &= e^{-\frac{i}{2}\omega_{ab}(\mathbf{S}^{ab}+L^{ab})}\,\theta^c\,e^{\frac{i}{2}\,\omega_{ab}(\mathbf{S}^{ab}+L^{ab})}\\
&= \theta^c - \frac{i}{2}\omega_{ab}\{\mathbf{S}^{ab},\theta^c\}_- + \cdots = \theta^c + \omega^c{}_a\theta^a + \cdots = \Lambda^c{}_a\theta^a\,,\\
x'^c &= \Lambda^c{}_a x^a\,,\quad \gamma'^c = \Lambda^c{}_a\gamma^a\,,\quad \tilde{\gamma}'^c = \Lambda^c{}_a\tilde{\gamma}^a\,.
\end{aligned}
\tag{9.81}
$$

Correspondingly one finds that compositions like $\gamma^a p_a$ and $\tilde{\gamma}^a p_a$, here p_a are p^x_a $(= i\frac{\partial}{\partial x^a})$, transform as scalars (remaining invariants), while $S^{ab}\,\omega_{abc}$ and $\tilde{S}^{ab}\,\tilde{\omega}_{abc}$ transform as vectors.

Also objects like

$$
R = \frac{1}{2}\,f^{\alpha[a}f^{\beta b]}\,(\omega_{ab\alpha,\beta} - \omega_{ca\alpha}\,\omega^c{}_{b\beta})
$$

and

$$
\tilde{R} = \frac{1}{2}\,f^{\alpha[a}f^{\beta b]}\,(\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha}\tilde{\omega}^c{}_{b\beta})
$$

from Eq. (9.1) transform with respect to the Lorentz transformations as scalars.

Making a choice of the Cartan subalgebra set of the algebra $\mathbf{S}^{ab}$, S^{ab} and $\tilde{S}^{ab}$, Eqs. (9.13, 9.17, 9.18),

$$
\begin{aligned}
&\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \cdots, \mathbf{S}^{d-1\;d}\,,\\
&S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\;d}\,,\\
&\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\;d}\,,
\end{aligned}
\tag{9.82}
$$

one can arrange the basic vectors so that they are eigenstates of the Cartan subalgebra, belonging to representations of $\mathbf{S}^{ab}$, or of S^{ab} and $\tilde{S}^{ab}$, with ab from Eq (9.82).

9.5 Appendix: Technique to generate spinor representations in terms of Clifford algebra objects

We shall briefly repeat the main points of the technique for generating spinor representations from Clifford algebra objects, following Ref. [16]. We advise the reader to look for details and proofs in this reference.

We assume the objects γ^a, Eq. (9.17), which fulfill the Clifford algebra, Eq (9.16).

$$\{\gamma^a, \gamma^b\}_+ = I \;\; 2\eta^{ab}, \quad \text{for} \quad a, b \quad \in \{0, 1, 2, 3, 5, \cdots, d\}, \tag{9.83}$$

for any d, even or odd. I is the unit element in the Clifford algebra, while $\{\gamma^a, \gamma^b\}_\pm = \gamma^a\gamma^b \pm \gamma^b\gamma^a$.

We accept the "Hermiticity" property for γ^a's, Eq. (9.20), $\gamma^{a\dagger} = \eta^{aa}\gamma^a$, leading to $\gamma^{a\dagger}\gamma^a = I$. Assuming the relation of Eq. (9.17) this last relations follow.

The Clifford algebra objects S^{ab} close the Lie algebra of the Lorentz group $\{S^{ab}, S^{cd}\}_- = i(\eta^{ad}S^{bc} + \eta^{bc}S^{ad} - \eta^{ac}S^{bd} - \eta^{bd}S^{ac})$. One finds from Eq.(9.20) that $(S^{ab})^\dagger = \eta^{aa}\eta^{bb}S^{ab}$ and that $\{S^{ab}, S^{ac}\}_+ = \frac{1}{2}\eta^{aa}\eta^{bc}$.

Recognizing that two Clifford algebra objects S^{ab}, S^{cd} with all indexes different commute, we select (out of many possibilities) the Cartan sub algebra set of the algebra of the Lorentz group as follows

$$\begin{aligned} S^{0d}, S^{12}, S^{35}, \cdots, S^{d-2\; d-1}, &\quad \text{if} \quad d = 2n, \\ S^{12}, S^{35}, \cdots, S^{d-1\; d}, &\quad \text{if} \quad d = 2n+1. \end{aligned} \tag{9.84}$$

To make the technique simple, we introduce the graphic representation [16] as follows

$$\begin{aligned} \stackrel{ab}{(k)}: &= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b), \\ \stackrel{ab}{[k]}: &= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b), \end{aligned} \tag{9.85}$$

where $k^2 = \eta^{aa}\eta^{bb}$. One can easily check by taking into account the Clifford algebra relation (Eq. (9.83)) and the definition of S^{ab} (Eq. (9.25)) that if one multiplies from the left hand side by S^{ab} the Clifford algebra objects $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$, it follows that

$$\begin{aligned} S^{ab} \stackrel{ab}{(k)} &= \frac{1}{2}k \stackrel{ab}{(k)}, \\ S^{ab} \stackrel{ab}{[k]} &= \frac{1}{2}k \stackrel{ab}{[k]}. \end{aligned} \tag{9.86}$$

This means that $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$ acting from the left hand side on anything (on a vacuum state $|\psi_0\rangle$, for example) are eigenvectors of S^{ab}.

We further find

$$\begin{aligned} \gamma^a \stackrel{ab}{(k)} &= \eta^{aa} \stackrel{ab}{[-k]}, & \gamma^b \stackrel{ab}{(k)} &= -ik \stackrel{ab}{[-k]}, \\ \gamma^a \stackrel{ab}{[k]} &= \stackrel{ab}{(-k)}, & \gamma^b \stackrel{ab}{[k]} &= -ik\eta^{aa} \stackrel{ab}{(-k)}. \end{aligned} \tag{9.87}$$

It follows that $S^{ac}\overset{ab}{(k)}\overset{cd}{(k)} = -\frac{i}{2}\eta^{aa}\eta^{cc}\overset{ab}{[-k]}\overset{cd}{[-k]}$, $S^{ac}\overset{ab}{[k]}\overset{cd}{[k]} = \frac{i}{2}\overset{ab}{(-k)}\overset{cd}{(-k)}$, $S^{ac}\overset{ab}{(k)}\overset{cd}{[k]} = -\frac{i}{2}\eta^{aa}\overset{ab}{[-k]}\overset{cd}{(-k)}$, $S^{ac}\overset{ab}{[k]}\overset{cd}{(k)} = \frac{i}{2}\eta^{cc}\overset{ab}{(-k)}\overset{cd}{[-k]}$. It is useful to deduce the following relations

$$
\begin{array}{llll}
\overset{ab}{(k)}\overset{ab}{(k)} = 0\,, & \overset{ab}{(k)}\overset{ab}{(-k)} = \eta^{aa}\overset{ab}{[k]}\,, & \overset{ab}{(-k)}\overset{ab}{(k)} = \eta^{aa}\overset{ab}{[-k]}\,, & \overset{ab}{(-k)}\overset{ab}{(-k)} = 0\,, \\
\overset{ab}{[k]}\overset{ab}{[k]} = \overset{ab}{[k]}\,, & \overset{ab}{[k]}\overset{ab}{[-k]} = 0\,, & \overset{ab}{[-k]}\overset{ab}{[k]} = 0\,, & \overset{ab}{[-k]}\overset{ab}{[-k]} = \overset{ab}{[-k]}\,, \\
\overset{ab}{(k)}\overset{ab}{[k]} = 0\,, & \overset{ab}{[k]}\overset{ab}{(k)} = \overset{ab}{(k)}\,, & \overset{ab}{(-k)}\overset{ab}{[k]} = \overset{ab}{(-k)}\,, & \overset{ab}{(-k)}\overset{ab}{[-k]} = 0\,, \\
\overset{ab}{(k)}\overset{ab}{[-k]} = \overset{ab}{(k)}\,, & \overset{ab}{[k]}\overset{ab}{(-k)} = 0\,, & \overset{ab}{[-k]}\overset{ab}{(k)} = 0\,, & \overset{ab}{[-k]}\overset{ab}{(-k)} = \overset{ab}{(-k)}\,.
\end{array}
\tag{9.88}
$$

We recognize in the first equation of the first row and the first equation of the second row the demonstration of the nilpotent and the projector character of the Clifford algebra objects $\overset{ab}{(k)}$ and $\overset{ab}{[k]}$, respectively.

Whenever the Clifford algebra objects apply from the left hand side, they always transform $\overset{ab}{(k)}$ *to* $\overset{ab}{[-k]}$, *never to* $\overset{ab}{[k]}$, *and similarly* $\overset{ab}{[k]}$ *to* $\overset{ab}{(-k)}$, *never to* $\overset{ab}{(k)}$.

We define in Eq. (9.62) a vacuum state $|\psi_{oc}>$ so that one finds

$$
< \overset{ab}{(k)}{}^{\dagger}\,\overset{ab}{(k)} >= 1\,, \qquad < \overset{ab}{[k]}{}^{\dagger}\,\overset{ab}{[k]} >= 1\,. \tag{9.89}
$$

Taking the above equations into account it is easy to find a Weyl spinor irreducible representation for d-dimensional space, with d even or odd. (We advise the reader to see Ref. [16].)

For d even, we simply set the starting state as a product of d/2, let us say, only nilpotents $\overset{ab}{(k)}$ for $d = 2(2n+1)$, Eq. (9.57), or nilpotents and one projector, Eq. (9.58), for $d = 4n$, one for each S^{ab} of the Cartan subalgebra elements (Eq. (9.84)), applying it on the vacuum state, Eq. (9.62). Then the generators S^{ab}, which do not belong to the Cartan subalgebra, applied to the starting state from the left hand side, generate all the members of one Weyl spinor.

$$
\begin{array}{c}
\overset{0d}{(k_{0d})}\overset{12}{(k_{12})}\overset{35}{(k_{35})}\cdots\overset{d-1\;d-2}{(k_{d-1\;d-2})}\,|\psi_{oc}>\,,\\
\overset{0d}{[-k_{0d}]}\overset{12}{[-k_{12}]}\overset{35}{(k_{35})}\cdots\overset{d-1\;d-2}{(k_{d-1\;d-2})}\,|\psi_{oc}>\,,\\
\overset{0d}{[-k_{0d}]}\overset{12}{(k_{12})}\overset{35}{[-k_{35}]}\cdots\overset{d-1\;d-2}{(k_{d-1\;d-2})}\,|\psi_{oc}>\,,\\
\vdots\\
\overset{od}{(k_{0d})}\overset{12}{[-k_{12}]}\overset{35}{[-k_{35}]}\cdots\overset{d-1\;d-2}{[-k_{d-1\;d-2}]}\,|\psi_{oc}>\,,\\
\text{for } d = 2(2n+1)\,, \quad n = \text{positive integer}\,.
\end{array}
\tag{9.90}
$$

$$
\begin{aligned}
&\stackrel{0d}{(k_{0d})}\stackrel{12}{(k_{12})}\stackrel{35}{(k_{35})}\cdots\stackrel{d-1\,d-2}{[k_{d-1\,d-2}]}\,|\psi_{oc}>,\\
&\stackrel{0d}{[-k_{0d}]}\stackrel{12}{[-k_{12}]}\stackrel{35}{(k_{35})}\cdots\stackrel{d-1\,d-2}{[k_{d-1\,d-2}]}\,|\psi_{oc}>,\\
&\stackrel{0d}{[-k_{0d}]}\stackrel{12}{(k_{12})}\stackrel{35}{[-k_{35}]}\cdots\stackrel{d-1\,d-2}{[k_{d-1\,d-2}]}\,|\psi_{oc}>,\\
&\vdots\\
&\stackrel{od}{(k_{0d})}\stackrel{12}{[-k_{12}]}\stackrel{35}{[-k_{35}]}\cdots\stackrel{d-1\,d-2}{[k_{d-1\,d-2}]}\,|\psi_{oc}>,\\
&\text{for } d=4n\,,\quad n=\text{positive integer}\,.
\end{aligned}
\tag{9.91}
$$

9.5.1 Technique to generate "families" of spinor representations in terms of Clifford algebra objects

When all 2^d states are considered as a Hilbert space, we found in this paper that for d even there are $2^{d/2-1}$ "families members" and $2^{d/2-1}$ "families" of spinors, which can be second quantized. (The reader is advised to se also Ref. [2,26,16,17,27,9].) We shall pay attention on only even d.

One Weyl representation form a left ideal with respect to the multiplication with the Clifford algebra objects. We proved in Ref. [9], and the references therein that there is the application of the Clifford algebra object from the right hand side, which generates "families" of spinors.

Right multiplication with the Clifford algebra objects namely transforms the state with the quantum numbers of one "family member" belonging to one "family" into the state of the same "family member" (into the same state with respect to the generators S^{ab} when the multiplication from the left hand side is performed) of another "family".

We defined in Ref.[17] the Clifford algebra objects $\tilde{\gamma}^a$'s as operations which operate formally from the left hand side (as γ^a's do) on any Clifford algebra object A as follows

$$
\tilde{\gamma}^a A = i(-)^{(A)} A\gamma^a\,, \tag{9.92}
$$

with $(-)^{(A)} = -1$, if A is an odd Clifford algebra object and $(-)^{(A)} = 1$, if A is an even Clifford algebra object.

Then it follows, in accordance with Eqs. (9.17, 9.18, 9.19), that $\tilde{\gamma}^a$ obey the same Clifford algebra relation as γ^a.

$$
(\tilde{\gamma}^a\tilde{\gamma}^b + \tilde{\gamma}^b\tilde{\gamma}^a)A = -ii((-)^{(A)})^2 A(\gamma^a\gamma^b + \gamma^b\gamma^a) = I\cdot 2\eta^{ab}A \tag{9.93}
$$

and that $\tilde{\gamma}^a$ and γ^a anticommute

$$
(\tilde{\gamma}^a\gamma^b + \gamma^b\tilde{\gamma}^a)A = i(-)^{(A)}(-\gamma^b A\gamma^a + \gamma^b A\gamma^a) = 0\,. \tag{9.94}
$$

We may write

$$
\{\tilde{\gamma}^a, \gamma^b\}_+ = 0, \quad \text{while} \quad \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = I\cdot 2\eta^{ab}\,. \tag{9.95}
$$

One accordingly finds

$$\tilde{\gamma}^a \overset{ab}{(k)} := -i \overset{ab}{(k)} \gamma^a = -i\eta^{aa} \overset{ab}{[k]}\,, \qquad \tilde{\gamma}^b \overset{ab}{(k)} := -i \overset{ab}{(k)} \gamma^b = -k \overset{ab}{[k]}\,,$$
$$\tilde{\gamma}^a \overset{ab}{[k]} := \ i \overset{ab}{[k]} \gamma^a = \ i \overset{ab}{(k)}\,, \qquad \tilde{\gamma}^b \overset{ab}{[k]} := \ i \overset{ab}{[k]} \gamma^b = -k\eta^{aa} \overset{ab}{(k)}\ . \tag{9.96}$$

If we define

$$\tilde{S}^{ab} = \frac{i}{4}\,[\tilde{\gamma}^a, \tilde{\gamma}^b] = \frac{1}{4}(\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a)\,, \tag{9.97}$$

it follows

$$\tilde{S}^{ab} A = A \frac{1}{4}(\gamma^b \gamma^a - \gamma^a \gamma^b)\,, \tag{9.98}$$

manifesting accordingly that $\tilde{S}^{ab}$ fulfil the Lorentz algebra relation as S^{ab} do. Taking into account Eq. (9.92), we further find

$$\{\tilde{S}^{ab}, S^{ab}\}_- = 0\,, \quad \{\tilde{S}^{ab}, \gamma^c\}_- = 0\,, \quad \{S^{ab}, \tilde{\gamma}^c\}_- = 0\,. \tag{9.99}$$

One also finds

$$\{\tilde{S}^{ab}, \Gamma\}_- = 0\,, \quad \{\tilde{\gamma}^a, \Gamma\}_- = 0\,, \quad \text{for } d \ \text{ even}\,,$$
$$\Gamma^{(d)} := (i)^{d/2} \prod_a (\sqrt{\eta^{aa}}\gamma^a)\,, \quad \text{if} \quad d = 2n\,, \tag{9.100}$$

where handedness Γ ($\{\Gamma, S^{ab}\}_- = 0$) is a Casimir of the Lorentz group, which means that in d even transformation of one "family" into another with either $\tilde{S}^{ab}$ or $\tilde{\gamma}^a$ leaves handedness Γ unchanged.

We advise the reader also to read [2] where the two kinds of Clifford algebra objects follow as two different superpositions of a Grassmann coordinate and its conjugate momentum.

We present for $\tilde{S}^{ab}$ some useful relations

$$\tilde{S}^{ab} \overset{ab}{(k)} = \frac{k}{2} \overset{ab}{(k)}, \qquad \tilde{S}^{ab} \overset{ab}{[k]} = -\frac{k}{2} \overset{ab}{[k]}, \qquad \tilde{S}^{ac} \overset{ab}{(k)}\overset{cd}{(k)} = \frac{i}{2}\eta^{aa}\eta^{cc} \overset{ab}{[k]}\overset{cd}{[k]},$$
$$\tilde{S}^{ac} \overset{ab}{[k]}\overset{cd}{[k]} = -\frac{i}{2} \overset{ab}{(k)}\overset{cd}{(k)}, \quad \tilde{S}^{ac} \overset{ab}{(k)}\overset{cd}{[k]} = -\frac{i}{2}\eta^{aa} \overset{ab}{[k]}\overset{cd}{(k)}, \quad \tilde{S}^{ac} \overset{ab}{[k]}\overset{cd}{(k)} = \frac{i}{2}\eta^{cc} \overset{ab}{(k)}\overset{cd}{[k]}\,. \tag{9.101}$$

We transform the state of one "family" to the state of another "family" by the application of $\tilde{S}^{ac}$ (formally from the left hand side) on a state of the first "family" for a chosen a, c. To transform all the states of one "family" into states of another "family", we apply $\tilde{S}^{ac}$ to each state of the starting "family". It is, of course, sufficient to apply $\tilde{S}^{ac}$ to only one state of a "family" and then use generators of the Lorentz group (S^{ab}) to generate all the states of one Dirac spinor d-dimensional space.

One must notice that nilpotents $\overset{ab}{(k)}$ and projectors $\overset{ab}{[k]}$ are eigenvectors not only of the Cartan subalgebra S^{ab} but also of $\tilde{S}^{ab}$. Accordingly only $\tilde{S}^{ac}$, which do not carry the Cartan subalgebra indices, cause the transition from one "family" to another "family".

The starting state of Eq. (9.90) can change, for example, to

$$\stackrel{0d}{[k_{0d}]}\stackrel{12}{[k_{12}]}\stackrel{35}{(k_{35})}\cdots\stackrel{d-1\;d-2}{(k_{d-1\;d-2})}\,, \tag{9.102}$$

if $\tilde{S}^{01}$ was chosen to transform the Weyl spinor of Eq. (9.90) to the Weyl spinor of another "family".

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