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Enumeration of genus–four maps by number of edges

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Abstract

An explicit form of the ordinary generating function for the number of rooted maps on a closed orientable surface of genus four with a given number of edges is given. An analytical formula for the number of unrooted maps of genus four with a given number of edges is obtained through the number of rooted ones. Both results are new.

Keywords: Enumeration, surface, orbifold, rooted map, unrooted map, Fuchsian group.

Math. Subj. Class.: 05C30, 57R18, 20H10

1 Introduction

By a *map* we mean a 2-cell decomposition of a compact orientable surface. A map is *rooted* if one of its *darts* (edge–vertex incidence pairs) is distinguished as a *root* and *unrooted* otherwise. Enumeration of maps up to orientation–preserving homeomorphism has attracted a lot of attention in the last decades. Enumeration of rooted maps on the plane by number of edges was done in the pioneering paper by W. T. Tutte [18]. Later, an analogous result for the torus was given by D. Arquès [2]. A structure for the generating function of the number of rooted maps on the surface of given genus g was suggested by E. A. Bender and E. R. Canfield [5]. By making use of their approach they obtained numerical tables of numbers of rooted maps for genera 2 and 3. The enumeration of rooted maps of given genus by

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number of vertices and faces was done in [3]. Also, a general algorithm to compute the generating function for such numbers was suggested. The results of this algorithm confirm the available data [22] obtained earlier by T. Walsh and A. Lehman for small numbers of edges. Enumeration of unrooted maps is a more complicated problem. For unrooted maps on the oriented sphere a closed analytical counting formula (of the type studied in the present paper) was given by V. A. Liskovets [12]. Counting algorithms suitable for maps either on the oriented sphere or on the general one were independently provided by N. C. Wormald [23, 24]. A general formula for counting unrooted maps through the number of rooted ones was obtained in [14]. In particular, the problem was completely solved for the torus and the orientable surfaces of genera 2 and 3. Different approaches to counting unrooted maps with prescribed properties were given in the papers [13, 8, 20].

The present paper provides two new results for maps on a closed orientable surface of genus 4. The first result is an explicit form of the ordinary generating function for the number of rooted maps of genus 4 with a given number of edges. The second result is an explicit analytical formula relating the number of unrooted maps of genus 4 with n edges with numbers of rooted maps of genus $g \leq 4$ and with $e \leq n$ edges.

2 Orientable combinatorial maps

By a (combinatorial) *map* we mean a triple $(D; R, L)$ composed of a finite set D and two permutations R and L, with L satisfying $L^2 = 1$, generating a transitive subgroup of the symmetric group S_D. The elements of D are called *darts* and the orbits of R, L and RL are respectively called *vertices*, *edges* and *faces*. Edges of size one are called *semiedges*. The genus g of a map $M = (D; R, L)$ is given by $2 - 2g = V + E + F - |D|$, where V is the number of vertices, E is the number of edges and F is the number of faces. If M has no semiedges (i.e. if L is fixed–point free), then $|D| = 2E$ and $2 - 2g = V - E + F$.

Combinatorial maps describe topological maps on orientable surfaces with a chosen global orientation; the permutation R represents the cyclic order of the edge-ends incident with each vertex encountered by a rotation around that vertex in the direction corresponding to that orientation. Hence they are determined up to orientation–preserving homeomorphisms of the surface leaving the set of vertices, of edges and of faces invariant. This gives rise to the following definition: two maps $(D_1; R_1, L_1)$ and $(D_2; R_2, L_2)$ are *isomorphic*, written $(D_1; R_1, L_1) \cong (D_2; R_2, L_2)$, if there is a bijection $\psi : D_1 \to D_2$ such that $\psi R_1 = R_2 \psi$ and $\psi L_1 = L_2 \psi$. The *isomorphism class* of the map $(D; R, L)$ is the set of maps $(D; R^\psi, L^\psi)$ such that $\psi \in S_D$ is a permutation of D. A *rooted map* is a 4-tuple $(D, r; R, L)$, where $r \in D$ and $(D; R, L)$ is a map. The dart r is called the *root*. Two rooted maps $(D, r; R, L)$ and $(D', r'; R', L')$ are *isomorphic* is there is an isomorphism (D, R, L) onto (D', R', L') taking root r to root r'. To each map $M = (D; R, L)$ there is an associated *closed* orientable surface (that is, a compact orientable surface without boundary) which can be constructed by associating a 2-cell to each orbit of the permutation RL. Hence M can be regarded as a topological map. In turn, any topological map on a closed orientable surface can be realized as a combinatorial map and two topological maps are related by an orientation-preserving homeomorphism if and only if the corresponding combinatorial maps are related by an isomorphism [10].

By enumerating unrooted maps we mean enumerating isomorphism classes of combinatorial maps, which is equivalent to enumerating topological maps up to orientation-preserving homeomorphism. Topological maps correspond to cellular embeddings of graphs. Since graphs were generally assumed to be without semiedges, we will interpret "maps on a closed orientable surface of genus g", or just "genus g orientable maps" as maps *without semiedges*.

The theory of maps presented in [10] gives a close relationship between maps and subgroups of a certain universal group. Denote by $\Delta = \Delta(\infty, \infty, 2)$ the group $\langle \alpha, \beta | \beta^2 = 1 \rangle \cong Z * Z_2$. Given a map $(D; R, L)$, the assignment $\alpha \mapsto R$ and $\beta \mapsto L$ extends to an epimorphism $\Phi : \Delta \to \langle R, L \rangle$. It follows that Δ acts on D by $z \cdot x = \Phi(z)x$ for $z \in \Delta$ and $x \in D$. The stabilizer $K \leq \Delta$ of a dart $x \in D$ has index $[\Delta : K] = |D|$. Conversely, each subgroup $K \leq \Delta$ of finite index determines a rooted map $M = (D, r; R, L)$, where D is the set of left cosets xK, $x \in \Delta$, $r = K$ is the trivial coset and the action of R and L is defined by left multiplication: $R(xK) = \alpha x K$, $L(xK) = \beta x K$. Moreover, M has no semiedges if and only if K is torsion–free.

2.1 Maps on orbifolds

In this paper we consider *maps on orbifolds*. This is a new and fruitful idea already used in previous papers [14, 15]. By an *oriented orbifold* O we mean an oriented surface S with a discrete subset of points $B = \{p_1, p_2, \ldots\}$ such that to each point p_i an integer $m_i \geq 2$ is assigned. The elements of B will be called *branch points* and the respective numbers $m_1, m_2, \ldots, m_i, \ldots$ will be called *branch indices*. If S is a compact connected orientable surface of genus g, then B is of finite cardinality $|B| = r$. In this case, the orbifold O is uniquely determined (up to orientation–preserving homeomorphism) by its signature $[g; m_1, m_2, \ldots, m_r]$, $1 < m_1 \leq m_2 \leq \ldots \leq m_r$. Hence we write $O = O[g; m_1, m_2, \ldots, m_r]$. The fundamental group $\pi_1(O)$ of O is an F-group (see [10]) defined by

$$\begin{aligned} \pi_1(O) &= F[g; m_1, m_2, \ldots, m_r] \\ &= \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g, e_1, \ldots, e_r | \\ &\qquad \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{r} e_j = 1, e_1^{m_1} = \cdots = e_r^{m_r} = 1 \rangle. \end{aligned} \tag{2.1}$$

A map on an orbifold O is a map on the underlying surface S_g of genus g satisfying the following three properties:

(P1) if $x \in B$, then x is either an internal point of a face, or a vertex, or an end-point of a semiedge (*free end*) which is not a vertex,

(P2) each face contains at most one branch point,

(P3) each free end of a semiedge is a branch point and the branch index of this point is 2.

Maps on orbifolds arise naturally when we take a quotient of an ordinary map on a closed surface by a finite group G of automorphisms. Then the numbers $m_1, \ldots, m_r$ are the orders of the stabilizers of the faces, vertices and edges under the action of G. Note that these stabilizers are always cyclic. Further information on maps on orbifolds can be found in [14] and [15].

An epimorphism $\pi_1(O) \to Z_\ell$ onto a cyclic group of order ℓ is called *order preserving* if it preserves the orders of the generators e_j, $j = 1, \ldots, r$. Equivalently, an order–preserving epimorphism $\pi_1(O) \to Z_\ell$ has a torsion–free kernel. We denote by $Epi_0(\pi_1(O), Z_\ell)$ the number of order–preserving epimorphisms $\pi_1(O) \to Z_\ell$.

For a technical reason it is convenient to modify the signature in the following way. Let

$$[g; m_1, m_2, \dots, m_r] = [g; \underbrace{2, \dots, 2}_{b_2\, times}, \underbrace{3, \dots, 3}_{b_3\, times}, \dots, \underbrace{\ell, \dots, \ell}_{b_\ell\, times}].$$

Then we write $[g; 2^{b_2}, 3^{b_3}, \dots, \ell^{b_\ell}]$ rather than $[g; m_1, m_2, \dots, m_r]$ listing only those j^{b_j} with $b_j > 0$.

Denote by $Orb(S_g/Z_\ell)$ the set of ℓ-tuples $[g; 2^{b_2}, 3^{b_3}, \dots, \ell^{b_\ell}]$ which are the signatures of cyclic orbifolds of type S_g/Z_ℓ for some S_g and Z_ℓ. By definition, the fundamental group $\pi_1(O)$ is uniquely determined by the signature of the orbifold O. Hence, for any $O \in Orb(S_g/Z_\ell)$, $O = [g; 2^{b_2}, 3^{b_3}, \dots, \ell^{b_\ell}]$, the group $\pi_1(O)$ is well defined.

3 Rooted map enumeration

Let $Q_g(z) = \sum_{n\geq 0} N_g(n) z^n$ be the ordinary generating function counting the number $N_g(n)$ of rooted maps on the orientable surface of genus g by number of edges (the exponent of z). This generating function satisfies an equation system presented by E. A. Bender and E. R. Canfield in [4]. Given a genus g, a closed expression for $Q_g(z)$ can in principle be computed from this equation system by induction. However the computational complexity is so high that up to 1998 exact solutions where only known for the first four genera, from 0 to 3. A common pattern for all the $Q_g(z)$, where g ranges over the positive integers, was proposed in [5]. Each $Q_g(z)$ is a rational function of a quadratic parameter of z, but this pattern leaves a polynomial of this parameter unknown. An upper bound for the polynomial degree is conjectured but not proved. The first proof of a more precise pattern, with a maximal degree for each unknown polynomial, is due to D. Arquès and the second author [3, 9], for the more general case of counting by number of vertices and faces. This section presents new results derived from this former work by focusing on counting by number of edges.

3.1 Generating functions counting rooted maps

The following result proves the conjecture in [5] and is an easy consequence of Theorem 1 of [3].

Theorem 3.1. *For any positive integer g, the ordinary generating function $Q_g(z)$ counting rooted maps on a closed orientable surface of genus g by number of edges (exponent of z) can be written*

$$Q_g(z) = z^{2g}(1-3m)^{-2}(1-2m)^{4-5g}(1-6m)^{3-5g} P_g(m),$$

where $m = \dfrac{1-\sqrt{1-12z}}{6}$ *and $P_g(m)$ is a polynomial of m of degree less than or equal to* $6g-6$.

The explicit formulae for polynomials $P_g(m)$, $g = 0, 1, 2, 3$ can be derived from the papers [19], [2], [5] and [9], respectively. The first step to counting unrooted genus–four maps by number of edges is given by the following proposition.

Proposition 3.2. *The polynomial P_g in Theorem 3.1 for $g = 4$ is given by the formula*

$$
\begin{aligned}
P_4(m) = \quad & 9(1-2m)^6 \\
& (41956066368m^{12} - 107657028288m^{11} + 128766120048m^{10} \\
& -95026128096m^9 + 48202134300m^8 - 17709582732m^7 \\
& +4855070265m^6 - 1025233956m^5 + 178608786m^4 \\
& -28633200m^3 + 4245462m^2 - 465894m + 25025).
\end{aligned} \tag{3.1}
$$

Proof. Theorem 5.1, Relation (6.1) and Propositions 6.1 and 6.2 from [9] make it possible to compute P_g from polynomials of lower degree. This result has been computed by a software developed by the second author in his Ph.D. thesis. The software does not directly compute the polynomials but rather the generating functions enumerating rooted maps by genus, along the principles presented in [5] and detailed in [9]. It has not been designed for efficiency but it successfully computes the generating functions for rooted maps up to genus 4. The genus–four formula was computed in less than one minute on an Intel Pentium at 1.4 GHz with 1.25 Gb of memory. In accordance with Theorem 3.1, the polynomial computed for P_4 is indeed of degree $6g-6=18$. □

3.2 Counting rooted genus–four maps by number of edges

Theorem 3.3. *The ordinary generating function $Q_4(z)$ counting rooted maps on a closed orientable surface of genus four by number of edges (the exponent of z) is*

$$
\begin{aligned}
Q_4(z) = \quad & 9z^8(1-3m)^{-2}(1-2m)^{-10}(1-6m)^{-17} \\
& (41956066368m^{12} - 107657028288m^{11} + 128766120048m^{10} \\
& -95026128096m^9 + 48202134300m^8 - 17709582732m^7 \\
& +4855070265m^6 - 1025233956m^5 + 178608786m^4 \\
& -28633200m^3 + 4245462m^2 - 465894m + 25025),
\end{aligned} \tag{3.2}
$$

where $m = \dfrac{1-\sqrt{1-12z}}{6}$.

Proof. Formula (3.2) follows from Theorem 3.1 and Proposition 3.2. □

Table 1 presents the coefficients of $Q_g(z) = \sum_{n\geq 0} N_g(n) z^n$. The first seven coefficients correspond to the ones found by Timothy Walsh in his Ph.D. thesis [21]. The remaining coefficients are new.

4 Unrooted map enumeration

4.1 Counting unrooted maps through rooted maps on orbifolds

The following theorem is the main result of [14].

Theorem 4.1. *The number $U_g(e)$ of unrooted maps with e edges on a closed orientable surface of genus g is given by the formula*

$$U_g(e) = \frac{1}{2e} \sum_{\ell m = 2e} \sum_{O \in Orb(S_g/Z_\ell)} Epi_0(\pi_1(O), Z_\ell)\nu_O(m),$$

where $\nu_O(m)$ is the number of rooted maps with m darts on the orbifold O.

n	The number $N_4(n)$ of rooted maps of genus 4 with n edges
8	225225
9	24635754
10	1495900107
11	66519597474
12	2416610807964
13	75981252764664
14	2141204115631518
15	55352670009315660
16	1334226671709010578
17	30347730709395639732
18	657304672067357799042
19	13652607304062788395788
20	273469313030628783700080
21	5306599156694095573465824
22	100128328831437989131706976
23	1842794650155970906232185656
24	33167202398202989127880734894
25	585079650671639944950451625580
26	10134917623511547808118654370114
27	172678013694177771071548169002188
28	2897912714075648947715005321906392
29	47963145773909943419634526762950192
30	783757995914247522485178250636927380
31	12657015244648210693716700196736399336
32	202177082281879102698899470748726765316
33	3196834110175421253323791465873251739560
34	50072299181065185108291501010224952255668
35	777384663760023780739632793721755383049272
36	11969638731261482998116895312223651253180480
37	182875502596501323216343759769794526714561664
38	2773716775724835345230901154059649970954877396
39	41781661724286164921640221635213792280118832368
40	625310196714095279935937237998816771771464314790
41	9301365625304817339752604766781541863133507845340
42	137556789724353166312824029682866215741796911453698
43	2023172807939725017933640132814869413798020476575564
44	29601998835280343256197223863418277211551813053748872
45	430981509422356688373368386557125320381885703792230800

Table 1: Enumeration of rooted maps of genus 4 by number of edges.

The number of rooted maps on the orbifold $O = O[g; 2^{b_2}, \dots, \ell^{b_\ell}]$ can be expressed through the number $N_g(n)$ of rooted maps on genus g surface by the following proposition given in [14].

Proposition 4.2. *Let* $O = O[g; 2^{b_2}, \dots, \ell^{b_\ell}]$ *be an orbifold,* $b_i \geq 0$ *for* $i = 2, \dots, \ell$. *Let*

$N_g(n)$ be the number of rooted maps of genus g with n edges. Then the number of rooted maps $\nu_O(m)$ with m darts on the orbifold O is

$$\nu_O(m) = \sum_{s=0}^{b_2} \binom{m}{s} \binom{\frac{m-s}{2}+2-2g}{b_2-s, b_3, \dots, b_\ell} N_g((m-s)/2),$$

where $N_g(n) = 0$ if n is not an integer.

Denote by $\mu(n)$, $\phi(n)$ and $\Phi(x,n)$ the Möbius, Euler and von Sterneck functions [1, 16]. The relationship between them is given by the formula

$$\Phi(x,n) = \frac{\phi(n)}{\phi(\frac{n}{(x,n)})}\, \mu\left(\frac{n}{(x,n)}\right),$$

where (x,n) is the greatest common divisor of x and n. It was shown by O. Hölder that $\Phi(x,n)$ coincides with the Ramanujan sum $\sum\limits_{\substack{1\le k\le n\\ (k,\,n)=1}} \exp(\frac{2\,ikx}{n})$. For the proof, see [1, p.164] and [16].

Recall that the Jordan multiplicative function $\phi_k(n)$ of order k can be defined (for more information see [7, p.199], [11, 17]) as

$$\phi_k(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d^k.$$

From [14] we have the following proposition.

Proposition 4.3. *Let $O = O[g; m_1, \dots, m_r]$ be an orbifold of signature $(g; m_1, \dots, m_r)$. Denote by $m = lcm(m_1, \dots, m_r)$ the least common multiple of $m_1, \dots, m_r$ and let ℓ be a multiple of m. Then the number of order-preserving epimorphisms of the orbifold fundamental group $\pi_1(O)$ onto a cyclic group Z_ℓ is given by the formula*

$$Epi_0(\pi_1(O), Z_\ell) = m^{2g}\phi_{2g}(\ell/m)E(m_1, m_2, \dots, m_r),$$

where

$$E(m_1, m_2, \dots, m_r) = \frac{1}{m}\sum_{k=1}^{m} \Phi(k,\, m_1)\cdot\Phi(k,\, m_2)\cdot\ldots\cdot\Phi(k,\, m_r),$$

$\phi_{2g}(\ell)$ is the Jordan multiplicative function of order $2g$, and $\Phi(k,\, m)$ is the von Sterneck function.

4.2 Counting unrooted genus–four maps by number of edges

Theorem 4.4. *The number $U_4(e)$ of unrooted maps on a closed orientable surface of genus four counted by the number of edges e is given by the formula*

$$\frac{1}{2e}\big(N_4(e) + 16\nu_{[2;2^2]}(e) + 4\nu_{[1;2^6]}(e) + \nu_{[0,2^{10}]}(e)$$

$$+80N_2(e/3) + 18\nu_{[1;3^3]}(2e/3) + 22\nu_{[0;3^6]}(2e/3)$$

$$+32\nu_{[1;4^2]}(e/2) + 8\nu_{[0;2,4^4]}(e/2) + 2\nu_{[0;2^4,4^2]}(e/2)$$

$$+52\nu_{[0;5^4]}(2e/5) + 32\nu_{[1;2^2]}(e/3) + 2\nu_{[0;2,6^3]}(e/3)$$

$$+2\nu_{[0;2^2,3^3]}(e/3) + 6\nu_{[0;3^2,6^2]}(e/3) + 2\nu_{[0;2^3,3,6]}(e/3)$$

$$+4\nu_{[0;2^2,8^2]}(e/4) + 18\nu_{[0;9^3]}(2e/9) + 12\nu_{[0;5,10^2]}(e/5)$$

$$+4\nu_{[0;2^2,5^2]}(e/5) + 4\nu_{[0;3,12^2]}(e/6) + 4\nu_{[0;4,6,12]}(e/6)$$

$$+8\nu_{[0;3,5,15]}(2e/15) + 8\nu_{[0;2,16^2]}(e/8) + 6\nu_{[0;2,9,18]}(e/9)\big),$$

where $\nu_O(m)$ is defined in Proposition 4.2 and $N_g(e)$ is the number of rooted maps of genus g with e edges.

Proof. O.V. Bogopol'skii [6] described all possible signatures of orbifolds of the type S_4/G, where S_4 is a surface of genus four and G is a finite group of homeomorphisms acting on S_4. In particular, $G = Z_\ell$ is cyclic of order ℓ only for $\ell = 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16$ and 18.

From this observation and Proposition 4.3 we get the following lemma.

Lemma 4.5. *Let $O \in Orb(S_4/Z_\ell)$. Then one of the following cases occurs:*

(1) $\ell = 1 : O = O[4;\emptyset]$ *with* $Epi_0(\pi_1(O), Z_\ell) = 1$;

(2) $\ell = 2 : O = O[2;2^2], O[1;2^6], O[0;2^{10}]$ *with* $Epi_0(\pi_1(O), Z_\ell) = 16, 4, 1$;

(3) $\ell = 3 : O = O[2;\emptyset], O[1;3^3], O[0;3^6]$ *with* $Epi_0(\pi_1(O), Z_\ell) = 80, 18, 22$;

(4) $\ell = 4 : O = O[1;4^2], O[0;2,4^4], O[0;2^4,4^2]$ *with* $Epi_0(\pi_1(O), Z_\ell) = 32, 8, 2$;

(5) $\ell = 5 : O = O[0;5^4]$ *with* $Epi_0(\pi_1(O), Z_\ell) = 52$;

(6) $\ell = 6 : O = O[1;2^2], O[0;2,6^3], O[0;2^2,3^3], O[0;3^2,6^2], O[0;2^3,3,6]$ *with* $Epi_0(\pi_1(O), Z_\ell) = 32, 2, 2, 6, 2$;

(7) $\ell = 8, 9 : O = O[0;2^2,8^2], O[0;9^3]$ *with* $Epi_0(\pi_1(O), Z_\ell) = 4, 18$;

(8) $\ell = 10 : O = O[0;5,10^2], O[0;2^2,5^2]$ *with* $Epi_0(\pi_1(O), Z_\ell) = 12, 4$;

(9) $\ell = 12 : O = O[0;3,12^2], O[0;4,6,12]$ *with* $Epi_0(\pi_1(O), Z_\ell) = 4, 4$;

(10) $\ell = 15, 16, 18 : O = O[0;3,5,15], O[0;2,16^2], O[0;2,9,18]$ *with* $Epi_0(\pi_1(O), Z_\ell) = 8, 8, 6$.

Now, the theorem follows from Lemma 4.5 and Theorem 4.1. □

We present the numbers thus obtained in Table 2.

n	The number $U_4(n)$ of unrooted maps of genus 4 with n edges
8	14118
9	1369446
10	74803564
11	3023693380
12	100692692173
13	2922359760376
14	76471600288836
15	1845089145736960
16	41694584320696782
17	892580319444417876
18	18258463136626650660
19	359279139700128276168
20	6836732826365623258492
21	126347598971804884131800
22	2275643837092089686415858
23	40060753264325317709454720
24	690983383296198882647616692
25	11701593013434174490416914028
26	194902261990612930685627941344
27	3197740994336653065511697474864
28	51748441322779568341478022803550
29	826950789205344386488852660387184
30	13062633265237461036677963280146184
31	204145407171745343738289312062076704
32	3159016910654361022421358641441865404
33	48436880457203352503593806713722630064
34	736357340898017428826654622692598290184
35	11105495196571768299465860194739273233104
36	166244982378631708320492461115910055656280
37	2471290575628396259735279442756616067154240
38	36496273364800465069054923670966215780880134
39	535662329798540575919393860220780811735355616
40	7816377458926190999203023224069385048268971894
41	113431288113473382192120378978151871847586693068
42	1637580830051823408486063085325625200796356325376
43	23525265208601453696903044755726884736237505268568
44	336386350400912991547696743715303742977503309390766
45	4788683438026185426370763920603749547029983530699104

Table 2: Enumeration of unrooted maps of genus 4 by number of edges.

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