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Dragan Marušič and Tomaž Pisanski

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A novel characterization of cubic Hamiltonian graphs via the associated quartic graphs*

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Abstract

We give a necessary and sufficient condition for a cubic graph to be Hamiltonian by analyzing Eulerian tours in certain spanning subgraphs of the quartic graph associated with the cubic graph by 1-factor contraction. This correspondence is most useful in the case when it induces a blue and red 2-factorization of the associated quartic graph. We use this condition to characterize the Hamiltonian I-graphs, a further generalization of generalized Petersen graphs. The characterization of Hamiltonian I-graphs follows from the fact that one can choose a 1-factor in any I-graph in such a way that the corresponding associated quartic graph is a graph bundle having a cycle graph as base graph and a fiber and the fundamental factorization of graph bundles playing the role of blue and red factorization. The techniques that we develop allow us to represent Cayley multigraphs of degree 4, that are associated to abelian groups, as graph bundles. Moreover, we can find a family of connected cubic (multi)graphs that contains the family of connected I-graphs as a subfamily.

Keywords: Generalized Petersen graphs, I-graphs, Hamiltonian cycles, Eulerian tours, Cayley multigraphs.

Math. Subj. Class.: 05C45, 05C25, 05C15, 05C76, 05C70, 55R10, 05C60

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1 Introduction

A graph is *Hamiltonian* if it contains a spanning cycle (*Hamiltonian cycle*). To find a Hamiltonian cycle in a graph is an NP–complete problem (see [12]). This fact implies that a characterization result for Hamiltonian graphs is hard to find. For this reason, most graph theorists have restricted their attention to particular classes of graphs.

In this paper we consider cubic graphs. In Section 2 we give a necessary and sufficient condition for a cubic graph to be Hamiltonian. Using this condition we can completely characterize the Hamiltonian I-graphs.

The family of I-graphs is a generalization of the family of generalized Petersen graphs. In [5], the generalized Petersen graphs were further generalized to I-graphs. Let n, p, q be positive integers, with $n \geq 3$, $1 \leq p, q \leq n-1$ and $p, q \neq n/2$. An I-graph $I(n,p,q)$ has vertex-set $V(I(n,p,q)) = \{v_i, u_i : 0 \leq i \leq n-1\}$ and edge-set $E(I(n,p,q)) = \{[v_i, v_{i+p}], [v_i, u_i], [u_i, u_{i+q}] : 0 \leq i \leq n-1\}$ (subscripts are read modulo n). The graph $I(n,p,q)$ is isomorphic to the graphs $I(n,q,p)$, $I(n,n-p,q)$ and $I(n,p,n-q)$. It is connected if and only if $\gcd(n,p,q) = 1$ (see [3]).

For $p = 1$ the I-graph $I(n,1,q)$ is known as a generalized Petersen graph and is denoted by $G(n,q)$. The Petersen graph is $G(5,2)$. It has been proved that $I(n,p,q)$ is isomorphic to a generalized Petersen graph if and only if $\gcd(n,p) = 1$ or $\gcd(n,q) = 1$ (see [3]). A connected I-graph which is not a generalized Petersen graph is called a *proper* I-graph. Recently, the class of I-graphs has been generalized to the class of GI-graphs (see [6]).

It is well known that the Petersen graph is not Hamiltonian. A characterization of Hamiltonian generalized Petersen graphs was obtained by Alspach [2].

Theorem 1.1 (Alspach, [2])**.** *A generalized Petersen graph $G(n,q)$ is Hamiltonian if and only if it is not isomorphic to $G(n,2)$ when $n \equiv 5 \pmod 6$.*

In this paper we develop a powerful theory that helps us to extend this result to all I-graphs.

Theorem 1.2. *A connected I-graph is Hamiltonian if and only if it is not isomorphic to $G(n,2)$ when $n \equiv 5 \pmod 6$.*

For the proof of the above main theorem, we developed techniques that are of interest by themselves and are presented in the following sections. In particular, we introduce *good Eulerian graphs* that are similar to *lattice diagrams* that were originally used by Alspach in his proof of Theorem 1.1.

Our theory also involves Cayley multigraphs. In Section 4 we show that Cayley multigraphs of degree 4, that are associated to abelian groups, can be represented as graph bundles [19]. By the results concerning the isomorphisms between Cayley multigraphs (see [7]), we can establish when two graph bundles are isomorphic or not (see Section 4.2). Combining the definition of graph bundles with Theorem 3.3, we can find a family of connected cubic (multi)graphs that contains the family of connected I-graphs as a subfamily (see Section 5).

2 Cubic graph with a 1-factor and the associated quartic graph with transitions

A cubic Hamiltonian graph has a 1-factor. In fact, it has at least three (edge-disjoint) 1-factors. Namely any Hamiltonian cycle is even and thus gives rise to two 1-factors and the remaining chords constitute the third 1-factor. The converse is not true. There are cubic graphs, like the Petersen graph, that have a 1-factor but are not Hamiltonian. Nevertheless, we may restrict our search for Hamiltonian graphs among the cubic graphs to the ones that possess a 1-factor. In this section, we give a necessary and sufficient condition for the existence of a Hamiltonian cycle in a cubic graph G possessing a 1-factor F.

Let G be a connected simple cubic graph and let F be one of its 1-factors. Denote by $X = G/F$ the graph obtained from G by contracting the edges of F. The graph X is connected, quartic, that is, regular of degree 4 and might have multiple edges (X has no loop since G is simple). We say that the quartic graph X is *associated with G and F*. Since X is even and connected, it is Eulerian. A path on three vertices with middle vertex v that is a subgraph of X is called a *transition at v*. Since any pair of edges incident at v defines a transition, there are $\binom{4}{2} = 6$ transitions at each vertex of X. For general graphs each vertex of valence d gives rise to $\binom{d}{2}$ transitions. In an Euler tour some transitions may be used, others are not used. We are interested in some particular Eulerian spanning subgraphs W. Note that any such graph is sub-quartic and the valence at any vertex of W is either 4 or 2. A vertex of valence 4 has therefore 6 transitions, while each vertex of valence 2 has $\binom{2}{2} = 1$ transition. Let Y be the complementary 2-factor of F in G. Note that the edges of Y are in one-to-one correspondence with the edges of X, while the edges of F are in one-to-one correspondence with the vertices of X. If a is an edge of Y, we denote by a' the corresponding edge in X. If e is an edge of F, the corresponding vertex of X will be denoted by x_e. Let u and v be the end-vertices of e and let a and b be the other edges incident with u and similarly c and d the edges incident with v. After contraction of e, the vertex x_e is incident with four edges: a', b', c', d'. By considering the pre-images of the six transitions at x_e, they fall into two disjoint classes. Transitions $a'b'$ and $c'd'$ are *non-traversing* while the other four transitions are *traversing transitions*. In the latter case the edge e has to be used to traverse from one edge of the pre-image transition to the other.

Let W be a spanning Eulerian subgraph of X. Transitions of X carry over to W. The 4-valent vertices of W keep the same six transitions, while each 2-valent vertex inherits a single transition. We say that W is *admissible* if the transition at each 2-valent vertex of W is traversing.

Let W be an admissible Eulerian subgraph of X. A closed walk in W that allows only non-traversing transitions at each 4-valent vertex of W is said to be a *closed walk with allowed transitions*. A closed walk with allowed transitions passing through a 4-valent vertex x_e of W might use both transitions $a'b'$, $c'd'$ or only one of the two non-traversing transitions. If it passes through a 2-valent vertex of W, then it uses traversing transitions. Hence, the underlying graph of a closed walk with allowed transitions might be a cycle. A partition of the edge-set of W into closed walks with allowed transitions is said to be *a tour with allowed transitions*. Each closed walk in the tour is a *component* of the tour.

Lemma 2.1. *Let G be a connected cubic graph with 1-factor F. There is a one-to-one correspondence between 2-factors T of G and admissible Eulerian subgraphs W of $X = G/F$ in such a way that the number of cycles of T is the same as the number of components of a tour with allowed transitions in W.*

Proof. Let T be a 2-factor of G and let $e = uv$ be an edge of the 1-factor F. Let W be the projection of T to $X = G/F$. We will use the notation introduced above. Hence the edge e and its end-vertices u and v project to the same vertex x_e of X. There are two cases:

Case 1: e belongs to T. In this case exactly one other edge, say a, incident with u and another edge, say c, incident with v belong to T. The other two edges (b and d) do not belong to T. This means that x_e is a 2-valent vertex with traversing transition.

Case 2: e does not belong to T. In this case both edges a and b incident with u belong to T and both edges c and d incident with v belong to T. In this case x_e is a 4-valent vertex with non-traversing transitions.

Clearly, W is an admissible Eulerian subgraph. Each component of the tour determined by W with transitions gives back a cycle of T. The correspondence between T and W is therefore established. □

An Eulerian tour in W with allowed transitions is said to be *good*. An admissible subgraph W of X possessing a good Eulerian tour is said to be a *good Eulerian subgraph*. In a good Eulerian subgraph W there are two extreme cases:

1. each vertex of W is 4-valent: this means that $W = X$; in this case the complementary 2-factor $Y = G - F$ is a Hamiltonian cycle and no edge of F is used;
2. each vertex of W is 2-valent: this means that W is a good Hamiltonian cycle in X. In this case F together with the pre-images of edges of W in G form a Hamiltonian cycle.

Theorem 2.2. *Let G be a connected cubic graph with 1-factor F. Then G is Hamiltonian if and only if $X = G/F$ contains a good Eulerian subgraph W.*

Proof. Clearly G is Hamiltonian if and only if it contains a 2-factor with a single cycle. By Lemma 2.1, this is true if and only if W is an admissible Eulerian subgraph possessing an Eulerian tour with allowed transitions. But this means W is good. □

Corollary 2.3. *Let G be a connected cubic graph with 1-factor F. Finding a good Eulerian subgraph W of $X = G/F$ is NP-complete.*

Proof. Since finding a good Eulerian subgraph is equivalent to finding a Hamiltonian cycle in a cubic graph, and the latter is NP-complete [12], the result follows readily. □

Also in [11] Eulerian graphs are used to find a Hamiltonian cycle (and other graph properties), but our method is different.

The results of this section may be applied to connected I-graphs. The obvious 1-factor F of an I-graph $I(n, p, q)$ consists of spokes. Let $Q(n, p, q)$ denote the quotient $I(n, p, q)/F$. We will call $Q(n, p, q)$ the *quartic graph associated* with $I(n, p, q)$.

Corollary 2.4. *Let $I(n, p, q)$ be a connected I-graph and let $Q(n, p, q)$ be its associated quartic graph. Then $I(n, p, q)$ is Hamiltonian if and only if $Q(n, p, q)$ contains a good Eulerian subgraph W.*

3 Special 1-factors and their applications

Let G be a cubic graph, F a 1-factor and Y the complementary 2-factor of F in G. Define an auxiliary graph $Y(G, F)$ having cycles of Y as vertices and having two vertices adjacent if and only if the corresponding cycles of Y are joined by one or more edges of F. If an edge of F is a chord in one of the cycles of Y, then the graph $Y(G, F)$ has a loop. We shall say that the 1-factor F is *special* if the graph $Y(G, F)$ is bipartite. A cubic graph with a special 1-factor will be called *special*. If F is a special 1-factor of G, then the edges of F join vertices belonging to distinct cycles of Y since $Y(G, F)$ is loopless.

Theorem 3.1. *Let G be a connected cubic graph with a 1-factor, and let F be one of its 1-factors and $X = G/F$ its associated quartic graph. Then X admits a 2-factor whose edges may be colored blue or red in such a way that the traversing transitions are exactly color-switching and non-traversing transitions are color-preserving if and only if G and F are special.*

Proof. Assume that F is a special 1-factor of G. Since $Y(G, F)$ is bipartite, we can bicolor the vertices of $Y(G, F)$: let one set of the bipartition be blue and the other red. This coloring induces a coloring on the edges of Y: for every blue vertex (respectively, red vertex) of $Y(G, F)$ we color in blue (respectively, in red) the edges of the corresponding cycle of Y. Since the edges of Y are in one-to-one correspondence with the edges of X, we obtain a 2-factorization of X into a blue 2-factor and red 2-factor. Since F is special, the edges of F are incident with vertices of G belonging to cycles of Y with different colors (a blue cycle and a red cycle). Therefore, a traversing transition is color-switching and a non-traversing transition is color-preserving.

Conversely, assume that X has a blue and red 2-factorization such that the traversing transitions are color-switching and non-traversing transitions are color-preserving. Since the edges of X are in one-to-one correspondence with the edges of Y, we can partition the cycles of Y into red cycles and blue cycles. Since the traversing transitions are color-switching and non-traversing transitions are color-preserving, the edges of F are incident with edges belonging to cycles of different colors. This means that the graph $Y(G, F)$ is bipartite, hence F is special. □

Proposition 3.2. *Let G and F be special and let W be any Eulerian subgraph of $X = G/F$ the associated quartic graph with a blue and red 2-factorization. Then W is admissible if and only if each 2-valent vertex is incident with edges of different colors.*

Proof. An Eulerian subgraph W is admissible if and only if each 2-valent vertex v in W is incident with edges forming a traversing transition at v. By Theorem 3.1, a traversing transition is color-switching. Hence, W is admissible if and only if the edges incident with v have different colors. □

Note that quartic graphs with a given 2-factorization can be put into one-to-one correspondence with special cubic graphs.

Theorem 3.3. *Any special cubic graph G with a special 1-factor F gives rise to the associated quartic graph with a blue and red 2-factorization. However, any quartic graph with a given 2-factorization determines back a unique special cubic graph by color-preserving splitting vertices and inserting a special 1-factor.*

Proof. By Theorem 3.1, a special cubic graph G with a special 1-factor F gives rise to the graph $X = G/F$ admitting a blue and red 2-factorization.

Conversely, it is well known that every quartic graph X possesses a 2-factorization, that is, the edges of X can be partitioned into a blue and red 2-factor. We use the blue and red 2-factors of X to construct a cubic graph G as follows: put in G a copy of the blue 2-factor and a copy of the red 2-factor; construct a 1-factor of G by joining vertices belonging to distinct copies. It is straightforward to see that G and F are special. □

We will now apply this theory to the I-graphs. In Section 7 we will see that this theory allows us to find a Hamiltonian cycle in a proper I-graph and also to find a family of special cubic graphs that contains the family of I-graphs as a subfamily (see Section 5).

Let $I(n,p,q)$ be an I-graph. A vertex v_i (respectively, u_i) is called an *outer vertex* (respectively, an *inner vertex*). An edge of type $[v_i, v_{i+p}]$ (respectively, of type $[u_i, u_{i+q}]$) is called an *outer edge* (respectively, an *inner edge*). An edge $[v_i, u_i]$ is called a *spoke*. The spokes of $I(n,p,q)$ determine a 1-factor of $I(n,p,q)$. The set of outer edges is called the *outer rim*, the set of inner edges is called the *inner rim*. As a consequence of the results proved in [3], the following holds.

Proposition 3.4. *Let $I(n,p,q)$, $n \geq 3$, $1 \leq p,q \leq n-1$, $p,q \neq n/2$, be an I-graph. Set $t = \gcd(n,q)$ and $s = n/t$. Then $t < n/2$ and $3 \leq s \leq n$. Moreover, $I(n,p,q)$ is connected if an only if $\gcd(t,p) = 1$ and $\gcd(s,p)$ is coprime with q. It is proper if and only if t and $\gcd(s,p)$ are different from 1.*

Proof. The integer t satisfies the inequality $t < n/2$, since t is a divisor of q and $q \leq n-1$, $q \neq n/2$; whence $3 \leq s \leq n$. By the results proved in [3], $I(n,p,q)$ is connected if and only if $\gcd(n,p,q) = 1$. Since $n = st$ and $q = t(q/t)$, the relation $\gcd(n,p,q) = 1$ can be written as $\gcd(st,p,t(q/t)) = 1$, whence $\gcd(t,p) = 1$ and $\gcd(s,p)$ is coprime with q. Also the converse is true, and therefore $I(n,p,q)$ is connected if and only if $\gcd(t,p) = 1$ and $\gcd(s,p)$ is coprime with q. A connected I-graph $I(n,p,q)$ is a generalized Petersen graph if and only if $\gcd(n,q) = 1$ or $\gcd(n,p) = 1$ (see [3]). By the previous results, $I(n,p,q)$ is a generalized Petersen graph if and only if $1 = \gcd(n,q) = t$ or $1 = \gcd(n,p) = \gcd(st,p) = \gcd(s,p)$. The assertion follows. □

The smallest proper I-graphs are $I(12,2,3)$ and $I(12,4,3)$. It is straightforward to see that the following result holds.

Lemma 3.5. *Let F be the 1-factor determined by the spokes of $I(n,p,q)$ and $X = Q(n,p,q)$ its associated quartic graph. Then F is special, the graph X is a circulant multigraph $Cir(n;p,q)$, the blue edges of X correspond to the inner rim and the red edges to the outer rim of $I(n,p,q)$.*

In the next section we introduce a class of graphs $X(s,t,r)$ and later show that it contains $Q(n,p,q)$ as its subclass.

4 Graphs $X(s,t,r)$

Let Γ be a group in additive notation with identity element 0_Γ. Let S be a list of not necessarily distinct elements of Γ satisfying the symmetry property $S = -S = \{-\gamma : \gamma \in S\}$. The Cayley multigraph associated with Γ and S, denoted by $Cay(\Gamma, S)$, is an undirected multigraph having the elements of Γ as vertices and edges of the form $[x, x+\gamma]$ with

$x \in \Gamma$, $\gamma \in S$. If Γ is a cyclic group of order n, then $Cay(\Gamma, S)$ is a circulant multigraph of order n. A Cayley multigraph $Cay(\Gamma, S)$ is regular of degree $|S|$ (in determining $|S|$, each element of S is considered according to its multiplicity in S). It is connected if and only if S is a set of generators of Γ. If the elements of S are pairwise distinct, then $Cay(\Gamma, S)$ is a simple graph and we will use the term Cayley graph. We are interested in connected Cayley multigraphs of degree 4. In this case we write S as the list $S = \{\pm\gamma_1, \pm\gamma_2\}$. A circulant multigraph of order n will be denoted by $Cir(n; \pm\gamma_1, \pm\gamma_2)$. If γ_i, with $i \in \{1, 2\}$, is an involution of Γ or the trivial element, then $\pm\gamma_i$ means that the element γ_i appears twice in the list S. Consequently, the associated Cayley multigraph has multiple edges or loops. We will denote by $o(\gamma_i)$ the order of γ_i. We will show that the Cayley multigraphs $Cay(\Gamma, \{\pm\gamma_1, \pm\gamma_2\})$ defined on a suitable abelian group Γ (and in particular the circulant multigraphs $Cir(n; \pm\gamma_1, \pm\gamma_2)$) can be given a different interpretation in terms of $X(s, t, r)$ graphs (see Figure 1) defined as follows.

Definition 4.1. Let $s, t \geq 1$, $0 \leq r \leq s - 1$ be integers. Let $X(s, t, r)$ be the graph with vertex-set $\{x^i_j : 0 \leq i \leq t-1, 0 \leq j \leq s-1\}$ and edge-set $\{[x^i_j, x^i_{j+1}] : 0 \leq i \leq t-1, 0 \leq j \leq s-1\} \cup \{[x^i_j, x^{i+1}_j] : 0 \leq i \leq t-2, 0 \leq j \leq s-1\} \cup \{[x^{t-1}_j, x^0_{j+r}] : 0 \leq j \leq s-1\}$ (the superscripts are read modulo t, the subscripts are read modulo s).

The graph $X(s, t, r)$ has edges of type $[x^i_j, x^i_{j+1}]$, $[x^i_j, x^{i+1}_j]$ or $[x^{t-1}_j, x^0_{j+r}]$. An edge of type $[x^i_j, x^i_{j+1}]$ will be called *horizontal*. An edge of type $[x^i_j, x^{i+1}_j]$ will be called *vertical*, an edge of type $[x^{t-1}_j, x^0_{j+r}]$ will be called *diagonal*. For $t = 1$, we say that the edges are horizontal and diagonal (a diagonal edge is an edge of type $[x^0_j, x^0_{j+r}]$). For $s = 1$, the horizontal edges are loops. For $(t, r) = (1, 0)$, the diagonal edges are loops. For $s = 2$ or $s > 2$ and (t, r)=$(1, 1)$, $(1, s/2)$, $(2, 0)$ the graph has multiple edges. For the other values of s, t, r, the graph $X(s, t, r)$ is a simple graph. A simple graph $X(s, t, r)$ is a graph bundle with a cycle fiber C_s over a cycle base C_t; the parameter r represents an automorphism of the cycle C_s that shifts the cycle r steps (see [19] for more details on graph bundles). In the literature a simple graph $X(s, t, r)$ is also called r-pseudo-cartesian product of two cycles (see for instance [10]). The definition of $X(s, t, r)$ suggests that the graph $X(s, t, r)$ is isomorphic to $X(s, t, s - r)$. The existence of this isomorphism can be also obtained from the following statement.

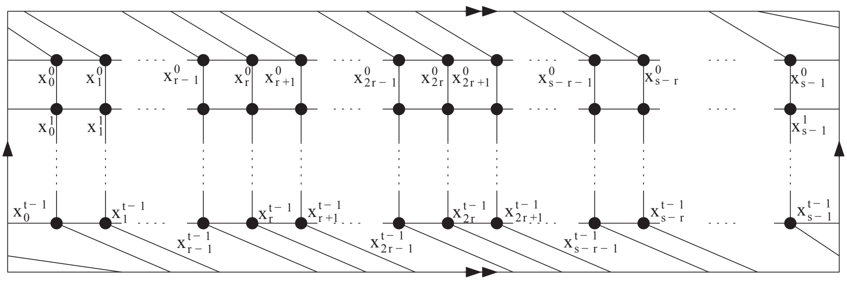


Figure 1: The graph $X(s, t, r)$ is embedded into torus with quadrilateral faces; it has a blue and red 2-factorization: the vertical and diagonal edges form the blue 2-factor, the horizontal edges form the red 2-factor.

Proposition 4.2. *Let $Cay(\Gamma, \{\pm\gamma_1, \pm\gamma_2\})$ be a connected Cayley multigraph of degree 4, where Γ is an abelian group, $o(\gamma_1) = s$ and $|\Gamma|/s = t$. Then $a\gamma_2 = r\gamma_1$ for some integer r, $0 \leq r \leq s-1$, if and only if $a = t$. Consequently, $Cay(\Gamma, \{\pm\gamma_1, \pm\gamma_2\})$ can be represented as the graph $X(s, t, r)$ or $X(s, t, s-r)$.*

Proof. We show that $G_1 = Cay(\Gamma, \{\pm\gamma_1, \pm\gamma_2\})$ and $G_2 = X(s, t, r)$ are isomorphic. Since γ_1 and γ_2 are generators of Γ, the elements of Γ can be written in the form $i\gamma_2 + j\gamma_1$, where $i\gamma_2 \in \langle\gamma_2\rangle$, $j\gamma_1 \in \langle\gamma_1\rangle$. Hence we can describe the elements of Γ by the left cosets of the subgroup $\langle\gamma_1\rangle$ in Γ. By this representation, we can see that the endvertices of an edge $[x, x \pm \gamma_1]$ of $Cay(\Gamma, \{\pm\gamma_1, \pm\gamma_2\})$ belong to the same left coset of $\langle\gamma_1\rangle$ in Γ; the endvertices of an edge $[x, x \pm \gamma_2]$ belong to distinct left cosets of $\langle\gamma_1\rangle$ in Γ. Therefore, $a\gamma_2 = r\gamma_1 \in \langle\gamma_1\rangle$ if and only if $a = t$, since $Cay(\Gamma, \{\pm\gamma_1, \pm\gamma_2\})$ is connected and $|\Gamma/\langle\gamma_1\rangle| = t$. Hence we can set $V(G_1) = \{i\gamma_2 + j\gamma_1 : 0 \leq i \leq t-1, 0 \leq j \leq s-1\}$ and $E(G_1) = \{[i\gamma_2 + j\gamma_1, (i+1)\gamma_2 + j\gamma_1], [i\gamma_2 + j\gamma_1, i\gamma_2 + (j+1)\gamma_1] : 0 \leq i \leq t-1, 0 \leq j \leq s-1\}$. The map $\varphi : V(G_1) \to V(G_2)$ defined by $\varphi(i\gamma_2 + j\gamma_1) = x^i_j$ is a bijection between $V(G_1)$ and $V(G_2)$. Moreover, if v_1, v_2 are adjacent vertices of G_1, that is, $v_1 = i\gamma_2 + j\gamma_1$ and $v_2 = (i+1)\gamma_2 + j\gamma_1$ (or $v_2 = i\gamma_2 + (j+1)\gamma_1$), then $\varphi(v_1) = x^i_j$, $\varphi(v_2) = x^{i+1}_j$ (or $\varphi(v_2) = x^i_{j+1}$) are adjacent vertices of G_2. In particular, if $v_1 = (t-1)\gamma_2 + j\gamma_1$ and $v_2 = t\gamma_2 + j\gamma_1 = r\gamma_1 + j\gamma_1 = (r+j)\gamma_1$, then $\varphi(v_1) = x^{t-1}_j$, $\varphi(v_2) = x^0_{r+j}$ are adjacent vertices of G_2. It is thus proved that φ is an isomorphism between G_1 and G_2. If we replace the element γ_1 by its inverse $-\gamma_1$, then G_1 is the graph $X(s, t, s-r)$. □

In what follows, we show that for $s, t \geq 1$ there exists a Cayley multigraph on a suitable abelian group that satisfies Proposition 4.2, that is, for every $s, t \geq 1$ the graph $X(s, t, r)$ can be represented as a Cayley multigraph. The proof is particularly easy when $t = 1$; $r = 0$; or $s = 2$. For these cases, the following holds.

Proposition 4.3. *The graph $X(s, 1, r)$, with $s \geq 1$, $0 \leq r \leq s-1$, is the circulant multigraph $Cir(s; \pm1, \pm r)$. The graph $X(s, t, 0)$, with $s, t \geq 1$, is the Cayley multigraph $Cay(\mathbb{Z}_s \times \mathbb{Z}_t, \{\pm(1,0), \pm(0,1)\})$. The graph $X(2, t, 1)$, with $t \geq 1$, is the circulant multigraph $Cir(2t; \pm t, \pm 1)$.*

Proof. For the graph $X(s, 1, r)$ we apply Proposition 4.2 with $\Gamma = \mathbb{Z}_{st}$, $\gamma_1 = 1$, $\gamma_2 = r$. For the graph $X(s, t, 0)$ we apply Proposition 4.2 with $\Gamma = \mathbb{Z}_s \times \mathbb{Z}_t$, $\gamma_1 = (1, 0)$, $\gamma_2 = (0, 1)$. For the graph $X(2, t, 1)$ we apply Proposition 4.2 with $\Gamma = \mathbb{Z}_{2t}$, $\gamma_1 = t$, $\gamma_2 = 1$. □

The following lemmas concern the graph $X(s, t, r)$ with $s \geq 3$, $t \geq 2$ and $0 < r \leq s-1$. They will be used in the proof of Proposition 4.6.

Lemma 4.4. *Let $a > 1$ be an integer and let $b \geq 1$ be a divisor of a. Let $\{[c]_b : 0 \leq c \leq b-1\}$ be the residue classes modulo b. Every equivalence class $[c]_b$ whose representative c is coprime with b contains at least one integer h, $1 \leq h \leq a-1$, such that* $\gcd(a, h) = 1$.

Proof. The assertion is true if $b = a$ (we set $h = c$). We consider $b < a$. Let $[c]_b$ be an equivalence class modulo b with $1 \leq c \leq b-1$ and $\gcd(c, b) = 1$. If c is coprime with a, then we set $h = c$ and the assertion follows. We consider the case $\gcd(c, a) \neq 1$. We denote by $\mathcal{P}$ the set of distinct prime numbers dividing a. We denote by $\mathcal{P}_b$ (respectively, by $\mathcal{P}_c$) the subset of $\mathcal{P}$ containing the prime numbers dividing b (respectively, c). Since b is a divisor of a (respectively, $\gcd(c, a) \neq 1$), the set $\mathcal{P}_b$ (respectively, $\mathcal{P}_c$) is non-empty.

Since c and b are coprime, the subsets $\mathcal{P}_b$, $\mathcal{P}_c$ are disjoint. We set $\mathcal{P}' = \mathcal{P} \setminus (\mathcal{P}_b \cup \mathcal{P}_c)$. The set $\mathcal{P}'$ might be empty. We denote by ω the product of the prime numbers in $\mathcal{P}'$ (if $\mathcal{P}'$ is empty, then we set $\omega = 1$) and consider the integer $h = c + \omega b \in [c]_b$. We show that $h < a$. Note that $\omega \leq a/(2b)$. More specifically, $(a/b) \geq (\prod_{p \in \mathcal{P}_c} p) \cdot \omega \geq 2\omega$, whence $\omega \leq a/(2b)$. Hence $h = c + \omega b \leq c + (a/2) < a$, since $c < b$ and $b \leq (a/2)$. One can easily verify that $\gcd(h, a) = \gcd(c + \omega b, a) = 1$, since no prime number in $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}_b \cup \mathcal{P}_c$ can divide $c + \omega b$. The assertion follows. □

Lemma 4.5. *Let $s \geq 3$, $t \geq 2$ and $\mathbb{Z}_{st/d_1}$ be the cyclic group of integers modulo st/d_1, where $d_1 \geq 1$ is a divisor of $d = \gcd(s, t)$. Let $\langle t/d_1 \rangle$ be the cyclic subgroup of $\mathbb{Z}_{st/d_1}$ generated by the integer t/d_1 and let $x + \langle t/d_1 \rangle$, $y + \langle t/d_1 \rangle$ be left cosets of $\langle t/d_1 \rangle$ in $\mathbb{Z}_{st/d_1}$. If $x, y \in \mathbb{Z}_{st/d_1}$ are congruent modulo d/d_1, then $t(x + \langle t/d_1 \rangle) = \{tx + \mu t^2/d_1 : 0 \leq \mu \leq s-1\}$ and $t(y + \langle t/d_1 \rangle) = \{ty + \mu' t^2/d_1 : 0 \leq \mu' \leq s-1\}$ represent the same subset of $\mathbb{Z}_{st/d_1}$.*

Proof. Set $x = y + \lambda d/d_1$ with $\lambda \in \mathbb{Z}$ and $t = sm' + m$ with $m' \in \mathbb{Z}$ and $0 \leq m \leq s-1$. Since $\gcd(s, t) = d$, then also $\gcd(s, m) = d$. Hence the integers $dt/d_1, mt/d_1 \in \mathbb{Z}_{st/d_1}$ generate the same cyclic subgroup of $\langle t/d_1 \rangle$ of order s/d. Since $t^2/d_1 = (sm'+m)t/d_1 \equiv mt/d_1 \pmod{st/d_1}$, each set $t(x + \langle t/d_1 \rangle)$, $t(y + \langle t/d_1 \rangle)$ consists of exactly s/d distinct elements of $\mathbb{Z}_{st/d_1}$, namely, $t(x + \langle t/d_1 \rangle) = \{tx + \mu mt/d_1 : 1 \leq \mu \leq s/d\}$, $t(y + \langle t/d_1 \rangle) = \{ty + \mu' mt/d_1 : 1 \leq \mu' \leq s/d\}$. Therefore, to prove that $t(x + \langle t/d_1 \rangle) = t(y + \langle t/d_1 \rangle)$, it suffices to show that every element of $t(x + \langle t/d_1 \rangle)$ is contained in $t(y + \langle t/d_1 \rangle)$. Consider $tx + \mu mt/d_1 \in t(x + \langle t/d_1 \rangle)$. Since $x = y + \lambda d/d_1$, we can write $tx + \mu mt/d_1 = t(y + \lambda d/d_1) + \mu mt/d_1$, whence $tx + \mu mt/d_1 = ty + \lambda dt/d_1 + \mu mt/d_1$. Since $\langle dt/d_1 \rangle = \langle mt/d_1 \rangle$, we can set $\lambda dt/d_1 + \mu mt/d_1 \equiv \mu' mt/d_1 \pmod{st/d_1}$, with $0 \leq \mu' \leq s/d$. Hence $tx + \mu mt/d_1 \equiv ty + \mu' mt/d_1 \pmod{st/d_1}$, that is, $tx + \mu mt/d_1 \in t(y + \langle t/d_1 \rangle)$. The assertion follows. □

Proposition 4.6. *Let $s \geq 3$, $t \geq 2$, $0 < r \leq s-1$, with $\gcd(s, t, r) = d_1$. The cyclic group $\mathbb{Z}_{st/d_1}$ of integers modulo st/d_1 contains an integer k such that $\gcd(k, t) = 1$ and $k \equiv r/d_1 \pmod{s/d_1}$. The graph $X(s, t, r)$ can be represented as the Cayley graph $Cay(\mathbb{Z}_{st/d1} \times \mathbb{Z}_{d_1}, \{\pm(t/d_1, 1), \pm(k, 0)\})$. In particular, if $d_1 = 1$ then $X(s, t, r)$ can be represented as the circulant graph $Cir(st; \pm t, \pm k)$.*

Proof. Set $d = \gcd(s, t)$. Since $\gcd(s, t, r) = d_1$, the integer r/d_1 is coprime with $\gcd(s/d_1, t/d_1) = d/d_1$. Hence, r/d_1 belongs to an equivalence class $[c]_{d/d_1}$ whose representative is coprime with d/d_1. By Lemma 4.4, the class $[c]_{d/d_1}$ contains an integer h, $1 \leq h < t$, such that $\gcd(h, t) = 1$. Consider the cyclic group $\mathbb{Z}_{st/d_1}$. Since $r/d_1 < s$, $h < t$, the integer r/d_1 and h belong to $\mathbb{Z}_{st/d_1}$. Hence we can apply Lemma 4.5 with $x = r/d_1$, $y = h$ and find that $t(r/d_1 + \langle t/d_1 \rangle) = t(h + \langle t/d_1 \rangle)$, that is, there exists an integer $k \in h + \langle t/d_1 \rangle$ such that $tk \equiv rt/d_1 \pmod{st/d_1}$. The integer k is coprime with t, since $\gcd(h, t) = 1$. The assertion follows from Proposition 4.2, by setting $\Gamma = \mathbb{Z}_{st/d_1} \times \mathbb{Z}_{d_1}$, $\gamma_1 = (t/d_1, 1)$, $\gamma_2 = (k, 0)$. Note: if $d_1 = 1$, then Γ is the cyclic group of order st, $\gamma_1 = t$, $\gamma_2 = k$ and $X(s, t, r)$ is the circulant graph $Cir(st; \pm t, \pm k)$. □

The result that follows is based on a well-known consequence of the Chinese Remainder Theorem. More specifically, it is known that if a, b, and n are positive integers, with $\gcd(a, n) = c \geq 1$, then the equation $ax \equiv b \pmod{n}$ admits a solution if and only if c is a divisor of b and in this case $x \equiv (a/c)^{-1}(b/c) \pmod{n/c}$ is a solution to the equation. The following holds.

Proposition 4.7. *Let $s, t \geq 1$, $0 \leq r \leq s-1$ and $d_1 = \gcd(s,t,r)$. If $r \neq 0$, then there exists an integer k, $0 < k < st/d_1$, such that $\gcd(k,t) = 1$ and $k \equiv r/d_1 \pmod{s/d_1}$. The graph $X(s,t,r)$, with $r \neq 0$, is isomorphic to the graph $X(st/\gcd(s,r), \gcd(s,r), r')$, where $r' \equiv \pm t(kd_1/\gcd(s,r))^{-1} \pmod{st/\gcd(s,r)}$. The graph $X(s,t,0)$ is isomorphic to the graph $X(t,s,0)$.*

Proof. We prove the assertion for $s \geq 3$, $t \geq 2$ and $0 < r \leq s-1$. The existence of the integer k follows from Proposition 4.6. By the same proposition, we can represent the graph $X(s,t,r)$ as $Cay(\mathbb{Z}_{st/d1} \times \mathbb{Z}_{d_1}, \{\pm(t/d_1, 1), \pm(k, 0)\})$. We apply Proposition 4.2 by setting $\Gamma = \mathbb{Z}_{st/d1} \times \mathbb{Z}_{d_1}$, $\gamma_1 = (k,0)$ and $\gamma_2 = (t/d_1, 1)$. Note that $\gcd(st/d_1, k) = \gcd(s/d_1, k) = \gcd(s,r)/d_1$, as k is coprime with t and $k \equiv r/d_1 \pmod{s/d_1}$. Whence the element $(k,0)$ has order

$$s' = st/(d_1 \gcd(st/d_1, k)) = st/\gcd(s,r) \text{ and } t' = |\Gamma/\langle(k,0)\rangle| = \gcd(s,r).$$

By Proposition 4.2, $\gcd(s,r)(t/d_1, 1) = r'(k,0)$ for some integer r', $1 \leq r' \leq st/\gcd(s, r)$. The integer r' is a solution to the equation $\gcd(s,r)(t/d_1) \equiv r'k \pmod{st/d_1}$. By the Chinese Remainder Theorem, $r' \equiv t(kd_1/\gcd(s,r))^{-1} \pmod{st/\gcd(s,r)}$. An easy calculation shows that $s' - r' \equiv -t(kd_1/\gcd(s,r))^{-1} \pmod{st/\gcd(s,r)}$. It is straightforward to see that $X(s,t,r)$ and $X(s',t',r')$, $X(s',t',s'-r')$ are isomorphic. Hence the assertion follows. For the remaining values of s, t, r, we represent the graph $X(s,t,r)$ as the Cayley multigraph in Proposition 4.3 and use Proposition 4.2. Note: if $r \neq 0$, then $k = r$; if $r = 0$, then set $\gamma_1 = (0,1)$, $\gamma_2 = (1,0)$ and apply Proposition 4.2. □

4.1 Fundamental 2-factorization of $X(s,t,r)$

From the definition of $X(s,t,r)$ one can see that the horizontal edges form a 2-factor (the *red 2-factor*) whose complementary 2-factor in $X(s,t,r)$ is given by the vertical and diagonal edges (the *blue 2-factor*). We say that the red and blue 2-factor constitute the *fundamental 2-factorization* of $X(s,t,r)$. A graph $X(s,t,r)$ can be represented as a Cayley multigraph $Cay(\Gamma, \{\pm\gamma_1, \pm\gamma_2\})$, where Γ and $\{\pm\gamma_1, \pm\gamma_2\}$ can be defined as in Proposition 4.3 or 4.6. From the proof of the propositions, one can see that the set of horizontal edges of $X(s,t,r)$ is the set $\{[x, x \pm \gamma_1] : x \in \Gamma\}$, the set of vertical and diagonal edges is the set $\{[x, x \pm \gamma_2] : x \in \Gamma\}$. The edges in $\{[x, x \pm \gamma_1] : x \in \Gamma\}$ will be called the *γ_1-edges* and the edges in the set $\{[x, x \pm \gamma_2] : x \in \Gamma\}$ will be called the *γ_2-edges*. The following result holds.

Proposition 4.8. *The red 2-factor of $X(s,t,r)$ has exactly t cycles of length s consisting of γ_1-edges. The blue 2-factor of $X(s,t,r)$ has exactly $\gcd(s,r)$ cycles of length $st/\gcd(s,r)$ consisting of γ_2-edges.*

Proof. It is straightforward to see that the red 2-factor has t horizontal cycles of length s (if $s = 1$, then each cycle is a loop; if $s = 2$, then each cycle is a dipole with 2 parallel edges). By the previous remarks, each cycle consists of γ_1-edges. The blue 2-factor of $X(s,t,r)$ corresponds to the red 2-factor of the graph $X(st/\gcd(s,r), \gcd(s,r), r')$ in Proposition 4.7. Hence it has $\gcd(s,r)$ cycles of length $st/\gcd(s,r)$ consisting of γ_2-edges. □

4.2 Isomorphisms between $X(s,t,r)$ graphs

We now ask when two graphs $X(s,t,r)$ and $X(s',t',r')$ are isomorphic. Our question is connected to the following well-known problem [7, 14]. Given two isomorphic Cayley multigraphs $Cay(\Gamma, S)$, $Cay(\Gamma', S')$ or, equivalently, given two Cayley representations (Γ, S), (Γ', S') of the same multigraph, determine whether (Γ, S) and (Γ', S') are equivalent. We recall that two Cayley representations of the same multigraph are said to be equivalent if there exists a permutation on the vertex-set of the multigraph that induces an isomorphism from the group Γ to the group Γ' and sends S onto S'. Two Cayley representations (Γ, S), (Γ, S') are equivalent if and only if there exists an automorphism σ of the group Γ that sends S onto S' (see [14]). The automorphism σ is called a *CI-isomorphism* (CI stands for Cayley Isomorphism). Ádám [1] considered this problem for circulant graphs and formulated a well-known conjecture which was disproved in [9]. He conjectured that two circulant graphs $Cir(n; S)$, $Cir(n; S')$ are isomorphic if and only if there exists an integer $m' \in \mathbb{Z}_n$, $\gcd(m', n) = 1$, such that $S' = \{m'x : x \in S\}$. Even though the conjecture was disproved, there are some circulant graphs for which it holds (see for instance [16]). In [7] the problem is studied for Cayley multigraphs of degree 4 which are associated to abelian groups. The results in [7] are described in terms of Ádám isomorphisms. An *Ádám isomorphism* from $Cay(\Gamma, S)$ to $Cay(\Gamma', S')$ is an isomorphism obtained from a permutation on the vertex-set of $Cay(\Gamma, S)$, that makes (Γ, S), (Γ', S') equivalent, and an automorphism of the graph $Cay(\Gamma', S')$. By the definition of equivalent Cayley representations, the existence of an Ádám isomorphism means that the groups Γ, Γ' are isomorphic and there exists an isomorphism between the groups that sends S onto S'. An Ádám isomorphism between $Cay(\Gamma, S)$ and $Cay(\Gamma, S')$ is a CI-isomorphism. Since the graphs $X(s, t, r)$ can be represented as Cayley multigraphs, we can extend the notion of Ádám isomorphism to the graphs $X(s, t, r)$. We will say that the graphs $X(s, t, r)$, $X(s', t', r')$ are *Ádám isomorphic* if the corresponding Cayley multigraphs $Cay(\Gamma, \{\pm\gamma_1, \pm\gamma_2\})$, $Cay(\Gamma', \{\pm\gamma_1', \pm\gamma_2'\})$, respectively, are Ádám isomorphic ($Cay(\Gamma, \{\pm\gamma_1, \pm\gamma_2\})$, $Cay(\Gamma', \{\pm\gamma_1', \pm\gamma_2'\})$ are described in Proposition 4.3 or 4.6). The following statements hold.

Proposition 4.9. *Every Ádám isomorphism between the graphs $X(s, t, r)$, $X(s', t', r')$ sends the fundamental 2-factorization of $X(s, t, r)$ onto the fundamental 2-factorization of $X(s', t', r')$.*

Proof. An Ádám isomorphism between the graphs $Cay(\Gamma, \{\pm\gamma_1, \pm\gamma_2\})$, $Cay(\Gamma', \{\pm\gamma_1', \pm\gamma_2'\})$ sends a γ_i-edge, $i = 1, 2$, of $Cay(\Gamma, \{\pm\gamma_1, \pm\gamma_2\})$ onto a $\tau(\gamma_i)$-edge of $Cay(\Gamma', \{\pm\gamma_1', \pm\gamma_2'\})$, where $\tau(\gamma_i) \in \{\pm\gamma_1', \pm\gamma_2'\}$. Since Proposition 4.8 holds, every Ádám isomorphism sends the red (respectively, the blue) 2-factor of $X(s, t, r)$ onto the red (respectively, the blue) 2-factor of $X(s', t', r')$ or vice versa. □

Proposition 4.10. *Let $s, t \geq 1$, $0 \leq r \leq s - 1$ and $\gcd(s, t, r) = d_1$. If $r \neq 0$, then there exists an integer k, $0 < k < st/d_1$, such that $\gcd(k, t) = 1$ and $k \equiv r/d_1 \pmod{s/d_1}$. The graphs $X(s, t, r)$, with $r \neq 0$, and $X(s', t', r')$ are Ádám isomorphic if and only if $s' = s$, $t' = t$, $r' = s - r$ or $s' = st/\gcd(s, r)$, $t' = \gcd(s, r)$ and $r' \equiv \pm t(kd_1/\gcd(s, r))^{-1} \pmod{st/\gcd(s, r)}$. The graphs $X(s, t, 0)$, and $X(s', t', r')$ are Ádám isomorphic if and only if $s' = t$, $t' = s$ and $r' \equiv 0 \pmod{t}$.*

Proof. We prove the assertion for $s \geq 3$, $t \geq 2$ and $r \neq 0$. The graph $X(s, t, r)$ is the Cayley graph $Cay(\mathbb{Z}_{st/d_1} \times \mathbb{Z}_{d_1}, \{\pm(t/d_1, 1), \pm(k, 0)\})$, since Proposition 4.6 holds. By Proposition 4.3 or 4.6, we can represent $X(s', t', r')$ as the Cayley multigraph $Cay(\Gamma', \{\pm\gamma_1', \pm\gamma_2'\})$. The graphs $X(s, t, r)$, $X(s', t', r')$ are Ádám isomorphic if and only if there exists an isomorphism τ between the groups $\mathbb{Z}_{st/d_1} \times \mathbb{Z}_{d_1}$ and Γ' that sends the set $\{\pm(t/d_1, 1), \pm(k, 0)\}$ onto the set $\{\pm\gamma_1', \pm\gamma_2'\}$. Without loss of generality, we can set

$\{\pm\gamma_1'\}$ $=$ $\{\pm\tau((t/d_1,\,1))\}$ and $\{\pm\gamma_2'\}=$ $\{\pm\tau((k,\,0))\}$. By the existence of τ we can identify the group Γ' with the group $\mathbb{Z}_{st/d_1} \times \mathbb{Z}_{d_1}$. Hence γ_1' and γ_2' are elements of $Z_{st/d_1} \times \mathbb{Z}_{d_1}$ of order s and $st/\gcd(s,r)$, respectively, since $(t/d_1, 1)$ and $(k, 0)$ have order s and $st/\gcd(s,r)$, respectively (see the proof of Proposition 4.6 and 4.7). It is an easy matter to prove that an element $(a,b) \in \mathbb{Z}_{st/d_1} \times \mathbb{Z}_{d_1}$ has order $o(a)\cdot o(b)/\gcd(o(a), o(b)) = st/(d_1 \gcd(st/d_1, a))$, since d_1 is a divisor of s and t. Hence (a,b) has order s if and only if $\gcd(st/d_1, a) = t/d_1$, that is, $(a,b) = (m't/d_1, b)$ where $m' \in \mathbb{Z}_{st/d_1}$, $\gcd(m', s) = 1$, b is an arbitrary element of $\mathbb{Z}_{d_1}$. The element (a,b) has order $st/\gcd(s,r)$ if and only if $\gcd(st/d_1, a) = \gcd(s,r)/d_1 = \gcd(s/d_1, k)$, since k is coprime with t and $k \equiv r/d_1 \pmod{s/d_1}$. Hence $Cay(\Gamma', \{\pm\gamma_1', \pm\gamma_2'\})$ is a graph of type $Cay(\mathbb{Z}_{st/d_1} \times \mathbb{Z}_{d_1}, \{\pm(m't/d_1, b), \pm(a, b')\})$, where $\gcd(m', s) = 1$, $\gcd(st/d_1, a) = \gcd(s/d_1, k)$, b and b' are suitable elements of $\mathbb{Z}_{d_1}$. Note that a is coprime with t and the relation $ta \equiv rm't/d_1 \pmod{st/d_1}$ holds, since τ is an isomorphism and $tk \equiv rt/d_1 \pmod{st/d_1}$. If we apply Proposition 4.2 to the graph $G_1 = Cay(\mathbb{Z}_{st/d_1} \times \mathbb{Z}_{d_1}, \{\pm(m't/d_1, b), \pm(a, b')\})$ by setting $\gamma_1 = (m't/d_1, b)$ (or $\gamma_1 = -(m't/d_1, b)$), then G_1 can be represented as the graph $X(s,t,r)$ or $X(s,t,s-r)$. The graph $X(s,t,r)$ is isomorphic to the graph $G_2 = X(s', t', r')$, where $s' = st/\gcd(s,r)$, $t' = \gcd(s,r)$, $r' \equiv \pm t(kd_1/\gcd(s,r))^{-1} \pmod{st/\gcd(s,r)}$, since Proposition 4.7 holds. Hence G_1 is isomorphic to G_2. The isomorphism between G_1 and G_2 can be obtained also by applying Proposition 4.7 to the graph G_1. For the remaining values of s, t, r we represent the graph $X(s,t,r)$ as the Cayley multigraph in Proposition 4.3 and apply the previous method. □

The results that follow are based on the following theorem of [7].

Theorem 4.11 ([7]). *Any two finite isomorphic connected undirected Cayley multigraphs of degree* 4 *coming from abelian groups are Ádám isomorphic, unless they are obtained with the groups and sets* $\mathbb{Z}_{4n}$, $\{\pm1, \pm(2n-1)\}$ *and* $\mathbb{Z}_{2n} \times \mathbb{Z}_2$, $\{\pm(1,0), \pm(1,1)\}$.

By Theorem 4.11 the existence of an isomorphism between two Cayley multigraphs of degree 4, that are associated to abelian groups, implies the existence of an Ádám isomorphism, unless they are the graphs $Cir(4n; \pm1, \pm(2n-1))$ and $Cay(\mathbb{Z}_{2n} \times \mathbb{Z}_2, \{\pm(1,0), \pm(1,1)\})$. The following statements are consequences of Theorem 4.11.

Corollary 4.12. *The graphs* $X(4n, 1, 2n-1)$ *and* $X(s', t', r')$ *are isomorphic if and only if* $s' = 4n$, $t' = 1$, $r' = 2n+1$ *or* $s' = 2n$, $t' = 2$, $r' \in \{2, 2n-2\}$. *Moreover, there is no isomorphism between* $X(4n, 1, 2n-1)$ *and* $X(2n, 2, 2)$ *that sends the fundamental* 2*-factorization of* $X(4n, 1, 2n-1)$ *onto the fundamental* 2*-factorization of* $X(2n, 2, 2)$.

Proof. The graph $X(4n, 1, 2n-1)$ is the graph $Cir(4n; \pm1, \pm(2n-1))$ (see Proposition 4.3). By Theorem 4.11, the graphs $X(4n, 1, 2n-1)$ and $X(s', t', r')$ could be Ádám isomorphic or not. If they are Ádám isomorphic, then $s' = 4n$, $t' = 1$, $r' = 2n+1$, since Proposition 4.10 holds. If they are not Ádám isomorphic, then $X(s', t', r')$ is the graph $Cay(\mathbb{Z}_{2n} \times \mathbb{Z}_2, \{\pm(1,0), \pm(1,1)\})$ (see Theorem 4.11). Hence $s' = 2n$, $t' = 2$, $r' \in \{2, 2n-2\}$ (see Proposition 4.6 and 4.10). The fundamental 2-factorization of $X(4n, 1, 2n-1)$ consists of two Hamiltonian cycles, whereas the fundamental 2-factorization of $X(2n, 2, 2)$ consists of two 2-factors whose connected components are two $2n$-cycles (see Proposition 4.8). Therefore no isomorphism between $X(4n, 1, 2n-1)$ and $X(2n, 2, 2)$ can send the fundamental 2-factorization of $X(4n, 1, 2n-1)$ onto the fundamental 2-factorization of $X(2n, 2, 2)$. □

Proposition 4.13. *Let $X(s,t,r)$, $X(s',t',r')$ be non-isomorphic to $X(4n,1,2n-1)$, $X(2n,2,2)$. Then $X(s,t,r)$ and $X(s',t',r')$ are isomorphic if and only if they are Ádám isomorphic, that is, if and only if the parameters s', t', r' satisfy Proposition 4.10.*

Proof. The assertion follows from Theorem 4.11, and Proposition 4.10. □

5 Special cubic graphs arising from $X(s,t,r)$ graphs

When we consider graphs $X(s,t,r)$ we assume we are given a fundamental 2-factorization. This, in turn, implies we may turn the graph $X(s,t,r)$ into a cubic one by appropriately splitting each vertex. We note in passing that the operation of vertex-splitting and its converse were successfully used in a different context in [20].

There are two complementary possibilities. Either $X(s,t,r)$ arises from an I-graph or not. We consider each case separately.

5.1 I-graphs arising from $X(s,t,r)$

In Theorem 3.3 we remarked that any special cubic graph with a blue and red 2-factorization gives rise to the associated quartic graph with a blue and red 2-factorization. In Lemma 3.5, we showed that a proper I-graph $I(n,p,q)$ is special and gives rise to the associated circulant graph $Q(n,p,q)$. The following holds.

Lemma 5.1. *The circulant graph $Cir(n;p,q) = Q(n,p,q)$ arising from a connected I-graph $I(n,p,q)$ by contracting the spokes is the graph $X(s,t,r)$ with $t = \gcd(n,q)$, $s = n/t \geq 3$ and $r \equiv \pm p(q/t)^{-1} \pmod s$.*

Proof. The result follows from Proposition 4.2 by setting $\Gamma = \mathbb{Z}_n$, $\gamma_1 = q$, $\gamma_2 = p$. Whence $tp = rq$ for some integer r, $0 \leq r \leq s-1$, that is, r is a solution to the equation $r(q/t) \equiv p \pmod s$. By the Chinese Remainder Theorem, $r \equiv p(q/t)^{-1} \pmod s$. □

Theorem 5.2. *The graph $X(s,t,r)$ arises from a connected I-graph by contracting the spokes if and only if $\gcd(s,t,r) = 1$ and $(t,r) \neq (2,0)$ for odd values of s. In this case, the graph $X(s,t,r)$ together with its fundamental 2-factorization, is in one-to-one correspondence with the I-graph $I(st,k,t)$, where $0 < k < st$, $\gcd(k,t) = 1$ and $k \equiv r \pmod s$ (in particular, $k = s$ if $r = 0$). If at least one of the integers k, t, $\gcd(s,r)$ is 1, then $X(s,t,r)$ corresponds to a generalized Petersen graph.*

Proof. Assume that $X(s,t,r)$ arises from the connected I-graph $I(n,p,q)$ by contracting the spokes. By Lemma 5.1, $t = \gcd(n,q)$, $s = n/t \geq 3$ and $r(q/t) \equiv p \pmod s$. Whence $(t,r) \neq (2,0)$ if s is odd, otherwise $p = 0$ (which is not possible). We show that $\gcd(s,t,r) = 1$. Suppose, on the contrary, that $\gcd(s,t,r) = d_1 \neq 1$, then d_1 is a divisor of $\gcd(t,p)$ since $r(q/t) \equiv p \pmod s$. That yields a contradiction, since $\gcd(t,p) = 1$ (see Proposition 3.4). Hence $\gcd(s,t,r) = 1$.

Assume that $\gcd(s,t,r) = 1$. We show that $X(s,t,r)$ arises from a connected I-graph by contracting the spokes. Since $\gcd(s,t,r) = 1$, the graph $X(s,t,r)$ can be represented as the circulant graph $Cir(st;\pm t,\pm k)$, where $0 < k < st$, $\gcd(t,k) = 1$ and $k \equiv r \pmod s$ (see Proposition 4.6). If $r = 0$, then we can set $k = s$, since Proposition 4.3 holds. The graph $I(st,k,t)$ is connected and it gives rise to the graph $X(s,t,r)$, since Lemma 5.1 holds. By Theorem 3.3, the graph $X(s,t,r)$, together with its fundamental 2-factorization, is in one-to-one correspondence with the I-graph $I(st,k,t)$. If $k = 1$ or $t = 1$, then

$X(s,t,r)$ corresponds to a generalized Petersen graph (see [3]). If $\gcd(s,r) = 1$ then $X(s,t,r)$ is isomorphic to $X(st,1,r')$ (see Proposition 4.10. By the previous remarks, the graph $X(st,1,r')$ corresponds to a generalized Petersen graph. The assertion follows. $\square$

It is straightforward to see that isomorphic $X(s,t,r)$ graphs give rise to isomorphic I-graphs and also the converse is true. By Corollary 4.12 and Proposition 4.13, the circulant graphs $X(s,t,r)$, $X(s',t',r')$ are isomorphic if and only if they are Ádám-isomorphic, that is, there exists an automorphism of the cyclic group of order $st = s't'$ that sends the defining set of the circulant graph $X(s,t,r)$ onto the defining set of the circulant graph $X(s',t',r')$. This fact is equivalent to the results proved in [13] about the isomorphism between I-graphs.

5.2 Special Generalized *I*-graphs

In this section we consider the special cubic graphs that correspond to the graphs $X(s,t,r)$ with $\gcd(s,t,r) \neq 1$, according to the correspondence described in Theorem 3.3. By Proposition 5.2, these special cubic graphs do not belong to the family of connected I-graphs. By Theorem 3.3 and Definition 4.1, we can define a family of special cubic graphs containing the family of connected I-graphs as a subfamily. We call this family *Special Generalized I-graphs*. This family is not contained in the family of GI-graphs [6].

Let $s \geq 1$, $t \geq 1$ and $0 \leq r \leq s-1$. We define a *Special Generalized I-graph* $SGI(st,s,t,r)$ as a cubic graph of order st with vertex-set $V = \{u_{i,j}, u'_{i,j} : 0 \leq i \leq t-1, 0 \leq j \leq s-1\}$ and edge-set $E = \{[u_{i,j}, u_{i,j+1}], [u_{i,j}, u'_{i,j}] : 0 \leq i \leq t-1, 0 \leq j \leq s-1\} \cup \{[u'_{i,j}, u'_{i+1,j}] : 0 \leq i \leq t-2, 0 \leq j \leq s-1\} \cup \{[u'_{t-1,j}, u'_{0,j+r}] : 0 \leq j \leq s-1\}$ (the addition $j+1$ and $j+r$ are considered modulo s). For $s = 1$ or $(t,r) = (1,0)$, a special generalized I-graph has loops. For $s = 2$ or $(t,r) = (2,0)$, it has multiple edges. For the other values of s, t, r, it is a simple cubic graph. We say that a vertex $u_{i,j}$ (respectively, u'_{ij}) is an *outer vertex* (respectively, an *inner vertex*). We say that an edge $[u_{i,j}, u_{i,j+1}]$ (respectively, $[u'_{i,j}, u'_{i+1,j}]$) is an *outer edge* (respectively, an *inner edge*). We say that an edge $[u_{i,j}, u'_{i,j}]$ is a *spoke*. The spokes constitute the special 1-factor. The graph arising from $SGI(st,s,t,r)$ by contracting the spokes is the graph $X(s,t,r)$. The horizontal edges of $X(s,t,r)$ correspond to the outer edges of $SGI(st,s,t,r)$, vertical and diagonal edges of $X(s,t,r)$ correspond to the inner edges of $SGI(st,s,t,r)$. A generalization of the proof of Proposition 5.2 gives the following statement.

Proposition 5.3. *Let $s \geq 1$, $t \geq 1$, $0 \leq r \leq s-1$ and $d_1 = \gcd(s,t,r)$. The graph $X(s,t,r)$, together with its fundamental 2-factorization, is in one-to-one correspondence with the graph $SGI(st,s,t,k)$ where $k = s$ if $r = 0$, otherwise $0 < k < st/d_1$, $\gcd(k,t) = 1$ and $k \equiv r/d_1 \pmod{s/d_1}$.*

By Corollary 4.12, the graphs $X(4n,1,2n-1)$ and $X(2n,2,2)$ are isomorphic, but no isomorphism between them sends the fundamental 2-factorization of $X(4n,1,2n-1)$ onto the fundamental 2-factorization of $X(2n,2,2)$. This fact means that the application of Theorem 3.3 to the graphs $X(4n,1,2n-1)$ and $X(2n,2,2)$ yields non-isomorphic special cubic graphs. As a matter of fact, Proposition 5.2 says that the graph $X(4n,1,2n-1)$ is in one-to-one correspondence with a connected I-graph, whereas $X(2n,2,2)$ does not correspond to any I-graph. For instance, for $n = 2$ the graph $X(8,1,3)$ is associated with the Möbius-Kantor graph of girth 6, [15, 17], while $X(4,2,2)$ arises from a graph of girth 4 (see Figure 2).

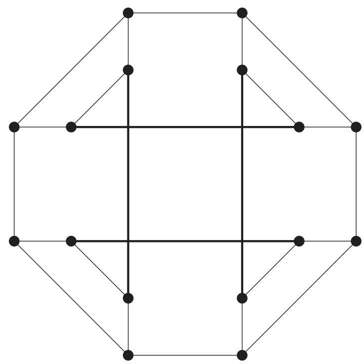


Figure 2: The cubic split $X(4, 2, 2)$ graph is $SGI(8, 4, 2, 2)$. The thick edges represent the special 1-factor.

6 Good Eulerian tours in $X(s, t, r)$ graphs

In this section we construct good Eulerian subgraphs of $X(s, t, r)$. For each $X(s, t, r)$ we denote by $W(s, t, r)$ the constructed good Eulerian subgraph. By Proposition 3.2, a spanning Eulerian subgraph W of $X(s, t, r)$ is admissible if and only if at each 2-valent vertex exactly one edge is horizontal. We consider $X(s, t, r)$ being embedded into the torus with quadrilateral faces. Hence any of its subgraphs may be viewed embedded in the same surface. A tour in W may be regarded as as a *straight-ahead walk* (or SAW) on the surface [18]. A good Eulerian tour of W is an Eulerian SAW that uses only allowed transitions, that is, the tour cannot switch from a horizontal to a vertical (or diagonal) edge when it visits a 4-valent vertex of W. For instance, the graph W in Figure 4(a) is an admissible subgraph of $X(5, 4, 3)$; the tour $\mathcal{E} = (x_0^0, x_0^1, x_1^1, x_2^1, x_3^1, x_4^1, x_4^0, x_1^3, x_1^2, x_1^1, x_1^0, x_2^0, x_2^1, x_2^2, x_2^3, x_1^3, x_0^3, x_0^2, x_1^2, x_2^2, x_3^2, x_4^2, x_4^3, x_3^3, x_3^2, x_3^1, x_3^0, x_4^0, x_1^0)$ is a good Eulerian SAW of W; hence W is a good Eulerian subgraph of $X(5, 4, 3)$.

If we delete the diagonal edges in $X(s, t, r)$, we obtain a spanning subgraph that we denote by $X'(s, t, r)$. Clearly $X'(s, t, r)$ is the cartesian product of a cycle C_s with a path P_t embedded in the torus or cylinder. If we further delete an edge in C_s we obtain a path P_s. We denote the cartesian product of P_s and P_t by $X''(s, t, r)$ and obtain a spanning subgraph of $X'(s, t, r)$ and $X(s, t, r)$. In order to simplify the constructions, we will seek to find good Eulerian subgraphs in $X'(s, t, r)$ or in $X''(s, t, r)$. In this case the resulting good Eulerian subgraph will be denoted by $W'(s, t, r)$ and $W''(s, t, r)$, respectively. This simplification makes sense, since neither $X'(s, t, r)$ nor $X''(s, t, r)$ depend on the parameter r. Hence any Eulerian subgraph $W'(s, t, r)$ or $W''(s, t, r)$ is good for any r.

6.1 Method of construction

We give some lemmas that will be used in the construction of a good Eulerian subgraph $W(s, t, r)$. Given a graph $X(s, t, r)$, for every row index i, $0 \leq i \leq t-1$, we denote by V_i the set of vertical edges $V_i = \{[x_j^i, x_j^{i+1}] : 0 \leq j \leq s-1\}$. For every column index j, $0 \leq j \leq s-1$, we denote by H_j the set of horizontal edges $H_j = \{[x_j^i, x_{j+1}^i] : 0 \leq i \leq t-1\}$.

Let H be a subgraph of $X(s, t, r)$. We say that H can be *expanded vertically (from row* *i)* if $|E(H) \cap V_i| = s-1$ or $s-2 > 0$ (for $s = 3$ we require $|E(H) \cap V_i| = 2$). We say that H can be *expanded horizontally (from column j)* if $|E(H) \cap H_j| = t-1$ or $t-2 > 0$ (for $t = 3$ we require $|E(H) \cap H_j| = 2$). The following statements hold.

Lemma 6.1. *Let $W(s, t_1, r)$ be a good Eulerian subgraph that can be expanded vertically. Then there exists a good Eulerian subgraph $W(s, t, r)$ for every $t \geq t_1$, $t \equiv t_1 \pmod 2$.*

Proof. We use the graph $W_1 = W(s, t_1, r)$ to construct a good Eulerian subgraph $W(s, t, r)$. By the assumptions, $|E(W_1) \cap V_i| = s-1$ or $s-2$ for some row index i, $0 \leq i \leq t-1$. By the symmetry properties of the graph $X(s, t_1, r)$, we can cyclically permute its rows so that we can assume $0 < i < t-1$. We treat separately the cases $|E(W_1) \cap V_i| = s-1$ and $|E(W_1) \cap V_i| = s-2$. Consider $|E(W_1) \cap V_i| = s-1$ and denote by $[x_a^i, x_a^{i+1}]$ the vertical edge of V_i which is missing in W_1. We can cyclically permute the columns of $X(s, t_1, r)$ and assume $a = 0$. We subdivide every vertical edge $[x_j^i, x_j^{i+1}]$, with $0 < j \leq s-1$, by inserting two new vertices, namely, y_j^i and y_j^{i+1} such that y_j^i is adjacent to x_j^i and y_j^{i+1} is adjacent to x_j^{i+1}, and we add two new vertices y_0^{i+1}, y_0^i between x_0^{i+1} and x_0^i in column 0. We now delete the edge $[y_{s-1}^i, y_{s-1}^{i+1}]$ and replace it with the path from y_{s-1}^i to y_{s-1}^{i+1} composed of the edges $[y_j^{i+1}, y_{j+1}^{i+1}]$, $[y_j^i, y_{j+1}^i]$, $0 \leq j \leq s-2$, and $[y_0^i, y_0^{i+1}]$. The resulting graph is a good Eulerian subgraph $W(s, t_1+2, r)$. We can iterate the process and find a good Eulerian subgraph $W(s, t, r)$ for every $t \geq t_1$, $t \equiv t_1 \pmod 2$. The case $|E(W_1) \cap V_i| = s-2$ can be treated analogously to the case $|E(W_1) \cap V_i| = s-1$. As an example, consider the graph $W''(6, 5, r)$ in Figure 3. It can be expanded vertically from row 1 and it yields a good Eulerian subgraph $W''(6, 7, r)$. □

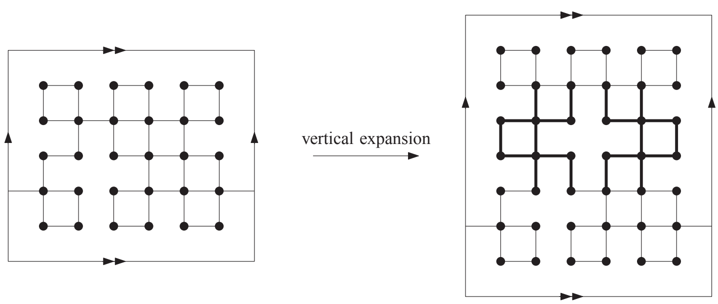
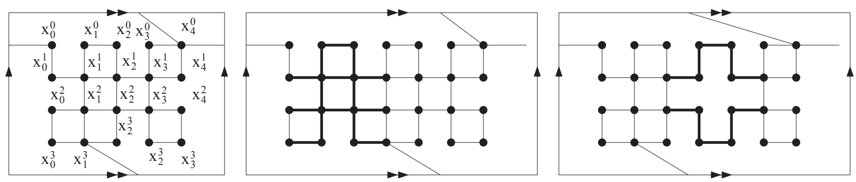


Figure 3: A vertical expansion of the good Eulerian subgraph $W'(6, 5, r)$ yields a good Eulerian subgraph $W(6, 7, r)$.

In the following lemma we consider horizontal expansions. In this case we have to pay attention to the diagonal edges of $W(s, t, r)$, if any exist. If $[x_j^{t-1}, x_{j+r}^0]$, where $j + r$ is considered modulo s, is a diagonal edge of $W(s, t, r)$, then we can assume $j < j+r$, since we can cyclically permute the columns of $W(s, t, r)$. Therefore we can say that a diagonal edge $[x_j^{t-1}, x_{j+r}^0]$ *crosses column ℓ* if $j \leq \ell < j+r$.

Lemma 6.2. *Let $W(s_1, t, r_1)$ be a good Eulerian subgraph that can be expanded horizontally from column ℓ. If no diagonal edge of $W(s_1, t, r_1)$ crosses column ℓ, then there exists a good Eulerian subgraph $W(s, t, r_1)$ for every $s \geq s_1$, $s \equiv s_1 \pmod 2$. If every diagonal* *edge crosses column ℓ, then there exists a good Eulerian subgraph $W(s_1 + r - r_1, t, r)$ for every $r \geq r_1$, $r \equiv r_1 \pmod 2$.*

Proof. We apply the method described in Lemma 6.1 to the edges in H_ℓ. If every diagonal edge of $W(s_1, t, r_1)$ crosses column ℓ, then by subdividing the edges of H_ℓ we can shift of $r - r_1$ steps the diagonal edges of $W(s_1, t, r_1)$. If no diagonal edge of $W(s_1, t, r_1)$ crosses column ℓ, then no diagonal edge is shifted. As an example, consider the graph $W(5, 4, 3)$ in Figure 4. If we expand horizontally the graph from column $\ell = 0$, then no diagonal edge crosses column ℓ and we obtain a good Eulerian subgraph $W(7, 4, 3)$. If we expand horizontally the graph from column $\ell = 2$, then every diagonal edge crosses column ℓ and we obtain a good Eulerian subgraph $W(7, 4, 5)$. □



(a) (b) (c)

Figure 4: A good Eulerian subgraph: (a) $W(5, 4, 3)$; (b) $W(7, 4, 3)$; (c) $W(7, 4, 5)$. The graphs $W(7, 4, 3)$ and $W(7, 4, 5)$ are obtained from $W(5, 4, 3)$ by an horizontal expansion from column 0 and column 2, respectively.

6.2 Constructions of good Eulerian subgraphs.

We apply the lemmas described in Section 6.1 to construct a good Eulerian subgraph $W(s, t, r)$. It is straightforward to see that the existence of loops in $X(s, t, r)$ excludes the existence of a good Eulerian subgraph $W(1, t, r)$ and $W(s, 1, 0)$. Analogously, the existence of horizontal parallel edges in $X(2, t, r)$ excludes the existence of a good Eulerian subgraph $W(2, t, r)$ with t odd and $W(2, t, 1)$ with t even, $t > 2$, (see Case 2 in the proof of Lemma 6.5 for a good Eulerian subgraph $W(2, 2, 1)$ and $W(2, t, 0)$ with t even). Hence we can consider $s \geq 3$ and $(t, r) \neq (1, 0)$. The following hold.

Proposition 6.3. *The graph $X(s, 1, r)$, $r \neq 0$, possesses a good Eulerian subgraph, unless $s = 6m + 5$, with $m \geq 0$, and $r \in \{2, s - 2, (s + 1)/2, (s - 1)/2\}$.*

Proof. By Proposition 4.3, the graph $X(s, 1, r)$ can be represented as the circulant multigraph $Cir(st; \pm 1, \pm r)$. By Proposition 5.2, the graph $X(s, 1, r)$ corresponds to the generalized Petersen graph $I(s, r, 1)$ or $G(s, r)$. In particular, the graph $X(6m + 5, 1, 2)$ corresponds to the generalized Petersen graph $G(6m + 5, 2)$. Hence $X(s, 1, r)$ has a good Eulerian subgraph, unless it is isomorphic to $X(6m + 5, 1, 2)$, since Theorems 1.1 and 2.2 hold. By Proposition 4.13, the graphs that are isomorphic to $X(6m+5, 1, 2)$ are $X(6m+5, 1, r')$, where $r' \in \{2, 6m + 3\}$ or $r' \equiv \pm 2^{-1} \pmod{6m + 5}$, that is, $r' \in \{3m + 3, 3m + 2\}$, since $r' < 6m + 5$. □

We can construct a good Eulerian subgraph $W(s, 1, r)$, $r \neq 0$, without using Theorem 1.1. More specifically, by Proposition 4.10 the graph $X(s, 1, r)$, with $r \neq 0$, is isomorphic to the graph $X(s/\gcd(s,r), \gcd(s,r), r')$, where $r' \equiv \pm r^{-1} \pmod{s}$. For $r \neq 0$ and $\gcd(s,r) > 1$, a construction of a good Eulerian subgraph can be found in the proof of Lemma 6.5. We can also provide an ad hoc construction for the case $\gcd(s,r) = 1$, but we prefer to omit this construction, since the existence of a good Eulerian subgraph $W(s,1,r)$, $r \neq 0$, is known (see Proposition 6.3) and the construction is based on the method of Lemma 6.5. We will show that the graph $X(6m+5,1,2)$, $m \geq 0$, has no good Eulerian subgraph, that is, the generalized Petersen graph is not Hamiltonian. The following statement is a consequence of Proposition 6.3 and it will be used in the proof of Lemma 6.5.

Proposition 6.4. *The graph $X(s,t,r)$, with $s \geq 3$, $t > 1$ and $\gcd(s,r) = 1$ has a good Eulerian subgraph.*

Proof. By Proposition 4.6, the graph $X(s,t,r)$ can be represented as the circulant graph $Cir(st; \pm t, \pm k)$, where $\gcd(k,t) = 1$ and $k \equiv r \pmod{s}$. By Proposition 4.10, the graph $X(s,t,r)$ is isomorphic to the graph $X(st,1,r')$, with $r' \neq 0$, since $\gcd(s,r) = 1$. If $st \not\equiv 5 \pmod{6}$, then the assertion follows from Proposition 6.3 (see Proposition 4.10). Consider $st \equiv 5 \pmod{6}$. We show that $X(s,t,r)$ is not isomorphic to $X(6m+5,1,2)$, $m \geq 0$. Suppose, on the contrary, that $X(s,t,r)$ is isomorphic to $X(6m+5,1,2)$. Then $X(st,1,r')= X(6m+5,1,r')$, where $r' \in \{2,\ st-2,\ (st+1)/2,\ (st-1)/2\}$ (see Proposition 6.3). By Proposition 4.10, the integer r' satisfies the relation $r' \equiv \pm tk^{-1} \pmod{st}$. Whence t is a divisor of r'. That yields a contradiction, since $r' \in \{2,\ st-2,\ (st+1)/2,\ (st-1)/2\}$ and t is coprime with the integers in $\{2,\ st-2,\ (st+1)/2,\ (st-1)/2\}$. □

Lemma 6.5. *Let $s \geq 3$, $t \geq 2$ and $0 \leq r \leq s-1$. There exists a good Eulerian subgraph $W(s,t,r)$, unless s is odd and $(t,r) = (2,0)$.*

Proof. We treat separately the cases: $t = 3$; s, t even; s even, t odd, $t \geq 5$; s odd, t even; s, t odd, $t \geq 5$. When we will speak of "vertical" and "horizontal" expansion we refer implicitly to Lemma 6.1 and 6.2, respectively.

Case 1: $t = 3$. This case is treated in Section 8, since it requires a lengthy description.

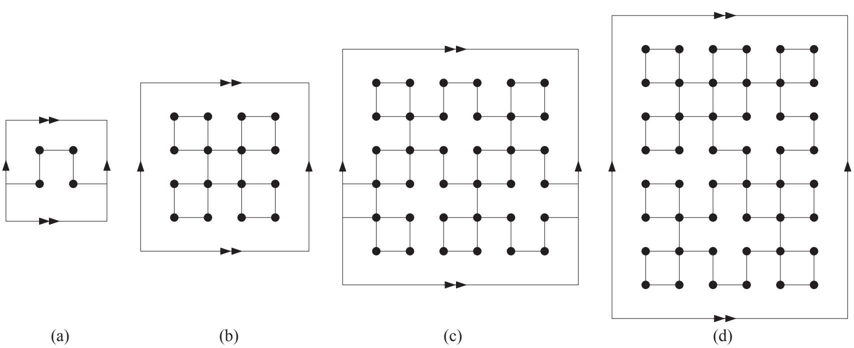


Figure 5: A good Eulerian subgraph: (a) $W'(2,2,r)$; (b) $W''(4,4,r)$; (c) $W'(6,6,r)$; (d) $W''(6,8,r)$.

Case 2: s **even,** t **even.** The graph $W''(6, 8, r)$ in Figure 5(d) can be expanded vertically from row 1 and horizontally from column 2. It yields a good Eulerian subgraph $W''(s, t, r)$ for every s, t even $s \geq 6$, $t \geq 8$. It remains to construct a good Eulerian subgraph $W''(s, t, r)$ for $s \geq 6$, $t = 2, 4, 6$ and $W''(4, t, r)$ for $t \geq 2$, t even. The graph $W'(2, 2, r)$ in Figure 5(a) can be expanded horizontally from column 0 or 1. It yields a good Eulerian subgraph $W'(s, 2, r)$ for every s even, $s \geq 2$. We expand horizontally the graph $W''(4, 4, r)$ in Figure 5(b) and obtain $W''(s, 4, r)$ for every s even, $s \geq 4$. We rotate $W''(s, 4, r)$ by 90 degrees clockwise (around a vertex) and obtain a good Eulerian subgraph $W''(4, t, r)$ for every t even, $t \geq 4$. We expand horizontally the graph $W''(6, 6, r)$ in Figure 5(c) from column 3 and obtain $W''(s, 6, r)$ for every s even, $s \geq 6$.

Case 3: s **even,** t **odd,** $t \geq 5$**.** The graph $W'(6, 5, r)$ in Figure 3 can be expanded vertically from row 2 and horizontally from column 3. It yields a good Eulerian subgraph $W'(s, t, r)$ for every s even, $s \geq 6$, t odd, $t \geq 5$. It remains to construct $W(4, t, r)$ with t odd, $t \geq 5$, $0 \leq r \leq 3$. Since $X(4, t, r)$ is isomorphic to $X(4, t, 4 - r)$, we can consider $0 \leq r \leq 2$. A good Eulerian subgraph for $W(4, t, 0)$, t odd, $t \geq 5$, can be obtained from $W(4, 3, 0)$ in Figure 6(a) by a vertical expansion from row 1. The existence of a good Eulerian subgraph $W(4, t, 1)$ follows from Proposition 6.4. By Proposition 4.10, the graph $X(4, t, 2)$ is isomorphic to the graph $X(2t, 2, r')$. By the results in Case 2, there exists a good Eulerian subgraph $W(2t, 2, r')$.

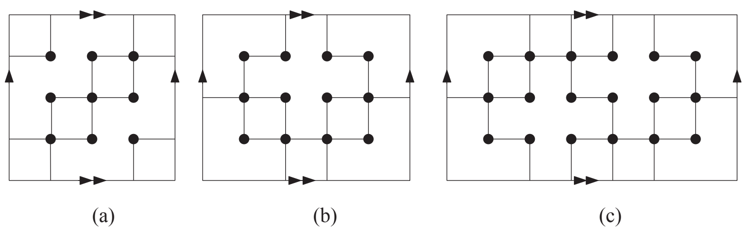


Figure 6: A good Eulerian subgraph: (a) $W(3, 3, 0)$; (b) $W(4, 3, 0)$; (c) $W(6, 3, 0)$.

Case 4: s **odd,** t **even.** By Proposition 4.10, the graph $X(s, t, r)$, with $r \neq 0$, is isomorphic to the graph $X(st/\gcd(s, r), \gcd(s, r), r')$, with $r' \neq 0$, or to $X(t, s, 0)$ if $r = 0$. If $r \neq 0$ and $\gcd(s, r) = 1$ or 3, then the existence of a good Eulerian subgraph follows from Proposition 6.4 or from the results in Case 1, respectively. Note that $st/\gcd(s, r) \geq 4$, since t is even and $0 < r \neq s - 1$. Hence, for $\gcd(s, r) \geq 5$, the existence of a good Eulerian subgraph follows from Case 3. Consider $r = 0$. There is no good Eulerian subgraph $W(s, 2, 0)$, because of the existence of parallel vertical edges. Consider $t \geq 4$. As remarked, the graph $X(s, t, 0)$ is isomorphic to the graph $X(t, s, 0)$. For $s \geq 5$ the existence of a good Eulerian subgraph $W(t, s, 0)$ follows from the results in Case 3. The existence of a good Eulerian subgraph $W(t, 3, 0)$ follows from Case 1.

Case 5: s **odd,** t **odd,** $t \geq 5$**.** A good Eulerian subgraph $W(s, t, 0)$ can be obtained from the graph $W(3, 3, 0)$ in Figure 6(a). If $r \in \{1, 2\}$, then the existence of a good Eulerian subgraph follows from Proposition 6.4. Consider $3 \leq r \leq s - 3$ and $s \geq 7$. Since $X(s, t, r)$ is isomorphic to $X(s, t, s - r)$ and s is odd, we can construct a good Eulerian subgraph $W(s,t,r)$ for every s, r odd, $s \geq 7$, $3 \leq r \leq s-4$. The graph $W(7,5,3)$ in Figure 10(c) can be expanded horizontally from column 4 and vertically from row 1 (or 2). It yields a good Eulerian subgraph $W(s,t,3)$ for every s, t odd, $s \geq 7$, $t \geq 5$. Since $s-r+3 \geq 7$, we can consider the graph $W(s-r+3,t,3)$ arising from $W(7,5,3)$ in Figure 10(c). We expand horizontally the graph $W(s-r+3,t,3)$ from column 2 and obtain a good Eulerian subgraph $W(s,t,r)$ for every s,t,r odd, $s \geq 7$, $t \geq 5$ and $3 \leq r \leq s-4$.

□

Proposition 6.6. *The graph $X(6m+5,1,2)$, $m \geq 0$, has no good Eulerian subgraph. Consequently, the generalized Petersen graph $G(6m+5,2)$ has no Hamiltonian cycle.*

Proof. We give a sketch of the proof by showing that $X(5,1,2)$ has no good Eulerian subgraph. Suppose, on the contrary, that W is a good Eulerian subgraph of $X(6m+5,1,2)$. Since the unique horizontal layer of W has an odd number of vertices, the graph W contains at least one path P_{2j+1} consisting of $2j$ horizontal edges. It is possible to prove that $2j = 2$ (if $2j > 2$, then W is not good). Without loss of generality we can set $P_{2j+1} = (x_0^0, x_1^0, x_2^0)$. Whence $[x_3^0, x_4^0] \in E(W)$ and no other horizontal edge of $X(5,1,2)$ belongs to $E(W)$. Moreover, $[x_1^0, x_3^0]$,$[x_1^0, x_4^0]$ are edges of W, since W is admissible and x_1^0 is 4-valent in W. Whence $[x_0^0, x_2^0] \in E(W)$ and each admissible tour of W contains the component $A = (x_3^0, x_4^0, x_4^0, x_3^0)$. That yields a contradiction, since A is not a spanning subgraph of $X(6m+5,1,2)$. Hence $X(5,1,2)$ has no good Eulerian subgraph. By Theorem 2.2, the graph $G(5,2)$ has no Hamiltonian cycle. The proof can be generalized to the case $G(6m+5,2)$ with $m > 0$. □

7 Characterization of Hamiltonian I-graphs

Now we are ready to prove the main theorem.

Proof of Theorem 1.2. By Theorem 1.1, a generalized Petersen graph is Hamiltonian if and only if it is not isomorphic to $G(6m+5,2)$, $m \geq 0$. We prove that a proper I-graph is Hamiltonian. By Lemma 3.5, a proper I-graph $I(n,p,q)$ is special and its associated quartic graph X is the circulant graph $Cir(n;p,q)$. By Lemma 5.1, the graph $Cir(n;p,q)$ can be represented as the graph $X(s,t,r)$, where $t = \gcd(n,q)$, $s = n/t \geq 3$, $r \equiv \pm p(q/t)^{-1} \pmod{s}$ and $(t,r) \neq (2,0)$ for odd values of s. By Lemma 6.5, the graph $X(s,t,r)$ has a good Eulerian subgraph. The assertion follows from Theorem 2.2. □

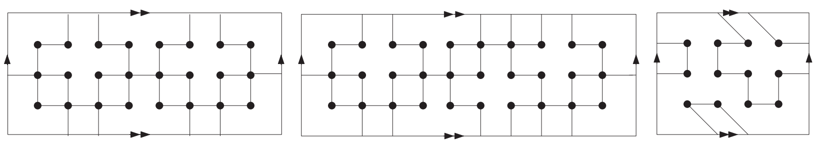
By Theorem 2.2 and Lemma 6.5, we can extend the result of Theorem 1.2, about the existence of a Hamiltonian cycle, to the special generalized I-graphs.

As a consequence of Theorem 1.2, a proper I-graph is 3-edge-colorable or, equivalently, 1-factorizable (because it is cubic and Hamiltonian). A widely studied property for 1-factorizable graphs is the property of admitting a perfect 1-factorization. We recall that a 1-factorization is perfect if the union of any pair of distinct 1-factors is a Hamiltonian cycle. Partial results are known for generalized Petersen graphs: $G(n,k)$ admits a perfect 1-factorization when $(n,k) = (3,1)$; $(n,k) = (n,2)$ with $n \equiv 3,4 \pmod{6}$; $(n,k) = (9,3)$; $(n,k) = (3d,d)$ with d odd; $(n,k) = (3d,k)$ with $k > 1$, d odd, $3d$ and k coprime (see [4]). So, it is quite natural to extend the same problem to proper I-graphs.

Some further problems can be considered: the generalization of the existence of good Eulerian tour to other graph bundles of a cycle over a cycle, the characterization of Hamiltonian GI-graphs or of Hamilton-laceable I-graphs. In [8], the authors proved by a computer search that all bipartite connected I-graphs on $2n \leq 200$ vertices are Hamilton-laceable.

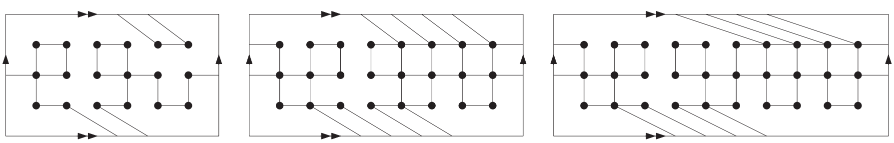
8 Appendix. Proof of Lemma 6.5

Case 1, $t = 3$. We expand horizontally the graph $W(3,3,0)$ in Figure 6(a) from column 0 and obtain a good Eulerian subgraph $W(s,3,0)$ for every s odd, $s \geq 3$. A good Eulerian subgraph $W(s,3,0)$ with s even can be obtained from the graphs $W(4,3,0)$ and $W(6,3,0)$ in Figure 6(b)-(c). As an example, the graph $W(8,3,0)$ in Figure 7(a) has been obtained by connecting two copies of the graph $W(4,3,0)$. The graph $W(10,3,0)$ in Figure 7(b) has been obtained by connecting the graphs $W(4,3,0)$ and $W(6,3,0)$. For $r = 1$ the existence of a good Eulerian subgraph $W(s,3,1)$ follows from Proposition 6.4. Hence we can consider $2 \leq r \leq s/2$, since $X(s,3,r)$ is isomorphic to $X(s,3,s-r)$. The graph $W(4,3,2)$ in Figure 7(c) can be expanded horizontally from column 3. It yields a good Eulerian subgraph $W(s,3,2)$ for every s even, $s \geq 4$. Since $s-r+2 \geq 4$, we can consider the graph $W(s-r+2,3,2)$ obtained from $W(4,3,2)$ in Figure 7(c). We expand horizontally $W(s-r+2,3,r)$ from column 1 and obtain a good Eulerian subgraph $W(s,3,r)$ for every s, r even, $s \geq 4$, $2 \leq r \leq s/2$. Analogously, the graphs $W(6,3,3)$, $W(8,3,3)$ and $W(10,3,5)$ in Figure 8 yield a good Eulerian subgraph $W(s,3,r)$ for every s even, r odd, $3 \leq r \leq s/2$. More specifically, we expand horizontally the graph $W(8,3,3)$ from column 7 and obtain a good Eulerian subgraph $W(s,3,3)$ for every even integer $s \geq 8$. The graph $W(10,3,5)$ can be expanded horizontally from column 9 (or 0). It yields a good Eulerian subgraph $W(s,3,5)$ for every even integer s, $s \geq 10$. Since $s-r+5 \geq 10$, we can consider the graph $W(s-r+5,3,5)$ obtained from $W(10,3,5)$ in Figure 8(c). We expand $W(s-r+5,3,5)$ from column 4 and obtain a good Eulerian subgraph $W(s,3,r)$ for every s even, $s \geq 10$, r odd, $5 \leq r \leq s/2$.



(a) (b) (c)

Figure 7: A good Eulerian subgraph: (a) $W(8,3,0)$; (b) $W(10,3,0)$; (c) $W(4,3,2)$.



(a) (b) (c)

Figure 8: A good Eulerian subgraph: (a) $W(6,3,3)$; (b) $W(8,3,3)$; (c) $W(10,3,5)$.

Consider s odd, $s \geq 5$. The graph $W(5,3,2)$ in Figure 9(a) can be expanded horizontally from column 4. It yields a good Eulerian subgraph $W(s,3,2)$ for every s odd,

$s \geq 5$. Analogously, the graph $W(9,3,4)$ in Figure 9(b) yields a good Eulerian subgraph $W(s,3,4)$ for every s odd, $s \geq 9$. The graph $W(13,3,6)$ in Figure 9(c) can be expanded horizontally from column 2 and column 10. It yields a good Eulerian subgraph $W(2r+1,3,r)$ with r even, $6 \leq r \leq s/2$. Since $s - 2r + 1 \geq 0$, we can expand $W(2r+1,3,r)$ from column $2r$ and find a good Eulerian subgraph $W(s,3,r)$ for every s odd, $s \geq 13$, r even, $r \geq 6$. It remains to construct a good Eulerian subgraph $W(s,3,r)$ with s, r odd, $s \geq 7$, $3 \leq r \leq s/2$. We use the graph $W(7,3,3)$ in Figure 10(a) to construct a good Eulerian subgraph $W(2r+1,3,r)$ with r odd, $r \geq 3$. As an example, the graph $W(11,3,5)$ in Figure 10(b) has been obtained by expanding horizontally the graph $W(7,3,3)$ from column $r = 3$ and $s-1 = 6$ and by adding new diagonal edges. If we iterate the process, then we obtain a good Eulerian subgraph $W(2r+1,3,r)$ with r odd, $r \geq 3$. The graph $W(2r+1,3,r)$ thus obtained can be expanded horizontally from column $2r$. It yields a good Eulerian subgraph $W(s,3,r)$ for every s, r odd, $s \geq 7$, $3 \leq r \leq s/2$. □

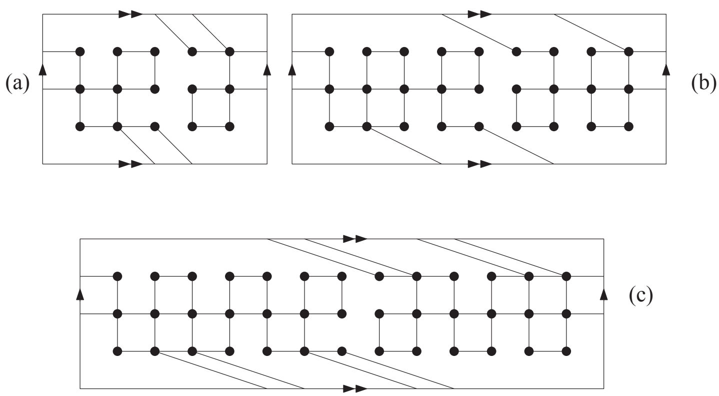


Figure 9: A good Eulerian subgraph: (a) $W(5,3,2)$; (b) $W(9,3,4)$; (c) $W(13,3,6)$

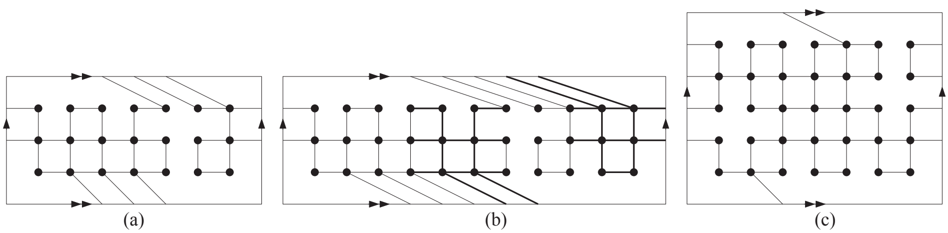


Figure 10: A good Eulerian subgraph: (a) $W(7,3,3)$; (b) $W(11,3,5)$; (c) $W(7,5,3)$. To obtain the graph $W(11,3,5)$ we expanded horizontally the graph $W(7,3,3)$ from column $r = 3$ and column $s-1 = 6$, then we added new diagonal edges (see the bold edges).

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The parameters of Fibonacci and Lucas cubes*

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Abstract

Motivated by the conjectures from Castro, et al. in 2011, in this paper we use integer programming formulations for computing the domination number, the 2-packing number and the independent domination number of Fibonacci cubes and Lucas cubes for $n \leq 13$.

Keywords: Fibonacci cubes, Lucas cubes, domination number, 2-packing number.

Math. Subj. Class.: 05C69, 05C25

1 Introduction

Hypercubes form one of the most applicable classes of graphs with many appealing properties. The n-cube Q_n is the graph whose vertices are all binary strings of length n, and two vertices are adjacent if they differ in exactly one position. The Fibonacci cubes were introduced as a model for interconnection networks [4, 2]. They offer challenging mathematical and computational problems, and admit a recursive decomposition into smaller Fibonacci cubes (see [5], [6], [8] for their structural properties). The Fibonacci cubes can be recognized in $O(m \log n)$ time (where n is the order and m the size of a given graph) [10]. The Lucas cubes [7] form a class of graphs closely related to the Fibonacci cubes, obtained by removing some vertices from the Fibonacci cubes.

Let Q_n be the n-dimensional hypercube. A Fibonacci string of length n is a binary string $b_1 b_2 \dots b_n$ with $b_i \cdot b_{i+1} = 0$ for $1 \leq i < n$. In other words, Fibonacci strings are binary strings that contain no consecutive ones. The Fibonacci cube Γ_n, for $n \geq 1$ is the subgraph of Q_n induced by the Fibonacci strings of length n. A Fibonacci string $b_1 b_2 \dots b_n$ is a Lucas string if $b_1 \cdot b_n = 0$. In other words, Lucas strings are binary strings

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that contain no consecutive ones circularly. The Lucas cube Λ_n, for $n \geq 1$ is the subgraph of Q_n induced by the Lucas strings of length n. It is well-known that $|V(\Gamma_n)| = F_{n+2}$, where F_n are the Fibonacci numbers: $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Similarly, $|V(\Lambda_n)| = L_n$ for $n \geq 1$, where L_n are the Lucas numbers: $L_0 = 2$, $L_1 = 1$, $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1$.

Let G be a graph. Set $D \subseteq V(G)$ is a dominating set if every vertex from $V(G)$ either belongs to D or is adjacent to some vertex from D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. A set $X \subseteq V(G)$ is called a 2-packing if $d(u, v) > 2$ for any two different vertices u and v of X. The 2-packing number $\rho(G)$ is the maximum cardinality of a 2-packing of G. It is well-known that for any graph G holds $\gamma(G) \geq \rho(G)$.

An independent set or stable set is a set of vertices in a graph, no two of which are adjacent. The independent domination number $i(G)$ of a graph G is the size of the smallest independent dominating set (or, equivalently, the size of the smallest maximal independent set). The minimum dominating set in a graph will not necessarily be independent, but the size of a minimum dominating set is always less than or equal to the size of a minimum maximal independent set, $\gamma(G) \leq i(G)$.

Pike and Zou in [9] obtained a lower bound for the domination number of Fibonacci cube of order n and determined the exact value of the domination number of Fibonacci cubes of order at most 8. Castro et al. in [1] obtained upper and lower bounds for the domination and 2-packing number of Fibonacci and Lucas cubes. Furthermore, the authors obtained the exact values for $\gamma(\Gamma_n)$ and $\gamma(\Lambda_n)$ for $n \leq 9$ and for $\rho(\Gamma_n)$ and $\rho(\Lambda_n)$ for $n \leq 10$.

In this paper we use integer programming method to compute the exact values of the domination, 2-packing and independent domination number of Fibonacci and Lucas cubes for $n \leq 13$, which resolves the conjecture from [1].

2 Main results

For each subset of the vertex set $S \subseteq V(G)$ define

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \in V \setminus S. \end{cases}$$

The neighborhood $N(v)$ of a vertex v in a graph G is the induced subgraph of G consisting of all vertices adjacent to v and all edges connecting two such vertices. Let $N[v] = N(v) \cup \{v\}$ denote the closed neighborhood of the vertex v.

The domination number of G can be formulated as the following $0-1$ integer programming problem:

$$\gamma(G) = \min \sum_{i=1}^{n} x_i \tag{2.1}$$

subject to

$$\sum_{j \in N[i]} x_j \geq 1, \tag{2.2}$$

$$x_i \in \{0, 1\}, \qquad \text{for all } 1 \leq i \leq n. \tag{2.3}$$

It is easy to see that the conditions (2.2) and (2.3) define dominating set S and vice versa [3]. For Fibonacci cube Γ_n this formulation has F_{n+2} variables and $2F_{n+2}$ constraints, while each condition from (2.2) contains at most n variables. For Lucas cube Λ_n this formulation has L_n variables and $2L_n$ constrains, while each condition from (2.2) contains at most n variables.

The 2-packing number of G can be formulated as the following $0-1$ integer programming problem:

$$\rho(G) = \max \sum_{i=1}^{n} x_i \tag{2.4}$$

subject to

$$\sum_{j \in N[i]} x_j \leq 1, \tag{2.5}$$

$$x_i \in \{0,1\}, \qquad \text{for all } 1 \leq i \leq n. \tag{2.6}$$

We will prove that the conditions (2.5) and (2.6) define 2-packing set S and vice versa. Let S be a 2-packing set. Since S does not contain two vertices on distance 1 or 2, for each $v \in V(G)$ there is at most one vertex from the closed neighborhood $N[v]$ which belongs to S. Assume now that the set S satisfies the condition (2.5) and let u and v be two vertices from S on distance 2. In that case for the shortest path vwu, we have $\sum_{j \in N[w]} x_j \geq 2$, which is impossible. Therefore, S is a 2-packing set.

The independent domination number G can be formulated as the following $0-1$ integer programming problem:

$$i(G) = \min \sum_{i=1}^{n} x_i \tag{2.7}$$

subject to

$$\sum_{j \in N[i]} x_j \geq 1, \tag{2.8}$$

$$(n-1)x_i + \sum_{j \in N(i)} x_j \leq n-1, \tag{2.9}$$

$$x_i \in \{0,1\}, \qquad \text{for all } 1 \leq i \leq n. \tag{2.10}$$

The conditions (2.8) and (2.10) define domination set S, while the condition (2.9) ensures the independence. For $x_i = 0$ we have always true $\sum_{j \in N(i)} x_j \leq n-1$, while for $x_i = 1$ we have $\sum_{j \in N(i)} x_j \leq 0$ which is equivalent to $\sum_{j \in N[i]} x_j = 1$. This proves that the formulation is correct. For Fibonacci cube Γ_n this formulation has F_{n+2} variables and $3F_{n+2}$ constraints, while each conditions from (2.8) and (2.9) contain at most n variables. For Lucas cube Λ_n this formulation has L_n variables and $3L_n$ constrains, while each condition from (2.8) and (2.9) contain at most n variables.

The tests were performed on the Intel Core 2 Duo T5800 2.0 GHz with 2 GB RAM running the Linux operating system and using CPLEX 8.1. The results are summarized in Tables 1 and 2. In Tables 3 and 4 we give some examples of dominating sets and 2-packing sets that were obtained during the computation of these values.

These results resolve the conjecture from [1] and support Problem 5.1 for $n \leq 12$.

n	1	2	3	4	5	6	7	8	9	10	11
$\lvert V(\Gamma_n)\rvert$	2	3	5	8	13	21	34	55	89	144	233
$\lvert E(\Gamma_n)\rvert$	1	2	5	10	20	38	71	130	235	420	744
$\gamma(\Gamma_n)$	1	1	2	3	4	5	8	12	17	25	
$\rho(\Gamma_n)$	1	1	2	2	3	5	6	9	14	20	29
$i(\Gamma_n)$	1	1	2	3	4	5	8	12	19	26	

Table 1: Parameters of small Fibonacci cubes.

n	1	2	3	4	5	6	7	8	9	10	11	12
$\lvert V(\Lambda_n)\rvert$	1	3	4	7	11	18	29	47	76	123	199	322
$\lvert E(\Lambda_n)\rvert$	0	2	3	8	15	30	56	104	189	340	605	1068
$\gamma(\Lambda_n)$	1	1	1	3	4	5	7	11	16	23	35	
$\rho(\Lambda_n)$	1	1	1	2	3	5	6	8	13	18	26	38
$i(\Lambda_n)$	1	1	1	3	4	5	8	11	17	24	35	

Table 2: Parameters of small Lucas cubes.

Dominating set	
$\Gamma(10)$	$\Lambda(11)$
(0, 1, 0, 1, 0, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 1, 0, 0, 0, 0)	(1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0), (1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0)
(1, 0, 1, 0, 0, 1, 0, 0, 0, 0), (1, 0, 0, 1, 0, 0, 1, 0, 0, 0)	(0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0), (0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0)
(0, 0, 0, 0, 1, 0, 1, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0)	(0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0)
(1, 0, 1, 0, 1, 0, 0, 1, 0, 0), (1, 0, 0, 1, 0, 1, 0, 1, 0, 0)	(1, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0), (0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0)
(1, 0, 0, 0, 0, 0, 0, 0, 1, 0), (0, 0, 1, 0, 0, 0, 0, 0, 1, 0)	(1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0)
(0, 1, 0, 0, 1, 0, 0, 0, 1, 0), (0, 0, 0, 1, 0, 1, 0, 0, 1, 0)	(0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0), (0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 0)
(0, 0, 1, 0, 0, 0, 1, 0, 1, 0), (0, 1, 0, 1, 0, 0, 1, 0, 1, 0)	(1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0)
(1, 0, 1, 0, 1, 0, 1, 0, 1, 0), (0, 0, 0, 1, 0, 0, 0, 0, 0, 1)	(0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0), (0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0)
(1, 0, 0, 0, 1, 0, 0, 0, 0, 1), (0, 0, 1, 0, 1, 0, 0, 0, 0, 1)	(0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0), (0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0)
(1, 0, 0, 0, 0, 1, 0, 0, 0, 1), (0, 1, 0, 0, 0, 0, 1, 0, 0, 1)	(1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0), (1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0)
(1, 0, 1, 0, 0, 0, 1, 0, 0, 1), (1, 0, 0, 0, 0, 0, 0, 1, 0, 1)	(1, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0), (0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0)
(0, 1, 0, 0, 1, 0, 0, 1, 0, 1), (0, 0, 1, 0, 0, 1, 0, 1, 0, 1)	(0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0), (1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0)
(0, 1, 0, 1, 0, 1, 0, 1, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1), (0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1)
	(0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1), (0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1)
	(0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 1), (0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1)
	(0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1), (0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1)
	(0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1), (0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1)
	(0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1)

Table 3: Examples of minimal dominating sets for $\Gamma(10)$ and $\Lambda(11)$

2-packaging set	
$\Gamma(11)$	$\Lambda(12)$
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), (1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), (1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0)
(0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0), (0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0)	(0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0), (1, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0)
(1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0), (0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0)	(0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0), (1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0)
(0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0), (0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0)	(0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0), (1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0)
(1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0), (0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0)	(0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0), (1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0)
(0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0), (0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0)	(0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0), (1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0)
(1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0), (0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0)	(0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 0), (0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0)
(0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0), (0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0)	(0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0), (1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0)
(1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0), (0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1)	(0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0), (1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0)
(1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1), (0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1)	(0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0), (1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0)
(0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1), (0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 1)	(1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0), (0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0)
(1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1), (0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1)	(0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0), (1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0)
(1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1), (1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1)	(0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0), (0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0)
(0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 1), (1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1)	(0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0), (0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1)
(1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1)	(0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1), (0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1)
	(0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1), (0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1)
	(0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1), (0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1)
	(0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1), (0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1)
	(0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1), (0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1)

Table 4: Examples of 2-packing sets for $\Gamma(11)$ and $\Lambda(12)$

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A characterization of plane Gauss paragraphs

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Abstract

Gauss first studied representations of self-intersecting curves in the plane using only lists of their crossings in the sequence as they occur when traversing a curve, i.e., representations using Gauss words. The characterisation of words that are Gauss words has been elusive for a long time, and only in recent decades have some good characterizations been established. Together with these, the interest in Gauss paragraphs, i.e., representations of sets of curves by sets of words listing their sequences of crossings, has came to light, and we are unaware of a (good) characterization of abstract sets of words that are Gauss paragraphs. We establish such a characterization and we show that characterizing Gauss paragraphs is algorithmically equivalent to characterizing Gauss words, as there exists a word W that can be obtained from a set of words P in linear time, such that P is a Gauss paragraph if and only if W is a Gauss word.

Keywords: Gauss words, Gauss codes, Gauss paragraphs, good characterization.

Math. Subj. Class.: 5C10, 57M15

1 Introduction

Gauss [5, 282-286] has studied representations of closed curves using lists of their crossings in the sequence obtained by following the curve. Clearly, each crossing appears exactly twice, and Gauss noticed that these two occurrences must have one an even and the other an odd index in the sequence, i.e. there has to be an odd number of letters between them. Gauss noted that the condition is not sufficient for curves with five or more crossings. The question of characterizing such words has not been solved until late 1960s, when Marx [9] and

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Treybig [13] gave algorithmic characterization of words that are Gauss words. Grünbaum [6] noted that they lack the aesthetic appeal of, for instance, Kuratowski theorem, an issue resolved by Lovász and Marx [8], who gave the first characterization satisfying Edmonds' criterion for "good characterization" [4].

Recently, the interest in *Gauss words*, i.e. the words that occur when the crossings of a self-intersecting curve are read in a sequence, has been renewed through several new good characterizations [3, 10, 11] and through introduction of Gauss paragraphs, sets of words corresponding in the same manner to sets of curves. The questions that arise in the bibliography are classified by Courcelle [2] into (i) Which (sets of) words over some alphabet are Gauss words (paragraphs), i.e. realizable as (sets of) (self)intersecting curves whose sequences of crossings are equal to specified (sets of) words, (ii) Which (sets of) curves can be uniquely reconstructed from their Gauss words (paragraphs) and (iii) What is the common structure of (sets of) curves having the same Gauss word (paragraph).

In our paper, we investigate the question (i) for Gauss paragraphs, and develop an efficient characterization of sets of words that can be realized with sets of (self)intersecting curves in the plane so that a Gauss paragraph of this set of curves equals the original set of words. The same problem was recently studied by Schellhorn [12], who extended virtual strings introduced by Turaev [14] from single close curve S^1 to sets of such curves and used them to characterize realizable Gauss paragraphs with a conjunction of seven technical conditions. In what follows, we give an elementary characterization that reduces the problem of realizability of a set of words to the problem of realizability of a single specific word obtainable from the set in linear time, avoiding the use of virtual strings. Besides showing that the problem of recognizing Gauss paragraphs is equivalent to recognizing Gauss words, the main improvement over Shcellhorn's characterization is the added algorithmic transparency.

2 Characterization of Gauss paragraphs

We first summarize some of the used notation. A *double-occurrence word* over an alphabet Σ is a word in which every letter of Σ appears exactly twice. The double-occurrence words that are Gauss words of some self-intersecting curve have been characterized by Rosenstiehl [10, 11] and de Fraysseix and de Mendez [3]. Rosenstiehl proved the following algebraic characterization of Gauss words.

Theorem 2.1. *[10, Theorem $2'$] A double-occurrence word W on a finite set Σ of letters is a Gauss word if, and only if,*

1. *any letter of W has an even number of interlaced letters;*
2. *any non-interlaced pair of letters has an even number of common interlaced letters;*
3. *the interlaced pairs having an even number of common interlaced letters form a separating set S, i.e. there exists $\Sigma' \subseteq \Sigma$, such that any pair of S has a letter of Σ' and a letter of $\Sigma \setminus \Sigma'$.*

The last condition of the theorem suggests it has a natural graph-theoretic formulation. We state it in terms of the *interlace-graph* G^W of a Gauss word W over the alphabet Σ, defined so that the letters of Σ are the vertex set, $V(G^W) = \Sigma$, and two vertices $u, v \in \Sigma$ are adjacent in G^W, $uv \in E(G^W)$, if and only if they interlace in W. A *cut* is a partition of the vertices of a graph into two disjoint subsets. Any cut determines a *cut-set*, the set of edges that have one endpoint in each subset of the partition.

Using these concepts, Theorem 2.1 can be stated as the following:

Theorem 2.2 ([3]). *Let W be a double-occurrence word over a finite alphabet Σ and let G^W be its interlace-graph. Then W is a Gauss word if and only if*

1. *each component of G^W is Eulerian;*
2. *if u and v are two nonadjacent vertices of G^W, then they have an even number of common neighbors;*
3. *the set $\{e = uv \mid u, v$ have an even number of common neighbors$\}$ is a cut-set in G^W.*

When studying sets of curves, a crossing may appear on different curves, so we need to relax the condition of double-occurrence. We define a *semi-double-occurence word* over an alphabet Σ to be a word, in which every letter of Σ appears at most twice. Then, a *double-occurence k-paragraph*[1] (shortly, *k-DOP*) over an alphabet Σ is a set of k semi-double-occurence words over Σ, such that each letter appears precisely twice in the union of all words of the paragraph. Further, a *mixed* crossing of a set of (self)intersecting curves in the plane is a crossing of two different curves, i.e. not a self-crossing of some word. Correspondingly, a *mixed* letter of a k-DOP P is a letter that appears in two different words of P. With $M(P)$ or just M, when the paragraph is clear from the context, we will denote the mixed letters of P.

Note that, in contrast to some knot-theoretic bibliography [1], our definition follows the original definition of Gauss, which does not encode over- or under-pass information that is required for knot-theoretic investigation. For us, the curves are embedded in the plane and each crossing is either a self-crossing of some curve, appearing twice in the same word, or is a crossing of two curves, appearing once in each corresponding word.

Finally, for a k-DOP $P = (w_1, \dots, w_k)$, we define its *intersection graph $G(P)$* as the graph whose vertices are words of P, $V(G(P)) = P$, in which two vertices are adjacent, iff the corresponding words share a letter of Σ.

Let P be k-DOP that contains $x \in M$. Then we will simplify notation and write $P = (xw_1, xw_2, \dots, w_k)$.

Lemma 2.3. *Let $P = (xw_1, xw_2, \dots, w_k)$ be a k-DOP and let $x \in M$ be a selected letter appearing in the first two words. Then P is a Gauss paragraph, if and only if the $(k-1)$-DOP $P^x = (xx'w_1xx'w_2, w_3, \dots, w_k)$ is a Gauss paragraph.*

Proof. Suppose first that P is a Gauss paragraph. Let π be a drawing that realizes P. In π, replace x in its small neighborhood by a digon xx' with incoming edges adjacent to x and outgoing to x' (see Figure 1). The resulting embedding is an embedding of $(k-1)$-DOP P^x, showing that P^x is a Gauss paragraph.

For the converse, suppose that P^x is a Gauss paragraph, realized in π. We will first prove that xx' is not a cut. Indeed, if xx' is a cut, then x and x' are not interlaced in π, a contradiction. Since x and x' induce a cycle and do not induce a cut, one of the faces of this cycle is empty and the other contains the full embedding. This implies that the out-edges and the in-edges come consecutive in the vertex rotation around the empty face. By

[1] As pointed out by one of the referees, a more natural name for this concept would be a sentence, as sentence is the next grammatical structure composed of words. Indeed we used *double-occurrence sentence and Gauss sentence* until a more thorough search through the bibliography [1] revealed that it was studied under the name *Gauss paragraph*.

contracting the empty face, x and x' become a single point. By rerouting the curves so that x is a crossing, we get a realization of P. □

Figure 1: Replacing x with digon xx' or vice versa.

We say that P^x from Theorem 2.3 is an x-reduction of P. With a sequence of reductions, we would like to obtain a single word. Let $x \in w_1 \cap w_2$. Since the letters appearing only in w_1 and w_2, after x-reduction appear in a common word, at most $(k-1)$ reductions reduce a Gauss paragraph to a single word, to which we can apply Theorem 2.2.

Let P be a k-DOP and $G(P)$ the intersection graph of semi-double-occurence words of P; its vertices are words and two words are adjacent if they have at least one letter in common. Let T be a tree in $G(P)$ and $w_1, \ldots, w_t$ the vertices of T, such that w_i has at most one neighbor in $\{w_{i+1}, \ldots, w_t\}$ and the connecting edge results from letter $m_i \in M$. Let $w^1 = w_1$. We define recursively $w^{i+1} = m_i m_i' w^i m_i m_i' w_{i+1}$, $i = 1, \ldots, t-1$. The T-reduction of P is $P^T = (w^t, w_{t+1}, \ldots, w_k)$. By induction, using the previous lemma as induction step, we get the following result:

Theorem 2.4. *Let $P = (w_1, \ldots, w_k)$ be a k-DOP and let T be a tree in $G(P)$ on t vertices $v_1, \ldots, v_t$, such that v_i has at most one neighbour in $\{v_{i+1}, \ldots v_t\}$. Then P is a Gauss paragraph, if and only if P^T is a Gauss paragraph.*

Applying this corollary to a spanning tree of G, we get the following characterization of k-DOPs that are Gauss paragraphs:

Corollary 2.5. *Let $P = (w_1, \ldots, w_k)$. Let T be a spanning tree in $G(P)$, and let W be the only word of P^T. Then P is a Gauss paragraph, iff W is a Gauss word, i.e. iff G^W satisfies the conditions of Theorem 2.2.*

It is clear that this corollary implies existence of a polynomial algorithm for determining whether a k-DOP is a Gauss paragraph, and thus satisfies Edmonds' criterion for a good characterization [4]: if A is the adjacency matrix of a graph G, then A^2 counts the number of length-two walks between any pair of vertices, i.e., the number of common neighbors, the crucial information required for verifying conditions of Theorem 2.2. The matrix A^2 can be computed in $O(|V(G)|^\omega)$ time, with $\omega < 2.376$, using fast matrix multiplication. This yields the dominating time-complexity term $O(|\Sigma|^\omega)$ of the k-DOP realizability verifying algorithm, as running time of the algorithm is dominated by the requirement to count the common neighbors of any pair of vertices of G^W, either adjacent or not. There are $|\Sigma| + k$ vertices of G^W, but the vertices x and x' have the same set of neighbors and are adjacent, hence with some preprocessing it suffices to check only a matrix of size $|\Sigma|$.

Note that the best known time complexity of exact counting of all triangles in a general graph with n vertices (which is equivalent to counting the common neighbors of just the adjacent pairs of graph's vertices) is $O(n^{\omega})$ [7], which indicates that the time complexity of checking realizability of a given k-DOW using conditions of Theorem 2.2 can hardly be improved, unless some detailed properties of the graph G^W are exploited in counting the common neighbors. However, as constructing the graph $G(P)$ can be done in time $O(|\Sigma|)$, its spanning tree T found in $O(k)$, and the T-reduction of P found in $O(k + |\Sigma|)$, then any improvement in checking realizability of a double-occurring word immediately translates into an improvement of checking realizability of k-DOP.

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On skew Heyting algebras

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Abstract

In the present paper we generalize the notion of a Heyting algebra to the non-commutative setting and hence introduce what we believe to be the proper notion of the implication in skew lattices. We list several examples of skew Heyting algebras, including Heyting algebras, dual skew Boolean algebras, conormal skew chains and algebras of partial maps with poset domains.

Keywords: Skew lattices, Heyting algebras, non-commutative algebra, intuitionistic logic.

Math. Subj. Class.: 06F35, 03G27

1 Introduction

Non-commutative generalizations of lattices were introduced by Jordan [11] in 1949. The current approach to such objects began with Leech's 1989 paper on skew lattices [13]. Similarly, skew Boolean algebras are non-commutative generalizations of Boolean algebras. In 1936 Stone proved that each Boolean algebra can be embedded into a field of sets [20]. Likewise, Leech showed in [14, 15] that each right-handed skew Boolean algebra can be embedded into a generic skew Boolean algebra of partial functions from a given set to the codomain $\{0, 1\}$. Bignall and Leech [5] showed that skew Boolean algebras play a central role in the study of discriminator varieties.

Though not using categorical language, Stone essentially proved in [20] that the category of Boolean algebras and homomorphisms is dual to the category of Boolean topological spaces and continuous maps. Generalizations of this result within the commutative setting yield Priestley duality [16, 17] between bounded distributive lattices and Priestley spaces, and Esakia duality [9] between Heyting algebras and Esakia spaces. (See [4] for details.) In a recent paper [10] on Esakia's work, Gehrke showed that Heyting algebras may be understood as those distributive lattices for which the embedding into their Booleanisation has a right adjoint. A recent line of research generalized the results of Stone and

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Priestley to the non-commutative setting. By results in [1] and [12], any skew Boolean algebra is dual to a sheaf of rectangular bands over a locally-compact Boolean space. A further generalization given in [2] showed that any strongly distributive skew lattice (as defined below) is dual to a sheaf (of rectangular bands) over a locally compact Priestley space.

While Boolean algebras provide algebraic models of classical logic, Heyting algebras provide algebraic models of intuitionistic logic. In the present paper we introduce the notion of a skew Heyting algebra. In passing to the non-commutative setting one needs to sacrifice either the top or the bottom of the algebra in order not to end up in the commutative setting. In the previous papers [1], [12] and [2] algebras with bottoms were considered, and hence the notion of distributivity was generalized to the notion of so-called strong distributivity. If one tried to define an implication operation in the setting of strongly distributive skew lattices with a bottom as a right adjoint to conjunction, that would force the skew lattice to also possess a top and hence be commutative, resulting in a usual Heyting algebra. In order to define implication in the skew lattice setting we consider the $\vee - \wedge$ duals of strongly distributive skew lattices with a bottom, namely, the co-strongly distributive skew lattices with a top. That is not surprising as a top plays a crucial role in logic. The category of co-strongly distributive skew lattices with a top is, of course, isomorphic to the category of strongly distributive skew lattices with a bottom. In choosing co-strongly distributive skew lattices with a top we follow the path paved by Bignall and Spinks in [6], and by Spinks and Veroff in [19] where dual skew Boolean algebras were introduced. For further reading on implications in skew Boolean algebras and their algebraic duals, see [7].

After reviewing some preliminary definitions and concepts in Section 2, in the next section we introduce the notion of a skew Heyting algebra, prove that such algebras form a variety and show that the maximal lattice image of a skew Heyting algebra is a generalized Heyting algebra (possibly without a bottom). Indeed, a co-strongly distributive skew lattice with a top is the reduct of a skew Heyting algebra, if and only if its maximal lattice image forms a generalized Heyting algebra. (See Theorem 3.5.) This leads to a number of useful corollaries and examples. We finish with Section 4 where we explore the consequences of our results to duality theory, and describe how skew Heyting algebras correspond to sheaves over local Esakia spaces.

2 Preliminaries

A *skew lattice* is an algebra $\mathbf{S} = (S; \wedge, \vee)$ of type $(2, 2)$ such that $\wedge$ and $\vee$ are both idempotent and associative and they satisfy the following absorption laws:

$$x \wedge (x \vee y) = x = x \vee (x \wedge y) \text{ and } (x \wedge y) \vee y = y = (x \vee y) \wedge y.$$

These identities are collectively equivalent to the pair of equivalences: $x \wedge y = x \Leftrightarrow x \vee y = y$ and $x \wedge y = y \Leftrightarrow x \vee y = x$.

On a skew lattice $\mathbf{S}$ one can define the *natural partial order* by stating that $x \leq y$ if and only if $x \vee y = y = y \vee x$, or equivalentely $x \wedge y = z = y \wedge x$, and the *natural preorder* by $x \preceq y$ if and only if $y \vee x \vee y = y$, or equivalentely $x \wedge y \wedge x = x$. *Green's equivalence relation $\mathcal{D}$* is then defined by

$$x \mathcal{D} y \text{ if and only if } x \preceq y \text{ and } y \preceq x. \tag{2.1}$$

Lemma 2.1. *([8]). For elements x and y elements of a skew lattice $\mathbf{S}$ the following are equivalent:*

(i) $x \leq y$,

(ii) $x \vee y \vee x = y$,

(iii) $y \wedge x \wedge y = x$.

Leech's First Decomposition Theorem for skew lattices states that the relation $\mathcal{D}$ is a congruence on a skew lattice $\mathbf{S}$, $\mathbf{S}/\mathcal{D}$ is the maximal lattice image of $\mathbf{S}$, and each congruence class is a maximal rectangular skew lattice in S [13]. Rectangular skew lattices are characterized by $x \wedge y \wedge z = x \wedge z$, or equivalentely $x \vee y \vee z = x \vee z$. We denote the $\mathcal{D}$-class containing an element x by $\mathcal{D}_x$.

It was proved in [13] that a skew lattice always forms a *regular band* for either of the operations $\wedge$, $\vee$, i.e. it satisfies the identities

$$x \wedge u \wedge x \wedge v \wedge x = x \wedge u \wedge v \wedge x \text{ and } x \vee u \vee x \vee v \vee x = x \vee u \vee v \vee x.$$

A *skew lattice with top* is an algebra $(S; \wedge, \vee, 1)$ of type $(2, 2, 0)$ such that $(S; \wedge, \vee)$ is a skew lattice and $x \vee 1 = 1 = 1 \vee x$, or equivalently $x \wedge 1 = x = 1 \wedge x$, holds for all $x \in S$. A skew lattice with bottom is defined dually and the bottom, if it exists, is usually denoted by 0.

Furthermore, a skew lattice is called *strongly distributive* if it satisfies the following identities:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z);$$

and it is called *co-strongly distributive* if it satisfies the identities:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \text{ and } (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z).$$

If a skew lattice $\mathbf{S}$ is either strongly distributive or co-strongly distributive then $\mathbf{S}$ is *distributive* in that it satisfies the identities

$$x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x) \text{ and } x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x).$$

A skew lattice $\mathbf{S}$ that is jointly strongly distributive and co-strongly distributive is *binormal*, i.e. $\mathbf{S}$ factors as a direct product of a lattice $\mathbf{L}$ and a rectangular skew lattice $\mathbf{B}$, $\mathbf{S} \cong \mathbf{L} \times \mathbf{B}$, with $\mathbf{L}$ in this case being distributive. (See [15] and [18].)

Applying duality to a result of Leech [15], it follows that a skew lattice S is co-strongly distributive if and only if $\mathbf{S}$ is jointly:

- *quasi-distributive*: the maximal lattice image $\mathbf{S}/\mathcal{D}$ is a distributive lattice,
- *symmetric*: $x \wedge y = y \wedge x$ if and only if $x \vee y = y \vee x$, and
- *conormal*: $x \vee y \vee z \vee w = x \vee z \vee y \vee w$.

If a skew lattice is conormal then given any $u \in S$ the set

$$u\uparrow = \{u \vee x \vee u \,|\, x \in S\} = \{x \in S \,|\, u \leq x\}$$

forms a (commutative) lattice for the induced operations $\wedge$ and $\vee$, cf. [15].

The following lemma is the dual of a well known result in skew lattice theory.

Lemma 2.2. *Let* $\mathbf{S}$ *be a conormal skew lattice and let* A *and* B *be* $\mathcal{D}$*-classes such that* $B \leq A$ *holds in the lattice* $\mathbf{S}/\mathcal{D}$*. Given* $b \in B$ *there exists a unique* $a \in A$ *such that* $b \leq a$*.*

Proof. First the uniqueness. If a and a' both satisfy the desired property then by Lemma 2.1 we have $a = b \vee a \vee b$ and likewise $a' = b \vee a' \vee b$. Now, using idempotency of $\vee$, conormality and the fact that $a \,\mathcal{D}\, a'$ we obtain:

$$a = b \vee a \vee b = b \vee a \vee a' \vee a \vee b = \\ b \vee a \vee a' \vee b = b \vee a' \vee a \vee a' \vee b = b \vee a' \vee b = a'.$$

To prove the existence of a take any $x \in A$ and set $a = b \vee x \vee b$. Then $a \in A$ and using the idempotency of $\vee$ we get:

$$b \vee a \vee b = b \vee (b \vee x \vee b) \vee b = b \vee x \vee b = a$$

which implies $b \leq a$. □

An important class of strongly distributive skew lattices that have a bottom is the class of skew Boolean algebras where by a *skew Boolean algebra* we mean an algebra $\mathbf{S} = (S; \wedge, \vee, \setminus, 0)$ where $(S; \wedge, \vee, 0)$ is a strongly distributive skew lattice with bottom 0, and $\setminus$ is a binary operation on S such that both $(x \wedge y \wedge x) \vee (x \setminus y) = x = (x \setminus y) \vee (x \wedge y \wedge x)$ and $(x \wedge y \wedge x) \wedge (x \setminus y) = 0 = (x \setminus y) \wedge (x \wedge y \wedge x)$. Given any element u of a skew Boolean algebra $\mathbf{S}$ the set

$$u{\downarrow} = \{u \wedge x \wedge u \,|\, x \in S\} = \{x \in S \,|\, x \leq u\}$$

is a Boolean algebra with top u and with $u \setminus x$ being the complement of $u \wedge x \wedge u$ in $u{\downarrow}$.

Recall that a *Heyting algebra* is an algebra $\mathbf{H} = (H; \wedge, \vee, \to, 1, 0)$ such that $(H, \wedge, \vee, 1, 0)$ is a bounded distributive lattice that satisfies the condition:

(HA) $x \wedge y \leq z$ iff $x \leq y \to z$.

Stated otherwise, $\forall y, z \in H$ the sublattice $\{x \in H \,|\, x \wedge y \leq z\}$ is nonempty and contains a top element to be denoted by $y \to z$. Thus, given a bounded distributive lattice $(H; \wedge, \vee, 1, 0)$, if a binary operation $\to$ exists that makes $(H; \wedge, \vee, \to, 1, 0)$ a Heyting algebra, then it is unique because it is already there implicitly. Indeed, given two isomorphic lattices, if either is the lattice reduct of a Heyting algebra then so is the other, and both are isomorphic as Heyting algebras.

Equivalently, (HA) can be replaced by the following set of identities:

(H1) $(x \to x) = 1$,

(H2) $x \wedge (x \to y) = x \wedge y$,

(H3) $y \wedge (x \to y) = y$,

(H4) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

Lemma 2.3. *In any Heyting algebra,* $x \to y = (x \vee y) \to y$*.*

A *generalized Heyting algebra* is an algebra $\mathbf{A} = (A; \wedge, \vee, \to, 1)$ such that the reduct $(A, \wedge, \vee, 1)$ is a distributive lattice with top 1, and condition (HA) holds. If it also has a bottom, it is a Heyting algebra. In general, each upset $u{\uparrow}$ forms a Heyting algebra. The identities above also characterize this more general class of algebras, which are often called *Brouwerian algebras*.

3 Skew Heyting algebras

A *skew Heyting lattice* is an algebra $\mathbf{S} = (S; \wedge, \vee, 1)$ of type $(2, 2, 0)$ such that:

- $(S; \wedge, \vee, 1)$ is a co-strongly distributive skew lattice with top 1. Each upset $u\sup$ is thus a bounded distributive lattice.
- for any $u \in S$ an operation $\to_u$ can be defined on $u{\uparrow}$ such that $(u{\uparrow}; \wedge, \vee, \to_u, 1, u)$ is a Heyting algebra with top 1 and bottom u.

Given a skew Heyting lattice $\mathbf{S}$, define $\to$ on S by setting

$$x \to y = (y \vee x \vee y) \to_y y.$$

A *skew Heyting algebra* is an algebra $\mathbf{S} = (S; \wedge, \vee, \to, 1)$ of type $(2, 2, 2, 0)$ such that $(S; \wedge, \vee, 1)$ is a skew Heyting lattice and $\to$ is the implication thus induced. A sense of global coherence for $\to$ on S is given by:

Lemma 3.1. *Let $\mathbf{S}$ be a skew Heyting lattice with $\to$ as defined above and let $x, y, u \in S$ be such that both $x, y \in u{\uparrow}$ hold. Then $x \to y = x \to_u y$.*

Proof. As x and y both lie in $u{\uparrow}$, they commute. By the definition of $\to$, $x \to y = (x \vee y) \to_y y \geq y$ by (H3). On the other hand, since $\to_u$ is the Heyting implication in the Heyting algebra $u{\uparrow}$ it follows that $x \to_u y = (x \vee y) \to_u y \geq y$. Thus y, $x \vee y$, $(x \vee y) \to_y y$ and $(x \vee y) \to_u y$ all lie iin the Heyting algebra $y{\uparrow}$. The maximal element characterization of both $(x \vee y) \to_y y$ and $(x \vee y) \to_u y$ forces both to agree. $\square$

We will use the axioms of Heyting algebras to derive an axiomatization of skew Heyting algebras. The reader should find most of the axioms of Theorem 3.2 below to be intuitively clear generalizations to the non-commutative case. However, two axioms should be given further explanation. Firstly, the u in axiom (SH4) below appears since upon passing to the non-commutative case, an element that is both below x and y with respect to the partial order $\leq$ no longer need exist. (We can have $x \wedge y \wedge x \leq x$ but not $x \wedge y \wedge x \leq y$ in general.) Similarly, axiom (SH0) is needed since in the non-commutative case it no longer follows from the other axioms, the reason being that in general $x \leq y \vee x \vee y$ need not hold.

Theorem 3.2. *Let $(S; \wedge, \vee, \to, 1)$ be an algebra of type $(2, 2, 2, 0)$ such that $(S; \wedge, \vee, 1)$ is a co-strongly distributive skew lattice with top 1. Then $(S; \wedge, \vee, \to, 1)$ is a skew Heyting algebra if and only if it satisfies the following axioms:*

(SH0) $x \to y = (y \vee x \vee y) \to y$.

(SH1) $x \to x = 1$,

(SH2) $x \wedge (x \to y) \wedge x = x \wedge y \wedge x$,

(SH3) $y \wedge (x \to y) = y$ *and* $(x \to y) \wedge y = y$,

(SH4) $x \to (u \vee (y \wedge z) \vee u) = (x \to (u \vee y \vee u)) \wedge (x \to (u \vee z \vee u))$.

Proof. Assume that $\mathbf{S}$ is a skew Heyting algebra.

(SH0). Both $x \to y$ and $(y \vee x \vee y) \to y$ are defined as $(y \vee x \vee y) \to_y y$. Thus they are equal.

(SH1). This is true because $1 \wedge x = x$ is true in $x{\uparrow}$.

(SH2). In $y\uparrow$ (H2) implies $(y \vee x \vee y) \wedge ((y \vee x \vee y) \to_y y) = (y \vee x \vee y) \wedge y = y$. Hence

$$x \wedge (y \vee x \vee y) \wedge (x \to y) \wedge x = x \wedge y \wedge x.$$

On the other hand,

$$x \wedge (y \vee x \vee y) \wedge (x \to y) \wedge x = x \wedge (y \vee x \vee y) \wedge x \wedge (x \to y) \wedge x = x \wedge (x \to y) \wedge x,$$

where we have used the regularity of $\wedge$ and the fact that $x \preceq y \vee x \vee y$.

(SH3). Both identities hold because $y \wedge (y \vee x \vee y) = y$ in $y\uparrow$. Thus $x \to y = (y \vee x \vee y) \to y \geq y$.

(SH4). First note that (SH4) is equivalent to

(SH4') $(u \vee x \vee u) \to (u \vee (y \wedge z) \vee u) = ((u \vee x \vee u) \to (u \vee y \vee u)) \wedge ((u \vee x \vee u) \to (u \vee z \vee u))$.

Indeed, (SH0) and the conormality of $\vee$ give both

$$(u \vee x \vee u) \to (u \vee w \vee u) = (u \vee x \vee w \vee u) \to (u \vee w \vee u)$$

and

$$x \to (u \vee w \vee u) = (u \vee x \vee w \vee u) \to (u \vee w \vee u)$$

so that

$$x \to (u \vee w \vee u) = (u \vee x \vee u) \to (u \vee w \vee u).$$

Hence it suffices to prove that (SH4') holds.

Observe that distributivity implies

$$(u \vee y \vee u) \wedge (u \vee z \vee u) = u \vee (y \wedge z) \vee u. \tag{3.1}$$

Since $u \vee x \vee u$, $u \vee y \vee u$, $u \vee z \vee u$ and $u \vee (y \wedge z) \vee u$ all lie in $u\uparrow$ we can simply compute in $u\uparrow$. Using (3.1) and axiom (H4) for Heyting algebras we obtain: $(u \vee x \vee u) \to (u \vee (y \wedge z) \vee u) = (u \vee x \vee u) \to ((u \vee y \vee u) \wedge (u \vee z \vee u)) = ((u \vee x \vee u) \to (u \vee y \vee u)) \wedge ((u \vee x \vee u) \to (u \vee z \vee u))$.

To prove the converse assume that (SH0)–(SH4) hold. Given arbitrary $u \in S$ and $x, y, z \in u\uparrow$ set $x \to_u y = x \to y$. Axiom (SH3) implies that $x \to y \in y\uparrow \subseteq u\uparrow$. Thus the restriction $\to_u$ of $\to$ to $u\uparrow$ is well defined. Since $u\uparrow$ is commutative with bottom u, axioms (SH1)–(SH4) for $\to$ respectively simplify to (H1)–(H4) for $\to_u$, making $\to_u$ the Heyting implication on $u\uparrow$. Axiom (SH0) assures that $\to$ is indeed the skew Heyting implication satisfying $a \to b = (b \vee a \vee b) \to_b b$, for any $a, b \in S$. $\square$

Corollary 3.3. *Skew Heyting algebras form a variety.*

In the remainder of the paper, given a skew Heyting algebra we shall simplify the notation $\to_u$ to $\to$ when referring to the Heyting implication in $u\uparrow$. Remarks made about Heyting algebras in Section 2 apply here also. Given a co-strongly distributive skew lattice $(S; \wedge, \vee, 1)$ with a top 1, if a binary operation $\to$ exists that makes $(S; \wedge, \vee, \to, 1)$ a skew Heyting algebra, then it is unique since it is already there implicitly. Hence, given two isomorphic skew lattices, if either is the reduct of a skew Heyting algebra, then so is the other and both are isomorphic as skew Heyting algebras.

Proposition 3.4. *The relation $\mathcal{D}$ defined in* (2.1) *is a congruence on any skew Heyting algebra.*

Proof. Let $(S; \wedge, \vee, \to, 1)$ be a skew Heyting algebra. Since $\mathcal{D}$ is a congruence for co-strongly distributive skew lattices with a top, we only need to prove $(a \to b)\ \mathcal{D}\ (c \to d)$ holds for any $a, b, c, d \in S$ satisfying $a\ \mathcal{D}\ c$ and $b\ \mathcal{D}\ d$. Without loss of generality we may assume $b \leq a$ and $d \leq c$. (Otherwise replace a by $b \vee a \vee b$ and c by $d \vee c \vee d$.)

To begin, define a map $\varphi : b\!\uparrow\ \to d\!\uparrow$ by setting $\varphi(x) = d \vee x \vee d$. We claim that φ is a lattice isomorphism of $(b\!\uparrow; \wedge, \vee)$ with $(d\!\uparrow; \wedge, \vee)$, with inverse $\psi : d\!\uparrow\ \to b\!\uparrow$ given by $\psi(y) = b \vee y \vee b$. It is easily seen that φ and ψ are inverses of each other. For instance, $\psi(\varphi(x)) = b \vee d \vee x \vee d \vee b$ equals $(b \vee d \vee b) \vee x \vee (b \vee d \vee b)$ by the regularity of $\vee$. But the latter reduces to $b \vee x \vee b$ because $b\mathcal{D}d$, and since $x \in b\!\uparrow$ it reduces further to x by Lemma 2.1, giving $\psi(\varphi(x)) = x$. φ must preserve $\wedge$ and $\vee$. Indeed distributivity gives:

$$\varphi(x \wedge x') = d \vee (x \wedge x') \vee d = (d \vee x \vee d) \wedge (d \vee x' \vee d) = \varphi(x) \wedge \varphi(x').$$

And the regularity gives:

$$\varphi(x \vee x') = d \vee (x \vee x') \vee d = (d \vee x \vee d) \vee (d \vee x' \vee d) = \varphi(x) \vee \varphi(x').$$

Thus φ (and ψ) is a lattice isomorphism of $b\!\uparrow$ with $d\!\uparrow$. But then φ and ψ are also isomorphisms of Heyting algebras. That is, e.g., $\varphi(x \to y) = \varphi(x) \to \varphi(y)$.

Next, observe that $x\ \mathcal{D}\ \varphi(x)$ for all $x \in b\!\uparrow$. Indeed, regularity gives:

$$\varphi(x) \vee x \vee \varphi(x) = (d \vee x \vee d) \vee x \vee (d \vee x \vee d) = d \vee x \vee d = \varphi(x)$$

and likewise $x \vee \varphi(x) \vee x = \psi(\varphi(x)) \vee \varphi(x) \vee \psi(\varphi(x)) = \psi(\varphi(x)) = x$. There is more: a is the unique element in its $\mathcal{D}$-class belonging to $b\!\uparrow$ and c is the unique element in the same $\mathcal{D}$-class belonging to $d\!\uparrow$ (since each upset $u\!\uparrow$ intersects any $\mathcal{D}$-class in at most one element). But $\varphi(a)$ in $d\!\uparrow$ behaves in the manner just like c, and so $\varphi(a) = c$. Since $b\ \mathcal{D}\ d$, $\varphi(b) = d \vee b \vee d = d$ and $\varphi(a \to b) = \varphi(a) \to \varphi(b) = c \to d$, thus giving $a \to b\ \mathcal{D}\ c \to d$. $\square$

Following [5] a *commutative subset* of a symmetric skew lattice is a non-empty subset whose elements both join and meet commute.

Theorem 3.5. *Given a co-strongly distributive skew lattice $(S; \wedge, \vee, 1)$ with top 1, an operation $\to$ can be defined on S making $(S; \wedge, \vee, \to, 1)$ a skew Heyting algebra if and only if the operation $\to$ can be defined on $S/\mathcal{D}$ making $(S/\mathcal{D}; \wedge, \vee, \to, \mathcal{D}_1)$ a generalized Heyting algebra.*

Proof. To begin, for any u in S, the upset $u\!\uparrow$ is a $\mathcal{D}$-class transversal of the principal filter $S \vee u \vee S$. Secondly, the induced homomorphism $\varphi : S \to S/\mathcal{D}$ is bijective on any commutative subset of S since distinct commuting elements of S lie in distinct $\mathcal{D}$-classes. It follows that for each u in S, φ restricts to an isomorphism of upsets, $\varphi_u : u\!\uparrow\ \cong \varphi(u)\!\uparrow$. Thus each upset $u\!\uparrow$ in S forms a Heyting algebra if and only if each upset $v\!\uparrow$ in $S/\mathcal{D}$, being some $\varphi(u)\!\uparrow$, must form a Heyting algebra. The theorem follows. $\square$

Comment. This result is a near-dual of the important fact that a strongly distributive skew lattice $\mathbf{S}$ with bottom 0 is the (necessarily unique) reduct of a skew Boolean algebra if and only if its lattice image $\mathbf{S}/\mathcal{D}$ is the reduct of a (necessarily unique) generalized Boolean algebra. ([15], 3.8.)

We next consider consequences of the above theorem. The first is on the "skew lattice side" of things and the next is more on the "Heyting side". But first recall the definitions of Green's relations $\mathcal{L}$ and $\mathcal{R}$ on a skew lattice:

$$x\mathcal{L}y \Leftrightarrow (x \wedge y = x \,\&\, y \wedge x = y, \text{ or equivalently } x \vee y = y \,\&\, y \vee x = x),$$
$$x\mathcal{R}y \Leftrightarrow (x \wedge y = y \,\&\, y \wedge x = x, \text{ or equivalently } x \vee y = x \,\&\, y \vee x = y).$$

Relations $\mathcal{L}$ and $\mathcal{R}$ are contained in the Green's relation $\mathcal{D}$ defined above and $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} = \mathcal{D}$ holds. A skew lattice is called *right-handed* if the relation $\mathcal{L}$ is trivial, in which case $\mathcal{D} = \mathcal{R}$, and it is called *left-handed* if the relation $\mathcal{R}$ is trivial, in which case $\mathcal{D} = \mathcal{L}$. By Leech's Second Decomposition Theorem [13] the relations $\mathcal{L}$ and $\mathcal{R}$ are congruences on any skew lattice $\mathbf{S}$, $\mathbf{S}/\mathcal{R}$ is left-handed, $\mathbf{S}/\mathcal{L}$ is right-handed and the following diagram is a pullback:

$$\begin{tikzcd}
\mathbf{S} \arrow[r] \arrow[d] & \mathbf{S}/\mathcal{R} \arrow[d] \\
\mathbf{S}/\mathcal{L} \arrow[r] & \mathbf{S}/\mathcal{D}
\end{tikzcd}$$

Corollary 3.6. *If $\mathbf{S} = (S; \wedge, \vee, 1)$ be a co-strongly distributive skew lattice with top 1, then the following are equivalent:*

1. *$\mathbf{S}$ is the reduct of a skew Heyting algebra $(S; \wedge, \vee, \rightarrow, 1)$.*
2. *$\mathbf{S}/\mathcal{L}$ is the reduct of a skew Heyting algebra $(S/\mathcal{L}; \wedge, \vee, \rightarrow, 1)$.*
3. *$\mathbf{S}/\mathcal{R}$ is the reduct of a skew Heyting algebra $(S/\mathcal{R}; \wedge, \vee, \rightarrow, 1)$.*

Moreover, both $\mathcal{L}$ and $\mathcal{R}$ are congruences on any skew Heyting algebra.

Proof. The equivalence of (i)–(iii) is due to the preceding theorem plus the fact that $\mathbf{S}/\mathcal{D}$, $(\mathbf{S}/\mathcal{L})/\mathcal{D}_{S/\mathcal{L}}$ and $(\mathbf{S}/\mathcal{R})/\mathcal{D}_{S/\mathcal{R}}$ are isomorphic. Next, the induced map $\rho : \mathbf{S} \to \mathbf{S}/\mathcal{L}$ is at least a homomorphism of co-strongly distributive skew lattices. By the argument of the preceding theorem, it induces isomorphisms between corresponding pairs of upsets, $u{\uparrow}$ in S and $\mathcal{L}_u{\uparrow}$ in $S/\mathcal{L}$. Thus given $x \to y = (y \vee x \vee y) \to_y y$ and $u \to v = (v \vee u \vee v) \to_v v$ with $x;,\mathcal{L};,u$ and $y;,\mathcal{L};,v$ in S, both $(y \vee x \vee y) \to_y y$ and $(v \vee u \vee v) \to_v v$ are mapped to the common $\mathcal{L}_{y \vee x \vee y} \to_{\mathcal{L}_y} \mathcal{L}_y$, making $x \to y;,\mathcal{L};,y \to v$ in $\mathbf{S}$. A similar argument applies to the induced map $\lambda : \mathbf{S} \to \mathbf{S}/\mathcal{R}$. $\square$

An alternative to the characterization of Theorem 3.2 is given by:

Corollary 3.7. *Every skew Heyting algebra satisfies:*

(SHA) $x \preceq y \to z$ *if and only if* $x \wedge y \preceq z$.

In particular, $x \to y = 1$ *iff* $x \preceq y$.

In general, an algebra $\mathbf{S} = (S; \wedge, \vee, \rightarrow, 1)$ of type $(2, 2, 2, 0)$ is a skew Heyting algebra if the following conditions hold:

1. *The reduct $(S; \wedge, \vee, 1)$ is a co-strongly distributive skew lattice with top 1.*
2. *$y \leq x \to y$ holds for all $x, y \in S$.*

3. $\mathbf{S}$ *satisfies axiom (SHA).*

Proof. Given that S is a skew Heyting algebra, since the induced epimorphism $\varphi : S \to S/\mathcal{D}$ is a homomorphism of skew Heyting algebras we have

$$x \preceq y \to z \text{ iff } \varphi(x) \leq \varphi(y) \to \varphi(z) \text{ iff } \varphi(x) \wedge \varphi(y) \leq \varphi(z) \text{ iff } x \wedge y \preceq z.$$

Conversely, let $\mathbf{S} = (S; \wedge, \vee, \to, 1)$ be an algebra of type $(2,2,2,0)$ satisfying (1)–(3). Suppose that x, y, z lie in a common upset $u \uparrow$. Since $\preceq$ is just $\leq$ in $u \uparrow$ nad $y \to z$ lies in $u \uparrow$ by (2) we have $x \leq y \to z$ iff $x \wedge y \leq z$ in $u \uparrow$. $(S; \wedge, \vee, 1)$ is thus at least a skew Heyting lattice. Now consider the derived implication $\to^*$ given by $x \to^* y = (y \vee x \vee y) \to_y y$. Both $y \to z$ and $y \to^* z$ satisfy (SHA) and thus are $\mathcal{D}$-equivalent. But since both lie in the sublattice $z\uparrow$, they must be equal. □

We have seen that each skew Heyting algebra is basically a co-strongly distributive skew lattice $\mathbf{S}$ with top, say 1, for which $\mathbf{S}/\mathcal{D}$ is a generalized Heyting algebra, in which case the Heyting structure on each upset $u\uparrow$ of $\mathbf{S}$ is obtained from that of the isomorphic upset $\mathcal{D}_u\uparrow$ in $\mathbf{S}/\mathcal{D}$. This suggests that all standard classes of generalized Heyting algebras yield classes of skew Heyting algebras whose maximal commutative images belong to the particular class. We consider several cases.

Case 1. Finite distributive lattices possess a well-defined Heyting algebra structure. Thus any finite co-strongly distributive skew lattice with a top, or more generally any co-strongly distributive skew lattice with a top and a finite maximal lattice image is the reduct of a unique skew Heyting algebra.

Case 2. All chains possessing a top 1 form Heyting algebras. Here things are simple:

$$x \to y = \begin{cases} 1; & \text{if } x \leq y. \\ y; & \text{otherwise.} \end{cases}$$

Thus all co-strongly distributive skew chains with a top are skew Heyting algebra reducts, where a *skew chain* is any skew lattice $\mathbf{S}$ where $\mathbf{S}/\mathcal{D}$ is a chain, i.e., $\preceq$ is a total preorder on $\mathbf{S}$. Here, given x, y in a common $u\uparrow$ one has:

$$x \to y = \begin{cases} 1; & \text{if } x \preceq y. \\ y; & \text{otherwise.} \end{cases}$$

Case 3. Dual generalized Boolean algebras. These are algebras $\mathbf{S} = (S; \wedge, \vee, \backslash\backslash, 1)$ where $(S; \wedge, \vee, 1)$ is a distributive lattice with top 1 and $\backslash\backslash$ is a binary operation on S such that $(y \vee x) \vee (y \backslash\backslash x) = 1$ and $(y \vee x) \wedge (y \backslash\backslash x) = y$ for all x, y in S. As with $\backslash$ for generalized Boolean algebras, $\backslash\backslash$ is uniquely determined. Moreover, in this case, $x \to y = y \backslash\backslash x$. A dual-skew Boolean algebra $\mathbf{S} = (S; \wedge, \vee, \backslash\backslash, 1)$ is an algebra such that $(S; \wedge, \vee, 1)$ is a co-strongly distributive skew lattice with top 1 and binary operation $\backslash\backslash$ such that:

$$(y \vee x \vee y) \vee (y \backslash\backslash x) = 1 = (y \backslash\backslash x) \vee (y \vee x \vee y);$$
$$(y \vee x \vee y) \wedge (y \backslash\backslash x) = y = (y \backslash\backslash x) \wedge (y \vee x \vee y).$$

The relevant diagram is:

$$\begin{tikzcd}[column sep=small] & 1 & \\ y \vee x \vee y \arrow[ur, dash] & & y \mathbin{\backslash\!\backslash} x \arrow[ul, dash] \\ & y \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

These dual algebras are, of course, precisely the co-strongly distributive skew lattices with a top whose maximal lattice images are the lattice reducts of dual-generalized Boolean algebras. Once again we follow the commutative case: $x \to y = y \mathbin{\backslash\!\backslash} x$ which now is $y \mathbin{\backslash\!\backslash} (y \vee x \vee y)$ in $y{\uparrow}$.

We thus have:

Corollary 3.8. *A co-strongly distributive skew lattice with a top* $\mathbf{S} = (S; \wedge, \vee, 1)$ *is the reduct of a uniquely determined skew Heyting algebra* $(S; \wedge, \vee, \backslash\!\backslash, 1)$ *if any one of the following conditions holds:*

1. $\mathbf{S}/\mathcal{D}$ *is finite.*

2. $\mathbf{S}$ *is a skew chain.*

3. $\mathbf{S}$ *is the reduct of a dual generalized Boolean algebra,* $\mathbf{S} = (S; \wedge, \vee, \backslash\!\backslash, 1)$.

Implicit in Case 3 is a basic duality that occurs for skew lattices. Given a skew lattice $\mathbf{S} = (S; \wedge, \vee)$, its (vertical) *dual* is the skew lattice $\mathbf{S}^{\updownarrow} = (S; \wedge^{\updownarrow}, \vee^{\updownarrow})$, where as binary functions, $\wedge^{\updownarrow} = \vee$ and $\vee^{\updownarrow} = \wedge$. Clearly $\mathbf{S}^{\updownarrow\updownarrow} = \mathbf{S}$ and any homomorphism $f : \mathbf{S} \to \mathbf{T}$ of skew lattices ia also a homomorphism from $\mathbf{S}^{\updownarrow}$ to $\mathbf{T}^{\updownarrow}$; moreover a skew lattice $\mathbf{S}$ is distributive (or symmetric) iff $\mathbf{S}^{\updownarrow}$ is thus. Either $\mathbf{S}$ or $\mathbf{S}^{\updownarrow}$ is strongly distributive iff the other is co-strongly distributive; more generally, $\mathbf{S}$ or $\mathbf{S}^{\updownarrow}$ is normal iff the other is co-normal. Also, one has a bottom element iff the other has a top element, both being the same element in S.

Expanding the signature, $(S; \wedge, \vee, \backslash, 0)$ is a skew Boolean algebra if and only if its dual $(S; \wedge^{\updownarrow}, \vee^{\updownarrow}, \backslash\!\backslash, 1)$ is a dual skew Boolean algebra where $\backslash$ and 0 are replaced by $\backslash\!\backslash$ and 1. Thus any skew Boolean algebra $(S; \wedge, \vee, \backslash, 0)$ induces a skew Heyting algebra $(S; \wedge^{\updownarrow}, \vee^{\updownarrow}, \to, 1)$ where $x \to y = y \setminus x$ and $1 =$ old 0. Standard examples of skew Boolean algebras thus give us:

Example 3.9. Given sets X and Y, the skew Heyting operations derived from the skew Boolean operations on the set $\mathcal{P}(X, Y)$ of all partial functions from X to Y are as follows.

skew Heyting operation	description	skew Boolean operation
$f \wedge g$	$f \cup (g\vert_{\mathrm{dom}g - \mathrm{dom}f})$	$f \vee g$
$f \vee g$	$g\vert_{\mathrm{dom}g \cap \mathrm{dom}f}$	$f \wedge g$
$f \to g$	$g\vert_{\mathrm{dom}g - \mathrm{dom}f}$	$g \setminus f$
1	$\emptyset$	0

Example 3.10. Given a surjective function $\pi : Y \to X$, let $Sec(\pi)$ denote the set of all *sections* of π, that is, functions f from subsets U of X to Y such that $\pi \circ f = id_{\mathrm{dom}(f)}$. Skew Heyting algebra operations and corresponding skew Boolean operations are defined on $Sec(\pi)$ using precisely the above descriptions. At first glance this seems to be just a subalgebra of the type of algebra in Example 1. The latter, however, is isomorphic to $Sec(\pi)$ where π is now the coordinate projection of $X \times Y$ onto X. Modifications of this example arise in the dualities of the next section.

It so happens that any right-handed (co-)strongly distributive skew lattice is isomorphic to a subset of partial functions in some $\mathcal{P}(X, Y)$ that is closed under the relevant $\wedge$ and $\vee$ operations above. (See [12] Section 3.7.) It follows that the skew lattice reduct of a skew Heyting algebra is isomorphic to some such partial function algebra. The difference of this more general case from that of the two examples above is that here $x \to y$ need not have a polynomial definition, unlike the co-Boolean case where $x \to y = y \setminus x$.

The following result is useful for computing in skew Heyting algebras.

Proposition 3.11. *Let* $\mathbf{S} = (S; \wedge, \vee, \to, 1)$ *be a skew Heyting algebra and* $x, y, z \in S$. *Then*

$$(x \vee y \vee x) \to z = (x \to z) \wedge (y \to z) \wedge (x \to z).$$

Proof. As $\mathbf{S}/\mathcal{D}$ is a generalized Heyting algebra and relation $\mathcal{D}$ respects all skew Heyting algebra operations, it follows that $(x \vee y \vee x) \to z \, \mathcal{D} \, (x \to z) \wedge (y \to z) \wedge (x \to z)$. However, both $(x \vee y \vee x) \to z$ and $(x \to z) \wedge (y \to z) \wedge (x \to z)$ are above z with respect to the natural partial order, and hence must be equal by Lemma 2.2. $\square$

A skew lattice $\mathbf{S}$ is *meet [join] complete* if each nonempty commutative subset possesses an infimum [a supremum] in S. It follows from the dual of [5] Proposition 2.10 that if $\mathbf{S}$ is a meet complete co-strongly distributive skew lattice with 1, then $\mathbf{S}$ is complete. We call a skew Heyting algebra *complete* if it is complete as a skew lattice.

4 Connections to duality

Dual skew Boolean algebras are order duals (upside-downs) to usually studied skew Boolean algebras. Skew Boolean algebras and dual skew Boolean algebras are categorically isomorphic. Right-handed (dual) skew Boolean algebras are dually equivalent to sheaves over locally compact Boolean spaces by results of [1] and [12], where a *locally compact Boolean space* is a topological space whose one-point-compactification is a Boolean space. The obtained duality asserts that any right- [left-]handed skew Boolean algebra is isomorphic to the skew Boolean algebra of compact open sections (i.e. sections over compact open subsets) of an étale map over some locally compact Boolean space. Let us note that the restriction to right- [left-]handed algebras is not a major restriction since Leech's Second Decomposition Theorem yields that any skew lattice is a pull back of a left-handed and a right-handed skew lattice over their common maximal lattice image [13]. The general two-sided case was also covered in [1].

Bounded distributive lattices are dual to Priestley spaces; in this duality each bounded distributive lattice is represented as the distributive lattice of all clopen (i.e. closed and open) upsets of a Priestley space. The Esakia duality established in [9] yields that Heyting algebras are dual to *Esakia spaces*, i.e. those Priestley spaces in which the downset of each clopen set is again clopen. Moreover, if $(X, \leq, \tau)$ is an Esakia space then given clopen subsets U and V in X the implication is defined by

$$U \to V = X \setminus {\downarrow}(U \setminus V).$$

Duality for strongly distributive skew lattices was recently established in [2]. It yields that right-handed strongly distributive skew lattices with bottom are dual to the sheaves over locally Priestley spaces, where by a *locally Priestley space* we mean an ordered topological space whose one-point-compactification is a Priestley space. Via the obtained duality each right-handed strongly distributive skew lattice with bottom is represented as a skew lattice of sections over copen (i.e. compact and open) downsets of a locally Priestley space, with the operations being defined as follows:

$$\begin{aligned}
0 &= \emptyset, \\
r \wedge s &= s|_{\mathrm{dom} r \cap \mathrm{dom} s}, \\
r \vee s &= r \cup s|_{\mathrm{dom} s - \mathrm{dom} r}, \\
r \setminus s &= r|_{\mathrm{dom} r - \mathrm{dom} s}.
\end{aligned}$$

Given a distributive lattice $\mathbf{L}$ denote by $\mathbf{L}^c$ the distributive lattice that is obtained from $\mathbf{L}$ by reversing the order. Denote by $\mathbf{DL}$ the category of all distributive lattices, by $\mathbf{PS}$ the category of all locally Priestley spaces and consider the functors:

$$^c: \begin{array}{ccc} \mathbf{DL} & \to & \mathbf{DL} \\ \mathbf{L} & \mapsto & \mathbf{L}^c \end{array} \quad \text{and} \quad ^r: \begin{array}{ccc} \mathbf{PS} & \to & \mathbf{PS} \\ (X, \leq) & \mapsto & (X, \geq). \end{array}$$

Restricting the functors ^{c} and r to the categories $\mathbf{HA}$ of all Heyting algebras and $\mathbf{ES}$ of all Esakia spaces, respectively, yields the following isomorphism of categories:

$$^c: \begin{array}{ccc} \mathbf{HA} & \to & \mathbf{cHA} \\ \mathbf{L} & \mapsto & \mathbf{L}^c \end{array} \quad \text{and} \quad ^r: \begin{array}{ccc} \mathbf{ES} & \to & \mathbf{cES} \\ (X, \leq) & \mapsto & (X, \geq), \end{array}$$

where $\mathbf{cHA}$ denotes the category of all *co-Heyting algebras* (defined as order-inverses of Heyting algebras) and $\mathbf{cES}$ denotes the category of all *co-Esakia spaces* the latter being introduced in [3] as Priestley spaces in which an upset of a clopen is again clopen.

We introduce the following categories:

$\mathbf{SDSL}$: strongly distributive skew lattices with 0,
$\mathbf{cSDSL}$: co-strongly distributive skew lattices with 1,
$\mathbf{SHA}$: skew Heyting algebras,
$\mathbf{cSHA}$: co-skew Heyting algebras,

with the latter being defined as the category of all algebras of the form $\mathbf{S}^c$, where $\mathbf{S}$ is a skew Heyting algebra and

$$^c: \begin{array}{ccc} \mathbf{SDSL} & \to & \mathbf{cSDSL} \\ \mathbf{S} & \mapsto & \mathbf{S}^c \end{array}$$

is the isomorphism of categories that turns a skew lattice upside-down. The restriction of the functor ^{c} to the categories $\mathbf{cSHA}$ and $\mathbf{SHA}$ yields the isomorphism:

$$^c: \begin{array}{ccc} \mathbf{cSHA} & \to & \mathbf{SHA} \\ \mathbf{S} & \mapsto & \mathbf{S}^c \end{array}.$$

The isomorphism of categories induces an isomorphism of concepts:

SHA	**cSHA**
$\wedge$	$\vee$
$\vee$	$\wedge$
1	0
strong codistributivity	strong distributivity
filter	ideal
prime filter	prime ideal

It follows from Theorem 3.5 that the skew Heyting algebra structure can be imposed exactly on those co-strongly distributive skew lattices with top whose maximal lattice image is a generalized Heyting algebra. Therefore the duality for right-handed skew Heyting algebras yields that they are dual to sheaves over *local Esakia spaces*, i.e. ordered topological spaces whose one-point-compactification is an Esakia space.

Let $(B, \leq)$ be an Esakia space, E a topological space and $p : E \to B$ a surjective étale map. Consider the set S of all sections of p over copen upsets in B, i.e. an element of S is a map $s : U \to E$, where U is a copen upset in B, that satisfies the property $p \circ s = \mathrm{id}_U$. A section $s \in S$ is considered to be below a section $r \in S$ when s extends r. The skew Heyting operations are defined on S by:

$$\begin{aligned} r \vee s &= s|_{\mathrm{dom} r \cap \mathrm{dom} s}, \\ r \wedge s &= r \cup s|_{\mathrm{dom} s \setminus \mathrm{dom} r}, \\ r \to s &= r|_{\uparrow(\mathrm{dom} s \setminus \mathrm{dom} r)} \\ 1 &= \emptyset. \end{aligned}$$

Theorem 4.1. *Let $p : E \to B$ be a surjective étale map over a local Esakia space B. Then the set S of all sections of p over copen upsets in B forms a skew Heyting algebra under the above operations.*

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A decomposition for Markov processes at an independent exponential time

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Abstract

The path of Markov process X run up to an independent exponential random time S_θ can be decomposed into the part prior to the last exit time from a point before S_θ, and the remainder up to S_θ. In this paper the laws of the two segments are identified under suitable assumptions using excursion theory.

Keywords: Markov processes, excursions, last exit decomposition, diffusions, Brownian motion.

Math. Subj. Class.: 60J25, 60J55, 60J60, 60J65, 60G51.

1 Introduction

Considering a Markov process X up to an independent exponential time S_θ with rate $\theta > 0$ has been used effectively to compute functionals of X. The computations are often based on decompositions of the path of X up to S_θ into fragments before and after the last exit time from a set before time S_θ. The known results described below are more general in the sense that the path is decomposed at the last exit from a set before either fixed times or random times belonging to a suitable family. Choosing an independent exponential time in some cases leads to simpler descriptions of the laws of the two fragments involved. They are often conditionally independent given suitable conditioning variables and their laws are related to laws of known processes.

Williams [27] uses a decomposition of Brownian motion with drift run up to an independent S_θ to prove result of Ray [23] on the distribution of local times in the space

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variable. In their investigations of Ray-Knight theorems for Brownian motion B at fixed times Biane and Yor [4] considered the pair of processes $(B_t\colon 0 \leq g_{S_\theta})$ and $(B_{S_\theta - t}\colon 0 \leq t \leq S_\theta - g_{S_\theta})$ where S_θ is an exponential random variable independent of B and $g_t = \sup\{s \leq t\colon B_s = 0\}$ is the last exit time from 0 before time t. Under P_0 the two processes are shown to be independent and their conditional laws given the local time $L(S_\theta)$ at zero of Brownian motion and B_{S_θ} respectively are identified. This decomposition has been exploited by Jeanblanc, Pitman and Yor [13] to derive Feynman-Kac formulae for Brownian motion. Salminen, Vallois and Yor [26] extend the decomposition for Brownian motion to linear diffusions on $[0, \infty)$ with 0 a recurrent point and use them to study the excursion of the diffusion straddling an independent exponential time.

For general Markov processes Pittenger and Shih [22] investigated the dependence of the fragments of the path of a càdlàg strong Markov process X before coterminal time $L^t \leq t$ and the fragment on the interval between L^t and t. Last exit times L^t_F before time t from a closed set F are coterminal times. It is shown that given a suitably defined σ-algebra $\mathcal{F}_{L^t}$, the conditional law of the process $(X_{L_t+s}\colon 0 \leq s \leq t - L^t)$ only depends on L_t and $X_{L^F_t}$ or X_{L^t-} and is an inhomogenous strong Markov process. Getoor and Sharpe [7] give related results. General and elegant treatments of last exit decompositions are given in Maisonneuve [18] and Pitman [21]. Kallenberg [14] proves that for Lévy processes the fragments considered by Pittenger and Shih are conditionally independent given $X_{L^t_F-}$ and L^t_F where the last exit time from a set F is an instance of a backward time. Under suitable conditions the laws of the two fragments are described.

Another example of considering a decomposition of a Markov process at last exit time from 0 before and independent exponential time S_θ is the proof of fluctuation equalities for Lévy processes given by Greenwood and Pitman [9]. If X is a Lévy process then it is known that the process reflected at the supremum defined by $Y_t = \sup_{s\leq t} X_s - X_t$ is a strong Markov process. See e.g. Bertoin [2], p.156. If Y is split at the last exit time from 0 before an independent exponential time S_θ the two fragments are independent and their laws can be described. This gives a direct proof of the infinite divisibility results needed to prove the fluctuation identities by Pečerskiĭ and Rogozin [20].

The setting of this paper is a strong Markov processes X with a recurrent point a. Last exit times from a are considered and the path of X is split at the last exit time from a before an independent exponential time S_θ. The two resulting fragments turn out to be independent and their laws are described.

2 Notation and statement

Let X be a càdlàg Markov process with state space $(E, \mathcal{E})$ which we will assume to be Lusinian. We will assume that the semigroup P_t maps Borel functions into Borel functions and that the process can be realized as the coordinate process on the Skorohod space Ω of paths which are right continuous with left limits. Assume that the Markov process has transition densities $p_t(x, y)$ with respect to a σ-finite measure ξ on $(E, \mathcal{E})$. The densities are assumed to be jointly continuous in all three variables for $t > 0$ which implies the Chapman-Kolmogorov equations

$$p_{t+s}(x, y) = \int_E p_t(x, z)p_s(z, y)\,\xi(dz)$$

for all $s, t > 0$ and $x, y \in E$.

To formulate the results the existence of a dual strong Markov process $\hat{X}$ on $(E, \mathcal{E})$ relative to the measure ξ will be assumed. This means that ξ is an invariant measure for both X and $\hat{X}$ and

$$P_x(X_t \in dy) = p_t(x,y)\xi(dy) \quad \text{and} \quad \hat{P}_x(\hat{X}_t \in dy) = \hat{p}_t(y,x)\xi(dy) \tag{2.1}$$

for all $t > 0, x, y \in E$ with $\hat{p}_t(x,y) = p_t(y,x)$. See [6] and the references therein for details. Denote by

$$r_\theta(x,y) = \int_0^\infty e^{-\theta t} p_t(x,y)\, dt \quad \text{and} \quad \hat{r}_\theta(x,y) = \int_0^\infty e^{-\theta t} \hat{p}_t(x,y)\, dt$$

the resolvent densities of X and $\hat{X}$ respectively. For the sake of simplicity it will be assumed that X and $\hat{X}$ have infinite lifetimes ζ under P_x and $\hat{P}_y$ for all $x, y \in E$ respectively.

The assumptions on X imply that it is possible to define bridge laws

$$P^t_{x,y}(\cdot) = P_x(\cdot | X_t = y) \tag{2.2}$$

for $t > 0$ and for $x, y \in E$. By Proposition 1 in Fitzsimmons, Pitman, Yor [6] for any $x, y \in E$ and $t > 0$ with $p_t(x,y) > 0$ there is a unique law $P^t_{x,y}$ on $(\Omega, \mathcal{F}_t)$ such that for any $\mathcal{F}_s$-measurable functional F for $0 \le s < t$

$$E^t_{x,y}(F) \cdot p_t(x,y) = E_x\left(F \cdot p_{t-s}(X_s, y)\right), \tag{2.3}$$

where $E^t_{x,y}$ and E_x are expectations with respect to measures $P^t_{x,y}$ and P_x respectively. The laws $P^t_{x,y}$ provide a regular version of the family of conditional distributions $P(\cdot|X_t = y)$. Furthermore by Corollary 1 in Fitzsimmons, Pitman, Yor [6] the law of the reversed bridge $(X_{(t-s)-} : 0 \le s < t)$ under $P^t_{x,y}$ has the law of the bridge of the dual process $\hat{P}^t_{y,x}$.

The subject of this paper is the law of the process X started at a and run to an independent exponential time S_θ with rate θ and conditioned on $\{X_{S_\theta} = b\}$. Conditionally on $\{S_\theta = t, X_t = b\}$ the law of the process will be the law of the bridge $P^t_{a,b}$ and the laws $P^t_{a,b}$ will serve as the regular version of the family of conditional distributions.

Assume that a is a recurrent point of the process X. Let $T_a = \inf\{t > 0 \colon X_{t-} = a \,\text{or}\, X_t = a\}$. Since a is assumed to be recurrent the assumptions imply that $P_b(T_a < \infty) = 1$ for all $b \in E$.

For $t > 0$ define the last exit time from a before time t as

$$g_t = \sup\{s \le t \colon X_{t-} = a \,\text{or}\, X_t = a\}.$$

Let $(L^a_t \colon t \ge 0)$ be the local time for the process X at a. We will assume that such a right continuous nondecreasing additive functional exists and only increases on the set $M = \{t \ge 0 \colon X_{t-} = a \,\text{or}\, X_t = a\}$. All the results will be valid for any choice of normalization of the local time. Let for $s \ge 0$

$$\tau_s = \inf\{t \ge 0 \colon L^a_t > s\} \tag{2.4}$$

be the right continuous inverse of the local time. From the strong Markov property of X it follows that $(\tau_s \colon s \ge 0)$ is a subordinator. Since we are assuming recurrence the local time at a will be unbounded and hence τ_s is well defined for all s. For simplicity we will assume that the set M has Lebesgue measure 0 almost surely. This means that the subordinator

$(\tau_s\colon s \geq 0)$ has no drift. From properties of subordinators, see Bertoin [2], Ch. 3, it follows that

$$E_a\left(e^{-\theta\tau_u}\right) = e^{-\psi(\theta)u}\,. \tag{2.5}$$

The notation $X(t)$, $L^a(t)$ and $g(t)$ will be used for X_t, L^a_t and g_t whenever necessary. The theorem to be proved is stated as follows.

Theorem 2.1. *Assume that a is a recurrent point for the process X and $p_t(a,b) > 0$ for all $t > 0$. Let $X_0 = a$ and assume $(L^a_t\colon t \geq 0)$ is the local time of X at a. If S_θ is an exponential random variable with parameter θ independent of X then, under the measure P_a:*

(i) The random variables $L^a(S_\theta)$ and $X(S_\theta)$ are independent with distributions

$$P_a(L^a(S_\theta) \in du) = \psi(\theta)\, e^{-\psi(\theta)u}du \quad \textit{and} \quad P_a(X(S_\theta) \in dy) = \theta r_\theta(a,y)dy \tag{2.6}$$

where $\psi(\theta)$ is the Laplace exponent defined in Equation 2.5.

(ii) The processes

$$(X_t\colon 0 \leq t \leq g_{S_\theta}) \quad \textit{and} \quad (X_{g_{S_\theta}+u}\colon 0 \leq u \leq S_\theta - g_{S_\theta})$$

are independent.

(iii) For bounded measurable functionals F and G

$$E_a\left[F(X_s\colon 0 \leq s \leq g_{S_\theta})|\, L_{S_\theta} = u\right] = \frac{E_a\left[F(X_s\colon 0 \leq s \leq \tau_u)\, e^{-\theta\tau_u}\right]}{E_a\left[e^{-\theta\tau_u}\right]} \tag{2.7}$$

$$\begin{aligned} E_a[G(X_{(S_\theta-s)-}\colon 0 \leq s \leq S_\theta - g_{S_\theta})|X_{S_\theta} = b] &= \\ = \frac{\hat{E}_b\left[G(\hat{X}_s\colon 0 \leq s \leq \hat{T}_a)\, e^{-\theta\hat{T}_a}\right]}{\hat{E}_b[e^{-\theta\hat{T}_a}]} \end{aligned} \tag{2.8}$$

where $\hat{E}_b$ refers to expectation under the law $\hat{P}_b$ of the dual process, and $\hat{T}_a = \inf\{t : \hat{X}_t = a \text{ or } \hat{X}_{t-} = a\}$ is the hitting time of a for $\hat{X}$.

The novelty lies in the fact that known special cases are covered by the more general Markov setup. Excursion arguments used are standard.

3 Excursion arguments

Let Π be a point process on an abstract space $(S, \mathcal{S})$ with mean measure Λ. If Π is a Poisson process then by Campbell's Theorem, see Kingman [16], p. 28, for any measurable $f \geq 0$

$$E\left(\exp\left(-\int_S f(x)\Pi(dx)\right)\right) = \exp\left(-\int_S (1-e^{-f(x)})\,\Lambda(dx)\right)\,. \tag{3.1}$$

Conversely, if Equation 3.1 holds for any measurable $f \geq 0$, then Π is a Poisson process with mean measure Λ.

Assume that Π is a Poisson process with mean measure Λ, and let $h\colon S \to [0,\infty)$ be a measurable function such that

$$\int_S (1-e^{-h(x)})\,\Lambda(dx) < \infty\,. \tag{3.2}$$

The random variable Σ_h defined by

$$\Sigma_h = \int_S h(x)\Pi(dx)$$

is almost surely finite and non-negative with

$$E\left(\exp(-\Sigma_h)\right) = \exp\left(-\int_S (1-e^{-h(x)})\Lambda(dx)\right)\,. \tag{3.3}$$

Define a new probability measure Q by

$$\frac{dQ}{dP} = \frac{\exp\left(-\Sigma_h\right)}{E\left[\exp\left(-\Sigma_h\right)\right]}\,. \tag{3.4}$$

The following lemma is known in the literature, see Proposition 2.1 in James [12] and the discussion therein, or Proposition 2.4 in Bertoin [3].

Lemma 3.1. *Under the measure Q, Π is a Poisson process with mean measure $e^{-h(x)}\cdot\Lambda(dx)$.*

Proof. It suffices to check that Equation 3.1 holds. Denote $c = 1/E(\exp(-\Sigma_h))$ and let $\Sigma_f = \int_S f(x)\Pi(dx)$ for a measurable funtion $f \geq 0$. One has

$$\begin{aligned}
E_Q\left[\exp(-\Sigma_f)\right] &= c\,E_P\left[\exp\left(-\Sigma_h\right)\cdot\exp\left(-\int_S f(x)\Pi(dx)\right)\right] \\
&= c\,E_P\left[\exp\left(-\int_S \left(f(x)+h(x)\right)\,\Pi(dx)\right)\right] \\
&= c\,\exp\left(-\int_S \left(1-e^{-f(x)-h(x)}\right)\,\Lambda(dx)\right) \\
&= c\,\exp\left(-\int_S \left[(1-e^{-f(x)})\,e^{-h(x)} + (1-e^{-h(x)})\right]\,\Lambda(dx)\right) \\
&= \exp\left(-\int_S (1-e^{-f(x)})\,e^{-h(x)}\,\Lambda(dx)\right)
\end{aligned}$$

□

Let X be a càdlàg strong Markov process. The set $M = \{t \geq 0\colon X_{t-} = a \quad\text{or}\quad X_t = a\}$ is closed under the assumptions. Since we are assuming recurrence of X the complement of M is a countable union of bounded open intervals. The segments of the path of X on these open intervals are called the excursions of X away from the point a. By definition the open intervals coincide with the complement of the range of the subordinator $(\tau_s\colon s>0)$ defined in Equation 2.4. Let U_δ be the space of càdlàg paths

$w\colon [0,\infty) \to E$ such that $w(0) = a$ and there is a $t > 0$ such that for $0 < s < t$ we have $w(s) \neq a, w(s-) \neq a$, and $w(t) = a$ or $w(t-) = a$ and $w(s) = \delta$ for $s > t$ where δ is the coffin state added to E. Let $\mathcal{U}_\delta$ be the σ-algebra generated by the coordinate maps in U_δ. Define the point process $(e_s\colon s > 0)$ of excursions of X in the sense of Itô as

$$e_s = \begin{cases} \delta & \text{if } \tau_s - \tau_{s-} = 0 \\ e_s(u) = X_{\tau_{s-}+u} & \text{for } u < \tau_s - \tau_{s-} \text{ and } \delta \text{ else.} \end{cases} \tag{3.5}$$

The process e is a Poisson process in the sense of Itô governed by the measure $\tilde{\lambda} \times n$ where $\tilde{\lambda}$ is a multiple of the Lebesgue measure on $[0,\infty)$ and n is the Itô excursion law. We can change the normalization of the local time, if needed, in order to ensure that $\tilde{\lambda}$ is the standard Lebesgue measure. See Rogers and Williams [25], Ch. 8 or Revuz and Yor [24], Ch. 8 for background on excursion theory.

The connection between excursion theory and the law of a Markov process run up to an independent exponential time S_θ is established through marking excursions. Let Π be a Poisson process on an abstract space $(S, \mathcal{S})$. If conditionally on Π each point $x \in \Pi$ is assigned a mark with probability $p(x)$ independently of all the other points in Π then the resulting marked and unmarked processes are both Poisson and are independent. If Λ is the mean measure of Π the marked process will have mean measure $p \cdot \Lambda$ and the unmarked process $(1-p)\cdot\Lambda$. See Kingman [16], Ch. 5 for definitions and proofs.

The excursion of the Markov process X straddling the independent exponential random time S_θ can be interpreted as the first marked excursion of X where conditionally on e marks are assigned to an excursion e with probability $1 - e^{-\theta R(e)}$ where $R(e)$ stands for the duration of the excursion i.e. the length of the open interval of M^c containing S_θ. Intuitively we can think that S_θ is the first point in a Poisson process N on $(0,\infty)$ with rate θ and independent of X. Excursions straddling a point of N are considered marked and other excursions are considered unmarked. By independence properties of Poisson processes conditionally on e the marks are assigned independently and an excursion of length $R(e)$ contains a point of the Poisson process with probability $1 - e^{R(e)}$ which follows from the Poisson distribution of points contained in the excursion interval. See Sec. 49 in Rogers and Williams [25] for definitions and proofs.

The following theorem is stated in slightly more general terms allowing the Poisson process $(e_s\colon s > 0)$ to be killed at a rate q. This would correspond to excursions of X that have infinte length. The conclusions of the theorem are well known, see Sec. 49.4 in Rogers and Williams [25].

Theorem 3.2. *Let $(e_s : s > 0)$ be a possibly killed excursion process of X from a recurrent point a in the sense of Itô, and let S_θ be an exponential random variable with parameter θ independent of X. Denote by $(L^a_t : t \geq 0)$ the local time process of X at a and $(e_s\colon s > 0)$ the associated excursion process governed by $\lambda \times n$ where λ is the Lebesgue measure.*

(i) The local time $L^a(S_\theta)$ during the excursion straddling S_θ is an exponential random variable with parameter

$$\int_U (1 - e^{-\theta R(e)})\, n(de; R < \infty) + q$$

where q is the rate of arrival of excursions with infinite lifetime. Moreover, it is independent of the excursion $e^* = e(L^a_{S_\theta})$ which may have infinite lifetime.

(ii) Given $L^a(g_{S_\theta}) = u$ the process of excursions $(e_s : 0 < s < u)$ is a Poisson process in the sense of Itô which is governed by the measure $\bar{n}$ given by $\bar{n}(de) = e^{-\theta R(e)}\, n(de; R(e) < \infty)$ where n is Itô's excursion law and $R(e)$ denotes the length of the excursion. Moreover, e^* is conditionally independent of $(e_s : 0 < s < u)$ given $\{L^a(S_\theta) = u\}$, and is independent of $L^a(S_\theta)$.

Proof. The first marked excursion in $(e_s\colon s > 0)$ will arrive at an exponential time. The processes of finite length excursions and those of infinte length are independent so the rates of arrivals add. The rate of arrivals of marked finite length excursions is by definition equal to

$$\int_U (1 - e^{-\theta R(e)})\, n(de; R < \infty)\,.$$

The two processes of marked and unmarked excursions are independent. This means that conditionally on $L^a(S_\theta) = u$ the process $(e_s : 0 < s < u)$ is a Poisson process on $(0,u) \times U_\delta$.

The first marked excursion is picked according to the normalized law $(1 - e^{-\theta R}) \cdot n$ irrespective of the local time $L^a(S_\theta)$. This and the independence of marked and unmarked excursions conclude the proof. □

4 Proofs

Recall that under the assumptions on X and $\hat{X}$ and if $p_t(a,b) > 0$ there is a measure $P^t_{a,b}$ corresponding to the bridge of X starting at a and conditioned to be b at time t. The family of $P^t_{a,b}$ is a family of regular conditional laws of X given $X_t = b$. If S_θ is an exponential random variable of rate θ then under P_a the family $P^t_{a,b}$ is a regular conditional law of X given $\{S_\theta = t, X_t = b\}$. The assumptions made on X and $\hat{X}$ also imply that X and $\hat{X}$ have no jumps at fixed times. See (3.18) in Getoor and Sharpe [8] for a proof. Let $\hat{T}_b = \inf\{t > 0\colon \hat{X}_t = b \text{ or } \hat{X}_{t-} = b\}$. Assume further that $\hat{P}_b(\hat{T}_b > 0) = 0$ which in conjunction with right continuity and strong Markov property implies that $X_{T_b} = b$.

Lemma 4.1. *Assume that $P_a(T_b < \infty) = 1$. Assume that $p_t(a,b) > 0$ for all $t > 0$. Then*

$$P_a(T_b \in ds | X(S_\theta) = b) = e^{-\theta s} P_a(T_b \in ds)\, \frac{r_\theta(b,b)}{r_\theta(a,b)}\,,$$

where $r_\theta(x,y)$ is the resolvent density.

Proof. By assumptions on b and $\hat{X}$ we have $P_a(T_b < S_\theta | X(S_\theta) = b) = 1$. For fixed $0 < s < t$ and a $\mathcal{F}_s$ measurable functional F by 2.6 in Fitzsimmons, Pitman and Yor [6]

$$E^t_{a,b}(F) \cdot p_t(a,b) = E_a\left(F \cdot p_{t-s}(X_s, b)\right)\,. \tag{4.1}$$

Choose an open neighbourhood $B \in \mathcal{E}$ of b such that $r_\theta(a,y) > 0$ for $y \in B$. Let

$F = 1(T_b \in ds)$. We have

$$
\begin{aligned}
&P_a(T_b \in ds, T_b < S_\theta, X_{S_\theta} \in B) = \\
&= \int_s^\infty \theta e^{-\theta t}\, dt \int_B p_t(a,y) P^t_{a,y}(T_b \in ds)\, \xi(dy) \\
&= \int_s^\infty \theta e^{-\theta t}\, dt \int_B P_a\,(T_b \in ds)\, p_{t-s}(b,y)\, \xi(dy) \\
&= e^{-\theta s} P_a(T_b \in ds) \int_B \xi(dy) \int_0^\infty \theta e^{-\theta u} p_u(b,y)\, du \\
&= e^{-\theta s} P_a(T_b \in ds)\, \theta \int_B r_\theta(b,y) \xi(dy) \\
&= e^{-\theta s} P_a(T_b \in ds) \int_B \frac{r_\theta(b,y)}{r_\theta(a,y)} \cdot \theta r_\theta(a,y)\, \xi(dy)
\end{aligned}
$$

It follows that

$$P_a(T_b \in ds | X(S_\theta) = b) = e^{-\theta s} P_a(T_b \in du)\, \frac{r_\theta(b,b)}{r_\theta(a,b)}\,.$$

□

Remark 4.2. Integrating the equation with respect to s over $(0, \infty)$ in Lemma 4.1 gives the well known formula

$$E_a(e^{-\theta T_b}) = \frac{r_\theta(a,b)}{r_\theta(b,b)}\,. \tag{4.2}$$

See e.g. Rogers and Williams [25], (50.7) on p. 293.

Let us consider the process $(X_t \colon 0 \le t \le T_b)$ given $\{X_{S_\theta} = b\}$. The following lemma gives the conditional distribution of this process given $X(S_\theta) = b$.

Lemma 4.3. *Assume that $P_a(T_b < \infty) = 1$. Assume that $p_t(a,b) > 0$ for all $t > 0$. Let F be a non-negative measurable functional of the process $(X_t \colon 0 \le t \le T_b)$. Then*

$$E_a(F | X(S_\theta) = b) = E_a\left(e^{-\theta T_b} \cdot F\right) \cdot \frac{r_\theta(b,b)}{r_\theta(a,b)}\,, \tag{4.3}$$

where $r_\theta(x,y)$ is the resolvent density.

Proof. As in Lemma 4.1 we compute for an open neighbourhood $B \in \mathcal{E}$ of b such that

$r_\theta(a,y) > 0$ and a bounded measurable functional F

$$\begin{aligned} & E_a\left[F \cdot 1(T_b < S_\theta, T_b \in ds, X_{S_\theta} \in B)\right] = \\ = \; & \int_s^\infty \theta e^{-\theta t}\, dt \int_B p_t(a,y) E^t_{a,y}(F \cdot 1(T_b \in ds))\, dy \\ = \; & \int_s^\infty \theta e^{-\theta t}\, dt \int_B E_a\left(F \cdot 1(T_b \in ds)\right) p_{t-s}(b,y)\, dy \\ = \; & e^{-\theta s} E_a\left(F \cdot 1(T_b \in ds)\right) \int_B dy \int_0^\infty \theta e^{-\theta u} p_u(b,y)\, du \\ = \; & e^{-\theta s} E_a\left(F \cdot 1(T_b \in ds)\right) \theta \int_B r_\theta(b,y)\, dy \\ = \; & e^{-\theta s} E_a\left(F \cdot 1(T_b \in ds)\right) \int_B \frac{r_\theta(b,y)}{r_\theta(a,y)} \cdot \theta r_\theta(a,y)\, dy \end{aligned}$$

This in conjunction with the distribution of T_b from Lemma 4.1 completes the proof. □

The conclusions of Lemma 4.3 apply equally to the dual process $\hat{X}$. Moreover under P_a the conditional law of the process $(X_{(S_\theta - t)-} \colon 0 \le t \le S_\theta)$ given $X_{S_\theta} = b$ is equal to the law of $\hat{X}$ started at b run to an independent exponential time and conditioned to be a at the end. This implies that under the assumptions on a for any bounded functional G

$$\begin{aligned} & E_a\left[G\Big(X_{(S_\theta - t)-} \colon 0 \le t \le S_\theta - g_{S_\theta}\Big)\Big| X_{S_\theta} = b\right] \\ & \qquad\qquad = \hat{E}_b\left[G\left(\hat{X}_t \colon 0 \le t \le \hat{T}_a\right) e^{-\theta \hat{T}_a}\right] \cdot \frac{\hat{r}_\theta(a,a)}{\hat{r}_\theta(b,a)} \end{aligned} \tag{4.4}$$

because the last exit time g_{S_θ} from a is the first hitting time of a for the reversed process. This is in accordance with Theorem 7.6 in Getoor and Sharpe [8] that excursions straddling a fixed time reversed and conditioned on the length are the excursions of the dual process. See also formula (3.12) in Ikeda, Nagasawa, Sato [11] who give the law of the process reversed from the lifetime of a killed Markov process. The case treated here considers killing at a constant rate.

We are now in position to give the proof of Theorem 2.1.

Proof. The second assertion in (i) is the definition of the resolvent density. Let $(e_s \colon s > 0)$ be the excursion process of X from a. Marked excursions arrive at an exponential rate so we know that $L^a(S_\theta)$ will be exponential. Since excursions are marked by an independent Poisson process, the event $\{L^a(S_\theta) > u\}$ is equal to the event that there is no mark in the interval $[0, \tau_u]$, and has conditional probability $e^{-\theta \tau_u}$. Integration gives

$$P_a(L^a(S_\theta) > u) = E_a\left(e^{-\theta \tau_u}\right) = e^{-u\psi(\theta)},$$

which by differentiation gives the density. To prove (ii) note that by Theorem 3.2 conditionally on $\{L^a(S_\theta) = u\}$ the process of excursions $(e_s \colon 0 < s < u)$ is independent of the excursion e^* straddling S_θ. Because marks to excursions are assigned by an independent Poisson process conditionally on $R(e^*) = r$ the mark is distributed at the distance U from the left endpoint with density

$$\theta e^{-\theta u}/(1 - e^{-\theta r})$$

on $[0, r]$ independently of the process of unmarked excursions and of the local time $L^a(S_\theta)$. So the excursion e^* together with the position of S_θ within the duration of e^* are independent of the process of unmarked excursions and of $L^a(S_\theta)$. This proves the independence of the two processes in (ii).

For the first assertion in (iii) note that conditionally on $L^a(S_\theta) = u$ the excursions of $(X_t : 0 \le t \le g_{S_\theta})$ from the point a are a Poisson process with excursion law $e^{-\theta R} \cdot n$ by Theorem 3.2 (i). On the other hand, if we let $(e_s : 0 < s < u)$ be the Poisson process of excursions of X from a and choose $h(e) = \theta\, R(e)$ in Lemma 3.1, under the new measure the process is still Poisson but with the mean measure $e^{-\Sigma_h} \cdot n$. But under the assumption that the set M has Lebesgue measure 0 we have $\Sigma_h = \theta \cdot \tau_u$.

The proof of the second formula in (iii) follows from Lemma 4.3 applied to the reversed process. □

Note the connection with Lemma 4.1 in Kallenberg [14] which states that for Lévy processes with continuous densities

$$P_a\left(F(X_s : 0 \le s \le \tau_u) | \tau_u = t\right) = P^t_{a,a}\left(F(X_s : 0 \le s \le t) | L_t = u\right) \tag{4.5}$$

where P_t refers to the law of the bridge of length t. Noting that Lemma 3.1 gives

$$P_a\left(g_{S_\theta} \in dt | L(S_\theta) = u\right) = \frac{e^{-\theta t} P_a(\tau_u \in dt)}{E_a\left(e^{-\theta \tau_u}\right)}\,. \tag{4.6}$$

Equations 4.5 and 4.6 imply part (ii) in Theorem 2.1.

5 Examples

5.1 Linear diffusions

Let X be a regular diffusion on an interval $I \subset \mathbb{R}$ with speed measure m. It is well known that X has a jointly continuous density $p(t, x, y)$ with respect to m:

$$P_x(X_t \in A) = \int_A p(t, x, y)\, m(dy)\,. \tag{5.1}$$

The density is symmetric in x and y which implies that for diffusions the dual process is the diffusion itself. Assume that the X has a recurrent point a and that the point a is not an atom of the speed measure m. This implies that the inverse local time at a has no drift. With such assumptions the conclusions of the Theorem 2.1 hold with $\hat{X} = X$. Moreover, it is known that

$$E_b\left(e^{-\theta T_a}\right) = \frac{r_\theta(b, a)}{r_\theta(a, a)} \quad \text{and} \quad E_a\left(e^{-\theta \tau_u}\right) = e^{-\frac{u}{r_\theta(a,a)}}\,. \tag{5.2}$$

See Rogers and Williams [25], Sec. 50.

As the first example one can take X to be Brownian motion and $a = 0$. All the assumptions are satisfied. It follows that the two processes $(B_t : 0 \le t \le g_{S_\theta})$ and $(B_{S_\theta - t} : 0 \le t \le S_\theta - g_{S_\theta})$ are independent. For the first process we get

$$E_0\left(F(B_t : 0 \le t \le g_{S_\theta}) | L(g_{S_\theta}) = l\right) = \frac{E_0\left(F(B_t : 0 \le t \le \tau_u) e^{-\theta \tau_u}\right)}{E_0\left(e^{-\theta \tau_u}\right)}\,. \tag{5.3}$$

It is well known that

$$E_0\left(e^{-\theta\tau_l}\right)=e^{-l\sqrt{2\theta}} \quad \text{and} \quad E_b\left(e^{-\theta T_0}\right)=\exp\left(-|b|\sqrt{2\theta}\right).$$

where $T_0=\inf\{t\geq 0\colon B_t=0\}$. The first assertion follows from (i) in Theorem 2.1 and the fact that $L_t\stackrel{\mathrm{d}}{=}|B_t|$, see e.g. Revuz and Yor [24], p. 289. The hitting time distribution is an elementary consequence of the reflection principle for Brownian motion. The law of the second process, given $B_{S_\theta}=a$, is described by

$$E(G(B_{S_\theta-t}\colon 0\leq t\leq S_\theta-g_{S_\theta})|B_{S_\theta}=b)=\frac{E_b\left(F(B_s\colon 0\leq s\leq T_0)e^{-\theta T_0}\right)}{E_b\left(e^{-\theta T_0}\right)}. \tag{5.4}$$

which yields the result first obtained by Biane and Yor in [4]. See Leuridan [17] for an alternative elementary proof and Yen and Yor [28], Ch 9. for an alternative proof.

Since the bridge laws for Brownian motion with drift $B_t^{(\mu)}=B_t+\mu t$ are exactly the same for all drifts the conditional law of $(B_t^{(\mu)}\colon 0\leq t\leq S_\theta)$ given $\{B_{S_\theta}^{(\mu)}=b\}$ does not depend on μ. This means that Equations 5.3 and 5.4 hold for Brownian motion with drift with B instead of $B^{(\mu)}$. The only change is that the resolvent density changes to that of Brownian motion with drift

$$r_\theta(a,b)=\frac{1}{\sqrt{2\theta+\mu^2}}e^{\mu(b-a)-|b-a|\sqrt{2\theta+\mu^2}}.$$

The skew Brownian motion $X^{(\alpha)}$ with parameter $\alpha\in(0,1)$ is constructed by independently flipping the excursions of $|B_t|$ up with probability α and down with probability $1-\alpha$. A pair of dual processes with respect to Lebesgue measure are the processes $X^{(\alpha)}$ and $X^{(1-\alpha)}$. Both processes behave like Brownian motion away from 0 and the distribution of their local time at a fixed point is equal to the distribution of the local time of Brownian motion $X^{(1/2)}$. From the known transition densities of $X^{(\alpha)}$, see p. 82 in Revuz and Yor [24], it follows

$$r_\theta(0,b)=\left(2\alpha 1(b>0)+2(1-\alpha)1(b\leq 0)\right)e^{-|b|\sqrt{2\theta}}.$$

The skew Brownian motion satisfies all the assumptions made on the Markov process X. Equation 2.7 holds with the same $E(e^{-\theta\tau_u})$ as in the case of Brownian motion. Equation 2.8 holds with X replaced by standard Brownian motion started at 0.

Let X be a Bessel process of dimension $\delta\in(0,2)$. Denote $\nu=\delta/2-1\in(-1,0)$. It is well known, see Revuz and Yor [24], that 0 is a recurrent point for X, satisfying all the assumptions and that the time X spends at 0 has Lebesgue measure 0. The results of Theorem 2.1 apply. Bessel processes are dual to themselves under the speed measure with density $\xi(dx)=-\frac{x^{2\nu+1}}{\nu}dx$ on $[0,\infty)$. Let $I_\nu(z)$ and $K_\nu(z)$ be the modified Bessel functions with index ν. With respect to ξ the transition density of X for $a,b>0$ is given by

$$p_t(a,b)=\frac{1}{t(ab)^\nu}e^{-\frac{a^2+b^2}{2t}}I_\nu\left(\frac{ab}{t}\right).$$

Using formula 15.55 in Oberhettinger [19] for $0<a\leq b$ we get that

$$r_\theta(b,a)=\frac{2}{a^\nu b^\nu}I_\nu(a\sqrt{2\theta})K_\nu(b\sqrt{2\theta}).$$

By Equation 4.2

$$E_a\left(e^{-\theta T_b}\right) = \left(\frac{b}{a}\right)^{\nu} \frac{K_\nu(a\sqrt{2\theta})}{K_\nu(b\sqrt{2\theta})}$$

in accordance with Theorem 3.1 in Kent [15]. By $P_a(T_0 < \infty) = 1$ and the continuity of paths we have $T_a \uparrow T_0$ as $a \downarrow 0$. Letting $a \to 0$ and taking into account that $K_\nu(z) \sim \frac{\Gamma(-\nu)}{2}(z/2)^\nu$ for $z \to 0$ we get

$$E_b\left(e^{-\theta T_0}\right) = \frac{2^{1+\nu}}{\Gamma(-\nu)}\left(b\sqrt{2\theta}\right)^{\nu/2} K_\nu\left(b\sqrt{2\theta}\right).$$

By Pitman, Barlow and Yor [1] there is a bicontinuous family of local times L_t^b of the process X such that

$$\int_0^t f(X_s)ds = \frac{1}{2}\int_0^\infty b^{\delta-1}L_t^b\,db$$

for bounded measurable functions f. With this choice the inverse local time is a stable subordinator of index $-\nu$ with Laplace transform

$$E_0\left(e^{-\theta\tau_u}\right) = \exp\left(-u\cdot\frac{2^{1+\nu}\theta^{-\nu}\Gamma(1+\nu)}{\Gamma(-\nu)}\right).$$

5.2 Lévy processes

For a Lévy process X the Lebesgue measure is invariant and the dual process is $-X$. If the process has continuous densities for $t > 0$, is recurrent and spends Lebesgue measure 0 at points the conclusions of Theorem 2.1 can be applied.

An example is provided by symmetric stable processes of index $\alpha \in (1,2)$. These processes are recurrent and by scaling property the inverse local time is a subordinator of index $1 - 1/\alpha$. See Bertoin [2], Ch. 8. The independence of $(X_t : 0 \le t \le g_{S_\theta})$ and $(X_{g_{S_\theta}+u} : 0 \le u \le S_\theta - g_{S_\theta})$ and scaling imply that given g_1 the process $(X_t : 0 \le t \le g_1)$ is conditionally independent of $(X_{g_1+u} : 0 \le u \le 1 - g_1)$. This means that the two processes

$$\left(\frac{X_{tg_1}}{\sqrt{g_1}} : 0 \le t \le 1\right) \quad \text{and} \quad \left(\frac{X_{g_1+u(1-g_1)}}{\sqrt{1-g_1}} : 0 \le u \le 1\right) \tag{5.5}$$

are independent. Scaling also implies that the inverse local time τ_u is a stable subordinator of index $1 - 1/\alpha$ with $E(e^{-\theta\tau_u}) = e^{-u\psi(\theta)} = e^{-c\theta^{1-1/\alpha}}$ for some constant c depending on the normalization of the local time. From Equation 2.7 we can compute

$$E\left(e^{-\lambda g_{S_\theta}}\,\middle|\,L(S_\theta) = u\right) = \frac{E\left(e^{-(\lambda+\theta)\tau_u}\right)}{E\left(e^{-\theta\tau_u}\right)}. \tag{5.6}$$

Using the form of $\psi(\theta)$ and unconditioning using Equation 2.6 gives

$$E\left(e^{-\lambda g_{S_\theta}}\right) = \left(\frac{\theta}{\lambda+\theta}\right)^{1-1/\alpha}.$$

It follows that $g_{S_\theta} \sim \Gamma(1 - 1/\alpha, \theta)$ and by independence $S_\theta - g_{S_\theta} \sim \Gamma(1/\alpha, \theta)$. Using scaling again this gives the arc-sine law $g_1 \sim \text{Beta}(1 - 1/\alpha, 1/\alpha)$. This result is due to Chaumont [5]. See also Bertoin [2], p. 230.

For another application let Y be a Lévy process. Assume $P(Y_0 = 0) = 1$ and define $\bar{Y}_t = \sup_{s \leq t} Y_s$. The reflected process $X = \bar{Y} - Y$ is a Feller process in the right continuous filtration $\mathcal{F}_t$ of X, see Ch. 6 in Bertoin [2]. Denote by L its local time 0 and let $\tau_u = \inf\{t \colon L_t > u\}$ be the right continuous inverse of L. Denote by $(e_u \colon u > 0)$ the excursion process attached to the process X. It is in general not possible to reconstruct $\bar{Y}$ from the exursion process of the reflected process X. As noted by Greenwood and Pitman in their remark on p. 899 in [10], however, the process of excursions can be extended into a two dimensional Poisson point process such that $\bar{X}$ can be reconstructed. The idea is to add to the excursion at time $u > 0$ the jump of the ladder height process $H_u = \bar{Y}_{\tau_u}$. Denote $J_u = H_u - H_{u-}$. The resulting point process $((e_u, J_u) \colon u > 0)$ is a Poisson point process in the sense of Itô in the space $U \times (0, \infty)$.

Let S_θ be an exponential random variable with rate θ independent of Y. If the assumptions of Theorem 3.2 are met the following conclusions can be made: (i) The pairs of random variables $(g_{S_\theta}, \bar{X}_{S_\theta})$ and $(S_\theta - g_{S_\theta}, \bar{Y}_{S_\theta} - Y_{S_\theta})$ are independent. This follows from Theorem 2.1. (ii). The random pair $(g_\theta, \bar{X}_\theta)$ is infinitely divisible. To prove this statement first recall a standard result about Lévy processes: if Z is a d-dimensional Lévy process and S_θ is an independent exponential random variable, then the random variable (S_θ, Z_{S_θ}) is infinitely divisible. See Bertoin, [2] p. 162. By Equation 2.8 applied to the reflected process $\tilde{X}$ we find that the law of $(g_{S_\theta}, \bar{X}_{S_\theta})$ given $\{\tilde{L}_\theta = u\}$ is just like the sum of the points of the process $((R_u, J_u) \colon u > 0)$ where R_u is the excursion length at local time u and J_u is the jump of the ladder height process H. This last two-dimensional process is a map of the extended excursion process $((e_u, J_u) \colon u > 0)$ and as such a Poisson point process on $(0, \infty)^2$. Sums of Poisson processes are infinitely divisible so it follows that $(g_{S_\theta}, \bar{X}_{S_\theta})$ given $\{L_{S_\theta} = u\}$ is infinitely divisible. But L_{S_θ} is exponentially distributed and infinite divisibility follows. The infinite divisibility of the pair $(S_\theta - g_{S_\theta}, \bar{Y}_{S_\theta} - Y_{S_\theta})$ follows by duality arguments. See Lemma 9 in Bertoin [2], p. 164.

The assertions about infinite divisibility and independence are true in general without additional assumptions on the reflected process X. See Greenwood and Pitman [9] for details.

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Euler's divergent series and an elementary model in Statistical Physics

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Abstract

We discuss the multiple integral of a multivariate exponential taken with respect either to the Lebesgue measure or to the discrete uniform Bernoulli measure. In the first case the integral is linked to Euler's everywhere divergent power series and its generalizations, while in the second case the integral is linked to a one-dimensional model of spin systems as encountered in physics.

Keywords: Euler divergent series, Abel-Plana Formula, Stirling numbers, spin system, Ising chain.

Math. Subj. Class.: 33E20, 28A35, 82B44, 82D30

1 Introduction

Consider the integral $(N \geq 1)$

$$Z_N(x) = \int_{\mathbb{R}^N} \exp\left(-H\left(\sum_{j=1}^{N} u_j\right) - x\prod_{j=1}^{N} u_j\right) \mu\left(\prod_{j=1}^{N} du_j\right) \tag{1.1}$$

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If μ is the Lebesgue measure on $(0,\infty)^N$ and $H = 1$, the integral is linked to the series

$$\sum_{n\geq 0}(-1)^n(n!)^{N-1}x^n$$

which, for $N = 2$, is attributed to Euler. If $N = 1$ the series reduces to $(1+x)^{-1}$ (convergent for $|x| < 1$) and for $N \geq 2$ it diverges for all $x \neq 0$.

If on the other hand, μ is the Bernoulli measure on the set $\{-1,1\}^N$ then the integral reads

$$Z_N(x) = \frac{1}{2^N}\sum_{\underline{u}\in\{\pm 1\}^N}\exp\left(-H\left(\sum_{j=1}^N u_j\right) - x\prod_{j=1}^N u_j\right)$$

and could represent a certain spin system described in Section 6.

2 Euler's divergent series

If μ is the Lebesgue measure on $(0,\infty)^N$, we suppose that $H > 0$ and $x \geq 0$. There is no loss of generality in the choice $H = 1$ in Formula 1.1; take new variables $v_j = Hu_j$. Integrate

$$Z_N(x) = \int_{(0,\infty)^N}\exp\left(-\left(\sum_{j=1}^N u_j\right) - x\prod_{j=1}^N u_j\right)\prod_{j=1}^N du_j$$

with respect to du_N to obtain

$$Z_N(x) = \int_{(0,\infty)^{N-1}}\frac{\exp\left(-\left(\sum_{j=1}^{N-1}u_j\right)\right)}{1+x\prod_{j=1}^{N-1}u_j}\prod_{j=1}^{N-1}du_j.$$

Suppose $N \geq 2$ since the case $N = 1$ is trivial. Z_N converges for all complex x outside the negative real axis $(-\infty, 0)$. Expand $\left(1+x\prod_{j=1}^{N-1}u_j\right)^{-1}$ into a formal power series

$$Z_N(x) = \int_{(0,\infty)^{N-1}}\exp\left(-\sum_{j=1}^{N-1}u_j\right)\sum_{n\geq 0}(-1)^n x^n\prod_{j=1}^{N-1}{u_j}^n\prod_{j=1}^{N-1}du_j.$$

If we accept to permute the summation with the integral, then

$$Z_N(x) = \sum_{n\geq 0}(-1)^n x^n\prod_{j=1}^{N-1}\int_{(0,\infty)}u_j^n\exp(-u_j)du_j = \sum_{n\geq 0}(-1)^n(n!)^{N-1}x^n.$$

What happens if $N = 1$ or 2? The case $N = 1$ is trivial yet interesting,

$$Z_1(x) = \int_0^\infty \exp(-u-xu)du = \frac{1}{1+x}\cdot$$

Expanding the integral with respect to x we obtain $Z_1(x) = \sum_{n\geq 0}(-1)^n x^n$ and for $x = 1$ we rediscover the well-known "equality" $\sum_{n\geq 0}(-1)^n = \frac{1}{2}\cdot$

$$\int_0^\infty \frac{\exp(-u)}{1+u} du = 0.5963\ldots$$

and therefore concluded

$$\sum_{n\geq 0} (-1)^n n! = 0.5963\ldots$$

a most astonishing equality! In his beautiful book G. H. Hardy [2] discusses in detail this case $N = 2$.

Remark 2.1. The constant $\int_0^\infty \frac{\exp(-u)}{1+u} du = 0.5963\ldots$ is called the Euler or the Euler-Gompertz constant (see [3], [1, Section 6.2], and in particular [1, Section 6.2.4] for the name "Gompertz"). Among the numerous results related to this constant we do not resist to write the following continued fraction expansion:

$$\int_0^\infty \frac{\exp(-u)}{1+u} du = \cfrac{1}{2 - \cfrac{1}{4 - \cfrac{4}{6 - \cfrac{9}{8 - \ddots}}}}$$

This continued fraction expansion is sometimes attributed to Stieltjes, but in [8] Stieltjes indicated that it was studied by Laguerre. We found indeed in [5, p. 154] that Laguerre considered e times the Prym function[1] $eQ(\alpha) = \int_1^\infty e^{1-x} x^{\alpha-1} dx$ and obtained as consecutive approximations of $eQ(0)$ the sequence

$$\frac{4}{7}, \frac{20}{34}, \frac{124}{209}, \frac{920}{2546}, \frac{7940}{13327}, \ldots$$

which are exactly the values of the first few truncatures of the above continued fraction (also see Laguerre [4, p. 77]). Of course $eQ(0) = \int_0^\infty \frac{\exp(-u)}{1+u} du = Z_2(1)$: it would thus be interesting to obtain such nice continued fraction expansions for the quantities $Z_N(1)$.

More generally, a formula given by Tannery in [9, p. 1699] or an easy rewriting of a formula given by Laguerre in [4, end of Page 75] reads

$$e^x \int_x^\infty \frac{e^{-t}}{t} dt = \cfrac{1}{x+1 - \cfrac{1}{x+3 - \cfrac{4}{x+5 - \cfrac{9}{x+7 - \ddots}}}}$$

But

$$Z_2(x) = \int_0^\infty \frac{\exp(-u)}{1+xu} du = \frac{1}{x} \int_0^\infty \frac{e^{-u}}{\frac{1}{x} + u} du = \frac{1}{x} e^{1/x} \int_{1/x}^\infty \frac{e^{-t}}{t} dt.$$

[1] Note that there seems to be a misprint in the formula given by Laguerre, where e^{1-x} is replaced by e^{-x}, see the original definition by Prym [7, p. 169].

Hence

$$Z_2(x) = \cfrac{1}{1+x-\cfrac{x^2}{1+3x-\cfrac{4x^2}{1+5x-\cfrac{9x^2}{1+7x-\ddots}}}}$$

3 The Borel operator

The sequence Z_N can be defined recursively by means of the so-called Borel operator

$$B : f \mapsto \int_0^\infty \exp(-u) f(ux) du.$$

The Borel operator applies the series $\sum_{n\geq 0} f^{(n)}(0)\frac{x^n}{n!}$ onto $\sum_{n\geq 0} f^{(n)}(0)x^n$.

Using the relation $Z_0(x) = \exp(-x)$ and $Z_{N+1} = BZ_N$, we see that the integral Z_N is therefore the Nth iterate B^N of $x \mapsto \exp(-x)$, or equivalently the $(N-1)$st iterate B^{N-1} of $x \mapsto (1+x)^{-1}$.

4 The Abel-Plana summation and the Γ function

In this section we study the behavior of Z_N when N goes to infinity. Note that for real $x \geq 0$, the sequence $N \mapsto Z_N(x)$ is bounded from above by 1 and furthermore it is increasing. Indeed let $\Delta_N(x) = Z_{N+1}(x) - Z_N(x)$ and $\Pi_N(x) = x\prod_{j=1}^N u_j$. Then

$$\Delta_N(x) = \int_{(0,\infty)^N} \exp\left(-\sum_{j=1}^N u_j\right)\left(\frac{1}{1+\Pi_N(x)} - \exp\left(-\Pi_N(x)\right)\right)\prod_{j=1}^N du_j.$$

Since $\frac{1}{1+t} - \exp(-t) \geq 0$, $Z_{N+1}(x) \geq Z_N(x)$ as claimed. Therefore $Z_N(x)$ tends to a limit which we now compute.

Theorem 4.1. *For all real $x \geq 0$, we have $\lim_{N\to\infty} Z_N(x) = 1$.*

Proof. Since the result is trivial for $x = 0$, we may assume $x > 0$. We note that $Z_N(x)$ can be written as a diverging series

$$Z_N(x) = \sum_{n\geq 0} (-1)^n f_N(n)$$

where $f_N : s \mapsto \Gamma(1+s)^{N-1}x^s$ is an analytic function on the half-plane $\Re(s) > -1$.

By blindly applying the Abel-Plana Formula (see [6, III, formula X]) to this series, we get

$$\begin{aligned} Z_N(x) &= -\int_{-1/2-i\infty}^{-1/2+i\infty} \frac{\Gamma(1+z)^{N-1}x^z}{2i\sin(\pi z)} dz \\ &= \int_{-\infty}^{+\infty} \frac{\Gamma(1/2+it)^{N-1}x^{-1/2+it}}{2\cosh(\pi t)} dt \end{aligned}$$

or by displacing the integration contour,

$$
\begin{aligned}
Z_N(x) &= 1-\int_{1/2-i\infty}^{1/2+i\infty}\frac{\Gamma(1+z)^{N-1}x^z}{2i\sin(\pi z)}dz && (4.1)\\
&= 1-\int_{-\infty}^{+\infty}\frac{\Gamma(3/2+it)^{N-1}x^{1/2+it}}{2\cosh(\pi t)}dt && (4.2)
\end{aligned}
$$

The convergence of the integrals is provided by the fact the Γ function decreases like $\exp(-\frac{\pi}{2}|z|)$ as z goes to $-1/2\pm i\infty$ (resp. $1/2\pm i\infty$), and $\sin(\pi z)$ increases like $\exp(\pi|z|)$.

Strictly speaking, the Abel-Plana Theorem only applies for $N=0$. However, by applying the Borel operator to the right-hand side and interverting the summations by Fubini's Theorem, we find that

$$F_N(x)=\int_{-\infty}^{+\infty}\frac{\Gamma(1/2+it)^{N-1}x^{-1/2+it}}{2\cosh(\pi t)}dt$$

satisfies the same recursion as $Z_N(x)$. Indeed,

$$
\begin{aligned}
BF_N(x) &= \int_0^{\infty}\exp(-u)\int_{-\infty}^{+\infty}\frac{\Gamma(1/2+it)^{N-1}(xu)^{-1/2+it}}{2\cosh(\pi t)}dtdu\\
&= \int_{-\infty}^{+\infty}\frac{\Gamma(1/2+it)^{N-1}x^{-1/2+it}}{2\cosh(\pi t)}\int_0^{\infty}u^{-1/2+it}\exp(-u)dudt.
\end{aligned}
$$

From the identity $\Gamma(1/2+it)=\int_0^\infty u^{-1/2+it}\exp(-u)du$, it follows

$$BF_N(x)=\int_{-\infty}^{+\infty}\frac{\Gamma(1/2+it)^{N}x^{-1/2+it}}{2\cosh(\pi t)}dt=F_{N+1}(x)$$

therefore $F_N=Z_N$.

Now since $|\Gamma(3/2+it)|\le\frac{\sqrt{\pi}}{2}<1$ for all $t\in\mathbb{R}$, the integral (4.2) converges to 0 when N goes to infinity for all real $x\geq 0$, thus we have proved:

$$\lim_{N\to\infty}Z_N(x)=1.$$

□

To conclude this section, we note that this formula for Z_N involves a single integral which is much more suitable for numerical computations than the original formula involving a multiple integral. Note also that N need not be an integer…

5 A differential equation

It might be worthwhile to mention that the function

$$Z_N(x)=\int_{(0,\infty)^N}\exp\left(-\left(\sum_{j=1}^N u_j\right)-x\prod_{j=1}^N u_j\right)\prod_{j=1}^N du_j$$

is a solution of a differential equation of order $N-1$ with polynomial coefficients.

Indeed, the shortest way to establish this is to introduce the linear operator U defined by $U(z) = (xz)'$. Clearly $U(x^n) = (n+1)x^n$ so that $U^k(x^n) = (n+1)^k x^n$. Then

$$\begin{aligned} U^{N-1} Z_N(x) &= U^{N-1} \sum_{n\geq 0} (-1)^n (n!)^{N-1} x^n \\ &= \sum_{n\geq 0} (-1)^n (n!)^{N-1} (n+1)^{N-1} x^n \\ &= \sum_{n\geq 0} (-1)^n ((n+1)!)^{N-1} x^n \ ; \\ x U^{N-1} Z_N(x) &= \sum_{n\geq 0} (-1)^n ((n+1)!)^{N-1} x^{n+1} \\ &= 1 - Z_N(x). \end{aligned}$$

The function $Z_N(x)$ is thus solution of the $(N-1)$-st order differential equation

$$xU^{N-1} y + y = 1$$

with initial conditions

$$y(0) = 1, y'(0) = -1, \ldots, y^{(N-2)}(0) = (-1)^{n-2} \left((N-2)!\right)^{N-1}.$$

The reader may well criticize the above proof since it involves divergent series. There is however no problem in justifying the result by applying the operator U to the integral representation of $Z_N(x)$; the calculations are just slightly more cumbersome.

Example 5.1. $Z_2(x), Z_3(x), Z_4(x)$ are respectively solution of the equations

$$\begin{aligned} x^2 y' + (x+1)y = 1 \\ x^3 y'' + 3x^2 y' + (x+1)y = 1 \\ x^4 y''' + 6x^3 y'' + 7x^2 y' + (x+1)y = 1 \end{aligned}$$

The reader will recognize the numbers above as the Stirling numbers of the second kind. This can be proved by noting that both families of numbers obey the formula

$$a_{n+1,k} = k a_{n,k} + a_{n,k-1}.$$

6 An unconventional spin system

We now assume that μ is the Bernoulli measure on $\{-1,+1\}^N$:

$$Z_N(x) = \frac{1}{2^N} \sum_{\underline{u} \in \{\pm 1\}^N} \exp\left(-H\left(\sum_{j=1}^N u_j \right) - x \prod_{j=1}^N u_j \right).$$

We interpret Z_N as the partition function of a certain spin system which we describe below. Conventional spin systems are discussed for example in C. J. Thompson [10].

Imagine an N-component particle, each component of which has a spin $u_j = \pm 1$, and which are instantaneously influenced by the $N-1$ others. The "total" spin of the particle, i.e., its sign is $\prod_{j=1}^{N} u_j$. A real external field H acts on the spins. The Hamiltonian attached to the spin system in state $\underline{u} = (u_1, u_2, \ldots, u_N)$ with external field $-H$ is then given by

$$x \prod_{j=1}^{N} u_j + H \sum_{i=1}^{N} u_j.$$

The behavior of the spin system is controlled by the partition function, in particular by its thermodynamical limit

$$\lim_{N\to\infty} \frac{\log Z_N(x)}{N}.$$

Theorem 6.1. *For all real $x \geq 0$,*

$$Z_N(x) = \cosh(x)\cosh(H)^N - (-1)^N \sinh(x)\sinh(H)^N.$$

Proof. By using the relation $\exp(-t) = \cosh(t) - \sinh(t)$, we write

$$Z_N(x) = \frac{1}{2^N} \sum_{\underline{u}\in\{\pm 1\}^N} \exp\left(-H\left(\sum_{j=1}^{N} u_j\right)\right)\left(\cosh(x\prod_{j=1}^{N} u_j) - \sinh(x\prod_{j=1}^{N} u_j)\right).$$

Since $\prod_{j=1}^{N} u_j = \pm 1$, $\cosh$ is even and $\sinh$ is odd, it follows that

$$Z_N(x) = \frac{1}{2^N} \sum_{\underline{u}\in\{\pm 1\}^N} \exp\left(-H\left(\sum_{j=1}^{N} u_j\right)\right)\left(\cosh(x) - \sinh(x)\prod_{j=1}^{N} u_j\right). \tag{6.1}$$

The following two formulas are easily proved by recursion on N:

$$\begin{aligned} \sum_{\underline{u}\in\{\pm 1\}^N} \exp\left(H\left(\sum_{j=1}^{N} u_j\right)\right) &= (2\cosh(H))^N \\ \sum_{\underline{u}\in\{\pm 1\}^N} \exp\left(H\left(\sum_{j=1}^{N} u_j\right)\right)\prod_{j=1}^{N} u_j &= (2\sinh(H))^N. \end{aligned}$$

From Equation (6.1) it follows:

$$Z_N(x) = \cosh(x)\cosh(H)^N - (-1)^N \sinh(x)\sinh(H)^N.$$

□

Remark 6.2. Theorem 6.1 above implies that $Z_N(x) \underset{N\to\infty}{\sim} \cosh(H)^N \cosh(x)$, so that

$$\lim_{N\to\infty} \frac{\log Z_N(x)}{N} = \log\cosh(H)$$

which happens to be independent of x and which is continuous with respect to H. The system has no critical value of the external field and therefore presents no phase transition.

7 A disturbed Ising chain

In the preceding section we described an unconventional spin system. We now turn to the most familiar one, namely the one dimensional Ising chain (see [10]) with Hamiltonian

$$H\sum_{j=1}^{N} u_j + J\sum_{j=1}^{N} u_j u_{j+1}$$

where J is a "coupling constant". Actually this Hamiltonian corresponds to the parameters $-H$ and $-J$ but that makes no essential difference for our computation.

We consider in fact a perturbed Ising chain with the additional term $x\prod_{j=1}^{N} u_j$. The Hamiltonian is therefore

$$\mathcal{H}(\underline{u}) = H\sum_{j=1}^{N} u_j + J\sum_{j=1}^{N} u_j u_{j+1} + x\prod_{j=1}^{N} u_j$$

and the partition function is now

$$Y_N = \frac{1}{2^N}\sum_{\underline{u}\in\{\pm 1\}^N} \exp(-\mathcal{H}(\underline{u}))$$

which we propose to compute where we need to specify u_{N+1}. Following most textbooks, we simplify the model by assuming that the chain is cyclic: $u_{N+1} = u_1$.

Theorem 7.1. *Define*

$$\begin{aligned}
\lambda_{\pm} &= \exp(-J)\cosh(H) \pm \left(\exp(-2J)\cosh(H)^2 + 2\sinh(2J)\right)^{\frac{1}{2}},\\
\overline{\lambda}_{\pm} &= \exp(-J)\sinh(H) \pm \left(\exp(-2J)\sinh(H)^2 - 2\sinh(2J)\right)^{\frac{1}{2}}.
\end{aligned}$$

Then

$$Y_N = \frac{1}{2^N}\cosh(x)(\lambda_+^N + \lambda_-^N) - \frac{(-1)^N}{2^N}\sinh(x)(\overline{\lambda}_+^N + \overline{\lambda}_-^N).$$

Proof. Observe as in Section 6 that

$$Y_N = \frac{\cosh x}{2^N}Y'_N - \frac{\sinh x}{2^N}Y''_N$$

where

$$\begin{aligned}
Y'_N &= \sum_{\underline{u}\in\{\pm 1\}^N} \exp\left(-H\sum_{j=1}^{N} u_j - J\sum_{j=1}^{N} u_j u_{j+1}\right),\\
Y''_N &= \sum_{\underline{u}\in\{\pm 1\}^N} \exp\left(-H\sum_{j=1}^{N} u_j - J\sum_{j=1}^{N} u_j u_{j+1}\right)\prod_{j=1}^{N} u_j.
\end{aligned}$$

The classical way to compute Y'_N is to introduce the 2×2 transfer matrix

$$L_1 = \begin{pmatrix} L_1(1,1) & L_1(1,-1) \\ L_1(-1,1) & L_1(-1,-1) \end{pmatrix}$$

where

$$L_1(u_1,u_2)=\exp\left(-\frac{H}{2}(u_1+u_2)-Ju_1u_2\right).$$

In other words

$$L_1=\begin{pmatrix}\exp(-H-J) & \exp(J)\\ \exp(J) & \exp(H-J)\end{pmatrix}.$$

Then

$$\begin{aligned} Y'_N &= \sum_{\underline{u}\in\{\pm1\}^N} L_1(u_1,u_2)L_1(u_2,u_3)\dots L_1(u_N,u_1)\\ &= \sum_{u_1\in\{\pm1\}} L_1^N(u_1,u_1)=\mathrm{Trace}(L_1^N)=\lambda_+^N+\lambda_-^N\end{aligned}$$

where λ_+ and λ_- are the eigenvalues of L_1, i.e., the solutions of

$$\lambda^2-2\lambda\exp(-J)\cosh(H)+\exp(-2J)-\exp(2J)=0.$$

Therefore

$$\lambda_\pm=\exp(-J)\cosh(H)\pm\left(\exp(-2J)\cosh(H)^2+2\sinh(2J)\right)^{\frac{1}{2}}.$$

The computation of Y''_N is quite similar. Let

$$L_2=\begin{pmatrix}L_2(1,1) & L_2(1,-1)\\ L_2(-1,1) & L_2(-1,-1)\end{pmatrix}$$

where

$$L_2(u_1,u_2)=u_1\exp\left(-\frac{H}{2}(u_1+u_2)-Ju_1u_2\right)$$

so that

$$L_2=\begin{pmatrix}\exp(-H-J) & \exp(J)\\ -\exp(J) & -\exp(H-J)\end{pmatrix}$$

then

$$\begin{aligned} Y''_N &= \sum_{\underline{u}\in\{\pm1\}^N} L_2(u_1,u_2)L_2(u_2,u_3)\dots L_2(u_N,u_1)\\ &= \sum_{u_1\in\{\pm1\}} L_2^N(u_1,u_1)=\mathrm{Trace}(L_2^N)=\overline{\lambda}_+^N+\overline{\lambda}_-^N\end{aligned}$$

where $\overline{\lambda}_+$ and $\overline{\lambda}_-$ are the eigenvalues of L_2, i.e., the solutions of

$$\lambda^2+2\lambda\exp(-J)\sinh(H)-\exp(-2J)+\exp(2J)=0.$$

Therefore

$$\overline{\lambda}_\pm=-\exp(-J)\sinh(H)\pm\left(\exp(-2J)\sinh(H)^2-2\sinh(2J)\right)^{\frac{1}{2}}.$$

Finally

$$Y_N=\frac{1}{2^N}\cosh(x)(\lambda_+^N+\lambda_-^N)-\frac{(-1)^N}{2^N}\sinh(x)(\overline{\lambda}_+^N+\overline{\lambda}_-^N).$$

□

Remark 7.2. The reader will easily verify that for $J = 0$ we obtain the value of Z_N computed in Section 6.

Remark 7.3. It is not difficult to see that $\max\{|\lambda_-|, |\overline{\lambda}_-|, |\overline{\lambda}_+|\} < \lambda_+$. Hence $Y_N \sim \frac{1}{2^N}\cosh(x)(\lambda_+^N)$ when N goes to infinity. This implies that the following limit exists, is continuous in both variables J and H, and is independent of x (as in Remark 6.2); therefore the system has no phase transition:

$$\begin{aligned}\lim_{N\to\infty}\frac{\log Y_N}{N} &= \log\frac{\lambda_+}{2}\\ &= \log\left(\frac{(\exp(-J)\cosh(H)}{2} + \frac{(\exp(-2J)\cosh(H)^2 + 2\,(\sinh(2J))^{\frac{1}{2}}}{2}\right).\end{aligned}$$

8 Conclusion and acknowledgements

This article illustrates a classical fact, namely that one formula may well lead to far distant and unexpected developments. Unifying themes is probably one of the most exciting aspects of mathematics.

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The distinguishing index of the Cartesian product of finite graphs*

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Abstract

The *distinguishing index* $D'(G)$ of a graph G is the least natural number d such that G has an edge colouring with d colours that is only preserved by the identity automorphism. In this paper we investigate the distinguishing index of the Cartesian product of connected finite graphs. We prove that for every $k \geq 2$, the k-th Cartesian power of a connected graph G has distinguishing index equal 2, with the only exception $D'(K_2^2) = 3$. We also prove that if G and H are connected graphs that satisfy the relation $2 \leq |G| \leq |H| \leq 2^{|G|}\big(2^{\|G\|} - 1\big) - |G| + 2$, then $D'(G \Box H) \leq 2$ unless $G \Box H = K_2^2$.

Keywords: Edge colouring, symmetry breaking, distinguishing index, Cartesian product of graphs.

Math. Subj. Class.: 05C15, 05E18

1 Introduction

We use standard graph theory notation (cf. [6]). In particular, $\mathrm{Aut}(G)$ denotes the automorphism group of a graph G.

An edge colouring *breaks an automorphism* $\varphi \in Aut(G)$ if φ does not preserve the colouring, i.e., there exists an edge of G that is mapped by φ to an edge of different colour. The *distinguishing index* $D'(G)$ of a graph G is the least natural number d such that G has an edge colouring with d colours that breaks all non-trivial automorphisms of G. Such a d-colouring is called *distinguishing*. This notion was introduced by Kalinowski and Pilśniak [10] as an analogue of the well-known *distinguishing number* $D(G)$ of a graph G defined by Albertson and Collins [1] as the least number of colours in a vertex colouring that breaks all non-trivial automorphisms of G. [1] As the distinguishing index is not defined for K_2, we assume henceforth that K_2 is not a connected component of any graph considered.

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[1] Fisher and Isaak [5] considered distinguishing edge colourings of complete bipartite graphs, but did not introduce any special notation or terminology.

The distinguishing index of several examples of graphs was exhibited in [10]. For instance, $D'(P_n) = D(P_n) = 2$, for $n \geq 3$; $D'(C_n) = D(C_n) = 2$, for $n \geq 6$, and $D'(C_n) = 3$, for $n = 3, 4, 5$. There exist graphs G for which $D'(G) < D(G)$, for instance $D'(K_n) = D'(K_{p,p}) = 2$, for any $n \geq 6$ and $p \geq 4$, while $D(K_n) = n$ and $D(K_{p,p}) = p+1$. It is also possible that $D'(G) > D(G)$. All trees satisfying this inequality were characterized in [10]. The following general upper bound of the distinguishing index was proved in [10].

Theorem 1.1. [10] *If G is a finite connected graph of order $n \geq 3$, then $D'(G) \leq D(G) + 1$. Moreover, if Δ is the maximum degree of G, then $D'(G) \leq \Delta$ unless G is a C_3, C_4 or C_5.*

The distinguishing index was also investigated for infinite graphs [2] and their Cartesian product [3].

The Cartesian product of graphs G and H is a graph, denoted $G \Box H$, whose vertex set is $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent if either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. Denote $G \Box G$ by G^2, and recursively define the *k-th Cartesian power* of G as $G^k = G \Box G^{k-1}$.

A non-trivial graph G is called *prime* if $G = G_1 \Box G_2$ implies that either G_1 or G_2 is K_1. It is easy to see that every non-trivial finite graph has a prime factorization with respect to the Cartesian product. For connected graphs it is also unique up to isomorphisms and the order of the factors, as has been shown by Sabidussi and Vizing (cf. [6]). Two graphs G and H are called *relatively prime* if K_1 is the only common factor of G and H.

Let v be a vertex of H. A *G^v-layer* (called also a *horizontal layer* of $G \Box H$) is the subgraph induced by the vertex set $\{(u, v) : u \in V(G)\}$. An *H^u-layer*, or *vertical* layer, is defined analogously for a vertex u of G. Clearly, each horizontal layer is isomorphic to G and each vertical one is isomorphic to H. Therefore, speaking of a specified layer of $G \Box H$, we shall usually use only one coordinate of a vertex. The same refers to edges.

We shall need knowledge of the structure of the automorphism group of the Cartesian product, which was determined by Imrich [7], and independently by Miller [11].

Theorem 1.2. [7, 11] *Suppose ψ is an automorphism of a connected graph G with prime factor decomposition $G = G_1 \Box G_2 \Box \ldots \Box G_r$. Then there is a permutation π of the set $\{1, 2, \ldots, r\}$ and there are isomorphisms $\psi_i \colon G_{\pi(i)} \mapsto G_i$, $i = 1, \ldots, r$, such that*

$$\psi(x_1, x_2, \ldots, x_r) = (\psi_1(x_{\pi(1)}), \psi_2(x_{\pi(2)}), \ldots, \psi_r(x_{\pi(r)})).$$

It follows in particular that every automorphism of the Cartesian product of two relatively prime graphs is a composition of a permutation of vertical layers generated by an automorphism of G and a permutation of horizontal layers generated by an automorphism of H. For additional results about the Cartesian product consult [6].

Our main results are extensions of theorems about the distinguishing number of Cartesian powers and of Cartesian products of connected graphs to the distinguishing index. The results (and some of the proofs) are inspired by a paper [8] by Imrich, Jerebic and Klavžar. In Section 2 we generalize a result of Imrich and Klavžar.

Theorem 1.3. [9] *Let G be a connected graph and $k \geq 2$. Then $D(G^k) = 2$ except for the graphs K_2^2, K_2^3, K_3^2 whose distinguishing number is three.*

The second result that we extend is also due to Imrich and Klavžar:

Theorem 1.4. [9] *Let G and H be connected, relatively prime graphs such that*

$$|G| \leq |H| \leq 2^{|G|} - |G| + 1.$$

Then $D(G \square H) \leq 2$.

In Section 3 we prove an analogous result (Theorem 3.4) for the distinguishing index of the Cartesian product of connected graphs, not necessarily relatively prime (let us note that, using our method of proof, Theorem 3.4 was already strengthened in [4] by omitting the assumption that G and H are relatively prime). We also obtain a slightly stronger result for trees (Theorem 3.1).

In proofs, we usually use colours $1, \ldots, d$. If $d = 2$, then we also use colours 0 and 1, or alternatively red and blue.

2 Distinguishing Cartesian powers

Let us start with the Cartesian powers of K_2. Recall that the k-dimensional hypercube is isomorphic to K_2^k and denoted by Q_k. As mentioned earlier, the distinguished index is not defined for $K_2 = Q_1$. Clearly, $D'(Q_2) = 3$ since $Q_2 = C_4$. The following result was proved in [13].

Theorem 2.1. [13] *If a graph G of order at least 7 contains a Hamiltonian path, then $D'(G) \leq 2$.*

Proposition 2.2. *If $k \geq 3$, then $D'(Q_k) = 2$.*

Proof. For $k \geq 3$ a hypercube Q_k is Hamiltonian and has at least eight vertices. Therefore, $D'(Q_k) = 2$ by Theorem 2.1. □

The distinguishing index of the square of cycles and for arbitrary powers of complete graphs with respect to the Cartesian, direct and strong products has been already considered by Pilśniak [12]. In particular, she proved that $D'(C_m^2) = 2$ for $m \geq 4$, and $D'(K_n^k) = 2$ for any $n \geq 4$ and $k \geq 2$.

Here we consider Cartesian powers of arbitrary connected graphs. We first prove some lemmas.

Lemma 2.3. *Let G and H be connected, relatively prime graphs with $D'(G) = D'(H) = 2$. Then $D'(G \square H) = 2$.*

Proof. We colour one G-layer and one H-layer with distinguishing 2-colourings. The remaining edges can be coloured arbitrarily. Such a colouring breaks all permutations of both horizontal and vertical layers. Since G and H are relatively prime, it follows from Theorem 1.2 that this colouring breaks all automorphisms of $G \square H$. □

Lemma 2.4. *Let G and H be two connected graphs where G is prime, $|G| \leq \|H\| + 1$ and $D'(H) = 2$. Then $D'(G \square H) = 2$.*

Proof. We first colour the H-layers of the graph $G \square H$. There are at least two H-layers, so we colour all edges of one layer blue, all edges of another one with a distinguishing red-blue colouring. If there are more H-layers, then we colour them such that each of them has a different number of blue edges (including the H-layers coloured previously). This is possible since $|G| \leq \|H\| + 1$. Next, we colour all edges in every G-layer red.

All automorphisms of the Cartesian product generated by the automorphisms of H are broken, since one H-layer assumes a distinguishing colouring. Also, no H-layers can be interchanged as every H-layer has different number of blue edges.

If H has a factor H' isomorphic to G, then $G \square H$ has an automorphism interchanging H' with G. However, since all G-layers have only red edges and there exists an H-layer with only blue edges, such an automorphism does not preserve this colouring. □

Lemma 2.5. *If H is a graph with $2 \leq D'(H) = d$, then*

$$2 \leq D'(H \square K_2) \leq d.$$

Proof. We colour the edges of one H-layer with a distinguishing d-colouring, and all the edges of the other H-layer with the same colour, say 1. Next, we colour all edges of K_2-layers with colour 2. Thus all automorphisms of the Cartesian product $H \square K_2$ generated by the automorphisms of H are broken, since one of the H-layers assumes a distinguishing colouring. Also, the two H-layers cannot be interchanged as they have different numbers of edges coloured with 1.

If H has a factor H' isomorphic to K_2, then $K_2 \square H$ has an automorphism interchanging H' with K_2. However, since all K_2-layers have only colour 2 and there exists an H-layer with all edges coloured with 1, such an automorphism does not preserve the colouring.

The equality for $d = 2$ is obvious since the prism of every graph has a non-trivial automorphism. □

We now consider the Cartesian powers of arbitrary connected graphs and continue with powers of connected prime graphs on at least three vertices.

Lemma 2.6. *If G is a connected prime graph with $|G| \geq 3$, then $D'(G^k) = 2$ for every $k \geq 2$.*

Proof. The proof goes by induction on k. Let $k = 2$. There are n horizontal and n vertical layers, where $n = |G|$.

Suppose first that G contains a cycle, i.e., $\|G\| \geq n$. We colour horizontal G-layers with two colours such that each of them has a different number of blue edges between 0 and $n - 1$. The other edges are coloured such that every vertical G-layer has a different number of blue edges between 1 to n. As every horizontal G-layer has a different number of blue edges they cannot be interchanged. The same is true for vertical G-layers. Therefore automorphisms of the Cartesian product generated by automorphisms of G are broken. Our colouring also breaks interchanging the factors, since there exists a completely red horizontal G-layer but no such vertical G-layer.

Assume now that G is a tree. Every tree has either a central vertex or a central edge fixed by every automorphism. In case of a tree with a central vertex v, we colour the edges of G^2 such that consecutive horizontal layers have $0, \ldots, n-1$ blue edges, and consecutive vertical layers have $0, \ldots, n - 1$ blue edges in such a way that the horizontal G^v-layer and the vertical G^v-layer have all edges coloured red and blue, respectively. The vertex (v, v) is fixed by every automorphism of G^2, hence this colouring is distinguishing. If G has a central edge $e_0 = uv$, we colour the edge $(u, u)(v, u)$ red and the remaining three edges of the subgraph $e_0 \square e_0$ blue. The vertical and horizontal G^v-layers have all edges blue and red, respectively. The remaining edges of G^2 are coloured as in the case of a tree with a central vertex. Such colouring forbids exchange of the horizonal layers with the vertical layers. Thus $D'(G^2) = 2$.

For the induction step, we apply Lemma 2.4 by taking $H = G^{k-1}$ since $|G| \leq \|G^{k-1}\| + 1$. $\square$

Let us now state the main theorem of this section that solves the problem of the distinguishing index of the k-th Cartesian power of a connected graph.

Theorem 2.7. *Let G be a connected graph and $k \geq 2$. Then*

$$D'(G^k) = 2$$

with the only exception: $D'(K_2^2) = 3$.

Proof. Let $G = G_1^{p_1} \square G_2^{p_2} \square \dots \square G_r^{p_r}$, where $p_i \geq 1, i = 1, \dots, r$, be the prime factor decomposition of G.

Assume first that $G_i \neq K_2$, $i = 1, 2, \dots, r$. Then for every i we have $D'(G_i^{kp_i}) = 2$ due to Lemma 2.6. By repetitive application of Lemma 2.3 we get $D'(G^k) = 2$ since $G_i^{kp_i}$ and $G_j^{kp_j}$ are relatively prime if $i \neq j$.

Suppose now that G has a factor isomorphic to K_2, say $G_1 = K_2$. If $p_1 \geq 2$, then $D'(K_2^{kp_1}) = 2$ and again $D'(G^k) = 2$ by Lemma 2.3 applied to $K_2^{kp_1}$ and $G_2^{p_2} \square \dots \square G_r^{p_r}$. The same argument applies in case $p_1 = 1$ and $k \geq 3$. Finally, if $p_1 = 1$ and $k = 2$ we apply Lemma 2.4 twice and we also get $D'(G^k) = 2$ unless $r = 1$, i.e., $G^k = K_2^2$. $\square$

3 Distinguishing Cartesian products

In this section we investigate sufficient conditions on the sizes of factors of the Cartesian product of two graphs to have the distinguishing index equal to two.

3.1 Trees

We begin with a result for trees. Observe first that, by Theorem 1.2, the Cartesian product of two non-isomorphic asymmetric trees is an asymmetric graph, so its distinguishing index is equal to 1.

Theorem 3.1. *Let T_m and T_n be trees of size m and n, respectively. If*

$$2 \leq m \leq n \leq 2^{2m+1} - \left\lceil \frac{m}{2} \right\rceil + 1,$$

then $D'(T_m \square T_n) \leq 2$.

Proof. If T_m is isomorphic to T_n, then the conclusion follows from Lemma2.6. Therefore, assume that T_m and T_n are non-isomorphic. Then they are relatively prime, and it is enough to prove the existence of a 2-colouring of edges of $T_m \square T_n$ that breaks the automorphisms generated by automorphisms of T_m and those generated by automorphisms of T_n.

In the proof we use the following well-known fact. In a rooted tree, if a parent vertex is fixed by every automorphism preserving a given colouring and its children cannot be permuted, then the children are also fixed.

Assume first that $n = 2^{2m+1} - \lceil \frac{m}{2} \rceil + 1$. We choose a root u_0 of T_m as follows. If T_m has a central vertex, then we take it as a root u_0. If T_m has a central edge, then we choose one of its end-vertices as u_0 and the other one as u_1. Then we choose an enumeration $u_0, \dots, u_m$ of vertices of the rooted tree T_m satisfying the following condition: if u_i is the parent of u_j, then $i < j$. We enumerate the edge $u_i u_j = e_j$. Thus $E(T_m) = \{e_1, \dots, e_m\}$. Let v_0 be a root of T_n chosen by the same rule as the root u_0 of T_m. Then we analogously enumerate vertices and edges of T_n to obtain $V(T_n) = \{v_0, \dots, v_n\}$, $E(T_n) = \{\varepsilon_1, \dots, \varepsilon_n\}$.

We begin by colouring the $T_m^{v_0}$-layer by putting colour 0 on the edges e_i, for $i = 1, \dots, \lceil \frac{m}{2} \rceil$, and colour 1 on the remaining edges of this layer. It is easy to see that we can choose such an enumeration of vertices, and hence of edges, that the root u_0 is fixed by every automorphism of T_m preserving this colouring. Indeed, this is obvious if u_0 is a central vertex; if $e_1 = u_0 u_1$ is a central edge of T_m, then we enumerate the vertices such that $u_1, \dots, u_{\lfloor \frac{m}{2} \rfloor}$ belong to the same subtree of $T_m - e_1$, therefore our colouring breaks all automorphisms of T_m reversing the end-vertices of e_1.

Then, we similarly colour the $T_n^{u_0}$-layer with 0 and 1 in such a way that the vertex (u_0, v_0) is fixed by every automorphism of $T_m \Box T_n$ preserving this partial colouring. Hence, the $T_m^{v_0}$-layer, as well as the $T_n^{u_0}$-layer, is mapped onto itself by every $\varphi \in \mathrm{Aut}(T_m \Box T_n)$ preserving this colouring.

Next, we colour the other layers. Consider the set S of all 2^{2m+1} binary sequences of length $2m+1$. Each $T_m^{v_i}$-layer with $i \geq 1$ is assigned a distinct sequence

$$s_i = (a_0, a_1, \dots, a_m, b_1 \dots, b_m)$$

from S, where $a_j, j = 0, \dots, m$, is the colour of the edge ε_i joining the vertex (u_j, v_i) with its parent in the $T_n^{u_j}$-layer (observe that a_0 has been already defined for all $i \geq 1$), and $b_j, j = 1, \dots, m$ is the colour of the edge of the $T_m^{v_i}$-layer corresponding to e_j. To break all permutations of T_n-layers we delete some sequences from the set S. First observe that the sum of each coordinate taken over all sequences in S is the same (and equal to 2^{2m}). Clearly, a $T_n^{u_j}$-layer and a $T_n^{u_{j'}}$-layer cannot be permuted whenever $j \leq \lceil \frac{m}{2} \rceil < j'$ since the edges e_j and $e_{j'}$ in the $T_m^{v_0}$-layer have different colours.

Consider the set $A = \{s^k \in S \ : \ k = 1, \dots, \lceil \frac{m}{2} \rceil - 1\}$, where $s^k = (a_0, a_1, \dots, a_m, b_1, \dots, b_m)$ is a sequence such that

$$a_j = a_{\lceil \frac{m}{2} \rceil + j} = 1, \quad j = 1, \dots, k,$$

and all other elements of s^k are equal to 0. Thus $|S \setminus A| = 2^{2m+1} - \lceil \frac{m}{2} \rceil + 1$. We use the set $S \setminus A$ to colour $T_m^{v_i}$-layers, $i = 1, \dots, 2^{2m+1} - \lceil \frac{m}{2} \rceil + 1$, hence the numbers of edges coloured with 1 is distinct for every pair of T_n-layers that could be permuted. Thus, all edges in $T_m \Box T_n$ are coloured, and we obtain a distinguishing 2-colouring of $T_m \Box T_n$, when $n = 2^{2m+1} - \lceil \frac{m}{2} \rceil + 1$.

Now, assume that the difference $l = 2^{2m+1} - \lceil \frac{m}{2} \rceil + 1 - n$ is positive. We have to choose l sequences from $S \setminus A$ that will not be used in the colouring. To do this we apply the idea of complementary pairs used in [8]. Denote $\overline{0} = 1, \overline{1} = 0$. A pair of sequences

$$(a_0, a_1, \dots, a_m, b_1, \dots, b_m), \quad (a_0, \overline{a_1}, \dots, \overline{a_m}, b_1, \dots, b_m)$$

from $S \setminus A$ is called *complementary*. When l is even, we choose $\frac{l}{2}$ complementary pairs. When l is odd, we choose the sequence $(0, \dots, 0) \in S \setminus A$ and $\frac{l-1}{2}$ complementary pairs. Again all permutations of layers in $T_m \Box T_n$ are broken by this colouring since for every pair of T_n-layers that could be permuted, the numbers of edges coloured with 1 is distinct, because $a_j + \overline{a_j} = 1$, $j = 1, \dots, m$. $\square$

The bound $2^{2m+1} - \lceil \frac{m}{2} \rceil + 1$ for the size of a larger tree is perhaps not sharp. However, it cannot be improved much since Proposition 3.2 below shows that the distinguishing index of the Cartesian product of a star $K_{1,n}$ of size n and a path P_m of order m is greater than 2 whenever $n > 2^{2m+1}$. It also shows that the distinguishing index of the Cartesian product of two graphs with widely different orders and sizes can be arbitrarily large.

Proposition 3.2. *If $m \geq 2$ and $n \geq 2$, then*

$$D'(K_{1,n} \square P_m) = \left\lceil \sqrt[2m-1]{n} \right\rceil$$

unless $m = 2$ and $n = r^3$ for some r. In the latter case $D'(K_{1,n} \square P_2) = r + 1$.

Proof. Let d be a positive integer such that $(d-1)^{2m-1} < n \leq d^{2m-1}$. Denote by v the central vertex of the star $K_{1,n}$, by $v_1, \dots, v_n$ its pendant vertices, and by $u_1, \dots, u_m$ consecutive vertices of P_m.

Suppose first that $m \geq 3$. Clearly, every automorphism of $K_{1,n} \square P_m$ maps the P_m^v-layer onto itself. We colour the first edge of this layer with 1 and all other edges of it with 2. Thus the P_m^v-layer is fixed by every automorphism, hence the $K_{1,n}$-layers cannot be permuted.

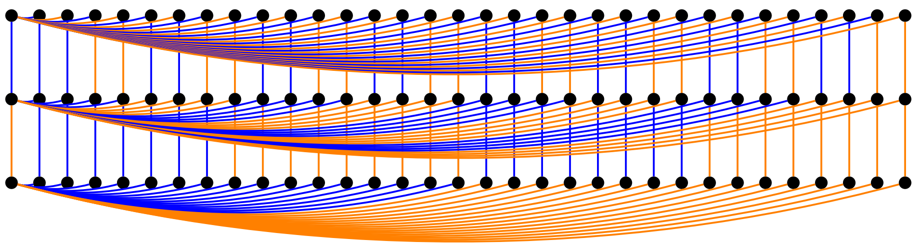

Figure 1: A distinguishing 2-colouring of $K_{1,32} \square P_3$

We want to show that the remaining edges of $K_{1,n} \square P_m$ can be coloured in such a way that P_m-layers also cannot be interchanged. Then all non-trivial automorphisms of $K_{1,n} \square P_m$ will be broken. Note that a colouring of the yet uncoloured edges can be fully described by defining a matrix M with $2m-1$ rows and n columns such that in the j-th column the initial $m-1$ elements are colours of consecutive edges of the $P_m^{v_j}$-layer, and the other m elements are colours of the edges of $K_{1,n}$-layers incident to consecutive vertices of the $P_m^{v_j}$-layer. It is easily seen that there exists a permutation of P_m-layers preserving colours if and only if M contains at least two identical columns. There are exactly d^{2m-1} sequences of length $2m-1$ with elements from the set $\{1, \dots, d\}$, hence there exists a colouring with d colours such that every column of M is distinct. Therefore, $D'(K_{1,n} \square P_m) \leq d = \lceil \sqrt[2m-1]{n} \rceil$. On the other hand, $n > (d-1)^{2m-1}$ so for every edge $(d-1)$-colouring of $K_{1,n} \square P_m$, the corresponding matrix has to contain two identical columns, therefore $D'(K_{1,n} \square P_m) > d - 1$. Figure 1 presents the case $n = 32$ and $m = 3$.

For $m = 2$, we colour the edges of $K_{1,n} \square P_2$ in the same way. The only difference is that every P_2-layer has only one edge, hence the two $K_{1,n}$-layers need not be fixed. This is the case when $n = d^3$, because then each element of $\{1, \ldots, d\}^3$ is a column in M, and there exists a permutation of columns of M which together with the transposition of rows of M defines a non-trivial automorphism of $K_{1,n} \Box P_2$ preserving the colouring. Thus we need an additional colour for one edge in a $K_{1,n}$-layer. When $n < d^3$, we put the sequence $(1, 1, 2)$ as the first column of M, and we do not use the sequence $(1, 2, 1)$ any more, thus breaking the transposition of the $K_{1,n}$-layers, and all automorphisms of $K_{1,n} \Box P_2$. □

Let us mention in passing that $D'(K_{1,n} \Box C_m) = \lceil \sqrt[2m]{n} \rceil$, unless $m \leq 5$ and $n = 2^{2m}$. In the latter case $D'(K_{1,n} \Box C_m) = \sqrt[2m]{n} + 1 = 3$. The proof can be led on the lines of the proof of Proposition 3.2.

3.2 Arbitrary factors

We now consider the Cartesian product of arbitrary connected graphs. We first formulate a result for relatively prime factors.

Lemma 3.3. *Let G and H be connected, relatively prime graphs such that*

$$3 \leq |G| \leq |H| \leq 2^{|G|}\big(2^{\|G\|} - 1\big) - |G| + 2.$$

Then $D'(G \Box H) \leq 2$.

Proof. Let $V(G) = \{u_1, \ldots, u_{|G|}\}$, $E(G) = \{e_1, \ldots, e_{\|G\|}\}$, $V(H) = \{v_1, \ldots, v_{|H|}\}$, $E(H) = \{\varepsilon_1, \ldots, \varepsilon_{\|H\|}\}$. Assume that v_1 is a root of a spanning tree T_H of the graph H, and the vertices of H are enumerated according to the rooted tree T_H, i.e., each child has an index greater than that of its parent. As G and H are relatively prime, the only automorphisms of $G \Box H$ are permutations of G-layers and H-layers.

We first colour the edges of the G^{v_1}-layer with a sequence

$$(b_1, \ldots, b_{\|G\|}) = (1, \ldots, 1).$$

We shall not use this sequence to colour the edges of any other G-layer any more. Thus the G^{v_1}-layer will be mapped onto itself by every automorphism of $G \Box H$ preserving the colouring.

From now on, we proceed similarly as in the proof of Theorem 3.1. For $i = 2, \ldots, n$, the G^{v_i}-layer will be assigned a distinct sequence of colours

$$(a_1, \ldots, a_{|G|}, b_1, \ldots, b_{\|G\|}),$$

where a_j is a colour of the edge joining the vertex (u_j, v_i) to its parent in the rooted tree T_H in the H^{u_j}-layer, and b_j is a colour of e_j in the G^{v_i}-layer. We have $2^{|G|}\big(2^{\|G\|} - 1\big)$ such sequences, as we excluded all sequences of the form $(a_1, \ldots, a_{|G|}, 1, \ldots, 1)$. Thus all permutations of G-layers are broken. To break permutations of H-layers, we also exclude all sequences $s^k = (a_1, \ldots, a_{|G|}, b_1, \ldots, b_{\|G\|})$ with $a_1 = \ldots = a_k = 1$ and $a_{k+1} = \ldots = a_{|G|} = b_1 = \ldots = b_{\|G\|} = 0$, for every $k = 1, \ldots, |G| - 1$. We have $2^{|G|}\big(2^{\|G\|} - 1\big) - (|G| - 1)$ sequences to colour $|H| - 1$ G-layers. Depending on the size of $|H|$, we also exclude a suitable number of complementary pairs of sequences

$$(a_1, \ldots, a_{|G|}, b_1, \ldots, b_{\|G\|}), \quad (\overline{a_1}, \ldots, \overline{a_{|G|}}, b_1, \ldots, b_{\|G\|})$$

and, possibly, a sequence $(0,\dots,0)$. Thus we obtain a colouring of $G\Box H$ with a set of sequences such that the number of 1's is distinct in any of the initial $|G|$ coordinates. Therefore, no permutation of H-layers preserves this colouring. Hence, this is a distinguishing 2-colouring of $G\Box H$. □

Finally, we state the main result of this section.

Theorem 3.4. *Let G and H be connected graphs such that*

$$2 \le |G| \le |H| \le 2^{|G|}\big(2^{\|G\|}-1\big) - |G| + 2.$$

Then $D'(G\Box H) \le 2$ unless $G = H = K_2$.

Proof. If $G = K_2$, then $|H| \le 4$. If $H \ne K_4$, then either $D'(H) = 2$ or H is a cycle or a star, and these cases were already settled in Section 2. To construct a distinguishing 2-colouring of $K_2\Box K_4$, colour one edge in one K_4-layer and two adjacent edges in the other K_4-layer red, and all remaining edges blue.

Let $|G| \ge 3$. The case when G and H are relatively prime was settled by Lemma 3.3. Therefore, we focus here on the situation when G and H have at least one common factor. Then $D'(G\Box H) \ge 2$, since the automorphism group of $G\Box H$ is non-trivial. Let $G = G_1^{k_1}\Box\dots\Box G_t^{k_t}$ and $H = H_1^{l_1}\Box\dots\Box H_s^{l_s}$ be the prime factor decompositions of G and H, respectively. Assume that the initial r factors are common, i.e., $G_i = H_i$ for $i = 1,\dots,r$. Denote

$$G_{II} = G_1^{k_1}\Box\dots\Box G_r^{k_r}, \qquad H_{II} = H_1^{l_1}\Box\dots\Box H_r^{l_r}.$$

Thus $G = G_I\Box G_{II}$ and $H = H_I\Box H_{II}$. We use the following notation

$$n_1 = |G_I|, \quad n_2 = |G_{II}|, \quad m_1 = |H_I|, \quad m_2 = |H_{II}|.$$

We first show that the distinguishing index of the Cartesian product

$$G_{II}\Box H_{II} = G_1^{l_i+k_1}\Box\dots\Box G_r^{l_r+k_r}$$

is equal to 2. If $G_{II}\Box H_{II} = K_2^2$, then $|H_I| \ge 2$ and the graphs $G_I\Box K_2^2$ and H_I satisfy the assumptions of Theorem 3.3, hence $D'(G\Box H) = 2$, unless $|G_I\Box K_2^2| > |H_I|$, that is $|H_I| < 4|G_I|$. In the latter case, we can also apply Theorem 3.3 for the graphs G_I and H_I which are relatively prime and satisfy the inequalities $|G_I| \le |H_I| \le 2^{|G_I|}(2^{\|G_I\|}-1) - |G_I| + 2$ unless $|G_I| = 2$ and $\le |H_I| \le 7$, i.e., $G\Box H = K_2^3\Box H_I'$, where H_I' is prime. So we can apply Proposition 2.2 and Lemma 2.4. In any case $D'(G\Box H) = 2$.

If $G_i^{l_i+k_i} \ne K_2^2$ for every $i = 1,\dots,r$, then $D'(G_i^{l_1+k_i}) = 2$ due to Theorem 2.7, and hence $D'(G_{II}\Box H_{II}) = 2$ by repeated application of Lemma 2.3. If $G_1^{l_1+k_1} = K_2^2$, then analogously $D'(G_2^{l_2+k_2}\Box\dots\Box G_r^{l_r+k_r}) = 2$, hence $D'(G_{II}\Box H_{II}) = 2$ by applying Lemma 2.5 twice.

Consider now the graphs $G' = G_I\Box G_{II}\Box H_{II}$ and $H' = H_I$. Clearly, they are relatively prime and

$$|H'| < |H| \le 2^{|G|}\big(2^{\|G\|}-1\big) - |G| + 2 < 2^{|G'|}\big(2^{\|G'\|}-1\big) - |G'| + 2.$$

If also $|G'| = n_1n_2m_2 \leq m_1 = |H'|$, then graphs G' and H' satisfy the conditions of Lemma 3.3, and consequently, $D'(G\Box H) = D'(G'\Box H') = 2$. Then suppose that $n_1n_2m_2 > m_1$. We consider two cases here.

Assume first that $n_1 \leq n_2m_2$, i.e., $|G_I| \leq |G_{II}\Box H_{II}|$. Hence, $|G_I| \leq \|G_{II}\Box H_{II}\| + 1$, and by repeated application of Lemma 2.4 we get $D'(G') = 2$. As $|H'| < |G'|$, we infer again from Lemma 2.4 that $D'(G\Box H) = D'(G'\Box H') = 2$.

In the second case, i.e., when $n_2m_2 < n_1$, suppose first that

$$m_1 = |H_I| \leq 2^{|G_I|}\big(2^{\|G_I\|} - 1\big) - |G_I| + 2.$$

Then $D'(G_I\Box H_I) \leq 2$ since the assumptions of Lemma 3.3 are satisfied whenever $|G_I| \leq |H_I|$. Recall that also $D'(G_{II}\Box H_{II}) = 2$ and graphs $G_I\Box H_I$ and $G_{II}\Box H_{II}$ are relatively prime. Hence $D'(G\Box H) = 2$ by Lemma 2.3. Otherwise, if $m_1 > 2^{|G_I|}\big(2^{\|G_I\|} - 1\big) - |G_I| + 2$, then we arrive at the sequence of inequalities

$$m_1 < n_1n_2m_2 \leq n_1^2 < 2^{n_1}(2^{n_1} - 1) - n_1 + 2 \leq 2^{|G_I|}\big(2^{\|G_I\|} - 1\big) - |G_I| + 2 < m_1,$$

which is impossible.

Then suppose that $|G_I| = n_1 > m_1 = |H_I|$ (and still $n_2m_2 < n_1$). Let $G'' = G_I$ and $H'' = G_{II}\Box H_I\Box H_{II}$. Clearly, $|G''| \leq |H''|$ since $|G| \leq |H|$. Moreover, we have

$$|H''| = n_2m_2m_1 < n_1m_1 < n_1^2 < 2^{|G''|}\big(2^{\|G''\|} - 1\big) - |G''| + 2.$$

It follows from Lemma 3.3 that $D'(G\Box H) = D'(G''\Box H'') = 2$. □

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$\chi_D(G)$, $|Aut(G)|$, and a variant of the Motion Lemma

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Abstract

The *Distinguishing Chromatic Number* of a graph G, denoted $\chi_D(G)$, was first defined in [5] as the minimum number of colors needed to properly color G such that no non-trivial automorphism ϕ of the graph G fixes each color class of G. In this paper,

1. We prove a lemma that may be considered a variant of the Motion lemma of [15] and use this to give examples of several families of graphs which satisfy $\chi_D(G) = \chi(G) + 1$.
2. We give an example of families of graphs that admit large automorphism groups in which every proper coloring is distinguishing. We also describe families of graphs with (relatively) very small automorphism groups which satisfy $\chi_D(G) = \chi(G) + 1$, for arbitrarily large values of $\chi(G)$.
3. We describe non-trivial families of bipartite graphs that satisfy $\chi_D(G) > r$ for any positive integer r.

Keywords: Distinguishing chromatic number, automorphism group of a graph, Motion Lemma, weak product of graphs.

Math. Subj. Class.: 05C15, 05C25, 05C76, 05C80

1 Introduction

For a graph $G = (V, E)$ let us denote by $Aut(G)$, its full automorphism group. A labeling of vertices of a graph $G, h : V(G) \to \{1, \ldots, r\}$ is said to be **distinguishing** (or r-distinguishing) provided no nontrivial automorphism of the graph preserves all of the vertex

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labels. The **distinguishing number** of the graph G, denoted by $D(G)$, is the minimum r such that G has an r-distinguishing labeling (see [1]).

Collins and Trenk introduced the notion of the **Distinguishing Chromatic Number** in [5], as the minimum number of colors r, needed to color the vertices of the graph so that the coloring is both proper and distinguishing. In other words, the Distinguishing Chromatic Number is the least integer r such that the vertex set can be partitioned into sets $V_1, V_2, \ldots, V_r$ such that each V_i is independent in G, and for every $I \neq \pi \in Aut(G)$ there exists some color class V_i such that $\pi(V_i) \neq V_i$.

The problem of determining the distinguishing chromatic number of a graph G, or at least good bounds for it, has been one of considerable interest in recent times ([4, 13, 5, 2, 3]). Clearly, the notion of the distinguishing chromatic number begins to get more interesting only if the graph admits a large group of automorphisms, in which case, it can vary substantially from the usual chromatic number. It is known (see [5]) that $\chi_D(G) = |V|$ if and only if G is complete multipartite. Consequently, it is simple to see that there exist graphs G with $\chi(G) = k, \chi_D(G) = l+k$, for any k, l, since for instance, a disjoint union of a clique of size k and $K_{1,l}$ achieves the same. Some upper bounds for $\chi_D(G)$ (for instance, a version of Brooks' theorem for the distinguishing chromatic number) appear in [4], which also includes the inequality $\chi_D(G) \leq D(G)\chi(G)$. However, in many interesting large naturally occurring families of graphs, we have $\chi_D(G) \leq \chi(G) + 1$ (see [3, 2, 5, 4]).

In this paper, we seek to address three aspects of the problem of determining $\chi_D(G)$ for a given graph G. Firstly, we prove a lemma that may be considered a variant of what is now well known as the motion lemma, first introduced in [15]. The motion lemma basically says that if every nontrivial automorphism of a graph moves 'many' vertices then its distinguishing number is small. A similar lemma also appears in the context of graph endomorphisms and 'endomorphism breaking' in [12]. As a result of our variant of the Motion lemma, we give examples of several families of graphs G satisfying $\chi_D(G) = \chi(G) + 1$.

Secondly, we contrast the relation between the size of the automorphism group $Aut(G)$ of a graph with its distinguishing chromatic number $\chi_D(G)$. A result describing an upper bound for $\chi_D(G)$ in terms of the prime factors of $|Aut(G)|$ appears in [4]. However, our perspective is somewhat different. We demonstrate families of vertex transitive graphs G with large chromatic number, and $\chi_D(G) = \chi(G) + 1$ even though $|Aut(G)|$ is not very large (we have $|Aut(G)| = O(|V|^{3/2})$). As a contrast, we also demonstrate a family of graphs with arbitrarily large chromatic number, with 'super large' automorphism groups for which *every* proper coloring of G with $\chi(G)$ colors is in fact distinguishing. This latter example also addresses a point raised in [3] and these contrasting results indicate that the relation between $|Aut(G)|$ and $\chi_D(G)$ can tend to be haphazard.

Finally, as we indicated earlier, while it is simple to give (the trivial) examples of graphs G with $\chi(G) = r, \chi_D(G) = r + s$, for any r, s, non-trivial examples are a little harder to come by. Clearly, adding a copy (not necessarily disjoint) of a large complete multipartite graph to an arbitrary graph achieves this goal but we shall consider such examples 'trivial' since the reason for the blowing-up of the distinguishing chromatic number is trivially attributed to the presence of the complete multipartite component. While it seems simple to qualitatively ascribe the notion of what constitutes a nontrivial example in this context, we find it a bit tedious to describe it precisely. Our last result in this paper describes what we would like to believe constitutes a nontrivial family of bipartite graphs G such that $\chi_D(G) > r$, for any $r \geq 2$. It turns out that large complete bipartite graphs do appear as induced subgraphs in our examples, but that alone does not guarantee that the distinguishing chromatic number necessarily increases. Furthermore, what makes these nontrivial in our opinion, is the fact that the distinguishing chromatic number of these graphs is more than what one might initially guess.

The rest of the paper is organized as follows. In Section 2, we state and prove what we regard as a variant of the motion lemma and use this to establish instances of families of graphs with $\chi_D(G) = \chi(G) + 1$ in Section 3. In Section 4, we describe two families of graphs — $\mathcal{G}_1$ and $\mathcal{G}_2$ — with rather contrasting properties. For $G \in \mathcal{G}_1$, we have $\chi_D(G) = \chi(G) + 1$ even though $|Aut(G)| = O(|V|^{3/2})$; for $G \in \mathcal{G}_2$, $|Aut(G)| = \omega(e^{|V|})$ and yet every proper $\chi(G)$ coloring of G is in fact distinguishing. In Section 5, we describe a family of bipartite graphs for which $\chi_D(G) > r$, for any $r \geq 2$. The Section 6 contains some concluding remarks and open questions.

2 A Variant of the Motion Lemma

Following [15], we recall that the motion of an automorphism $\phi \in Aut(G)$ is defined as

$$m(\phi) := \{v \in G : \phi(v) \neq v\}$$

and the motion of a graph G is defined as

$$m(G) := \min_{\substack{\phi \in Aut(G) \\ \phi \neq I}} m(\phi).$$

The Motion lemma of [15] states that for a graph G, if $m(G) > 2\log_2 |Aut(G)|$ then G is $2-$distinguishable. We prove a slightly more general criterion to obtain a similar conclusion for the distinguishing chromatic number.

For a graph G with full automorphism group $Aut(G)$, let $\mathcal{G} \subset Aut(G)$ be a subgroup of the automorphism group. For $A \in \mathcal{G}$ and $S \subseteq V(G)$ we define $Fix_A(S) = \{v \in S : A(v) = v\}$ and $F_A(S) = |Fix_A(S)|$. Let $F(S) := \max_{\substack{A \in \mathcal{G} \\ A \neq I}} F_A(S)$.

Definition 2.1. The *Orbit of a vertex v with respect to an automorphism A* is the set

$$Orb_A(v) := \{v, Av, A^2v, \ldots A^{k-1}v\}$$

where $A^k v = v$.

Lemma 2.2 (A variant of the motion lemma)**.** *Let C be a proper coloring of the graph G with $\chi(G)$ colors and let C_1 be a color class in C. Let $\mathcal{G}$ be the subgroup of $Aut(G)$ consisting of all automorphisms that fix the color class C_1. For each $A \in \mathcal{G}$, let θ_A denote the total number of distinct orbits induced by the automorphism A in the color class C_1. If for some integer $t \geq 2$,*

$$f(\mathcal{G}) = \sum_{A \in \mathcal{G}} t^{\theta_A - |C_1|} < r$$

where r is the least prime dividing $|\mathcal{G}|$, then $\chi_D(G) \leq \chi(G) + t - 1$. In particular, if $F(C_1) < |C_1| - 2\log_t |\mathcal{G}|$ then this conclusion holds.

Remark: Instead of $\mathcal{G}$, one can consider the subgroup of $\mathcal{G}$ consisting of all nontrivial automorphisms that fix all the color classes of the proper coloring C. The proof in that case is identical to the present one.

Proof. Let 1 be the color assigned in the color class C_1 and suppose that the symbols $2, \dots, t$ denote labels different from the color labels of the vertices in the proper coloring C. For each $v \in C_1$, pick uniformly and independently, an element in $\{1, 2, \dots, t\}$ and color v using that color. Keep the coloring of all other vertices intact. This creates $t-1$ additional color classes. This new coloring C' of G is clearly proper; we claim that with positive probability, it is also distinguishing.

For $A \in \mathcal{G}$, let B_A denote the event that A fixes every color class. Observe that if A fixes a color class containing a vertex v, then all other vertices in the set $Orb_A(v)$ are also in the same color class. Moreover the probability that $Orb_A(v)$ is in the same color class of v, equals $t^{1-|Orb_A(v)|}$. Then

$$\mathbb{P}(B_A) = \prod_{\theta_A} t^{1-|Orb_A(v)|} = t^{\theta_A - |C_1|}$$

Let $\mathcal{N} \subset \mathcal{G}$ denote the set of all automorphisms which fixes every color class in C' and let $N = |\mathcal{N}|$. Then note that the expectation

$$\mathbb{E}(N) \leq \sum_{A \in \mathcal{G}} \frac{1}{t^{|C_1| - \theta_A}} \tag{2.1}$$

By the hypothesis of the lemma, $\mathbb{E}(N) \leq f(\mathcal{G}) < r$, hence with positive probability $N < r$. Since $\mathcal{N}$ is in fact a subgroup of $\mathcal{G}$, N divides $|\mathcal{G}|$, so with positive probability, the coloring C' is such that $\mathcal{N} = \{I\}$, which implies that C' is distinguishing. Note that C' is a coloring with $\chi(G) + t - 1$ colors.

In particular, since $\theta_A \leq F(C_1) + \frac{|C_1| - F(C_1)}{2}$ it follows from Equation (2.1) that

$$\mathbb{E}(N) \leq \sum_{A \in \mathcal{G}} t^{\frac{F(C_1) - |C_1|}{2}} = |\mathcal{G}| t^{\frac{F(C_1) - |C_1|}{2}}.$$

Thus, if $F(C_1) < |C_1| - 2 \log_t |\mathcal{G}|$ then there exists a distinguishing proper $\chi(G) + t - 1$ coloring of the graph. □

3 Examples

3.1 Levi graphs

In this subsection, we restrict our attention to Desarguesian projective planes and consider the Levi graphs of these projective planes, which are the bipartite incidence graphs corresponding to the set of points and lines of the projective plane. It is well known [11] that the theorem of Desargues is valid in a projective plane if and only if the plane can be constructed from a three dimensional vector space over a skew field, which in the finite case reduces to the three dimensional vector spaces over finite fields.

In order to describe the graphs we are interested in, we set up some notation. Let $\mathbb{F}_q$ denote the finite field of order q, and let us denote the vector space $\mathbb{F}_q^3$ over $\mathbb{F}_q$ by V. Let $\mathcal{P}$ be the set of 1-dimensional subspaces of V and $\mathcal{L}$, the set of 2-dimensional subspaces of V. We shall refer to the members of these sets by points and lines respectively. The **Levi graph** of order q, denoted by LG_q, is a bipartite graph defined as follows: $V(LG_q) = \mathcal{P} \cup \mathcal{L}$, where this describes the partition of the vertex set; a point p is adjacent to a line l if and only if $p \in l$.

The Fundamental Theorem of Projective Geometry [11] states that the full group of automorphisms of the projective plane $PG(2, \mathbb{F}_q)$ is induced by the group of all non-singular semi-linear transformations $P\Gamma L(\mathbb{F}_q^3)$. If $q = p^n$ for a prime number p, $P\Gamma L(\mathbb{F}_q^3) \cong PGL(\mathbb{F}_q^3) \rtimes Gal(\mathbb{F}_q/\mathbb{F}_p)$. In particular, if q is a prime, we have $P\Gamma L(\mathbb{F}_q^3) \cong PGL(\mathbb{F}_q^3)$. The upshot is that LG_q admits a large group of automorphisms, namely, $P\Gamma L(\mathbb{F}_q^3)$.[1]

We first show that the distinguishing chromatic number for the Levi graphs LG_q is precisely 3 in almost all the cases. This is reminiscent of the result of [6] for the distinguishing number of affine spaces.

Theorem 3.1. $\chi_D(LG_q) = 3$ *for all prime powers* $q \geq 5$.

Proof. Firstly, we consider the case when $q \geq 5$ and q is prime and show that $\chi_D(LG_q) \leq 3$. Consider a 2-coloring of LG_q by assigning color 1 to the point set $\mathcal{P}$ and color 2 to the line set $\mathcal{L}$. It is easy to see that an automorphism of LG_q that maps $\mathcal{P}$ into itself and $\mathcal{L}$ into itself corresponds to an automorphism of the underlying projective plane, and hence any such automorphism is necessarily in $PGL(\mathbb{F}_q^3)$ (by the preceding remarks).

In order to use Lemma 2.2, set $\mathcal{G} = PGL(\mathbb{F}_q^3)$ and observe that every $A \in PGL(\mathbb{F}_q^3)$, which is not the identity, fixes at most $q + 2$ points of LG_q. Hence

$$\theta_A \leq q + 2 + \frac{(q^2 + q + 1) - (q + 2)}{2} = \frac{q^2 + 2q + 3}{2}.$$

Consequently,

$$f(\mathcal{G}) < \frac{(q^8 - q^6 - q^5 + q^3)}{t^{(q^2+1)/2}} + 1 \tag{3.1}$$

Case 1: $q \geq 7$.
For $q = 7, t = 2$, the right hand side of Equation (3.1) is approximately 1.16. Since the right hand side of inequality Equation (3.1) is monotonically decreasing in q, it follows that $f(\mathcal{G}) < 2$ for $q \geq 7$, hence by Lemma 2.2, LG_q admits a proper distinguishing $3-$coloring. In particular, $\chi_D(LG_q) = 3$, for $q \geq 7$, since clearly, $\chi_D(LG_q) > 2$.
Case 2: $q = 5$.
In this case, for $t = 2$ we actually calculate $f(\mathcal{G})$ using the open source Mathematics software SAGE to obtain $f(\mathcal{G}) \approx 1.2$; see the Appendix for the code with relevant explanations. Again in this case, $\chi_D(LG_5) = 3$.

We also can prove that $\chi_D(LG_2) = 4$ and $\chi_D(LG_3) \leq 5$; these proofs are included in the Appendix for the sake of completeness.

If $q = p^n$ for $n \geq 2$ and a prime p, we note that the cardinality of the automorphism group of $PG(2, \mathbb{F}_q)$ equals

$$n|PGL(\mathbb{F}_q^3)| \leq \log_2(q)|PGL(\mathbb{F}_q^3)|.$$

As in the prime case, we have

$$f(\mathcal{G}) \leq \frac{\log_2 q(q^8 - q^6 - q^5 + q^3)}{t^{\frac{q^2+1}{2}}} + 1.$$

For $q = 8$ and $t = 2$ the right hand side is approximately 1.01. By the same arguments as in the preceding section, it follows that $\chi_D(LG_q) = 3$. □

[1] It follows that this group is contained in the full automorphism group. The full group is larger since it also includes maps induced by isomorphism of the projective plane with its dual.

For $q = 4$ we calculate $f(\mathcal{G}) \approx 1.2.$ for $q = 4$, and $t = 3$ using the same SAGE code in the case $q = 5$ to make the actual computation, so we have $\chi_D(LG_4) \leq 4$. We believe that $\chi_D(LG_4) = 3$ though our methods fall short of proving this.

3.2 Levi graphs of order one

Suppose $n, k \in \mathbb{N}$ and $2k < n$, consider the bipartite graphs $G = G(L, R, E)$ where $L := \binom{[n]}{k-1}$ corresponds to the set of $k-1$ subsets of $[n]$, $R := \binom{[n]}{k}$ corresponds to the k subsets of $[n]$, and $u \in L, v \in R$ are adjacent if and only if $u \subset v$. We shall refer to these graphs as *Levi Graphs of order one* and we shall denote them by $LG_1(k, n)$, or sometimes, simply LG_1. Note that for each $u \in L, v \in R$ we have $d(u) = n - k + 1$ and $d(v) = k$.

It is clear that $S_n \subset Aut(LG_1)$. But in fact $Aut(LG_1) = S_n$, and this is a fairly routine exercise, so we skip these details.

We shall use Lemma 2.2 to determine the distinguishing chromatic number of $LG_1(k, n)$. Following the notation of the lemma, set $F_\sigma := \{v \in R : \sigma(v) = v\}$ for $\sigma \in S_n$ and let $F = \max\limits_{\substack{\sigma \in S_n \\ \sigma \neq I}} |F_\sigma|$.

Lemma 3.2. *For $n > 4$, $F \leq \binom{n-2}{k-2} + \binom{n-2}{k}$ and equality is attained if and only if σ is a transposition (ij) for some $i \neq j$.*

Proof. Firstly, it is easy to see that if $\sigma = (12)$ then $|F_\sigma| = \binom{n-2}{k-2} + \binom{n-2}{k}$, so it suffices to show that for any π that is not of the above form, $|F_\pi| < |F_\sigma|$.

Suppose not, i.e., suppose $\pi \in S_n$ is not an involution and $|F_\pi|$ is maximum. Write $\pi = O_1 O_2 \ldots O_t$ as a product of disjoint cycles with $|O_1| \geq |O_2| \geq \cdots \geq |O_t|$. Then either $|O_1| > 2$, or $|O_1| = |O_2| = 2$. If $|O_1| > 2$, then suppose without loss of generality, let $O_1 = (123 \cdots)$ If $h \in F_\pi$ then either $\{1, 2\} \subset h$ or $\{1, 2\} \cap h = \emptyset$. In either case we observe that $h \in F_\sigma$ as well. Therefore $F_\pi \subsetneq F_\sigma$. Furthermore, note that σ fixes the set $g = \{1, 2, 4, \ldots, k + 1\}$, while π does not. Hence $|F_\sigma| > |F_\pi|$, contradicting that $|F_\pi|$ is maximum. If $|O_1| = |O_2| = 2$, again without loss of generality let $O_1 = (12), O_2 = (34)$. Again, $h \in F_\pi$ implies that either $\{1, 2\} \subset h$ or $\{1, 2\} \cap h = \emptyset$, so once again, $h \in F_\sigma \Rightarrow h \in F_\pi$. Furthermore, $\{1, 2, 3, 5, \ldots, k + 1\} \in F_\sigma \cap \overline{F_\pi}$, which contradicts the maximality of $|F_\pi|$. □

For $k \geq 2$ define $n_0(k) := 2k + 1$ for $k \geq 3$ and $n_0(2) := 6$.

Theorem 3.3. $\chi_D(LG_1(k, n)) = 3$ *for $k \geq 2$ for $n \geq n_0(k)$.*

Proof. We deal with the cases $k = 2, k = 3$ first, and then consider the general case of $k > 3$.

For $k = 2$, let $A = \{(1, 2), (2, 3), (2, 4), (3, 4), (4, 5), (5, 6), \ldots, (n - 1, n)\}$, and consider the coloring with the color classes being $L, A, R \setminus A$. Consider the graph G with $V(G) = [n]$ and $E(G) = A$. Observe that the only automorphism G admits is the identity. Since a nontrivial automorphism that preserves all the color classes of this coloring must in fact be a nontrivial automorphism of G, it follows that the coloring described is indeed distinguishing.

If $k = 3$, note that the coloring described by the sets $R, A, L \setminus A$ is proper and distinguishing for the very same reason.

For the case $k \geq 4$, we use Lemma 2.2 with $t = 2$ and $\mathcal{G} = Aut(LG_1)$. From Lemma 3.2 we have $F \leq \binom{n-2}{k-2} + \binom{n-2}{k}$. Let $C_1 = R$ be the color class to be parted randomly and assign color 3 to all vertices in $L = \binom{[n]}{k-1}$. Then we have,

$$f(\mathcal{G}) \leq |Aut(LG_1)|2^{\frac{1}{2}(F-|C_1|)} + 1, \tag{3.2}$$

where $|C_1| = \binom{n}{k}$.

Therefore from Equation 3.2, we have

$$f(\mathcal{G}) \leq \frac{n!}{2^K} + 1,$$

where $K = \frac{\binom{n}{k}-\binom{n-2}{k-2}-\binom{n-2}{k}}{2}$. For $n > 2k$ it is not hard to show that $\frac{n!}{2^K} < 1$ for $n \geq n_0(k)$, so, by Lemma 2.2 we are through. $\square$

3.3 Weak product of graphs

The distinguishing chromatic number of a Cartesian product of graphs has been studied in [3]. The fact that any graph can be uniquely (upto a permutation of the factors) factorized into prime graphs with respect to the Cartesian product plays a pivotal role in determining the full automorphism group. In contrast, an analogous theorem for the weak product only holds under certain restrictions. In this subsection, we consider the n-fold weak product of certain graphs and consider the problem of determining their distinguishing chromatic number.

To recall the definition again, the weak product (or Direct product as it is sometimes called) of graphs G, H denoted $G \times H$, is defined as follows: $V(G \times H) = V(G) \times V(H)$. Vertices $(g_1, h_1), (g_2, h_2)$ are adjacent if and only if $\{g_1, g_2\} \in E(G)$ and $\{h_1, h_2\} \in E(H)$. We first collect a few basic results on the weak product of graphs following [9]. For more details we refer the interested reader to the aforementioned handbook.

Define an equivalence relation R on $V(G)$ by setting xRy if and only if $N(x) = N(y)$ where $N(x)$ denotes the set of neighbors of x. A graph G is said to be $R - thin$ if each equivalence class of R is a singleton, i.e., no distinct $x, y \in V(G)$ have the same set of neighbors. A graph G is **prime with respect to the weak product**, or simply prime, if it is nontrivial and $G \cong G_1 \times G_2$ implies that either G_1 or G_2 equals K_1^s, where K_1^s is a single vertex with a loop on it. Observe that $K_1^s \times G \cong G$.

Before we state our main theorem of this subsection, we state two useful results regarding the weak product of graphs. If G is connected, nontrivial, and non-bipartite then the same holds for $G^{\times n}$. This is a simple consequence of a theorem of Weischel (see [9] for more details). The other useful result is the following theorem which also appears in [9].

Theorem 3.4. *Suppose ϕ is an automorphism of a connected nonbipartite $R - thin$ graph G that has a prime factorization $G \cong G_1 \times G_2 \times \ldots \times G_k$. Then there exist a permutation π of $\{1, 2, \ldots, k\}$, together with isomorphisms $\phi_i : G_{\pi(i)} \to G_i$, such that*

$$\phi(x_1, x_2, \ldots, x_k) = (\phi_1(x_{\pi(1)}), \phi_2(x_{\pi(2)}), \ldots, \phi_k(x_{\pi(k)})).$$

We are now in a position to state our main result regarding the distinguishing chromatic number for a weak product of prime graphs. An analogous result for the cartesian product of graphs, under milder assumptions, appears in [3].

Theorem 3.5. *Let G be a connected, nonbipartite, $R-thin$, prime graph on at least 3 vertices. Denote by $G^{\times n}$ the n-fold product of G, i.e., $G^{\times n} := \overbrace{G \times G \times \ldots \times G}^{n\text{-times}}$. Suppose further that G admits a proper $\chi(G)$ coloring with a color class C_1 such that no non-trivial automorphism of G fixes every vertex of C_1. Then $\chi_D(G^{\times n}) \leq \chi(G)+1$ for $n \geq 4$.*

Proof. Let G be connected, non-bipartite, $R-thin$, and prime. We first claim that

$$Aut(G^{\times n}) \cong Aut(G) \wr S_n,$$

the wreath product of $Aut(G)$ and S_n. To see this, note that if G is $R-thin$ one can easily check that $G^{\times n}$ is also $R-thin$. Moreover since every connected non-bipartite nontrivial graph admits a unique prime factorization for the weak product (see [9]), it is a simple application of Theorem 3.4 to see that $Aut(G^{\times n}) \cong Aut(G) \wr S_n$. This proves the claim.

Suppose $\chi(G) = r$ and let $\{C_i : i \in [r]\}$ be a proper coloring of G. Then $C_i \times G^{\times n-1}, i \in [r]$ is a proper r coloring of the graph $G^{\times n}$, so $\chi(G^{\times n}) \leq r$. On the other hand, the map $g \to (g, g \ldots, g)$ is a graph embedding of G in $G^{\times n}$, so $\chi(G^{\times n}) = r$. Let us denote the aforementioned color classes of $G^{\times n}$ by $C_i', i \in [r]$. We claim that $\chi_D(G^{\times n}) \leq r+1$ and show this as a consequence of Lemma 2.2.

By hypothesis there exist a color class, say C_1 in G such that no nontrivial automorphism fixes each $v \in C_1$. Consider $C_1' = C_1 \times G^{\times n-1}$ and for each element in C_1' assign a value from $\{1, r+1\}$ uniformly and independently at random. This describes a proper $(r+1)-$coloring of $G^{\times n}$. By Lemma 2.2, we have

$$f(\mathcal{G}) \leq n!|Aut(G)|^n 2^{\frac{F-T}{2}} + 1 \tag{3.3}$$

where $T = |C_1 \times G^{\times n-1}|$, $\mathcal{G} = n!|Aut(G)|^n$ and F is as in Lemma 2.2.

Claim: If there exists a nontrivial automorphism of $G^{\times n}$ which fixes each color class $C_i', i = 1 \ldots, r$, then it cannot also fix each vertex of C_1'.

To prove the claim, suppose ψ is an automorphism of $G^{\times n}$ which fixes C_i' for each $i \in [r]$, and also fixes C_1' point-wise. By Theorem 3.4, there exist $\phi_1, \phi_2, \ldots, \phi_n \in Aut(G)$ and $\pi \in S_n$ such that

$$\psi(x_1, x_2, \ldots, x_n) = (\phi_1(x_{\pi(1)}), \phi_2(x_{\pi(2)}), \ldots, \phi_n(x_{\pi(n)})) \tag{3.4}$$

for all $(x_1, x_2, \ldots, x_n) \in G^{\times n}$. Now note that if ψ fixes C_1' point-wise then ϕ_1 fixes C_1 point-wise. Indeed,

$$\begin{aligned} \psi(x_1, x_2, \ldots, x_n) &= (x_1, x_2, \ldots, x_n) \\ \iff (\phi_1(x_{\pi(1)}), \phi_2(x_{\pi(2)}), \ldots, \phi_n(x_{\pi(n)})) &= (x_1, x_2, \ldots, x_n) \\ \iff \phi_i(x_{\pi(i)}) &= x_i \quad \text{for all } i \in [r]. \end{aligned} \tag{3.5}$$

Since Equation (3.5) holds for all vertices $(x_1, x_2, \ldots, x_n) \in G^{\times n}$ with $x_1 \in C_1$ and $x_i \in G$, $2 \leq i \leq n$, we conclude that $\pi = I$, $\phi_i = I$, for $2 \leq i \leq n$, and ϕ_1 acts trivially on C_1. But then by the hypothesis on G, it follows that $\phi_1 = I$ in G and hence $\psi = I$.

We now show that $F \leq (|C_1| - 2)|G|^{n-1}$.

We adopt similar notations as in Lemma 2.2 and for simplicity, let us denote $|G| = m$. For $\psi \in Aut(G^{\times n})$ we shall write $\psi = (\phi_1, \phi_2, \ldots, \phi_n : \pi)$ to denote the map

$$\psi(x_1, x_2, \ldots, x_n) = (\phi_1(x_{\pi(1)}), \phi_2(x_{\pi(2)}), \ldots, \phi_n(x_{\pi(n)}))$$

as in Equation (3.4) (see Theorem 3.4). Suppose ψ fixes the vertex $(x_1, x_2, \ldots, x_n) \in G^{\times n}$. In particular we have $x_{\pi(i)} = \phi_i^{-1}(x_i)$ for all i. It then follows that for all k, we have

$$\phi^{-1}_{\pi^k(i)}\left(x_{\pi^k(i)}\right) = x_{\pi^{k+1}(i)}$$

for each i. Consequently, if π has t cycles in its disjoint cycle representation then ψ can fix at most $|C_1|m^{t-1}$ vertices in C_1'.

If $\pi \neq I$, then $t < n$, and in this case, since $m \geq 3, n \geq 4$, we have $|C_1|m^{t-1} \leq (|C_1| - 2)m^{n-1}$. If $\pi = I$, then ψ is non-trivial if and only if $\phi_i \neq I$ for some i. In this case $\phi_i(x_i) = x_i$ for all i, so $(x_1, x_2, \ldots, x_n)$ is fixed by ψ if and only if $x_i \in Fix_{\phi_i}$ for all i. Consequently, $F_{\psi'} = \prod_{i=1}^{n} F_{\phi_i}$. Observe that if ϕ_i is not a transposition then it moves at least three vertices, say x, y and z in G. In particular, ψ does not fix any vertex of the form $(x_1, x_2, \ldots, g, \ldots, x_n)$, where $g \in \{x, y, z\}$ and appears in the i^{th} position. Thus, it follows that

$$F_\psi \leq |C_1|m^{n-2}(m-3).$$

If ϕ_i is a transposition for some $i > 1$ then it is easy to see that $F_\psi \leq (|C_1| - 3)m^{n-1} < (|C_1| - 2)m^{n-1}$. Finally, if ϕ_1 is a transposition, then again $F \leq (|C_1| - 2)m^{n-1}$. This proves the claim.

Setting $F = (|C_1| - 2)m^{n-1}$, $T = |C_1|m^{n-1}$ in Equation (3.3) gives us

$$f(\mathcal{G}) \leq \frac{n!|Aut(G)|^n}{2^{m^{n-1}}} + 1.$$

It is a simple calculation to see that the first term in the above expression is less than 1 for all $m \geq 3$ and $n \geq 4$. This completes the proof. □

Corollary 3.6. $\chi_D(K_r^{\times n}) = r + 1$ *for* $n \geq 4$*, and* $r \geq 3$.

Proof. First note that for $r \geq 3$, K_r is prime, non-bipartite, and $R - thin$. Hence by Theorem 3.5 it follows that $\chi_D(K_r^{\times n}) \leq r+1$. A result of Greenwell and Lovász (see [8]) tells us that all proper $r-$colorings for $K_r^{\times n}$ are induced by colorings of the factors K_r. In particular, it implies that $\chi_D(K_r^{\times n}) > r$. □

4 $\chi_D(G)$ versus $|Aut(G)|$

As indicated in the introduction, one aspect of the problem of the distinguishing chromatic number of particular interest is the contrasting behavior of the distinguishing chromatic number vis-á-vis the size of the automorphism group. Our sense of contrast here is to describe the size of the automorphism group as a function of the order of the graph.

First, note that one can give somewhat trivial examples of graphs with $\chi_D(G) > \chi(G) > k$ for any k and with a very small automorphism group as follows. Start with an arbitrary rigid graph, i.e. a graph with no non-trivial automorphism, with chromatic number larger than k. Now fix an edge $e = \{x, y\}$ and 'blow up' the vertices x, y by small disjoint subsets X, Y respectively, and replace the edge e by the complete bipartite graph on the sets X, Y. The new graph satisfies $\chi_D(G) > \chi(G)$ rather trivially and one can ensure that by picking small subsets X, Y we can ensure that the full automorphism group is not too large either.

In some sense, these examples are not very interesting because the fact that the distinguishing chromatic number exceeds the chromatic number for these graphs is attributable to a 'local' reason. It however becomes a more intriguing problem if we insist that the graph is also vertex transitive.

Our first theorem in this section gives examples of vertex transitive graphs that admit 'small' automorphism groups, and yet have $\chi_D(G) > \chi(G)$ and with arbitrarily large values of $\chi(G)$.

Theorem 4.1. *Given $k \in \mathbb{N}$, there exists a sequence of graphs G_{n_i} satisfying*

1. *$\chi(G_{n_i}) > k$,*
2. *$\chi_D(G_{n_i}) > \chi(G_{n_i})$,*
3. *G_{n_i} is vertex transitive and $|Aut(G_{n_i})| = O(n_i^{3/2})$.*

Proof. Let $q \geq k$ be prime and suppose $S \subset \mathbb{F}_q$ is a subset of size $\frac{q-1}{2}$. We define the graph G_S as follows: The vertices of $V(G_S)$ are the points of the affine plane $AG(2,q)$; $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in G_S if and only if $v_1 \neq u_1$ and $(v_2 - u_2)(v_1 - u_1)^{-1} \in S$. We denote $(v_2 - u_2)(v_1 - u_1)^{-1}$ by $s(u,v)$. For $\alpha, \beta \in \mathbb{F}_q$, consider the set $l_\alpha^\beta := \{(\beta + x, \beta + x\alpha) : x \in \mathbb{F}_q\}$. We shall call the sets l_α^β as *lines* in what follows. Observe that, for each $\alpha \in S$ and $\beta \in \mathbb{F}_q$, the sets l_α^β is a clique of size q, so $\chi(G_S) \geq q$. We shall denote the independent sets [2] $\{(\beta, x+\beta) : x \in \mathbb{F}_q\}$ by l_∞^β. Similarly, if $\alpha \notin S$ the set l_α^β is an independent set of size q, the collection $\{l_\alpha^\beta : \beta \in \mathbb{F}_q\}$ describes a proper q-coloring of G_S, hence $\chi(G_S) = q$.
Claim: $\chi_D(G_S) > q$. Let $C = \{C_1, C_2, \ldots, C_q\}$ be a proper $q-$coloring of G_S. We claim that each C_i is a line, i.e., for each $1 \leq i \leq q$ we have $C_i = l_\alpha^\beta$ for some $\alpha \notin S, \beta \in \mathbb{F}_q$.

Observe that for $\alpha \in S$, the collection $\mathcal{C} = \{l_\alpha^\beta | \beta \in \mathbb{F}_q\}$ partitions the vertex set of G_S into cliques of size q. Therefore, in any proper q-coloring of G_S, each color class contains exactly q vertices.

Next, we recall a result of Rédei [14] which states that for a prime number q if $X \subset AG(2,q)$ such that $|X| = q$ and X is not a line then the set $S(X) = \{s(x,y) | x \neq y, x, y \in X\}$ has size at least $\frac{q+3}{2}$.

If a color class C_i is not a line then by the theorem of Rédei, $|S(C_i)| \geq \frac{(q+3)}{2}$ and since $|S| = \frac{q-1}{2}$ this implies that $S(C_i) \cap S \neq \emptyset$. But then this contradicts that C_i is independent in G_S.

In particular, any proper q-coloring C of G_S must be a partition of the form $\{l_\alpha^\beta : \beta \in \mathbb{F}_q\}$ with $\alpha \in (\mathbb{F}_q \cup \{\infty\}) \setminus S$. Then the map

$$\begin{aligned} \phi_\alpha(x,y) &= (x+1, y+\alpha) \qquad \text{if } \alpha \neq \infty, \\ \phi_\infty(x,y) &= (x, y+1) \end{aligned}$$

is a nontrivial automorphism that fixes each color class of C. This establishes that $\chi_D(G_S) > q$ and proves the claim.

Now, we shall show that for a suitable choice of S, G_S has a relatively small automorphism group. Our choice of subset S shall be a uniformly random subset of $\mathbb{F}_q$.

Note that our earlier proof of the claim in fact shows that any maximum independent set corresponds to a line in $AG(2,q)$. We now make the observation that all maximum sized

[2] These are independent in G_S since $\infty \notin \mathbb{F}_q$.

cliques also correspond to certain lines in $AG(2,q)$. Indeed, suppose X is a maximum clique of size q which does not correspond to a line. Again, by Rédei's theorem we have $|S(X)| \geq \frac{q+3}{2}$. Since X is a clique, $S(X) \subset S$, but this contradicts the fact that $|S| = \frac{q-1}{2}$.

Consequently, if $\phi \in Aut(G_S)$ then since maximum cliques (respectively, maximum independent sets) are mapped into maximum cliques (resp. maximum independent sets), it follows that ϕ is a bijective map on $\mathbb{F}_q^2$ which maps affine lines into affine lines in $AG(2,q)$ (as a consequence of [14]). Hence, it follows that $Aut(G_S) \subset AGL(2,q)$ (see [11]). In other words, any $\phi \in Aut(G_S)$ can be written as $A + \bar{b}$ for some $A \in Aut_0(G_S)$ and $\bar{b}(= \phi(0,0)) \in \mathbb{F}_q^2$, where $Aut_0(G_S) \subset Aut(G_S)$ is the subgroup of automorphisms which fix the vertex $(0,0) \in V(G_S)$.

The following lemma shows that for a random choice of the set S, the automorphism group $Aut(G_S)$ is not very large.

Lemma 4.2. *Suppose S is picked uniformly at random from the set of all $\frac{q-1}{2}$ subsets of $\mathbb{F}_q$. Then asymptotically almost surely, $Aut(G_S) = \{\lambda I + \bar{b} : \lambda \in \mathbb{F}_q^*, \quad \bar{b} \in V(G_S)\}$. Consequently, $|Aut(G_S)| = q^2(q-1)$ asymptotically almost surely.*

Here by the phrase asymptotically almost surely we mean that the probability that $Aut(G_S) = \{\lambda I + \bar{b} : \lambda \in \mathbb{F}_q^*, \bar{b} \in V(G_S)\}$ approaches 1 as $q \to \infty$.

Proof. Since we have already observed that $Aut(G_S) \subset AGL(2,q)$, every $\phi \in Aut(G_S)$ can be written in the form $\phi(x,y) = A(x,y) + (b_1,b_2)$ for some $b_1, b_2 \in \mathbb{F}_q$ and $A \in Aut_0(G_S)$. Here, $A \in GL(2,q)$ corresponds to a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $a,b,c,d \in \mathbb{F}_q$ with $ad - bc \neq 0$.

We introduce the symbol ∞ and adopt the convention that $a + \infty = \infty, a \cdot \infty = \infty$ for $a \neq 0$, and $\frac{a}{0} = \infty$ for $a \neq 0$. For $\phi \in Aut(G_S)$, define a map $f_\phi : \mathbb{F}_q \cup \{\infty\} \to \mathbb{F}_q \cup \{\infty\}$ as follows:

$$\begin{aligned} f_\phi(\alpha) &= \frac{d\alpha + c}{a + b\alpha}, \qquad \text{if } \alpha \neq -\frac{a}{b}, \\ f_\phi\left(\frac{-a}{b}\right) &= \infty, \\ f_\phi(\infty) &= \frac{d}{b}. \end{aligned}$$

Observe that f_ϕ is trivial if and only if $b = c = 0$ and $a = d$.

Let $x = (x_1, x_2), y = (y_1, y_2)$ be two adjacent vertices in G_S. Since $\phi(x)$ is adjacent to $\phi(y)$, we have

$$s(\phi(x), \phi(y)) = \frac{c(y_1 - x_1) + d(y_2 - x_2)}{a(y_1 - x_1) + b(y_2 - x_2)} = \frac{d \cdot s(x,y) + c}{b \cdot s(x,y) + a}.$$

Observe that $y_1 - x_1$ is nonzero since $s(x,y) \in S$. Therefore we have,

$$s(\phi(x), \phi(y)) = f_\phi(s(x,y)). \tag{4.1}$$

Also note that if $\phi \neq \lambda I$ then for $\mu \in \mathbb{F}_q$ and $k \in \mathbb{N}$, setting

$$f_\phi^{(k)}(\mu) := \underbrace{f_\phi \circ f_\phi \circ \cdots \circ f_\phi}_{k-\text{fold}}(\mu) = \mu$$

yields a quadratic equation in μ, so there are at most two values of $\mu \in \mathbb{F}_q$ satisfying $f_\phi^{(k)}(\mu) = \mu$. In other words, for each positive integer k, the map f_ϕ admits at most two orbits of size k. Moreover if $A \in Aut(G_S)$ then by equation (4.1), $f_A(S) = S$.

Consider the event E: There exist a nontrivial automorphism $A \in Aut_0(G_S)$ such that f_A is not the identity map. Observe that E is the union of the events E_A where the event E_A is described as follows: For any $A \in GL(2, q)$ where $A \neq \lambda I$, $\lambda \neq 0$, S is the union of f_A orbits. Recall that f_A is not the trivial map if and only if $A \neq \lambda I$ for any $\lambda \neq 0$.

By a favorable automorphism, we shall mean an automorphism $A \in Aut_0(G_S)$, $A \neq \lambda I$ such that S is union of f_A orbits. By the preceding discussion, it follows that a favorable automorphism of G_S induces a partition Λ of $\frac{q-1}{2}$ in which there are at most two parts of any size. Therefore the number of favorable automorphisms is at most twice the number of integer partition of $\frac{q-1}{2}$ in which there are at most two parts of any size which is clearly less than $2p(\frac{q-1}{2})$, where $p(n)$ denotes the partition function. By the asymptotics of the partition function of Hardy-Ramanujan (see [10]),

$$p(t) \sim \frac{1}{4t\sqrt{3}} \exp\left(\pi\sqrt{\frac{2t}{3}}\right),$$

where $t = (q-1)/2$. So in particular, for any $A \in Aut_0(G_S)$ the probability that f_A is nontrivial is less than $p(t)\binom{q}{t}^{-1}$. Consequently,

$$\mathbb{P}(E) \leq (q^2-1)(q^2-q)\frac{2p(t)}{\binom{q}{t}} \to 0 \text{ as } q \to \infty.$$

Hence asymptotically almost surely, every $S \subset \mathbb{F}_q$ satisfies $Aut_0(G_S) = \{\lambda I : \lambda \in \mathbb{F}_q\}$. The second statement follows trivially from this conclusion and this completes the proof of the lemma. ☐

Resuming the proof of the theorem, let S be a subset of $\mathbb{F}_q$ of size $\frac{q-1}{2}$ such that $Aut(G_S) = \{\lambda I + \bar{b} : \lambda \in \mathbb{F}_q^*, \quad \bar{b} \in V(G_S)\}$; such a choice for S exists by the preceding lemma. For such S, the distinguishing chromatic number of G_S is greater than its chromatic number. Furthermore, since G_S admits all translations in $AG(2, q)$ as automorphisms it follows that it is vertex transitive. ☐

In fact, the graph G_S satisfies $\chi_D(G_S) = \chi(G_S) + 1$ as we shall see now.

Theorem 4.3. *Let $S \subset \mathbb{F}_q$ be a set of size $\frac{q-1}{2}$ such that $Aut_0(G_S) = \{\lambda I : \lambda \in \mathbb{F}_q\}$. Then $\chi_D(G_S) = q + 1$.*

Proof. For $1 \neq \gamma \notin S$, consider the coloring of G_S described by the color classes $\{l_\gamma^\beta : \beta \in \mathbb{F}_q\}$. Assign the color $q+1$ to only the vertex $(0,0) \in V(G_S)$. This forms a $q+1$ coloring of $G_S$which is obviously a proper coloring. To show that this is distinguishing, let ϕ be a color fixing nontrivial automorphism of G_S. By Theorem 4.2, ϕ maps (x, y) to $(ax + b_1, ay + b_2)$ for some $a, b_1, b_2 \in \mathbb{F}_q$. Since ϕ fixes $(0,0)$ we have $b_1 = b_2 = 0$ and $a \neq 1$. This implies $\phi = aI$ and hence it is not color fixing; indeed ϕ maps $(1,1)$ to (a,a) and $(a,a) \notin l_\gamma^1$. ☐

Our second result in this section describes a family of graphs with very large automorphism groups - much larger than exponential in $|V(G)|$, but for which $\chi_D(G) = \chi(G)$. As was proven in [2], we already know that the Kneser graphs $K(n, r)$ with $r \geq 3$ satisfy the same. However, one might also expect that in such cases, distinguishing proper colorings are perhaps rare, or at the very least, that there do exist minimal proper, non-distinguishing colorings of G. It turns out that even this is not true.

Theorem 4.4. *Let $\overline{K(n,r)}$ denote the complement of the Kneser graph, i.e., the vertices of $\overline{K(n,r)}$ correspond to r element subsets of $[n]$ and two vertices are adjacent if and only if their intersection is non-empty. Then for $n \geq 2r$ and $r \geq 3$ $\chi_D(\overline{K(n,r)}) = \chi(\overline{K(n,r)})$. Moreover, every proper coloring of $\overline{K(n,r)}$ is in fact distinguishing.*

Proof. First, observe that since $Aut(K(n,r)) \simeq S_n$ for $n \geq 2r$, the full automorphism group of $\overline{K(n,r)}$ is also S_n.

Consider a proper coloring c of $\overline{K(n,r)}$ into color classes $C_1, C_2, \ldots, C_t$. Note that for any two vertices v_1, v_2 in the same color class, $v_1 \cap v_2 = \emptyset$. If possible, let $\sigma \in S_n$ be a non-trivial automorphism which fixes C_i for each i. Without loss of generality let $\sigma(1) = 2$. Observe that for the vertex $v_1 = (1, 2, \ldots, r)$, its color class has no other vertex containing 1 or 2, so σ maps $\{1, 2, , \ldots, r\}$ to $\{1, 2, \ldots, r\}$. Again, with the vertex $v_2 = \{1, 3, \ldots, r+1\}$, which is in color class $C_2 \neq C_1$, σ maps v_2 into $\{2, \sigma(3), \ldots, \sigma(r+1)\} \neq v_2$, so $\sigma(v_2) \cap v_2 = \emptyset$ by assumption. However, since $\sigma(i) \in \{1, 2, \ldots, r\}$ for each $3 \leq i \leq r$ this yields a contradiction. □

5 Bipartite graphs with large $\chi_D(G)$

In this section we describe a family of bipartite graphs whose distinguishing chromatic number is greater than any integer k, where $k \geq 4$.

As we described in the introduction, the sense of non-triviality of these examples arises from a couple of factors. Our examples contain several copies of $K_{r,s}$ as induced subgraphs. That by itself does not imply that the distinguishing chromatic number is at least $r + s$ but it is suggestive. What makes these families nontrivial is the fact that the distinguishing chromatic number of these graphs is in fact $r + s + 1$.

Again, in order to describe these graphs, let $q \geq 5$ be a prime power, and let $\Pi := (\mathcal{P}, \mathcal{L})$ be a Desarguesian projective plane of order q. As is customary, we denote by $[r]$, the set $\{1, 2 \ldots, r\}$.

The graph which we denote $LG_q \otimes K_{r,s}$ has vertex set $V(LG_q \otimes K_{r,s}) = (\mathcal{P} \times [r]) \sqcup (\mathcal{L} \times [s])$, and for $p \in \mathcal{P}, l \in \mathcal{L}$, and $(i, j) \in [r] \times [s]$ we have (p, i) adjacent to (l, j) if and only if $p \in l$. Another way to describe this graph goes as follows. The weak product $LG_q \times K_{r,s}$ is bipartite and consists of two isomorphic bipartite components. The graph $LG_q \otimes K_{r,s}$ is one of the connected components.

For each point p there are r copies of p in the graph $LG_q \otimes K_{r,s}$; we call the set $\{(p, i) | i \in [r]\}$ the fiber of p, and denoted it by $F(p)$. Similarly we denote by $F(l)$, the set $F(l) = \{(l, i) : i \in [s]\}$, and shall call this the fiber of l. Each vertex (p, i) (resp. (l, j)) of $LG_q \otimes K_{r,s}$ has degree $r(q + 1)$ (resp. $s(q + 1)$).

Theorem 5.1. $\chi_D(LG_q \otimes K_{r,s}) = r + s + 1$, *where $r, s \geq 2$ and $q \geq 5$ is a prime number.*

Proof. Firstly, we show that $\chi_D(LG_q \otimes K_{r,s}) > r + s$.

If possible, let C be an $(r + s)$-proper distinguishing coloring of $LG_q \otimes K_{r,s}$ and let $C_i, i \in [r + s]$ be the color classes of C in $LG_q \otimes K_{r,s}$. We claim:

1. For each $p \in \mathcal{P}$, each vertex of $F(p)$ gets a distinct color. The same also holds for each $l \in \mathcal{L}$ and each vertex of $F(l)$.
2. If $\mathcal{C}_{\mathcal{P}}$ and $\mathcal{C}_{\mathcal{L}}$ denote the sets of colors on the vertices of $\bigcup_{p \in \mathcal{P}} F(p)$ and $\bigcup_{l \in \mathcal{L}} F(l)$ respectively, then $\mathcal{C}_{\mathcal{P}} \cap \mathcal{C}_{\mathcal{L}} = \emptyset$ and $|\mathcal{C}_{\mathcal{P}}| = r, |\mathcal{C}_{\mathcal{L}}| = s$. Consequently, for each i, either $F(p) \cap C_i \neq \emptyset$ for each $p \in \mathcal{P}$ or $F(l) \cap C_i \neq \emptyset$ for each $l \in \mathcal{L}$.

We shall first prove each of the claims made above.

1. For $p \in \mathcal{P}$ suppose $F(p)$ contains two elements, say (p, i) and (p, j), with the same color. Consider the map ϕ that swaps (p, i) with (p, j) and fixes all other vertices. It is easy to see that ϕ is a graph automorphism which fixes each color class C_i contradicting the assumption that C is distinguishing. The argument for the part regarding vertices in the fiber $F(l)$ is identical.
2. Let $l \in \mathcal{L}$ and $p \in l$. By claim 1 each vertex in $F(p)$ has a distinct color. Since $|F(p)| = r$ we may assume without loss of generality let (p, i) gets color i for $i \in [r]$. In that case, no vertex of $F(l)$ can be colored using any color in $[r]$. Furthermore, by the same reasoning as above, each vertex of $F(l)$ is colored using a distinct color, so we may assume again that (l, i) is colored $r + i$ for $i = 1, 2 \ldots, s$. Since there is a unique line through any two points, no vertex of the form (p', j) gets a color in $\bigcup_{i=r+1}^{r+s} C_i$. Similarly, no vertex of the form (l', j) belongs to $\bigcup_{i=1}^{r} C_i$. Therefore, all points and their fibers belongs to $\bigcup_{i=1}^{r} C_i$ and all lines with their fibers belongs to $\bigcup_{i=r+1}^{r+s} C_i$.

   From claims 1 and 2 above, we conclude that for each $p \in \mathcal{P}$, $C_i \cap F(p) \neq \emptyset$ for $i \in [r]$. Otherwise, since $|F(p)| = r$, there exist an $i \in [r]$ such that $|C_i \cap F(p)| \geq 2$, contradicting claim 1. Similar arguments show that for each $l \in \mathcal{L}$, $C_{i+r} \cap F(l) \neq \emptyset$ for $i \in [s]$.

To show C is not a distinguishing coloring we produce a nontrivial automorphism of $LG_q \otimes K_{r,s}$ which fixes each C_i for $i = 1, 2, \ldots, r + s$. We first set up some terminology. For $i \in [r]$, we call a vertex in the fiber of p its i^{th} vertex if its color is i and shall denote it p^i. Similarly, we shall call a vertex in the fiber of l its i^{th} point if its color is $i + r$ and shall denote it by l^i.

Let $\psi \in Aut(LG_q)$ be a nontrivial automorphism such that $\psi(\mathcal{P}) = \mathcal{P}$ so that it also satisfies $\psi(\mathcal{L}) = \mathcal{L}$. Let σ be defined on $V(LG_q \otimes K_{r,s})$ by $\sigma(v^i) = \psi(v)^i$ for $v \in \mathcal{P} \sqcup \mathcal{L}$. It is clear that σ is a color preserving map. Moreover σ preserves adjacency in $LG_q \otimes K_{r,s}$; indeed, v is adjacent to w in LG_q if and only if $F(v) \cup F(w)$ forms a $K_{r,s}$ as a subgraph of $LG_q \otimes K_{r,s}$ and $\psi \in Aut(LG_q)$. Therefore σ is a nontrivial automorphism which fixes the color classes, thereby showing that $\chi_D(LG_q \otimes K_{r,s}) > r + s$.

We now claim that $\chi_D(LG_q \otimes K_{r,s}) \leq r + s + 1$. For $1 \leq i \leq r - 1$, assign the color i to the points $\{(p, i) : p \in \mathcal{P}\}$ and for $r + 1 \leq j \leq r + s$ let $\{(l, j) : l \in \mathcal{L}\}$ be colored j. Recall that LG_q admits a distinguishing 3-coloring in which every vertex of $\mathcal{L}$ is given the same color, and the point set $\mathcal{P}$ is partitioned into $\mathcal{P}_1, \mathcal{P}_2$ that correspond to the other two color classes (Theorem 3.1). We split the set $\{(p, r) : p \in \mathcal{P}\}$ into $C_r := \{(p, r) | p \in \mathcal{P}_1\}$ and $C_{r+s+1} := \{(p, r) : p \in \mathcal{P}_2\}$ and designate these sets as color classes r and $r + s + 1$ respectively.

It is easy to see that the above coloring is proper since adjacent vertices get different colors. To see that it is distinguishing, let μ be a nontrivial automorphism which fixes each color class. Since μ fixes each color class as a set, and μ is nontrivial, in particular, μ fixes the set $\{(p,r) : p \in \mathcal{P}\}$, and also fixes each set $\{(l,i) : l \in \mathcal{L}\}$ for $r+1 \le i \le r+s$, so in particular, μ induces a nontrivial automorphism, ν, on $LG_q = C_r \cup C_{i+r}$ for each $i \in [s]$, which is non-distinguishing. But this contradicts Theorem 3.1, and so we are through. $\square$

6 Concluding Remarks

- It is possible to consider other Levi graphs arising out of other projective geometries (affine planes, incidence bipartite graphs of 1-dimensional subspaces versus k dimensional subspaces in an n dimensional vector space for some k etc). Many of our results and methods work in those contexts as well and it should be possible to prove similar results there as well, as long as the full automorphism group is not substantially larger. For instance, in the case of the incidence graphs of k sets versus l-sets of $[n]$, it is widely believed (see [7], chapter 1) that in most cases, the full automorphism group of the generalized Johnson graphs is indeed S_n though it is not known with certainty.

- As stated earlier, we believe that $\chi_D(LG_4) = 3$ though we haven't been able to show the same. Similarly, we believe $\chi_D(LG_3) = 4$. One can, by tedious arguments considering several cases, show that a monochromatic 3-coloring of LG_3 is not a proper distinguishing coloring. For related details on what a monochromatic coloring is, see the Appendix.

- We were able to show $\chi_D(K_r^{\times n}) = r+1$ since in this case, all proper r colorings of $K_r^{\times n}$ are of a specific type. For an arbitrary (prime) graph H, it is not immediately clear if $\chi_D(H^{\times n}) > \chi(H)$. It would be interesting to find some characterization of graphs H with $\chi_D(H^{\times n}) = \chi(H) + 1$ for large n.

- For a given $k \in \mathbb{N}$, we obtained nontrivial examples of family of vertex-transitive graphs G with arbitrarily large chromatic number which have $\chi_D(G) > \chi(G)$ and with $|Aut(G)|$ somewhat small. It is an interesting question to seek infinite families of vertex-transitive graphs G with $\chi_D(G) > \chi(G) > k$ for any prefixed k, while $|Aut(G)| = O_k(|G|)$.

- While we have attempted to construct non-trivial families of bipartite graphs with large distinguishing chromatic number, it would be interesting to construct nontrivial examples of graphs with arbitrary chromatic number, and arbitrarily large distinguishing chromatic number.

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7 Appendix

7.1 The Levi graph LG_2

Firstly, we remark that the upper bound $\chi_D(G) \le 2\Delta - 2$ whenever G is bipartite and $G \not\cong K_{\Delta-1,\Delta}, K_{\Delta,\Delta}$, which appears in [13], gives $\chi_D(LG_q) \le 2q$. In particular, $\chi_D(LG_2) \le 4$. We shall show that in fact $\chi_D(LG_2) = 4$.

We first set up some notation, let $\{e_1, e_2, e_3\}$ be the standard basis of the vector space V with $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. For $g, h, k \in F_q$, a vector $v \in V$ is denoted by (g, h, k) if $v = ge_1 + he_2 + ke_3$. A point $p \in \mathcal{P}$ is denoted by (g, h, k) if $p =< ge_1 + he_2 + ke_3 >$. Thus, there are q^2 points in the form $(1, h, k)$ such that $h, k \in \mathbb{F}_q$, q points in the form of $(0, 1, k)$ such that $k \in \mathbb{F}_q$ and finally the point $(0, 0, 1)$ to account for a total of $q^2 + q + 1$ points in $PG(2, \mathbb{F}_q)$.

We start with the following definition.

Definition 7.1. A coloring of the Levi graph is said to be **monochromatic** if all the vertices in one set of the vertex partition have the same color.

Lemma 7.2. *LG_2 does not have a proper distinguishing monochromatic 3-coloring.*

Proof. Assume that LG_2 has a proper distinguishing monochromatic 3-coloring. Without loss of generality let the line set $\mathcal{L}$ be colored with a single color, say red. Call the remaining two colors blue and green, say, which are the colors assigned to the vertices in $\mathcal{P}$. We shall refer to the set of points that are assigned a particular color, say green, as the color class $Green$. By rank of a color class $\mathcal{C}$ (denoted $r(\mathcal{C})$), we mean the rank of the vector subspace generated by $\mathcal{C}$. Observe that a nontrivial linear map T that fixes the color class $Green$, must necessarily also fix the color class $Blue$, so any such linear map would correspond to an automorphism that preserves each color class.
For any 2-coloring of $\mathcal{P}$ (which has 7 points), one of the two color class has fewer than four points. Without loss of generality, assume that this is the color class $Green$. Firstly, if $r(Green) \le 2$ then consider a basis B of V which contains a maximal linearly independent set of points in color class $Green$. If $r(Green) = 2$, then the linear map T obtained by swapping the elements of the color class $Green$ in B, and fixing every other basis element is a non-trivial linear transformation of V which necessarily fixes the color class $Green$. If $r(Green) = 1$, then consider the map T which fixes the green point of B and swaps the other two (necessarily Blue) is a nontrivial linear transform that fixes the color class $Green$. Finally, if $r(Green) = 3$, then let T be the map that swaps two of them and fixes the third. Again, this map is a nontrivial linear map that fixes every color class. □

We now set up some notation. Denote the points in LG_2 by e_1, e_2, e_3, $e_1 + e_2$, $e_1 + e_3$, $e_2 + e_3$ and $e_1 + e_2 + e_3$ (see Figure1) and denote the lines in the following way:

1. $l_1 : \langle e_1, e_2 \rangle$ the line (two dimensional subspace) spanned by e_1 and e_2.
2. $l_2 : \langle e_1, e_3 \rangle$.
3. $l_3 : \langle e_2, e_3 \rangle$.
4. $l_4 : \langle e_1, e_2 + e_3 \rangle$.
5. $l_5 : \langle e_2, e_1 + e_3 \rangle$.
6. $l_6 : \langle e_3, e_1 + e_2 \rangle$.

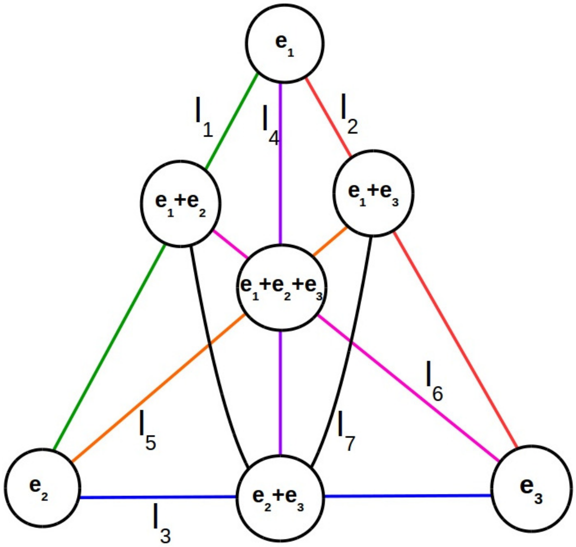

Figure 1: Fano plane

7. $l_7 : \langle e_1 + e_3, e_2 + e_3 \rangle$.

Theorem 7.3. $\chi_D(LG_2) = 4$.

Proof. By the remark at the beginning of the section, we have $\chi_D(LG_2) \leq 4$, so it suffices to show $\chi_D(LG_2) > 3$. We first claim that if LG_2 has a proper distinguishing 3-coloring, then three linearly independent points (points corresponding to three linearly independent vectors) get the same color.

Suppose the claim is false. Then each monochrome set $\mathcal{C}$ of points satisfies $r(\mathcal{C}) \leq 2$. Since any set of four points contains three linearly independent points and $|V(LG_2)| = 7$, a 3-coloring yields a monochrome set of points of size exactly three. Denote this set by E and observe that E in fact corresponds to a line $l_E \in \mathcal{L}$. Since any two lines intersect, no line is colored the same as the points of E. If $p, p' \in \mathcal{P} \setminus E$ are colored differently, then the line $l_{p,p'}$ cannot be colored by any of the three colors contradicting the assumption. Consequently, every point in $\mathcal{P} \setminus E$ must be colored the same if the coloring were to be proper. But then this gives a color class with four points which contains three linearly independent points contradicting that the claim was false. Without loss of generality, suppose e_1, e_2, e_3 are all colored red. Since l_7 contains the points $e_1 + e_2, e_2 + e_3$ and $e_1 + e_3$, these three points cannot all have different colors. Hence at least two of these three points are in the same color class.

Without loss of generality, assume that $e_1 + e_2$ and $e_2 + e_3$ have the same color. Now observe that the map σ defined by $\sigma(e_1) = e_3, \sigma(e_3) = e_1, \sigma(e_2) = e_2$, induces an automorphism of LG_2 that fixes every color class within $\mathcal{P}$. Furthermore σ swaps l_1 with l_3 and l_4 with l_6 and fixes all the other lines. If the sets of lines $\{l_4, l_6\}$ and $\{l_1, l_3\}$ are both monochrome in $\mathcal{L}$, then note that σ fixes every color class contradicting that the coloring in question is distinguishing. Thus we consider the alternative, i.e., the possibilities that the lines l_1 and l_3 (resp. l_4 and l_6) are in different color classes, and in each of those cases produce a non-trivial automorphism fixing every color class.

Case I : l_4 and l_6 have different colors, say blue and green respectively. In this case, the point set witnesses at most two colors and none of the points of $\mathcal{P} \setminus \{e_1 + e_3\}$ can be colored blue or green. Moreover, by Lemma 7.2, all the seven points cannot be colored red (note that e_1, e_2, e_3 are colored red). Consequently, $e_1 + e_3$ is colored, say blue, and all the other points are colored red. The l_7, l_5 and l_2 are all colored green since all these three lines contain the point $e_1 + e_3$. As mentioned above, we shall in every case that may arise, describe a non-trivial automorphism σ that fixes each color class. As before, we shall only describe its action on the set $\{e_1, e_2, e_3\}$.
Sub case 1 : l_1 is colored blue. Then $\sigma(e_1) = e_1, \sigma(e_2) = e_2 + e_3, \sigma(e_3) = e_3$ fixes $e_1 + e_3$, swaps l_1 with l_4 and fixes l_3. Consequently, it fixes every color class.
Sub case 2 : l_1 is colored green and l_3 is colored blue. In this case, $\sigma(e_1) = e_2, \sigma(e_2) = e_1, \sigma(e_3) = e_1+e_2+e_3$ does the job. *Sub case 3 :* l_1 and l_3 are both colored green. In this case, the only line which is colored blue is l_4. Then $\sigma(e_1) = e_2 + e_3, \sigma(e_2) = e_2, \sigma(e_3) = e_1 + e_2$, does the job.
From the above it follows that l_4 and l_6 cannot be in different color classes. So, we now consider the other possibility, namely that l_1 and l_3 are in different color classes.

Case II: l_6 and l_4 have the same color but l_1 and l_3 are in different color classes, say blue and green respectively. Here we first note that $e_1 + e_2$ and $e_2 + e_3$ are necessarily red because they belong to l_1 and l_3 respectively. Again, we are led to three subcases:
Sub case 1 : $e_1 + e_3$ and $e_1 + e_2 + e_3$ are both colored blue. Here, it is a straightforward check to see that every $l \neq l_1$ is colored green. Then, one can check that $\sigma(e_1) = e_1 + e_2, \sigma(e_2) = e_2, \sigma(e_3) = e_3$ fixes every color class.
Sub case 2 : The point $e_1 + e_3$ is colored red and $e_1 + e_2 + e_3$ is colored blue. Again, one can check in a straightforward manner, that for all $3 \leq i \leq 6$, l_i is colored green. If l_2 is blue then $\sigma(e_2) = e_3, \sigma(e_3) = e_2, \sigma(e_1) = e_1$ does the job. If l_2 is colored green, $\sigma(e_1) = e_2, \sigma(e_2) = e_1, \sigma(e_3) = e_3$ does the job.
Sub case 3 : $e_1+e_2+e_3$ is colored red and e_1+e_3 is colored blue. Here we first observe that l_2, l_3, l_5, l_7 are all necessarily green. Also, by the underlying assumption (characterizing Case II), l_4, l_6 bear the same color. In this case, $\sigma(e_1) = e_1+e_2, \sigma(e_3) = e_2+e_3, \sigma(e_2) = e_2$, does the job. This exhausts all the possibilities, and hence we are through. □

7.2 The Levi graph LG_3

As remarked earlier, it is not too hard to show that $\chi(LG_q) \leq 6$, so the same holds for $q = 3$ as well. The next theorem shows an improvement on this result.

Theorem 7.4. $\chi_D(LG_3) \leq 5$.

Proof. As indicated earlier we denote the points $p \in \mathcal{P}$ as mentioned in the beginning of this section. A line corresponding to the subspace $\{(x, y, z) \in \mathcal{P} : ax + by + cz = 0\}$ is denoted $(\mathbf{a}, \mathbf{b}, \mathbf{c})$. We color the graph using the colors $1, 2, 3, 4, 5$ as in Figure 2 (the color is indicated in a rectangular box corresponding to the vertex) It is straightforward to check that the coloring is proper. For an easy check we provide below, a table containing adjacencies of each $p \in \mathcal{P}$.

Here the first row lists all the points in the projective plane of order 3. The column corresponding to the vertex $p \in \mathcal{P}$ lists the set of lines $l \in \mathcal{L}$ such that $p \in l$, so that the

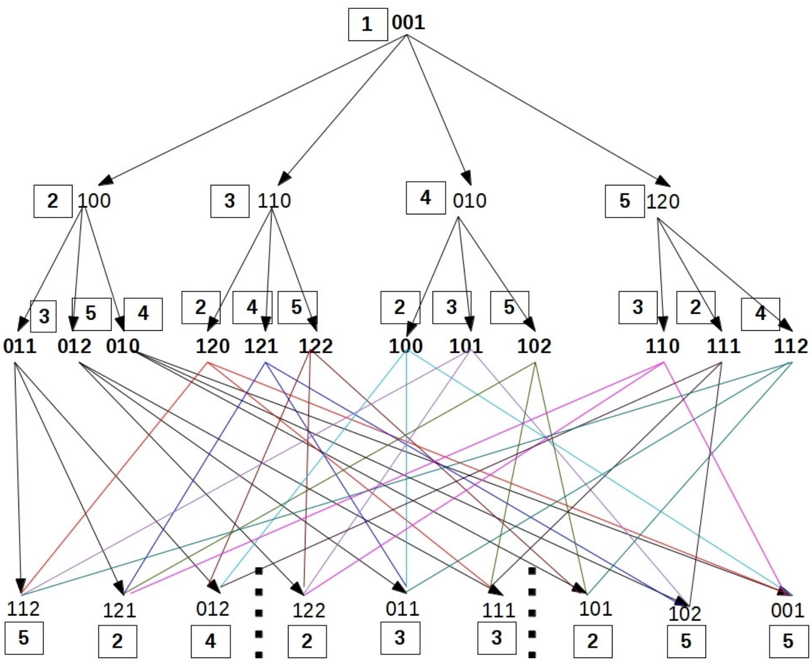

Figure 2: LG_3

Points →	100	110	010	120	112	121	012	122	011	111	101	102	001
Lines	**001**	**001**	**001**	**001**	**011**	**011**	**011**	**012**	**012**	**012**	**010**	**010**	**010**
↓	**011**	**120**	**100**	**110**	**120**	**121**	**122**	**122**	**121**	**120**	**122**	**121**	**120**
	012	**121**	**101**	**111**	**101**	**102**	**100**	**101**	**100**	**102**	**102**	**101**	**100**
	010	**122**	**102**	**112**	**112**	**110**	**111**	**110**	**112**	**111**	**112**	**111**	**110**

columns are the adjacency lists for the vertices in $\mathcal{P}$. To see that this coloring is distinguishing, firstly, observe that the line $\mathbf{001}$ is the only vertex with color 1. Therefore, any automorphism ϕ that fixes every color class necessarily fixes this line. Consequently, the points on $\mathbf{001}$ are mapped by ϕ onto themselves. Since each point on $\mathbf{001}$ bears a different color, it follows that ϕ fixes each $p \in \mathbf{001}$. In particular, for $1 \leq i \leq 4$, ϕ maps each set $\{l_{i1}, l_{i2}, l_{i3}\}$ onto itself. Here, $\{l_{ij}, 1 \leq j \leq 3\}$ denotes the set of lines adjacent to the i^{th} point of $\mathbf{001}$. But again note that by the coloring indicated, the vertices l_{ij} and $l_{ij'}$ have different colors for each i, so $\phi(l_{ij}) = l_{ij}$ for each pair (i, j) with $1 \leq i \leq 4, 1 \leq j \leq 3$. Now it is a straightforward check to see that $\phi = I$. □

7.3 SAGE code to calculate $f(\mathcal{G})$ when $\mathcal{G} = PGL(\mathbb{F}_q^3)$

SAGE has inbuilt functions that allow us to list the elements of $PGL(\mathbb{F}_q^3)$ and to write down the disjoint cycle decomposition of a given permutation. For $\sigma \in PGL(\mathbb{F}_q^3)$ the disjoint cycle decomposition, including its fixed points gives θ_σ, the total number of distinct orbits induced by σ as in Lemma 2.2. Now it is easy to calculate $\mathbb{P}(B_\sigma)$ and sum over $PGL(\mathbb{F}_q^3)$. The SAGE code that we used for the calculation is given below, note that the text in the square bracket is an explanation of the corresponding line in the code.

```
#!/usr/bin/env sage -python
import sys
from sage.all import *
p = int(input("Enter p = " ))
t = float(input("Enter t = " ))
etot = p**2 + p + 1 [Total number of points in LG_p.]
y = 0
G = PGL(3,p) [Automorphism group of PG(3, p).]
n = int(G.order()) [cardinality of G.]
for i in range(1, n):
    g = G[i]
    s = Set(g.cycle_tuples(singletons=True)) [Set of disjoint cycles of g ∈ G including singleton.]
    og = s.cardinality() [Gives total number of distinct orbits induced by ‘g′. That is θ_g in Lemma 2.2.]
    ex = float(etot - og) [|C − 1| − θ_g as in the proof of Lemma 2.2.]
    pg = 1 / float(t**ex)
    y = float(pg + y)
print "tot prob is",y
```

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Coherent configurations over copies of association schemes of prime order

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Abstract

Let G be a group acting faithfully and transitively on Ω_i for $i = 1, 2$. A famous theorem by Burnside implies the following fact: If $|\Omega_1| = |\Omega_2|$ is a prime and the rank of one of the actions is greater than two, then the actions are equivalent, or equivalently $|(\alpha, \beta)^G| = |\Omega_1| = |\Omega_2|$ for some $(\alpha, \beta) \in \Omega_1 \times \Omega_2$.

In this paper we consider a combinatorial analogue to this fact through the theory of coherent configurations, and give some arithmetic sufficient conditions for a coherent configuration with two homogeneous components of prime order to be uniquely determined by one of the homogeneous components.

Keywords: Coherent configurations, association schemes, prime order, symmetric designs.

Math. Subj. Class.: 05C15, 05C10

1 Introduction

A famous theorem by Burnside states that each transitive permutation group of prime degree with rank greater than two is Frobenius or regular. Since any Frobenius group of prime degree is a subgroup of one-dimensional affine group, it follows that such a permutation group is uniquely determined by its rank and degree up to equivalence of group actions. Especially, if a group acts faithfully, transitively but not 2-transitively on each of two sets of the same prime size, then the two actions are equivalent. Let us formulate this fact in the following two paragraphs.

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Let G be a group acting transitively on Ω_i for $i = 1, 2$. Then G acts on $\Omega_i \times \Omega_j$ by

$$(\alpha, \beta)^g = (\alpha^g, \beta^g) \qquad \text{for } (\alpha, \beta) \in \Omega_i \times \Omega_j \quad \text{and} \quad g \in G,$$

for all $i, j = 1, 2$. It is well-known that (e.g., see [6, Lemma 1.6B]) the following are equivalent:

(a) The action of G on Ω_1 is equivalent to that on Ω_2;

(b) There exists $(\alpha, \beta) \in \Omega_1 \times \Omega_2$ such that $G_\alpha = G_\beta$;

(c) There exists $(\alpha, \beta) \in \Omega_1 \times \Omega_2$ such that $|(\alpha, \beta)^G| = |\Omega_1| = |\Omega_2|$.

Note that the rank of the action of G on Ω_i is equal to the number of orbits of G on $\Omega_i \times \Omega_i$, and if G acts faithfully on Ω_i, then G can be identified with a permutation group of Ω_i.

Suppose that G acts faithfully on Ω_i with $i = 1, 2$ and $|\Omega_1| = |\Omega_2|$ is a prime. Then, as mentioned in the first paragraph, these actions are equivalent if the rank of one of the actions is greater than two, and so there exists an orbit R of G on $\Omega_1 \times \Omega_2$ such that $|R| = |\Omega_1| = |\Omega_2|$.

In this paper we consider a combinatorial analogy to this fact through the theory of coherent configurations. The concept of coherent configurations was first introduced by Higman who published a series of papers (e.g., [11], [12], [13]) to associate a lot of important criterions with group actions.

Here we define a coherent configuration, its intersection numbers and its fibers according to the notations as in [7].

Definition 1.1. Let V be a finite set and $\mathcal{R}$ a partition of $V \times V$. We say that the pair $\mathcal{C} = (V, \mathcal{R})$ is a *coherent configuration* if it satisfies the following:

1. The diagonal relation Δ_V is a union of elements of $\mathcal{R}$ where we denote $\{(u, u) \mid u \in U\}$ by Δ_U for a set U.
2. For each $R \in \mathcal{R}$ its transpose $R^t = \{(u, v) \mid (v, u) \in R\}$ is an element of $\mathcal{R}$.
3. For all $R, S, T \in \mathcal{R}$ there exists a constant c_{RS}^T such that

$$c_{RS}^T = |R(u) \cap S^t(v)| \text{ for all } (u, v) \in T,$$

where we denote by $T(w)$ the set $\{z \in V \mid (w, z) \in T\}$ for $w \in V$ and $T \in \mathcal{R}$.

The constants c_{RS}^T are called the *intersection numbers*. A subset X of V is called a *fiber* of $\mathcal{C}$ if $\Delta_X \in \mathcal{R}$. We denote the set of all fibers of $\mathcal{C}$ by $\mathrm{Fib}(\mathcal{C})$. By Definition 1.1(i), V is partitioned into the fibers of $\mathcal{C}$, and by Definition 1.1(i),(iii), $\mathcal{R}$ is partitioned into

$$\Big\{\mathcal{R}_{X,Y} \mid X, Y \in \mathrm{Fib}(\mathcal{C})\Big\} \text{ where } \mathcal{R}_{X,Y} = \Big\{R \in \mathcal{R} \mid R \subseteq X \times Y\Big\}.$$

Let U be a union of fibers of $\mathcal{C}$. Then the pair

$$\Big(U, \{R \in \mathcal{R} \mid R \subseteq U \times U\}\Big),$$

is also a coherent configuration, which is denoted by $\mathcal{C}_U$.

For $R \in \mathcal{R}_{X,Y}$ we denote $c_{RR^t}^{\Delta_X}$ by d_R. Then, by two-way counting we have

$$|R| = d_R|X| = d_{R^t}|Y|. \tag{1.1}$$

For $X \in \mathrm{Fib}(\mathcal{C})$, $\mathcal{C}_X$ is nothing but an *association scheme*, i.e., a coherent configuration with only one fiber (see [2] or [20] for its background). For short we shall write $\mathcal{R}_{X,X}$ as $\mathcal{R}_X$ and $\mathcal{C}_X$ is called a *homogeneous component* of $\mathcal{C}$.

A general question here is formulated as follows: what can be said about the coherent configuration if its homogeneous components are known. For example, it is a well-known fact that the coherent configuration corresponds to a system of linked block designs if $|\mathcal{R}_X| = 2$ for all $X \in \mathrm{Fib}(\mathcal{C})$. After the seminal Hanaki-Uno theorem on association schemes of prime order (see [10] or Theorem 3.1), it seems quite natural to ask on a possible structure of a coherent configuration each homogeneous component of which is of prime order. The following is our first main result answering to this question:

Theorem 1.2. *Let $X, Y \in \mathrm{Fib}(\mathcal{C})$ such that $|X| = |Y|$ is a prime. Then $|\mathcal{R}_{X,Y}| \in \{1, |\mathcal{R}_X|\}$. In particular, if $|\mathcal{R}_{X,Y}| > 1$, then*

$$|\mathcal{R}_{X,Y}| = |\mathcal{R}_X| = |\mathcal{R}_Y|.$$

In order to state our second main theorem we need to recall the following observation. Let G be a group acting on a finite set Ω. Then G acts on $\Omega \times \Omega$ componentwise, and an orbit of G on $\Omega \times \Omega$ is called an *orbital* (or 2-orbit) of G. We denote the set of orbitals of G by $\mathcal{O}_G$. Then it is well-known that $\mathcal{C}_G = (\Omega, \mathcal{O}_G)$ is a coherent configuration, and $\mathrm{Fib}(\mathcal{C}_G)$ is the set of orbits of G on Ω. In this sense, a coherent configuration is a combinatorial object to generalize the orbitals of a group action.

Now we assume that $\mathcal{C} = (V, \mathcal{R})$ is a coherent configuration with exactly two fibers X, Y. Then (1.1) proves the equivalence of the first two statements of the following (see [16] for the remaining):

(d) There exists $R \in \mathcal{R}_{X,Y}$ such that $|R| = |X| = |Y|$.

(e) $1 \in \{d_R \mid R \in \mathcal{R}_{X,Y}\} \cap \{d_R \mid R \in \mathcal{R}_{Y,X}\}$.

(f) $\mathcal{C}$ is isomorphic to $\mathcal{C}_X \bigotimes \mathcal{T}_2$ where $\mathcal{T}_n = \Big(\{1, 2, \ldots, n\}, \big\{\{(i,j)\} \mid 1 \le i, j \le n\big\}\Big)$ (see Section 2 for the definition of isomorphism and $\bigotimes$).

We notice the following:
(d) is a combinatorial analogy to (c), and such R is a matching between X and Y; (e) is a simple arithmetic condition on intersection numbers; (f) implies that $\mathcal{C}_X$ and $\mathcal{C}_Y$ are isomorphic, and $\mathcal{C}$ is uniquely determined by $\mathcal{C}_X$.

In this paper we aim to obtain the analogous conclusion (d)–(f) to (a)–(c). The following is our second main result to generalize the fact as in the first paragraph under certain arithmetic conditions on intersection numbers:

Theorem 1.3. *Suppose that $\mathcal{C} = (V, \mathcal{R})$ is a coherent configuration with exactly two fibers X, Y satisfying*

$$|X| = |Y| \text{ is a prime, } |\mathcal{R}_{X,X}| > 2 \text{ and } |\mathcal{R}_{X,Y}| > 1. \tag{1.2}$$

Then there exists $R \in \mathcal{R}_{X,Y}$ such that $|R| = |X| = |Y|$ if one of the following conditions holds with $k = \dfrac{|X| - 1}{|\mathcal{R}_{X,X}| - 1}$:

(i) $|\mathcal{R}_{X,X}| > k^2(k + e - 2)$ *where e is the number of prime divisors of k;*

(ii) $k \in \{q, 2q, 3q\}$ *for some prime power* q*;*

(iii) $k = 4q$ *for some prime power* q *with* $3 \nmid q + 1$.

Let us show the reason why we exclude the case of $|\mathcal{R}_{X,X}| = 2$. Each symmetric design induces the coherent configuration with exactly two fibers and eight relations (see [14] or [16, Example 1.3]), and if the design is a non-trivial one on a prime number of points, like the Fano plane, then the induced coherent configuration does not satisfy (d)–(f).

Of course, if $|\mathcal{R}_{X,Y}| = 1$, then none of (d)–(f) hold, while $\mathcal{C}$ is the direct sum of $\mathcal{C}_X$ and $\mathcal{C}_Y$ (see [16] for the definition of direct sum).

Remark 1.4. Applying Theorem 1.3 for $\mathcal{C}_{X\cup Y}$ with $|X| < 100$ we obtain the same conclusion as Theorem 1.3 except for the case $(|X|, k) = (71, 35)$ (see Section 5 for the details).

Suppose that

$$(|X|, k) = (71, 35) \text{ and } 1 \notin \{d_R \mid R \in \mathcal{R}_{X,Y}\}. \tag{1.3}$$

Then by Theorem 1.2, $|\mathcal{R}_{X,Y}| = 3$. The three elements of $\mathcal{R}_{X,Y}$ must form three symmetric designs whose parameters (v, k, λ) are $(71, 35, 17)$, $(71, 21, 6)$ and $(71, 15, 3)$, respectively. Though each of such symmetric designs exists (see [1], [3], [5], [9] and [17] or [4, II.6.24,VI.16.30]), it does not guarantee the existence of a coherent configuration satisfying (1.3).

In [14], Higman gave a result to eliminate the case of $(|X|, k) = (71, 35)$ as in the previous paragraph. But, the proof given in [14, (3.2)] contains a serious gap, so the result may not be recognized to be true, while we have not found any counterexample. We would be able to disprove [14, (3.2)] if there exists a coherent configuration satisfying (1.3).

In Section 2 we prepare several basic results on intersection numbers and introduce the concepts of complex products and equitable partitions. In Section 3 we give a proof of Theorem 1.2. In Section 4 we give a proof of Theorem 1.3. We add Section 5 for the elimination of coherent configurations on at most 200 points satisfying (1.2).

2 Preliminaries

Throughout this section we assume that $\mathcal{C} = (V, \mathcal{R})$ is a coherent configuration.

Let $\mathcal{C}_i = (V_i, \mathcal{R}_i)$ be a coherent configurations, $i = 1, 2$.

An *isomorphism* from $\mathcal{C}_1$ to $\mathcal{C}_2$ is defined to be a bijection $\psi : V_1 \cup \mathcal{R}_1 \longrightarrow V_2 \cup \mathcal{R}_2$ such that for all $u, v \in V_1$ and $R \in \mathcal{R}_1$,

$$(u, v) \in R \Longleftrightarrow \big(\psi(u), \psi(v)\big) \in \psi(R).$$

We say that $\mathcal{C}_1$ is *isomorphic* to $\mathcal{C}_2$ and denote it by $\mathcal{C}_1 \simeq \mathcal{C}_2$ if there exists an isomorphism from $\mathcal{C}_1$ to $\mathcal{C}_2$.

We set

$$\mathcal{R}_1 \otimes \mathcal{R}_2 = \Big\{R_1 \otimes R_2 \mid R_1 \in \mathcal{R}_1,\ R_2 \in \mathcal{R}_2\Big\},$$

where

$$R_1 \otimes R_2 = \Big\{\big((u_1, u_2), (v_1, v_2)\big) \mid (u_1, v_1) \in R_1,\ (u_2, v_2) \in R_2\Big\}.$$

Then $\big(V_1 \times V_2, \mathcal{R}_1 \otimes \mathcal{R}_2\big)$ is a coherent configuration called the *tensor product* of $\mathcal{C}_1$ and $\mathcal{C}_2$ and denoted by $\mathcal{C}_1 \bigotimes \mathcal{C}_2$.

Following [20] we define the *complex product* on the power set of $\mathcal{R}$. For all subsets $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{R}$ we define the complex product $\mathcal{S}\mathcal{T}$ of $\mathcal{S}$ and $\mathcal{T}$ to be the subset

$$\Big\{R \in \mathcal{R} \mid \exists (S,T) \in \mathcal{S} \times \mathcal{T}; c_{ST}^{R} > 0\Big\}.$$

The complex product is an associative binary operation on the power set of $\mathcal{R}$ where the proof is parallel to that for association schemes (see [20]). For convenience we shall write $\mathcal{S}\{T\}$, $\{S\}\mathcal{T}$ and $\{S\}\{T\}$ as $\mathcal{S}T$, $S\mathcal{T}$ and ST, respectively.

In this paper we need intersection numbers c_{RS}^{T} for $R \in \mathcal{R}_{X,Y}$, $S \in \mathcal{R}_{Y,Z}$ and $T \in \mathcal{R}_{X,Z}$ under the assumption $|X| = |Y| = |Z|$. The following is a collection of simplified equations on such intersection numbers (see [19] or [16, Lemma 2.2] for general formed equations [1]). For $\mathcal{U} \subseteq \mathcal{R}$ we shall write $d_{\mathcal{U}}$ instead of $\sum_{U \in \mathcal{U}} d_U$.

Lemma 2.1. *For all X, Y, $Z \in \mathrm{Fib}(\mathcal{C})$ with $|X| = |Y| = |Z|$ and all $R \in \mathcal{R}_{X,Y}$, $S \in \mathcal{R}_{Y,Z}$ and $T \in \mathcal{R}_{X,Z}$ we have the following:*

1. $d_R d_S = \displaystyle\sum_{T \in \mathcal{R}_{X,Z}} c_{RS}^{T} d_T$;
2. $c_{RS}^{T} d_T = c_{TS^t}^{R} d_R = c_{R^tT}^{S} d_S$ *and* $\mathrm{lcm}(d_R, d_S) \mid c_{RS}^{T} d_T$;
3. $|\{U \in \mathcal{R} \mid c_{RS}^{U} > 0\}| \le \gcd(d_R, d_S)$, *i.e.*, $|RS| \le \gcd(d_R, d_S)$;
4. $|X| = d_{\mathcal{R}_{X,X}} = d_{\mathcal{R}_{X,Y}}$.

The following lemmata were proved in [18, Lemma 2.3, Lemma 2.2][2]:

Lemma 2.2. *For all $S, T \in \mathcal{R}_{X,Y}$ with $|X| = |Y|$, we have*

$$SS^t \cap TT^t \subseteq \{\Delta_X\} \text{ if and only if } c_{S^tT}^{R} \le 1 \text{ for each } R \in \mathcal{R}.$$

Lemma 2.3. *Let $Z \in \mathrm{Fib}(\mathcal{C})$ such that $|Z|$ is a prime. Then for each $R \in \mathcal{R}_Z \setminus \{\Delta_Z\}$ we have:*

1. $d_R = k$ *where* $k = \dfrac{|Z| - 1}{|\mathcal{R}_Z| - 1}$;
2. $\sum_{S \in \mathcal{R}_Z} c_{SS^t}^{R} = k - 1$.

According to [8] or [15] we define an equitable partition of a homogeneous component.

Definition 2.4. Let $X \in \mathrm{Fib}(\mathcal{C})$ and $\Pi = \{C_1, C_2, \dots, C_m\}$ be a partition of X, i.e.,

$$X = \bigcup_{i=1}^{m} C_i,\ \ C_i \cap C_j \ne \emptyset \text{ if } i \ne j, \text{ and } C_i \ne \emptyset \text{ for each } i = 1, 2, \dots, m.$$

An element of Π is called a *cell*. We say that Π is an *equitable partition* of $\mathcal{C}_X$ if, for all $i, j = 1, 2, \dots, m$ and each $R \in \mathcal{R}_X$, $|R(x) \cap C_j|$ is constant whenever $x \in C_i$.

---

[1] We missed to assume that all fibers of $\mathcal{C}$ have the same size at Lemma 2.2 in [16] where the lemma is used only for such coherent configurations in [16].

[2] Though it is a statement for association schemes, a parallel way to the proof can be applied for balanced coherent configurations.

For example, $\{X\}$ and $\{\{x\} \mid x \in X\}$ are equitable partitions of $\mathcal{C}_X$.

For each $Y \in \mathrm{Fib}(\mathcal{C})$ and each $y \in Y$ we define

$$\Pi_y := \big\{T(y) \mid T \in \mathcal{R}_{Y,X}\big\}. \tag{2.1}$$

Then Π_y is an equitable partition of $\mathcal{C}_X$, since

$$|R(x) \cap S(y)| = c_{RS^t}^T \text{ whenever } x \in T(y).$$

3 Proof of Theorem 1.2

In [10] Hanaki and Uno proved the following brilliant theorem:

Theorem 3.1. *All non-principal irreducible characters of an association scheme of prime order are algebraic conjugate and of degree one.*

The following proposition is obtained as a consequence of the previous theorem:

Proposition 3.2. *Let $\mathcal{C} = (V, \mathcal{R})$ be an association scheme of prime order and Π be an equitable partition of $\mathcal{C}$. Then $|\Pi| \equiv 1 \mod |\mathcal{R}| - 1$.*

Proof. Let $\mathcal{A}$ denote the adjacency algebra of $\mathcal{C}$ over $\mathbb{C}$. Then the subspace W spanned by the characteristic vectors of the cells in Π is a left $\mathcal{A}$-module with respect to the ordinary matrix product. Since $\mathcal{A}$ is semi-simple, W is a direct sum of irreducible submodules.

Note that the subspace spanned by the all-one vector is an $\mathcal{A}$-submodule of W affording the principal character, and its multiplicity is one.

Since the character afforded by W is integral valued, it is left invariant from any algebraic conjugate action. It follows from Theorem 3.1 that all non-principal irreducible submodules of W have the same multiplicity, say m. Since

$$\dim_{\mathbb{C}}(W) = |\Pi| \text{ and } \dim_{\mathbb{C}}(\mathcal{A}) = |\mathcal{R}|,$$

it follows that

$$|\Pi| = 1 + m(|\mathcal{R}| - 1).$$

□

Proof of Theorem 1.2. Let $\mathcal{C} = (V, \mathcal{R})$ be a coherent configuration with $X, Y \in \mathrm{Fib}(\mathcal{C})$ such that $|X| = |Y|$ is a prime. Recall that Π_y is an equitable partition of $\mathcal{C}_X$ where $y \in Y$. By (2.1), $|\Pi_y| = |\mathcal{R}_{X,Y}|$. Then it follows from Proposition 3.2 that

$$|\mathcal{R}_{X,Y}| \equiv 1 \mod |\mathcal{R}_X| - 1.$$

Since $|\mathcal{R}_{X,Y}| \leq |\mathcal{R}_X|$ (see [13, p.223] or [16, Proposition 2.7]), $|\mathcal{R}_{X,Y}| \in \{1, |\mathcal{R}_X|\}$. Applying the first statement for $\mathcal{C}_Y$ with $|\mathcal{R}_{X,Y}| \leq |\mathcal{R}_Y|$, we obtain the second statement.

□

4 Proof of Theorem 1.3

For the remainder of this paper we assume that $\mathcal{C} = (V, \mathcal{R})$ is a coherent configuration with $X, Y \in \mathrm{Fib}(\mathcal{C})$ such that

$$m = |X| = |Y| \text{ is a prime, } r = |\mathcal{R}_X| > 2 \text{ and } |\mathcal{R}_{X,Y}| > 1.$$

By Theorem 1.2, we have

$$r = |\mathcal{R}_X| = |\mathcal{R}_{X,Y}| = |\mathcal{R}_Y|.$$

For the remainder of this paper we set

$$k = \frac{m-1}{r-1}.$$

By Lemma 2.3(i) the multi-set $(d_R \mid R \in \mathcal{R}_Z)$ with $Z \in \{X, Y\}$ coincides with $(1, k, \ldots, k)$ by a suitable ordering. In this section we aim to show that $1 \in \{d_R \mid R \in \mathcal{R}_{X,Y}\}$, which implies that the multi-set $(d_R \mid R \in \mathcal{R}_{X,Y})$ coincides with $(1, k, \ldots, k)$ by a suitable ordering, since the complex product SR is a singleton with $d_{SR} = d_S$ whenever $S \in \mathcal{R}_X$ and $d_R = 1$ by Lemma 2.1(iii).

Lemma 4.1. *For all $S, T \in \mathcal{R}_{X,Y}$ with $S \neq T$ we have the following:*

(i) $d_S d_S \equiv d_S \mod k$;

(ii) $d_S d_T \equiv 0 \mod k$.

Proof. (i) Applying Lemma 2.1(i) for S and S^t with $d_S = d_{S^t}$ and $c_{SS^t}^{\Delta_X} = d_S$, we obtain that

$$d_S d_S = d_S + k \sum_{\substack{T \in \mathcal{R}_{X,X} \\ T \neq \Delta_X}} c_{SS^t}^T.$$

(ii) Applying Lemma 2.1(i) for S and T^t with $d_T = d_{T^t}$ and $\Delta_X \notin ST^t$, we obtain that

$$d_S d_T = k \sum_{T \in \mathcal{R}_{X,X}} c_{ST^t}^T.$$

□

We set

$$\mathcal{S}_1 := \{T \in \mathcal{R}_{X,Y} \mid k \nmid d_T\}, \quad \mathcal{S}_2 := \{T \in \mathcal{R}_{X,Y} \mid d_T = k\} \text{ and}$$

$$\mathcal{S}_3 := \{T \in \mathcal{R}_{X,Y} \mid k \mid d_T,\ k < d_T\}.$$

Lemma 4.2. *Let $k = p_1^{\alpha_1} \cdots p_e^{\alpha_e}$ where p_i are the distinct prime divisors of k and α_i are positive integers. Then we have the following:*

1. *For each $i = 1, \ldots, e$ there exists a unique $S \in \mathcal{R}_{X,Y}$ such that $p_i \nmid d_S$;*
2. $|\mathcal{S}_1| \leq e$;
3. $k|\mathcal{S}_3| + d_{\mathcal{S}_1} \leq 1 + k(e-1)$.

Proof. (i) By Lemma 2.1(iv) and Lemma 2.3(i),

$$m = 1 + (r-1)k \equiv 1 \mod p_i.$$

Since $m = d_{\mathcal{R}_{X,Y}}$, there exists an $S \in \mathcal{R}_{X,Y}$ such that $p_i \nmid d_S$. The uniqueness of such S is a direct consequence of Lemma 4.1(ii).

(ii) The correspondence given in (i) gives a function from $\{p_1, p_2, \dots, p_e\}$ to $\mathcal{S}_1$. It remains to show that this function is onto.

Let $S \in \mathcal{S}_1$. By the definition of $\mathcal{S}_1$, there exists p_i such that ${p_i}^{\alpha_i}$ does not divide d_S. By Lemma 4.1(i),

$$d_S d_S \equiv d_S \mod k.$$

Therefore $d_S(d_S - 1)$ is divided by k. Since d_S and $d_S - 1$ are relatively prime, $p_i^{\alpha_i} \nmid d_S$ implies that $p_i \nmid d_S$. It follows from (i) that d_S lies in the range of the function.

(iii) Note that $r = |\mathcal{S}_1| + |\mathcal{S}_2| + |\mathcal{S}_3|$ and

$$m = \sum_{S \in \mathcal{R}_{X,Y}} d_S = \sum_{i=1}^{3} d_{\mathcal{S}_i} \geq d_{\mathcal{S}_1} + k|\mathcal{S}_2| + 2k|\mathcal{S}_3|.$$

Since $k|\mathcal{S}_2| + k|\mathcal{S}_3| = k(r - |\mathcal{S}_1|)$ and $m = 1 + k(r-1)$, it follows that

$$1 + k(|\mathcal{S}_1| - 1) \geq d_{\mathcal{S}_1} + k|\mathcal{S}_3|.$$

By (ii), we have

$$1 + k(e-1) \geq d_{\mathcal{S}_1} + k|\mathcal{S}_3|.$$

This completes the proof of (iii). □

Lemma 4.3. *We have* $\max\{d_S \mid S \in \mathcal{R}_{X,Y}\} \leq k \cdot \min\{d_S \mid S \in \mathcal{R}_{X,Y}\}$.

Proof. Let $S, T \in \mathcal{R}_{X,Y}$ such that

$$d_S = \min\{d_S \mid S \in \mathcal{R}_{X,Y}\} \text{ and } d_T := \max\{d_S \mid S \in \mathcal{R}_{X,Y}\}.$$

Then $T \in RS$ for some $R \in \mathcal{R}_X$ since $T \in \mathcal{R}_X S$. Applying Lemma 2.1(i) we have $d_T \leq k d_S$. □

For $S \in \mathcal{R}_{X,Y}$ we define

$$\mathcal{U}_S := \left\{R \in \mathcal{R}_X \mid R^t R \cap SS^t = \{\Delta_X\}\right\}.$$

Lemma 4.4. *For each $S \in \mathcal{R}_{X,Y}$ we have the following:*

1. *$r - |\mathcal{U}_S| \leq (d_S - 1)(k-1)$.*
2. *If $R \in \mathcal{U}_S - \{\Delta_X\}$, then k divides d_T for each $T \in RS$.*
3. *If $\mathcal{U}_S S \cap \mathcal{S}_2 = \emptyset$, then $r < d_S(k + e - 2)$.*

Proof. (i) Note that

$$\mathcal{R}_X - \mathcal{U}_S = \bigcup_{R_1 \in SS^t - \{\Delta_X\}} \{R \in \mathcal{R}_X \mid R_1 \in R^t R\}.$$

By Lemma 2.1(iii) with $c_{SS^t}^{\Delta_X} > 0$,

$$|SS^t - \{\Delta_X\}| \leq d_S - 1.$$

It follows from Lemma 2.3(ii) that

$$|\{R \in \mathcal{R}_X \mid R_1 \in R^t R\}| \leq \sum_{R \in \mathcal{R}} c_{R^t R}^{R_1} = k - 1.$$

This implies that

$$r - |\mathcal{U}_S| = |\mathcal{R}_X - \mathcal{U}_S| \leq (d_S - 1)(k - 1).$$

(ii) It is an immediate consequence of Lemma 2.1(ii) and Lemma 2.2.

(iii) Suppose that

$$\mathcal{U}_S S \cap \mathcal{S}_2 = \emptyset.$$

Then we have

$$\mathcal{U}_S S \subseteq \mathcal{R}_{X,Y} - \mathcal{S}_2.$$

It follows from (ii) that

$$(\mathcal{U}_S - \{\Delta_X\})S \subseteq \mathcal{S}_3.$$

By Lemma 4.2(iii) and Lemma 4.3,

$$d_{\mathcal{S}_3} \leq d_S k |\mathcal{S}_3| \leq d_S[1 + k(e-1) - d_{\mathcal{S}_1}]. \tag{4.1}$$

On the other hand, applying Lemma 2.2 and Lemma 2.1(iv) for the first inequality and (i) for the second one,

$$d_{\mathcal{U}_S S} \geq 1 + (|\mathcal{U}_S| - 1)k - d_S \geq 1 + [r - (d_S - 1)(k - 1) - 1]k. \tag{4.2}$$

Since $(\mathcal{U}_S - \{\Delta_X\})S \subseteq \mathcal{S}_3$,

$$d_{\mathcal{U}_S S} - d_S \leq d_{(\mathcal{U}_S - \{\Delta_X\})S} \leq d_{\mathcal{S}_3}.$$

It follows from (4.1) and (4.2) that

$$1 + [r - (d_S - 1)(k - 1) - 1]k - d_S \leq d_S[1 + k(e-1) - d_{\mathcal{S}_1}],$$

and hence,

$$r \leq \frac{d_S}{k}[2 + k(e-1) - d_{\mathcal{S}_1}] - \frac{1}{k} + (d_S - 1)(k - 1) + 1.$$

Thus,

$$r \leq d_S[\frac{2}{k} + e - 1 - \frac{d_{\mathcal{S}_1}}{k} + k - 1] - k + 2 - \frac{1}{k} < d_S(k + e - 2).$$

This completes the proof of (iii). □

Proposition 4.5. *If $r > k^2(k + e - 2)$ where e is the number of prime divisors of k, then $1 \in \{d_S \mid S \in \mathcal{R}_{X,Y}\}$.*

Proof. We claim that
$$\min\{d_S \mid S \in \mathcal{R}_{X,Y}\} \leq k.$$
If not, then
$$1 + k(r-1) = m = \sum_{S \in \mathcal{R}_{X,Y}} d_S > kr,$$
a contradiction.

By Lemma 4.3,
$$\max\{d_S \mid S \in \mathcal{R}_{X,Y}\} \leq k^2.$$
Applying the contraposition of Lemma 4.4(iii) we have $\mathcal{U}_S S \cap \mathcal{S}_2 \neq \emptyset$ for each $S \in \mathcal{R}_{X,Y}$, and hence, $T \in RS$ for some $R \in \mathcal{U}_S$ and $T \in \mathcal{S}_2$. Since $d_T = k$ and $c_{RS}^T = 1$ by Lemma 2.2, d_S divides k for each $S \in \mathcal{R}_{X,Y}$. This implies that $|\mathcal{S}_3| = 0$.

We claim $|\mathcal{S}_1| = 1$. Suppose not. Since $1 + (r-1)k = m = d_{\mathcal{S}_1} + k(r - |\mathcal{S}_1|)$,
$$1 + k|\mathcal{S}_1| \leq k + \sum_{S \in \mathcal{S}_1} d_S \leq k + k/2 + k/2 + (|\mathcal{S}_1| - 2)k,$$
a contradiction. By the claim we have $\mathcal{S}_1 = \{S\}$ for some $S \in \mathcal{R}_{X,Y}$. Since
$$1 + k(r-1) = m = k|\mathcal{S}_2| + d_S = k(r-1) + d_S,$$
we have $d_S = 1$. This completes the proof. ☐

Lemma 4.6. *If $S, T \in \mathcal{R}_{X,Y}$ with $ST^t = \{R\}$, then*
$$c_{RR^t}^{R_1} \geq d_T \textit{ for each } R_1 \in SS^t \textit{ and } c_{R^tR}^{R_2} \geq d_S \textit{ for each } R_2 \in TT^t.$$

Proof. Let $y \in Y$, $x_1, x_2 \in S^t(y)$ and $z \in T^t(y)$. Note that $(x_i, z) \in R$ for $i = 1, 2$ since $ST^t = \{R\}$. Since $z \in T^t(y)$ is arbitrarily taken, we have $T^t(y) \subseteq R(x_1) \cap R(x_2)$, which proves the first statement. By the symmetric argument the second statement can be proved. ☐

Proposition 4.7. *There exist no $S, T \in \mathcal{R}_{X,Y}$ such that*
$$ST^t = \{R\},\ d_S + d_T \geq k + 1 \textit{ and } 1 < d_S < d_T. \tag{4.3}$$

Proof. Suppose that $S, T \in \mathcal{R}_{X,Y}$ satisfies (4.3).

We claim that $SS^t = \{\Delta_X, R_1\}$ for some $R_1 \in \mathcal{R}_X - \{\Delta_X\}$. Suppose not, i.e., $SS^t - \{\Delta_X\}$ has at least two elements R_1, R_2. By Lemma 2.1(i),
$$k^2 = d_R d_{R^t} \geq k + c_{RR^t}^{R_1} d_{R_1} + c_{RR^t}^{R_2} d_{R_2} = k + c_{RR^t}^{R_1} k + c_{RR^t}^{R_2} k.$$
It follows from Lemma 4.6 and $d_S + d_T \geq k + 1$ that
$$k^2 \geq k(k+2),$$
a contradiction.

We claim that $SS^t \cap TT^t = \{\Delta_X, R_1\}$. Suppose not, i.e., $SS^t \cap TT^t = \{\Delta_X\}$. Then, by Lemma 2.2, $c_{ST^t}^R = 1$. It follows from Lemma 2.1(i) that $k = d_R = d_S d_T$, which contradicts $d_S + d_T \geq k + 1$ and $1 < d_S < d_T$.

We claim that $R = R^t$. Suppose not, i.e., $R \neq R^t$. Then, by Lemma 2.3(ii),

$$k - 1 = \sum_{R_2 \in \mathcal{R}_X} c_{R_2 R_2^t}^{R_1} \geq c_{RR^t}^{R_1} + c_{R^t R}^{R_1} \geq d_S + d_T \geq k + 1,$$

a contradiction.

We claim that $TT^t = \{\Delta_X, R_1\}$. If $R_2 \in TT^t - \{\Delta_X, R_1\}$, then $c_{RR}^{R_2} \geq d_S$ by Lemma 4.6 with $R = R^t$. By Lemma 2.1(i),

$$k^2 = d_R d_R \geq k + c_{RR}^{R_1} k + c_{RR}^{R_2} k,$$

which implies that $k \geq 1 + d_T + d_S$, a contradiction to $d_S + d_T \geq k + 1$.

We claim that $c_{R_1 R_1^t}^{R_1} \geq d_T - 2$. By the previous claim, for all $z_1, z_2 \in T^t(y)$ with $z_1 \neq z_2$ we have $(z_1, z_2) \in R_1$. Thus,

$$c_{R_1 R_1^t}^{R_1} = |R_1(z_1) \cap R_1(z_2)| \geq |T^t(y) - \{z_1, z_2\}| \geq d_T - 2.$$

Since $c_{R_1 R_1^t}^{R_1} + c_{RR^t}^{R_1} \geq d_T - 2 + d_T \geq k$ by Lemma 4.6, it follows from Lemma 2.3(ii) that $R = R_1$. Thus, $c_{RR^t}^{R} = k - 1$ since $1 < d_S$ and

$$S^t(y) \cup T^t(y) \setminus \{x_1, x_2\} \subseteq R(x_1) \cap R(x_2) \text{ for } x_1, x_2 \in S^t(y).$$

Since $\{\Delta_X, R\}$ is closed under the complex product, $1 + k$ divides $|X|$. Since $|X|$ is a prime, it follows that $\{\Delta_X, R\} = \mathcal{R}_X$, and hence $|\mathcal{R}_X| = 2$, a contradiction. □

Lemma 4.8. *Suppose that $k = 4q$ for some prime power q and $1 \notin \{d_S \mid S \in \mathcal{R}_{X,Y}\}$. Then $|\mathcal{S}_3| = 0$, $|\mathcal{S}_1| = 2$, and $\{d_S \mid S \in \mathcal{S}_1\} = \{3q, q + 1\}$.*

Proof. By Lemma 4.2(iii) and the assumption, $|\mathcal{S}_3| = 0$. By Lemma 4.2(ii), $|\mathcal{S}_1| \leq 2$. Let $S \in \mathcal{S}_1$. Then, by Lemma 4.1, $d_S \equiv 1 \mod q$. By the assumption, $1 < d_S < 4q$. Since $d_S \leq d_{\mathcal{S}_1} \leq 1 + 4q$ Lemma 4.2(iii), it follows from Lemma 4.1 that

$$d_S \in \{q + 1, 3q + 1\}.$$

Let $T \in \mathcal{R}_{X,Y}$ with $S \neq T$. Since $d_S d_T \equiv 0 \mod 4q$ by Lemma 4.1, $q \mid d_T$. Since $m = 1 + k(r - 1) = d_{\mathcal{S}_1} + d_{\mathcal{S}_2} = d_S + d_T + k(r - 2)$, we have $d_S + d_T = k + 1$. Therefore, we conclude from Proposition 4.7 that $\{d_S \mid S \in \mathcal{S}_1\} = \{3q, q + 1\}$. □

Proof of Theorem 1.3. (i) is a direct consequence of Proposition 4.5.

(ii) Suppose on the contrary that

$$1 \notin \{d_S \mid S \in \mathcal{R}_{X,Y}\}.$$

Note that $e \leq 2$ if $k \in \{q, 2q, 3q\}$ for some prime power q. By Lemma 4.2(iii), $|\mathcal{S}_3| = 0$, and $d_{\mathcal{S}_1} \leq k + 1$. Since

$$1 + k(r - 1) = d_{\mathcal{S}_1} + d_{\mathcal{S}_2} \leq k + 1 + d_{\mathcal{S}_2},$$

we have $d_{\mathcal{S}_2} \geq k(r - 2)$, and, hence, $|\mathcal{S}_2| \geq r - 2$.

Suppose $k = q$. Then the statement follows from Lemma 4.2(iii) since $e = 1$.

Suppose $k = 2q$. Then $|\mathcal{S}_1| \leq 2$ and $\{d_S \mid S \in \mathcal{S}_1\} = \{q, q+1\}$ by Lemma 4.2(ii),(iii) and Lemma 4.1. Without loss of generality we assume that

$$\mathcal{S}_1 = \{S, T\}, d_S = q + 1 \text{ and } d_T = q.$$

Since q and $q + 1$ are relatively prime, it follows from Lemma 2.1(iii) that $ST^t = \{R\}$ for some $R \in \mathcal{R}$, which contradicts Proposition 4.7.

Suppose $k = 3q$. Then we have either

$$\{d_S \mid S \in \mathcal{S}_1\} = \{q, 2q + 1\} \text{ or } \{d_S \mid S \in \mathcal{S}_1\} = \{2q, q + 1\}.$$

The first case is done by Proposition 4.7.

For the last case we assume that $\mathcal{S}_1 = \{S, T\}$, $d_S = q + 1$ and $d_T = 2q$. By Lemma 2.1(i),(ii), $SS^t = \{\Delta_X, R\}$ for some $R \in \mathcal{R}$ with $R = R^t$. This implies that $k = d_R$ is even since $|X|$ is an odd prime, so q is a power of two. Thus, d_S and d_T are relatively prime. Therefore, the statement follows from Lemma 2.1(iii) and Proposition 4.7.

(iii) Suppose $k = 4q$. Then, by Lemma 4.8, $\{d_S \mid S \in \mathcal{R}_{X,Y}\} = \{q, 3q + 1\}$ or $\{d_S \mid S \in \mathcal{R}_{X,Y}\} = \{3q, q + 1\}$. The statement follows from the assumption and Proposition 4.7. □

5 Appendix

In this section we show how Theorem 1.3 is applied to small configurations $\mathcal{C}_{X\cup Y}$ with $|X| = |Y| < 100$.

First, we denote by $\mathcal{M}$ the set of primes m less than 100.

Second, we take the set $\mathcal{K}$ of positive integers k such that

$$k \mid m - 1 \text{ for some } m \in \mathcal{M} \text{ with } k < m - 1 \text{ and}$$

$$k \notin \{q, 2q, 3q \mid q \text{ is a prime power}\} \cup \{4q \mid q \text{ is a prime power with } 3 \nmid q + 1\}.$$

Then $\mathcal{K} = \{20, 30, 35, 44\}$.

Lemma 5.1. *If $k = 20$, then $1 \in \{d_S \mid S \in \mathcal{R}_{X,Y}\}$.*

Proof. Suppose not. By Lemma 4.8, $\{d_S \mid S \in \mathcal{S}_1\} = \{15, 6\}$. Let $S \in \mathcal{R}_{X,Y}$ with $d_S = 6$. By Lemma 2.1(ii), $6 \mid c_{SS^t}^R k$ for $R \in SS^t \setminus \{\Delta_X\}$. Thus, $3 \mid c_{SS^t}^R$, which contradicts Lemma 2.1(ii). □

Lemma 5.2. *Suppose that each element of $\mathcal{R}_Y = \{\Delta_Y, R, R'\}$ is symmetric and $\Pi_x = \{C_1, C_2, C_3\}$ is the equitable partition of $(Y, \mathcal{R}_Y)$ as in Section 2 for $x \in X$. We define*

$$\{\beta_{ij}\}_{1\leq i,j\leq 3} \text{ and } \{\gamma_{ij}\}_{1\leq i,j\leq 3}$$

such that $\beta_{ij} = |R(y) \cap C_j|$ with $y \in C_i$ and $\gamma_{ij} := |R'(y) \cap C_j|$ with $y \in C_i$. Then we have the following:

1. *For each i we have $\sum_{j=1}^{3} \beta_{ij} = k$;*
2. *For all i, j with $i \neq j$ we have $\beta_{ij} + \gamma_{ij} = |C_j|$;*
3. *For each i we have $\beta_{ii} + \gamma_{ii} = |C_i| - 1$;*

4. *For all* i, j *we have* $|C_i|\beta_{ij} = \beta_{ji}|C_j|$;
5. *We have* $\beta_{11} + \beta_{22} + \beta_{33} = k - 1$.

Proof. The first four statements can be proved by checking the definition of equitable partitions and using a double-way counting for $(C_i \times C_j) \cap R$.

Let $\mathcal{A}$ be the adjacency algebra of $\mathcal{C}_Y$ and W the subspace spanned by the characteristic vectors of the cells of Π_x. Then W is a left $\mathcal{A}$-module corresponding to the algebra homomorphism defined by $A_R \mapsto (\beta_{ij})$, $A_{R'} \mapsto (\gamma_{ij})$.

We claim that W affords the regular character. Let χ be the character afforded by W, i.e., the value of the adjacency matrix of R is equal to $\sum_{i=1}^{3} \beta_{ii}$. Note that the character afforded by W is integral valued but not a sum of principal character. Since $\dim(W) = 3$, it follows that χ is the sum of irreducible characters of $\mathcal{A}$. This implies that χ is the regular character of $\mathcal{A}$, and, hence, the trace of the matrix (β_{ij}) is equal to $k - 1$ by Lemma 2.3(ii) with Lemma 2.1(ii). □

Proposition 5.3. *If* $(k, m) = (30, 61)$, *then* $1 \in \{d_S \mid S \in \mathcal{R}_{X,Y}\}$.

Proof. Suppose not.

By Lemma 4.2(ii),(iii), $|\mathcal{S}_1| \leq 3$ and $|\mathcal{S}_3| \leq 1$. if $|\mathcal{S}_3| = 1$, then $2k \leq d_{\mathcal{S}_3} < m = 2k + 1$, a contradiction. Thus, $|\mathcal{S}_3| = 0$.

Since $|\mathcal{S}_2| \leq 1$, it follows from Lemma 4.1 that the following are only possible cases of $\{d_S \mid S \in \mathcal{R}_{X,Y}\}$:

$$\{30, 25, 6\};\ \{30, 15, 16\};\ \{30, 10, 21\};\{15, 36, 10\};\{15, 6, 40\}.$$

The first three cases do not occur by Proposition 4.7 since each of them contains a pair of relatively prime numbers.

Note that $\{|C_i| \mid i = 1, 2, 3\} = \{d_S \mid S \in \mathcal{R}_{X,Y}\}$ where $\Pi_x = \{C_1, C_2, C_3\}$ as in Lemma 5.2. Without loss of generality we may assume that

$$C_i = S_i(x) \text{ for } i = 1, 2, 3.$$

From now on we shall use Lemma 5.2 many times without mentioning.

Suppose that

$$(|C_1|, |C_2|, |C_3|) = (10, 15, 36).$$

Since $|C_2|\beta_{23} = |C_3|\beta_{32}$, we have $12 \mid \beta_{23}$. If $\beta_{23} \in \{0, 36\}$, then $|S_2 S_3^t| = 1$, which contradicts Proposition 4.5. Replacing $R \in \mathcal{R}_Y$ by R' if necessary we may assume that $\beta_{23} = 24$, and hence, $\beta_{32} = 10$.

Since

$$|C_1|\beta_{13} = |C_3|\beta_{31},$$

we have $18 \mid \beta_{13}$. If $\beta_{13} \in \{0, 36\}$, then $|S_3 S_1^t| = 1$, which contradicts Proposition 4.7. Thus, $\beta_{13} = 18$, and, hence, $\beta_{31} = 5$.

By Lemma 5.2(i),

$$\beta_{33} = 15,\ \beta_{21} + \beta_{22} = 6 \text{ and } \beta_{11} + \beta_{12} = 12.$$

By Lemma 5.2(v), $\beta_{11} + \beta_{22} = 23$. Thus, $\beta_{12} + \beta_{21} = 19$, which contradicts $10\beta_{12} = 15\beta_{21}$. Therefore, $(d_S, d_T, d_U) = (10, 15, 36)$ does not occur.

Suppose
$$(|C_1|, |C_2|, |C_3|) = (15, 6, 40).$$
Since $|C_2|\beta_{23} = |C_3|\beta_{32}$, we have $20 \mid \beta_{23}$. If $\beta_{23} \in \{0, 40\}$, then $|S_2 S_3^t| = 1$, which contradicts Proposition 4.7. We may assume that $\beta_{23} = 20$, and hence, $\beta_{32} = 3$.

Since
$$|C_1|\beta_{12} = |C_2|\beta_{21},$$
we have $5 \mid \beta_{21}$. By Lemma 5.2(i),
$$\beta_{21} + \beta_{22} + 20 = 30.$$
Thus, $5 \mid \beta_{22}$. Replacing $R \in \mathcal{R}_Y$ by R' if necessary we may assume that $\beta_{22} = 5$, and hence, $\beta_{21} = 5$ and $\beta_{12} = 2$.

By Lemma 5.2(i),(v), we have
$$\beta_{11} + \beta_{13} = 28,\ \beta_{31} + \beta_{33} = 27 \text{ and } \beta_{11} + \beta_{33} = 24.$$
Thus, $\beta_{13} + \beta_{31} = 31$, which contradicts $15\beta_{13} = 40\beta_{31}$.

This completes the proof. □

Proposition 5.4. *If* $(k, m) = (44, 89)$*, then* $1 \in \{d_S \mid S \in \mathcal{R}_{X,Y}\}$.

Proof. Suppose not. By Lemma 4.8, the following is a unique possible case of $\{d_S \mid S \in \mathcal{R}_{X,Y}\}$:
$$\{12, 33, 44\}.$$
Without loss of generality we may assume that
$$C_i = S_i(y) \text{ for } i = 1, 2, 3 \text{ and } (|C_1|, |C_2|, |C_3|) = (12, 33, 44).$$
Since $12\beta_{12} = 33\beta_{21}$, $\beta_{12} \in \{0, 11, 22, 33\}$. Proposition 4.7 forces $\beta_{12} \in \{11, 22\}$, and we may assume that $\beta_{12} = 22$ by replacing $R \in \mathcal{R}_Y$ by R'. Then $\beta_{21} = 8$.

Note that 11 divides β_{13} and so does β_{11} by Lemma 5.2(i). We divide our consideration into the following two cases $\beta_{11} = 11$ or 0.

Suppose $\beta_{11} = 11$. Then $\beta_{13} = 11$ and $\beta_{31} = 3$. By Lemma 5.2(i),(v),
$$\beta_{22} + \beta_{23} = 36,\ \beta_{32} + \beta_{33} = 41 \text{ and } \beta_{22} + \beta_{33} = 32.$$
Therefore, $\beta_{23} + \beta_{32} = 45$, which contradicts $33\beta_{23} = 44\beta_{32}$.

Suppose $\beta_{11} = 0$. Then
$$\beta_{13} = 22 \text{ and } \beta_{31} = 6.$$
By Lemma 5.2(i),(v),
$$\beta_{22} + \beta_{23} = 36,\ \beta_{32} + \beta_{33} = 38 \text{ and } \beta_{22} + \beta_{33} = 43.$$
Therefore, $\beta_{23} + \beta_{32} = 34$, which contradicts $33\beta_{23} = 44\beta_{32}$.

This completes the proof. □

Lemma 5.5. *If* $(k, m) = (35, 71)$ *then* $\{d_S \mid S \in \mathcal{R}_{X,Y}\} = \{15, 21, 35\}$.

Proof. Applying Lemma 4.2(i),(iii) and Lemma 4.1 we conclude that $\{15, 21, 35\}$ is a unique case of $\{d_S \mid S \in \mathcal{R}_{X,Y}\}$. □

We notice that the lemmata given in this section justify the elimination given in Introduction.

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The strong metric dimension of generalized Sierpiński graphs with pendant vertices

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Abstract

Let G be a connected graph of order n having $\varepsilon(G)$ end-vertices. Given a positive integer t, we denote by $S(G,t)$ the t-th generalized Sierpiński graph of G. In this note we show that if every internal vertex of G is a cut vertex, then the strong metric dimension of $S(G,t)$ is given by

$$\dim_s(S(G,t)) = \frac{\varepsilon(G)\left(n^t - 2n^{t-1} + 1\right) - n + 1}{n-1}.$$

Keywords: Strong metric dimension, Sierpiński graphs.

Math. Subj. Class.: 05C12, 05C76

1 Introduction

For two vertices u and v in a connected graph G, the interval $I_G[u,v]$ between u and v is defined as the collection of all vertices that belong to some shortest $u-v$ path. A vertex w strongly resolves two vertices u and v if $v \in I_G[u,w]$ or $u \in I_G[v,w]$. A set S of vertices in a connected graph G is a *strong metric generator* for G if every two vertices of G are strongly resolved by some vertex of S. The smallest cardinality of a strong metric generator of G is called *strong metric dimension* and is denoted by $\dim_s(G)$. After the publication of

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the first paper [16], the strong metric dimension has been extensively studied. The reader is invited to read, for instance, the following works [10, 11, 12, 13, 15] and the references cited therein. For some basic graph classes, the strong metric dimension is easy to compute. For instance, $\dim_s(G) = n-1$ if and only if G is the complete graph of order n. For the cycle C_n of order n the strong dimension is $\dim_s(C_n) = \lceil n/2 \rceil$ and if T is a tree with $l(T)$ leaves, its strong metric dimension equals $l(T)-1$ (see [16]).

Given a connected graph G and two vertices $x, y \in V(G)$, we denote by $d_G(x, y)$ the distance from x to y. A vertex u of G is *maximally distant* from v if for every vertex w in the open neighborhood of u, $d_G(v, w) \le d_G(u, v)$. If u is maximally distant from v and v is maximally distant from u, then we say that u and v are *mutually maximally distant*. The *boundary* of $G = (V, E)$ is defined as $\partial(G) = \{u \in V :$ there exists $v \in V$ such that u, v are mutually maximally distant$\}$. For some basic graph classes, such as complete graphs K_n, complete bipartite graphs $K_{r,s}$, cycles C_n and hypercube graphs Q_k, the boundary is simply the whole vertex set. It is not difficult to see that this property holds for all 2-antipodal[1] graphs and also for all distance-regular graphs. Notice that the boundary of a tree consists exactly of the set of its leaves. A vertex of a graph is a *simplicial vertex* if the subgraph induced by its neighbors is a complete graph. Given a graph G, we denote by $\sigma(G)$ the set of simplicial vertices of G. Notice that $\sigma(G) \subseteq \partial(G)$.

We use the notion of strong resolving graph introduced in [13]. The *strong resolving graph*[2] of G is a graph G_{SR} with vertex set $V(G_{SR}) = \partial(G)$ where two vertices u, v are adjacent in G_{SR} if and only if u and v are mutually maximally distant in G. There are some families of graph for which its resolving graph can be obtained relatively easily. For instance, we emphasize the following cases.

- If $\partial(G) = \sigma(G)$, then $G_{SR} \cong K_{|\partial(G)|}$. In particular, $(K_n)_{SR} \cong K_n$ and for any tree T with $l(T)$ leaves, $(T)_{SR} \cong K_{l(T)}$.
- For any 2-antipodal graph G of order n, $G_{SR} \cong \bigcup_{i=1}^{\frac{n}{2}} K_2$. In particular, $(C_{2k})_{SR} \cong \bigcup_{i=1}^{k} K_2$.
- $(C_{2k+1})_{SR} \cong C_{2k+1}$.

A set S of vertices of G is a *vertex cover* of G if every edge of G is incident with at least one vertex of S. The *vertex cover number* of G, denoted by $\alpha(G)$, is the smallest cardinality of a vertex cover of G. Oellermann and Peters-Fransen [13] showed that the problem of finding the strong metric dimension of a connected graph G can be transformed to the problem of finding the vertex cover number of G_{SR}.

Theorem 1.1. [13] *For any connected graph G,* $\dim_s(G) = \alpha(G_{SR})$.

It was shown in [13] that the problem of computing $\dim_s(G)$ is NP-hard. This suggests finding the strong metric dimension for special classes of graphs or obtaining good bounds on this invariant. In this note we study the problem of finding exact values or sharp bounds for the strong metric dimension of Sierpiński graphs with pendant vertices.

[1]The diameter of $G = (V, E)$ is defined as $D(G) = \max_{u,v \in V}\{d(u, v)\}$. We recall that $G = (V, E)$ is 2-antipodal if for each vertex $x \in V$ there exists exactly one vertex $y \in V$ such that $d_G(x, y) = D(G)$.

[2]In fact, according to [13] the strong resolving graph G'_{SR} of a graph G has vertex set $V(G'_{SR}) = V(G)$ and two vertices u, v are adjacent in G'_{SR} if and only if u and v are mutually maximally distant in G. So, the strong resolving graph defined here is a subgraph of the strong resolving graph defined in [13] and can be obtained from the latter graph by deleting its isolated vertices.

2 Preliminaries on generalized Sierpiński graphs

Let G be a non-empty graph of order n and vertex set $V(G)$. We denote by $V^t(G)$ the set of words of size t on alphabet $V(G)$. The letters of a word u of length t are denoted by $u_1u_2 \dots u_t$. The concatenation of two words u and v is denoted by uv. Klavžar and Milutinović introduced in [6] the graph $S(K_n, t)$ whose vertex set is $V^t(K_n)$, where $\{u, v\}$ is an edge if and only if there exists $i \in \{1, \dots, t\}$ such that:

$$\text{(i) } u_j = v_j, \text{ if } j < i; \text{ (ii) } u_i \neq v_i; \text{ (iii) } u_j = v_i \text{ and } v_j = u_i \text{ if } j > i.$$

When $n = 3$, those graphs are exactly Tower of Hanoi graphs. Later, those graphs have been called Sierpiński graphs in [7] and they were studied by now from numerous points of view. The reader is invited to read, for instance, the following recent papers [2, 5, 4, 7, 8, 9] and references therein. This construction was generalized in [3] for any graph G, by defining the t-th *generalized Sierpiński graph* of G, denoted by $S(G, t)$, as the graph with vertex set $V^t(G)$ and edge set defined as follows. $\{u, v\}$ is an edge if and only if there exists $i \in \{1, \dots, t\}$ such that:

(i) $u_j = v_j$, if $j < i$;

(ii) $u_i \neq v_i$ and $\{u_i, v_i\} \in E(G)$;

(iii) $u_j = v_i$ and $v_j = u_i$ if $j > i$.

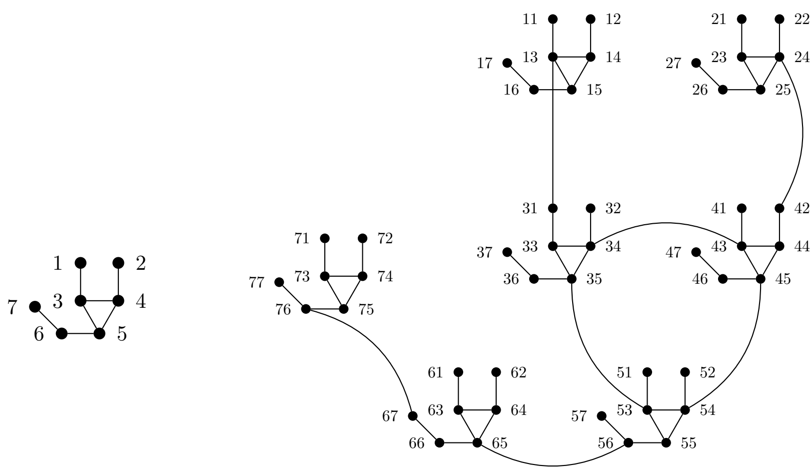

Figure 1: A graph G and the generalized Sierpiński graph $S(G, 2)$

Figure 1 shows a graph G and the Sierpiński graph $S(G, 2)$, while Figure 2 shows the Sierpiński graph $S(G, 3)$.

Notice that if $\{u, v\}$ is an edge of $S(G, t)$, there is an edge $\{x, y\}$ of G and a word w such that $u = wxyy \dots y$ and $v = wyxx \dots x$. In general, $S(G, t)$ can be constructed recursively from G with the following process: $S(G, 1) = G$ and, for $t \geq 2$, we copy n times $S(G, t-1)$ and add the letter x at the beginning of each label of the vertices belonging to the copy of $S(G, t-1)$ corresponding to x. Then for every edge $\{x, y\}$ of

G, add an edge between vertex $xyy\ldots y$ and vertex $yxx\ldots x$. See, for instance, Figure 2. Vertices of the form $xx\ldots x$ are called *extreme vertices*. Notice that for any graph G of order n and any integer $t \geq 2$, $S(G,t)$ has n extreme vertices and, if x has degree $d(x)$ in G, then the extreme vertex $xx\ldots x$ of $S(G,t)$ also has degree $d(x)$. Moreover, the degrees of two vertices $yxx\ldots x$ and $xyy\ldots y$, which connect two copies of $S(G,t-1)$, are equal to $d(x)+1$ and $d(y)+1$, respectively.

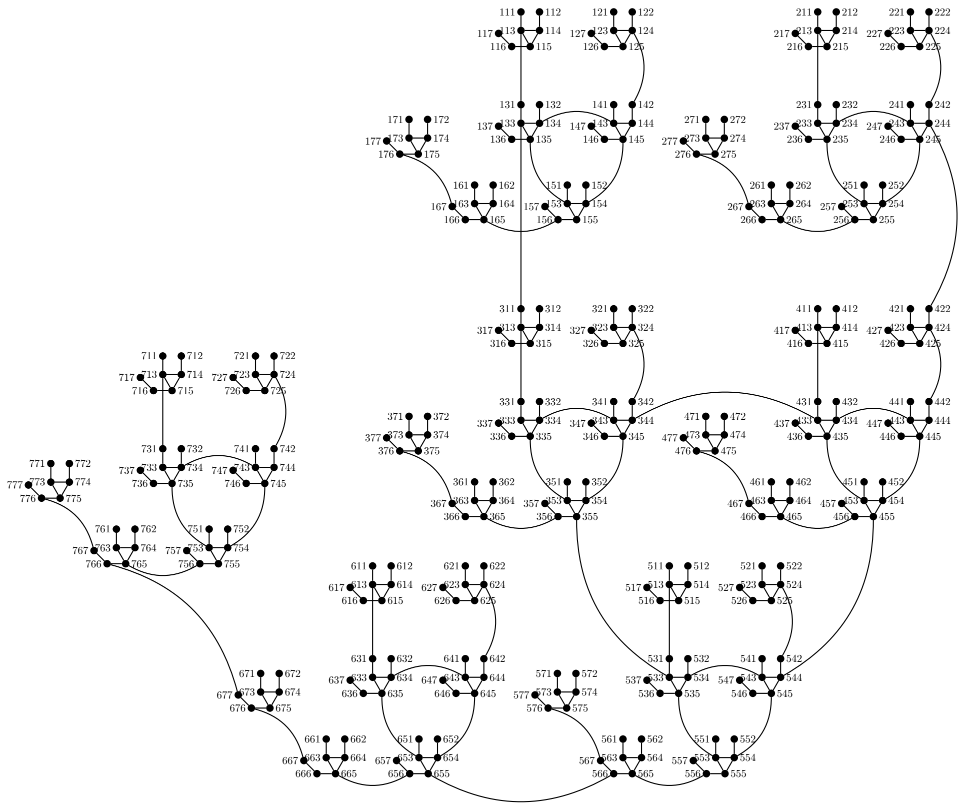

Figure 2: The generalized Sierpiński graph $S(G,3)$ with the base graph G shown in Figure 1.

To the best of our knowledge, [14] is the first published paper studying the generalized Sierpiński graphs. In that article, the authors obtained closed formulae for the Randić index of polymeric networks modelled by generalized Sierpiński graphs. In this note we consider the case where every internal vertex of G is a cut vertex and we obtain a closed formula for the strong metric dimension of $S(G,t)$.

3 The strong metric dimension of $S(G,t)$

The following basic lemma will become an important tool to prove our main results.

Lemma 3.1. *Let G be a connected graph. If v is a cut vertex of G, then $v \notin \partial(G)$.*

Proof. Let $v \in V(G)$ be a cut vertex and $x \in V(G)-\{v\}$. Let G_1 be the connected component of $G-\{v\}$ containing x and let G_2 be a connected component of $G-\{v\}$ different from G_1. Since there exists $y \in V(G_2)$ which is adjacent to v in G and $d_G(x,v) < d_G(x,y)$, we conclude that x and v are not mutually maximally distant in G. □

An *end-vertex* is a vertex of a graph that has exactly one edge incident to it, while a *support vertex* is a vertex adjacent to an end-vertex.

Theorem 3.2. *Let G be a connected graph and let $\varepsilon(G)$ be the number of end-vertices of G. Then,*

$$\dim_s(G) \geq \varepsilon(G) - 1.$$

Moreover, if every vertex of degree greater than one is a cut vertex, then the bound is achieved.

Proof. Let G be a connected graph. Since the set $\Omega(G)$ of end-vertices of G is a subset of $\partial(G)$ and the subgraph of G_{SR} induced by $\Omega(G)$ is a clique, we conclude that $\alpha(G_{SR}) \geq \varepsilon(G) - 1$. Hence, by Theorem 1.1 we obtain the lower bound.

Now, if every vertex of degree greater than one is a cut vertex, by Lemma 3.1 we have that $\partial(G)$ is equal to the set of end-vertices of G. Then $G_{SR} \cong K_{|\varepsilon(G)|}$ and so Theorem 1.1 leads to $\dim_s(G) = \varepsilon(G) - 1$. □

From now on, we will say that a vertex of degree greater than one in a graph G is an *internal vertex* of G. We shall show that if every internal vertex of G is a cut vertex, then the bound above is achieved for $S(G,t)$. To begin with, we state the following lemma.

Lemma 3.3. *Let G be a graph of order n having $\varepsilon(G)$ end-vertices. For any positive integer t, the number of end-vertices of $S(G,t)$ is*

$$\varepsilon(S(G,t)) = \frac{\varepsilon(G)\left(n^t - 2n^{t-1} + 1\right)}{n-1}.$$

Proof. In this proof, we denote by $\mathrm{Sup}(G)$ the set of support vertices of G. Also, if $x \in \mathrm{Sup}(G)$, then $\varepsilon_G(x)$ will denote the number of end-vertices of G which are adjacent to x.

Let $t \geq 2$. For any $x \in V(G)$, we denote by $S_x(G,t-1)$ the copy of $S(G,t-1)$ corresponding to x in $S(G,t)$, *i.e.*, $S_x(G,t-1)$ is the subgraph of $S(G,t)$ induced by the set $\{xw : \ w \in V^{t-1}(G)\}$, which is isomorphic to $S(G,t-1)$. To obtain the result, we only need to determine the contribution of $S_x(G,t-1)$ to the number of end-vertices of $S(G,t)$, for all $x \in V(G)$. By definition of $S(G,t)$, there exists an edge of $S(G,t)$ connecting the vertex $xy\ldots y$ of $S_x(G,t-1)$ with the vertex $yx\ldots x$ of $S_y(G,t-1)$ if and only if x and y are adjacent in G. Hence, an end-vertex $xy\ldots y$ of $S_x(S(G,t-1)$ is adjacent in $S(G,t)$ to a vertex $yx\ldots x$ of $S_y(G,t-1)$ if and only if y is an end-vertex of G and x is its support vertex. Thus, if $x \in \mathrm{Sup}(G)$, then the contribution of $S_x(G,t-1)$ to the number of end-vertices of $S(G,t)$ is $\varepsilon(S(G,t-1)) - \varepsilon_G(x)$ and, if $x \notin \mathrm{Sup}(G)$, then the contribution of $S_x(G,t-1)$ to the number of end-vertices of $S(G,t)$ is $\varepsilon(S(G,t-1))$. Then we obtain,

$$\begin{aligned}\varepsilon(S(G,t)) &= (n - |\mathrm{Sup}(G)|)\varepsilon(S(G,t-1)) + \sum_{x\in \mathrm{Sup}(G)} (\varepsilon(S(G,t-1)) - \varepsilon_G(x)) \\ &= n\varepsilon(S(G,t-1)) - \varepsilon(G).\end{aligned}$$

Now, since $\varepsilon(S(G,1)) = \varepsilon(G)$, we have that

$$\varepsilon(S(G,t)) = \varepsilon(G)\left(n^{t-1} - n^{t-2} - \cdots - n - 1\right) = \varepsilon(G)\left(n^{t-1} - \frac{\left(n^{t-1}-1\right)}{n-1}\right).$$

Therefore, the result follows. □

The following result is a direct consequence of Theorem 3.2 and Lemma 3.3.

Theorem 3.4. *Let G be a connected graph of order n having $\varepsilon(G)$ end-vertices and let t be a positive integer. Then*

$$\dim_s(S(G,t)) \geq \frac{\varepsilon(G)\left(n^t - 2n^{t-1} + 1\right) - n + 1}{n-1}.$$

As we will show in Theorem 3.6, the bound above is tight.

Lemma 3.5. *Let G be a connected graph and let t be a positive integer. If every internal vertex of G is a cut vertex, then every internal vertex of $S(G,t)$ is a cut vertex.*

Proof. As above, for any $x \in V(G)$, we denote by $S_x(G,t-1)$ the copy of $S(G,t-1)$ corresponding to x in $S(G,t)$. We proceed by induction on t. Let $S(G,1) = G$ be a connected graph such that every internal vertex is a cut vertex and assume that every internal vertex of $S(G,t-1)$ is a cut vertex. We differentiate two cases for any internal vertex xw of $S(G,t)$, where $x \in V(G)$ and $w \in V^{t-1}(G)$.

Case 1. w **has degree one in** $S(G,t-1)$**.** In this case xw has degree two in $S(G,t)$. Hence, xw is adjacent to x_1w', for some $x_1 \in V(G)-\{x\}$, and then $w = x_1x_1\ldots x_1$, $w' = xx\ldots x$, x_1 is an end-vertex of G and x is the support of x_1. As a result, $\{xw, x_1w'\}$ is the only edge connecting vertices in $S_{x_1}(G,t-1)$ to vertices outside the subgraph $S_{x_1}(G,t-1)$. Therefore, xw is a cut vertex of $S(G,t)$.

Case 2. w **is a cut vertex of** $S(G,t-1)$**.** In this case, we take two connected components C_1 and C_2 obtained by removing w from $S(G,t-1)$. Suppose, for contradiction purposes, that xw is not a cut vertex of $S(G,t)$. Then there exist two neighbours x_1, x_k of x and a sequence of subgraphs $S_{x_1}(G,t-1)$, $S_{x_2}(G,t-1), \ldots, S_{x_k}(G,t-1)$ such that $x_1\ldots x_1 \in V(C_1)$, $x_k\ldots x_k \in V(C_2)$ and there exists an edge of $S(G,t)$ connecting $S_{x_i}(G,t-1)$ to $S_{x_{i+1}}(G,t-1)$, for all $i \in \{1,2,\ldots,k\}$. Note that the only vertices connecting $S_{x_i}(G,t-1)$ and $S_{x_{i+1}}(G,t-1)$ are $x_ix_{i+1}x_{i+1}\ldots x_{i+1}$ and $x_{i+1}x_ix_i\ldots x_i$, where x_i and x_{i+1} are adjacent in G. Hence, $x, x_1, x_2, \ldots, x_k, x$ is a cycle in G, and so there is a cycle in $S(G,t-1)$ of the form P_{xx_1},$P_{x_1x_2}$,$P_{x_2x_3}, \ldots, P_{x_{k-1}x_k}$, P_{x_kx}, where $P_{x_ix_{i+1}}$ is the path of order 2^{t-1} from $x_ix_i\ldots x_i$ to $x_{i+1}\,x_{i+1}\ldots x_{i+1}$ composed by binary words on alphabet $\{x_i, x_{i+1}\}$ (the paths P_{xx_1} and P_{x_kx} are defined by analogy) and we identify the vertex $x_ix_i\ldots x_i$ of two consecutive paths $P_{x_{i-1}x_i}$ and $P_{x_ix_{i+1}}$ to form the cycle. As a result, there are two disjoint paths from $x_1x_1\ldots x_1$ to $x_kx_k.\ldots x_k$, which contradicts the fact that $x_1x_1\ldots x_1 \in V(C_1)$ and $x_kx_k.\ldots x_k \in C_2$. Therefore, xw is a cut vertex of $S(G,t)$.

According to the two cases above, we conclude the proof by induction. □

Our next result is obtained from Theorem 3.2 and Lemma 3.5.

Theorem 3.6. *Let G be a connected graph of order n having $\varepsilon(G)$ end-vertices and let t be a positive integer. If every internal vertex of G is a cut vertex, then*

$$\dim_s(S(G,t)) = \frac{\varepsilon(G)\left(n^t - 2n^{t-1} + 1\right) - n + 1}{n-1}.$$

Obviously, if the base graph is a tree, then we can apply the formula above. In particular, we would emphasize the following particular case of this result, where $K_{1,r}$ denotes the star graph of r leaves and P_r denotes the path graph of order r.

Corollary 3.7. *For any integers $r, t \geq 2$,*

- $\dim_s(S(K_{1,r},t)) = (r+1)^{t-1}(r-1)$.
- $\dim_s(S(P_r,t)) = \dfrac{2r^t - 4r^{t-1} - r + 3}{r-1}$.

Let G be a graph of order n and let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be a family of graphs. The corona product graph $G \odot \mathcal{H}$ is defined as the graph obtained from G and $\mathcal{H}$ by taking one copy of G and joining by an edge each vertex of H_i with the i^{th}-vertex of G. These graphs were defined by Frucht and Harary in [1].

Corollary 3.8. *Let G be a graph of order n and let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be a family of empty graphs of order n_i, respectively. Then for any positive integer t,*

$$\dim_s(S(G \odot \mathcal{H},t)) = \frac{n'(n+n')^{t-1}(n+n'-2) - n + 1}{n+n'-1},$$

where $n' = \sum_{i=1}^{n} n_i$.

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Remarks on the thickness of $K_{n,n,n}$

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Abstract

The thickness $\theta(G)$ of a graph G is the minimum number of planar subgraphs into which G can be decomposed. In this paper, we provide a new upper bound for the thickness of the complete tripartite graphs $K_{n,n,n}$ $(n \geq 3)$ and obtain $\theta(K_{n,n,n}) = \lceil \frac{n+1}{3} \rceil$, when $n \equiv 3 \pmod 6$.

Keywords: Thickness, complete tripartite graph, planar subgraphs decomposition.

Math. Subj. Class.: 05C10

1 Introduction

The *thickness* $\theta(G)$ of a graph G is the minimum number of planar subgraphs into which G can be decomposed. It was defined by Tutte [11] in 1963, derived from early work on biplanar graphs [2, 10]. It is a classical topological invariant of a graph and also has many applications to VLSI design, graph drawing, etc. Determining the thickness of a graph is NP-hard [7], so the results about thickness are few. The only types of graphs whose thicknesses have been determined are complete graphs [1, 3], complete bipartite graphs [4] and hypercubes [5]. The reader is referred to [6, 8] for more background on the thickness problems.

In this paper, we study the thickness of complete tripartite graphs $K_{n,n,n}$, $(n \geq 3)$. When $n = 1, 2$, it is easy to see that $K_{1,1,1}$ and $K_{2,2,2}$ are planar graphs, so the thickness of both ones is one. Poranen proved $\theta(K_{n,n,n}) \leq \lceil \frac{n}{2} \rceil$ in [9] which was the only result about the thickness of $K_{n,n,n}$, as far as the author knows. We will give a new upper bound for $\theta(K_{n,n,n})$ and provide the exact number for the thickness of $K_{n,n,n}$, when n is congruent to 3 mod 6, the main results of this paper are the following theorems.

Theorem 1.1. *For $n \geq 3$, $\theta(K_{n,n,n}) \leq \lceil \frac{n+1}{3} \rceil + 1$.*

Theorem 1.2. *$\theta(K_{n,n,n}) = \lceil \frac{n+1}{3} \rceil$ when $n \equiv 3$ (mod 6).*

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2 The proofs of the theorems

In [4], Beineke, Harary and Moon determined the thickness of complete bipartite graph $K_{m,n}$ for almost all values of m and n.

Lemma 2.1. *[4] The thickness of $K_{m,n}$ is $\left\lceil \frac{mn}{2(m+n-2)} \right\rceil$ except possibly when m and n are odd, $m \le n$ and there exists an integer k satisfying $n = \left\lfloor \frac{2k(m-2)}{m-2k} \right\rfloor$.*

Lemma 2.2. *For $n \ge 3$, $\theta(K_{n,n,n}) \ge \left\lceil \frac{n+1}{3} \right\rceil$.*

Proof. Since $K_{n,2n}$ is a subgraph of $K_{n,n,n}$, we have $\theta(K_{n,n,n}) \ge \theta(K_{n,2n})$. From Lemma 2.1, the thickness of $K_{n,2n}$ $(n \ge 3)$ is $\left\lceil \frac{n+1}{3} \right\rceil$, so the lemma follows. □

For the complete tripartite graph $K_{n,n,n}$ with the vertex partition (A, B, C), where $A = \{a_0, \ldots, a_{n-1}\}$, $B = \{b_0, \ldots, b_{n-1}\}$ and $C = \{c_0, \ldots, c_{n-1}\}$, we define a type of graphs, they are planar spanning subgraphs of $K_{n,n,n}$, denoted by $G[a_i b_{j+i} c_{k+i}]$, in which $0 \le i, j, k \le n-1$ and all subscripts are taken modulo n. The graph $G[a_i b_{j+i} c_{k+i}]$ consists of n triangles $a_i b_{j+i} c_{k+i}$ for $0 \le i \le n-1$ and six paths of length $n-1$, they are

$$a_0 b_{j+1} c_{k+2} a_3 b_{j+4} c_{k+5} \ldots a_{3i} b_{j+3i+1} c_{k+3i+2} \cdots,$$

$$c_k a_1 b_{j+2} c_{k+3} a_4 b_{j+5} \ldots c_{k+3i} a_{3i+1} b_{j+3i+2} \cdots,$$

$$b_j c_{k+1} a_2 b_{j+3} c_{k+4} a_5 \ldots b_{j+3i} c_{k+3i+1} a_{3i+2} \cdots,$$

$$a_0 c_{k+1} b_{j+2} a_3 c_{k+4} b_{j+5} \ldots a_{3i} c_{k+3i+1} b_{j+3i+2} \cdots,$$

$$b_j a_1 c_{k+2} b_{j+3} a_4 c_{k+5} \ldots b_{j+3i} a_{3i+1} c_{k+3i+2} \cdots,$$

$$c_k b_{j+1} a_2 c_{k+3} b_{j+4} a_5 \ldots c_{k+3i} b_{j+3i+1} a_{3i+2} \cdots.$$

Equivalently, the graph $G[a_i b_{j+i} c_{k+i}]$ is the graph with the same vertex set as $K_{n,n,n}$ and edge set

$$\{a_i b_{j+i-1}, a_i b_{j+i}, a_i b_{j+i+1}, a_i c_{k+i-1}, a_i c_{k+i}, a_i c_{k+i+1} \mid 1 \le i \le n-2\}$$

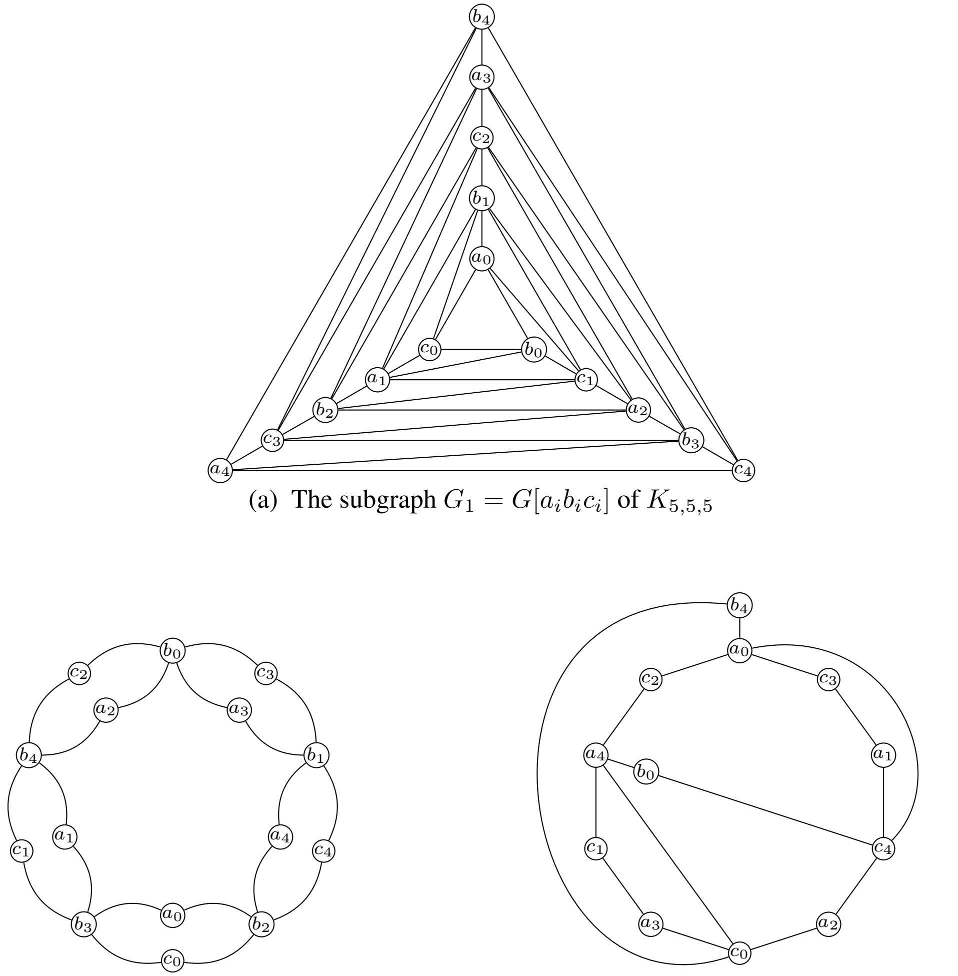
$$\cup \{b_{j+i} c_{k+i-1}, b_{j+i} c_{k+i}, b_{j+i} c_{k+i+1} \mid 1 \le i \le n-2\}$$

$$\cup \{a_0 b_j, a_0 b_{j+1}, a_{n-1} b_{j+n-2}, a_{n-1} b_{j+n-1}\}$$

$$\cup \{a_0 c_k, a_0 c_{k+1}, a_{n-1} c_{k+n-2}, a_{n-1} c_{k+n-1})\}$$

$$\cup \{b_j c_k, b_j c_{k+1}, b_{j+n-1} c_{k+n-2}, b_{j+n-1} c_{k+n-1}\}.$$

Figure 1(a) illustrates the planar spanning subgraph $G[a_i b_i c_i]$ of $K_{5,5,5}$.



(a) The subgraph $G_1 = G[a_ib_ic_i]$ of $K_{5,5,5}$

(b) The subgraph G_2 of $K_{5,5,5}$

(c) The subgraph G_3 of $K_{5,5,5}$

Figure 1: A planar subgraphs decomposition of $K_{5,5,5}$

Theorem 2.3. *When $n = 3p + 2$ (p is a positive integer), $\theta(K_{n,n,n}) \leq p + 2$.*

Proof. When $n = 3p + 2$ (p is a positive integer), we will construct two different planar subgraphs decompositions of $K_{n,n,n}$ according to p is odd or even, in which the number of planar subgraphs is $p + 2$ in both cases.

Case 1. p is odd. Let $G_1, \ldots, G_p$ be p planar subgraphs of $K_{n,n,n}$ where $G_t = G[a_ib_{i+3(t-1)}c_{i+6(t-1)}]$, for $1 \leq t \leq \frac{p+1}{2}$; and $G_t = G[a_ib_{i+3(t-1)}c_{i+6(t-1)+2}]$, for $\frac{p+3}{2} \leq t \leq p$ and $p \geq 3$. From the structure of $G[a_ib_{j+i}c_{k+i}]$, we get that no two edges in $G_1, \ldots, G_p$ are repeated. Because subscripts in $G_t, 1 \leq t \leq p$ are taken modulo n, $\{3(t-1) \text{ (mod n)} \mid 1 \leq t \leq p\} = \{0, 3, 6, \ldots, 3(p-1)\}$, $\{6(t-1) \text{ (mod n)} \mid 1 \leq t \leq \frac{p+1}{2}\} = \{0, 6, \ldots, 3(p-1)\}$ and $\{6(t-1)+2 \text{ (mod n)} \mid \frac{p+3}{2} \leq t \leq p\} = \{3, 9, \ldots, 3(p-2)\}$, the subscript sets of b and c in G_t, $1 \leq t \leq p$ are the same, i.e.,

$\{i+3(t-1) \text{ (mod n)} \mid 1 \leq t \leq p\}$

$$= \{i+6(t-1) \text{ (mod n)} \mid 1 \leq t \leq \frac{p+1}{2}\} \cup \{i+6(t-1)+2 \text{ (mod n)} \mid \frac{p+3}{2} \leq t \leq p\}.$$

Furthermore, if there exists $t \in \{1,\dots,p\}$ such that a_ib_j is an edge in G_t, then a_ic_j is an edge in G_k for some $k \in \{1,\dots,p\}$. If the edge a_ib_j is not in any G_t, then neither is the edge a_ic_j in any G_t, for $1 \leq t \leq p$.

From the construction of G_t, the edges that belong to $K_{n,n,n}$ but not to any G_t, $1 \leq t \leq p$, are

$$a_0b_{3(t-1)-1},\ \ a_0c_{3(t-1)-1},\quad 1 \leq t \leq p \tag{1}$$

$$a_{n-1}b_{3(t-1)},\ \ a_{n-1}c_{3(t-1)},\quad 1 \leq t \leq p \tag{2}$$

$$a_ib_{i-3},\ \ a_ib_{i-2},\quad 0 \leq i \leq n-1 \tag{3}$$

$$a_ic_{i-3},\ \ a_ic_{i-2},\quad 0 \leq i \leq n-1 \tag{4}$$

$$b_ic_{i+3(t-1)-1},\ \ b_ic_{i+3(t-1)},\quad 0 \leq i \leq n-1 \text{ and } t = \frac{p+3}{2} \tag{5}$$

$$b_{3(t-1)}c_{6(t-1)-1},\ \ b_{3(t-1)-1}c_{6(t-1)},\quad 1 \leq t \leq \frac{p+1}{2} \tag{6}$$

$$b_{3(t-1)}c_{6(t-1)+1},\ \ b_{3(t-1)-1}c_{6(t-1)+2},\quad \frac{p+3}{2} \leq t \leq p \text{ and } p \geq 3 \tag{7}$$

Let G_{p+1} be the graph whose edge set consists of the edges in (3) and (5), and G_{p+2} be the graph whose edge set consists of the edges in (1), (2), (4), (6) and (7). In the following, we will describe plane drawings of G_{p+1} and G_{p+2}.

(a) A planar embedding of G_{p+1}.
Place vertices $b_0, b_1, \dots, b_{n-1}$ on a circle, place vertices a_{i+3} and $c_{i+\frac{n+1}{2}}$ in the middle of b_i and b_{i+1}, join each of a_{i+3} and $c_{i+\frac{n+1}{2}}$ to both b_i and b_{i+1}, we get a planar embedding of G_{p+1}. For example, when $p=1$, $n=5$, Figure 1(b) shows the subgraph G_2 of $K_{5,5,5}$.

(b) A planar embedding of G_{p+2}.
Firstly, we place vertices $c_0, c_1, \dots, c_{n-1}$ on a circle, join vertex a_{i+3} to c_i and c_{i+1}, for $0 \leq i \leq n-1$, so that we get a cycle of length $2n$. Secondly, join vertex a_{n-1} to $c_{3(t-1)}$ for $1 \leq t \leq p$, with lines inside of the cycle. Let ℓ_t be the line drawn inside the cycle joining a_{n-1} with $c_{6(t-1)-1}$ if $1 \leq t \leq \frac{p+1}{2}$ or with $c_{6(t-1)+1}$ if $\frac{p+3}{2} \leq t \leq p$ $(p \geq 3)$. For $1 \leq t \leq p$, insert the vertex $b_{3(t-1)}$ in the line ℓ_t. Thirdly, join vertex a_0 to $c_{3(t-1)-1}$ for $1 \leq t \leq p$, with lines outside of the cycle. Let ℓ'_t be the line drawn outside the cycle joining a_0 with $c_{6(t-1)}$ if $1 \leq t \leq \frac{p+1}{2}$ or with $c_{6(t-1)+2}$ if $\frac{p+3}{2} \leq t \leq p$ $(p \geq 3)$. For $1 \leq t \leq p$, insert the vertex $b_{3(t-1)-1}$ in the line ℓ'_t. In this way, we can get a planar embedding of G_{p+2}. For example, when $p=1$, $n=5$, Figure 1(c) shows the subgraph G_3 of $K_{5,5,5}$.

Summarizing, when p is an odd positive integer and $n = 3p+2$, we get a decomposition of $K_{n,n,n}$ into $p+2$ planar subgraphs $G_1, \dots, G_{p+2}$.

Case 2. p is even. Let $G_1, \dots, G_p$ be p planar subgraphs of $K_{n,n,n}$ where $G_t = G[a_ib_{i+3(t-1)}c_{i+6(t-1)+3}]$, for $1 \leq t \leq \frac{p}{2}$; and $G_t = G[a_ib_{i+3(t-1)}c_{i+6(t-1)+2}]$, for $\frac{p+2}{2} \leq t \leq p$. With a similar argument to the proof of Case 1, we can get that the subscript sets of b and c in G_t, $1 \leq t \leq p$ are the same, i.e.,

$\{i+3(t-1) \text{ (mod n)} \mid 1 \leq t \leq p\}$

$$= \{i+6(t-1)+3 \text{ (mod n)} \mid 1 \leq t \leq \frac{p}{2}\} \cup \{i+6(t-1)+2 \text{ (mod n)} \mid \frac{p+2}{2} \leq t \leq p\}.$$

From the construction of G_t, $G_{\frac{p}{2}}$ and $G_{\frac{p+2}{2}}$ have $n-2$ edges in common, they are $b_{i+3(\frac{p+2}{2}-1)}c_{i+6(\frac{p+2}{2}-1)+1}$, $1 \le i \le n-1$ and $i \ne n-4$, we can delete them in one of these two graphs to avoid repetition.

The edges that belong to $K_{n,n,n}$ but not to any G_t, $1 \le t \le p$, are

$$a_0 b_{3(t-1)-1},\ \ a_0 c_{3(t-1)-1}, \quad 1 \le t \le p \tag{8}$$

$$a_{n-1} b_{3(t-1)},\ \ a_{n-1} c_{3(t-1)}, \quad 1 \le t \le p \tag{9}$$

$$a_i b_{i-3},\ \ a_i b_{i-2}, \quad 0 \le i \le n-1 \tag{10}$$

$$a_i c_{i-3},\ \ a_i c_{i-2}, \quad 0 \le i \le n-1 \tag{11}$$

$$b_i c_{i-1},\ \ b_i c_i,\ \ b_i c_{i+1}, \quad 0 \le i \le n-1 \tag{12}$$

$$b_{3(t-1)} c_{6t-4}, \quad 1 \le t \le \frac{p}{2} \tag{13}$$

$$b_{3(t-1)} c_{6t-5}, \quad \frac{p+2}{2} < t \le p \tag{14}$$

$$b_{3(t-1)-1} c_{6t-3}, \quad 1 \le t < \frac{p}{2} \tag{15}$$

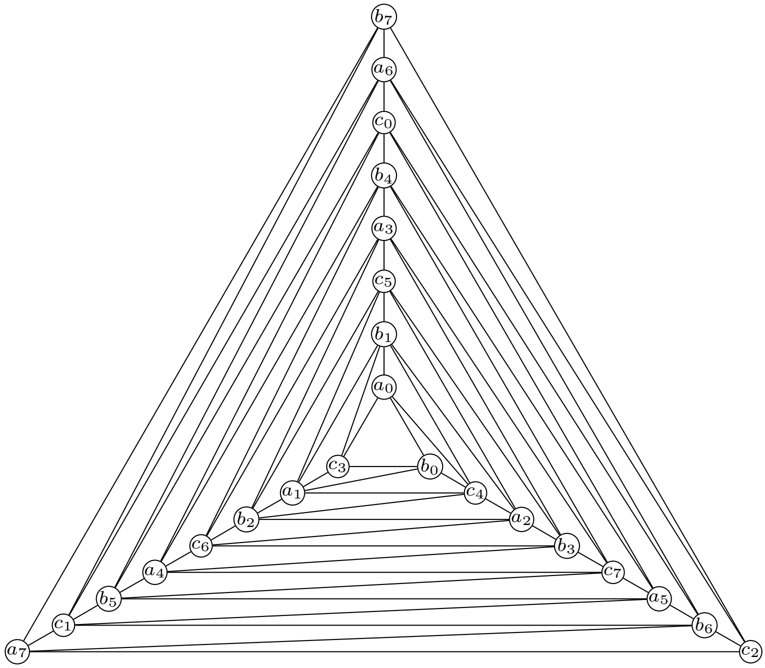
$$b_{3(t-1)-1} c_{6t-4}, \quad \frac{p+2}{2} \le t \le p \tag{16}$$

Let G_{p+1} be the graph whose edge set consists of the edges in $(10), (11)$ and (12), and G_{p+2} be the graph whose edge set consists of the edges in $(8), (9), (13), (14), (15)$ and (16). We draw G_{p+1} in the following way. Firstly, place vertices $b_0, c_0, b_1, c_1, \dots, b_{n-1}$, c_{n-1} on a circle C, join vertex c_i to b_i and b_{i+1}, we get a cycle of length $2n$. Secondly, place vertices $a_0, a_2, \dots, a_{n-2}$ on a circle C' in the unbounded region defined by the circle C such that C is contained in the closed disk defined by C', place vertices $a_1, a_3, \dots, a_{n-1}$ on a circle C'' contained in the bounded region of C. Join a_i to $b_{i-3}, b_{i-2}, c_{i-3}$, and c_{i-2}, join b_i to c_{i+1}. We can get a planar embedding of G_{p+1}, so it is a planar graph. G_{p+2} is also planar because it is a subgraph of a graph homeomorphic to a dipole (two vertices joined by some edges). For example, when $p = 2$, $n = 8$, Figure 2(c) and Figure 2(d) show the subgraphs G_3 and G_4 of $K_{8,8,8}$ respectively.

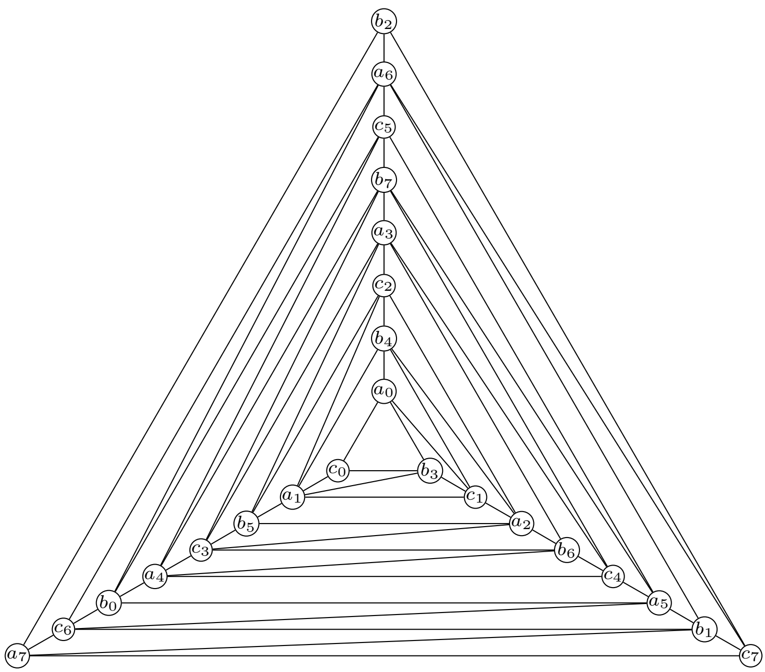
Summarizing, when p is an even positive integer and $n = 3p + 2$, we obtain a decomposition of $K_{n,n,n}$ into $p + 2$ planar subgraphs $G_1, \dots, G_{p+2}$.

Theorem follows from Cases 1 and 2. $\square$

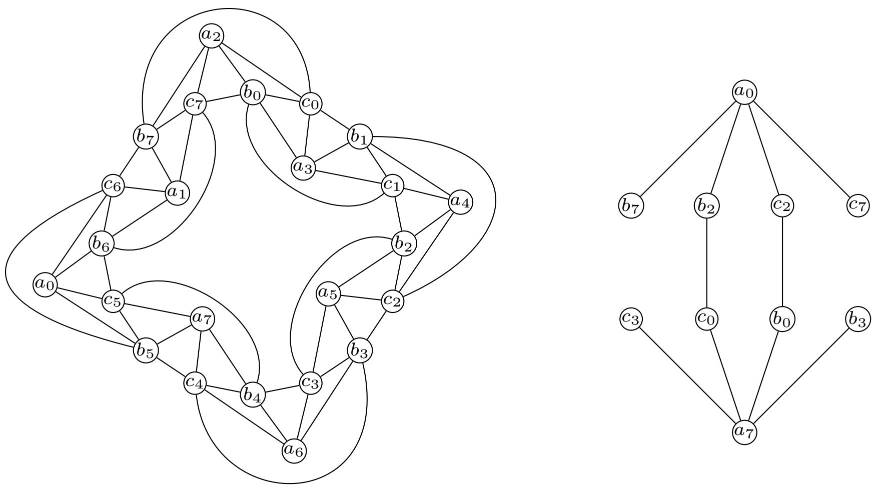
From the proof of Theorem 2.3, we draw planar subgraphs decompositions of $K_{5,5,5}$ and $K_{8,8,8}$ as illustrated in Figure 1 and Figure 2 respectively.



(a) The subgraph $G_1 = G[a_i b_i c_{i+3}]$ of $K_{8,8,8}$



(b) The subgraph $G_2 - b_4c_0 - b_5c_1 - b_6c_2 - b_0c_4 - b_1c_5 - b_2c_6$ of $K_{8,8,8}$ in which $G_2 = G[a_i b_{i+3} c_i]$



(c) The subgraph G_3 of $K_{8,8,8}$ (d) The subgraph G_4 of $K_{8,8,8}$

Figure 2: A planar subgraphs decomposition of $K_{8,8,8}$

Proof of Theorem 1.1. Because graph $K_{n-1,n-1,n-1}$ is a subgraph of $K_{n,n,n}$, $\theta(K_{n-1,n-1,n-1}) \leq \theta(K_{n,n,n})$, by Theorem 2.3, $\theta(K_{n,n,n}) \leq p+2$ also holds, when $n = 3p$ or $n = 3p+1$ (p is a positive integer), the theorem follows. □

Proof of Theorem 1.2. When $n = 3p$ is odd, i.e., $n \equiv 3 \pmod 6$, we decompose $K_{n,n,n}$ into $p+1$ planar subgraphs $G_1, \ldots, G_{p+1}$, where $G_t = G[a_i b_{i+3(t-1)} c_{i+6(t-1)}]$, for $1 \leq t \leq p$. With a similar argument to the proof of Theorem 2.3, we can get that the subscript sets of b and c in G_t, $1 \leq t \leq p$ are the same, i.e.,

$$\{i + 3(t-1) \text{ (mod n)} \mid 1 \leq t \leq p\} = \{i + 6(t-1) \text{ (mod n)} \mid 1 \leq t \leq p\}.$$

If the edge $a_i b_j$ is in G_t for some $t \in \{1, \ldots, p\}$, then there exists $k \in \{1, \ldots, p\}$ such that $a_i c_j$ is in G_k. If the edge $a_i b_j$ is not in any G_t, then neither is the edge $a_i c_j$ in any G_t, for $1 \leq t \leq p$.

From the construction of $G_t = G[a_i b_{i+3(t-1)} c_{i+6(t-1)}]$, we list the edges that belong to $K_{n,n,n}$ but not to any G_t, $1 \leq t \leq p$, as follows.

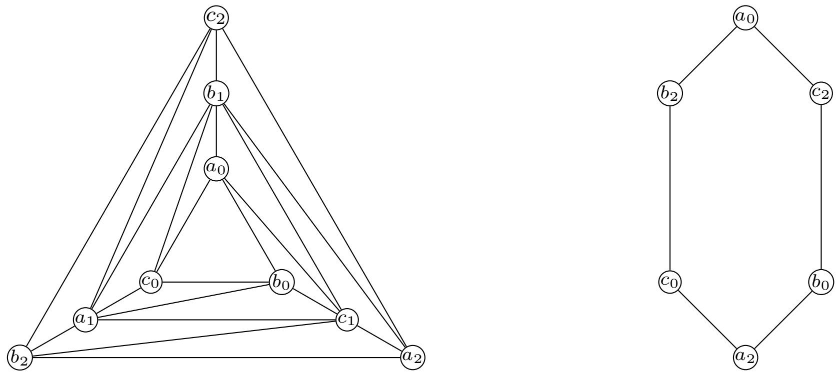
$$a_0 b_{3(t-1)-1}, \ a_0 c_{6(t-1)-1}, \quad 1 \leq t \leq p \tag{17}$$

$$a_{n-1} b_{3(t-1)}, \ a_{n-1} c_{6(t-1)}, \quad 1 \leq t \leq p \tag{18}$$

$$b_{3(t-1)} c_{6(t-1)-1}, \ b_{3(t-1)-1} c_{6(t-1)}, \quad 1 \leq t \leq p \tag{19}$$

Let G_{p+1} be the graph whose edge set consists of the edges in (17), (18) and (19). It is easy to see that G_{p+1} is homeomorphic to a dipole and it is a planar graph.

Summarizing, when p is an odd positive integer and $n = 3p$, we obtain a decomposition of $K_{n,n,n}$ into $p+1$ planar subgraphs $G_1, \ldots, G_{p+1}$, therefor $\theta(K_{n,n,n}) \leq p+1$. Combining this fact and Lemma 2.2, the theorem follows. □

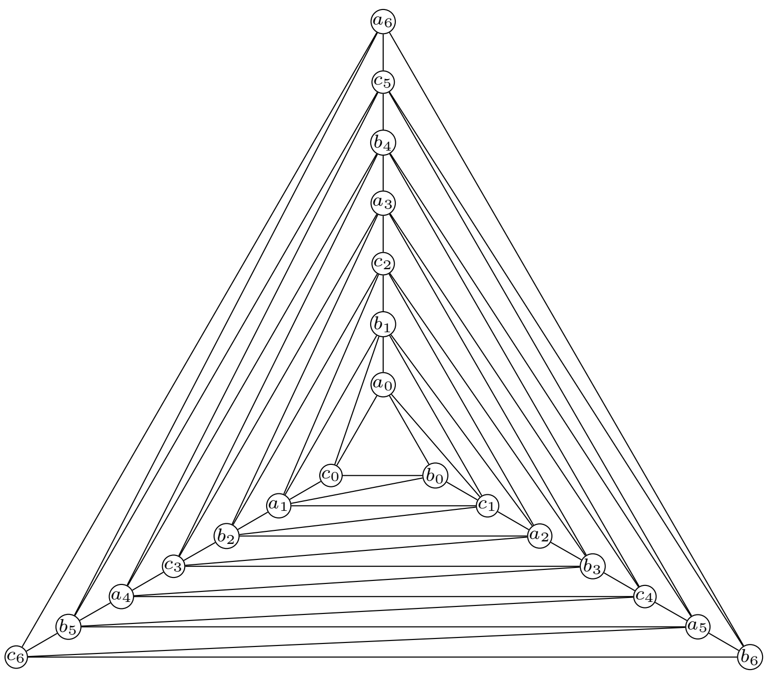


(a) The subgraph $G_1 = G[a_ib_ic_i]$ of $K_{3,3,3}$ (b) The subgraph G_2 of $K_{3,3,3}$

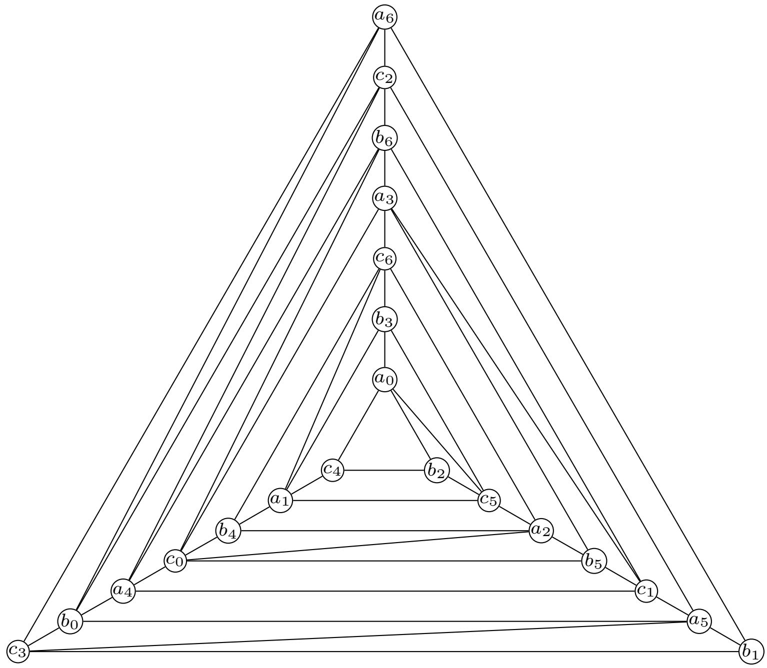
Figure 3: A planar subgraphs decomposition of $K_{3,3,3}$

According to the proof of Theorem 1.2, we draw a planar subgraphs decomposition of $K_{3,3,3}$ as shown in Figure 3.

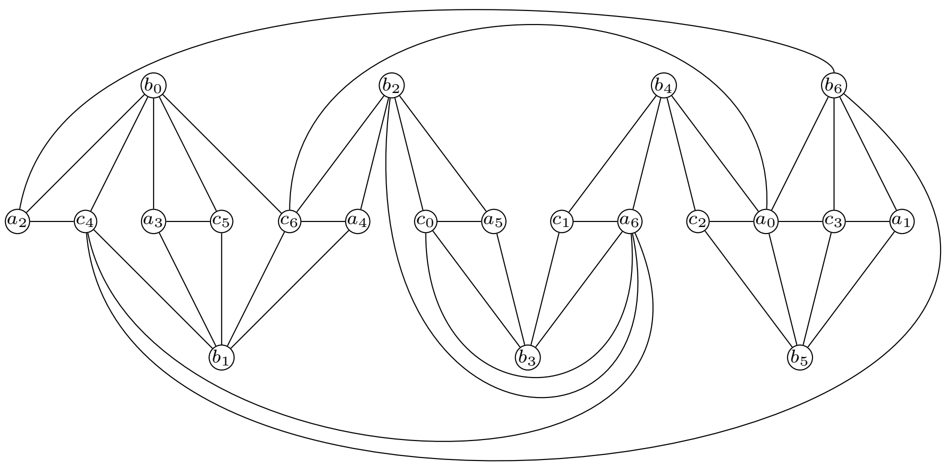
For some other $\theta(K_{n,n,n})$ with small n, combining Lemma 2.2 and Poranen's result mentioned in Section 1, we have $\theta(K_{4,4,4}) = 2, \theta(K_{6,6,6}) = 3$. Since there exists a decomposition of $K_{7,7,7}$ with three planar subgraphs as shown in Figure 4, Lemma 2.2 implies that $\theta(K_{7,7,7}) = 3$. We also conjecture that the thickness of $K_{n,n,n}$ is $\lceil \frac{n+1}{3} \rceil$ for all $n \geq 3$.



(a) The subgraph $G_1 = G[a_ib_ic_i]$ of $K_{7,7,7}$



(b) The subgraph
$G_2 - a_1b_2 - a_2b_3 - a_3b_4 - a_4b_5 - a_5b_6 - b_0c_1 - b_1c_2 - b_3c_4 - b_4c_5 - b_5c_6$ of $K_{7,7,7}$ in which $G_2 = G[a_ib_{i+2}c_{i+4}]$



(c) The subgraph G_3 of $K_{7,7,7}$

Figure 4: A planar subgraphs decomposition of $K_{7,7,7}$

Acknowledgements

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Automorphism group of the balanced hypercube*

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Abstract

Huang and Wu in [IEEE Transactions on Computers 46 (1997), pp. 484–490] introduced the balanced hypercube BH_n as an interconnection network topology for computing systems. In this paper, we completely determine the full automorphism group of the balanced hypercube. Applying this, we first show that the n-dimensional balanced hypercube BH_n is arc-transitive but not 2-arc-transitive whenever $n \geq 2$. Then, we show that BH_n is a lexicographic product of an n-valent graph X_n and the null graph with two vertices, where X_n is a $\mathbb{Z}_2^{n-1}$-regular cover of the n-dimensional hypercube Q_n.

Keywords: Automorphism group, balanced hypercube, Cayley graph, arc-transitive.

Math. Subj. Class.: 05C25, 20B25

1 Introduction

The hypercube is widely known as one of the most popular interconnection networks for parallel computing systems. As a variant of the hypercube, the balanced hypercube was proposed by Huang and Wu [8] to enhance some properties of the hypercube. An n-dimensional balanced hypercube, denoted by BH_n, is defined as follows.

Definition 1.1. For $n \geq 1$, BH_n has 4^n vertices, and each vertex has a unique n-component vector on $\{0, 1, 2, 3\}$ for an address, also called an n-bit string. A vertex $(a_0, a_1, \ldots, a_{n-1})$ is connected to the following $2n$ vertices:

$$\begin{cases} ((a_0+1)(\operatorname{mod} 4), a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1}), \\ ((a_0-1)(\operatorname{mod} 4), a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1}), \end{cases}$$

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$$\begin{cases} ((a_0+1)(\text{mod } 4), a_1, \ldots, a_{i-1}, (a_i + (-1)^{a_0})(\text{mod } 4), a_{i+1}, \ldots, a_{n-1}), \\ ((a_0-1)(\text{mod } 4), a_1, \ldots, a_{i-1}, (a_i + (-1)^{a_0})(\text{mod } 4), a_{i+1}, \ldots, a_{n-1}), \end{cases}$$

for $1 \leq i \leq n-1$.

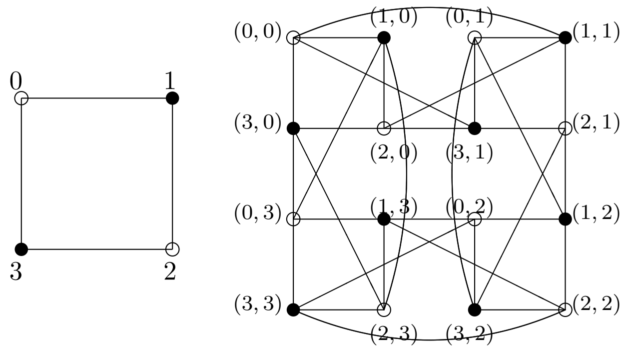

Figure 1: Two balanced hypercubes: BH_1 and BH_2

By now, various properties of the balanced hypercube, such as, Hamiltonian laceability, bipanconnectivity, super connectivity etc. have been extensively investigated in the literature [7, 8, 9, 14, 16, 17, 18, 19]. In many situations, it is highly desired to use interconnection networks which are highly symmetric. This often simplifies the computational and routing algorithms. It has been shown that the balanced hypercube is vertex-transitive and arc-transitive (see [14, 22]). When dealing with the symmetry of graphs, the goal is to gain as much information as possible about the structure of the full automorphism groups. Recently, several publications have been put into investigation of automorphism groups of Cayley graphs having connection with interconnection networks (see, for example, [5, 10, 20, 21]).

In [22], it was proved that BH_n is an arc-transitive Cayley graph.

Definition 1.2. For $n \geq 1$, let H_n be an abelian group defined as follows:

$$H_n \quad = \langle y \rangle \times \langle z_1 \rangle \times \langle z_2 \rangle \times \ldots \times \langle z_{n-1} \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \ldots \times \mathbb{Z}_4.$$

The *generalized dihedral group of* H_n, denoted by $\text{Dih}(H_n)$, is the semi-direct product of H_n by a group $\langle x \rangle$ of order 2 with the involution x inverting every element in H_n. Let $G_n = \text{Dih}(H_n) = H_n \rtimes \langle x \rangle$ and let $S = \{x, xy, xz_i, xyz_i \mid i = 1, 2, \ldots, n-1\}$. Let Γ_n be the following Cayley graph:

$$\Gamma_n = \text{Cay}(G_n, S). \tag{1.1}$$

Proposition 1.3. [22, Theorem 3.7] *For each* $n \geq 1$*,* $BH_n \cong \Gamma_n$ *is arc-transitive.*

Definition 1.4. Let L_n be a subgroup of H_n defined by

$$L_n \quad = \langle z_1 \rangle \times \langle z_2 \rangle \times \ldots \times \langle z_{n-1} \rangle \cong \underbrace{\mathbb{Z}_4 \times \mathbb{Z}_4 \times \ldots \times \mathbb{Z}_4}_{n-1}.$$

Let $T_n = \mathrm{Dih}(L_n) = L_n \rtimes \langle x \rangle$. Clearly, T_n is a subgroup of G_n of index 2. Set $\Omega = \{x, xz_i \mid i = 1, 2, \ldots, n-1\}$, and define X_n as the following Cayley graph:

$$X_n = \mathrm{Cay}(T_n, \Omega). \tag{1.2}$$

For convenience, in what follows we shall always let $\Gamma_n = BH_n$. In [3], the authors proved the following result.

Proposition 1.5. [3, Theorem 3.4] *For each $n \geq 1$, $BH_n \cong X_n[2K_1]$, where X_n is defined as following:*

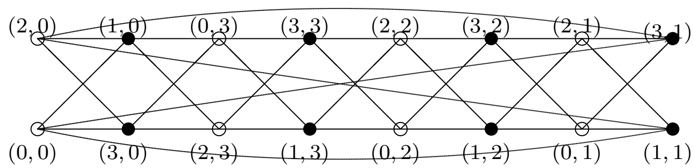

Figure 2: Another layout of BH_2

By Proposition 2.1, it is easy to see that $\mathrm{Aut}(BH_n) = \mathbb{Z}_2 \wr \mathrm{Aut}(X_n)$ *. So, to determine the automorphism group of BH_n, the key is to determine the automorphism group of X_n. In this paper, we prove that X_n is a 2-arc-transitive normal Cayley graph, and $\mathrm{Aut}(X_n) = R(T_n) \rtimes \mathrm{Aut}(T_n, \Omega) \cong T_n \rtimes S_n$.

As the automorphism group of BH_n is known, one may ask: Does BH_n have a stronger symmetry property? In this paper, we show that BH_n is arc-transitive but not 2-arc-transitive.

As another application, we prove that X_n is a $\mathbb{Z}_2^{n-1}$-regular cover of the hypercube Q_n. This, together with the fact $BH_n \cong X_n[2K_1]$, gives a theoretical explanation of the relationship between BH_n and Q_n.

2 Preliminaries

In this section, we shall introduce some notations, terminology and preliminary results. Throughout this paper only undirected simple connected graphs without loops and multiple edges are considered. Unless stated otherwise, we follow Bondy and Murty [2] for terminology and definitions.

Let n be a positive integer. Denote by $\mathbb{Z}_n$ the cyclic group of order n, by S_n the symmetric group of degree n and by $K_{n,n}$ the complete bipartite graph of order $2n$ and valency n, respectively. We also use nK_1, K_n and C_n to denote the null graph, the complete graph and the cycle with n vertices, respectively.

In a parallel computing system, processors are connected based on a specific interconnection network. An interconnection network is usually represented by a graph in which vertices represent processors and edges represent links between processors. Let G be a simple undirected connected graph. We denote by $\mathrm{Aut}(G)$ the full automorphism group of G, and by $V(G)$ and $E(G)$ the sets of vertices and edges of G, respectively. For $u, v \in V(G)$, denote by $\{u, v\}$ the edge incident to u and v in G. For a vertex v in a graph G, use $N_G(v)$ to denote the neighborhood of v, that is, the set of vertices adjacent to v.

*One can also obtain this by using [4, Theorem 5.7]. We thank a referee for pointing out this.

An *s-arc* in a graph G is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of G such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. A graph G is said to be *s-arc-transitive* if $\mathrm{Aut}(G)$ is transitive on the set of s-arcs in G. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. A graph G is *edge-transitive* if $\mathrm{Aut}(G)$ acts transitively on $E(G)$. Clearly, every arc-transitive graph is both edge-transitive and vertex-transitive.

2.1 Wreath products of groups

For a set V and a group G with identity element 1, an *action* of G on V is a mapping $V \times G \to V$, $(v, g) \mapsto v^g$, such that $v^1 = v$ and $(v^g)^h = v^{gh}$ for $v \in V$ and $g, h \in G$. The *kernel* of G acting on V is the subgroup of G fixing V pointwise. For two groups K, H, if H acts on K (as a set) such that $(xy)^h = x^h y^h$ for any $x, y \in K$ and $h \in H$, then H is said to act on K as a group. In this case, we use $K \rtimes H$ to denote the *semi-direct product* of K by H with respect to the action.

Let H be a permutation group on a finite set Δ. For convenience, let $\Delta = \{1, 2, \cdots, n\}$. Let G be a permutation group on a finite set Φ, and let

$$N = \underbrace{G \times G \times \cdots \times G}_{n \text{ times}}.$$

We define the action of H on N as following:

$$\forall h \in H, (g_1, g_2 \cdots, g_n)^h = (g_{1^{h^{-1}}}, g_{2^{h^{-1}}}, \cdots, g_{n^{h^{-1}}}), g_i \in G, i = 1, 2, \cdots, n.$$

Then the semi-direct product of N by H with respect to this action is called the *wreath product* of G and H, denoted by $G \wr H$. Clearly,

$$G \wr H = \{(g_1, g_2, \cdots, g_n; h) \mid g_i \in G, h \in H\}.$$

Moreover, $G \wr H$ can be viewed as a permutation group on $\Phi \times \Delta$ as following:

$$(x, i)^{(g_1, g_2, \cdots, g_n; h)} = (x^{g_i}, i^h).$$

Let G and H be two graphs. The *lexicographic product* $G[H]$ is defined as the graph with vertex set $V(G) \times V(H)$ and for any two vertices $(u_1, v_1), (u_2, v_2) \in V(G) \times V(H)$, they are adjacent in $G[H]$ if and only if either $u_1 = u_2$ and v_1 is adjacent to v_2 in H, or u_1 is adjacent to u_2 in G. In view of [13, Theorem.], we have the following.

Proposition 2.1. [13, Theorem.] *Let X and Y be two graphs. Then* $\mathrm{Aut}(X[Y]) = \mathrm{Aut}(Y) \wr \mathrm{Aut}(X)$ *if and only if*

(1) *if there are two distinct vertices $u, v \in V(X)$ such that $N_X(u) = N_X(v)$, then Y is connected;*

(2) *if there are two distinct vertices $u, v \in V(X)$ such that $N_X(u) \cup \{u\} = N_X(v) \cup \{v\}$, then the complement $\overline{Y}$ of Y is connected.*

2.2 Cayley graphs

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if

$G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular. Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. A Cayley graph $\mathrm{Cay}(G, S)$ is connected if and only if S generates G. Given a $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg, x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$, called the *right regular representation* of G, is a permutation group isomorphic to G. It is well-known that $R(G) \leq \mathrm{Aut}(\mathrm{Cay}(G, S))$. So, $\mathrm{Cay}(G, S)$ is vertex-transitive. In general, a vertex-transitive graph X is isomorphic to a Cayley graph on a group G if and only if its automorphism group has a subgroup isomorphic to G, acting regularly on the vertex set of X (see [1, Lemma 16.3]).

For two inverse closed subsets S and T of a group G not containing the identity 1, if there is an $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$ then S and T are said to be *equivalent*, denoted by $S \equiv T$. The following proposition is easy to obtain.

Proposition 2.2. *If S and T are equivalent then* $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$.

A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$ (see [15]). Let $\mathrm{Cay}(G, S)$ be a Cayley graph on a group G with respect to a subset S of G. Set $A = \mathrm{Aut}(\mathrm{Cay}(G, S))$ and $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$.

Proposition 2.3. [15, Proposition 1.5] *The Cayley graph* $\mathrm{Cay}(G, S)$ *is normal if and only if* $A_1 = \mathrm{Aut}(G, S)$*, where A_1 is the stabilizer of the identity* 1 *of G in A.*

2.3 Covers of graphs

An important tool in studying symmetry properties of graphs is the covering technique. An epimorphism $\wp : \widetilde{X} \to X$ of graphs is called a *regular covering projection* if there is a semiregular subgroup $\mathrm{CT}(\wp)$ of the automorphism group $\mathrm{Aut}(\widetilde{X})$ of $\widetilde{X}$ whose orbits in $V(\widetilde{X})$ coincide with the *vertex fibers* $\wp^{-1}(v)$, $v \in V(X)$, and the arc and edge orbits of $\mathrm{CT}(\wp)$ coincide with the *arc fibers* $\wp^{-1}((u, v))$, $u \sim v$, and the *edge fibers* $\wp^{-1}(\{u, v\})$, $u \sim v$, respectively. In particular, we call the graph $\widetilde{X}$ a regular *cover* of the graph X. The semiregular group $\mathrm{CT}(\wp)$ is the *covering transformation group*. If $\mathrm{CT}(\wp)$ is isomorphic to an abstract group N then we speak of $\widetilde{X}$ as a regular *N-cover* of X. For more results on the covering of graphs, we refer the reader to [6, 12].

Let X be a connected k-valent graph and let $G \leq \mathrm{Aut}(X)$ act transitively on the 2-arcs of X. Let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N in $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [11, Theorem 9], we have the following.

Proposition 2.4. *If N has more than two orbits in $V(X)$, then N is semiregular on $V(X)$, X_N is a k-valent graph with G/N as a 2-arc-transitive group of automorphisms, and X is a regular N-cover of X_N.*

3 Automorphism group of the balanced hypercube

In this section, we shall determine the full automorphism group of the balanced hypercube. From Proposition 1.5 we know that $\Gamma_n \cong X_n[2K_1]$, and by Proposition 2.1, $\mathrm{Aut}(\Gamma_n) \cong \mathbb{Z}_2 \wr \mathrm{Aut}(X_n)$. So, the key step is to determine the automorphism group of X_n.

Lemma 3.1. *For $n \geq 1$, X_n is a 2-arc-transitive normal Cayley graph, and furthermore,* $\mathrm{Aut}(X_n) = R(T_n) \rtimes \mathrm{Aut}(T_n, \Omega)$, *where* $R(T_n) \cong T_n = \mathrm{Dih}(\mathbb{Z}_4^{n-1})$ *and* $\mathrm{Aut}(T_n, \Omega) \cong S_n$.

Proof. Clearly, $X_1 \cong K_2$ and $X_2 \cong C_8$. It is easy to see that the statement is true for these two cases. In what follows, assume that $n \geq 3$. We first prove the following two claims.

Claim 1 $\mathrm{Aut}(T_n, \Omega) \cong S_n$.

Since Ω generates T_n, $\mathrm{Aut}(T_n, \Omega)$ acts faithfully on Ω, and hence $\mathrm{Aut}(T_n, \Omega) \leq S_n$.

It is easy to verify that $xz_1, z_1^{-1}z_i(2 \leq i \leq n-1), z_1^{-1}$ generate T_n and they satisfy the same relations as $x, z_i(1 \leq i \leq n-2), z_{n-1}$. This implies that the map

$$\alpha : x \mapsto xz_1, z_i \mapsto z_1^{-1}z_{i+1}(1 \leq i \leq n-2), z_{n-1} \mapsto z_1^{-1},$$

induces an automorphism of T_n. Clearly, for each $1 \leq i \leq n-2$, $(xz_i)^\alpha = xz_{i+1}$, and $x \mapsto xz_1$ and $(xz_{n-1})^\alpha = x$. This implies that α cyclicly permutates the elements in Ω, and so $\alpha \in \mathrm{Aut}(T_n, \Omega)$.

Similarly, for each $2 \leq i \leq n-1$, we define a map as the following:

$$\beta_i : x \mapsto x, z_1 \mapsto z_i, z_i \mapsto z_1, z_j \mapsto z_j(1 \leq i, j \leq n-1 \text{ and } i \neq j).$$

Then β_i induces an automorphism of T_n, and furthermore, β_i interchanges xz_1 and xz_i and fixes all other elements in Ω. Hence, $\beta_i \in \mathrm{Aut}(T_n, \Omega)$ and by elementary group theory, we obtain that the subgroup generated by $\beta_i(2 \leq i \leq n-1)$ is isomorphic to S_{n-1}. Since S_{n-1} is maximal in S_n, one has $\langle \alpha, \beta_i \mid 2 \leq i \leq n-1 \rangle \cong S_n$. It follows that $\mathrm{Aut}(T_n, \Omega) = \langle \alpha, \beta_i \mid 2 \leq i \leq n-1 \rangle \cong S_n$.

Claim 2 *For any xz_i, there are $(n-2)$ 6-cycles in X_n passing through the 2-arc $(x, 1, xz_i)$, namely, $C^{i,j} = (1, x, z_j^{-1}, xz_iz_j^{-1}, z_j^{-1}z_i, xz_i, 1)$ with $j \neq i$ and $1 \leq j \leq n-1$.*

By Claim 1, $\mathrm{Aut}(T_n, \Omega)$ acts 2-transitively on Ω. It is well-known that a vertex-transitive graph is 2-arc-transitive if and only if the vertex-stabilizer $\mathrm{Aut}(X_n)_v$ is 2-transitive on the set of vertices adjacent to v. So, X_n is 2-arc-transitive. To prove the claim, it suffices to show that the statement is true for the case when $i = 1$.

First, for any $2 \leq j \leq n-1$, one may easily check that $C^{1,j} = (1, x, z_j^{-1}, xz_1z_j^{-1}, z_1z_j^{-1}, xz_1, 1)$ is a 6-cycle passing through the 2-arc $(x, 1, xz_1)$. Let C' be an arbitrary 6-cycle passing through $(x, 1, xz_1)$. Then there exist $s_1, s_2, t_1, t_2 \in \Omega$ such that $C' = (1, x, s_1x, s_2s_1x = t_2t_1xz_1, t_1xz_1, xz_1, 1)$, where $s_1 \neq x$, $s_2 \neq s_1$, $t_1 \neq xz_1$ and $t_1 \neq t_2$. Clearly, $s_1 = xz_j$ for some $1 \leq j \leq n-1$. In the rest of the proof of Claim 2 the following well-known fact will be frequently used.

Fact *Every element in $\langle z_1 \rangle \times \langle z_2 \rangle \times \ldots \times \langle z_{n-1} \rangle$ can be uniquely written in the following form*

$$z_1^{a_1}z_2^{a_2}\ldots z_{n-1}^{a_{n-1}}, a_i \in \mathbb{Z}_4(1 \leq i \leq n-1).$$

If $s_2 = x$, then $xxz_jx = t_2t_1xz_1$. It follows that $z_jx = t_2t_1xz_1$ and hence $z_jz_1 = t_2t_1$. If $t_2 = x$, then $t_1 = xz_k$ for some $1 \leq k \leq n-1$, and so $z_jz_1 = z_k$. By Fact, this is impossible. If $t_2 = xz_\ell$ for some $1 \leq \ell \leq n-1$, then we have either $t_1 = x$ or $t_1 = xz_p$ for some $1 \leq p \leq n-1$. For the former, we have $z_jz_1 = z_\ell^{-1}$, and for the latter, we have $t_2t_1 = xz_\ell xz_p = z_\ell^{-1}z_p = z_jz_1$. From the above Fact, both of these cannot happen.

If $s_2 = xz_i$ for some $1 \le i \le n-1$, then $xz_ixz_jx = t_2t_1xz_1$. It follows that $z_i^{-1}z_jx = t_2t_1xz_1$ and hence $z_i^{-1}z_jz_1 = t_2t_1$. If $t_1 = xz_k$ and $t_2 = xz_p$ for some $1 \le k, p \le n-1$, then $t_2t_1 = z_p^{-1}z_k = z_i^{-1}z_jz_1$. This is also impossible. If $t_1 = x$ and $t_2 = xz_p$ for some $1 \le p \le n-1$, then $t_2t_1 = z_p^{-1} = z_i^{-1}z_jz_1$. This is also impossible. So, we must have $t_1 = xz_k$ and $t_2 = x$ for some $1 \le k \le n-1$. Then $t_2t_1 = z_k = z_i^{-1}z_jz_1$. Clearly, $s_1 \ne s_2$. Then $z_k = z_j$ and $z_i = z_1$. That is $s_2 = xz_1$, $t_2 = x$, $t_1 = s_1 = xz_j$. It follows that $C' = C^{1,j} = (1, x, z_j^{-1}, xz_1z_j^{-1}, z_j^{-1}z_1, xz_1, 1)$.

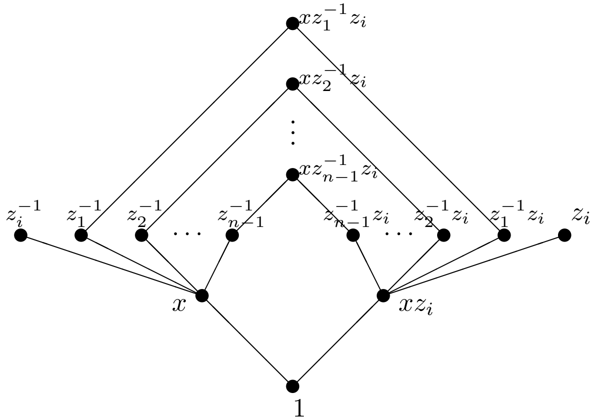

Figure 3: 6-cycles passing through $(x, 1, xz_i)$

Now we are ready to complete the proof. Let $A = \mathrm{Aut}(X_n)$ and let A_1 be the stabilizer of the identity 1 in A. Let A_1^* be the kernel of A_1 acting on Ω. Then A_1^* fixes every element in Ω. For any xz_i $(1 \le i \le n-1)$, by Claim 2, there are exactly $(n-2)$ 6-cycles in X_n passing through the 2-arc $(x, 1, xz_i)$, namely, $C^{i,j} = (1, x, z_j^{-1}, xz_iz_j^{-1}, z_j^{-1}z_i, xz_i, 1)$ with $j \ne i$ and $1 \le j \le n-1$ (see Fig. (3)). Note that the neighborhood of x is $\{1, z_i^{-1} \mid 1 \le i \le n-1\}$ and the neighborhood of xz_i is $\{1, z_i, z_j^{-1}z_i \mid 1 \le i, j \le n-1, j \ne i\}$. Since there are no 6-cycles passing through $z_i^{-1}, x, 1, xz_i$ and z_i, it follows that A_1^* fixes z_i^{-1} and z_i $(1 \le i \le n-1)$.

By [3, Lemma 4.2], X_n has girth 6, and so $C^{i,j}$ is the unique 6-cycle passing through $z_j^{-1}, x, 1, xz_i, z_j^{-1}z_i$. As A_1^* fixes $z_j^{-1}, x, 1$ and xz_i, A_1^* must fix $z_j^{-1}z_i$. By the arbitrariness of i, j, we obtain that A_1^* fixes every vertex of the set $\{z_i^{-1}, z_i, z_j^{-1}z_i \mid 1 \le i, j \le n-1, j \ne i\}$ which is just the set of vertices at distance 2 from the identity 1. By the vertex-transitivity and connectivity of X_n, A_1^* fixes all vertices of X_n. It follows that $A_1^* = 1$, and so A_1 acts faithfully on Ω. Therefore, $A_1 \lesssim S_n$. By Claim 1, $\mathrm{Aut}(T_n, \Omega) \cong S_n$, and since $\mathrm{Aut}(T_n, \Omega) \le A_1$, one has $\mathrm{Aut}(T_n, \Omega) = A_1$. By Proposition 2.3, X_n is normal, and so $A = R(T_n) \rtimes \mathrm{Aut}(T_n, \Omega)$. $\square$

Now we are ready to determine the automorphism group of BH_n.

Theorem 3.2. *For* $n \ge 1$, $\mathrm{Aut}(BH_n) = \mathbb{Z}_2 \wr (T_n \rtimes S_n)$.

Proof. By Proposition 1.5, $BH_n \cong X_n[2K_1]$. By Proposition 2.1, $\mathrm{Aut}(BH_n) \cong \mathbb{Z}_2 \wr \mathrm{Aut}(X_n)$. From Theorem 3.1 we obtain that $\mathrm{Aut}(X_n) = R(T_n) \rtimes \mathrm{Aut}(T_n, \Omega) \cong T_n \rtimes S_n$. It follows that $\mathrm{Aut}(BH_n) = \mathbb{Z}_2 \wr (T_n \rtimes S_n)$. $\square$

4 Related results

As the automorphism group of BH_n is known, we can obtain more information about the symmetry properties of BH_n. By Proposition 1.3, BH_n is arc-transitive, and by Theorem 3.1, X_n is 2-arc-transitive. It is natural to ask: whether BH_n has much stronger symmetry property? We answer this in negative by showing that BH_n is not 2-arc-transitive.

Theorem 4.1. *For $n \geq 2$, BH_n is arc-transitive but not 2-arc-transitive.*

Proof. Suppose, by way of contradiction, that BH_n is 2-arc-transitive. Recall that $BH_n = \mathrm{Cay}(G_n, S)$. Then the vertex-stabilizer $\mathrm{Aut}(BH_n)_1$ of the identity 1 of G_n in $\mathrm{Aut}(BH_n)$ is 2-transitive on S. That is, for any two distinct ordered pairs from $S \times S$, say (u_1, v_1) and (u_2, v_2), there exists $\alpha \in \mathrm{Aut}(BH_n)_1$ such that $(u_1, v_1)^\alpha = (u_2, v_2)$. In particular, there exists $\alpha \in \mathrm{Aut}(BH_n)_1$ such that $(x, xy)^\alpha = (x, xz_1)$. This implies that x and xz_1 have the same neighborhood because x and xy have the same neighborhood. However, from [22, Lemma 3.8], we see that xy is the unique vertex which has the same neighborhood as x, a contradiction. □

By Proposition 1.5, $BH_n \cong X_n[2K_1]$. As a consequence of Theorem 3.1, we can also prove that X_n is a $\mathbb{Z}_2^{n-1}$-regular cover of the hypercube Q_n. This reveals the relationship between the balanced hypercube BH_n and the hypercube Q_n.

Lemma 4.2. *For $n \geq 1$, let $N = \mathbb{Z}_2^n$. Let $G = \mathrm{Cay}(N, S)$ be a connected n-valent Cayley graph. Then G is isomorphic to the n-dimensional hypercube Q_n.*

Proof. It is well-known that Q_n is a Cayley graph on N with respect to the subset

$$T = \{(1, 0, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots, (0, 0, 0, \cdots, 1)\}.$$

Viewing N as an n-dimensional vector space on the field $\mathbb{Z}_2$, one may see that T is a basis of N. Since G is an n-valent Cayley graph, one has $|S| = n$, and since G is connected, one has $N = \langle S \rangle$. This means that S is also a basis of N. So, there is an automorphism of N which maps S to T. By Proposition 2.2, $G \cong Q_n$, as desired. □

Theorem 4.3. *For $n \geq 3$, X_n is a $\mathbb{Z}_2^{n-1}$-regular cover of Q_n.*

Proof. By Theorem 3.1, $R(T_n)$ is normal in $\mathrm{Aut}(X_n)$. Remember that $T_n = \mathrm{Dih}(L_n) = L_n \rtimes \langle x \rangle$, where

$$L_n = \langle z_1 \rangle \times \ldots \times \langle z_{n-1} \rangle \cong \underbrace{\mathbb{Z}_4 \times \ldots \times \mathbb{Z}_4}_{n-1 \text{ times}},$$

and x is an involution inverting every element in L_n. Set $Z = \langle R(z_1^2) \rangle \times \ldots \times \langle R(z_{n-1}^2) \rangle$. Then

$$Z \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{n-1 \text{ times}},$$

and Z is just the center of $R(T_n)$. It follows that Z is characteristic in $R(T_n)$. Since $R(T_n) \trianglelefteq \mathrm{Aut}(X_n)$, one has $Z \trianglelefteq \mathrm{Aut}(X_n)$. Consider the quotient graph Y_n of X_n relative to Z. Clearly, Z is semiregular on the vertex-set of X_n, and so it has more than 2 orbits on $V(X)$. Since X_n is 2-arc-transitive, by Proposition 2.4, Y_n is also an n-valent graph with

$\mathrm{Aut}(X_n)/Z$ as a 2-arc-transitive automorphism group, and X_n is a Z-regular cover of Y_n. To complete the proof, it suffices to prove that $Y_n \cong Q_n$.

Noting that $Z \trianglelefteq R(T_n)$ and $R(T_n)$ is regular on $V(X_n)$, $R(T_n)/Z$ is regular on $V(Y_n)$. It follows that Y_n is a Cayley graph on $R(T_n)/Z$. As $R(T_n) = \mathrm{Dih}(L_n)$, one has $R(T_n)/Z \cong \mathbb{Z}_2^n$. Since Y_n has valency n, by Lemma 4.2, one has $Y_n \cong Q_n$. □

Conclusion

In [14], the authors introduced the balanced hypercube to enhance some properties of the hypercube. Graph symmetry is an important factor in the design of an interconnection network. In 1997, it has been shown that the balanced hypercube is vertex-transitive. Recently, it was shown that the balanced hypercube is also arc-transitive. However, the full automorphism group of the balanced hypercube remained unknown. In this paper, we solve this problem. As applications, we first analyze the symmetry properties of the balanced hypercube and show that the balanced hypercube is not 2-arc-transitive. Then, we give a theoretical explanation of the relationship between the balanced hypercube and the hypercube.

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Uniquely colorable Cayley graphs

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Abstract

It is shown that the chromatic number $\chi(G) = k$ of a uniquely colorable Cayley graph G over a group Γ is a divisor of $|\Gamma| = n$. Each color class in a k-coloring of G is a coset of a subgroup of order n/k of Γ. Moreover, it is proved that $(k-1)n$ is a sharp lower bound for the number of edges of a uniquely k-colorable, noncomplete Cayley graph over an abelian group of order n. Finally, we present constructions of uniquely colorable Cayley graphs by graph products.

Keywords: Vertex coloring, color classes, Cayley graph.

Math. Subj. Class.: 05C15, 05C25

1 Introduction

A proper k-coloring of an undirected graph $G = (V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$ is a map $f : V \to C$ from V into a set C with $|C| = k$ elements ('colors') such that any two adjacent vertices are assigned different colors. If not otherwise stated a k-coloring is always understood to be a proper k-coloring. A graph G is k-colorable if it admits a k-coloring. The chromatic number $\chi(G)$ is the smallest number k for which G is k-colorable. An optimal coloring of G is a $\chi(G)$-coloring of G. The color class of the coloring $f : V \to C$ with respect to color $c \in C$ consists of all vertices $x \in V$ with $f(x) = c$. If f is a k-coloring of G, then the color classes of f constitute a partition of V into at most k disjoint stable sets which means that any two elements of these sets are nonadjacent. The graph G is uniquely colorable if every optimal coloring of G induces the

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same partition into color classes. If G is uniquely colorable, then we mean by the color classes of G the color classes of an optimal coloring of G.

Let us point out some previous work on uniquely colorable graphs. Harary et al. [11] construct new ones from given uniquely colorable graphs. Bollobas [4] presents a lower bound for the minimal degree $\delta(G)$ which forces G to be uniquely colorable. Hillar and Windfeldt [13] give an algebraic characterization of uniquely k-colorable graphs, which partly originates in ideas of Lovász [16] and Bayer [3]. They also apply commutative algebra to develop an algorithm for recognizing unique colorability. Xu [19] establishes a sharp lower bound for the number of edges of a uniquely k-colorable graph on n vertices:

$$|E| \;\geq\; (k-1)n \;-\; \binom{k}{2}. \tag{1.1}$$

Daneshgar [7] and Daneshgar, Naserasr [8] concentrate on cliques in uniquely colorable graphs. Special classes of uniquely colorable graphs are investigated by Akbari et al. [1], Chao and Chen [5], Chartrand and Geller [6].

The Cayley graph $G = \mathrm{Cay}(\Gamma, S)$ over the finite (multiplicative) group Γ with shift set (or symbol) $S \subseteq \Gamma$ has vertex set $V = V(G) = \Gamma$ and edge set

$$E = E(G) = \{\{x, y\} : x, y \in \Gamma, xy^{-1} \in S\}.$$

To avoid loops we demand that the unit element $e \in \Gamma$ is not in S. To make G undirected we require that S is self-inverse, $S^{-1} = S$, which means that $s \in S$ always implies $s^{-1} \in S$. For general properties of Cayley graphs we refer to Godsil and Royle [9]. Circulant graphs are Cayley graphs over cyclic groups. We represent the cyclic group of order n by the additive group Z_n of integers modulo n, $Z_n = \{0, 1, \dots, n-1\}$. A well-known circulant graph is the unitary Cayley graph

$$X_n \;=\; \mathrm{Cay}(Z_n, U_n) \text{ with } U_n \;=\; \{x \in Z_n : \; \gcd(x, n) = 1\}.$$

Here $\gcd(x, n)$ denotes the greatest common divisor of x and n and U_n is the set of multiplicative units of Z_n considered as a ring. In [15] we proved for $n > 1$ that $\chi(X_n) = p$, where p is a smallest prime divisor of n. Bašić and Ilić [2] remarked in passing that X_n is uniquely p-colorable. This remark encouraged us to look closer at uniquely colorable Cayley graphs in general.

In this paper we show that the chromatic number $\chi(G) = k$ of a uniquely colorable Cayley graph G over a group Γ is a divisor of the number of elements $|\Gamma| = n$ of Γ. Each color class of G is a coset of some subgroup of order n/k of Γ. For a uniquely colorable, noncomplete Cayley graph over an abelian group the estimate (1.1) on its number of edges can be improved to $|E| \geq (k-1)n$. For every divisor k of n, $1 < k < n$, we construct a uniquely k-colorable circulant graph on n vertices with the minimal number of $(k-1)n$ edges. In the final section, extending a result of Greenwell and Lovász [10], we present a general method for constructing uniquely colorable graphs by graph products, which can especially be applied to Cayley graphs.

2 Necessary conditions

A graph $G = (V, E)$ is transitive if for any two vertices $x, y \in V$ there is an automorphism τ of G with $\tau(x) = y$. Transitive graphs are regular. We call G weakly transitive if we require the existence of an automorphism τ of G with $\tau(x) = y$ only for adjacent vertices x and y.

Lemma 2.1. *Let the graph $G = (V, E)$ be weakly transitive und uniqely k-colorable. Then $\chi(G) = k$ is a divisor of $|V| = n$ and every color class of G has n/k elements.*

Proof. We may assume $k > 1$. Let $C_1,\ C_2$ be an arbitrary pair of color classes of G. Since $\chi(G) = k$ there exists a pair $x,\ y$ of adjacent vertices $x \in C_1$ and $y \in C_2$. As G is weakly transitive we know that there is an automorphism τ of G with $\tau(x) = y$. Every automorphism of a uniquely colorable graph G maps each color class of G to another color class of G. Therefore, $x \in C_1$, $y \in C_2$ and $\tau(x) = y$ imply $\tau(C_1) = C_2$ and $|C_1| = |C_2|$. Every color class C of G has the same number of $|C|$ elements. As the color classes partition the vertex set V into k disjoint sets of equal size $|C|$, we have $|V| = n = k|C|$, which proves the lemma. □

Let $G = \mathrm{Cay}(\Gamma, S)$ be a Cayley graph. Define the bijection $\tau_a : \Gamma \to \Gamma$ for $a \in \Gamma$ by $\tau_a(x) = xa$. We verify for $x, y \in \Gamma$:

$$x, y \text{ adjacent in } G \ \Leftrightarrow\ xy^{-1} \in S \ \Leftrightarrow\ (xa)(ya)^{-1} \in S \ \Leftrightarrow\ \tau_a(x), \tau_a(y) \text{ adjacent in } G.$$

For $a = x^{-1}y$ we have $\tau_a(x) = y$. This shows that $H(\Gamma) = \{\tau_a : a \in \Gamma\}$ is a subgroup of the automorphism group $\mathrm{Aut}(G)$ that operates transitively on the vertices of G. As Cayley graphs are transitive, Lemma 2.1 can especially be applied to Cayley graphs.

Theorem 2.2. *For a uniquely colorable Cayley graph $G = \mathrm{Cay}(\Gamma, S)$ the following statements are true.*

1. *The chromatic number $\chi(G) = k$ divides the number $|V(G)| = |\Gamma| = n$ of vertices of G.*
2. *Every color class C of G is a left coset of a subgroup $U(C) \subseteq \Gamma$ of order $|U(C)| = \frac{n}{k}$.*
3. *For any two distinct color classes C_1 and C_2 of G there exists an element $\gamma \in \Gamma$ such that $U(C_2) = \gamma U(C_1)\gamma^{-1}$. If Γ is abelian, then every color class C of G has the same subgroup $U(C)$.*

Proof. 1. This is a consequence of Lemma 2.1.

2. Suppose that $C = \{a_1, \ldots, a_r\},\ r = n/k$, is a color class of G. Define

$$U \ =\ U(C) \ =\ \{a_i^{-1}a_j \,:\, i, j \in \{1, \ldots, r\}\}.$$

We prove that U is a subgroup of Γ.

The unit element $e = a_i^{-1}a_i$ belongs to U. For $x = a_i^{-1}a_j \in U$ we have $x^{-1} = a_j^{-1}a_i \in U$. Assume that $x = a_i^{-1}a_j \in U$ and $y = a_s^{-1}a_t \in U$. We are going to show $xy \in U$. The automorphism τ_x of G maps a_i to a_j, $\tau_x(a_i) = a_i x = a_j$. From the unique colorability of G we conclude $\tau_x(C) = C$ and analogously $\tau_y(C) = C$. For arbitrary $\zeta \in C$ we have

$$\begin{aligned} \tau_x(\zeta) \ &=\ \zeta x =\ \zeta_1 \in C, \\ \tau_y(\zeta_1) \ &=\ \zeta_1 y \ =\ \zeta xy \ =\ \zeta_2 \in C, \\ xy \ &=\ \zeta^{-1}\zeta_2 \in U. \end{aligned}$$

Next, we show $C = a_1U$, the left coset of U represented by a_1. For every $a_i \in C$ we have $a_i = a_1(a_1^{-1}a_i) \in a_1U$, which implies $C \subseteq a_1U$. Suppose

$$z \in a_1U,\ z = a_1a_i^{-1}a_j = a_1x,\ x = a_i^{-1}a_j \text{ for some } i, j \in \{1, \ldots, r\}.$$

As above we see $\tau_x(C) = C$. Therefore, $z = a_1x = \tau_x(a_1) \in C,\ C = a_1U$.

3. Let $C_1 = aU_1$ and $C_2 = bU_2$ be different color classes of G, $U_1 = U(C_1)$, $U_2 = U(C_2)$. For the automorphism τ_d of G with $d = a^{-1}b$ we have $\tau_d(a) = b$. The unique colorability of G implies $\tau_d(C_1) = C_2$, hence

$$C_2 = C_1d,\ bU_2 = aU_1a^{-1}b$$

and therefore

$$U_2 = \zeta U_1 \zeta^{-1} \text{ with } \zeta = b^{-1}a.$$

If Γ is abelian, we conclude $U_2 = U_1$. □

Corollary 2.3. *If $G = \mathrm{Cay}(Z_n, S)$ is a uniquely colorable circulant graph, then $\chi(G) = k$ is a divisor of n. The color classes of G are the residue classes modulo k in Z_n. If S is extended by elements $s' \in Z_n$, $s' \not\equiv 0$ modulo k, to a self-inverse set S', then $G' = \mathrm{Cay}(Z_n, S')$ is also a uniquely colorable graph with $\chi(G') = k$.*

Proof. According to Theorem 2.2, the color classes of G are the cosets of a subgroup $U \subseteq Z_n$, $|U| = n/k$. The (additive) cyclic group Z_n has exactly one subgroup of order n/k that is $\langle k\rangle = \{0, k, \ldots, (n/k - 1)k\}$, the cyclic subgroup generated by k. The cosets of $\langle k\rangle$ are the residue classes modulo k in Z_n. The graph $G' = \mathrm{Cay}(Z_n, S')$ is constructed from G by adding edges between different color classes. So the graph remains uniquely colorable with the same chromatic number. □

Problem 2.4. Is there a uniquely colorable Cayley graph over a nonabelian group such that different color classes are left cosets of different subgroups?

Theorem 2.5. *Let $G = \mathrm{Cay}(\Gamma, S)$ be a uniquely colorable Cayley graph over the abelian group Γ, $|\Gamma| = n$, $\chi(G) = k < n$. Then we have:*
The subgraph of G induced by any two color classes of G is uniquely colorable and regular of degree $l \geq 2$. Moreover, $|E(G)| \geq (k-1)n$. This bound is sharp.

Proof. The subgraph induced by any color classes of G must be uniquely colorable because otherwise G would not have this property. Consider arbitrary different color classes C and D of G. According to Theorem 2.2(3) they are cosets $C = aU$, $D = bU$ of the same subgroup $U = \{u_1, \ldots, u_r\} \subseteq \Gamma$, $r = n/k$. Without loss of generality let au_1 be a vertex of maximum degree l in the subgraph $G_1 = G(C \cup D)$ induced by $C \cup D$ in G. The neighbors of au_1 in G_1 must lie in bU. Let these be $bu_{i_1}, \ldots, bu_{i_l}$. For $u \in U$ we apply the automorphism τ_u of G defined by $\tau_u(x) = xu$ to au_1 and its neighbors in G_1 and conclude:

$$au_1u \in aU \text{ is adjacent to } bu_{i_1}u, \ldots, bu_{i_l}u \in bU \text{ for every } u \in U.$$

As au_1u runs through all elements of aU for $u \in U$, we see that all vertices in aU must have the same degree l in G_1. The same holds for the vertices of bU since the r vertices of bU have rl edges in G_1 and the maximum degree of G_1 is l.

It is easy to see (cf. Theorem 1 in [11]) that the subgraph $G_1 = G(C \cup D)$ induced by any two color classes C, D of G must be connected. This implies

$$l\frac{n}{k} = |E(G_1)| \geq |V(G_1)| - 1 = 2\frac{n}{k} - 1$$

so that

$$l \geq 2 - \frac{k}{n} > 1.$$

As l is an integer we have $l \geq 2$. This implies for $|S|$, the degree of regularity of G, $|S| \geq 2(k-1)$. Finally, we estimate the number of edges of G:

$$|E(G)| = \frac{1}{2}|S|n \geq (k-1)n.$$

Examples in the next section (see Corollary 3.4) will show that this bound is sharp. □

3 Uniquely colorable Cayley graphs with few edges

For the next theorem recall that the clique number $\omega(G)$ of a graph G is the largest number of vertices in a complete subgraph of G. The clique number $\omega(\overline{G})$ of the complementary graph $\overline{G}$ of G is also known as the independence number or stability number of G.

Theorem 3.1. *Let U be a subgroup of the (additive) abelian group Γ, $|U| = |\Gamma|/k$, $k > 1$ a divisor of $|\Gamma|$. Moreover, let $\{r_1, \ldots, r_k\}$ be a system of distinct representatives of the cosets of U in Γ. Define*

$$S = \{r_i - r_j : \ i, j \in \{1, \ldots, k\}, \ i \neq j\} \textit{ and } G = \mathrm{Cay}(\Gamma, S).$$

Then we have:

1. $\chi(G) = \omega(G) = k$.
2. $\chi(\overline{G}) = \omega(\overline{G}) = \frac{|\Gamma|}{k}$.
3. *The cosets of U in Γ are the color classes of an optimal coloring of G.*

Proof. From the definition of the representatives $r_1, \ldots, r_k$ we deduce $S \cap U = \emptyset$. Suppose that $x, \ y$ belong to the same coset $r_i + U$, $1 \leq i \leq k$. Then we can find elements $u_1, \ u_2 \in U$ such that $x = r_i + u_1$ and $y = r_i + u_2$. Now $x - y = u_1 - u_2 \in U$ implies $x - y \notin S$, which means that x and y are not adjacent in G. The cosets of U partition the vertex set Γ of G into k stable sets, i.e. sets of pairwise nonadjacent vertices. So we have

$$\omega(G) \leq \chi(G) \leq k.$$

On the other hand $r_1, \ldots, r_k$ induce a clique of size k in G. This proves claims 1 and 3.

Let $U = \{u_1, \ldots, u_t\}$, $t = |\Gamma|/k$. The sets

$$K_j = \{r_i + u_j : \ i = 1, \ldots, k\}, \ 1 \leq j \leq t,$$

induce cliques of size k in G, and therefore stable sets of size k in $\overline{G}$. To show that these sets are pairwise disjoint, we assume $x \in K_{j_1} \cap K_{j_2}$ for $j_1 \neq j_2$. We can find $i_1, i_2 \in \{1, \ldots, k\}$ such that

$$x = r_{i_1} + u_{j_1} = r_{i_2} + u_{j_2}.$$

Hence,

$$r_{i_1} - r_{i_2} = u_{j_2} - u_{j_1} \in U.$$

From $S \cap U = \emptyset$ we deduce $i_1 = i_2$, which implies $j_1 = j_2$ contrary to our assumption. The sets K_j, $1 \le j \le t$, constitute a partition of the vertex set Γ of $\overline{G}$ into $t = |\Gamma|/k$ stable sets of $\overline{G}$. Therefore, we have

$$\omega(\overline{G}) \le \chi(\overline{G}) \le \frac{|\Gamma|}{k}.$$

Finally, claim 2 follows from the fact that every coset of U induces a clique of size $t = |\Gamma|/k$ in $\overline{G}$. □

Theorem 3.1 gives a first impression of what symbol sets may potentially yield uniquely colorable Cayley graphs. The next example, however, shows that the symbol set structure mentioned there is not sufficient in general for unique colorability.

Example 3.2. We consider the integers modulo 12, $\Gamma = Z_{12} = \{0, 1, \dots, 11\}$. Let $U = \langle 4 \rangle = \{0, 4, 8\}$ be the cyclic subgroup of Z_{12} generated by 4. Then we have $k = |\Gamma|/|U| = 4$ and $\{r_1, r_2, r_3, r_4\} = \{0, 1, 6, 7\}$ as a system of distinct representatives for the cosets of U. We define

$$S = \{r_i - r_j : i, j \in \{1, \dots, 4\},\ i \ne j\} = \{1, 5, 6, 7, 11\} \text{ and } G = \mathrm{Cay}(\Gamma, S).$$

According to Theorem 3.1 the cosets of U in Γ,

$$\{0, 4, 8\},\ \{1, 5, 9\},\ \{2, 6, 10\},\ \{3, 7, 11\},$$

are the color classes of an optimal coloring of G. But there is another partition of Z_{12} into four stable sets of G:

$$\{0, 2, 4\},\ \{1, 3, 5\},\ \{6, 8, 10\},\ \{7, 9, 11\}.$$

Therefore, G is not uniquely colorable.

A more careful choice of the system of representatives will improve the situation.

Theorem 3.3. *Let k be a divisor of n, $1 < k < n$,*

$S_{k,n} = \{1, 2, \dots, k-1\} \cup \{n-1, n-2, \dots, n-(k-1)\}$, *and* $G_{k,n} = \mathrm{Cay}(Z_n, S_{k,n})$.

Then the circulant graph $G_{k,n}$ is uniquely colorable with

$$\chi(G_{k,n}) = \omega(G_{k,n}) = k \ \text{ and } \ \chi(\overline{G_{k,n}}) = \omega(\overline{G_{k,n}}) = \frac{n}{k}. \tag{3.1}$$

The residue classes modulo k in Z_n are the maximal stable sets of $G_{k,n}$ and the color classes of an optimal coloring of $G_{k,n}$.

Proof. The integers $r_1 = 0$, $r_2 = 1, \dots, r_k = k-1$ constitute a system of distinct representatives for the cosets of the subgroup $U = \langle k \rangle$ generated by k in Z_n. Modulo n we have:

$$S_{k,n} = \{r_i - r_j : i, j \in \{1, 2, \dots, k\},\ i \ne j\}.$$

Now Theorem 3.1 implies (3.1) and the fact that the cosets of U, i.e. the residue classes modulo k in Z_n, are the color classes of an optimal coloring of $G_{k,n}$. Let M be a stable set with a maximal number of vertices in $G_{k,n}$. We have $|M| = n/k$ by (3.1). For every $x \in M$ the consecutive integers $x+1, \dots,\ x+k-1$ (modulo n) are adjacent to x and therefore not in M. This implies that M is the residue class $x + \langle k \rangle$ in Z_n.

Let F be an optimal coloring of $G_{n,k}$, i.e. a coloring of the vertices of $G_{k,n}$ with k colors. Every color class of F must be a maximal stable set of $G_{n,k}$ with n/k elements. We have just shown that these sets are the cosets of $U = \langle k \rangle$ in Z_n. Therefore, $G_{k,n}$ is uniquely colorable. $\square$

The graph $G_{k,n} = \mathrm{Cay}(Z_n, S_{k,n})$ is regular of degree $|S_{k,n}| = 2(k-1)$. This implies $|E(G_{k,n})| = (k-1)n$. Hence we immediately obtain:

Corollary 3.4. *For every divisor k of n, $1 < k < n$, the graph $G_{k,n}$ defined in Theorem 3.3 is a uniquely k-colorable, circulant graph with n vertices and the minimal number of $|E(G_{k,n})| = (k-1)n$ edges.*

Example 3.5. Let $X_n = \mathrm{Cay}(Z_n, U_n)$ be the unitary Cayley graph on n vertices, $U_n = \{x \in Z_n :\ \gcd(x,n) = 1\}$. Suppose that p is the smallest prime divisor of n, $1 < p < n$. According to Theorem 3.3 we define

$$S_{p,n} = \{1, 2, \dots, p-1\} \cup \{n-1, n-2, \dots, n-(p-1)\} \text{ and } G_{p,n} = \mathrm{Cay}(Z_n, S_{p,n}).$$

Then $G_{p,n}$ is uniquely colorable and $\chi(G_{p,n}) = \chi(X_n) = p$. The unitary Cayley graph X_n results from $G_{p,n}$ by adding additional edges between different color classes of $G_{p,n}$. So X_n and $G_{p,n}$ are both uniquely colorable with the same color classes.

Problem 3.6. Is necessarily $\chi(G) = \omega(G)$ for every circulant uniquely colorable Cayley graph?

4 Constructing uniquely colorable graphs by graph products

The direct product $X \times Y$ of graphs X and Y has as its vertex set the cartesian product $V(X) \times V(Y)$. Vertices $(x_1, y_1),\ (x_2, y_2)$ of $X \times Y$ are adjacent if x_1 is adjacent to x_2 in X and y_1 is adjacent to y_2 in Y. If $X = \mathrm{Cay}(\Gamma_1, S_1)$ and $Y = \mathrm{Cay}(\Gamma_2, S_2)$ are Cayley graphs, then $X \times Y$ is a Cayley graph $\mathrm{Cay}(\Gamma, S)$ over the direct product $\Gamma = \Gamma_1 \times \Gamma_2$ with shift set $S = S_1 \times S_2$. A product $X \times Y$ of connected graphs is connected if both factors have at least two vertices and at least one factor is not bipartite (see [14]). Every proper n-coloring $f : V(X) \to Z_n$ of X induces a proper n-coloring $F : V(X) \times V(Y) \to Z_n$ of $X \times Y$ by $F(x, y) = f(x)$ for every $x \in V(X),\ y \in V(y)$. As the same is true for Y instead of X, we immediately see

$$\chi(X \times Y) \leq \min\{\chi(X), \chi(Y)\}.$$

A famous conjecture of Hedetniemi ([12], [17]) states that always equality occurs. We denote by $2K_2$ the graph consisting of two disjoint edges. A graph X is $2K_2$-free if it has no induced subgraph $2K_2$. D. Turzik [18] showed that Hedetniemi's conjecture is true if one of the factors is $2K_2$-free.

Lemma 4.1. *Let the graph X be $2K_2$-free and let $c : V(X) \times V(Y) \to Z_n$ be a proper n-coloring of $X \times Y$. For $y \in V(Y)$ define the map $c_y : V(X) \to Z_n$ by*

$$c_y(x) = c(x, y) \text{ for every } x \in V(X).$$

If every c_y, $y \in V(Y)$, is an improper coloring of X, then $\chi(Y) \leq n$.

Proof. The map c_y is an improper coloring of X means that there are adjacent vertices $x_1,\ x_2$ of X such that $c_y(x_1) = c_y(x_2)$. Let $\varphi(y)$ be the least value $c_y(x_1)$ such that there are adjacent vertices $x_1,\ x_2$ of X with $c_y(x_1) = c_y(x_2)$. We show that φ is a proper n-coloring of Y.

Let $y_1,\ y_2$ be adjacent vertices of Y. Assume $\varphi(y_1) = \varphi(y_2)$. Then we find two pairs x_1, x_2 and x_3, x_4 of adjacent vertices in X such that

$$c_{y_1}(x_1) = c_{y_1}(x_2) = \varphi(y_1) = \varphi(y_2) = c_{y_2}(x_3) = c_{y_2}(x_4),$$

$$c(x_1, y_1) = c(x_2, y_1) = c(x_3, y_2) = c(x_4, y_2). \tag{4.1}$$

As $x_1, \ldots, x_4$ do not induce a subgraph $2K_2$ in X, either $\{x_1, x_2\} \cap \{x_3, x_4\} = D \neq \emptyset$ or $D = \emptyset$ and there is an edge between $\{x_1, x_2\}$ and $\{x_3, x_4\}$. Suppose e.g. $D = \emptyset$ and x_1, x_3 are adjacent. Then (x_1, y_1) and (x_3, y_2) are adjacent vertices of $X \times Y$. But now $c(x_1, y_1) = c(x_3, y_2)$ in (4.1) contradicts the fact that c is a proper coloring of $X \times Y$. Similarly, the other cases lead to a contradiction. □

The following theorem extends a result of Greenwell and Lovász [10].

Theorem 4.2. *Let the graph X be uniquely n-colorable and $2K_2$-free. If Y is a connected graph with chromatic number $\chi(Y) > n$, then $X \times Y$ is uniquely n-colorable.*

Proof. We know $\chi(X \times Y) = m \leq \chi(X) = n$. Let $c : V(X) \times V(Y) \to Z_m$ be an arbitrary proper m-coloring of $X \times Y$. For $y \in Y$ define $c_y : V(X) \to Z_m$ by

$$c_y(x) = c(x, y) \text{ for every } x \in V(X).$$

If c_y is an improper m-coloring of X for every $y \in Y$, then Lemma 2.1 implies $\chi(Y) \leq m \leq n$ contradicting $\chi(Y) > n$. We conclude that there is a vertex y of Y such that c_y is a proper m-coloring of X. Moreover, $m \leq n = \chi(X)$ implies $m = n$. Let u be any neighbor of y in Y. Assume that there is a vertex x_1 in X such that $c_u(x_1) \neq c_y(x_1)$. As c_y is a proper n-coloring of the uniquely n-colorable graph X, all n colors except $c_y(x_1)$ appear in the range of c_y at the neighbors of x_1. In particular, we find a neighbor x_2 of x_1 with $c_y(x_2) = c_u(x_1)$, $c(x_2, y) = c(x_1, u)$. But this is impossible, because (x_2, y) is adjacent to (x_1, u) in $X \times Y$ and c is a proper coloring of this graph. Therefore, we have

$$c_u(x) = c_y(x) \text{ for every } x \in V(X).$$

We may repeat the above argument for every neighbor of u. Continuing this way we reach every vertex in the connected graph Y and achieve the following result:

$$c(x, y_1) = c(x, y_2) \text{ for every } y_1, y_2 \in V(Y) \text{ and every } x \in V(X).$$

This implies that the color classes $C_1, \ldots, C_n$ of the arbitrary n-coloring c of $X \times Y$ are given by the uniquely determined color classes $D_1, \ldots, D_n$ of X,

$$C_i = D_i \times Y, \text{ for } i = 1, \ldots, n.$$

This means that $X \times Y$ is uniquely n-colorable. □

In the following subsections we present some graph candidates for the application of Theorem 4.2.

4.1 Complete multipartite graphs

We call a graph X a complete m-partite graph if its vertex set $V(X)$ can be partitioned into m nonempty, disjoint subsets ('color classes') such that each vertex is adjacent to every vertex which is not in his own class. Obviously, these graphs are uniquely m-colorable and $2K_2$-free. If a complete m-partite graph is regular, then all color classes must have the same size k. Such a graph can be represented as a Cayley graph over $Z_m \times Z_k$.

Corollary 4.3. *Let X_i be a complete m_i-partite graph for $i = 1, \ldots, r$, $r \geq 2$, and $2 \leq m_1 \leq m_2 \ldots \leq m_r$. Then $X = X_1 \times X_2 \times \ldots \times X_r$ has chromatic number $\chi(X) = m_1$. The graph X is uniquely m_1-colorable if and only if $m_1 < m_2$.*

Proof. We have $\chi(X) \leq \min\{m_1, \ldots, m_r\} = m_1$. If we take one vertex from each color class of X_i we get a clique Q_i of size m_i in X_i. Assume that Q_i has vertex set $\{1, 2, \ldots, m_i\}$. Then the tuples $(a, a, \ldots, a)$ with the r-fold entry $a \in \{1, 2, \ldots, m_1\}$ define a clique of size m_1 in X. Thus we see $\chi(X) = m_1$.

If $m_1 < m_2$ we set $Y = X_2 \times \ldots \times X_r$. This graph is connected with $\chi(Y) = m_2 > m_1 = \chi(X_1)$. Therefore, we may apply Theorem 4.2 to the product $X_1 \times Y$ and conclude that it is uniquely m_1-colorable.

If $m_1 = m_2 = m$, let f_1 be an m-coloring of X_1 and f_2 be an m-coloring of X_2. The colorings of X induced by f_1 and by f_2 are distinct optimal colorings of X. $\square$

4.2 Complementary graphs of compass graphs

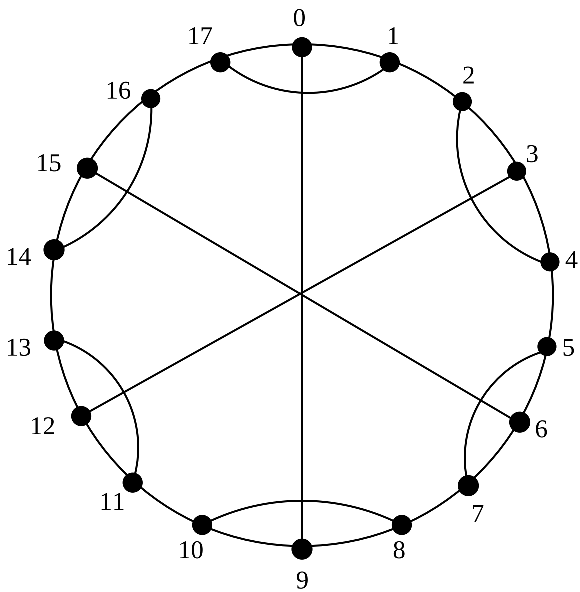

Figure 1

The compass graph $\mathrm{CS}(k, P)$ is regular of degree 3 and has $n = 6k$ vertices, $k \geq 2$. The vertices $0, 1, \ldots, n-1$ are arranged in this order along a hamiltonian cycle. Every vertex x divisible by 3 forms a triangle with the adjacent vertices $x \pm 1 \mod n$. By P we denote a partition of $Z_m = \{0, 1, \ldots, m-1\}$, $m = 2k$, in 2-element subsets which do not consist of two consecutive integers modulo m. For every $\{a, b\} \in P$ we connect the vertices $3a$ and $3b$ by an edge. Figure 1 displays $\mathrm{CS}(3, P)$ with $P = \{\{0, 3\}, \{1, 4\}, \{2, 5\}\}$.

Obviously, every compass graph $\mathrm{CS}(k, P)$ does not contain an induced cycle C_4 of length 4. This means for the complementary graph $\overline{\mathrm{CS}}(k, P)$ that it does not contain an induced $2K_2$. The maximal cliques of $\mathrm{CS}(k, P)$ are given by its triangles, which in $\overline{\mathrm{CS}}(k, P)$ define the maximal stable sets. To achieve an optimal coloring of $\overline{\mathrm{CS}}(k, P)$ we must take the sets of vertices $\{x, x-1, x+1 \mod n\}$, $x \equiv 0 \mod 3$, as color classes. The graph $\overline{\mathrm{CS}}(k, P)$ is uniquely $2k$-colorable. These graphs are candidates for the graph X in Theorem 4.2.

It seems to be difficult to decide generally which compass graphs are Cayley graphs. The graph in Figure 1 is the only Cayley compass graph with 18 vertices. Similarly, we found that there is a unique Cayley compass graph with 12, 24, 42, 48 or 54 vertices. But there is definitely no such graph with 30 or 36 vertices. Again, we found a compass graph with 60 vertices, which is a Cayley graph over the alternating group A_5. But we do not know if it is unique.

Infinite sequences of $2K_2$-free, uniquely colorable Cayley graphs can be constructed by the following operations. The k-fold join, $\mathrm{join}(k, G)$, of a graph G consists of k disjoint copies $G_1, \ldots, G_k$ of G. For every $i < j$ every vertex of G_i is connected by an edge to every vertex of G_j. Let the $n \times n$-matrix A be an adjacency matrix of G and J_k the $k \times k$-matrix with all entries equal to 1. The Kronecker product $J_k \times A$ is the $(kn) \times (kn)$-matrix which results from J_k by replacing every entry by A. The k-fold clone, $\mathrm{clone}(k, G)$, is the graph with adjacency matrix $J_k \times A$. We leave the proof of the following statement as an exercise for the reader.

Proposition 4.4. *If the Cayley graph G is $2K_2$-free and uniquely colorable then join(k, G) and clone(k, G) are $2K_2$-free, uniquely colorable Cayley graphs for every integer $k \geq 2$.*

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Non-negative spectrum of a digraph

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Abstract

Digraphs are considered by means of eigenvalues of the matrix AA^T, and similarly A^TA, where A is the adjacency matrix of a digraph. The common spectrum of these matrices is called *non-negative spectrum* or N*-spectrum* of a digraph. Several properties of the N-spectrum are proved. The notion of cospectrality is generalized, and some examples of cospectral (multi)(di)graphs are constructed.

Keywords: Digraph, non-negative spectrum, multigraph, cospectrality, isomorphism.

Math. Subj. Class.: 05C20

1 Introduction

Spectral (di)graph theory means usage of linear algebra tools and techniques in the study of (di)graphs. It is a very well developed mathematical field (see [8] or [6]) with many applications (see, for example, [2] and [15]).

For any (di)graph matrix M, one can build a spectral (di)graph subtheory, and then be able to study (di)graphs by means of eigenvalues of the matrix M. We will denote these eigenvalues M-eigenvalues. In general case, in order to avoid confusion, to any notion in the corresponding subtheory a prefix 'M' should be added. Frequently used graph matrices are the adjacency matrix A, the Laplacian $L = D - A$ and the signless Laplacian $Q = D + A$, where D is a diagonal matrix of vertex degrees. The spectral (di)graph theory then consolidates all these particular subtheories together with interaction tools.

In this paper, digraphs are considered by means of eigenvalues of the matrix AA^T, and similarly A^TA, where A is the adjacency matrix of a digraph. The common spectrum of these matrices is denoted N*-spectrum* and called *non-negative spectrum* of a digraph. According to [5], the N-spectrum of a digraph was not considered in the mathematical

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literature so far. Since the matrices AA^T and A^TA appear in applications (see, for example, [11] and [12]), we believe that introduced notion and presented results could be useful to mathematicians and informaticians. Namely, N-spectrum can facilitate the examination of digraphs since frequently used adjacency matrix of a digraph is not symmetric in general case, and therefore its spectrum consists of complex numbers. It is well known that digraphs serve as models for different processes and phenomena in computer sciences, where some spectrally based techniques are used in investigations. By this approach some new conclusions and comparisons of existing results could be made.

The paper is organized as follows:

In Section 2 basic digraph terminology is given and some elementary facts related to the matrices AA^T and A^TA and their spectrum are pointed out. Since this paper represents the first mathematical paper on the N-spectrum, elementary observations useful for further work are presented in Section 3. In Section 4 the effect of certain digraph operations and transformations on the N-spectrum is studied. One family of N-cospectral digraphs is determined in this section. Structural similarity (i.e. values and layout of entries in the matrix) of the matrix AA^T of some digraph with the adjacency or the signless Laplacian matrix of some multigraph, has motivated us to generalize the notion of cospectrality in Section 5. The study of cospectrality with respect to different (multi)(di)graph matrices could be useful in finding connections between different spectral subtheories that are based on these matrices, and, what is more important, in finding new pairs of cospectral (multi)(di)graphs in particular spectral subtheory. That way, certain pairs of multigraps that are cospectral with respect to the adjacency matrix are found. The study of spectral subtheory based on the signless Laplacian matrix is currently used (see, for example, [7]), so the paper is concluded with some examples of digraphs and multigraphs whose N- and Q-spectrum, respectively, are the same.

2 Preliminaries

Let $D = (V(D), E(D))$ be a *digraph of order* n with the set of *vertices* $V(D) = \{v_1, v_2, \ldots, v_n\}$. The set of *edges* $E(D)$ consists of ordered pairs of vertices, and we suppose that the *loops*, i.e. the edges of the form (v_i, v_i) are permitted, but multiple edges are not. The *adjacency matrix* $A = [a_{ij}]$ of D is the binary matrix of order n, such that $a_{ij} = 1$, if there is an edge from v_i to v_j, and otherwise $a_{ij} = 0$.

If $e = (v_i, v_j)$ is the edge of D, we say that v_i is the *initial vertex* of e, while v_j is the *terminal vertex*. The vertex $v_j \in V(D)$ is the *out-neighbour* of the vertex $v_i \in V(D)$ if there is the edge $(v_i, v_j) \in E(D)$. The vertex $v_k \in V(D)$ is the *in-neighbour* of the vertex $v_i \in V(D)$ if there is the edge $(v_k, v_i) \in E(D)$. The *out-degree* of vertex v_i, denoted by $outdeg_D(v_i)$ or $d_D^+(v_i)$, is the number of edges of which it is the initial vertex, while the *in-degree* of v_i, denoted by $indeg_D(v_i)$ or $d_D^-(v_i)$, is the number of edges of which v_i is the terminal vertex. A loop at some vertex contributes 1 to both the in-degree and the out-degree of that vertex.

Let us suppose that the edges of D are ordered as $e_1, e_2, \ldots, e_m$. The *in-incidence matrix* of D is the n by m matrix $B_{in} = [b_{ij}]$ such that $b_{ij} = 1$ if $e_j = (v_k, v_i)$ for some vertex v_k, and otherwise $b_{ij} = 0$. The *out-incidence* matrix $B_{out} = [g_{ij}]$ of the digraph D is the n by m matrix such that $g_{ij} = 1$ if $e_j = (v_i, v_l)$ for some vertex v_l, and otherwise $g_{ij} = 0$. It is a matter of routine to check that $A = B_{out}B_{in}^T$ holds.

The characteristic polynomial $\det(\lambda I - A)$ of A is the *characteristic polynomial* of the digraph D, and the eigenvalues of A are the *eigenvalues* of D. For the remaining notation and terminology related to digraphs, and also graphs, we refer the reader to [5], [2], [3], [1], [8] and [6].

In this paper we are interested in the structural characteristics of a digraph D related to the spectrum of matrices AA^T and A^TA, where A is the adjacency matrix of D. The matrices AA^T and A^TA are non-negative, square and symmetric. One can easily check that these matrices are positive semi-definite (see, for example, [14]), which means that their eigenvalues are non-negative.

The entries of the matrices AA^T and A^TA are characterised by the following proposition (see [12]):

Proposition 2.1. *The (i, j)-entry of the matrix AA^T (A^TA) of D is equal to the number of common out-neighbours (in-neighbours) of v_i and v_j. Diagonal entries of the matrix AA^T (A^TA) represent out-degrees (in-degrees) of the vertices of D.*

According to the previous observations, one can introduce the following notation: $N_{out} = AA^T$ and $N_{in} = A^TA$. The characteristic polynomial $\det(\lambda I - N_{in})$ of N_{in} is the N_{in}-*characteristic polynomial* of D, while the characteristic polynomial $\det(\lambda I - N_{out})$ of N_{out} is the N_{out}-*characteristic polynomial* of D. Since the spectrum of N_{out} and N_{in} is the same (see [14]), it can be denoted by the single name - the N-*spectrum*. Therefore, the characteristic polynomials $N(x)$ of these matrices can be named the N-*polynomials*. However, we underline that through the investigation we mainly considered $N_{out}(D)$ matrix of D, whose spectrum is denoted by $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_n$. The N-*spectral radius* $\rho_N(D)$ of D is defined to be the spectral radius of $N_{out}(D)$, and similarly $N_{in}(D)$.

Remark 2.2. For the N-spectrum $\eta_1, \eta_2, \ldots, \eta_n$ of a digraph D with m edges the following holds:

- The numbers $\eta_1, \eta_2, \ldots, \eta_n$ are real and non-negative,
- $\eta_1 + \eta_2 + \ldots + \eta_n = m$,
- D is consisted of only isolated vertices if and only if $\eta_1 = \eta_2 = \ldots = \eta_n = 0$.

3 Some basic results

In this section we give some elementary results that we will use in the subsequent sections.

Let us remind you that a digraph D is r-*regular* if the in-degree and the out-degree of each its vertex are equal to r. By use of the basic combinatorial principles for counting one can easily check that the row sum for each row of the matrix $N_{out}(D)$ is equal to $r + r(r-1) = r^2$. Now, we can prove the following lemma:

Lemma 3.1. *N-spectral radius $\rho_N(D)$ of a r-regular digraph D of order n is r^2.*

Proof. Since $N_{out}(D)$ is the square, non-negative matrix with equal row sums, according to Theorem of Frobenius (see [4]) the spectral radius of this matrix is r^2. □

Remark 3.2. The eigenvector that corresponds to the N-eigenvalue r^2 of a r-regular digraph D is all-1 vector.

Example 3.3. The complete digraph of order n is the digraph $\overset{\leftrightarrow}{K}_n$ in which for each pair of vertices there is an edge, including a loop at each vertex. The N-characteristic polynomial of this digraph is:

$$N_{\overset{\leftrightarrow}{K}_n}(x) = (x - n^2)x^{n-1},$$

and thus its N-spectrum is: n^2, $[0]^{n-1}$.

Here, and in the further text, an eigenvalue η of the multiplicity k is denoted by $[\eta]^k$.

Let us now consider connected digraphs whose vertices do not have the common out-neighbours. If $D = (V(D), E(D))$ is such a digraph, then $indeg_D(v_i) \leq 1$ must hold for each vertex $v_i \in V(D)$.

Let us remind you that a *rooted oriented tree*, briefly rooted tree, is an oriented tree with a specific vertex v_1, called the *root*, such that for every other vertex v_j the path connecting v_1 to v_j is a directed path from v_1 to v_j. This means that D is connected, $indeg(v_1) = 0$ and $indeg_D(v_i) = 1$ for every other vertex v_i of D, and vice versa according to Theorem 15.2 from [1]. It is obvious that vertices of a rooted tree do not have the common out-neighbours.

If in a digraph D whose vertices do not have the common out-neighbours there are at least two vertices such that their in-degrees are equal to 0, D would not have been connected, i.e. D would consist of at least two connected components.

Since in a rooted tree there is unique vertex v_1 such that $indeg(v_1) = 0$, one can add one extra edge to obtain a digraph where there is no pair of vertices with common out-neighbours. We distinguish two possibilities: this extra edge is a loop at v_1, i.e. (v_1, v_1) or it is an edge (v_x, v_1), for exactly one vertex v_x of a rooted tree. Hence, we can say that a resulting digraph is a unicyclic digraph derived from a rooted tree (Figure 1).

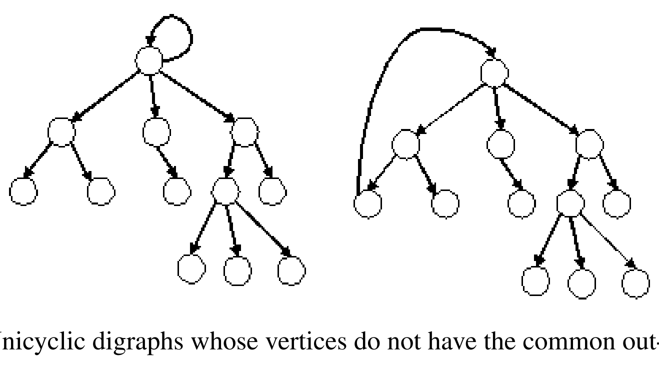

Figure 1: Unicyclic digraphs whose vertices do not have the common out-neighbours

That way, the following proposition is proved:

Proposition 3.4. *D is a connected digraph whose vertices do not have the common out-neighbours if and only if it is a rooted tree or a unicyclic digraph that can be derived from a rooted tree.*

Remark 3.5. Since the matrix $N_{out}(D)$ of a connected digraph D such that there is no pair of vertices with the common out-neighbours in D is the diagonal matrix of vertex degrees, the N-spectrum of D is: $outdeg_D(v_1), outdeg_D(v_2), \ldots, outdeg_D(v_n)$.

Remark 3.6. The converse digraph $Conv(D)$ of a digraph D is obtained by reversing the direction of each edge of D (see [2]). So, a digraph whose vertices do not have the common in-neighbours is the converse digraph of a rooted tree or of a unicyclic digraph that can be derived from a rooted tree.

Example 3.7. The N-characteristic polynomial of a rooted tree D is:

$$N_D(x) = x^l \prod_{v_i \in U(D)} (x - outdeg(v_i)),$$

where l is the number of vertices v_x such that $outdeg_D(v_x) = 0$, while $U(D) \subset V(D)$ is the set of vertices whose out-degree is at least 1.

The digraph $\overrightarrow{P_n}$ is the special case of a rooted oriented tree. If $V(\overrightarrow{P_n}) = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices of this digraph, then its set of edges consists of the pairs of vertices (v_i, v_{i+1}), for $i = 1, 2, \ldots, n-1$. The N-characteristic polynomial of $\overrightarrow{P_n}$ is:

$$N_{\overrightarrow{P_n}}(x) = x(x-1)^{n-1}.$$

1-regular digraph $\overrightarrow{C_n}$ is the special case of a unicyclic digraph derived from a rooted tree. Its N-characteristic polynomial is:

$$N_{\overrightarrow{C_n}}(x) = (x-1)^n. \tag{3.1}$$

4 Some digraph operations and transformations

We open this section with the result related to the N-spectrum of the complement of a given regular digraph.

The *complement* $D^C = (V(D^C), E(D^C))$ of a digraph $D = (V(D), E(D))$ has the vertex set $V(D^C) = V(D)$ and $e \in E(D^C)$ if and only if $e \notin E(D)$. Also, there is a loop at vertex v_i in D^C if and only if there is no loop at v_i in D. Similarly to the proof of Theorem 2.1.2 from [6] for regular graphs we can prove the following:

Proposition 4.1. *If the N-eigenvalues of a r-regular digraph D of order n are $\eta_i(D)$, $i = 1, 2, \ldots, n$, then the N-eigenvalues of D^C are $\eta_1(D^C) = (n-r)^2$ and $\eta_i(D^C) = \eta_i(D)$, $i = 2, 3, \ldots, n$.*

Proof. If A_D is the adjacency matrix of D and J is all-1 matrix, we find:

$$N_{out}(D^C) = J^2 - A_D J - J A_D^T + A_D A_D^T = (n-2r)J + N_{out}(D),$$

because the row sum for each row of A_D is equal to r. □

Let us denote by $D^{'}$ the digraph obtained from a connected digraph D by deleting the edge (v_i, v_j). Then we have: $N_{out}(D) = N_{out}(D^{'}) + M$. Here, $M = [m_{pq}]$ is the square matrix of order n such that $m_{ii} = 1$ and $m_{il} = m_{li} = 1$ for each pair of vertices v_i, v_l such that $(v_i, v_j), (v_l, v_j) \in E(D)$, where $l \in \{1, 2, \ldots, n\} \setminus \{i\}$.

Theorem 4.2. (Interlacing theorem - edge version) *Let D be a connected digraph of order n whose N-spectrum is $\eta_1(D) \geq \eta_2(D) \geq \cdots \geq \eta_n(D)$, and there is at least one* *vertex v_j in D such that $indeg_D(v_j) = 1$. Let $D^{'}$ be a digraph obtained from D by deleting an edge (v_i, v_j). If the N-eigenvalues of $D^{'}$ are $\eta_1(D^{'}) \geq \eta_2(D^{'}) \geq \cdots \geq \eta_n(D^{'})$, then*

$$\eta_1(D) \geq \eta_1(D^{'}) \geq \eta_2(D) \geq \eta_2(D^{'}) \geq \ldots \eta_n(D) \geq \eta_n(D^{'}) \geq 0.$$

Proof. Since the spectrum of the matrix M consists of $[1]$ and $[0]^{n-1}$, the proof follows from Courant-Weyl inequalities (see, for example [6]). □

Remark 4.3. By considering N_{in} matrix of a digraph, one can prove that the previously given Interlacing theorem holds also for a connected digraph D in which there is at least one vertex v_j such that $outdeg_D(v_j) = 1$, and for its subdigraph $D^{'}$ obtained from D by deleting an edge (v_j, v_i), for some vertex v_i.

In general case, such the N-eigenvalue interlacing does not hold. Namely, we have the following example.

Example 4.4. For the digraph D that is depicted on Figure 2 and the digraph $D^{'}$ that is obtained from D by deleting the edge $(1, 3)$, the N-interlacing property holds, i.e. for the N-spectra of these digraphs we have the following inequalities: $4.390 \geq 3.879 \geq 1.838 \geq 1.653 \geq 1 \geq 1 \geq 0.544 \geq 0.468 \geq 0.228 \geq 0$.

On the other hand, the N-eigenvalues of the digraphs D_1 (Figure 2) and $D_1^{'}$, that is obtained from D_1 by deleting the edge $(1, 3)$, are $5.303 \geq 1.697 \geq 1 \geq 1 \geq 0$, and similarly $4.115 \geq 1.764 \geq 1 \geq 1 \geq 0.139$, so the N-interlacing property does not hold in this case.

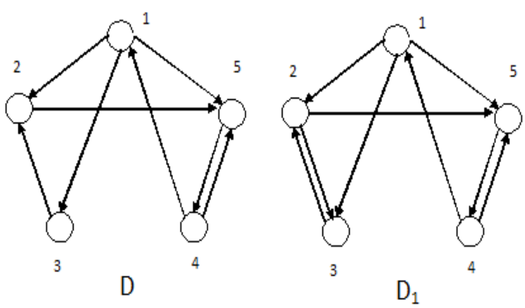

Figure 2: Digraphs D and D_1 from Example 4.4

Now, we will consider a digraph D^* obtained from a connected digraph D by adding a *pendant edge* at the vertex v_i of D (i.e. an edge of the form (v_x, v_i) such that $indeg_{D^*}(v_x) = 0$ and $outdeg_{D^*}(v_x) = 1$, or an edge of the form (v_i, v_x) such that $indeg_{D^*}(v_x) = 1$ and $outdeg_{D^*}(v_x) = 0$).

The following statement obviously holds.

Proposition 4.5. *Let D^* denotes a digraph obtained from a connected digraph D of order n by adding a pendant edge (v_{n+1}, v_i) at the vertex v_i such that $indeg_D(v_i) = 0$. Then the N-characteristic polynomial of D^* is: $N_{D^*}(x) = (x - 1)N_D(x)$.* □

Let us denote by D_{v_k} a digraph obtained from a digraph D by deleting the vertex v_k, and let $\mu_D(v_i, v_j) = 1$, if $(v_i, v_j) \in E(D)$, and otherwise $\mu_D(v_i, v_j) = 0$, for $i, j \in \{1, 2, \ldots, n\}$.

Definition 4.6. The digraph $D^{out}_{(v_k,v_i)}$ is the out-(v_k,v_i)-shrinking of D if for the edge (v_k,v_i) in $E(D)$, $V(D^{out}_{(v_k,v_i)})=V(D_{v_k})$ and

$$E(D^{out}_{(v_k,v_i)})=E(D_{v_k})\cup\{(v_j,v_i)|\mu_D(v_j,v_k)=1,\ \text{for each } j\neq k\}.$$

It is obvious that $D^{out}_{(v_k,v_i)}$ is a multidigraph in general case, and that if $indeg_D(v_i)=1$ then the matrix $N_{out}(D^{out}_{(v_k,v_i)})$ equals the matrix obtained from $N_{out}(D)$ by deleting the k-th row and the k-th column.

Theorem 4.7. *Let D^* denotes a digraph obtained from a connected digraph D of order n by adding the pendant edge (v_j,v_i) at the vertex v_i such that $(v_k,v_i)\in E(D)$ and $indeg_D(v_i)=1$. Then*

$$N_{D^*}(x)=(x-1)N_D(x)-N_{D^{out}_{(v_k,v_i)}}(x),$$

where $N_{D^{out}_{(v_k,v_i)}}(x)$ is the N-characteristic polynomial of the digraph $D^{out}_{(v_k,v_i)}$ that is the out-(v_k,v_i)-shrinking of a digraph D.

Proof. Since $indeg_{D^*}(v_i)=2$, we have

$$N_{out}(D^*)=\begin{pmatrix} N_{out}(D) & r\\ r^T & 1\end{pmatrix}_{(n+1)\times(n+1)},$$

where $r=(0,\ldots,0,\underbrace{1}_{k},0,\ldots,0)^T$ is the vector of order n. The only no null coordinate of the vector r corresponds to the common out-neighbour of v_k and v_j. By expanding the determinant of the matrix $xI-N_{out}(D^*)$ by the last row we get:

$$N_{D^*}(x)=\det\left(xI-N_{out}(D^*)\right)=(x-1)N_D(x)+(-1)^{(n+1)+k}\cdot\det\left(M|r\right),$$

where the matrix M is obtained from $xI-N_{out}(D)$ by deleting the k-th column. Now, by expanding the determinant of the matrix $(M|r)$ by the last column, we have:

$$\begin{aligned}N_{D^*}(x)=&(x-1)N_D(x)+(-1)^{(n+1)+k}(-1)^{k+n}\det\left(xI-M^{'}\right)=\\&(x-1)N_D(x)-\det\left(xI-M^{'}\right),\end{aligned}$$

where $M^{'}$ is obtained from the matix $N_{out}(D)$ by deleting the k-th row and k-th column. □

The *line digraph* $L(D)$ of a digraph D (see, for example [5]) is the digraph whose vertices are the edges $e_1,e_2,\ldots,e_m$ of D such that there is an edge from e_i to e_j in $L(D)$ if and only if the terminal vertex of e_i equals the initial vertex of e_j in D. If an edge e_p is a loop at some vertex of D, then it becomes a loop at e_p in $L(D)$.

Some results on adjacency spectra and energies of iterated line graphs are exposed in [13]. On the similar way, we can define iterated line digraphs. If $D=L^0(D)$ is a digraph and $L(D)=L^1(D)$ is its line digraph, then $L^k(D)$, $k=2,3,\ldots$ defined recursively by the formula $L^k(D)=L\left(L^{k-1}(D)\right)$ are the *iterated line digraphs* of D. The line digraph of an r-regular digraph is also r-regular digraph. More precisely, the line digraph $L^1(D)$ of an r-regular digraph D of order n is the $r_1=r$-regular digraph of order $n_1=nr$. Consequently, $L^k(D)$, $k=2,3,\ldots$ is the $r_k=r$-regular digraph of order $n_k=rn_{k-1}=r^kn$, where n_{k-1} is the order of the digraph $L^{k-1}(D)$.

Theorem 4.8. *The N-eigenvalues of the line digraph $L(D)$ of a r-regular digraph D are:* $[r^2]^n$, $[0]^{(r-1)n}$.

Proof. We will determine the N-characteristic polynomial $N_{L(D)}$ of $L(D)$ related to the $N_{out}(L(D))$ matrix.

As $L = B_{in}^T B_{out}$ is the adjacency matrix of the line digraph $L(D)$ of D (see [5]), where B_{in} and B_{out} are the in-incidence matrix and the out-incidence matrix of D, respectively, we find: $N_{out}(L(D)) = rB_{in}^T B_{in}$. Here, we have that the diagonal matrix whose entries are the out(in)-degrees of vertices in D is: $\Delta = rI = B_{in}B_{in}^T = B_{out}B_{out}^T$.

According to Lemma 8.2.3. from [10] we get:

$$\det\left(I - B_{in}B_{in}^T\right) = \det\left(I - B_{in}^T B_{in}\right),$$

i.e.

$$\det\left(I_n - x^{-1} r I_n\right) = \det\left(I_m - x^{-1}\frac{1}{r}N_{out}(L(D))\right).$$

Furthermore we have:

$$x^{m-n}\det\left(xI_n - rI_n\right) = \det\left(xI_m - \frac{1}{r}N_{out}(L(D))\right),$$

and also

$$\det\left((x-r+1)I_n - I_n\right) = x^{n-m}\frac{1}{r^m}\det\left(rxI_m - N_{out}(L(D))\right).$$

According to (3.1) we find:

$$N_{\vec{C}_n}(x-r+1) = x^{n-m}\frac{1}{r^m}N_{L(D)}(rx),$$

i.e.

$$N_{L(D)}(x) = x^{m-n}(x-r^2)^n,$$

and the proof follows. □

Therefore the N-spectrum of the line digraph $L^k(D)$ of a r-regular digraph D of order n consists of $\left[r^2\right]^{n_k} = \left[r^2\right]^{nr^k}$ and $[0]^{(r-1)n_k} = [0]^{(r-1)r^k n}$, and hence we have the following corollary:

Corollary 4.9. *Let D_1 and D_2 be two r-regular digraphs of order n (not necessary N-cospectral). Then for all $k \geq 1$ digraphs $L^k(D_1)$ and $L^k(D_2)$ are N-cospectral.*

This way, we found a family of N-cospectral mates (i.e. the digraphs whose N-spectra are the same). We will continue examination of cospectrality in the next section.

5 Cospectrality relation

Let $\mathcal{D}_M^n$ be the set of (multi)(di)graphs D of order n with the associated spectrum $\sigma_M(D)$ related to some (multi)(di)graph matrix M. Let us introduce the relation $\rho \subseteq \mathcal{D}_{M_1}^n \times \mathcal{D}_{M_2}^n$ between sets $\mathcal{D}_{M_1}^n$ and $\mathcal{D}_{M_2}^n$, for some (multi)(di)graph matrices M_1 and M_2 in the following way: we say that the (multi)(di)graph D_1 is in the relation ρ with the (multi)(di)graph D_2, i.e. $D_1 \rho D_2$ if and only if $\sigma_{M_1}(D_1) = \sigma_{M_2}(D_2)$. So, the relation ρ is the *cospectrality relation*, while D_1 and D_2 form an (M_1, M_2)-*cospectral mate*. That way, we can generalize the notion of cospectrality:

Definition 5.1. Let M_1 and M_2 be some (multi)(di)graph matrices. If the (multi)(di)graph $D_1 \in \mathcal{D}^n_{M_1}$ is in the cospectrality relation ρ with the (multi)(di)graph $D_2 \in \mathcal{D}^n_{M_2}$, i.e. the M_1-spectrum of a (multi)(di)graph D_1 is equal to the M_2-spectrum of a (multi)(di)graph D_2, then D_1 and D_2 are called (M_1, M_2)-cospectral (multi)(di)graphs.

It is obvious that ρ is the equivalence relation on the set $\mathcal{D}^n_M$, in which case (multi)(di)-graphs D_1 and D_2 such that $D_1 \rho D_2$ are M*-cospectral.* As a result of the composition of the cospectrality relations, one can get some new pairs of cospectral (multi)(di)graphs, as follows.

Let us consider the set $\mathcal{D}^n_N$ of digraphs D of order n with the associated N-spectrum $\sigma_N(D)$. Clearly, N is related to N_{out} or N_{in} matrix of a digraph. Let us denote by $\mathcal{G}^n_{A+}$ and $\mathcal{G}^n_{A-}$ the sets of *out-multigraphs* and *in-multigraphs*, respectively with the corresponding adjacency spectra. The in-multigraph $M^-_D \in \mathcal{G}^n_{A-}$ and the out-multigraph $M^+_D \in \mathcal{G}^n_{A+}$ are associated to a digraph $D \in \mathcal{D}^n_N$ in the following way:

Definition 5.2. The in-multigraph $M^-_D = (V(M^-_D), E(M^-_D))$ of a digraph D is the multigraph such that $V(M^-_D) = V(D)$, $\{v_i, v_j\} \in E(M^-_D)$ if and only if there is a vertex $v_k \in V(D)$ such that $(v_k, v_i),\ (v_k, v_j) \in E(D)$, and for each edge (v_k, v_i) in D there is a loop at v_i in M^-_D.

Definition 5.3. The out-multigraph $M^+_D = (V(M^+_D), E(M^+_D))$ of a digraph D is the multigraph such that $V(M^+_D) = V(D)$, $\{v_i, v_j\} \in E(M^+_D)$ if and only if there is a vertex v_k such that $(v_i, v_k),\ (v_j, v_k) \in E(D)$, and for each edge (v_i, v_k) in D there is a loop at v_i in M^+_D.

According to the previous definitions, one can notice the cospectrality relation, say ρ_-, between sets $\mathcal{G}^n_{A-}$ and $\mathcal{D}^n_N$, and similarly the cospectrality relation, say ρ_+, between sets $\mathcal{D}^n_N$ and $\mathcal{G}^n_{A+}$. As the result of the composition of relations ρ_+ and ρ_- the pairs of A-cospectral multigraphs M^-_D and M^+_D are getting. That way we have:

Theorem 5.4. *Multigraphs M^-_D and M^+_D are A-cospectral.*

So, the exposed construction is a way for obtaining new pairs of cospectral and not necessarily isomorphic multigraphs.

Example 5.5. The adjacency matrix of the in-multigraph M^-_D, and similarly the out-multigraph M^+_D, that is associated to the digraph D (which is depicted on Figure 3) is:

$$A(M^-_D) = N_{in}(D) = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and } A(M^+_D) = N_{out}(D) = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix}.$$

Remark 5.6. Multigraphs M^-_D and M^+_D associated to a digraph D are simple graphs only in the case when digraph D is a set of isolated vertices. If we permit existence of single loops (i.e. loops of multiplicity one) in a simple graph, the primary digraph D can be $\overrightarrow{C_n}$ or $\overrightarrow{P_n}$. In this case, multigraphs M^-_D and M^+_D are the sets of isolated loops or the disjoint unions of isolated loops and a single isolated vertex, and therefore M^-_D and M^+_D are not only A-cospectral but also isomorphic.

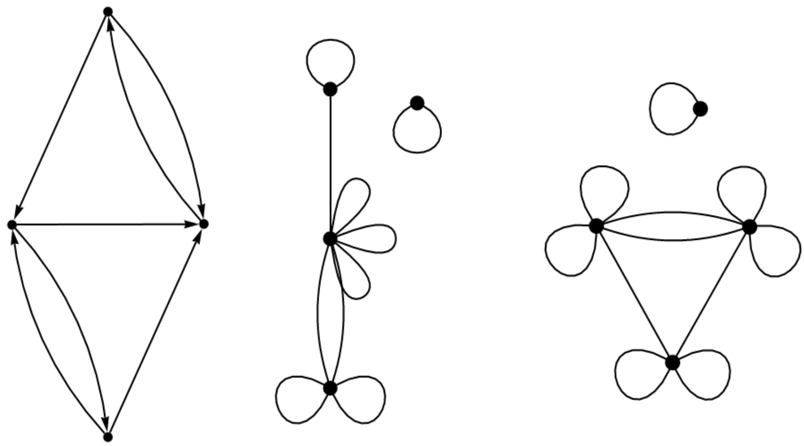

Figure 3: Digraph D from Example 5.5 and associated multigraphs M_D^- and M_D^+, respectively

There are many examples where the multigraphs M_D^- and M_D^+ associated to a given digraph D are isomorphic, so the investigation of such multigraphs can be the subject of future research. If a primary digraph D is such that if $(v_i, v_j) \in E(D)$ then also $(v_j, v_i) \in E(D)$, for all $v_i,\ v_j \in V(D)$, it is obvious that the associated multigraphs M_D^- and M_D^+ will be isomorphic. We also have:

Proposition 5.7. *Multigraphs M_D^- and M_D^+ associated to a digraph D of prime order, $n > 2$, with circulant adjacency matrix are isomorphic.*

Proof. Since $N_{in}(D)$ and $N_{out}(D)$ are circulant matrices with the same eigenvalues, according to Theorem 1 from [9] they are permutationally similar. □

For an integer $n \geq 2$ and a set $S \subseteq \{1, 2, \dots, n-1\}$ the circulant digraph $C_n(S)$ is a digraph such that $V(C_n(S)) = \{1, 2, \dots, n\}$ and $E(C_n(S)) = \{(i, i + j \,(\mathrm{mod}\, n)) : 1 \leq i \leq n,\ j \in S\}$. Circulant digraphs are of great interest in the graph and digraph theory and their applications (see [2]).

Proposition 5.8. *Multigraphs M_D^- and M_D^+ associated to a circulant digraph $C_n(S)$ are isomorphic.*

Proof. Since the converse digraph $Conv(C_n(S))$ of $C_n(S)$ is isomorphic to $C_n(S)$ (according to Proposition 2.14.1 from [2]) and since $N_{in}(C_n(S)) = N_{out}(Conv(C_n(S)))$, and similarly $N_{out}(C_n(S)) = N_{in}(Conv(C_n(S)))$, the proof follows. □

Example 5.9. The matrix $N_{out}(D)$ of the 2-regular digraph D that is depicted on Figure 4 structurally corresponds to the signless Laplacian matrix $Q(M)$ of the 2-regular graph M, also depicted on Figure 4, i.e.

$$N_{out}(D) = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = Q(M).$$

That way, one can notice the cospectrality relation $\rho \subseteq \mathcal{D}_N^n \times \mathcal{G}_Q^n$ between set $\mathcal{D}_N^n$ of digraphs D of order n with the associated N-spectrum $\sigma_N(D)$ and the set $\mathcal{G}_Q^n$ of multigraphs M of order n with the associated Q-spectrum $\sigma_Q(M)$.

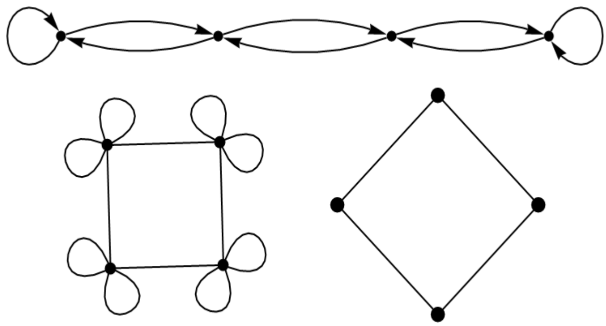

Figure 4: Triplet of (N, A, Q)-cospectral digraph D, multigraph $M_D^- = M_D^+$ and graph M, respectively

This one and similar examples have motivated us to examine some new (N, Q)-cospectral mates. Furthermore, the multigraph M that makes (Q, N)-cospectral mate with a given digraph D can be used in determining some isomorphic multigraphs M_D^- and M_D^+, as follows:

Proposition 5.10. *Let D be a connected r-regular digraph of order n. If $N_{out}(D) = Q(M)$ holds for some multigraph M, then $r = 0$ or $r = 2$.*

Proof. We have $N_{out}(D) = rI + C$, where row sum of C is $r(r-1)$ for each row.

If $N_{out}(D)$ is the signless Laplacian matrix of some multigraph without loops, then $r = r(r-1)$ holds, which implies $r = 0$ or $r = 2$. On the other hand, if $N_{out}(D)$ is the signless Laplacian matrix of a multigraph with loops, then the number of loops at some vertex is $(r - (r-1)r)/2$, which means that $r = 0$ or $r = 2$. $\square$

Remark 5.11. The statement from the previous proposition also holds in the case of the matrix $N_{in}(D)$. Beside that, the multigraph M is the connected r-regular multigraph without loops. Therefore, we conclude that multigraphs M_D^- and M_D^+ associated to some 2-regular digraph D are isomorphic.

In order to examine (N, Q)-cospectrality, we will introduce some binary digraph operations. Still, according to the nature and the mutual relationships between entries of matrices $N_{out}(D)$ and $Q(M)$ of some digraph D and some multigraph M, respectively, one can suspect poor variety in terms of the structure and the order (i.e. number of vertices) of the (N, Q)-cospectral mates (that could be obtained by direct comparing of these matrices).

Let $D_1 = (V(D_1), E(D_1))$ and $D_2 = (V(D_2), E(D_2))$ be two disjoint digraphs (i.e. digraphs with no common vertices nor edges).

Definition 5.12. The out-join $D_1 \nabla_{out} D_2$ of two disjoint digraphs $D_1 = (V(D_1), E(D_1))$ and $D_2 = (V(D_2), E(D_2))$ is the digraph $D = (V(D), E(D))$ such that $V(D) =$

$V(D_1) \cup V(D_2)$ and $E(D) = E(D_1) \cup E(D_2) \cup \{(u,v)|u \in V(D_1), v \in V(D_2)\}$, for each $u \in V(D_1)$ and $v \in V(D_2)$.

It is obvious that this digraph operation is not commutative, i.e. $D_1 \nabla_{out} D_2 \neq D_2 \nabla_{out} D_1$. $N_{out}(D)$ matrix of the digraph D which is obtained by out-joining is:

$$N_{out}(D) = N_{out}(D_1 \nabla_{out} D_2) = \begin{pmatrix} A_1 & J \\ O & A_2 \end{pmatrix} \begin{pmatrix} A_1^T & O^T \\ J^T & A_2^T \end{pmatrix} = \begin{pmatrix} N_{out}(D_1) + JJ^T & (A_2 J^T)^T \\ A_2 J^T & N_{out}(D_2) \end{pmatrix},$$

where A_1 and A_2 are the adjacency matrices of digraphs D_1 and D_2, respectively, while J is all-1 matrix. Each entry of the j-th row of the matrix $A_2 J^T$ is equal to $outdeg_{D_2}(u_j)$, where $u_j \in V(D_2)$.

In the same way one can define:

Definition 5.13. The in-join $D_1 \nabla_{in} D_2$ of two disjoint digraphs $D_1 = (V(D_1), E(D_1))$ and $D_2 = (V(D_2), E(D_2))$ is the digraph $D = (V(D), E(D))$ such that $V(D) = V(D_1) \cup V(D_2)$ and $E(D) = E(D_1) \cup E(D_2) \cup \{(v,u)|v \in V(D_2), u \in V(D_1)\}$, for each $u \in V(D_1)$ and $v \in V(D_2)$.

Definition 5.14. The join $D_1 \nabla D_2$ of two disjoint digraphs $D_1 = (V(D_1), E(D_1))$ and $D_2 = (V(D_2), E(D_2))$ is the digraph D with the vertex set $V(D) = V(D_1) \cup V(D_2)$, whose set of edges is $E(D) = (E(D_1 \nabla_{out} D_2) \cup E(D_1 \nabla_{in} D_2)) \setminus (E(D_1) \cup E(D_2))$.

Proposition 5.15. *Let $D = D_1 \nabla_{out} D_2$ be the digraph obtained by out-joining two connected disjoint digraphs D_1 and D_2 of orders n_1 and n_2, respectively. If $N_{out}(D) = Q(M)$ holds for some multigraph M, then:*

1. *D_1 is an isolated vertex, while D_2 is a unicyclic digraph derived from a rooted tree.*
2. *$D_1 = \overset{\leftrightarrow}{K}_1$, while D_2 is a rooted tree;*
3. *D_2 is an isolated vertex, and:*
   - *(a) if $n_1 = 1$, then $D_1 = \overset{\leftrightarrow}{K}_1$,*
   - *(b) if $n_1 = 2$, then D_1 is any of digraphs depicted on Figure 5,*
   - *(c) if $n_1 = 3$, then D_1 is 1-regular digraph,*
   - *(d) if $n_1 \geq 4$, then there is no digraph D_1 such that the statement given by the proposition holds.*

Proof. Let us denote by $V(D_1) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(D_2) = \{v_1, v_2, \ldots, v_{n_2}\}$ the sets of vertices of digraphs D_1 and D_2, respectively.

If $N_{out}(D) = [n_{ij}]$ is the signless Laplacian matrix of some multigraph M, then by observing its rows $n_1+1, n_1+2, \ldots, n_1+n_2$, one can conclude that the number:

$$(1-n_1)\, outdeg_{D_2}(v_p) - \sum_{q=1, q\neq p}^{n_2} n_{pq}(N_{out}(D_2)),$$

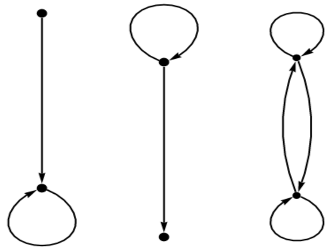

Figure 5: Digraphs from Proposition 5.15

for each $p = 1, 2, \ldots, n_2$, is zero or even positive integer. This means that $n_1 = 1$ and D_2 is a digraph such that there are no vertices with the common out-neighbours or D_2 is an isolated vertex.

In the former case, by observing rows $1, 2, \ldots, n_1$ of $N_{out}(D)$, one concludes that:

$$outdeg_{D_1}(u_k) + n_2 - m_2,$$

for each $k = 1, 2, \ldots, n_1$, is zero or even positive integer. Here m_2 is the number of edges of D_2, and the proof for statements 1. and 2. follows.

If D_2 is an isolated vertex, then by observing rows $1, 2, \ldots, n_1$ of $N_{out}(D)$, we get that:

$$outdeg_{D_1}(u_k) - \sum_{l=1, l\neq k}^{n_1} n_{kl}(N_{out}(D_1)) - n_1 + 2, \tag{5.1}$$

for each $k = 1, 2, \ldots, n_1$, is zero or even positive integer. Let us consider the structure of D_1.

If $n_1 = 1$ or $n_1 = 2$, statements (a) and (b) follows from (5.1) by direct computation.

If $n_1 = 3$, then $3 \geq outdeg_{D_1}(u_k) \geq 1$ must hold for each $k = 1, 2, 3$. Let us suppose that $outdeg_{D_1}(u_1) = 3$. This implies $indeg_{D_1}(u_1) = indeg_{D_1}(u_2) = indeg_{D_1}(u_3) = 1$, and since the out-degree of u_2 and u_3 must be at least 1, (5.1) will be a negative number for at least one k. One can analyse the case when $outdeg_{D_1}(u_1) = 2$ the same way. And finally, if $outdeg_{D_1}(u_1) = 1$, (5.1) is a non-negative integer if and only if $\sum_{l=2}^{3} n_{1l}(N_{out}(D_1)) = 0$. Since the out-degree of each vertex in D_1 must be at least 1, D_1 is 1-regular digraph.

Now, we will prove that there is no digraph D_{n_1} of order $n_1 \geq 4$ such that (5.1) is zero or even positive integer. The proof will be carried out by use of the mathematical induction on the number of vertices n_1 of D_{n_1}.

If $n_1 = 4$, analogously as in the case when $n_1 = 3$, one can show that there is at least one vertex, for example u_k, in D_4 such that $outdeg_{D_4}(u_k) < \sum_{l=1, l\neq k}^{4} n_{kl}(N_{out}(D_4)) + 2$, where $k \in \{1, 2, 3, 4\}$. Let us suppose that in a digraph D_s of order $s > 4$ there is at least one vertex such that (5.1) is a negative number. Let us consider a digraph D_{s+1} of order $s + 1$. By deleting an arbitrary vertex of D_{s+1} we get a digraph D_s of order s, where, according to the inductive hypothesis, we can find at least one vertex, say u_x, such that

$$outdeg_{D_s}(u_x) < \sum_{q=1, q\neq p}^{s} n_{xq}(N_{out}(D_s)) + s - 2.$$

If we return the removed vertex and all edges that are incident to it, we get the following inequalities:

$$outdeg_{D_{s+1}}(u_x) \leq outdeg_{D_s}(u_x) + 1 <$$

$$\sum_{q=1,q\neq i}^{s} n_{xq}(N_{out}(D_s)) + s - 2 + 1 \leq \sum_{q=1,q\neq p}^{s+1} n_{xq}(N_{out}(D_{s+1})) + s - 1.$$

Hence, according to the principle of the mathematical induction, when D_2 is an isolated vertex, there is no digraph D_1 of order $n_1 \geq 4$ such that $N_{out}(D) = N_{out}(D_1 \nabla_{out} D_2) = Q(M)$. □

Proposition 5.16. *Let $D = D_1 \nabla D_2$ be the digraph obtained by joining two connected disjoint digraphs D_1 and D_2 of orders n_1 and n_2, respectively. If $N_{out}(D)$ is the signless Laplacian matrix of some multigraph, then:*

1. *D_1 is an isolated vertex, while D_2 is any of digraphs depicted on Figure 6;*
2. *$D_1 = D_2 = \overset{\leftrightarrow}{K}_1$;*
3. *there are no digraphs D_1 and D_2 of orders $n_1, n_2 \geq 3$ such that the statement given by the proposition holds.*

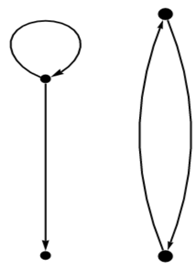

Figure 6: Digraphs from Proposition 5.16

Proof. Let us denote by $V(D_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(D_2) = \{u_1, u_2, \ldots, u_{n_2}\}$ the sets of vertices of digraphs D_1 and D_2, respectively. We have:

$$N_{out}(D) = N_{out}(D_1 \nabla D_2) = \begin{pmatrix} A_1 & J^T \\ J & A_2 \end{pmatrix} \begin{pmatrix} A_1^T & J^T \\ J & A_2^T \end{pmatrix} =$$

$$\begin{pmatrix} N_{out}(D_1) + J^T J & A_1 J^T + J^T A_2^T \\ (A_1 J^T + J^T A_2^T)^T & N_{out}(D_2) + J J^T \end{pmatrix},$$

where A_1 and A_2 are the adjacency matrices of digraphs D_1 and D_2, respectively.

If $N_{out}(D) = [n_{ij}]$ is the signless Laplacian matrix of some multigraph, we have:

$$(1 - n_2)\, outdeg_{D_1}(v_i) + (2 - n_1)\, n_2 - \sum_{j=1,j\neq i}^{n_1} n_{ij}(N_{out}(D_1)) - m_2 = 2w_1, \quad (5.2)$$

for some non-negative integer w_1 and $i = 1, 2, \ldots, n_1$, and

$$(1 - n_1)\, outdeg_{D_2}(u_k) + n_1\,(2 - n_2) - \sum_{l=1,l\neq k}^{n_2} n_{ij}(N_{out}(D_2)) - m_1 = 2w_2, \quad (5.3)$$

for some non-negative integer w_2 and $k = 1, 2, \ldots, n_2$, where m_1 and m_2 are the numbers of edges of digraphs D_1 and D_2, respectively.

First, let us prove that $n_1 < 3$. Since (5.2) means that:

$$(1 - n_2)\, outdeg_{D_1}(v_i) \geq (n_1 - 2)\, n_2 + \sum_{j=1, j\neq i}^{n_1} n_{ij}(N_{out}(D_1)) + m_2$$

holds for each $i = 1, 2, \ldots, n_1$, if we suppose that $n_1 \geq 3$, we get:

$$0 \geq 1 + \sum_{j=1, j\neq i}^{n_1} n_{ij}(N_{out}(D_1)) + m_2,$$

that is a contradiction. In the same way, one can prove that $n_2 < 3$.

Statements 1. and 2. from the proposition one can get by direct analysis of (5.2) and (5.3). $\square$

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Cycle bases of reduced powers of graphs

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Abstract

We define what appears to be a new construction. Given a graph G and a positive integer k, the *reduced kth power of* G, denoted $G^{(k)}$, is the configuration space in which k indistinguishable tokens are placed on the vertices of G, so that any vertex can hold up to k tokens. Two configurations are adjacent if one can be transformed to the other by moving a single token along an edge to an adjacent vertex. We present propositions related to the structural properties of reduced graph powers and, most significantly, provide a construction of minimum cycle bases of $G^{(k)}$.

The minimum cycle basis construction is an interesting combinatorial problem that is also useful in applications involving configuration spaces. For example, if G is the state-transition graph of a Markov chain model of a stochastic automaton, the reduced power $G^{(k)}$ is the state-transition graph for k identical (but not necessarily independent) automata. We show how the minimum cycle basis construction of $G^{(k)}$ may be used to confirm that state-dependent coupling of automata does not violate the principle of microscopic reversibility, as required in physical and chemical applications.

Keywords: Graph products, Markov chains, cycle spaces.

Math. Subj. Class.: 05C76, 60J27

1 Introduction

Time-homogenous Markov chains [19] are used as a mathematical formalism in applications as diverse as computer systems performance analysis [21], queuing theory in operations research [18], simulation and analysis of stochastic chemical kinetics [12], and biophysical modeling of ion channel gating [10].

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Many properties of a Markov chain, such its rate of mixing and its steady-state probability distribution, can be numerically calculated using its transition matrix [24]. A continuous-time Markov chain $X(t)$ ($t \geq 0$) with a finite number of states $\{1, \ldots, \eta\}$ is defined by an initial probability distribution, $\pi_i(0) = \Pr\{X(0) = i\}$, and a transition matrix $Q = (q_{ij})$ where $1 \leq i, j \leq \eta$, $q_{ij} \geq 0$ for $i \neq j$ and $q_{ii} = -\sum_{j \neq i} q_{ij}$, so called because, for $i \neq j$, $q_{ij} = \lim_{dt \to 0} \Pr\{X(t + dt) = j | X(t) = i\}/dt$. The requirement that Q has zero row sums, $\sum_j q_{ij} = 0$, corresponds to conservation of probability, $\sum_i \pi_i(t) = 1$, in the ordinary differential equation initial value problem, $d\boldsymbol{\pi}/dt = \boldsymbol{\pi} Q$ with initial condition $\boldsymbol{\pi}(0)$, solved by the time-dependent discrete probability distribution $\boldsymbol{\pi}(t) = (\pi_1(t), \ldots, \pi_\eta(t))$ where $\pi_i(t) = \Pr\{X(t) = i\}$.

A CTMC with a single communicating class of $\eta < \infty$ states is irreducible, positive recurrent, and has a unique steady-state probability distribution that solves $\bar{\boldsymbol{\pi}} Q = \mathbf{0}$ subject to $\sum_i \bar{\pi}_i = 1$ (by the Perron-Frobenius theorem). The Perron vector and steady-state distribution $\bar{\boldsymbol{\pi}}$ is the limiting probability distribution of the Markov chain, $\lim_{t \to \infty} \|\boldsymbol{\pi}(t) - \bar{\boldsymbol{\pi}}\| = 0$, for any initial condition satisfying conservation of probability, $\sum_i \pi_i(0) = 1$. In general, the calculation of steady-state distributions and other properties for Markov chains with η states requires algorithms of $\mathcal{O}(\eta^3)$ complexity.

Many open questions in the physical and biological sciences involve the analysis of systems that are naturally modeled as a collection of interacting stochastic automata [3,17,23]. Unfortunately, representing a stochastic automata network as a single *master* Markov chain suffers from the computational limitation that the aggregate number of states is exponential in the number of components. For example, the transition matrix for k coupled stochastic automata, each of which can be represented by an v-state Markov chain, has $\eta = v^k$ states and requires algorithms of $\mathcal{O}(v^{3k})$ complexity.

Many results are relevant to overcoming combinatorial state-space explosions of coupled stochastic automata. For example, memory-efficient numerical methods may use ordinary Kronecker representations of the master transition matrix $Q = \sum_\ell \bigotimes_{n=1}^{k} R_{\ell n}$ where the $R_{\ell n}$ are size v, and many are identity matrices, eliminating the need to generate and store the size v^k transition matrix [9]. Kronecker representations may be generalized to allow for matrix operands whose entries are functions that describe state-dependent transition rates, i.e., $Q = \bigoplus_{n=1}^{k} F_n$ and $F_n(i,j) : \times_{n=1}^{k} X_n \to \mathbb{R}$ where $X_n(t)$ is the state of the nth automata [5]. Hierarchical Markovian models may be derived in an automated manner and leveraged by multi-level numerical methods [7].

Redundancy in master Markov chains for interacting stochastic automata can often be eliminated without approximation. Both lumpability at the level of individual automata and model composition have been extensively researched, though the latter reduces the state space in a manner that eliminates Kronecker structure [4, 6, 13]. To see this, consider k identical and indistinguishable stochastic automata, each with v states, that interact via transition rates that are functions of the global state, that is, $Q = \bigoplus^{k} F$ where $F(i,j) : \times_{\ell=1}^{v} n_\ell \to \mathbb{R}$ where $n_\ell(t) = \sum_{n=1}^{k} \mathcal{I}\{X_n(t) = \ell\}$ is the number of automata in state ℓ. As defined Q, is size v^k, however, states may be lumped using symmetry in the model specification to yield an equivalent master Markov chain of size $\eta = \binom{k+v-1}{k}$. Although model reduction in this spirit is intuitive and widely used in applications, the mathematical structure of the transition graphs resulting from such contractions does not appear to have been extensively studied.

More concretely, let G represent the transition graph for an v-state automaton with transition matrix $Q = (q_{ij})$. As required in many applications, we assume that Q is irreducible and that state transitions are reversible ($q_{ij} > 0 \Leftrightarrow q_{ji} > 0$, $i \neq j$). Thus, the transition graph G corresponding to Q is simple (unweighted, undirected, no loops or multiple edges) and connected (by the irreducibility of Q). The transition graph G has adjacency matrix $A(G) = (a_{ij})$ where $a_{ii} = 0$, and for $i \neq j$, $a_{ij} = 0$ when $q_{ij} = 0$ and $a_{ij} = 1$ when $q_{ij} > 0$.

The transition graph for the master Markov chain for k automata with transition graphs G_n is the Cartesian graph product $G_1 \Box G_2 \Box \cdots \Box G_k$. If these k automata are identical, the transition graph for the master Markov chain is the kth Cartesian power of G, that is, the k-fold product $G^k = G \Box G \Box \cdots \Box G$. The focus of this paper is the ***k*-th reduced power** of G, i.e., the transition graph of the contracted master Markov chain for k indistinguishable (but not necessarily independent) v-state automata with isomorphic transition graphs.

The remainder of this paper is organized as follows. In Sections 2–3 we formally define the reduced power of a graph and interpret it as particular configuration space. Sections 4–6 present our primary result, the construction of minimal cycle bases of reduced graph powers. Section 7 explicates the relevance of these minimal cycle bases to applications that do not allow state-dependent coupling of automata to introduce nonequilibrium steady states.

2 Reduced Cartesian powers of a graph

There are several equivalent formulations of the reduced power of a graph. For the first formulation, recall that given graphs G and H, their *Cartesian product* is the graph $G \Box H$ whose vertex set is the Cartesian product $V(G) \times V(H)$ of the vertex sets of G and H, and whose edge set is

$$E(G \Box H) \quad = \quad \big\{(x,u)(y,v) \mid xy \in E(G) \text{ and } u = v, \ \textbf{or} \ \ x = y \text{ and } uv \in E(H)\big\}.$$

This product is commutative and associative [14]. For typographical efficiency we may abbreviate a vertex (x, y) of $G \Box H$ as xy if there is no danger of confusion.

The kth *Cartesian power* of a graph G is the k-fold product $G^k = G \Box G \Box \cdots \Box G$. The symmetric group S_k acts on G^k by permuting the factors. Specifically, for a permutation $\pi \in S_k$ the map

$$(x_1, x_2, \ldots, x_k) \mapsto (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)})$$

is an automorphism of G^k. The kth *reduced power* of G is the graph that has as vertices the orbits of this action, with two orbits being adjacent if G^k has an edge joining one orbit to the other. Said more succinctly, the reduced kth power is the quotient G^k/S_k of G^k by its S_k action.

Figure 1 shows a graph G next to $G^2 = G \Box G$. The S_2 action on G^2 has as orbits the singletons $\{aa\}$, $\{bb\}$, $\{cc\}$, $\{dd\}$, along with the pairs $\{ab, ba\}$, $\{ac, ca\}$, $\{ad, da\}$, $\{bc, cb\}$, $\{bd, db\}$, and $\{cd, dc\}$. Let us identify a singleton orbit such as $\{aa\}$ with the monomial $aa = a^2$, and a paired orbit such as $\{ab, ba\}$ with the monomial ab (with $ab = ba$). The reduced power $G^{(2)}$ appears on the right of Figure 1. Note that two monomials xy and uv are adjacent in $G^{(2)}$ provided that xy and uv have a common factor, and the remaining two factors are adjacent vertices of G.

As each monomial xy corresponds uniquely to the 2-multiset $\{x, y\}$ of vertices of G, we can also define the reduced power $G^{(2)}$ as follows. Its vertices are the 2-multisets of vertices of G, with two multisets being adjacent precisely if they agree in one element, and the other elements are adjacent in G.

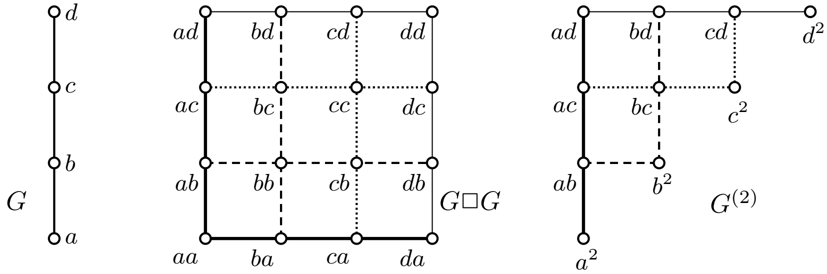

Figure 1: A graph G, the Cartesian square $G^2 = G\Box G$, and the reduced power $G^{(2)}$. For each $x \in V(G)$, the vertices $\{xv \mid v \in V(G)\}$ induce a subgraph $Gx \cong G$ of $G^{(2)}$. These subgraphs are shown dashed, dotted and solid in $G^{(2)}$. Note Gx and Gy intersect precisely at vertex xy if $x \neq y$.

In general, higher reduced powers $G^{(k)}$ can be understood as follows. Suppose $V(G) = \{a_1, a_2, \dots, a_v\}$. Any vertex of $G^{(k)}$ is the S_k-orbit of some $x = (x_1, x_2, \dots, x_k) \in V(G^k)$. For each index $1 \leq i \leq v$, say x has $n_i \geq 0$ coordinates equal to a_i. Then $\sum_{i=1}^{v} n_i = k$, and the S_k-orbit of x consists precisely of those k-tuples in $V(G^k)$ having n_i coordinates equal to a_i, for $1 \leq i \leq v$. This orbit – this vertex of $G^{(k)}$ – can then be identified with either the degree-k monomial

$$a_1^{n_1} a_2^{n_2} \cdots a_v^{n_v},$$

or with the k-multiset

$$\{\underbrace{a_1, a_1, \dots, a_1}_{n_1} \mid \underbrace{a_2, a_2, \dots, a_2}_{n_2} \mid \dots \;\; \dots \mid \underbrace{a_v, a_v, \dots, a_v}_{n_v}\}, \tag{2.1}$$

where $v-1$ dividing bars are inserted for clarity. We will mostly use the monomial notation for $V(G^{(k)})$, but will also employ the multiset phrasing when convenient. Let us denote the set of monic monomials of degree k, with indeterminates $V(G)$, as $M_k(G)$, with $M_0(G) = \{1\}$. The above, together with the definition of the Cartesian product, yields the following.

Definition 2.1. For a graph G with vertex set $\{a_1, a_2, \dots, a_v\}$, the **reduced kth power** $G^{(k)}$ is the graph whose vertices are the monomials $a_1^{n_1} a_2^{n_2} \cdots a_v^{n_v} \in M_k(G)$. For edges, if $a_i a_j$ is an edge of G, and $f(a_1, a_2, \dots a_v) \in M_{k-1}(G)$, then $a_i f(a_1, a_2, \dots, a_v)$ is adjacent to $a_j f(a_1, a_2, \dots, a_v)$.

Figure 2 shows the three-cycle $G = C_3$ and its reduced second and third powers. Figure 3 shows the five-cycle and its reduced second and third powers.

The reduced power $G^{(k)}$ is not to be confused with the *symmetric power* of G, for which each vertex represents a k-subset of $V(G)$, and two k-subsets are joined if and only if their symmetric difference is an edge of G [1, 2].

The multiset notation (2.1) gives a quick formula for the number of vertices of reduced kth powers. This presentation describes the multiset as a list of length $k+v-1$ involving k symbols a_i, $1 \leq i \leq k$, and $v-1$ separating bars. We can count the multisets by choosing k slots for the a_i's and filling in the remaining slots with bars. Therefore, when $|V(G)| = v$,

$$\left|V\left(G^{(k)}\right)\right| = \binom{k+v-1}{k}. \tag{2.2}$$

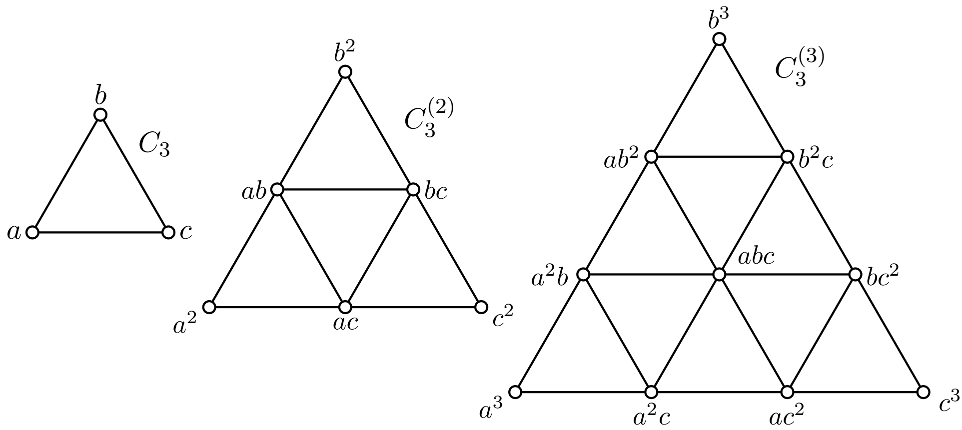

Figure 2: The three-cycle C_3 and its second and third reduced powers $C_3^{(2)}$ and $C_3^{(3)}$.

The number of vertices in G^k that are identified with vertex $a_1^{n_1}a_2^{n_2}\cdots a_v^{n_v} \in V(G^{(k)})$ in the quotient $G^{(k)} = G^k/S_k$ is given by the multinomial coefficient $\binom{k}{n_1\ n_2\ \dots\ n_v}$.

Definition 2.1 says that for each edge a_ia_j of G, and for each monomial $f \in M_{k-1}(G)$, there is an edge of $G^{(k)}$ from a_if to a_jf. Because there are $\binom{k+m-2}{k-1}$ such monomials f,

$$\left|E\big(G^{(k)}\big)\right| = |E(G)| \cdot \binom{k+v-2}{k-1}. \tag{2.3}$$

3 Reduced graph powers as configuration spaces

The reduced power $G^{(k)}$ is the transition graph of the contracted master Markov chain for k identical and indistinguishable v-state automata, each with transition graph G. Consequently, an intuitive way of envisioning $G^{(k)}$ is to imagine it as a configuration space in which k indistinguishable tokens are placed on the vertices of G, so that any vertex can hold up to k tokens. The monomial $a_1^{n_1}a_2^{n_2}\cdots a_v^{n_v}$ then represents the configuration in which n_i tokens are placed on each vertex a_i. Two configurations are adjacent if one can be transformed to the other by moving a single token along an edge of G to an adjacent vertex. In this way $G^{(k)}$ is interpreted as the space of all such configurations. See [11] for a related construction in which no vertex can hold more than one token.

The reduced power $G^{(k)}$ may also be interpreted as the *reachability graph* for a fundamental class of stochastic Petri nets with k tokens, $v = |V(G)|$ places, and $2|E(G)|$ flow relations (directed arcs) between places [8, 22]. The arc from place a_i (origin) to place a_j (destination) has firing rate n_iq_{ij} given by the product of transition rate q_{ij} and the number n_i of tokens in the origin place. That is, the $a_i \to a_j$ firing time is the minimum of n_i exponentially distributed random variables with expectation $1/q_{ij}$. The $a_i \to a_j$ firing rate per token will be denoted $q_{ij}[a_1^{n_1}a_2^{n_2}\cdots a_v^{n_v}]$ when it is a function of the global state (token configuration) of the stochastic Petri net.

The token interpretation can be helpful in deducing properties of reduced powers, such as the following.

Proposition 3.1. *The vertex $a_1^{n_1} a_2^{n_2} \cdots a_v^{n_v}$ of $G^{(k)}$ has degree*

$$\deg\left(a_1^{n_1} a_2^{n_2} \cdots a_v^{n_v}\right) = \sum_{n_i \geq 1} \deg_G(a_i).$$

Proof. The configuration $a_1^{n_1} a_2^{n_2} \cdots a_v^{n_v}$ can be transformed to an adjacent configuration only by moving a token on some vertex a_i (with $n_i \geq 1$) to an adjacent vertex. □

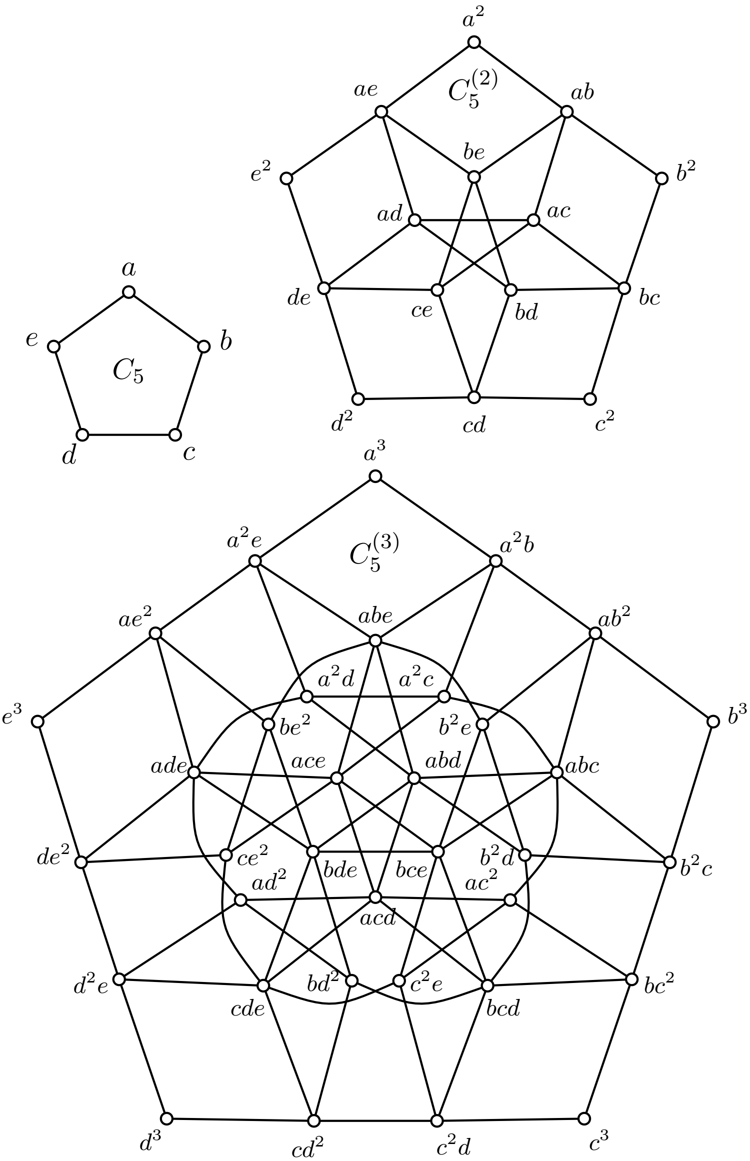

Figure 3: The five-cycle C_5 and its second and third reduced powers $C_5^{(2)}$ and $C_5^{(3)}$.

4 Cycle bases and minimum cycle bases

Here we quickly review the fundamentals of cycle spaces and bases. The following is condensed from Chapter 29 of [14].

For a graph G, its *edge space* $\mathscr{E}(G)$ is the power set of $E(G)$ viewed as a vector space over the two-element field $\mathbb{F}_2 = \{0, 1\}$, where the zero vector is $0 = \emptyset$ and addition is symmetric difference. Any vector $X \in \mathscr{E}(G)$ is viewed as the subgraph of G induced on X, so $\mathscr{E}(G)$ is the set of all subgraphs of G without isolated vertices. Thus $E(G)$ is a basis for $\mathscr{E}(G)$, and $\dim(\mathscr{E}(G)) = |E(G)|$. The *vertex space* $\mathscr{V}(G)$ of G is the power set of $V(G)$ as a vector space over $\mathbb{F}_2$. It is the set of all edgeless subgraphs of G and its dimension is $|V(G)|$.

We define a linear *boundary map* $\delta_G : \mathscr{E}(G) \to \mathscr{V}(G)$ by declaring that $\delta_G(xy) = x+y$ on the basis $E(G)$. The subspace $\mathscr{C}(G) = \ker(\delta_G)$ is called the *cycle space* of G. It contains precisely the subgraphs in $\mathscr{E}(G)$ whose vertices all have even degree (that is, the Eulerian subgraphs). Because every such subgraph can be decomposed into edge-disjoint cycles, each in $\mathscr{C}(G)$, we see that $\mathscr{C}(G) \subseteq \mathscr{E}(G)$ is spanned by the cycles in G.

The dimension of $\mathscr{C}(G)$, denoted $\beta(G)$, is called the (first) *Betti number* of G. If G is connected, the rank theorem applied to δ_G yields

$$\beta(G) = |E(G)| - |V(G)| + 1. \tag{4.1}$$

A basis for the cycle space is called a *cycle basis*. To make a cycle basis of a connected graph G, take a spanning tree T, so the set $S = E(G) - E(T)$ has $|E(G)| - |V(G)| + 1 = \beta(G)$ edges. For each $e \in S$, let C_e be the unique cycle in $T + e$. Then the set $\mathscr{B} = \{C_e \mid e \in S\}$ is linearly independent. As $\mathscr{B}$ has cardinality $\beta(G)$, it is a basis (see Figure 4).

The elements of a cycle basis are naturally weighted by their number of edges. The *total length* of a cycle basis $\mathscr{B}$ is the number $\ell(\mathscr{B}) = \sum_{C \in \mathscr{B}} |C|$. A cycle basis with the smallest possible total length is called a *minimum cycle basis*, or *MCB*.

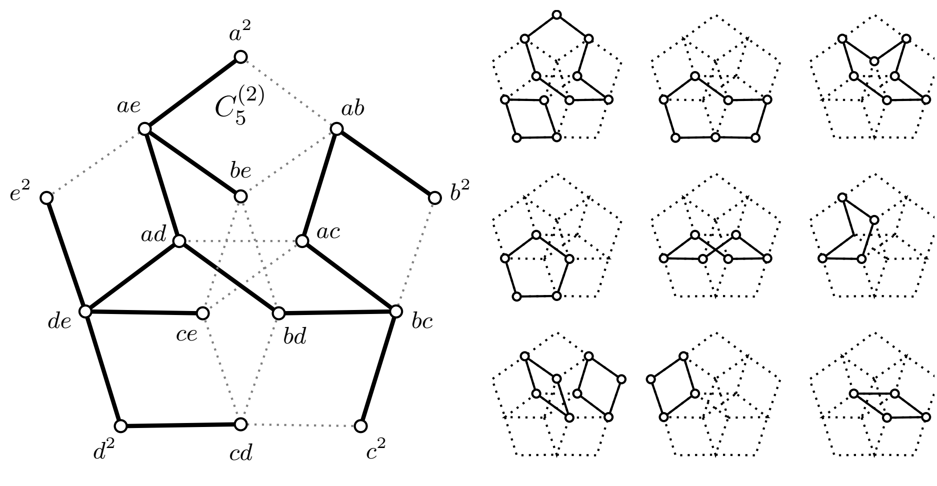

Figure 4: A spanning tree T of $G = C_5^{(2)}$. The set $S = E(G) - E(T)$ has $\beta(G) = 25 - 15 + 1 = 11$ edges. For each $e \in S$, let C_e be the unique cycle in $T + e$. The set $\{C_e \mid e \in S\}$ is a cycle basis for G, but not a minimum cycle basis (see Figure 5).

The cycle space is a weighted matroid where each element C has weight $|C|$. Hence the Greedy Algorithm [20] always terminates with an MCB: Begin with $\mathscr{M} = \emptyset$; then append shortest cycles to it, maintaining independence of $\mathscr{M}$, until no further shortest cycles can be appended; then append next-shortest cycles, maintaining independence, until no further such cycles can be appended; and so on, until $\mathscr{M}$ is a maximal independent set. Then $\mathscr{M}$ is an MCB.

Here is our primary criterion for determining if a cycle basis is an MCB. (See Exercise 29.4 of [14].)

Proposition 4.1. *A cycle basis $\mathscr{B} = \{B_1, B_2, \ldots, B_{\beta(G)}\}$ for a graph G is an MCB if and only if every $C \in \mathscr{C}(G)$ is a sum of basis elements whose lengths do not exceed $|C|$.*

For graphs G and H, a *weak homomorphism* $\varphi : G \to H$ is a map $\varphi : V(G) \to V(H)$ having the property that for each xy of G, either $\varphi(x)\varphi(y)$ is an edge of H, or $\varphi(x) = \varphi(y)$. Such a map induces a linear map $\varphi^* : \mathscr{E}(G) \to \mathscr{E}(H)$ defined on the basis $E(G)$ as $\varphi^*(xy) = \varphi(x)\varphi(y)$ provided $\varphi(x) \neq \varphi(y)$, and $\varphi^*(xy) = 0$ otherwise. Similarly we define $\varphi^*_V : \mathscr{V}(G) \to \mathscr{V}(H)$ as $\varphi^*_V(x) = \varphi(x)$ on the basis $V(G)$. Thus we have the following commutative diagram. (Check it on the basis $E(G)$.)

$$\begin{tikzcd}
\mathscr{E}(G) \arrow[r, "\varphi^*"] \arrow[d, "\delta_G"'] & \mathscr{E}(H) \arrow[d, "\delta_H"] \\
\mathscr{V}(G) \arrow[r, "\varphi^*_V"] & \mathscr{V}(H)
\end{tikzcd}$$

From this, φ^* restricts to a map $\mathscr{C}(G) \to \mathscr{C}(H)$ on cycle spaces, because if $C \in \mathscr{C}(G)$, then $\delta_G(C) = 0$, whence $\delta_H\varphi^*(C) = \varphi^*_V\delta_G(C) = 0$, meaning $\varphi^*(C) \in \ker(\delta_H) = \mathscr{C}(H)$. Certainly if φ is a graph isomorphism, then φ^* is a vector space isomorphism.

Of special interest will be the projections $p_i : G^k \to G$, where $p_i(x_1, x_2, \ldots, x_k) = x_i$. These are weak homomorphisms and hence induce linear maps $p_i^* : \mathscr{C}(G^k) \to \mathscr{C}(G)$.

Another important map is the natural projection $\eta : G^k \to G^{(k)}$ sending each k-tuple $x = (x_1, x_2, \ldots, x_k)$ to the monomial representing the S_k-orbit containing x. This map η^* also is a weak homomorphism, inducing a linear map $\eta^* : \mathscr{C}(G^k) \to \mathscr{C}(G^{(k)})$.

Lemma 4.2. *If G is connected, the map $\eta^* : \mathscr{C}(G^k) \to \mathscr{C}(G^{(k)})$ is surjective.*

Proof. Because any element of $\mathscr{C}(G^{(k)})$ is an edge-disjoint union of cycles, it suffices to show that any cycle $C = f_0f_1 \cdots f_nf_0 \in \mathscr{C}(G^{(k)})$ equals $\eta^*(C')$ for some $C' \in \mathscr{C}(G^k)$. For each index i, let $x_iy_{i+1} \in E(G^k)$ be an edge for which $\eta^*(x_iy_{i+1}) = \eta(x_i)\eta(y_{i+1}) = f_if_{i+1}$. (Each x_i, y_i is a k-tuple, and index arithmetic is modulo n.) Note that $\eta(x_i) = \eta(y_i)$, meaning x_i and y_i are in the same S_k-orbit, that is, y_i equals x_i with its coordinates permuted.

We will argue that each pair y_i, x_i can be joined by a path P_i in G^k, with $\eta^*(P_i) = 0$. This will prove the lemma because then

$$C' = P_0 + x_0y_1 + P_1 + x_1y_2 + P_2 + \ldots + P_n + x_ny_0 \ \in \ \mathscr{C}(G^k)$$

satisfies $\eta^*(C') = C$.

Consider two vertices $(\ldots a \ldots b \ldots)$ and $(\ldots b \ldots a \ldots)$ of G^k that are identical except for the transposition of two coordinates a and b. Take a path $a = v_0 v_1 \cdots v_q = b$ from a to b in G. Now form the following two paths in G^k

$$\begin{aligned} Q &= (\ldots a \ldots b \ldots)(\ldots v_1 \ldots b \ldots)(\ldots v_2 \ldots b \ldots) \ldots (\ldots b \ldots b \ldots) \\ R &= (\ldots b \ldots a \ldots)(\ldots b \ldots v_1 \ldots)(\ldots b \ldots v_2 \ldots) \ldots (\ldots b \ldots b \ldots). \end{aligned}$$

Concatenation of Q with the reverse of R is a path from $(\ldots a \ldots b \ldots)$ to $(\ldots b \ldots a \ldots)$. Moreover $\eta^*(Q+R) = 0$ because the images of the jth edges of Q and R are always equal; hence the edges cancel in pairs. As y_i and x_i differ only by a sequence of transpositions of their coordinates, the above construction can be used to build up a path P_i from y_i to x_i with $\eta(P_i) = 0$. $\square$

We have seen that the projections $p_i : G^k \to G$ induce linear maps $\mathscr{C}(G^k) \to \mathscr{C}(G)$. But there seems to be no obvious way of defining a projection $G^{(k)} \to G$. Still, it is possible to construct a natural linear map $p^* : \mathscr{C}(G^{(k)}) \to \mathscr{C}(G)$. To do this, recall that any edge of $G^{(k)}$ has form $af\ bf$ where $ab \in E(G)$ and $f \in M_{k-1}(G)$. We begin by defining p^* on the edge space. Put $p^*(af\ bf) = ab$ for each edge $af\ bf$ in the basis $E(G^{(k)})$ and extend linearly to a map $p^* : \mathscr{E}(G^{(k)}) \to \mathscr{E}(G)$. Note that $\sum_{i=1}^{k} p_i^* = p^* \circ \eta^*$. (Confirm it by checking it on the basis $E(G^k)$ of $\mathscr{E}(G^k)$.) Now, if $X \in \mathscr{C}(G^{(k)})$, then Lemma 4.2 guarantees $X = \eta^*(Y)$ for some Y in the cycle space of G^k. Then $p^*(X) = p^*(\eta^*(Y)) = \sum_{i=1}^{k} p_i^*(Y) \in \mathscr{C}(G)$.

We now have a linear map $p^* : \mathscr{C}(G^{(k)}) \to \mathscr{C}(G)$ for which $p^*(af\ bf) = ab$.

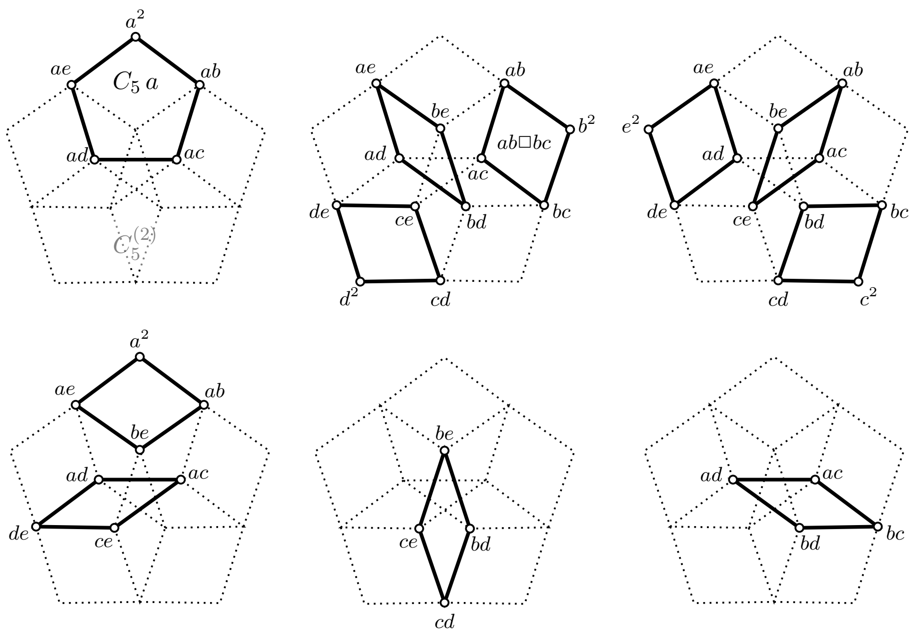

Figure 5: The union $\{C_5 a\} \cup \mathscr{B}$ is an MCB for $\mathscr{C}(C_5^{(2)}) = \mathscr{C}(C_5\,a) \bigoplus \mathscr{S}(C_5^{(2)})$.

5 Decomposing the cycle space of a reduced power

This section explains how to decompose the cycle space of a reduced power into the direct sum of particularly simple subspaces.

To begin, notice that if f is a fixed monomial in $M_{k-1}(G)$, then there is an embedding $G \to G^{(k)}$ defined as $x \mapsto xf$. Let us call the image of this map Gf. Notice that Gf is an induced subgraph of $G^{(k)}$ and is isomorphic to G.

Proposition 5.1. *For any fixed $f \in M_{k-1}(G)$, we have $\mathscr{C}(G^{(k)}) = \mathscr{C}(Gf) \bigoplus \ker(p^*)$.*

Proof. Consider the map $p^* : \mathscr{C}(G^{(k)}) \to \mathscr{C}(G)$. Its restriction $\mathscr{C}(Gf) \to \mathscr{C}(G)$ is a vector space isomorphism. The proof now follows from elementary linear algebra. □

Next we define a special type of cycle in a reduced power. Given distinct edges ab and cd of G and any $f \in M_{k-2}(G)$, we have a square in $G^{(k)}$ with vertices acf, bcf, bdf, adf. Let us call such a square a *Cartesian square*, and denote it as $(ab\Box cd)f$. See Figure 6.

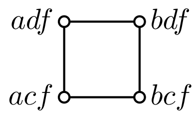

Figure 6: A Cartesian square $(ab\Box cd)f$ in $G^{(k)}$ with $k \geq 2$.

We regard this as a cycle in the cycle space; it is the subgraph of $G^{(k)}$ that is precisely the sum of edges $acf\,bcf + bcf\,bdf + bdf\,adf + adf\,acf$. (Observe that this sum is zero if and only if $ab = cd$.) We remark that although a subgraph Gf may have squares, they are not *Cartesian squares* because they do not have the form specified above. Define the *square space* $\mathscr{S}(G^{(k)})$ to be the subspace of $\mathscr{C}(G^{(k)})$ that is spanned by the Cartesian squares.

If S is a Cartesian square, then $p^*(S) = 0$, so $\mathscr{S}(G^{(k)}) \subseteq \ker(p^*)$. In the remainder of the paper we will show that in fact $\mathscr{S}(G^{(k)}) = \ker(p^*)$, so that Proposition 5.1 gives $\mathscr{C}(G^{(k)}) = \mathscr{C}(Gf) \bigoplus \mathscr{S}(G^{(k)})$. Simultaneously we will craft a simple MCB for $G^{(k)}$ by concatenating MCBs of $\mathscr{C}(Gf)$ and $\mathscr{S}(G^{(k)})$. See Figure 5 for an example.

6 Cycle bases for reduced powers

This section describes a simple cycle basis for the reduced kth power of a graph G. If G has no triangles, this cycle basis will be an MCB. (We do not consider MCBs in the cases that G has triangles because the applications we have in mind do not involve such situations. Constructing MCBs when G has triangles would be an interesting research problem.)

Let G be a connected graph with v vertices and e edges. Recall that by Equations (2.2) and (2.3), the graph $G^{(k)}$ has $\binom{k+v-1}{k}$ vertices, identified with the monomials $M_k(G)$, and $e\binom{k+v-2}{k-1}$ edges. Thus any cycle basis has dimension

$$\beta(G^{(k)}) = e\binom{k+v-2}{k-1} - \binom{k+v-1}{k} + 1. \tag{6.1}$$

We first examine the square space. Any pair of distinct edges ab and cd of G corresponds to a Cartesian square $(ab\Box cd)f$, where $f \in M_{k-2}(G)$, so there are $\binom{e}{2}\binom{k+v-3}{k-2}$

such squares. But this set of squares may not be independent. Our first task will be to construct a linearly independent set of Cartesian squares.

To begin, put $V(G) = \{a_1, a_2, \dots, a_v\}$. Let T be a rooted spanning tree of G with root a_1, and arrange the indexing so its order respects a breadth-first traversal of T, that is, for each i the vertex a_i is not closer to the root than any a_j for which $j < i$ (see Figure 7).

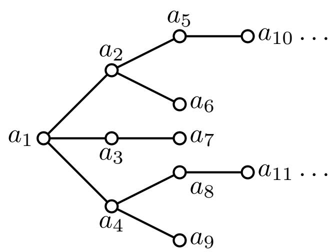

Figure 7: A rooted spanning tree T of G with $V(G) = \{a_1, a_2, \dots, a_v\}$, root a_1, and indexing that respects a breadth-first traversal of T.

With this labeling, any edge of T is uniquely determined by its endpoint a_j that is furthest from the root. For each $2 \leq i \leq v$, let e_j be the edge of T that has endpoints a_i and a_j, with a_j further from the root than a_i. Let $M_{k-2}(a_1, a_2, \dots a_j)$ denote the monic monomials of degree $k-2$ in indeterminates $a_1, a_2, \dots, a_j$, with $1 \leq j \leq v$. Define the following sets of Cartesian squares in $G^{(k)}$.

$$\begin{aligned} \Upsilon &= \{(e_i \Box e_j)f \mid 2 \leq i < j \leq v, f \in M_{k-2}(a_1, a_2, \dots, a_j)\}, \\ \Omega &= \{(a_\ell a_m \Box e_j)f \mid a_\ell a_m \in E(G) - E(T), 2 \leq j \leq v, f \in M_{k-2}(a_1, a_2, \dots, a_j)\}. \end{aligned}$$

Shortly we will show that $\Upsilon \cup \Omega$ is linearly independent. But first a few quick informal words about why we would expect this to be the case. Suppose $k \geq 3$ and take three distinct edges $a_i a_j$, $a_\ell a_m$ and $a_p a_q$ in G, and let $f \in M_{k-3}(G)$. Figure 8 indicates that these edges result in a cube in the kth reduced power. Each of the six square faces of this cube is in the square space. But the faces are dependent because any one of them is a sum of the others. Call a square face such as $(a_i a_j \Box a_\ell a_m) a_q f$ a "top square" of a cube if the monomial $a_q f$ involves an indeterminate a_t with $t > \max\{i, j, \ell, m\}$. Sets Υ and Ω are constructed so as to contain no top squares.

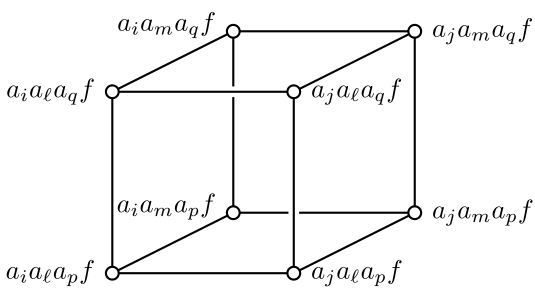

Figure 8: A Cartesian cube $(a_i a_j \Box a_\ell a_m \Box a_p a_q)f$ in the reduced power $G^{(k)}$.

(A configuration of the type illustrated in Figure 8 may not always be a cube in the combinatorial sense. The reader is cautioned that if $a_i a_j$, $a_\ell a_m$ and $a_p a_q$ are the edges of a triangle in G, then two of the diagonally opposite vertices of the "cube" are the same, as in $K_3^{(3)}$, shown in Figure 2. Here there is only one cube, which takes the form of a central vertex connected to the six vertices of a hexagon. This will cause no difficulties in what follows, even if we entertain the possibility that G does indeed have triangles.)

There is another kind of dependency that is ruled out in the definition of Υ and Ω, and we now sketch it. First, imagine G^2. Consider two cycles A and B in G each having exactly one edge not in T, say $a_i a_j$ and $a_\ell a_m$, respectively. Envision $A \square B$ is as a torus in G^2 with square faces, each edge shared by two faces. In adding up all the faces, the edges cancel in pairs, giving 0, so the squares are dependent. Removing the face $a_i a_j \square a_\ell a_m$ removes the dependency. Such squares $a_i a_j \square a_\ell a_m$ show up in $G^{(2)} f \subseteq G^{(k)}$ as squares $(a_i a_j \square a_\ell a_m) f$ with $a_i a_j, a_p a_q \in E(G) - E(T)$. Sets Υ and Ω contain no such squares.

Proposition 6.1. *The set* $\mathscr{B} = \Upsilon \cup \Omega$ *is linearly independent.*

Proof. We first show that Υ is linearly independent. Let $X = \sum (e_i \square e_j) f_n$ be a sum of elements of Υ. Form the forest $F \subseteq T$ consisting of all edges e_i and e_j that appear as edges of a squares in this sum, and let ab be an edge of F for which b is a leaf. Then any term $(a_\ell a_m \square ab) f_n$ of the sum is the unique square in the sum containing the edge $a_\ell b f_n \; a_m b f_n$. Because no term can cancel this edge, we get $X \neq 0$, so Υ is linearly independent.

To see that Ω is linearly independent, consider a sum $X = \sum (a_\ell a_m \square e_j) f_n$ of squares in Ω. Again form a forest $F \subseteq T$ of the edges e_j and let ab be as before. Then any term $(a_\ell a_m \square ab) f_n$ is the unique square in the sum containing the edge $a_\ell b f_n \; a_m b f_n$. Then $X \neq 0$ because no other term in the sum can cancel this edge; hence Ω is linearly independent.

Now we argue that the spans of Υ and Ω have zero intersection. By the previous paragraph, any nonzero linear combination of squares in Ω has edges of form $(a_\ell a_m \square ab) f_n$, with $a_\ell a_m \in E(G) - E(T)$. But no linear combination of squares in Υ has such edges. Hence the spans have zero intersection, so $\mathscr{B}$ is linearly independent. $\square$

Our next task is to show that $\mathscr{B}$ is actually a basis for the square space. In fact, we will show more: it is also a basis for $\ker(p^*)$, and $\mathscr{S}(G^{(k)}) = \ker(p^*)$. Our dimension counts will involve finding $|\Upsilon|$ and $|\Omega|$, and for this we use the following formulas. The first is standard; both are easily verified with induction.

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{r+n}{r} = \binom{r+n+1}{r+1} \tag{6.2}$$

$$0\binom{r}{r} + 1\binom{r+1}{r} + 2\binom{r+2}{r} + \cdots + n\binom{r+n}{r} = n\binom{r+n+1}{r+1} - \binom{r+n+1}{r+2} \tag{6.3}$$

Take an edge e_j of T with $3 \leq j$. From its definition, Υ has $(j-2)\binom{k+j-3}{k-2}$ squares of form $(e_i \square e_j) f$. We reckon as follows, using Equations (6.2) and (6.3) as appropriate.

$$\begin{aligned}
|\Upsilon| &= \sum_{j=3}^{v}(j-2)\binom{k+j-3}{k-2} \\
&= \sum_{j=1}^{v}(j-2)\binom{k+j-3}{k-2}+1 \\
&= \sum_{j=1}^{v}(j-1)\binom{k+j-3}{k-2}-\sum_{j=1}^{v}\binom{k+j-3}{k-2}+1 \\
&= (v-1)\binom{k+v-2}{k-1}-\binom{k+v-2}{k}-\binom{k+v-2}{k-1}+1 \\
&= (v-1)\binom{k+v-2}{k-1}-\binom{k+v-1}{k}+1. \qquad (6.4)
\end{aligned}$$

Now, given and edge e_j of T with $2 \leq j$, the set Ω has $\beta(G)\binom{k+j-3}{k-2}$ squares of form $(a_\ell a_m \Box e_j)f$. Consequently

$$\begin{aligned}
|\Omega| &= \beta(G)\sum_{j=2}^{v}\binom{k+j-3}{k-2} \\
&= \beta(G)\left(\sum_{j=1}^{v}\binom{k+j-3}{k-2}-1\right) \\
&= \beta(G)\binom{k+v-2}{k-1}-\beta(G). \qquad (6.5)
\end{aligned}$$

Proposition 6.2. *The set $\mathscr{B} = \Upsilon \cup \Omega$ is a basis for the square space of the reduced kth power of G. Moreover, the square space equals* $\ker(p^*)$.

Proof. By Proposition 6.1, the set $\mathscr{B}$ is linearly independent; and it is a subset of the square space by construction. We saw earlier that the square space is a subspace $\ker(p^*)$. To finish the proof we show that $\ker(p^*)$ has dimension $|\mathscr{B}|$. By the rank theorem applied to the surjective map $p^* : \mathscr{C}(G^{(k)}) \to \mathscr{C}(G)$ we have $\dim\ker(\varphi^*) = \beta(G^{(2)}) - \beta(G)$. This with Equations (6.1), (6.4) and (6.5), as well as the fact that $(v-1)+\beta(G) = e$, gives

$$\begin{aligned}
|\mathscr{B}| &= |\Upsilon| + |\Omega| \\
&= (v-1)\binom{k+v-2}{k-1}-\binom{k+v-1}{k}+1+\beta(G)\binom{k+v-2}{k-1}-\beta(G) \\
&= e\binom{k+v-2}{k-1}-\binom{k+v-1}{k}+1-\beta(G) \\
&= \beta(G^{(2)})-\beta(G) \\
&= \dim\ker(p^*).
\end{aligned}$$

Therefore $\mathscr{B}$ is a basis for both $\mathscr{S}(G^{(k)})$ and $\ker(p^*)$. $\Box$

If $k = 2$, then $\mathscr{B} = \{ab\Box cd \mid ab, cd \in E(G)\} - \{ab\Box cd \mid ab, cd \in E(G)-E(T)\}$, so $|\mathscr{B}| = \binom{e}{2} - \binom{\beta(G)}{2}$. It is interesting to note that if $\beta(G) \leq 1$, then $\binom{\beta(G)}{2} = 0$ and $\mathscr{B}$ consists of all squares in the square space; in all other cases it has fewer squares.

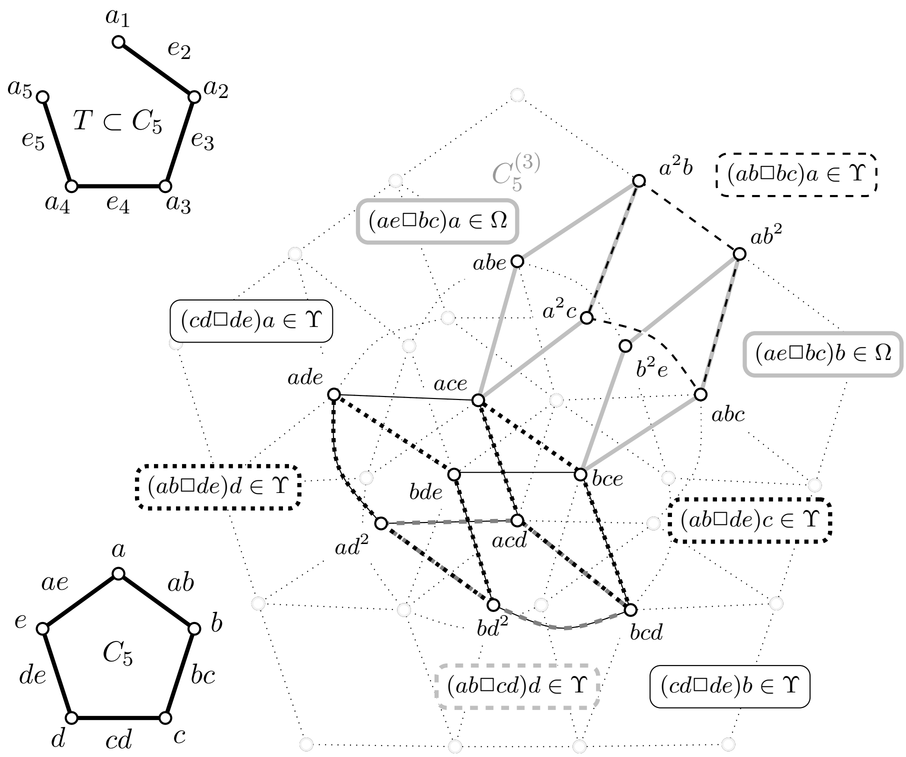

Figure 9: With T as indicated, the sets of squares Υ and Ω form a basis $\mathscr{B} = \Upsilon \cup \Omega$ of the square space of $C_5^{(3)}$. Here $\Upsilon = \{(ab\square bc)f \mid f \in \{a,b,c\}\} \cup \{(ab\square cd)f, (bc\square cd)f \mid f \in \{a,b,c,d\}\} \cup \{(ab\square de)f, (bc\square de)f, (cd\square de)f \mid f \in \{a,b,c,d,e\}\}$. Also $\Omega = \{(ae\square ab)f \mid f \in \{a,b\}\} \cup \{(ae\square bc)f \mid f \in \{a,b,c\}\} \cup \{(ae\square cd)f \mid f \in \{a,b,c,d\}\} \cup \{(ae\square de)f \mid f \in \{a,b,c,d,e\}\}$. Note $|\Upsilon| = 24$ and $|\Omega| = 14$. The square $(ab\square cd)e \notin \mathscr{B}$ is the "top square" of the Cartesian cube $ab\square cd\square de$.

We now can establish the main result of this section, namely a construction of an MCB for the reduced kth power. Take an $f \in M_{k-1}(G)$. Propositions 5.1 and 6.2 say

$$\mathscr{C}(G^{(k)}) = \mathscr{C}(Gf) \bigoplus \mathscr{S}(G^{(k)}). \tag{6.6}$$

To any cycle $C = c_1c_2\dots c_n$ in G, there corresponds cycle $Cf = c_1f\, c_2f \dots c_nf$ in $G^{(k)}$.

Theorem 6.3. *Take a cycle basis $\mathscr{C} = \{C_1, C_2, \dots, C_{\beta(G)}\}$ for G, and let $\mathscr{B}$ be the basis for $\mathscr{S}(G^{(k)})$ constructed above. Fix $f \in M_{k-1}(G)$ and put $\mathscr{C}f = \{C_1f, C_2f, \dots, C_{\beta(G)}f\}$. Then $\mathscr{C}f \cup \mathscr{B}$ is a cycle basis for $G^{(k)}$. If $\mathscr{C}$ is an MCB for G, and G has no triangles, then this basis is an MCB for $G^{(k)}$.*

Proof. That this is a cycle basis follows immediately from Equation (6.6).

Now suppose $\mathscr{C}$ is an MCB for G, and that G has no triangles. It is immediate that $G^{(k)}$ has no triangles either. The proof is finished by applying Proposition 4.1. Take any $C \in \mathscr{C}(G^{(k)})$, and write it as

$$C = \sum_{i \in I} G_i + \sum_{j \in J} B_j \,,$$

where the G_i are from $\mathscr{C} f$ and the B_j are from $\mathscr{B}$. According to Proposition 4.1, it suffices to show that C has at least as many edges as any term in this sum. Certainly C is not shorter than any square B_j (by the triangle-free assumption). To see that it is not shorter than any G_i in the sum, apply p^* to the above equation to get

$$p^*(C) = \sum_{i \in I} p^*(G_i)\,.$$

Because $p^* : \mathscr{C}(Gf) \to \mathscr{C}(G)$ is an isomorphism, the terms $p^*(G_i)$ are part of an MCB for G, and thus $|p^*(C)| \geq |p^*(G_i)| = |G_i|$ for each i, by Proposition 4.1. Also $|C| \geq |p^*(C)|$ (as some edges may cancel in the projection) so $|C| \geq |G_i|$. □

Although Theorem 6.3 gives a simple MCB for reduced powers of a graph that has no triangles, the constructed basis is definitely *not* minimum if triangles are present. Several different phenomena account for this. Consider the case $k = 2$. First, if G has triangles, then for each vertex x of G, the second reduced power contains a copy Gx of G. These copies are pairwise edge-disjoint; an MCB would have to capitalize on triangles in each of these copies at the expense of squares in the square space. Moreover, as Figure 2 demonstrates, some of the squares in the square space will actually be sums of two triangles. The figure also shows that for a triangle $\Delta = abc$ in G, we do not get just the three triangles Δa, Δb and Δc, but also a fourth triangle $ab\,bc\,ca$ not belonging to any Gx. We do not delve into this problem here.

7 Discussion

We have defined what appears to be a new construction, the kth reduced power of a graph, $G^{(k)}$, and have presented a theorem for construction of minimal cycle bases of $G^{(k)}$.

When G is the transition graph for a Markov chain, $G^{(k)}$ is the transition graph for the configuration space of k identical and indistinguishable v-state automata with transition graph G. Symmetry of model composition allows for interactions among stochastic automata, so long as the transition rates q_{ij} for $i, j \in \{1, 2, \cdots, v\}$, $i \neq j$ are constant or functions of the number of automata $n_\ell(t)$ in each state, $0 \leq n_\ell(t) \leq k$, $1 \leq \ell \leq v$. $G^{(k)}$ does not pertain if transition rates depend on the state of any particular automaton, $X_n(t) \in \{1, 2, \cdots, v\}$, $n \in \{1, 2, \cdots, k\}$, as this violates indistinguishability.

For concreteness, consider a stochastic automata network composed of three identical automata, each with transition graph C_5 and generator matrix,

$$Q = \begin{pmatrix} \diamond & q_{ab}[\cdot] & 0 & 0 & q_{ae} \\ q_{ba} & \diamond & q_{bc} & 0 & 0 \\ 0 & q_{cb} & \diamond & q_{cd} & 0 \\ 0 & 0 & q_{dc} & \diamond & q_{de} \\ q_{ea} & 0 & 0 & q_{ed} & \diamond \end{pmatrix} \tag{7.1}$$

where $\diamond$'s indicate the values required for zero row sum, $q_{ii} = -\sum_{j\neq i} q_{ij} < 0$, and $q_{ab}[\cdot]$ indicates a functional transition rate that depends on the global state of the three automata. Assume constant transition rates $q_{bc} = q_{cd} = q_{de} = q_{ea} = \mu > 0$ and $q_{ba} = q_{cb} = q_{dc} = q_{ed} = q_{ae} = \nu > 0$. Further assume that the automata may influence one another through the state-dependent transition rate,

$$q_{ab}[\cdot] = \lambda + \alpha\,(n_a[\cdot] - 1) + \beta\, n_b[\cdot] + \gamma\, n_c[\cdot] + \delta\, n_d[\cdot] + \epsilon\, n_e[\cdot], \tag{7.2}$$

where $\alpha, \beta, \gamma, \delta, \epsilon \geq 0$ and $[\cdot]$ denotes the global state $a_1^{p_1} a_2^{p_2} \cdots a_v^{p_v}$ that is the functional transition rate's argument. The transition rate $q_{ab} : M_k(a_1, a_2, \cdots, a_v) \to \mathbb{R}$ is a function of the global state via $n_\ell : M_k(a_1, a_2, \cdots, a_v) \to \mathbb{N}$ defined by $n_\ell[a_1^{p_1} a_2^{p_2} \cdots a_v^{p_v}] = p_\ell$. The three automata are uncoupled when $\alpha, \beta, \gamma, \delta, \epsilon = 0$ because this eliminates the dependence of $q_{ab}[\cdot]$ on the global state.

(In this model specification, coupling an isolated component automaton to itself is equivalent to absence of coupling. Because $q_{ab}[\cdot]$ is the rate of an $a \to b$ transition, $q_{ab}[\cdot]$ is only relevant when the isolated automaton is in state a. This functional transition rate has the property that $q_{ab}[a] = \lambda$ when $\alpha, \beta, \gamma, \delta, \epsilon > 0$ because $n_x[y] = 1$ for $x = y$ and 0 otherwise.)

The transition matrix for the master Markov chain $Q^{(3)}$ is defined by the model specification in the previous paragraph. For example, the transition rate from global state ad^2 to global state abd is $q^{(3)}[ad^2, abd] = 2\mu$ because $n_d[ad^2] = 2$ and $q_{db} = \mu$ is not a function of the global state. Other examples are $q^{(3)}[c^3, c^2d] = n_c[c^3]q_{cd} = 3\nu$, $q[a^2c, a^2d] = n_c[a^2c]q_{cd} = \nu$,

$$\begin{aligned}
q^{(3)}[abe, b^2e] &= n_a[abe]q_{ab}[abe] \\
&= \lambda + \alpha\,(n_a[abe] - 1) + \beta\, n_b[abe] + \gamma\, n_c[abe] + \delta\, n_d[abe] + \epsilon\, n_e[abe] \\
&= \lambda + \beta + \epsilon \\
q^{(3)}[a^3, a^2b] &= n_a[a^3]q_{ab}[a^3] \\
&= 3\left(\lambda + \alpha\,(n_a[a^3] - 1) + \beta\, n_b[a^3] + \gamma\, n_c[a^3] + \delta\, n_d[a^3] + \epsilon\, n_e[a^3]\right) \\
&= 3\,(\lambda + 2\alpha) \\
q^{(3)}[a^2c, abc] &= n_a[a^2c]q_{ab}[a^2c] \\
&= 2\left(\lambda + \alpha\,(n_a[a^2c] - 1) + \beta\, n_b[a^2c] + \gamma\, n_c[a^2c] + \delta\, n_d[a^2c] + \epsilon\, n_e[a^2c]\right) \\
&= 2\,(\lambda + \alpha + \gamma)\,.
\end{aligned}$$

This process of unpacking the model specification yields a master Markov chain with $\eta = \binom{k+v-1}{k} = \binom{3+5-1}{3} = 35$ states. The master Markov chain has 210 transition rates $q_{ij} > 0$ corresponding (in pairs) to the $5\binom{3+5-2}{3-1} = 105$ edges of the master transition graph $C_5^{(3)}$.

The construction of minimal cycle bases of $G^{(k)}$ provided by Theorem 6.3 is especially relevant to stochastic automata networks that arise in physical chemistry and biophysics [15]. For many applications in these domains, the principle of microscopic reversibility requires that the stationary distribution of *uncoupled* automata satisfying global balance, $\bar{\boldsymbol{\pi}} Q = \mathbf{0}$ subject to $\sum_i \bar{\pi}_i = 1$, also satisfies a stronger condition known as detailed balance,

$$\bar{\pi}_i \sum_{i\neq j} q_{ij} = \sum_{j\neq i} q_{ji}\bar{\pi}_j.$$

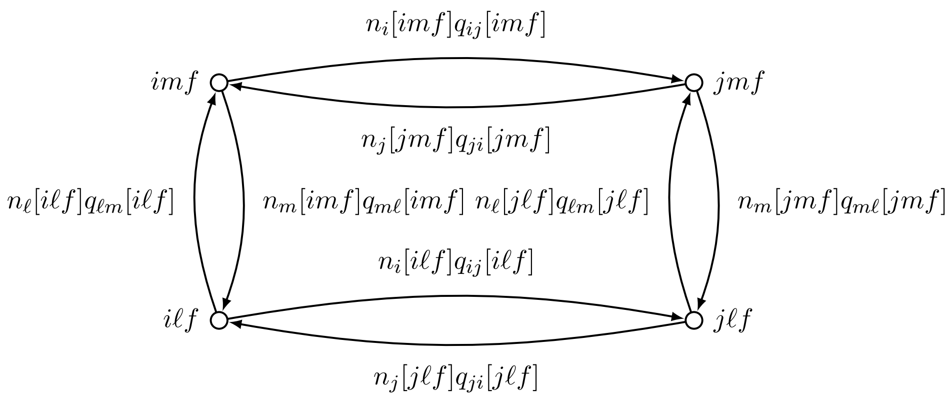

Figure 10: Many cycles of the directed, weighted transition graph for a master Markov chain for k coupled v-state automata correspond to Cartesian squares $(ij\square\ell m)f$ of the minimal cycle basis for the undirected, unweighted transition graph $G^{(k)}$, where $i, j, \ell, m \in \{a_1, a_2, \cdots, a_v\}$ and $f \in M_{k-2}(a_1, a_2, \cdots, a_v)$.

In other words, nonequilibrium steady states are forbidden. Markov chains have this property when the transition rates satisfy the Kolmogorov criterion, namely, equality of the product of rate constants in both directions around any cycle in the transition matrix Q [16]. For an isolated automaton with transition graph C_5 and transition matrix (7.1), the Komologorov criterion is

$$q_{ab}[a]\, q_{bc}\, q_{cd}\, q_{de}\, q_{ea} = q_{ae}\, q_{ed}\, q_{dc}\, q_{cb}\, q_{ba}. \tag{7.3}$$

Substituting the transition rates of the model specification, both those that are constant as well as $q_{ab}[a] = \lambda$ (7.2), yields the following condition on model parameters,

$$\lambda\mu^4 = \nu^5, \tag{7.4}$$

that ensures the stationary distribution of an isolated automaton will satisfy detailed balance.

By constructing the minimal cycle basis of $C_5^{(3)}$, we may verify that the master Markov chain for three *uncoupled* automata, each with transition graph C_5, also exhibits microscopic reversibility under the same parameter constraints.

To see this, recall that the minimal cycle basis of $C_5^{(3)}$ has 39 linearly independent cycles. Microscopic reversibility for the master Markov chain for three uncoupled automata requires that, given (7.4) and $\alpha, \beta, \gamma, \delta, \epsilon = 0$, 39 Komolgorov criteria are satisfied, each corresponding to a C_i in the MCB for $C_5^{(3)}$.

One cycle in the MCB for $C_5^{(3)}$ takes the form $C_5 f$ for fixed $f \in M_2(a, b, c, d, e)$. The Kolmogorov criterion for this cycle is

$$\begin{aligned} &n_a[af]q_{ab}[af] \cdot n_b[bf]q_{bc}[bf] \cdot n_c[cf]q_{cd}[cf] \cdot n_d[df]q_{de}[df] \cdot n_e[ef]q_{ea}[ef] \\ &= n_a[af]q_{ae}[af] \cdot n_e[ef]q_{ed}[ef] \cdot n_d[df]q_{dc}[df] \cdot n_c[cf]q_{cb}[cf] \cdot n_b[bf]q_{ba}[bf], \end{aligned}$$

where, for typographical efficiency, here and below, we drop the superscripted (3) on the transition rates $q^{(3)}[\cdot,\cdot]$ of $Q^{(3)}$. Canceling identical terms of the form $n_x[xf]$ gives

$$q_{ab}[af]\cdot q_{bc}[bf]\cdot q_{cd}[cf]\cdot q_{de}[df]\cdot q_{ea}[ef] = q_{ae}[af]\cdot q_{ed}[ef]\cdot q_{dc}[df]\cdot q_{cb}[cf]\cdot q_{ba}[bf].$$

When this expression is evaluated, the result is another instance of (7.4), which is satisfied by assumption.

The remaining 38 C_i in the MCB for $C_5^{(3)}$ are Cartesian squares (see Figure 10) that yield Kolmogorov criteria of the form,

$$\begin{aligned} &n_i[imf]q_{ij}[imf]\cdot n_m[jmf]q_{m\ell}[jmf]\cdot n_j[j\ell f]q_{ji}[j\ell f]\cdot n_\ell[i\ell f]q_{\ell m}[i\ell f] \\ &= n_m[imf]q_{m\ell}[imf]\cdot n_i[i\ell f]q_{ij}[i\ell f]\cdot n_\ell[j\ell f]q_{\ell m}[j\ell f]\cdot n_j[jmf]q_{ji}[jmf], \end{aligned}$$

where $f\in M_1(a,b,c,d,e)$. For $x\neq y$, $n_x[xyf] = n_x[x]+n_x[y]+n_x[f] = 1+n_x[f]$, so this criterion simplifies to

$$\begin{aligned} &(1+n_i[f])q_{ij}[imf]\cdot(1+n_m[f])q_{m\ell}[jmf]\cdot(1+n_j[f])q_{ji}[j\ell f]\cdot(1+n_\ell[f])q_{\ell m}[i\ell f] \\ &= (1+n_m[f])q_{m\ell}[imf]\cdot(1+n_i[f])q_{ij}[i\ell f]\cdot(1+n_\ell[f])q_{\ell m}[j\ell f]\cdot(1+n_j[f])q_{ji}[jmf]. \end{aligned}$$

Canceling identical terms of the form $(1+n_x[f])$ gives

$$q_{ij}[imf]\,q_{m\ell}[jmf]\,q_{ji}[j\ell f]\,q_{\ell m}[i\ell f] = q_{m\ell}[imf]\,q_{ij}[i\ell f]\,q_{\ell m}[j\ell f]\,q_{ji}[jmf] \tag{7.5}$$

for $(ij\Box\ell m)f\in\mathscr{B}=\Upsilon\cup\Omega$ with $f\in M_1(a_1,a_2,\ldots,a_v)$. When the automata are not coupled, $\alpha,\beta,\gamma,\delta,\epsilon=0$, the transition rates are not functions of the global state, and every factor on the left hand side has an equal partner on the right. Consequently, the 38 squares of $\mathscr{B}$ correspond to cycles in $Q^{(3)}$ that satisfy Komolgorov criteria.

We have shown that every cycle in the MCB for $C_5^{(3)}$, given by $C_5a\cup\mathscr{B}$, corresponds to a cycle in $Q^{(3)}$ that satisfies a Komolgorov criterion. For every cycle in $Q^{(3)}$, there is a representative in the cycle space $\mathscr{C}(C_5^{(3)})$ that is a linear combination (over the field $\mathbb{F}_2$) of elements of the MCB. It follows that every cycle in the master Markov chain satisfies the Komolgorov criterion. Thus, we conclude that the master Markov chain for three *uncoupled* automata exhibits microscopic reversibility provided an isolated automaton has this property. This property is expected, and yet important for model verification.

In many applications, it is important to establish whether or not model composition (i.e., the process of coupling the automata) results in a master Markov chain with nonequilibrium steady states, in spite of the fact that an isolated component automaton satisfies detailed balance. Such nonequilibrium steady states may be objects of study or, alternatively, the question may be relevant because the master Markov chain is not physically meaningful when model composition introduces the possibility of nonequilibrium steady states [15].

Our construction of minimal cycle bases of reduced graph powers provides conditions sufficient to ensure that model composition does not introduce nonequilibrium steady states. In general, it is sufficient that (7.5) hold of every Cartesian square $(ij\Box\ell m)f$ of the MCB for the undirected, unweighted transition graph $G^{(k)}$. In the example under discussion, many of these Komolgorov criteria do not involve the functional transition rate $q_{ab}[\cdot]$; these conditions are satisfied without placing any constraints on the coupling parameters $\alpha,\beta,\gamma,\delta,\epsilon$. The remaining constraints take the form

$$q_{ab}[amf]\,q_{m\ell}[bmf]\,q_{ba}[b\ell f]\,q_{\ell m}[a\ell f] = q_{m\ell}[amf]\,q_{ab}[a\ell f]\,q_{\ell m}[b\ell f]\,q_{ba}[bmf] \tag{7.6}$$

for $\ell m \in \{bc, cd, de, ae\}$. The Cartesian squares of concern are elements of the set $\{(ab\Box\ell m)f \mid \ell m \in \{bc, cd, de\}\} \subset \Upsilon$ and $(ae\Box ab)f \in \Omega$. Note that $\ell m \neq ab$ and, consequently, $q_{m\ell}[bmf] = q_{m\ell}[amf]$, $q_{\ell m}[a\ell f] = q_{\ell m}[b\ell f]$ and $q_{ba}[b\ell f] = q_{ba}[bmf] = \nu$. Thus, (7.6) simplifies to

$$q_{ab}[a\ell f] = q_{ab}[amf] \qquad \ell m \in \{bc, cd, de, ae\}. \tag{7.7}$$

To see how this requirement constrains the coupling parameters $\alpha, \beta, \gamma, \delta, \epsilon$, we expand both sides of (7.7) using (7.2), for example,

$$\begin{aligned} q_{ab}[a\ell f] &= \lambda + \alpha(n_a[a\ell f] - 1) + \beta n_b[a\ell f] + \gamma n_c[a\ell f] + \delta n_d[a\ell f] + \epsilon n_e[a\ell f] \\ &= \lambda + \alpha n_a[\ell f] + \beta n_b[\ell f] + \gamma n_c[\ell f] + \delta n_d[\ell f] + \epsilon n_e[\ell f] \end{aligned}$$

where we used $n_a[a\ell f] = 1 + n_a[\ell f]$. Subtracting both sides of (7.7) by $\lambda + \alpha n_a[f] + \beta n_b[f] + \gamma n_c[f] + \delta n_d[f] + \epsilon n_e[f]$ and using $n_x[\ell f] = n_x[\ell] + n_x[f]$ we obtain

$$\alpha n_a[\ell] + \beta n_b[\ell] + \gamma n_c[\ell] + \delta n_d[\ell] + \epsilon n_e[\ell] = \alpha n_a[m] + \beta n_b[m] + \gamma n_c[m] + \delta n_d[m] + \epsilon n_e[m]$$

for $\ell m \in \{bc, cd, de, ae\}$. These four equations yield four parameter constraints that ensure detailed balance in the master Markov chain for the three coupled stochastic automata, for example, $\ell m = bc$ gives

$$\alpha n_a[b] + \beta n_b[b] + \gamma n_c[b] + \delta n_d[b] + \epsilon n_e[b] = \alpha n_a[c] + \beta n_b[c] + \gamma n_c[c] + \delta n_d[c] + \epsilon n_e[c],$$

which implies that $\beta = \gamma$. Substituting $\ell m = cd$, de and ae, we find $\gamma = \delta$, $\delta = \epsilon$ and $\alpha = \epsilon$, respectively. We conclude that $\alpha = \beta = \gamma = \delta = \epsilon$.

In our example, the three automata are coupled when one or more of $\alpha, \beta, \gamma, \delta, \epsilon$ is positive. The analysis of Cartesian squares in the MCB for $C_5^{(3)}$ shows that coupling the three automata in the manner specified by (7.2) *will* introduce nonequilibrium steady states unless the coupling parameters are equal. This result is intuitive because $\sum_i n_i[\cdot] = k = 3$ and, consequently, equal coupling parameters $\alpha = \beta = \gamma = \delta = \epsilon$ correspond to a functional transition rate that, for every global state, evaluates to the constant $q_{ab}[\cdot] = \lambda + \alpha(k-1) = \lambda + 2\alpha$.

The simplicity of this parameter constraint is a consequence of evaluating (7.5) in the context of the example model specification. In general, the resulting constraints may be more complex and less restrictive. Any choice of model parameters that simultaneously satisfies

$$q_{ij}[imf]\, q_{m\ell}[jmf]\, q_{ji}[j\ell f]\, q_{\ell m}[i\ell f] = q_{m\ell}[imf]\, q_{ij}[i\ell f]\, q_{\ell m}[j\ell f]\, q_{ji}[jmf]$$

for $(ij\Box\ell m)f \in \mathscr{B} = \Upsilon \cup \Omega$ with $f \in M_{k-2}(a_1, a_2, \ldots, a_v)$ are conditions sufficient to ensure that the process of model composition (i.e., coupling k identical and indistinguishable v-state automata) does not introduce a violation of microscopic reversibility.

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Dan Archdeacon (11 May 1954 to 18 February 2015)

Professor Dan Archdeacon was a world-renowned mathematician, an intellectually engaging teacher, and a highly respected colleague. His research interests were in graph theory, combinatorics, and theoretical computer science. He published over seventy refereed papers in these fields, a majority of them in topological graph theory. His dissertation entitled "A Kuratowski Theorem for the Projective Plane" contains a proof of a ground-breaking and highly cited theorem which gives the extension of the Kuratowski theorem for the projective plane. This result has not been superseded.

His service to the mathematical community is widely appreciated. For over a decade, he was an editor of the Journal of Combinatorial Theory, Series B, and then he managed the offices of Journal of Graph Theory as its managing editor. He also served as a referee to over 30 journals in his field, and with Jeff Dinitz, he organised seven workshops in the Vermont Summer School on Combinatorics and Graph Theory. He started and maintained an online compendium of open problems in topological graph theory, which was an inspiration to several generations of the researchers in the field.

For most of his career, Dan taught at the University of Vermont, where he was named a University Scholar for the 2003/04 academic year. He was a Fulbright Teaching Fellow at the Riga Commerce School (Latvia), and visiting professor at the University of Auckland (New Zealand), Yokohama National University (Japan), the Technical University of Denmark, and the Open University (UK). He was an invited speaker at mathematics conferences across the globe. His talks were entertaining and always well attended.

Some of us had the privilege of working with Dan for more than two decades, on projects that resulted in many joint papers. Doing research with Dan was always a fantastic experience, both mathematically and socially. Dan was a very quick and sharp thinker. When tackling a research problem he quickly plunged into the absolute depth of the matter, but kept thinking in terms of a bigger picture of the situation at all stages. He was a true visionary, outlining avenues of research followed by others. At the same time he was a person with a great sense of humour, highly intelligent and very pleasant. He will be missed by an entire generation of scholars.

Dan visited Slovenia twice. In June 1999, he participated at the 4th Slovenian International Graph Theory Conference at Lake Bled, speaking of the representativity of planar graphs. During his second visit, in January 2014, he was already fighting cancer, but with the same optimism that he showed in his energy and passion for research. He spoke about toroidal triangulations being geometric, and initiated a problem on generalisation of Gauss words, which led to his last paper, published in this issue of Ars Mathematica Contemporanea.

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The award is named in memory of Dr Petra Šparl, a talented woman mathematician with a promising future who worked in graph theory and combinatorics, but died mid-career in 2016 after a battle with cancer.

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