IMFM
Institute of Mathematics, Physics and Mechanics
Jadranska 19, 1000 Ljubljana, Slovenia

Preprint series
Vol. 51 (2013), 1191
ISSN 2232-2094

MINUS PARTIAL ORDER IN RICKART RINGS

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Ljubljana, October 16, 2013

Minus partial order in Rickart rings[☆]

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Abstract

The minus partial order is already known for complex matrices and bounded linear operators on Hilbert spaces. We extend this notion to Rickart rings, and thus we generalize some well-known results.

Keywords: Rickart ring, minus partial order
2000 MSC: 16B99, 47A05

1. Introduction and motivation

Let $\mathcal{A}$ be a ring with the unit 1. If $M \subseteq \mathcal{A}$, then the right annihilator of M is denoted by $M^\circ = \{x \in \mathcal{A} : (\forall m \in M)\, mx = 0\}$, and the left annihilator of M is denoted by ${}^\circ M = \{x \in \mathcal{A} : (\forall m \in M)\, xm = 0\}$. M° is the right ideal of $\mathcal{A}$, and ${}^\circ M$ is a left ideal of $\mathcal{A}$. Particularly, if $a \in \mathcal{A}$ and $M = \{a\}$, then we shortly use $a^\circ = \{a\}^\circ$ and ${}^\circ a = {}^\circ\{a\}$.

The set of idempotents of $\mathcal{A}$ is denoted by $\mathcal{A}^\bullet = \{p \in \mathcal{A} : p^2 = p\}$.

[☆]This work is a part of a bilateral project between Serbia and Slovenia: "Preservers, operator and matrix equations with applications".

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[1]The authors are supported by the Ministry of Education, Science and Technological Development of Serbia, under grant no. 174007.

[2]The author is supported by the Ministry of Science of Slovenia, under grant no. BI-RS/12-13-001.

A ring $\mathcal{A}$ is a Rickart ring, if for every $a \in \mathcal{A}$ there exist some $p, q \in \mathcal{A}^\bullet$ such that $a^\circ = p\mathcal{A}$ and ${}^\circ a = \mathcal{A}q$. Note that if $\mathcal{A}$ is a Rickart ring, then $\mathcal{A}$ has a unity element. The proof is similar to that used for Rickart *-rings [1].

Let H be a Hilbert space, $\mathcal{L}(H)$ the set of all bounded linear operators on H, and let $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the range and the null-space of $A \in \mathcal{L}(H)$. If $M \subseteq H$, we will denote by $\overline{M}$ the norm-closure of M in H. For a finite dimensional Hilbert space H Hartwig [3] defined a partial order in $\mathcal{L}(H)$ in the following way:

$$A \preceq B \quad \text{if and only if} \quad \operatorname{rank}(B-A) = \operatorname{rank}(B) - \operatorname{rank}(A).$$

He also observed that there exists another equivalent definition of this order, namely

$$A \preceq B \quad \text{if and only if} \quad A^- A = A^- B \ \text{ and } \ AA^- = BA^-,$$

where A^- is a generalized inner inverse of A, i.e. $AA^-A = A$. The partial order $\preceq$ is thus usually called the minus partial order.

In [6] Šemrl extended the minus partial order in $\mathcal{L}(H)$ for an arbitrary Hilbert space H. The notion of a rank of an operator (equivalently, a rank of a finite complex matrix) can not be applied for bounded linear operators on general Hilbert spaces. Moreover, $A \in \mathcal{L}(H)$ has a generalized inner inverse if and only if its image is closed. Since Šemrl could not use the notion of rank of an operator and since he did not want to restrict his attention only to closed range operators, he found a new approach how to extend the minus partial order. He introduced another equivalent definition of the minus partial order: for $A, B \in \mathcal{L}(H)$, where H is a finite dimensional Hilbert space, we have $A \preceq B$ if and only if there exist idempotent operators $P, Q \in \mathcal{L}(H)$ such that $\mathcal{R}(P) = \mathcal{R}(A)$, $\mathcal{N}(Q) = \mathcal{N}(A)$, $PA = PB$ and $AQ = BQ$. Recall that the range of an idempotent operator $P \in \mathcal{L}(H)$, where H can be a general Hilbert space, is closed. Using the same equations, only adding the closure on $\mathcal{R}(A)$ Šemrl extended the concept of the minus partial order in $\mathcal{L}(H)$ for an arbitrary Hilbert space H:

Definition 1.1. *Let H be a Hilbert space and let $A, B \in \mathcal{L}(H)$. Then $A \preceq B$ if and only if there exists idempotents $P, Q \in (\mathcal{L}(H))^\bullet$ such that the following hold:*

(1) $\mathcal{R}(P) = \overline{\mathcal{R}(A)}$;
(2) $\mathcal{N}(A) = \mathcal{N}(Q)$;

(3) $PA = PB$;
(4) $AQ = BQ$.

Šemrl proved that $\preceq$ is indeed a partial order in $B(H)$. Moreover, it is proved in [6] that Šemrl's definition is a proper extension of Hartwig's definition of the minus partial order of matrices. Also, in [5], the minus partial order is generalized on Banach space operators which have generalized inverses.

We prove the following result, which allows us to consider the algebraic version of the minus partial order. First, we need the following auxiliary statement.

Lemma 1.2. *Let H, K, L, N be Hilbert spaces, $A_1 \in \mathcal{L}(H, L)$ and $A_2 \in \mathcal{L}(K, L)$. Then the following statements are equivalent:*
(1) *For every $B \in \mathcal{L}(L, N)$ the following equivalence holds: ($BA_1 = 0$ and $BA_2 = 0$) if and only if $B = 0$;*
(2) $\overline{\mathcal{R}(A_1) + \mathcal{R}(A_2)} = L$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $\overline{\mathcal{R}(A_1) + \mathcal{R}(A_2)} \neq L$. Then there exists a non-trivial closed subspace L_1 of L, such that $\overline{\mathcal{R}(A_1) + \mathcal{R}(A_2)} \oplus L_1 = L$. Let $B_1 \in \mathcal{L}(L_1, N)$ be any non-zero bounded linear operator, and define $B \in \mathcal{L}(L, N)$ as follows:

$$Bx = \begin{cases} 0, & x \in \overline{\mathcal{R}(A_1) + \mathcal{R}(A_2)}, \\ B_1 x, & x \in L_1. \end{cases}$$

Obviously, $B \neq 0$, $BA_1 = 0$ and $BA_2 = 0$.
(2) $\Longrightarrow$ (1): Obvious. □

Definition 1.3. *Let H be a Hilbert space and let $A, B \in \mathcal{L}(H)$. Then we write $A \preceq B$ if and only if there exist idempotent operators $P, Q \in \mathcal{L}(H)$ such that the following hold:*
(1) $^{\circ}A = \mathcal{L}(H) \cdot (I - P)$;
(2) $A^{\circ} = (I - Q) \cdot \mathcal{L}(H)$;
(3) $PA = PB$;
(4) $AQ = BQ$.

Theorem 1.4. *The minus partial order given by Definition 1.1 is on $\mathcal{L}(H)$ equivalent to the order given by Definition 1.3.*

Proof. First, let us prove that Definition 1.1 implies Definition 1.3. Let $\preceq$ be the order defined with Definition 1.1 and suppose $A \preceq B$, $A, B \in \mathcal{L}(H)$. Let P be a projection from H onto $\overline{\mathcal{R}(A)}$ and let $I - Q$ be a projection from H onto $\mathcal{N}(A)$. This is the same choice of projections as in [6], so statements (3) and (4) of this theorem hold.

Let $D \in \mathcal{L}(H)$. Then $D(I-P)A = 0$, since $\mathcal{R}(A) \subset \mathcal{R}(P) = \mathcal{N}(I-P)$. Thus, $\mathcal{L}(H) \cdot (I-P) \subset {}^\circ A$. On the other hand, suppose that $D \in {}^\circ A$. Then $DA = 0$ and $\mathcal{R}(P) \subset \mathcal{N}(D)$ since $\mathcal{N}(D)$ is closed. Thus, $\mathcal{N}(P^*) \supset \mathcal{R}(D^*)$, or equivalently $\mathcal{R}(I-P^*) \supset \mathcal{R}(D^*)$. By the Douglas theorem [2], there exists some $C \in \mathcal{L}(H)$ such that $D^* = (I-P^*)C$, implying $D = C^*(I-P)$. Hence, ${}^\circ A = \mathcal{L}(H) \cdot (I-P)$.

If $D \in \mathcal{L}(H)$, since $\mathcal{R}(I-Q) = \mathcal{N}(A)$, we get $A(I-Q)D = 0$. Thus, $(I-Q) \cdot \mathcal{L}(H) \subset A^\circ$. On the other hand, let $D \in A^\circ$. Then $\mathcal{R}(D) \subset \mathcal{N}(A) = \mathcal{R}(I-Q)$. Again, using the Douglas theorem [2], we conclude that there exists some $C \in \mathcal{L}(H)$ such that $D = (I-Q)C$. It follows that $A^\circ = (I-Q) \cdot \mathcal{L}(H)$.

Let us now prove that Definition 1.3 implies Definition 1.1. Suppose $A \preceq B$, $A, B \in \mathcal{L}(H)$, where $\preceq$ is the order defined with Definition 1.3.

From (1) we get $(I-P)A = 0$, so $\mathcal{R}(A) \subset \mathcal{N}(I-P) = \mathcal{R}(P)$, and consequently $\overline{\mathcal{R}(A)} \subset \mathcal{R}(P)$. Since $H = \mathcal{R}(P) \oplus \mathcal{N}(P)$, every operator from $\mathcal{L}(H)$ has a 2×2 matrix form with respect to this decomposition. Particularly, from $\mathcal{R}(A) \subset \mathcal{R}(P)$ we obtain the following:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(P) \\ \mathcal{N}(P) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(P) \\ \mathcal{N}(P) \end{bmatrix}.$$

Now we use the fact ${}^\circ A = \mathcal{L}(H) \cdot (I-P)$. Notice that $B \in \mathcal{L}(H) \cdot (I-P)$ if and only if $B = B(I-P)$. If

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(P) \\ \mathcal{N}(P) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(P) \\ \mathcal{N}(P) \end{bmatrix},$$

then $B = B(I-P)$ is equivalent to $B_1 = 0$ and $B_3 = 0$. On the other hand, $BA = 0$ if and only if

$$\begin{bmatrix} B_1A_1 & B_1A_2 \\ B_3A_1 & B_3A_2 \end{bmatrix} = 0.$$

So we have the equivalence:

$$(B_1A_1 = 0, B_1A_2 = 0, B_3A_1 = 0, B_3A_2 = 0) \iff (B_1 = 0, B_3 = 0).$$

From Lemma 1.2 we know that $\overline{R(A_1) + \mathcal{R}(A_2)} = \mathcal{R}(P)$. Since A_1 and A_2 act on different subspaces, we actually have $\overline{\mathcal{R}(A)} = \mathcal{R}(P)$.

Now, from the condition (2) of this theorem, using the result we have just proved, the following hold:

$$\begin{aligned} A^\circ = (I-Q)\cdot\mathcal{L}(H) \quad &\iff \quad {}^\circ(A^*) = \mathcal{L}(H)\cdot(I-Q^*) \iff \mathcal{R}(Q^*) = \overline{\mathcal{R}(A^*)} \\ &\iff \quad \mathcal{R}(I-Q) = \mathcal{N}(Q) = \mathcal{N}(A). \end{aligned}$$

Hence, $A \preceq B$. □

2. Results in rings

Previous Theorem 1.4 suggests the following definition of the minus partial order. Since some preliminary results can be proved in a general setting, we shall in this section use that $\mathcal{A}$ is a ring with the unit 1.

Definition 2.1. *Let $\mathcal{A}$ be a ring with the unit 1, and let $a, b \in \mathcal{A}$. Then we write $a \preceq b$ if and only if there exists idempotents $p, q \in \mathcal{A}^\bullet$ such that the following hold:*

(1) ${}^\circ a = \mathcal{A}(1-p)$;
(2) $a^\circ = (1-q)\mathcal{A}$;
(3) $pa = pb$;
(4) $aq = bq$.

We call $\preceq$ the minus partial order on $\mathcal{A}$. In the next section we will prove that when $\mathcal{A}$ is a Rickart ring, $\preceq$ is indeed a partial order.

Notice that from (1) we obtain $(1-p)a = 0$ so $a = pa$. Similarly, $a = aq$.

We need some auxiliary results.

Lemma 2.2. *Let $p, q \in \mathcal{A}^\bullet$. Then*

(i) $\mathcal{A}(1-p) = {}^\circ p$;

(ii) $(1-q)\mathcal{A} = q^\circ$.

Proof. We have $\mathcal{A}(1-p) \subseteq {}^\circ p$, since $(1-p)p = 0$. Suppose that for $u \in \mathcal{A}$, $up = 0$. Then $u = u(1-p) \in \mathcal{A}(1-p)$. The proof of (ii) is similar. □

It follows that we can replace the conditions (1) and (2) of Definition 2.1 by the conditions ${}^\circ a = {}^\circ p$ and $a^\circ = q^\circ$ respectively.

Lemma 2.3. *Let $p \in \mathcal{A}^\bullet$ and $a \in \mathcal{A}$. Then*

(i) $({}^\circ p)^\circ = p\mathcal{A}$;

(ii) ${}^\circ a = \mathcal{A}(1-p) \Longleftrightarrow ({}^\circ a)^\circ = ({}^\circ p)^\circ$.

Proof. (i): By Lemma 2.2,

$$({}^\circ p)^\circ = (\mathcal{A}(1-p))^\circ = \{u \in \mathcal{A} : (\forall x \in \mathcal{A})\, x(1-p)u = 0\} = \{u \in \mathcal{A} : (1-p)u = 0\} = p\mathcal{A}$$

(ii): The "only if" part follows from Lemma 2.2. Now, suppose that $({}^\circ a)^\circ = ({}^\circ p)^\circ$ i.e. $({}^\circ a)^\circ = p\mathcal{A}$. Let $u \in {}^\circ a$. As $p \in p\mathcal{A} = ({}^\circ a)^\circ$ we have $up = 0$ so $u = u(1-p) \in \mathcal{A}(1-p)$. On the other hand, suppose that $u \in \mathcal{A}(1-p)$ i.e. $up = 0$. As $a \in ({}^\circ a)^\circ = p\mathcal{A}$ we have $a = pa$ so $ua = upa = 0$. □

In the same manner we obtain the following lemma.

Lemma 2.4. *Let $q \in \mathcal{A}^\bullet$ and $a \in \mathcal{A}$. Then*

(i) ${}^\circ(q^\circ) = \mathcal{A}q$;

(ii) $a^\circ = (1-q)\mathcal{A} \Longleftrightarrow {}^\circ(a^\circ) = {}^\circ(q^\circ)$.

It follows that we can replace the conditions (1) and (2) of Definition 2.1 by the conditions $({}^\circ a)^\circ = p\mathcal{A}$ and ${}^\circ(a^\circ) = \mathcal{A}q$ respectively.

Our definition of order $\preceq$ is a proper extension of well known partial order on the set of idempotents.

Theorem 2.5. *Let $a, b \in \mathcal{A}^\bullet$. Then $a \preceq b$ if and only if $ab = ba = a$.*

Proof. Suppose that $a, b \in \mathcal{A}^\bullet$ and $ab = ba = a$. By Lemma 2.2 we have ${}^\circ a = \mathcal{A}(1-a)$, $a^\circ = (1-a)\mathcal{A}$ and by assumption $aa = ab$, $aa = ba$ so $a \preceq b$. Now suppose that $a \preceq b$. There exist $p, q \in \mathcal{A}^\bullet$ as in Definition 2.1 so $ab = (pa)b = (pb)b = pb = a$ and $ba = b(aq) = b(bq) = bq = a$. □

Recall that von Neumann regular ring is a ring $\mathcal{A}$ such that for every $a \in \mathcal{A}$ there exists an $x \in \mathcal{A}$ such that $axa = a$. The following theorem shows that, when $\mathcal{A}$ is a von Neumann regular ring, $\preceq$ order coincides with well known minus partial order which is defined by $a \leq b$ if there exists an $x \in \mathcal{A}$ such that $ax = bx$ and $xa = xb$ where $axa = a$. Thus, the minus partial order in von Neumann regular ring is defined in the same way as in $\mathcal{L}(H)$ where H is a finite dimensional Hilbert space.

Theorem 2.6. *Suppose that $\mathcal{A}$ is a von Neumann regular ring with the unit 1 and let $a, b \in \mathcal{A}$. Then $a \preceq b$ if and only if $a \leq b$.*

Proof. Suppose that $a \preceq b$ and let $p, q \in \mathcal{A}^\bullet$ be as in Definition 2.1. Since $\mathcal{A}$ is von Neumann regular ring, there exists an $x \in \mathcal{A}$ such that $axa = a$. Set $y = qxp$. We have $aya = a(qxp)a = axa = a$, $ay = aqxp = bqxp = by$, $ya = qxpa = qxpb = yb$ so $a \leq b$.

Now suppose that $a \leq b$. There exists an $x \in \mathcal{A}$ such that $axa = a$, $ax = bx$, $xa = xb$. Set $p = ax$ and $q = xa$. Then $p \in \mathcal{A}^\bullet$ and $1 - p \in {}^\circ a$. On the other hand if $u \in {}^\circ a$ then $up = u(ax) = 0$, so $u = u(1-p)$, ${}^\circ a = \mathcal{A}(1-p)$. Moreover, $pa = axa = axb = pb$. Similarly, $q \in \mathcal{A}^\bullet$, $a^\circ = (1-q)A$, $aq = bq$, so $a \preceq b$. □

Since we can not use decompositions of spaces induced by projections, we have to use idempotents appropriately.

Remark 1. We say that equality $1 = e_1 + e_2 + \cdots + e_n$, where $e_1, e_2, \ldots, e_n \in \mathcal{A}^\bullet$, is a decomposition of the identity of the ring $\mathcal{A}$ if e_i and e_j are orthogonal for $i \neq j$, i.e. $e_i e_j = 0$ for $i \neq j$. Let $1 = e_1 + \cdots + e_n$ and $1 = f_1 + \cdots + f_n$ be two decompositions of the identity of a ring $\mathcal{A}$. For any $x \in \mathcal{A}$ we have

$$x = 1 \cdot x \cdot 1 = (e_1 + \cdots + e_n)x(f_1 + \cdots + f_n) = \sum_{i,j=1}^{n} e_i x f_j,$$

and above sum defines a decomposition of $\mathcal{A}$ into a direct sum of groups:

$$\mathcal{A} = \bigoplus_{i,j=1}^{n} e_i \mathcal{A} f_j. \tag{2.1}$$

It is convenient to write x as a matrix

$$x = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}_{e \times f},$$

where $x_{ij} = e_i x f_j \in e_i \mathcal{A} f_j$.

If $x = (x_{ij})_{e\times f}$ and $y = (y_{ij})_{e\times f}$, then it is obvious that $x + y = (x_{ij} + y_{ij})_{e\times f}$. Moreover, if $1 = g_1 + \cdots + g_n$ is a decomposition of the identity of $\mathcal{A}$ and $z = (z_{ij})_{f\times g}$, then, by the orthogonality of idempotents involved, $xz = \left(\sum_{k=1}^{n} x_{ik}z_{kj}\right)_{e\times g}$. Thus, if we have decompositions of the identity of $\mathcal{A}$, then the usual algebraic operations in $\mathcal{A}$ can be interpreted as simple operations between appropriate $n\times n$ matrices over $\mathcal{A}$.

When $e_i = f_i$, $i = \overline{1,n}$, the decomposition (2.1) is known as the two-sided Peirce decomposition of the ring $\mathcal{A}$, [4]. When $n = 2$, $e_1 = p$ and $f_1 = q$ then we write

$$x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{p\times q}$$

We prove the following result.

Theorem 2.7. *Let $\mathcal{A}$ be a ring with the unit, and let $a, b \in \mathcal{A}$. Then $a \preceq b$ if and only if there exists idempotents $p, q \in \mathcal{A}^\bullet$ such that the following three conditions hold:*

(1) $a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q}$ *and* $b = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p\times q}$*;*

(2) *If $z \in \mathcal{A}p$ and $za_1 = 0$, then $z = 0$;*

(3) *If $z \in q\mathcal{A}$ and $a_1 z = 0$, then $z = 0$.*

Proof. Suppose that $a \preceq b$ and let $p, q \in \mathcal{A}^\bullet$ be corresponding idempotents. As we have seen $a = pa = aq = paq$, so

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q}.$$

Let

$$b = \begin{bmatrix} b_4 & b_2 \\ b_3 & b_1 \end{bmatrix}_{p\times q}.$$

From $p(b-a) = 0$ we get

$$\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} b_4 - a_1 & b_2 \\ b_3 & b_1 \end{bmatrix}_{p\times q} = \begin{bmatrix} p(b_4 - a_1) & pb_2 \\ 0 & 0 \end{bmatrix}_{p\times q} = 0,$$

implying that $p(b_4 - a_1) = 0$ and $pb_2 = 0$. Since $pa_1 = a_1$, $pb_4 = b_4$, and $pb_2 = b_2$, we get $a_1 = b_4$ and $b_2 = 0$. Analogously, from $(b-a)q = 0$ we get $b_3 = 0$. Thus, the statement (1) of this theorem is proved.

In order to prove the statement (2), suppose that $z \in \mathcal{A}p$ and $za_1 = 0$. Since $a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q}$ we get $za = 0$, so $z \in {}^\circ a = \mathcal{A}(1-p) = {}^\circ p$, i.e. $z = zp = 0$. The statement (3) can be proved proved in the same manner.

Now, we suppose that there exists idempotents $p, q \in \mathcal{A}^\bullet$ such that statements (1) – (3) of this theorem hold. We immediately obtain

$$p(a-b) = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} 0 & 0 \\ 0 & -b_1 \end{bmatrix}_{p\times q} = 0 \text{ and } (a-b)q = \begin{bmatrix} 0 & 0 \\ 0 & -b_1 \end{bmatrix}_{p\times q} \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}_{q\times q} = 0.$$

Now, we prove that ${}^\circ a = \mathcal{A}(1-p)$. If $y \in \mathcal{A}(1-p)$, then $y = \begin{bmatrix} 0 & y_2 \\ 0 & y_4 \end{bmatrix}_{q\times p}$. It is easy to see that $ya = 0$. Thus, we established $\mathcal{A}(1-p) \subseteq {}^\circ a$.

To prove the opposite inclusion, suppose that $z \in {}^\circ a$. Then $z = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}_{q\times p}$ and

$$0 = za = \begin{bmatrix} z_1 a_1 & 0 \\ z_3 a_1 & 0 \end{bmatrix}_{q\times q}.$$

We conclude $z_1 a_1 = z_3 a_1 = 0$. Since $z_1, z_3 \in \mathcal{A}p$, (2) shows that $z_1 = z_3 = 0$. Thus, $z = \begin{bmatrix} 0 & z_2 \\ 0 & z_4 \end{bmatrix}_{q\times p} \in \mathcal{A}(1-p)$. Hence, we proved ${}^\circ a \subseteq \mathcal{A}(1-p)$.

In the same manner we can prove that $a^\circ = (1-q)\mathcal{A}$. □

3. Minus partial order in Rickart rings

The idempotents in Definition 2.1 need not be unique. Write

$$\mathrm{LP}(a) := \{p \in \mathcal{A}^\bullet : {}^\circ a = \mathcal{A}(1-p)\},$$
$$\mathrm{RP}(a) := \{q \in \mathcal{A}^\bullet : a^\circ = (1-q)\mathcal{A}\}.$$

When $\mathcal{A}$ is Rickart ring then LP(a) and RP(a) are nonempty. Lemma 2.2 gives characterizations $\mathrm{LP}(a) = \{p \in \mathcal{A}^\bullet : {}^\circ a = {}^\circ p\}$ and $\mathrm{RP}(a) = \{q \in \mathcal{A}^\bullet : a^\circ = q^\circ\}$.

Lemma 3.1. *Let $a \in \mathcal{A}$, $p \in LP(a)$ and $q \in RP(a)$. (Such idempotents exist if $\mathcal{A}$ is a Rickart ring.) Then*

(i) $LP(a) = \left\{ \begin{bmatrix} p & p_1 \\ 0 & 0 \end{bmatrix}_{p\times p} : p_1 \in p\mathcal{A}(1-p) \right\}$;

(ii) $RP(a) = \left\{ \begin{bmatrix} q & 0 \\ q_1 & 0 \end{bmatrix}_{q\times q} : q_1 \in (1-q)\mathcal{A}q \right\}.$

Proof. From $(1-p)a = 0 = a(1-q)$ we conclude that $a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q}$. If $p' = \begin{bmatrix} p & p_1 \\ 0 & 0 \end{bmatrix}_{p\times p}$ then $p'^2 = p'$ and

$$(1-p')a = \begin{bmatrix} 0 & -p_1 \\ 0 & 1-p \end{bmatrix}_{p\times p} \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} = 0. \tag{3.1}$$

It is easily seen that $u = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}_{p\times p} \in {}^\circ a$ if and only if $u_1 = u_3 = 0$. From $\begin{bmatrix} 0 & u_2 \\ 0 & u_4 \end{bmatrix}_{p\times p} \begin{bmatrix} p & p_1 \\ 0 & 0 \end{bmatrix}_{p\times p} = 0$ we conclude ${}^\circ a \subseteq {}^\circ p'$. Now, (3.1) and Lemma 2.2 give ${}^\circ a = \mathcal{A}(1-p')$, that is $p' \in \mathrm{LP}(a)$.

Suppose now that $p' = \begin{bmatrix} p_2 & p_1 \\ p_3 & p_4 \end{bmatrix}_{p\times p} \in \mathrm{LP}(a)$. Then ${}^\circ p' = {}^\circ a = {}^\circ p$, so

$$0 = (1-p')p = \begin{bmatrix} p-p_2 & -p_1 \\ -p_3 & 1-p-p_4 \end{bmatrix}_{p\times p} \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} p-p_2 & 0 \\ -p_3 & 0 \end{bmatrix}_{p\times p}$$

and hence $p_2 = p$ and $p_3 = 0$. From ${}^\circ a \subseteq {}^\circ p'$ it follows

$$0 = \begin{bmatrix} 0 & u_2 \\ 0 & u_4 \end{bmatrix}_{p\times p} \begin{bmatrix} p & p_1 \\ 0 & p_4 \end{bmatrix}_{p\times p} = \begin{bmatrix} 0 & u_2p_4 \\ 0 & u_4p_4 \end{bmatrix}_{p\times p},$$

so $u_4p_4 = 0$, for every $u_4 \in (1-p)\mathcal{A}(1-p)$. Setting $u_4 = 1-p$ we get $p_4 = 0$. Thus, the statement (i) of the theorem is proved. In the same manner we can prove the statement (ii). □

Corollary 3.2. *Let $a, b \in \mathcal{A}$. Suppose that $a \preceq b$ and let $p, q \in \mathcal{A}^\bullet$ be corresponding idempotents. Then*

$$\{p' \in LP(a) : a = p'b\} = \left\{ \begin{bmatrix} p & p_1 \\ 0 & 0 \end{bmatrix}_{p\times p} : p_1 \in p\mathcal{A}(1-p),\ p_1b_1 = 0 \right\} \tag{3.2}$$

$$\{q' \in RP(a) : a = bq'\} = \left\{ \begin{bmatrix} q & 0 \\ q_1 & 0 \end{bmatrix}_{q\times q} : q_1 \in (1-q)\mathcal{A}q,\ b_1q_1 = 0 \right\},$$

where b_1 is as in Theorem 2.7.

Proof. We will prove only the equality (3.2); the proof of the other one is analogous. Since $a \preceq b$, Theorem 2.7 gives

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q}, \quad b = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p\times q}.$$

If p' belongs to the set on the right hand side of (3.2) then, by Lemma 3.1, $p' \in \mathrm{LP}(a)$. Also,

$$p'b = \begin{bmatrix} p & p_1 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p\times q} = \begin{bmatrix} a_1 & p_1 b_1 \\ 0 & 0 \end{bmatrix}_{p\times q} = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} = a.$$

To prove the opposite inclusion, suppose that $p' \in \mathrm{LP}(a)$ and $a = p'b$. Lemma 3.1 leads to $p = \begin{bmatrix} p & p_1 \\ 0 & 0 \end{bmatrix}_{p\times p}$. Now, $a = p'b$ gives

$$\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} = a = p'b = \begin{bmatrix} a_1 & p_1 b_1 \\ 0 & 0 \end{bmatrix}_{p\times q},$$

so $p_1 b_1 = 0$. □

However, to prove that $\preceq$ is actually a partial order, we need the assumption that $\mathcal{A}$ is Rickart ring.

We now prove the main result of this section.

Theorem 3.3. *Let $\mathcal{A}$ be a Rickart ring. Then $\preceq$ is a partial order in $\mathcal{A}$.*

Proof. Since $\mathcal{A}$ is a Rickart ring, for any $a \in \mathcal{A}$ there exist idempotents $p, q \in \mathcal{A}^\bullet$, such that ${}^\circ a = \mathcal{A}(1-p)$ and $a^\circ = (1-q)\mathcal{A}$. Now the reflexivity of $\preceq$ follows.

To prove the antisymmetry, suppose that $a \preceq b$ and $b \preceq a$. Then

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q}, \quad b = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p\times q}, \tag{3.3}$$

and there exist $r, s \in \mathcal{A}^\bullet$ such that $b = ra = as$. Let $r = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}_{p\times p}$. We have

$$\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p\times q} = b = ra = \begin{bmatrix} r_1 a_1 & 0 \\ r_3 a_1 & 0 \end{bmatrix}_{p\times q},$$

so $b_1 = 0$ and hence $a = b$.

We have to show transitivity. Let $a \preceq b$ and $b \preceq c$. Then there exist idempotents $p, q, r, s \in \mathcal{A}^\bullet$ such that a and b have the matrix forms as in (3.3), ${}^\circ a_1 = {}^\circ p$, $a_1^\circ = q^\circ$ and ${}^\circ b = \mathcal{A}(1-r) = {}^\circ r$, $b^\circ = (1-s)\mathcal{A} = s^\circ$, $b = rc = cs$. Suppose that

$$r = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}_{p\times p} \quad \text{and} \quad c = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_{p\times q}.$$

Since

$$0 = (1-r)b = \begin{bmatrix} p - r_1 & -r_2 \\ -r_3 & 1-p-r_4 \end{bmatrix}_{p\times p} \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p\times q} = \begin{bmatrix} (p-r_1)a_1 & -r_2 b_1 \\ -r_3 a_1 & (1-p-r_4)b_1 \end{bmatrix}_{p\times q},$$

${}^\circ a_1 = {}^\circ p$ shows that $0 = (p - r_1)p = p - r_1$ and $0 = r_3 p = r_3$. Also, $r_2 b_1 = 0$. From $b = rc$ we conclude that

$$\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p\times q} = \begin{bmatrix} p & r_2 \\ 0 & r_4 \end{bmatrix}_{p\times p} \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_{p\times q} = \begin{bmatrix} c_1 + r_2 c_3 & c_2 + r_2 c_4 \\ r_4 c_3 & r_4 c_4 \end{bmatrix}_{p\times q},$$

so

$$a_1 = c_1 + r_2 c_3 \quad \text{and} \quad 0 = c_2 + r_2 c_4. \tag{3.4}$$

Let $p' = \begin{bmatrix} p & r_2 \\ 0 & 0 \end{bmatrix}_{p\times p}$. From Corollary 3.2 it follows that

$${}^\circ a = \mathcal{A}(1 - p') \quad \text{and} \quad a = p'b. \tag{3.5}$$

Since,

$$p'c = \begin{bmatrix} c_1 + r_2 c_3 & c_2 + r_2 c_4 \\ 0 & 0 \end{bmatrix}_{p\times q},$$

(3.4) shows that

$$p'c = a. \tag{3.6}$$

Similar consideration shows that if $s = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}_{q\times q}$ than $s_1 = q$, $s_2 = 0$ and that for $q' = \begin{bmatrix} q & 0 \\ s_3 & 0 \end{bmatrix}_{q\times q}$ we have

$$a^\circ = (1 - q')\mathcal{A}, \quad a = bq' = cq'. \tag{3.7}$$

By definition, from (3.5)–(3.7) we obtain that $a \preceq c$. □

Moreover, from the proof of Theorem 3.3 it follows that if $a \preceq b$ and $b \preceq c$ then there exist common idempotents p' and q' showing that $a \preceq b$ and $a \preceq c$.

Theorem 3.4. *Let $\mathcal{A}$ be a Rickart ring and $a, b \in \mathcal{A}$. Then $a \preceq b$ if and only if there exist decompositions of the identity of the ring $\mathcal{A}$*

$$1 = e_1 + e_2 + e_3, \quad 1 = f_1 + f_2 + f_3$$

such that the following five conditions hold:

(1) $a = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e \times f}$ *and* $b = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e \times f}$;

(2) *If $z \in \mathcal{A}e_1$ and $za_1 = 0$, then $z = 0$;*

(3) *If $z \in f_1\mathcal{A}$ and $a_1 z = 0$, then $z = 0$;*

(4) *If $z \in \mathcal{A}e_2$ and $zb_1 = 0$, then $z = 0$;*

(5) *If $z \in f_2\mathcal{A}$ and $b_1 z = 0$, then $z = 0$.*

Proof. "If" part follows from Theorem 2.7. Now, suppose that $a \preceq b$. By Theorem 2.7 we have

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p \times q} \quad \text{and} \quad b = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p \times q},$$

such that $z = 0$ whenever $z \in \mathcal{A}p$ and $za_1 = 0$, and $z = 0$ whenever $z \in q\mathcal{A}$ and $a_1 z = 0$. $\mathcal{A}$ is a Rickart ring so there exist $r, s \in \mathcal{A}^\bullet$ such that ${}^\circ b_1 = \mathcal{A}(1-r) = {}^\circ r$ and $b_1^\circ = (1-s)\mathcal{A} = s^\circ$. Notice that $pr = 0$. Indeed, since $b_1 \in (1-p)\mathcal{A}(1-q)$, we have $pb_1 = 0$ and therefore $p \in {}^\circ b_1 = {}^\circ r$. Let $r' := r - rp(1-r) = r - rp = r(1-p)$. We have $r' \in \mathcal{A}^\bullet$ since $r'^2 = (r-rp)(r-rp) = r - rp = r'$. Our next claim is that ${}^\circ b_1 = {}^\circ r'$. If $ub_1 = 0$ then $ur = 0$ so $ur' = 0$. On the other hand, $(1-r')b_1 = (1-r+rp)rb_1 = 0$, due to $b_1 = rb_1$. Thus

$$ {}^\circ b_1 = {}^\circ r' = \mathcal{A}(1-r'). \tag{3.8}$$

Next, $pr = 0$ implies $pr' = 0$. Moreover, $r'p = r(1-p)p = 0$. Set $e_1 = p$, $e_2 = r'$, and $e_3 = 1 - p - r'$. Then $1 = e_1 + e_2 + e_3$ is decomposition of the identity of the ring $\mathcal{A}$ and from (3.8) it follows that $zb_1 = 0$ implies $z = 0$ when $z \in \mathcal{A}e_2$.

Now, set $f_1 = q$, $f_2 = (1-q)s$ and $f_3 = 1 - f_1 - f_2$. With similar consideration we can show that $1 = f_1 + f_2 + f_3$ is the decomposition of the identity of the ring $\mathcal{A}$ and that condition (5) of this theorem holds. Of course, statements (2) and (3) are satisfied by Theorem 2.7 since $e_1 = p$ and $f_1 = q$. As ${}^\circ b_1 = \mathcal{A}(1-r)$ and $b_1^\circ = (1-s)\mathcal{A}$ we have $e_2 b f_2 = r(1-p)b(1-q)s = r b_1 s = b_1$. The statement (1) is proved. □

Note that the statements (1)–(5) of the previous theorem are equivalent to

$$e_1 \in \mathrm{LP}(a),\ e_2 \in \mathrm{LP}(b-a),\ f_1 \in \mathrm{RP}(a),\ f_2 \in \mathrm{RP}(b-a).$$

Corollary 3.5. *Suppose that $\mathcal{A}$ is a Rickart ring and $a, b \in \mathcal{A}$. Then $a \preceq b$ if and only if $b - a \preceq b$.*

We conclude this section with one more characterization of minus partial order.

Theorem 3.6. *Let $\mathcal{A}$ be a Rickart ring and $a, b \in \mathcal{A}$. Then $a \preceq b$ if and only if there exists idempotents $e_1 \in LP(a)$, $e_2 \in LP(b-a)$, $f_1 \in RP(a)$, $f_2 \in RP(b-a)$ such that $e_1 e_2 = 0$ and $f_2 f_1 = 0$.*

Proof. The "only if" part follows from Theorem 3.4. In order to prove "if" part suppose that $e_1 \in \mathrm{LP}(a)$, $e_2 \in \mathrm{LP}(b-a)$ and $e_1 e_2 = 0$. Then $e_1 a = a$ and $e_1 b = e_1 a + e_1(b-a) = a + e_1 e_2 (b-a) = a$. Similarly, from $f_2 f_1 = 0$ where $f_1 \in \mathrm{RP}(a)$, $f_2 \in \mathrm{RP}(b-a)$ it follows that $a f_1 = a$ and $b f_1 = a f_1 + (b-a) f_1 = a + (b-a) f_2 f_1 = a$. By definition, $a \preceq b$. □

Under the notation of Theorem 3.4 it is easy to check that

$$e_1 + e_2 \in \mathrm{LP}(b) \quad \text{and} \quad f_1 + f_2 \in \mathrm{RP}(b).$$

Also, one can show

$$(e_1 + e_2)\mathcal{A} = e_1\mathcal{A} \oplus e_2\mathcal{A} \quad \text{and} \quad \mathcal{A}(f_1 + f_2) = \mathcal{A}f_1 \oplus \mathcal{A}f_2.$$

We have not proof for the opposite implication:

If there exist $e_1 \in \mathrm{LP}(a)$, $e_2 \in \mathrm{LP}(b-a)$, $e_3 \in \mathrm{LP}(b)$, $f_1 \in \mathrm{RP}(a)$, $f_2 \in \mathrm{RP}(b-a)$, $f_3 \in \mathrm{RP}(b)$ and if

$$e_3\mathcal{A} = e_1\mathcal{A} \oplus e_2\mathcal{A} \quad \text{and} \quad \mathcal{A}f_3 = \mathcal{A}f_1 \oplus \mathcal{A}f_2,$$

then $a \preceq b$. This can be suggested as an open problem.

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