Jurij Vega and Ballistics

Jurij Vega in balistika

Dr. Mihael Perman and Dr. Marko Razpet

Institute for Mathematics, Physics and Mechanics
University of Ljubljana

Abstract

The ballistic curve is the solution of a system of nonlinear differential equations if the resistance of the air is assumed to be proportional to the square of the velocity. In Chapter 4, vol 4. of his lectures on mathematics Jurij Vega (1754–1802) presents a careful analysis of the problem. The underlying assumptions from physics are examined, and then differential equations are derived. There is no closed form solution but the equations can be transformed into simpler forms that are more amenable to finding approximate solutions.

Vega suggests various numerical procedures. Perhaps the most original part is the use of a result that can be traced back to Euler's work on ballistics. The arclength covered by the cannon ball can be expressed explicitly as a function of the angle of the trajectory at a given moment. Vega develops an ingenious way to discretize the differential equations and find a numerical solution. The comparison to numerical solutions obtained by computer using Runge-Kutta methods shows that Vega's method gives superb results.

Povzetek

Balistična krivulja je rešitev sistema nelinearnih diferencialnih enačb, če privzamemo, da je zračni upor sorazmeren kvadratu hitrosti. V četrtem poglavju četrte knjige svojih matematičnih predavanj Jurij Vega (1754–1802) predstavlja podrobno analizo problema. Razdelane so osnovne fizikalne predpostavke, nato sta izpeljani diferencialni enačbi. Rešitev v sklenjeni obliki ni, vendar pa je enačbe mogoče pretvoriti v preprostejšo obliko, ki ji je lažje najti približne rešitve.

Vega predlaga različne numerične postopke. Morda najbolj izvirna je uporaba rezultata, ki ga lahko zasledimo v Eulerjevem delu na področju balistike. Dolžino loka, ki ga opiše topovska krogla, lahko izrazimo eksplicitno kot funkcijo kota, pod katerim se giblje krogla v danem trenutku. Vega razvije inovativen način, s katerim je mogoče diskretizirati diferencialni enačbi in najti numerično rešitev. Primerjava z numeričnimi rešitvami, ki jih pridobimo z računalnikom z uporabo metode Runge-Kutta, pokaže, da daje Vegova metoda izvrstne rezultate.

1. Introduction

The mathematician Jurij Vega 1754–1802 was a prolific mathematical writer. He is best known for his monumental work on logarithmic tables and his military exploits. From the perspective of a mathematician, however, his lectures on mathematics that were published between 1782 and 1802 in four volumes are perhaps even more important. Shortly after starting his teaching career at the imperial Artillery School in Vienna in 1782 Vega set out to write systematic and clear textbooks. They became an instant success and were reprinted many times over the following decades. See Vega's bibliography in [**3**]. It is interesting to note that the last edition of volume 2 was published in 1848. The informal style and carefully chosen examples are enjoyable to read even today.

In this paper we examine Vega's work on the ballistic curve, which appears as Chapter 4 in the last volume of his lectures published first in 1800 and then again in 1819. The volume is dedicated to hydrostatics, the motion of fluids and to the motion of objects in fluids. In his treatment of ballistics Vega sets out with a careful examination of the physical assumptions about the resistance of the medium to the motion of an immersed object. He acknowledges that there is some disagreement about the exact magnitude of the force opposing the motion, but finally settles for the quadratic law. He limits his investigation to spherical bodies, and as experimental evidence presents an experiment that was conducted at Newton's urging in London in 1710 and 1719.

The most intriguing part is the treatment of the ballistic curve. The basic equations are readily derived from Newton's laws. They are then skillfully transformed in such a way as to obtain a single differential equation. Vega correctly observes that there is no closed form solution and sets out to suggest various approximations. Perhaps the most surprising result is the derivation of an explicit formula for the arclength along the trajectory as a function of its slope. This result can be traced back to Euler's work on ballistics [**2**]. This analytic representation is then exploited to give a surprisingly accurate numerical procedure for finding the ballistic curve. One of the main aims of this paper is to translate the text into modern notation and to check the results from a modern viewpoint. Along the way it is necessary to unravel older notation, but once that is done the presentation is wonderfully clear, and, we dare say, better than many calculus textbooks used today. Vega also provides fully worked-out examples illustrating the solutions from a practical viewpoint. At the end we reproduce Vega's calculations and examine the accuracy of his approximations using modern computers.

2. Physical assumptions

Vega's computations are based on the assumption that the force exerted by the resistance of the medium on a moving object is proportional to the square of its velocity. If the density of the medium is denoted by ρ_a, the velocity by $\vec{v}$ and the cross-section by S, the force $\vec{R}$ is given as

$$\vec{R} = -\frac{1}{2}k\rho_a S v^2 \frac{\vec{v}}{v}$$

where $v = |\vec{v}|$. The proportionality factor k depends on the geometry of the moving body and is in general difficult to compute. In §117, however, Vega derives the explicit form of the resistance law for a moving ball. If the diameter of the ball is denoted by D, the force is given by

$$\vec{R} = -\frac{\pi}{16}\rho_a D^2 v\,\vec{v} \tag{2.1}$$

In order to further simplify the formulae, the mass of the cannon ball is denoted by M. Clearly

$$M = \frac{4\pi}{3}\left(\frac{D}{2}\right)^3 \rho,$$

where ρ is the mass density of the ball, and introducing the ratio $N = \rho/\rho_a$ one gets

$$M = \frac{\pi}{6}D^3 N\rho_a.$$

It follows that

$$\frac{\vec{R}}{M} = -\frac{3}{8DN}v\vec{v}.$$

Introducing the parameter

$$a = \frac{4}{3}DN \tag{2.2}$$

which has the dimension of length, the force can finally be expressed as

$$\frac{\vec{R}}{M} = -\frac{1}{2a} v\vec{v}. \tag{2.3}$$

It is interesting to note that Vega gives experimental evidence for the laws of resistance. First he solves the elementary differential equations describing the velocity of a ball moving vertically under the influence of gravity. In equation (4.34) in [4] the distance x covered by the falling ball by the time t is given as

$$x = bt + 2a \log\left(\frac{1 + h^{-bt/a}}{2}\right) \tag{2.4}$$

where h is the notation for Euler's e and b is an abbreviation for

$$b = \sqrt{\frac{4ga(N-1)}{N}}.$$

The calculations are then compared to data recorded in two experiments conducted at Newton's urging in London in 1710 and 1719. Balls were dropped from the dome of St. Paul's cathedral and the times it took them to reach the ground were recorded. In the first case glass balls were chosen and dropped from 220 feet. Vega computes the known height from the times using (2.4). Table 1 reproduces the results.

Weight of Ball	Diamet. of Ball	Recorded Duration of Fall		Computed Height	
Lon. Grain	Lon. Inches	Second.	Tertia.	Feet	Inches
510	5.1	8	12	226	11
642	5.2	7	42	230	9
599	5.1	7	42	227	10
515	5.0	7	57	224	5
483	5.0	8	12	225	5
641	5.2	7	42	230	7

TABELA / TABLE 1. Rezultati eksperimenta v Londonu iz leta 1710 / The data from the 1710 experiment in London

From the table it follows that the computed heights slightly overstate the height. In the second experiment, swine bladders were blown up in a wooden mold to give them the shape of a near perfect ball. The bladders were then dropped from 272 feet. Table 2 gives the data in this case.

In this second case the agreement between the computed and known height is more convincing except in the last case, which can most probably be attributed to measurement error. Vega offers no further comment on the experimental evidence

Weight of Ball	Diamet. of Ball	Observed Duration of Fall	Computed Height	
Grain	Inch.	Seconds	Feet	Inches
128	5.28	19	271	11
156	5.19	17	272	1.05
137.5	5.3	18.5	272	7
97.5	5.26	22	277	4
99.125	5	21.125	282	0

TABELA / TABLE 2. Rezultati eksperimenta v Londonu iz leta 1719 / The data from the 1719 experiment in London

in his §164. It is quite likely that this was deliberate because such discrepancies would defeat the purpose of finding accurate numerical methods to approximate the solutions of subsequent equations.

3. Vega's differential equations

Having dealt with physical assumptions, Vega turns to ballistics. The simple cases of motion in a straight line are dealt with first. In particular all the differential equations for vertical fall taking into account the resistance of the medium and the buoyancy are carefully solved. Then section 4.3 turns to the main problem: the ballistic curve. It is assumed that a cannon ball with mass density ρ and diameter D is fired at an angle μ and initial velocity v. The density of the air is denoted by ρ_a. The problem is to find the trajectory of the cannon ball under the quadratic law of air resistance. A few simplifying assumptions are also made. The air is assumed to be homogenous and at rest, the buoyancy is assumed to have a negligible effect, and there is no correction for the rotation of the earth. The question is translated into mathematics by placing the cannon at the origin of a coordinate system. The trajectory will lie in a plane determined by the vector of the initial velocity. The x-axis will represent the ground and the force $\vec{G}$ of gravity will point in the direction opposite to the y-axis as $\vec{G} = -Mg\vec{j}$, where $\vec{j} = (0, 1)$. The trajectory will be described by a vector function

$$\vec{r} = \vec{r}(t) = (x(t), y(t))$$

of time. As before, M is the mass of the cannon ball and g is the gravity constant. By Newton's second law we obtain the equation:

$$M\ddot{\vec{r}} = \vec{R} + \vec{G}.$$

Dividing by the mass M, the equation becomes

$$\ddot{\vec{r}} = -\frac{1}{2a} v\vec{v} - g\vec{j}. \tag{3.1}$$

Taking into account that

$$\vec{v} = (\dot{x}, \dot{y}) \quad \text{and} \quad v = \sqrt{\dot{x}^2 + \dot{y}^2},$$

and writing (3.1) componentwise, we obtain a system of differential equations

$$\ddot{x} = -\frac{1}{2a} v\dot{x} \tag{3.2}$$

$$\ddot{y} = -\frac{1}{2a} v\dot{y} - g \tag{3.3}$$

with the initial conditions

$$x(0) = 0, \ y(0) = 0, \ \dot{x}(0) = c\cos\mu, \ \dot{y}(0) = c\sin\mu.$$

The system of differential equations (3.2) and (3.3) has a unique solution satisfying the initial conditions. The solution, however, cannot be expressed in closed form using elementary functions. Approximate solutions can be found, but first the basic equations have to be transformed.

From (3.2) and (3.3) one can easily derive the identity

$$\frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{y}}{\dot{x}}\right) = -\frac{g}{\dot{x}}. \tag{3.4}$$

Let p be the slope of the trajectory at time t. We have

$$p = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}},$$

From (3.4) one obtains

$$\dot{p}\dot{x} = -g. \tag{3.5}$$

Denote the arc length from the starting point O to the position at time t by

$$s = s(t) = \int_0^t v(\tau)\, d\tau.$$

From

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1+p^2} \quad \text{and} \quad \dot{s} = \frac{\mathrm{d}s}{\mathrm{d}t} = v$$

and taking into account (3.2) we obtain

$$\sqrt{1+p^2} = \frac{\mathrm{d}s}{\mathrm{d}x} = \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{v\dot{x}}{\dot{x}^2} = -\frac{2a\ddot{x}}{\dot{x}^2}.$$

Multiplying the equations

$$\sqrt{1+p^2} = -\frac{2a\ddot{x}}{\dot{x}^2} \tag{3.6}$$

$$\dot{p} = -\frac{g}{\dot{x}} \tag{3.7}$$

we obtain

$$\sqrt{1+p^2}\,\dot{p} = 2ag\frac{\ddot{x}}{\dot{x}^3} = -ag\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\dot{x}^2}\right).$$

In other words, the equation is

$$\sqrt{1+p^2}\,\mathrm{d}p = -ag\,\mathrm{d}\left(\frac{1}{\dot{x}^2}\right).$$

Because

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\mathrm{d}p}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\dot{p}}{\dot{x}} = \frac{\dot{p}\dot{x}}{\dot{x}^2} = -\frac{g}{\dot{x}^2}, \tag{3.8}$$

the differential equation can be rewritten as:

$$\sqrt{1+p^2}\,\mathrm{d}p = a\mathrm{d}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right). \tag{3.9}$$

The initial conditions imply that

$$p(0) = \tan\mu \quad \text{and} \quad \frac{\mathrm{d}p}{\mathrm{d}x}(0) = -\frac{g}{c^2\cos^2\mu}. \tag{3.10}$$

One can, in principle, integrate the equation (3.9), but the resulting differential equation linking p and $\mathrm{d}p/\mathrm{d}x$ is too cumbersome for further calculations.

Vega suggests various approximations to the solution of the equation above. For small initial angles (Vega considers angles up to $15°$ to be small) one can argue that the term p^2 is small enough so that the term $\sqrt{1+p^2}$ in (3.9) can be ignored. Integrating the simplified equation

$$\mathrm{d}p = a\mathrm{d}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) \tag{3.11}$$

one obtains a nonhomogeneous first order linear differential equation

$$p - \tan\mu = a\left(\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{g}{c^2\cos^2\mu}\right).$$

Rewriting as

$$\frac{\mathrm{d}p}{p - \tan\mu - \frac{ag}{c^2\cos^2\mu}} = \frac{\mathrm{d}x}{a}$$

and using the initial conditions in (3.10), the solution is obtained as

$$\log\left(\frac{\tan\mu + \frac{ag}{c^2\cos^2\mu} - p}{\frac{ag}{c^2\cos^2\mu}}\right) = \frac{x}{a}, \tag{3.12}$$

Rewriting again we find

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p = \tan\mu + \frac{ag}{c^2\cos^2\mu} - \frac{ag}{c^2\cos^2\mu}e^{\frac{x}{a}}.$$

Because $y(0) = 0$, yet another integration gives Vega's approximation to the ballistic trajectory as

$$y = \left(\tan\mu + \frac{ag}{c^2\cos^2\mu}\right)x - \frac{a^2g}{c^2\cos^2\mu}\left(e^{\frac{x}{a}} - 1\right). \tag{3.13}$$

An interesting question arising at this point is the quality of this approximation. Figure 1 shows the numerical solution of the system of equations (3.2) and (3.3) using the Runge-Kutta algorithm with step size $0.1\ s$ (solid line) and the data $\mu = 15°$, $c = 400\ m/s$, and $a = 800\ m$. The dashed line shows Vega's approximate trajectory. As is to be expected, Vega's approximation slightly overstates the range of the cannon.

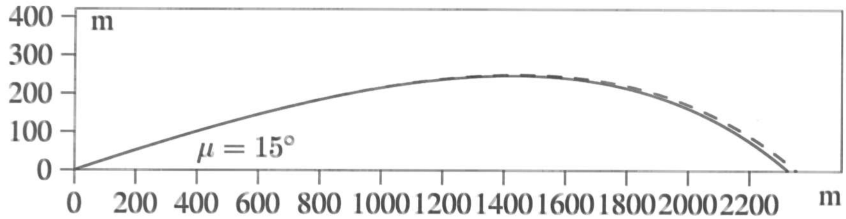

Figure 1

Numerical calculations show that the range of the cannon ball given the physical assumptions is $2{,}323.5\ m$. The highest elevation is $246\ m$ attained at $x = 1{,}437\ m$. Vega's range is $2{,}353.5\ m$ with the highest elevation $249.3\ m$ attained when $x = 1{,}447\ m$.

As an improvement to his first approximations – in particular, when the angles are not small – Vega suggests taking $p = \tan(\mu/2)$ in equation (3.9). He argues that this quantity is approximately the average angle of the trajectory. The basic equation is

$$\mathrm{d}p = a\cos(\frac{\mu}{2})\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)$$

which means that one only has to change the parameter a to $a\cos(\mu/2)$ in all the calculations.

In §152 in [4] Vega sets out to calculate the range of a cannon from the characteristics of the cannon ball, the angle, and the initial velocity. To this end one sets $y = 0$ in (3.13), which leads to the transcendental equation

$$\frac{e^{\frac{x}{a}} - 1}{\frac{x}{a}} = 1 + \frac{c^2}{ag}\sin\mu\cos\mu = 1 + \frac{c^2}{2ag}\sin 2\mu.$$

For practical purposes Vega compiled a ballistic table giving the values of the function $n \mapsto (e^n - 1)/n$ for n ranging from 0.01 to 10.00 with step size 0.01. For purposes of interpolation, the table also gives the differences of subsequent entries. Taking $n = x/a$ or $n = x/(a\cos(\mu/2))$, and using the table, one finds with a, μ and c given the equation

$$\frac{e^n - 1}{n} = 1 + \frac{c^2}{2ag}\sin 2\mu \quad \text{or} \quad \frac{e^n - 1}{n} = 1 + \frac{c^2}{2ag}\frac{\sin 2\mu}{\cos\frac{\mu}{2}}.$$

Once n is found, the range is computed as $x = na$ or $x = na\cos\mu/2$.

4. An alternative numerical procedure

The equations (3.2) and (3.3) cannot be solved in closed form. Surprisingly, however, in §173 in [4] Vega finds a simple expression for the arc-length as a function of the slope of the trajectory. Let φ be the angle between the x-axis and the tangent to the trajectory at a given point. Recall that $p = \tan\varphi$. A simple calculation gives $\mathrm{d}s/\mathrm{d}x = \sqrt{1+p^2} = 1/\cos\varphi$, which leads to

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\mathrm{d}p}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}x} = \frac{\mathrm{d}\tan\varphi}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}x} = \frac{1}{\cos^3\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}s}. \tag{4.1}$$

Taking derivatives with respect to t on both sides and using (3.9) gives

$$\frac{\mathrm{d}\varphi}{\cos^3\varphi} = a\mathrm{d}\left(\frac{1}{\cos^3\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right).$$

For convenience, introduce the function F as

$$F(\varphi) = \int_0^\varphi \frac{\mathrm{d}\vartheta}{\cos^3\vartheta} = \frac{\sin\varphi}{2\cos^2\varphi} + \frac{1}{2}\log\tan\left(\frac{\varphi}{2}+\frac{\pi}{4}\right), \quad -\frac{\pi}{2} < \varphi < \frac{\pi}{2}. \tag{4.2}$$

Obviously, $F(0) = 0$, $\mathrm{d}F/\mathrm{d}\varphi = 1/\cos^3\varphi$ and $F(-\varphi) = -F(\varphi)$. At the beginning when $s = 0$ one has $\varphi = \mu$ and consequently the expression $\mathrm{d}p/\mathrm{d}x = \mathrm{d}\varphi/(\cos^3\varphi\mathrm{d}s)$ takes the value $-g/(c^2\cos^2\mu))$. Integrating, one obtains

$$\int_\mu^\varphi \frac{\mathrm{d}\vartheta}{\cos^3\vartheta} = \frac{a}{\cos^3\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}s} + \frac{ag}{c^2\cos^2\mu}. \tag{4.3}$$

Using the function F defined in (4.2), the equation can be written in simpler form as a first order nonhomogeneous linear differential equation

$$F(\varphi) - F(\mu) = a\frac{\mathrm{d}F}{\mathrm{d}s} + \frac{ag}{c^2\cos^2\mu}.$$

Rewriting as

$$ds = a\frac{dF}{F(\varphi) - F(\mu) - \frac{ag}{c^2\cos^2\mu}},$$

and integrating, taking into account that $s(\mu) = 0$, one finally obtains

$$s = s(\varphi) = a\log\left(\frac{ag}{c^2\cos^2\mu} - (F(\varphi) - F(\mu))\right) - a\log\left(\frac{ag}{c^2\cos^2\mu}\right),$$

The last expression can be simplified to

$$s(\varphi) = a\log\left(1 + \frac{c^2\cos^2\mu}{ag}(F(\mu) - F(\varphi))\right). \tag{4.4}$$

From this expression, the arclength to the highest point of the trajectory when $\varphi = 0$ can easily be computed as

$$s(\varphi) = a \log\left(1 + \frac{c^2 \cos^2 \mu}{ag} F(\mu)\right).$$

Vega used this analytic result for an alternative approach to numerical calculation of the trajectory which does not rely on ignoring certain terms in the differential equations. He observed that the angle between the tangent to the trajectory and the x-axis will decrease steadily from the initial angle μ to $-\pi/2$. The idea is to first choose an arithmetic progression of angles with small enough step size

$$\mu = \varphi_0 > \varphi_1 > \ldots > \varphi_{k-1} > \varphi_k > \ldots$$

Denote the increments of the arclength between the subsequent angles by Δs_k:

$$\Delta s_k = s(\varphi_k) - s(\varphi_{k-1}),\ k = 1, 2, 3, \ldots$$

Suppose that the segment of the trajectory between the angles φ_{k-1} and φ_k starts in $T_{k-1}(x_{k-1}, y_{k-1})$ and ends in $T_k(x_k, y_k)$. The distance between these two points is approximately Δs_k. The smaller the step size $\Delta\varphi_k = \varphi_k - \varphi_{k-1}$ between subsequent angles, the smaller will be the error of this approximation. See Figure 2.

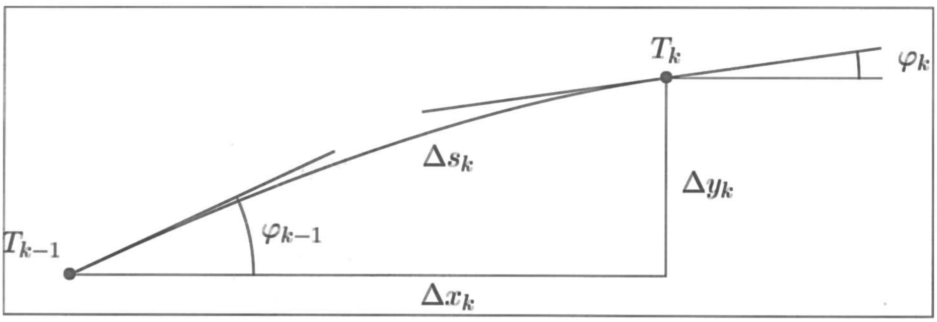

Figure 2

The increments in the x and y directions defined as

$$\Delta x_k = x_k - x_{k-1} \quad \text{and} \quad \Delta y_k = y_k - y_{k-1}$$

can be approximated quite accurately by

$$\Delta x_k = \Delta s_k \cos\left(\frac{\varphi_{k-1} + \varphi_k}{2}\right) \quad \text{and} \quad \Delta y_k = \Delta s_k \sin\left(\frac{\varphi_{k-1} + \varphi_k}{2}\right).$$

The position of the cannon ball when the angle is φ_n can be calculated as

$$x = \sum_{k=1}^{n} \Delta x_k \quad \text{and} \quad y = \sum_{k=1}^{n} \Delta y_k.$$

As an example take $\mu = 30°$, $c = 400\ m/s$, and $a = 800\ m$. The dashed line in Figure 3 shows the approximate solution from Section 2. There is virtually no difference, however, between the numerical solutions obtained by Vega's second method and the solution obtained by the Runge-Kutta algorithm with step size 0.1. The agreement is amazing!

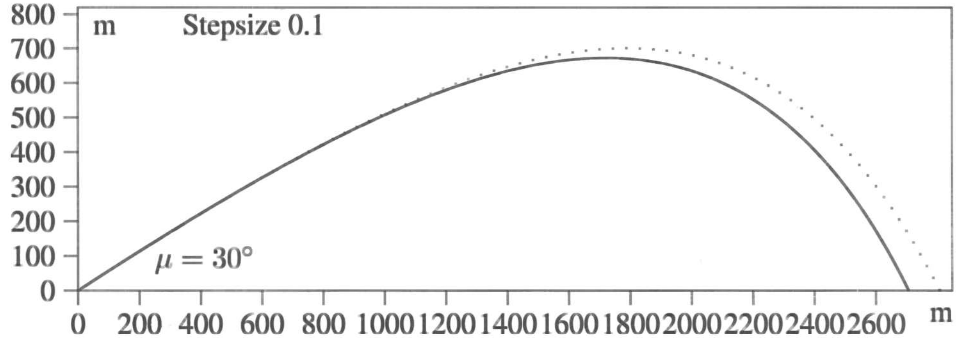

Figure 3

This time Vega's calculations give that the highest point on the trajectory is 672.6 m when $x = 1{,}725\ m$ and the range is $2{,}707\ m$. The approximate formulae from Section 2 would give the numbers 700.7 m at $x = 1{,}791\ m$ and the range $2{,}808.5\ m$ which overstates the cannon's capability.

Just as the arc-length can be given explicitly as a function of the angle φ, it is also possible to give an explicit formula for the velocity of the ball as a function of the tangent angle φ. Using the equation (3.8) one can express v^2 as

$$v^2 = \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)^2 \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = -g(1+p^2)\frac{\mathrm{d}x}{\mathrm{d}p} = -\frac{g}{\cos^2\varphi} \cdot \frac{\mathrm{d}x}{\mathrm{d}p}.$$

Substituting (4.1) for $\mathrm{d}x/\mathrm{d}p$ it follows that

$$v^2 = -\frac{g}{\cos^2\varphi}\cos^3\varphi\,\frac{\mathrm{d}s}{\mathrm{d}\varphi} = -\frac{g\cos\varphi}{\frac{\mathrm{d}\varphi}{\mathrm{d}s}}.$$

Taking into account (4.3) one finally gets

$$v^2 = v^2(\varphi) = \frac{ag}{\cos^2\varphi} \cdot \frac{1}{F(\mu) - F(\varphi) + \frac{ag}{c^2\cos^2\mu}}.$$

When $\varphi = 0$ we get the velocity at the highest point of the trajectory as

$$v^2 = \frac{ag}{F(\mu) + \frac{ag}{c^2 \cos^2 \mu}}.$$

As a final example recall that

$$v^2 = -g(1+p^2)\frac{\mathrm{d}x}{\mathrm{d}p} = -g\frac{\mathrm{d}x}{\mathrm{d}\varphi}.$$

This gives

$$\mathrm{d}x = -\frac{1}{g}v^2(\varphi)\,\mathrm{d}\varphi,$$

and from $\mathrm{d}y = (\mathrm{d}y/\mathrm{d}x)\,\mathrm{d}x = \tan\varphi\,\mathrm{d}x$

$$\mathrm{d}y = -\frac{1}{g}v^2(\varphi)\tan\varphi\,\mathrm{d}\varphi.$$

In principle one gets a parametric form of the ballistic curve by integration, c. f. [1],

$$\begin{aligned} x = x(\varphi) &= -\frac{1}{g}\int_{\mu}^{\varphi} v^2(\vartheta)\,\mathrm{d}\vartheta, \\ y = y(\varphi) &= -\frac{1}{g}\int_{\mu}^{\varphi} v^2(\vartheta)\tan\vartheta\,\mathrm{d}\vartheta. \end{aligned}$$

The function $v^2(\varphi)$, however, cannot be integrated in an elementary way.

Finally, let us consider the angle φ as a function of time. From $\mathrm{d}x = \dot{x}\,\mathrm{d}t = v\cos\varphi\,\mathrm{d}t = -(v^2/g)\,\mathrm{d}\varphi$ one obtains

$$\mathrm{d}t = -\frac{v(\varphi)}{g\cos\varphi}\,\mathrm{d}\varphi,$$

Integrating, one obtains

$$t = t(\vartheta) = -\frac{1}{g}\int_{\mu}^{\varphi}\frac{v(\vartheta)}{\cos\vartheta}\,\mathrm{d}\vartheta. \tag{4.5}$$

Inverting (4.5) gives the angle as a function of time. Figure 4 shows the graph of the velocity v versus the angle φ as well as the individual components $\dot{x}$ and $\dot{y}$ of the velocity. The initial conditions are given as $\mu = 30°$, $c = 400\ m/s$, and $a = 800\ m$.

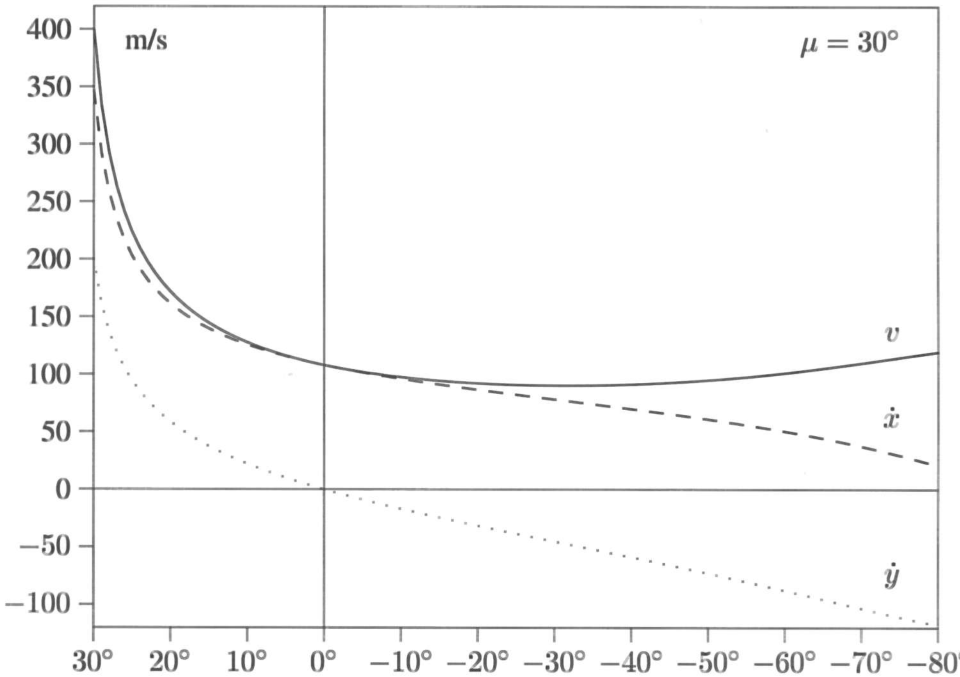

Figure 4

5. Conclusions

Vega's work on the ballistic curve is a wonderful example of how he was able to combine theoretical work with practical considerations. He is careful with physical assumptions, even citing experiments to confirm the quadratic law of medium resistance. The elementary differential equations may have been standard in Vega's time. The work on the numerical aspects of the ballistic curve can accurately be described as genuine research even by today's standards. The discovery that the distance covered by the cannonball can be explicitly described as a function of the angle and the use of this fact to numerical ends is by no means obvious. The observation that one can use this analytic result for practical purposes is another wonderful observation that would amuse any numerical analyst today: discretizing over the angles avoids approximating the derivatives, which is known to be likely to cause numerical problems.

References

[1] T. Anđelić, R. Stojanović, *Racionalna mehanika*, Zavod za izdavanje udžbenika SR Srbije, Beograd 1966.

[2] L. Euler, *Recherches sur la véritable courbe que décrivent le corps jetés dans l'air ou dans un autre fluide quelconque*, Memoires de l'Academie royale des Sciences et belles lettres, vol. IX, p. 321–355, 1755.

[3] J. Povšič, *Bibliografija Jurija Vege*, Slovenska akademija znanosti in umetnosti, Ljubljana, 1974.

[4] G. Vega, *Vorlesungen über die Mathematik, Vierter Band*, Bey Franz Tendler, Buchhändler am Graben im Trattnerhofe, Wien, 1819.

[5] G. Vega, *Vorlesungen über die Mathematik, Dritter Band*, Bey Franz Tendler, Buchhändler am Graben im Trattnerhofe, Wien, 1819.