



11 Unified Description of Quarks and Leptons in a Multi-spinor Field Formalism

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Abstract. Multi-spinor fields which behave as triple-tensor products of the Dirac spinors and form reducible representations of the Lorentz group describe three families of ordinary quarks and leptons in the visible sector and an additional family of exotic dark quarks and leptons in the dark sector of the Universe. Apart from the ordinary set of the gauge and Higgs fields in the visible sector, another set of gauge and Higgs fields belonging to the dark sector are assumed to exist. Two sectors possess channels of communication through gravity and a bi-quadratic interaction between the two types of Higgs fields. A candidate for the main component of the dark matter is a stable dark hadron with spin $3/2$, and the upper limit of its mass is estimated to be $15.1 \text{ GeV}/c^2$.

Povzetek. Avtor obravnava spinorje, ki se obnašajo kot trojni tenzorski produkt Diracovih spinorjev in tvorijo nerazcepno upodobitev Lorentzove grupe. Z njimi uspe opisati tri izmerjene družine kvarkov in leptonov, upodobitve pa ponudijo obstoj še četrte družine (imenuje jo "nevidno"). Umeriotvenim poljem izmerjenih članov družin in Higgsovega skalarnega polja doda ustrezna umeritvena vektorska polja in skalarno polje, s katerimi se sklopi "nevidna" družina. Gravitacija in bikvadratna interakcij poskrbita za interakcijo med med obema tipoma Higgsovih polj. Kot kandidata za poglavitno sestavino temne snovi predlaga hadron s spinom $3/2$. Zgornjo mejo za njegovo maso oceni na $15.1 \text{ GeV}/c^2$.

11.1 Introduction

The Standard Model (SM) has been accepted as an almost unique effective scheme for phenomenology of particle physics in the energy region around and lower than the electroweak scale. Nevertheless it should be considered that the SM is still in an incomplete stage, since its fermionic and Higgs parts are full of unknowns. It is not yet possible to answer the question why quarks and leptons exist in the modes of three families with the color and electroweak gauge symmetry $G = \text{SU}_c(3) \times \text{SU}_L(2) \times \text{U}_Y(1)$, and we have not yet found definite rules to determine their interactions with the Higgs field. It is also a crucial issue to inquire whether the SM can be extended so as to accommodate the degrees of freedom of dark matter.

To extend the SM to a more comprehensive theory which can elucidate its unknown features, we introduce the algebra, called *triplet algebra*, consisting of

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triple-tensor products of the Dirac algebra and construct a unified field theory with the multi-spinor field, called *triplet field*, which behaves as triple-tensor products of the four-component Dirac spinor [1,2]. The chiral triplet fields forming reducible representations of the Lorentz group include the three families of ordinary quarks and leptons and also an additional fourth family of exotic quarks and leptons which are assumed to belong to the dark sector of the Universe.

The bosonic part of the theory consists of the ordinary gauge and Higgs fields of the G symmetry and also the dark gauge and Higgs fields of the new $G_\star = \text{SU}_{c\star}(3) \times \text{SU}_R(2) \times \text{U}_{Y_\star}(1)$ symmetry. While the gauge fields of the G symmetry interact with the ordinary quarks and leptons of the three families, the gauge fields of the G_\star symmetry are presumed to interact exclusively with the quarks and leptons of the fourth family in the dark sector. The gauge fields of the extra color symmetry $\text{SU}_{c\star}(3)$ work to confine the dark quarks into dark hadrons. Apart from the ordinary Higgs field ϕ which breaks the electroweak symmetry $G_{EW} = \text{SU}_L(2) \times \text{U}_Y(1)$ at the scale Λ , another Higgs field ϕ_\star is assumed to exist to break the left-right twisted symmetry $G_{EW\star} = \text{SU}_R(2) \times \text{U}_{Y_\star}(1)$ at the scale Λ_\star ($\Lambda_\star > \Lambda$).

Our theory predicts existence of a stable dark hadron with spin 3/2 as a candidate for the main component of the dark matter. From a heuristic argument, we estimate the upper limit of its mass to be $15.1 \text{ GeV}/c^2$.

11.2 Triplet field and triplet algebra

To describe all fermionic species of the SM, we introduce the triplet field $\Psi(x)$ which behaves as triple-tensor products of the Dirac spinors as

$$\Psi_{abc} \sim \psi_a \psi_b \psi_c \quad (11.1)$$

where ψ is the four-component Dirac spinor. Operators acting on the triplet field belong to the triplet algebra A_T composed of the triple-tensor products of the Dirac algebra $A_\gamma = \langle \gamma_\mu \rangle = \{1, \gamma_\mu, \sigma_{\mu\nu}, \gamma_5 \gamma_\mu, \gamma_5\}$ as follows:

$$\begin{aligned} A_T &= \{p \otimes q \otimes r : p, q, r \in A_\gamma\} \\ &= \langle \gamma_\mu \otimes 1 \otimes 1, 1 \otimes \gamma_\mu \otimes 1, 1 \otimes 1 \otimes \gamma_\mu \rangle. \end{aligned} \quad (11.2)$$

The triplet algebra A_T is too large for all its elements to acquire physical meanings. To extract its subalgebras being suitable for physical description in the SM energy region, we impose the criterion [1] that the subalgebra bearing physical interpretation is closed and irreducible under the action of the permutation group S_3 which works to exchange the order of A_γ elements in the tensor product. With this criterion, the triplet algebra can be decomposed into three mutually commutative subalgebras, i.e., an external algebra defining external properties of fermions and two internal subalgebras that have the respective roles of prescribing family and color degrees of freedom.

The four elements

$$\Gamma_\mu = \gamma_\mu \otimes \gamma_\mu \otimes \gamma_\mu \in A_T, \quad (\mu = 0, 1, 2, 3) \quad (11.3)$$

satisfy the anti-commutation relations $\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\eta_{\mu\nu} I$ where $I = 1 \otimes 1 \otimes 1$. With them, let us construct an algebra A_Γ by

$$A_\Gamma = \langle \Gamma_\mu \rangle = \{ I, \Gamma_\mu, \Sigma_{\mu\nu}, \Gamma_5 \Gamma_\mu, \Gamma_5 \} \quad (11.4)$$

where $\Sigma_{\mu\nu} = -\frac{i}{2}(\Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu) = \sigma_{\mu\nu} \otimes \sigma_{\mu\nu} \otimes \sigma_{\mu\nu}$ and $\Gamma_5 = -i\Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 = \Gamma^5 = \gamma_5 \otimes \gamma_5 \otimes \gamma_5$. The algebra A_Γ being isomorphic to the original Dirac algebra A_γ fulfills the S_3 criterion and works to specify the external characteristics of the triplet field. Namely, we postulate that the operators $M_{\mu\nu} = \frac{1}{2}\Sigma_{\mu\nu}$ generate the Lorentz transformations for the triplet field $\Psi(x)$ in the four dimensional Minkowski spacetime $\{x^\mu\}$ where we exist as observers. The subscripts of operators Γ_μ are related and contracted with the superscripts of the spacetime coordinates x^μ .

Under the proper Lorentz transformation $x'^\mu = \Omega^\mu{}_\nu x^\nu$, the triplet field and its adjoint field $\bar{\Psi}(x) = \Psi^\dagger(x)\Gamma_0$ are transformed as

$$\Psi'(x') = S(\Omega)\Psi(x), \quad \bar{\Psi}'(x') = \bar{\Psi}(x)S^{-1}(\Omega) \quad (11.5)$$

where the transformation matrix is given by

$$S(\Omega) = \exp\left(-\frac{i}{2}M_{\mu\nu}\omega^{\mu\nu}\right) \quad (11.6)$$

with the angles $\omega^{\mu\nu}$ in the μ - ν planes. The Lorentz invariant scalar product is formed as

$$\bar{\Psi}(x)\Psi(x) = \sum_{abc} \bar{\Psi}_{abc}(x)\Psi_{abc}(x). \quad (11.7)$$

For discrete transformations such as space inversion, time reversal and the charge conjugation, the present scheme retains exactly the same structure as the ordinary Dirac theory. The chirality operators are given by

$$L = \frac{1}{2}(I - \Gamma_5), \quad R = \frac{1}{2}(I + \Gamma_5) \in A_\Gamma \quad (11.8)$$

which are used to assemble algebras for electroweak symmetries.

Note that the Dirac algebra A_γ possesses two $\mathfrak{su}(2)$ subalgebras

$$A_\sigma = \{\sigma_1 = \gamma_0, \sigma_2 = i\gamma_0\gamma_5, \sigma_3 = \gamma_5\} \quad (11.9)$$

and

$$A_\rho = \{\rho_1 = i\gamma_2\gamma_3, \rho_2 = i\gamma_3\gamma_1, \rho_3 = i\gamma_1\gamma_2\} \quad (11.10)$$

which are commutative and isomorphic with each other. By taking the triple-tensor products of elements of the respective subalgebras A_σ and A_ρ in A_Γ , we are able to construct two sets of commutative and isomorphic subalgebras with compositions “ $\mathfrak{su}(3)$ plus $\mathfrak{u}(1)$ ” which satisfy the criterion of S_3 irreducibility. Those algebras are postulated to have the roles to describe internal family and color degrees of freedom of the triplet fields.

11.3 Algebra for extended family degrees of freedom

From the elements of the algebra $A_\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ in Eq.(11.9), we can make up eight elements of A_T as follows:

$$\left\{ \begin{array}{l} \pi_1 = \frac{1}{2} (\sigma_1 \otimes \sigma_1 \otimes 1 + \sigma_2 \otimes \sigma_2 \otimes 1), \\ \pi_2 = \frac{1}{2} (\sigma_1 \otimes \sigma_2 \otimes \sigma_3 - \sigma_2 \otimes \sigma_1 \otimes \sigma_3), \\ \pi_3 = \frac{1}{2} (1 \otimes \sigma_3 \otimes \sigma_3 - \sigma_3 \otimes 1 \otimes \sigma_3), \\ \pi_4 = \frac{1}{2} (\sigma_1 \otimes 1 \otimes \sigma_1 + \sigma_2 \otimes 1 \otimes \sigma_2), \\ \pi_5 = \frac{1}{2} (\sigma_1 \otimes \sigma_3 \otimes \sigma_2 - \sigma_2 \otimes \sigma_3 \otimes \sigma_1), \\ \pi_6 = \frac{1}{2} (1 \otimes \sigma_1 \otimes \sigma_1 + 1 \otimes \sigma_2 \otimes \sigma_2), \\ \pi_7 = \frac{1}{2} (\sigma_3 \otimes \sigma_1 \otimes \sigma_2 - \sigma_3 \otimes \sigma_2 \otimes \sigma_1), \\ \pi_8 = \frac{1}{2\sqrt{3}} (1 \otimes \sigma_3 \otimes \sigma_3 + \sigma_3 \otimes 1 \otimes \sigma_3 - 2\sigma_3 \otimes \sigma_3 \otimes 1). \end{array} \right. \quad (11.11)$$

which are proved to obey the commutation and anti-commutation relations of the Lie algebra $\mathfrak{su}(3)$ as

$$[\pi_j, \pi_k] = 2f_{jkl}^{(3)}\pi_l, \quad \{\pi_j, \pi_k\} = \frac{4}{3}\delta_{jk}\Pi_{(v)} + 2d_{jkl}^{(3)}\pi_l \quad (11.12)$$

where

$$\Pi_{(v)} = \frac{1}{4} (3I - 1 \otimes 1 \otimes \sigma_3 \otimes \sigma_3 - 1 \otimes \sigma_3 \otimes 1 \otimes \sigma_3 - 1 \otimes \sigma_3 \otimes \sigma_3 \otimes 1) \quad (11.13)$$

and

$$\Pi_{(d)} = \frac{1}{4} (I + 1 \otimes 1 \otimes \sigma_3 \otimes \sigma_3 + 1 \otimes \sigma_3 \otimes 1 \otimes \sigma_3 + 1 \otimes \sigma_3 \otimes \sigma_3 \otimes 1) \quad (11.14)$$

are projection operators satisfying the relations

$$\Pi_{(a)}\Pi_{(b)} = \delta_{ab}\Pi_{(a)}, \quad \Pi_{(a)}\pi_j = \delta_{av}\pi_j \quad (11.15)$$

for $(a, b = v, d)$ and $(j = 1, \dots, 8)$.

Here we impose a crucial postulate that the operators $\Pi_{(v)}$ and $\Pi_{(d)}$ work to divide the triplet field into the orthogonal component fields as

$$\Psi_{(v)}(x) = \Pi_{(v)}\Psi(x), \quad \Psi_{(d)}(x) = \Pi_{(d)}\Psi(x) \quad (11.16)$$

which represent, respectively, fundamental fermionic species belonging to the visible and dark sectors of the Universe. The visible part $\Psi_{(v)}(x)$ can be further decomposed into the sum of the three component fields as follows:

$$\Psi_{(v)}(x) = \sum_{j=1,2,3} \Psi_j(x) = \sum_{j=1,2,3} \Pi_j \Psi(x) \quad (11.17)$$

where the projection operators Π_j are defined symmetrically by

$$\begin{cases} \Pi_1 = \frac{1}{4} (I + 1 \otimes 1 \otimes \sigma_3 \otimes \sigma_3 - 1 \otimes \sigma_3 \otimes 1 \otimes \sigma_3 - 1 \otimes \sigma_3 \otimes \sigma_3 \otimes 1), \\ \Pi_2 = \frac{1}{4} (I - 1 \otimes 1 \otimes \sigma_3 \otimes \sigma_3 + 1 \otimes \sigma_3 \otimes 1 \otimes \sigma_3 - 1 \otimes \sigma_3 \otimes \sigma_3 \otimes 1), \\ \Pi_3 = \frac{1}{4} (I - 1 \otimes 1 \otimes \sigma_3 \otimes \sigma_3 - 1 \otimes \sigma_3 \otimes 1 \otimes \sigma_3 + 1 \otimes \sigma_3 \otimes \sigma_3 \otimes 1). \end{cases} \quad (11.18)$$

which obey the relations $\sum_{j=1}^3 \Pi_j = \Pi_{(v)}$ and $\Pi_j \Pi_k = \delta_{jk} \Pi_j$. The component fields $\Psi_j(x)$ ($j = 1, 2, 3$) are interpreted to be the fields for three ordinary families of quarks and leptons in interaction modes.

With the operators π_j and $\Pi_{(a)}$, let us construct the set of the $\mathfrak{su}(3)$ and $u(1)$ algebras by

$$A_{(v)} = \{ \Pi_{(v)}, \pi_1, \pi_2, \dots, \pi_8 \}, \quad A_{(d)} = \{ \Pi_{(d)} \} \quad (11.19)$$

which are closed and irreducible under the action of S_3 permutation. The algebras $A_{(v)}$ and $A_{(d)}$ specify, respectively, the characteristics of the three ordinary families of the visible sector and the exotic family of the dark sector. Accordingly, the set $A_f = \{A_{(v)}, A_{(d)}\}$ is the algebra of operators specifying the family structure in the triplet field theory.

Rich varieties observed in flavor physics are presumed in the SM to result from the Yukawa couplings of quarks and leptons with the Higgs field. In low energy regime of flavor physics, quarks and leptons manifest themselves in both of the dual modes of electroweak interaction and mass eigen-states. It is the elements of the algebra $A_{(v)}$ in Eq.(11.19) that determine the structures of the Yukawa coupling constants which brings about varieties in the mass spectra and the electroweak mixing matrices.

It is the algebra $A_{(d)}$ consisting of the single element $\Pi_4 \equiv \Pi_{(d)}$ and the fourth component field $\Psi_4 \equiv \Psi_{(d)}$ of the triplet field that determine characteristics of exotic quarks and leptons belonging to the dark sector of the Universe. The projection operators Π_j ($j = 1, 2, 3, 4$) satisfy the relations $\sum_{j=1}^4 \Pi_j = I$ and $\Pi_j \Pi_k = \delta_{jk} \Pi_j$.

11.4 Algebra for extended color degrees of freedom

In parallel with the arguments in the preceding section, it is possible to construct another set of “ $\mathfrak{su}(3)$ plus $u(1)$ ” subalgebras from the algebra $A_p = \{ \rho_1, \rho_2, \rho_3 \}$ in Eq.(11.10). Replacing σ_a with ρ_a in Eq.(11.11), we obtain a new set of operators λ_j ($j = 1, \dots, 8$) in place of π_j . Likewise, corresponding to $\Pi_{(a)}$ ($a = v, d$) in Eqs.(11.13) and (11.14), we obtain operators $\Lambda^{(a)}$ ($a = q, \ell$) which play respective roles to project out component fields representing the quark-like and lepton-like modes of the triplet field.

Then, from Eqs.(11.12) and (11.15), we find that the operators λ_j and $\Lambda^{(a)}$ satisfy the relations

$$[\lambda_j, \lambda_k] = 2f_{jkl}^{(3)} \lambda_l, \quad \{\lambda_j, \lambda_k\} = \frac{4}{3} \delta_{jk} \Lambda^{(q)} + 2d_{jkl}^{(3)} \lambda_l \quad (11.20)$$

and

$$\Lambda^{(a)} \Lambda^{(b)} = \delta^{ab} \Lambda^{(a)}, \quad \Lambda^{(a)} \lambda_j = \delta^{aq} \lambda_j \quad (11.21)$$

for $(a, b = q, \ell)$ and $(j = 1, \dots, 8)$. The operators λ_j and $\Lambda^{(a)}$ enable us to build up the new set of $\mathfrak{su}(3)$ and $u(1)$ algebras as follows:

$$A^{(q)} = \{ \Lambda^{(q)}, \lambda_1, \lambda_2, \dots, \lambda_8 \}, \quad A^{(\ell)} = \{ \Lambda^{(\ell)} \}. \quad (11.22)$$

It is readily proved that these algebras $A^{(a)}$ satisfy the criterion of S_3 irreducibility and are commutative with the algebras A_Γ , $A_{(v)}$ and $A_{(d)}$.

The operator for the “baryon number minus lepton number” defined by

$$Q_{B-L} = \frac{1}{3} \Lambda^{(q)} - \Lambda^{(\ell)} \quad (11.23)$$

$$= -\frac{1}{3} (1 \otimes 1 \otimes \rho_3 \otimes \rho_3 + 1 \otimes \rho_3 \otimes 1 \otimes \rho_3 + 1 \otimes \rho_3 \otimes \rho_3 \otimes 1)$$

obeys the minimal equation

$$(Q_{B-L} + I) \left(Q_{B-L} - \frac{1}{3} I \right) = 0 \quad (11.24)$$

and has the eigenvalues $\frac{1}{3}$ and -1 . Therefore, $\Lambda^{(q)}\Psi$ and $\Lambda^{(\ell)}\Psi$ form, respectively, the quark-like and lepton-like modes of the triplet field.

At this stage, we construct the generators for extended color gauge symmetries $SU_c(3)$ and $SU_{c*}(3)$ which act, respectively, to the visible and dark fields $\Psi_{(v)}$ and $\Psi_{(d)}$. Combining the elements of the core algebras $A^{(q)}$ with the projection operators $\Pi_{(a)}$, we can make up the operators as

$$\Lambda_{(a)}^{(q)} = \Pi_{(a)} \Lambda^{(q)}, \quad \lambda_{(a)j} = \Pi_{(a)} \lambda_j \quad (11.25)$$

which form the algebras

$$A_{(a)}^{(q)} = \{ \Lambda_{(a)}^{(q)}, \lambda_{(a)j} : j = 1, \dots, 8 \} \quad (11.26)$$

where $a = v, d$. The elements of the algebras $A_{(a)}^{(q)}$ satisfy the commutation and anti-commutation relations

$$[\lambda_{(a)j}, \lambda_{(a)k}] = 2f_{jkl}^{(3)} \lambda_{(a)l}, \quad \{\lambda_{(a)j}, \lambda_{(a)k}\} = \frac{4}{3} \delta_{jk} \Lambda_{(a)}^{(q)} + 2d_{jkl}^{(3)} \lambda_{(a)l} \quad (11.27)$$

for $a = v, d$ and $j, k, l = 1, 2, \dots, 8$.

In this formalism, the quark-like species in the visible and dark sectors are presumed to be confined separately by different color gauge interactions associated with the groups $SU_c(3)$ and $SU_{c*}(3)$. Those gauge groups are defined by the exponential mappings of the algebras $A_{(a)}^{(q)}$ ($a = v, d$) as

$$SU_c(3) \times SU_{c*}(3) = \left\{ \exp \left(-\frac{i}{2} \sum_{a=v, d} \sum_j \lambda_{(a)j} \theta_{(a)}^j(x) \right) \Lambda^{(q)} \right\} \quad (11.28)$$

where $\theta_{(a)}^j(x)$ are arbitrary real functions of space-time. In addition to the ordinary gauge fields $A_{\mu}^{(3)j}(x)$ with coupling constant $g^{(3)}$ of the $SU_c(3)$ symmetry, our theory necessitates the new gauge fields $A_{\mu}^{(3)j}(x)$ with coupling constant $g_{\star}^{(3)}$ of the $SU_{c\star}(3)$ symmetry.

For the lepton-like species also, we have to introduce the algebras

$$A_{(a)}^{(\ell)} = \{ \Lambda_{(a)}^{(\ell)} \equiv \Pi_{(a)} \Lambda^{\ell} \}. \quad (11.29)$$

with $a = v, d$, which act to the visible and dark sectors, respectively. The set $A^c = \{ A_{(v)}^{(q)}, A_{(v)}^{(\ell)}; A_{(d)}^{(q)}, A_{(d)}^{(\ell)} \}$ is the algebra of operators characterizing the extended color degrees of freedom in the triplet field theory. The operators $Q_{B-L}^{(a)}$ ($a = v, d$) of “baryon number minus lepton number” in the visible and dark sectors are defined, respectively, by

$$Q_{B-L}^{(v)} = \Pi_{(v)} Q_{B-L}, \quad Q_{B-L}^{(d)} = \Pi_{(d)} Q_{B-L}. \quad (11.30)$$

11.5 Fundamental representation of the group $G \times G_{\star}$

The triplet field Ψ with $4 \times 4 \times 4$ spinor-components is decomposed as

$$\Psi = \Psi_{(v)} + \Psi_{(d)} = \left[\Psi_{(v)}^{(q)} + \Psi_{(v)}^{(\ell)} \right] + \left[\Psi_{(d)}^{(q)} + \Psi_{(d)}^{(\ell)} \right] \quad (11.31)$$

where $\Psi_{(a)}^{(c)} = \Lambda^{(c)} \Pi_{(a)} \Psi$ ($a = v, d; c = q, \ell$) are the four component fields of Dirac-type with degrees of freedom of $(3 + 1)$ -families and $(3 + 1)$ -colors. Hence the triplet field Ψ which is considered to be the basic unit of fermionic species has no more freedom.

In order to incorporate the Weinberg-Salam symmetry G_{EW} and its left-right twisted symmetry $G_{EW\star}$ in the visible and dark sectors, respectively, we have to postulate that there exists a two-storied compound field Ψ consisting of two triplet fields. The compound field Ψ forms the fundamental representations of the gauge group $G \times G_{\star}$ as

$$\Psi = \Psi_L + \Psi_R = \left(\begin{array}{c} \Psi_{(v)} \\ \Psi_{(d)} \end{array} \right)_L \begin{array}{c} U_{(d)} \\ D_{(d)} \end{array} + \left(\begin{array}{c} U_{(v)} \\ D_{(v)} \end{array} \right)_R \Psi_{(d)} \quad (11.32)$$

where $\Psi_{(v)L}, U_{(v)L}$ and $D_{(v)L}$ ($\Psi_{(d)R}, U_{(d)R}$ and $D_{(d)R}$) are, respectively, the chiral multi-spinor fields of the doublet, the up singlet and the down singlet of the $SU_L(2)$ ($SU_R(2)$) symmetry. We interpret that all fermionic species in the visible and dark sectors of the Universe are described by the component fields of the single compound field Ψ .

To name the fermionic species in the dark sector, let us assign new symbols u_{\star} and d_{\star} for up and down *dark quarks*, and v_{\star} and e_{\star} for up and down *dark leptons*. Then, the quark parts of the chiral compound fields Ψ_L and Ψ_R are schematically expressed, respectively, by

$$\Psi_{(v)}^{(q)} = \left(\begin{array}{ccc} u & c & t \\ d & s & b \end{array} \right)_L, \quad U_{(d)}^{(q)} = (u_{\star})_L, \quad D_{(d)}^{(q)} = (d_{\star})_L \quad (11.33)$$

and

$$U_{(v)}^{(q)} = (u \ c \ t)_R, \ D_{(v)}^{(q)} = (d \ s \ b)_R, \ \Psi_{(d)}^{(q)} = \begin{pmatrix} u_* \\ d_* \end{pmatrix}_R. \quad (11.34)$$

Similarly, the lepton parts have the following expressions as

$$\Psi_{(v)}^{(\ell)} = \begin{pmatrix} \nu_e & \nu_\mu & \nu_\tau \\ e & \mu & \tau \end{pmatrix}_L, \ U_{(d)}^{(\ell)} = (\nu_*)_L, \ D_{(d)}^{(\ell)} = (e_*)_L \quad (11.35)$$

and

$$U_{(v)}^{(\ell)} = (\nu_e \ \nu_\mu \ \nu_\tau)_R, \ D_{(v)}^{(\ell)} = (e \ \mu \ \tau)_R, \ \Psi_{(d)}^{(\ell)} = \begin{pmatrix} \nu_* \\ e_* \end{pmatrix}_R. \quad (11.36)$$

The kinetic and gauge parts of the Lagrangian density for all fermions can now be written down in terms of the chiral compound fields Ψ_L and Ψ_R by

$$\mathcal{L}_{kg} = \bar{\Psi}_L i\Gamma^\mu \mathcal{D}_\mu \Psi_L + \bar{\Psi}_R i\Gamma^\mu \mathcal{D}_\mu \Psi_R \quad (11.37)$$

in which the covariant derivatives act as follows:

$$\begin{aligned} i\mathcal{D}_\mu \Psi_L = & \left\{ i\partial_\mu - \left[g^{(3)} A_\mu^{(3)j} \frac{1}{2} \lambda_{(v)j} + g^{(2)} A_\mu^{(2)j} \frac{1}{2} \tau_{Lj} + g^{(1)} A_\mu^{(1)} \frac{1}{2} Y \right] \Pi_{(v)} \right. \\ & \left. - \left[g_\star^{(3)} A_{\star\mu}^{(3)j} \frac{1}{2} \lambda_{(d)j} + g_\star^{(1)} A_{\star\mu}^{(1)} \frac{1}{2} Y_\star \right] \Pi_{(d)} \right\} \Psi_L \end{aligned} \quad (11.38)$$

and

$$\begin{aligned} i\mathcal{D}_\mu \Psi_R = & \left\{ i\partial_\mu - \left[g^{(3)} A_\mu^{(3)j} \frac{1}{2} \lambda_{(v)j} + g^{(1)} A_\mu^{(1)} \frac{1}{2} Y \right] \Pi_{(v)} \right. \\ & \left. - \left[g_\star^{(3)} A_{\star\mu}^{(3)j} \frac{1}{2} \lambda_{(d)j} + g_\star^{(2)} A_{\star\mu}^{(2)j} \frac{1}{2} \tau_{Rj} + g_\star^{(1)} A_{\star\mu}^{(1)} \frac{1}{2} Y_\star \right] \Pi_{(d)} \right\} \Psi_R \end{aligned} \quad (11.39)$$

where $A_\mu^{(2)j}$ and $A_\mu^{(1)}$ ($A_{\star\mu}^{(2)j}$ and $A_{\star\mu}^{(1)}$) are gauge fields of the G_{EW} ($G_{EW\star}$) symmetry with coupling constants $g^{(2)}$ and $g^{(1)}$ ($g_\star^{(2)}$ and $g_\star^{(1)}$). The operators $\frac{1}{2}\tau_{Lj}$ ($\frac{1}{2}\tau_{Rj}$) are the generators of the visible (dark) $SU_L(2)$ ($SU_R(2)$) symmetry, and the hypercharge Y (Y_\star) in the visible (dark) sector can be expressed by

$$Y = Q_{B-L}^{(v)} + y, \quad Y_\star = Q_{B-L}^{(d)} + y_\star \quad (11.40)$$

in which y (y_\star) takes 0, 1 and -1 for the doublet Ψ , the up singlet U and the down singlet D .

To break down the gauge symmetries G_{EW} and $G_{EW\star}$, we require two types of Higgs doublets φ and φ_\star which, respectively, have the hypercharges ($Y = 1, Y_\star = 0$) and ($Y = 0, Y_\star = 1$). The Lagrangian density of the Yukawa interaction is given as follows:

$$\mathcal{L}_Y = \bar{\Psi}_L \{ \text{Higgs fields} \} \Psi_R + \text{h.c.}$$

$$= \bar{\Psi}_{(v)} \tilde{\varphi} \mathcal{Y}_U U_{(v)} + \bar{\Psi}_{(v)} \varphi \mathcal{Y}_D D_{(v)} + y_{u\star} \bar{U}_{(d)} \tilde{\varphi}_\star^\dagger \Psi_{(d)} + y_{d\star} \bar{D}_{(d)} \varphi_\star^\dagger \Psi_{(d)} + \text{h.c.} \quad (11.41)$$

where $\tilde{\varphi} = i\tau_{L2}\varphi^\star$ and $\tilde{\varphi}_\star = i\tau_{R2}\varphi_\star^\star$. The operators \mathcal{Y}_U and \mathcal{Y}_D consisting of elements of the algebra $A_{(v)}$ in Eq.(11.19) determine the patterns of Yukawa

interactions among the fermions in the visible sector [2], and y_{u*} and y_{d*} are the Yukawa coupling constants of the fermions in the dark sector.

The Lagrangian density of the visible and dark Higgs fields takes the form

$$\mathcal{L}_H = (D^\mu \varphi)^\dagger (D_\mu \varphi) + (D^\mu \varphi_*)^\dagger (D_\mu \varphi_*) - V_H \quad (11.42)$$

in which the covariant derivatives act as follows:

$$iD_\mu \varphi = \left(i\partial_\mu - g^{(2)} A_\mu^{(2)a} \frac{1}{2} \tau_{La} - g^{(1)} A_\mu^{(1)} \frac{1}{2} \right) \varphi \quad (11.43)$$

and

$$iD_\mu \varphi_* = \left(i\partial_\mu - g_*^{(2)} A_{*\mu}^{(2)a} \frac{1}{2} \tau_{Ra} - g_*^{(1)} A_{*\mu}^{(1)} \frac{1}{2} \right) \varphi_*. \quad (11.44)$$

The Higgs potential is generally given by

$$V_H = V_0 - \mu^2 \varphi^\dagger \varphi + \lambda (\varphi^\dagger \varphi)^2 - \mu_*^2 \varphi_*^\dagger \varphi_* + \lambda_* (\varphi_*^\dagger \varphi_*)^2 + 2\lambda_I (\varphi_*^\dagger \varphi_*) (\varphi^\dagger \varphi) \quad (11.45)$$

where λ , λ_* and λ_I are the constants of self-coupling and bi-quadratic mutual interaction.

11.6 A scenario for dark matter

If the quarks u_* and d_* acquire close masses like the u and d quarks of the first family when the G_{EW*} symmetry is spontaneously broken at the energy scale Λ_* ($\Lambda_* > \Lambda$), many kinds of dark nuclei and a variety of dark elements come necessarily into existence. Thereby, the dark sector with such quarks u_* and d_* is destined to follow a rich and intricate path of thermal history of evolution like the visible sector of our Universe.

Here we consider a situation that, just like the case of the t and b quarks of the third family, the dark up quark u_* is much heavier than the dark down quark d_* as [1,3]

$$m_{u_*} \gg m_{d_*} + m_{e_*} + m_{\nu_*}. \quad (11.46)$$

In such a case, the u_* quark disappears quickly through the process $u_* \rightarrow d_* + \bar{e}_* + \nu_*$ leaving the d_* quark as the main massive components of the dark sector. Consequently, the gauge fields $A_{*\mu}^{(3)}$ of $SU_{c*}(3)$ symmetry act to confine the d_* quark into the dark color-singlet hadron

$$\Delta_*^- = [d_* d_* d_*] = \frac{1}{\sqrt{6}} \epsilon_{ijk} d_*^i d_*^j d_*^k \quad (11.47)$$

which has the dark electric charge $Q_* = -1$ and the spin angular momentum $\frac{3}{2}$ due to the Fermi statistics.

In this scenario, Δ_*^- is the stable dark hadron which can exist as the only nucleus in the dark sector. Therefore, no rich nuclear reaction can occur and only a meager history of thermal evolution can take place. The stable atom which can exist in the matter-dominant stage of the dark sector is limited to be

$$\bar{H}_* = (\Delta_*^- + \bar{e}_*). \quad (11.48)$$

It can be speculated that the dark molecule

$$(\bar{H}_*)^2 = \bar{H}_* \bar{H}_* \quad (11.49)$$

in the spin 1 or 0 states can be stable entities prevailing over a broad spatial region of the late stage of the Universe. These features seem to be consistent with the characteristics of the dark matter which has the tendency to spread out rather monotonically over broad spatial regions as inferred by the observations of gravitational lensing.

11.7 Discussion

By generalizing the concept of Dirac spinor, we have developed a unified theory of multi-spinor fields that can describe the whole spectra of fermionic species in the visible and dark sectors of the Universe. The physical subalgebras of the triplet algebra satisfying the criterion of the S_3 irreducibility have the unique feature of the “ $\mathfrak{su}(3)$ plus $\mathfrak{u}(1)$ ” structure for both of the color and family degrees of freedom. The triplet compound field in Eq.(11.32) possessing the component fields of the three visible and one dark family modes with the tri-color quarks and colorless leptons enables us to formulate a unified theory that can describe the flavor physics in the visible sector and cosmological phenomena related to both the visible and dark sectors.

To develop the present theory further, it is necessary to examine the thermal history of the dark sector and to confirm that the present scheme is consistent with the well-established standard theory of astrophysics and cosmology. In this note, we make heuristic analyses to estimate the mass of the stable dark hadron Δ_* from the cosmological parameters for the densities of the cold dark matter and baryonic matter determined by WMAP [4] and Planck [5] observations.

The visible fermionic and bosonic fields can interact with the dark fermionic and bosonic fields through virtual loop corrections induced by the bi-quadratic interaction between the Higgs fields φ and φ_* in Eq.(11.45). Therefore, it is possible to assume that quanta of all fields of the visible and dark sectors constitute a common soup of inseparable phase in an early reheating period. Expansion of the Universe decreasing its temperature breaks both of the dark electroweak symmetry G_{EW*} and the visible electroweak symmetry G_{EW} symmetry in the quantum soup.

At present, there exists no reliable theory which can describe consistently the cosmic baryogenesis. So we set a simple working hypothesis ¹ that the process of baryogenesis takes place cooperatively through the two-step breakdowns of G_{EW*} and G_{EW} symmetries in such a way that the excess of quark numbers is created and preserved equally for all families in the visible and dark sectors. The quarks getting heavy masses decay down to the lighter quarks. While the u_* quark disappears leaving only the d_* quark in the dark sector, four types of the heavy quarks decay into the u and d quarks which have almost degenerate masses in the visible sector. The $SU_{C*}(3)$ gauge interaction works to confine the d_* quarks

¹ Technical details of some scenarios of ‘Two-Step Electroweak Baryogenesis’ can be found in the articles [6,7].

into the dark hadron Δ_* with 4 spin degrees of freedom, and the $SU_c(3)$ gauge interaction confine the u and d quarks into the nucleon possessing 2 iso-spin (proton and neutron) states with 2 spin degrees of freedom.

Remark that the quark numbers are separately conserved in the dark and visible sectors after the decoupling of the two sectors. Therefore, the observed ratio of the densities of cold dark matter and baryonic matter can be identified with the ratio of energies (masses/ c^2) stored by the stable particles in the dark and visible sectors. By assuming that the dark hadron Δ_* is the dominant component of the cold dark matter, we obtain the following relation for the masses of Δ_* and nucleon, m_{Δ_*} and m_N , as

$$2m_{\Delta_*} : 6m_N = \Omega_c h^2 : \Omega_b h^2 = 0.11889 : 0.022161 \quad (11.50)$$

where the values for the cosmological parameters $\Omega_c h^2$ and $\Omega_b h^2$ taken from the Table 10 of the reference [5] are the Planck best-fit including external data set. Consequently, the upper limit of the mass of the dark hadron Δ_* is estimated to be $m_{\Delta_*} = 16.1m_N = 15.1\text{GeV}/c^2$.

Until now no affirmative result has been found by either of the direct and indirect dark matter searches. The recent observation by the LUX group [8] has proved that the background-only hypothesis is consistent with their data on spin-independent WIMP-nucleon elastic scattering with a minimum upper limit on the cross section of $7.6 \times 10^{-46}\text{cm}^2$ at a WIMP mass of $33\text{GeV}/c^2$. More stringent experiments must be performed to confirm the possibility of scenarios for dark matter including stable particles with comparatively small masses such as the dark hadron Δ_* of our theory.

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