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Almost all Cayley maps are mapical regular representations

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Abstract

Cayley maps are combinatorial structures built upon Cayley graphs on a group. As such the original group embeds in their group of automorphisms, and one can ask in which situation the two coincide (one then calls the Cayley map a mapical regular representation or MRR) and with what probability. The first question was answered by Jajcay. In this paper we tackle the probabilistic version, and prove that as groups get larger the proportion of MRRs among all Cayley Maps approaches 1.

Keywords: Regular representation, Cayley map, automorphism group, asymptotic enumeration, graphical regular representation, GRR.

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1 Introduction

In this first section we define Cayley graphs and maps, give some context and state our main theorem. In the second section we prove the theorem. In the third one we prove a slight variation of the result in which Cayley maps are considered up to isomorphism.

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1.1 Cayley graphs

We consider only finite groups and finite graphs in this paper. As usual a *graph* Γ is an ordered pair $(V,\ E)$ with V a finite non-empty set and with E a collection of 2-subsets of V. An *automorphism* of a graph is a permutation on V that preserves the set E, and a *path* on a graph is a sequence $v_1, v_2, \dots, v_n$ of adjacent vertices, i.e. $\{v_i, v_{i+1}\} \in E$ for all i. The *neighbourhood* of a vertex v is the set $\Gamma(v) = \{w \in V | \{v, w\} \in E\}$ of all vertices connected to it by an edge.

Let R be a group and let S be an inverse-closed subset of R, that is, $S = \{s^{-1} \mid s \in S\}$. The *Cayley graph* $\mathrm{Cay}(R, S)$ is the graph with $V = R$ and with $\{r, t\} \in E$ if and only if $tr^{-1} \in S$, i.e. $E = \{\{r, sr\} | s \in S, r \in R\}$. The condition $S = S^{-1}$ is imposed to guarantee that $tr^{-1} \in S$ if and only if $rt^{-1} \in S$. A path $r_0, r_1, \dots, r_n$ in a Cayley graph can be specified equivalently by its starting vertex r_0 together with the unique sequence of elements $s_1, s_2, \dots, s_n$ from S such that $r_{i+1} = s_{i+1} r_i$. Usually one is interested in connected Cayley graphs, where for any two vertices there is at least one path connecting them. This is equivalent to the requirement that S is a set of generators for the group. We shall assume so throughout this paper.

A *graphical regular representation* (GRR) for a group R is a graph whose automorphism group is the group R acting regularly on the vertices of the graph. (A permutation group R is *regular* if it is transitive and if the identity element of R is the only element fixing some point of the domain.) It is an easy observation that the right regular action of R on itself preserves the edges, so R embeds in $\mathrm{Aut}(\mathrm{Cay}(R, S))$.[1] A GRR for R is therefore a Cayley graph on R that admits no other automorphisms.

The main thrust of much of the work through the 1970s was to determine which groups admit GRRs. This question was ultimately answered by Godsil in [2]. It was conjectured by Babai and Godsil that, except for two natural families of groups, GRRs not only exist, but they are abundant, that is, with probability tending to 1 as $|R| \to \infty$, a Cayley graph on R is a GRR. The first author reported the recent progress in [7, 8, 9, 10] on the Babai-Godsil conjecture at the SIGMAP 2022 conference at the University of Alaska Fairbanks. During this conference, Robert Jajcay has suggested a similar investigation for Cayley maps.[2] We now give some background on Cayley maps, state Jajcay's question and state our main result.

1.2 Graph maps and Cayley maps

Let $\Gamma := (V, E)$ be a graph. Given $v \in V$, we let $\Gamma(v)$ denote the neighbourhood of v in Γ. A *rotation* on Γ is a set $\rho := (\rho_v)_{v \in V}$, where each $\rho_v \colon \Gamma(v) \to \Gamma(v)$ is a cyclic ordering [3] of $\Gamma(v)$. A *map* is a pair (Γ, ρ), where Γ is a connected graph and ρ is a rotation of Γ.

The idea behind maps is that they represent a CW complex structure on an orientable surface whose 1-skeleton is the given graph, essentially an embedding of the graph in an orientable surface disconnecting it into disks. See for instance [3] for details. The ρ_v are the cyclic orderings of the edges incident to v in the embedding.

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[1] We let automorphisms act on the right, so we will write x^{φ} to denote the image of the vertex x under the automorphism φ, and we shall take $x^{\varphi\psi}$ to mean $(x^{\varphi})^{\psi}$.

[2] During the preparation of this paper, Xia and Zheng have announced a solution to the Babai-Godsil conjecture, see [11].

[3] A cyclic ordering on a (finite) set is a permutation with no fixed points and a single cycle in its cycle decomposition.

Intuitively, an automorphism of a map (Γ, ρ) is a pair: an automorphism of the graph and an oriented homeomorphism of the surface that are compatible through the embedding. Combinatorially this translates to an automorphism of Γ (a permutation of the vertices preserving the edges) which also preserves the rotation ρ. In order to make this idea precise, we make a slight detour. Let $\mathrm{Aut}(\Gamma)$ be the automorphism group of Γ and let $R(\Gamma)$ be the collection of all rotations of Γ. Now, $\mathrm{Aut}(\Gamma)$ has a natural action on $R(\Gamma)$:

$$\begin{aligned} R(\Gamma) \times \mathrm{Aut}(\Gamma) &\longrightarrow R(\Gamma) \\ (\rho, \varphi) &\longmapsto \rho^{(\varphi)}, \end{aligned}$$

where $\rho^{(\varphi)}_{v^\varphi} = \varphi^{-1}\rho_v\varphi$, for all $v \in V$. In other words, the rotation $\rho^{(\varphi)}$ at the vertex v^φ takes u^φ to w^φ when ρ_v takes u to w. Now, an *automorphism* of a map $M = (\Gamma, \rho)$ is an automorphism φ of the graph Γ such that $\rho^{(\varphi)} = \rho$, that is,

$$\rho_{v^\varphi} = \varphi^{-1}\rho_v\varphi, \text{ for each vertex } v \text{ of } \Gamma. \tag{1.1}$$

It is well known [1] that, if the underlying graph is connected, a map automorphism is determined uniquely by its value on an oriented edge (i.e. an ordered pair of adjacent vertices). We recall briefly the reason: suppose φ is a map automorphism, w_0, w_1 are adjacent vertices mapped to w_0^φ and w_1^φ respectively and $w_0, w_1, \ldots, w_t$ is a path in the graph. We can describe the path as a sequence of left and right turns, or with a closer analogy as the exits to take at consecutive roundabouts. There must be natural numbers n_i for $i \in \{1, \ldots, t-1\}$ such that $w_{i+1} = w_{i-1}^{\rho_{w_i}^{n_i}}$. Thus the path $\varphi(w_0), \varphi(w_1), \ldots, \varphi(w_t)$ is uniquely determined by the relations

$$w_{i+1}^\varphi = w_{i-1}^{\varphi\rho_{w_i^\varphi}^{n_i}} \text{ for } i \in \{1, \ldots, t-1\}.$$

In other words the automorphism group of a map on a connected graph acts semiregularly on the set of oriented edges.

Let now R be a group and S as above an inverse-closed set of generators excluding the identity. For every cyclic ordering $\mathfrak{r}\colon S \to S$, we define the *Cayley map* $CM(R, S, \mathfrak{r}) = (\Gamma, \rho)$ as follows: Γ is the Cayley graph $\mathrm{Cay}(R, S)$ and, for every $g \in R$ and for every x lying in the neighbourhood $\Gamma(g)$ of the vertex g,

$$\begin{aligned} \rho_g\colon\ \Gamma(g) &\longrightarrow \Gamma(g) \\ x &\longmapsto \rho_g(x) := g\mathfrak{r}(g^{-1}x). \end{aligned}$$

This is the unique map with the prescribed rotation $\mathfrak{r}$ around the identity vertex $e \in R$ such that the right regular action of the group on the Cayley graph preserves the rotation.

Combinatorially, we may think of a Cayley map as just a triple $(R, S, \mathfrak{r})$, where

- R is a finite group,
- $S \subseteq R \setminus \{e\}$ is a generating set with $S = S^{-1}$, and
- $\mathfrak{r}\colon S \to S$ is a cyclic ordering.

1.3 Mapical regular representations and the question of Jajcay

Given a Cayley map $CM(R, S, \mathfrak{r})$, the right regular representation of R is contained in the automorphism group of $CM(R, S, \mathfrak{r})$. Analogously to GRRs, we say that $CM(R, S, \mathfrak{r})$ is a *mapical regular representation* (or MRR for short) if

$$\mathrm{Aut}(CM(R, S, \mathfrak{r})) \cong R.$$

As far as we are aware, this definition was coined by Robert Jajcay in [5]. Theorem 7 in [5] shows that each finite group not isomorphic to $\mathbb{Z}_3$ or $\mathbb{Z}_2^2$ possesses an MRR. Observe that $CM(R, S, \mathfrak{r})$ is a MRR if and only if the only automorphism of $CM(R, S, \mathfrak{r})$ fixing a vertex is the identity.

Once that the existence of MRRs is established it is fairly natural to investigate the abundance of MRRs among Cayley maps. Indeed, Robert Jajcay has asked whether, as $|R| \to \infty$, the proportion of MRRs among Cayley maps on R tends to 1.

One could argue for different approaches in counting Cayley maps. In the present paper we mainly first takle *labelled* Cayley maps, where two Cayley maps $CM(R, S, \mathfrak{r})$ and $CM(R, S', \mathfrak{r}')$ over the same group are considered to be the same if and only if $S = S'$ and $\mathfrak{r} = \mathfrak{r}'$. In the last section we show that our methods are trivially adapted to *unlabelled* Cayley maps, which are reasonable isomorphism classes one might be interested in. In both cases we manage to answer Jajcay's question in the affirmative.

Theorem 1.1. *As $|R| \to \infty$, the proportion of MRRs among labelled Cayley maps on R tends to* 1.

Theorem 1.2. *As $|R| \to \infty$ the proportion of (equivalence classes of) MRRs among unlabelled Cayley maps on R tends to* 1.

Xia and Zheng [11] have recently announced a positive solution of the Babai-Godsil conjecture. This means that, except for abelian groups of exponent greater than 2 and for generalized dicyclic groups, with probability tending to 1 as $|R| \to \infty$, a Cayley graph on R is a GRR. There are some relations between our work and the work in [11]; for instance, both results depend upon a theorem on group generation due to Lubotzky [6]. However, there is no direct implication between our Theorem 1.1 and the main result in [11]; for instance, a positive solution of the Babai-Godsil conjecture does not imply the veracity of Theorem 1.1. Indeed, the number of Cayley maps on a fixed Cayley graph $\mathrm{Cay}(R, S)$ is $(|S| - 1)!$, thus most Cayley maps have almost all the group as connection set of the underlying Cayley graph, while a random Cayley graph has roughly $|R|/2$ elements in its connection set. More precisely: the two questions consider different marginal probability distributions on the space of Cayley graphs.

2 Proof of main theorem

In this section, we let R be a finite group and we let r denote its order.

We explore the inclusions $R \leq \mathrm{Aut}(\mathrm{CM}(R, S, \mathfrak{r})) \leq \mathrm{Sym}(R)$. Our strategy is proving a necessary condition for intermediate subgroups between R and $\mathrm{Sym}(R)$ to be automorphism groups of Cayley maps, bound the number of subgroups satisfying this condition and then bound the number of pairs $(S, \mathfrak{r})$ compatible with at least one of them.

The following lemma is essentially a restatement of insights in [4].

Lemma 2.1. *For any Cayley map $CM(R, S, \mathfrak{r})$, the stabilizer $\mathrm{Aut}(CM(R, S, \mathfrak{r}))_e$ of the identity vertex e is cyclic of order less than $|R|$. If $\mathrm{Aut}(CM(R, S, \mathfrak{r}))_e = \langle \gamma \rangle$, then $S^\gamma = S$ and the restriction $\gamma|_S$ has the same order as γ and it is a power of $\mathfrak{r}$.*

Proof. An automorphism fixing e sends its neighbourhood $\Gamma(e) = S$ to itself.

Since the action on oriented edges is semiregular, an element of the stabilizer is uniquely determined by its action on S, i.e. the restriction mapping

$$\begin{aligned} \mathrm{Aut}(CM(R, S, \mathfrak{r}))_e &\longrightarrow \mathrm{Sym}(S) \\ \varphi &\longmapsto \quad \varphi|_S \end{aligned}$$

is injective.

Moreover, if $\varphi \in \mathrm{Aut}(CM(R, S, \mathfrak{r}))_e$, then from (1.1) we have $\mathfrak{r} = \varphi^{-1}\mathfrak{r}\varphi$, i.e., $\varphi|_S \in \mathbf{C}_{\mathrm{Sym}(S)}(\mathfrak{r})$. From standard computations in permutation groups, we have $\mathbf{C}_{\mathrm{Sym}(S)}(\mathfrak{r}) = \langle \mathfrak{r} \rangle$. Thus $\mathrm{Aut}(CM(R, S, \mathfrak{r}))_e$ is isomorphic to a subgroup of a cyclic group, hence $\mathrm{Aut}(CM(R, S, \mathfrak{r}))_e$ is cyclic and all its elements restricted to S are powers of $\mathfrak{r}$. □

Until now, we have adopted the view that a group R with r elements can be embedded into $\mathrm{Sym}(r)$ using the usual right regular representation. It is convenient for our exposition to consider the equivalent formulation "R is a regular subgroup of $\mathrm{Sym}(r)$", here regular means that for any two points in $\{1, \ldots, r\}$ there exists a unique permutation in R sending the first to the second.

Lemma 2.2. *For every regular subgroup R of $\mathrm{Sym}(r)$, the number of subgroups G of $\mathrm{Sym}(r)$ with*

- *$R < G$ and*
- *G_1 cyclic and $|G_1| \leq r - 1$ (where G_1 is the stabiliser of 1 in G)*

is at most $2^{7(\log_2 r)^2 + 12\log_2 r}$.

Proof. Given G_0 and G_1 two abstract groups and $H_0 \leq G_0$, $H_1 \leq G_1$, we write $(G_0, H_0) \sim (G_1, H_1)$ if there exists a group isomorphism $\phi\colon G_0 \to G_1$ with $H_0^\phi = H_1$. Clearly, $\sim$ defines an equivalence relation. We denote by $[(G, H)]$ the $\sim$-equivalence class containing (G, H). Now consider

$$\mathcal{M} = \{[(G, H)] \mid G \text{ is } (\log_2 r + 1)\text{-generated}, H \leq G, |G| \leq r(r-1), \text{ and } H \text{ is cyclic}\}.$$

CLAIM 1: We have

$$|\mathcal{M}| \leq 2^{4(\log_2(r))^2 + 12\log_2 r}. \tag{2.1}$$

Proof of Claim 1. From [6, Theorem 1] together with [6, Remark 3(1)] we get that the number of isomorphism classes of groups of order N that are d-generated is at most $N^{2(d+1)\log_2(N)} = 2^{2(d+1)(\log_2(|N|))^2}$. In particular, applying this theorem with $d := \log_2(r) + 1$ and with $N \leq r(r-1)$, we get that the number of groups G that are $(\log_2(r) + 1)$-generated and of order at most $r(r-1)$ is at most $2^{4(\log_2(r)+2)\log_2 r} \cdot r^2$ (observe that the second factor counts the number of choices for N: the cardinality of G). Now, let G be a group of order at most $r(r-1)$. Since G has at most $|G| < r^2$ cyclic subgroups H, our claim is proved. □

Now, let R be a regular subgroup of $\mathrm{Sym}(r)$ and let $\mathcal{S}_R$ be the set of subgroups G of $\mathrm{Sym}(r)$ with $R < G$, with G_1 cyclic and with $|G_1| \leq r-1$. Since $G = RG_1$ and since R, as any group of order r, needs at most $\log_2 r$ generators, we deduce that G needs at most $\log_2(r)+1$ generators.

CLAIM 2: We have
$$|\mathcal{S}_R| \leq 2^{3(\log_2 r)^2}|\mathcal{M}|. \tag{2.2}$$

Proof of Claim 2. Every $G \in \mathcal{S}_R$ determines an element of $\mathcal{M}$ via the mapping $\varphi\colon G \mapsto [G, G_1]$.

We show that there are at most $2^{3(\log_2 r)^2}$ elements of $\mathcal{S}_R$ having the same image via φ, from which (2.2) immediately follows. We argue by contradiction and we let $G^1, \dots, G^\ell \in \mathcal{S}_R$ with $\varphi(G^i) = \varphi(G^1)$, for every $i \in \{1, \dots, \ell\}$, where $\ell > 2^{3(\log_2 r)^2}$. Thus there exists a group isomorphism $\phi_i\colon G^1 \to G^i$ with $(G^i)_1 = ((G^1)_1)^{\phi_i}$. Therefore the permutation representation of G^1 on the coset space $G^1/(G^1)_1$ is permutation isomorphic to the permutation representation of G^i on the coset space $G^i/(G^i)_1$. Thus G^1 and G^i are conjugate via an element of $\mathrm{Sym}(r)$, that is, $G^1 = (G^i)^{\sigma_i}$ for some $\sigma_i \in \mathrm{Sym}(r)$. Now, as G^1 acts transitively on $\{1, \dots, r\}$, replacing σ_i by an element of the form $g_i\sigma_i$ (for some $g_i \in G^1$), we may assume that σ_i fixes 1, that is, $1^{\sigma_i} = 1$.

As $R \leq G^i$ for every i, we get that $R^{\sigma_1}, \dots, R^{\sigma_\ell}$ are ℓ regular subgroups of G^1. Since R is $\log_2(r)$-generated, we see that G^1 contains at most $|G^1|^{\log_2(r)} \leq r^{2\log_2 r} = 2^{2(\log_2 r)^2}$ distinct subgroups of order r. In particular, since $\ell > 2^{3(\log_2 r)^2}$, we see that $R^{\sigma_{i_1}} = \cdots = R^{\sigma_{i_t}}$ for some $t > 2^{(\log_2(r))^2}$ and some subset $\{i_1, \dots, i_t\}$ of size t of $\{1, \dots, \ell\}$. Therefore $\sigma_{i_1}\sigma_{i_j}^{-1}$ normalises R. As $1^{\sigma_{i_1}\sigma_{i_j}^{-1}} = 1$, $\sigma_{i_1}\sigma_{i_j}^{-1}$ is an automorphism of R, for every $j \in \{1, \dots, t\}$. Since R has at most $|R|^{\log_2(r)} = 2^{(\log_2(r))^2}$ automorphisms, we get $\sigma_{i_1}\sigma_{i_j}^{-1} = \sigma_{i_1}\sigma_{i_{j'}}^{-1}$ for two distinct indices j and j'. Thus $\sigma_{i_j} = \sigma_{i_{j'}}$ and $G^{i_j} = (G^1)^{\sigma_{i_j}^{-1}} = (G^1)^{\sigma_{i_{j'}}^{-1}} = G^{i_{j'}}$, which is a contradiction. □

From (2.1) and (2.2), we have
$$|\mathcal{S}_R| \leq 2^{7(\log_2 r)^2 + 12\log_2 r},$$
and the proof of this lemma immediately follows. □

It remains to estimate the number of Cayley maps on a group R compatible with a fixed intermediate subgroup G with cyclic point stabilizer H.

Lemma 2.3. *For every pair of subgroups R and H of $\mathrm{Sym}(r)$ such that R is regular and $H = \langle\gamma\rangle$ is non-identity, cyclic of order less than r and fixing the point 1, let*
$$\mathcal{R}_\gamma = \{(S, \mathfrak{r}) | S \subseteq \{2, \dots, r\},\ \mathfrak{r} \text{ cyclic ordering on } S,\ \gamma \in \mathrm{Aut}(CM(R, S, \mathfrak{r}))\}$$
be the set of all Cayley maps on R admitting γ as an automorphism. Then $|\mathcal{R}_\gamma| \leq (r-1)\frac{r}{2}\lfloor r/2\rfloor!2^r$.

Proof. Let l be the order of γ. From Lemma 2.1, if $(S, \mathfrak{r}) \in \mathcal{R}_\gamma$, then $S^\gamma = S$; thus S is a union of H-orbits. Moreover, $\gamma|_S$ is a power of $\mathfrak{r}$; hence $\gamma|_S$ is a product of k disjoint cycles all of the same length l fixing no point in S. Clearly $kl = |S| < r$. For a fixed S (and hence k and l), $\mathfrak{r} \in \mathbf{C}_{\mathrm{Sym}(S)}(\gamma|_S)$. From routine computations, $\mathbf{C}_{\mathrm{Sym}(S)}(\gamma|_S)$ is isomorphic to the wreath product $C_l \wr \mathrm{Sym}(k)$. Thus, given S, there are at most $l^k k!$ choices for $\mathfrak{r}$.

If n_l is the number of cycles of length l in the cycle decomposition of γ, then there are $\binom{n_l}{l}$ choices for S such that $\gamma|_S$ decomposes in k cycles of length l.

Putting everything together, we have

$$|\mathcal{R}_\gamma| \leq \sum_{l=2}^{r-1} \sum_{k=1}^{n_l} \binom{n_l}{k} k! l^k. \tag{2.3}$$

Of course $ln_l \leq |S| < r$ and hence $n_l < r/l$.

In what follows, we use the generalized binomial coefficient $\binom{x}{k} = \frac{1}{k!}\prod_{i=0}^{k-1}(x-i)$. Observe that $\binom{x}{k}$ is increasing in the real variable $x \geq k$. Elementary computations show the inequality

$$\frac{\binom{\frac{r}{l+1}}{k} k!(l+1)^k}{\binom{\frac{r}{l}}{k} k! l^k} = \prod_{i=0}^{k-1} \frac{r - i(l+1)}{r - il} \leq 1.$$

This gives that the summands appearing in (2.3) are non-increasing in l and hence they can be estimated with $l = 2$. We deduce

$$|\mathcal{R}_\gamma| \leq \sum_{l=2}^{r-1} \sum_{k=1}^{\lfloor \frac{r}{l} \rfloor} \binom{\lfloor \frac{r}{l} \rfloor}{k} k! l^k \leq \sum_{l=2}^{r-1} \sum_{k=1}^{\lfloor \frac{r}{l} \rfloor} \binom{\frac{r}{l}}{k} k! l^k \leq \sum_{l=2}^{r-1} \sum_{k=1}^{\lfloor \frac{r}{l} \rfloor} \binom{\frac{r}{2}}{k} k! 2^k.$$

Furthermore, an easy computation shows that (for $0 \leq k \leq x$) $\binom{x}{k+1} - \binom{x}{k} \geq 0$ if and only if $k < \frac{x}{2}$. Thus we can estimate generalized binomial coefficients with an "almost central binomial coefficient": $\binom{\frac{r}{2}}{k} \leq \binom{\frac{r}{2}}{\lfloor \frac{r}{4} \rfloor}$. Thus

$$\begin{aligned}|\mathcal{R}_\gamma| &\leq \sum_{l=2}^{r-1} \sum_{k=1}^{\lfloor \frac{r}{2} \rfloor} \binom{\frac{r}{2}}{\lfloor \frac{r}{4} \rfloor} k! 2^k \leq \sum_{l=2}^{r-1} \sum_{k=1}^{\lfloor \frac{r}{2} \rfloor} \binom{\frac{r}{2}}{\lfloor \frac{r}{4} \rfloor} \left\lfloor \frac{r}{2} \right\rfloor ! 2^{\lfloor \frac{r}{2} \rfloor} \\ &\leq (r-1) \left\lfloor \frac{r}{2} \right\rfloor 2^{\frac{r}{2}} \left\lfloor \frac{r}{2} \right\rfloor ! 2^{\lfloor \frac{r}{2} \rfloor} \leq (r-1) \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{r}{2} \right\rfloor ! 2^r. \qquad \square\end{aligned}$$

Proof of Theorem 1.1. Notice that there are $(r-2)!$ Cayley maps with $S = R \setminus \{e\}$ (this is just the number of cyclic orderings $\mathfrak{r}$), the total number of Cayley maps must be greater than that, so combining Lemmas 2.2 and 2.3, we deduce that the fraction of Cayley maps on R admitting a group of automorphisms larger than R is less than

$$\frac{((r-1)\frac{r}{2} \lfloor r/2 \rfloor ! 2^r)(2^{7(\log_2 r)^2 + 12 \log_2 r})}{(r-2)!},$$

which goes to 0 when $r \to \infty$. $\square$

3 Unlabelled version

We have so far implicitly considered a probability distribution which is uniform on *labelled* Cayley graphs on a fixed group R. But of course it can also make sense to not distinguish between maps on the same group that are mapped to one another by a group automorphism. We can show quite easily that Theorem 1.2 still holds. To be precise we consider two

Cayley maps $\mathrm{CM}(R, S_0, \mathfrak{r}_0)$ and $\mathrm{CM}(R, S_1^\alpha, \mathfrak{r}_1)$ on the same group R and we say that they are *equivalent* if there exists a group automorphism α of R such that $\mathrm{CM}(R, S_0, \mathfrak{r}_0) = \mathrm{CM}(R, S_1, \alpha \circ \mathfrak{r}_1 \circ \alpha^{-1})$. We will call these equivalence classes *unlabelled Cayley maps*.

Proof of Theorem 1.2. This is a minor adaptation of the proof of Theorem 1.1. Of course unlabelled Cayley maps are at most in the same number as their labelled counterparts, so we can still apply Lemmas 2.2 and 2.3 to deduce that those admitting automorphisms other than those given by the action of R are fewer than $((r-1)\frac{r}{2}\lfloor r/2 \rfloor !2^r)(2^{7(\log_2 r)^2+12\log_2 r})$, where $r = |R|$. Moreover each equivalence class of labelled Cayley maps can contain at most $|\operatorname{Aut}(R)|$ elements. Using nothing more than the classic estimate $|\operatorname{Aut}(R)| \le r^{\log_2 r}$ we can deduce there are at least $\frac{(r-2)!}{r^{\log_2 r}}$ unlabelled Cayley maps with $S = R \setminus \{e\}$. Then the fraction of non MRRs among unlabelled Cayley maps is bounded by

$$\frac{((r-1)\frac{r}{2}\lfloor r/2 \rfloor !2^r)(2^{7(\log_2 r)^2+12\log_2 r})r^{\log_2 r}}{(r-2)!},$$

which again goes to 0 when $r \to \infty$. □

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Factorizing the Rado graph and infinite complete graphs*

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Abstract

Let $\mathcal{F} = \{F_\alpha : \alpha \in \mathcal{A}\}$ be a family of infinite graphs, together with Λ. The Factorization Problem $FP(\mathcal{F}, \Lambda)$ asks whether $\mathcal{F}$ can be realized as a factorization of Λ, namely, whether there is a factorization $\mathcal{G} = \{\Gamma_\alpha : \alpha \in \mathcal{A}\}$ of Λ such that each Γ_α is a copy of F_α.

We study this problem when Λ is either the Rado graph R or the complete graph $K_\aleph$ of infinite order $\aleph$. When $\mathcal{F}$ is a countably infinite family, we show that $FP(\mathcal{F}, R)$ is solvable if and only if each graph in $\mathcal{F}$ has no finite dominating set. We also prove that $FP(\mathcal{F}, K_\aleph)$ admits a solution whenever the cardinality of $\mathcal{F}$ coincides with the order and the domination numbers of its graphs.

For countable complete graphs, we show some non existence results when the domination numbers of the graphs in $\mathcal{F}$ are finite. More precisely, we show that there is no factorization of $K_\mathbb{N}$ into copies of a k-star (that is, the vertex disjoint union of k countable stars) when $k = 1, 2$, whereas it exists when $k \geq 4$, leaving the problem open for $k = 3$.

Finally, we determine sufficient conditions for the graphs of a decomposition to be arranged into resolution classes.

Keywords: Factorization problem, resolution problem, Rado graph, infinite graphs.

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1 Introduction

We assume that the reader is familiar with the basic concepts in (infinite) graph theory, and refer to [10] for further details.

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In this paper all graphs will be simple, namely, without multiple edges or loops. As usual, we denote by $V(\Lambda)$ and $E(\Lambda)$ the vertex set and the edge set of a simple graph Λ, respectively. We say that Λ is finite (resp. infinite) if its vertex set is so, and refer to the cardinality of $V(\Lambda)$ and $E(\Lambda)$ as the order and the size of Λ, respectively. Note that in the finite case $|E(\Lambda)| \leq \binom{|V(\Lambda)|}{2}$, whereas if Λ is infinite, then its order, which is a cardinal number, is greater than or equal to its size. We use the notation K_v for any complete graph of order v, and denote by K_V the complete graph whose vertex set is V.

Given a subgraph Γ of a simple graph Λ, we denote by $\Lambda \setminus \Gamma$ the graph obtained from Λ by deleting the edges of Γ. If Γ contains all possible edges of Λ joining any two of its vertices, then Γ is called an induced subgraph of Λ (in other words, an induced subgraph is obtained by vertex deletions only). Instead, if $V(\Gamma) = V(\Lambda)$, then Γ is called a spanning subgraph or a factor of Λ (hence, a factor is obtained by edge deletions only). If Γ is also h-regular, then we speak of an h-factor. We recall that a set D of vertices of Λ is dominating if all other vertices of Λ are adjacent to some vertex of D. The minimum size of a dominating set of Λ is called the domination number of Λ. Finally, we say that Λ is locally finite if its vertex degrees are all finite.

A decomposition of Λ is a set $\mathcal{G} = \{\Gamma_1, \Gamma_2, \ldots\}$ of subgraphs of Λ whose edge-sets partition $E(\Lambda)$. If the graphs Γ_i are all isomorphic to a given subgraph Γ of Λ, then we speak of a Γ-decomposition of Λ. When Γ and Λ are both complete graphs, we obtain 2-designs. More precisely, a K_k-decomposition of K_v is equivalent to a 2-$(v, k, 1)$ design.

Classically, the graphs Γ_i and Λ are taken to be finite, and the same usually holds for the parameters v and k of a 2-design. However, there has been considerable interest in designs on a infinite set of v points, mainly when $k = 3$. In this case, we obtain infinite Steiner triple systems whose first explicit constructions were given in [12, 13]. Further results concerning the existence of rigid, sparse, and perfect countably Steiner triple systems can be found in [6, 7, 11]. Results showing that any Steiner system can be extended are given in [1, 15]. The existence of large sets of Steiner triple systems for every infinite v (and more generally, of infinite Steiner systems) can be found in [4]. Also, infinite versions of topics in finite geometry, including infinite Steiner triple systems and infinite perfect codes are considered in [3]. A more comprehensive list of results on infinite designs can be found in [9].

When each graph of a decomposition $\mathcal{G}$ of Λ is a factor (resp. h-factor), we speak of a factorization (resp. h-factorization) of Λ. Also, when the factors of $\mathcal{G}$ are all isomorphic to Γ, we speak of a Γ-factorization of Λ. A factorization of K_v into factors whose components are copies of K_k is equivalent to a resolvable 2-$(v, k, 1)$ design.

In this paper, we consider the Factorization Problem for infinite graphs, which is here stated in its most general version.

Problem 1.1. Let Λ be a graph of order $\aleph$ and let $\mathcal{F} = \{F_\alpha : \alpha \in \mathcal{A}\}$ be a family of (non-empty) infinite graphs, not necessarily distinct, each of which has order $\aleph$, with $\aleph \geq |\mathcal{A}|$.

The Factorization Problem $FP(\mathcal{F}, \Lambda)$ asks for a factorization $\mathcal{G} = \{\Gamma_\alpha : \alpha \in \mathcal{A}\}$ of Λ such that Γ_α is isomorphic to F_α, for every $\alpha \in \mathcal{A}$. If Λ is the complete graph of order $\aleph$, we simply write $FP(\mathcal{F})$. If in addition to this each F_α is isomorphic to a given graph F and $|\mathcal{A}| = \aleph$, we write $FP(F)$. [1]

[1]Since in this case the factorization problem can be seen as a generalization of the Oberwolfach problem, in [8] the problem $FP(F)$ was denoted by $OP(F)$.

Note that the graphs of $\mathcal{F}$ are allowed to have zero degree vertices. This means that if F_α contains isolated vertices, then Γ_α is a copy of F_α covering all vertices of $K_\aleph$ (hence a factor of $K_\aleph$) and having the same number of isolated vertices as F_α. Otherwise, F_α has no zero degree vertices, hence Γ_α is a factor of K_α in the usual sense, that is, a spanning subgraph without isolated vertices.

As far as we know, there are only four papers dealing with the Factorization Problem for infinite complete graphs, and two of them, concern classic designs. In [14] it is shown that there exists a resolvable 2-design whenever $v = |\mathbb{N}|$ and k is finite; these designs have, in addition, a cyclic automorphism group G acting sharply transitively on the vertex set; briefly they are G-regular. In [9] it is shown that every infinite 2-design with $k < v$ is necessarily resolvable, and when $k = v$, both resolvable and non-resolvable designs exist. We point out that in [9, 14] both these results are proven more generally for t-designs whenever $t \geq 2$ is finite.

Furthermore, in [2] the authors construct a G-regular 1-factorization of a countably infinite complete graph for every finitely generated abelian infinite group G. Finally, [8] proves the following.

Theorem 1.2. *Let F be a graph whose order is the cardinal number $\aleph$. $FP(F)$ has a G-regular solution whenever the following two conditions hold:*

(1) *F is locally finite,*

(2) *G is an involution free group of order $\aleph$.*

Note that this result generalizes the one obtained in [14] to any complete graph of infinite order $\aleph$, blocks of any size less than $\aleph$, and groups G not necessarily cyclic. Furthermore, Theorem 1.2 can also be seen as a generalization of the result in [2] to complete graphs of any infinite order.

When dealing with infinite graphs, a central role is played by the Rado graph R (see [16]), named after Richard Rado who gave one of its first explicit constructions. Indeed, R is the unique countably infinite random graph, and it can be constructed as follows: $V(R) = \mathbb{N}$ and a pair $\{i, j\}$ with $i < j$ is an edge of R if and only if the i-th bit of the binary representation of j is one. R shows many interesting properties, such as the universal property: every finite or countable graph can be embedded as an induced subgraph of R.

When replacing the concept of induced subgraph with the dual one of factor, a weaker result holds. Indeed, in [5] it is pointed out that a countable graph F can be embedded as a factor of R if and only if the domination number of F is infinite. In the same paper, it is further shown that $FP(\mathcal{F}, R)$ has a solution whenever $\mathcal{F}$ is infinite and each of its graphs is locally finite. Note that a locally finite countable graph has infinite domination number, but the converse is not true: for example, the Rado graph is not locally finite and it has no finite dominating set (indeed, for every $D = \{i_1, \ldots, i_t\} \subset \mathbb{N}$, there exists an integer $j \in \mathbb{N}$ whose binary representation has 0 in positions $i_1, \ldots, i_t$, which means that j is adjacent with no vertex of D).

In this paper, we extend this result to any countable family $\mathcal{F}$ of admissible graphs. More precisely, we prove the following. We point out that throughout the paper, any countable family (or graph) is understood to be infinite.

Theorem 1.3. *Let $\mathcal{F}$ be a countable family of countable graphs. Then, $FP(\mathcal{F}, R)$ has a solution if and only if the domination number of each graph of $\mathcal{F}$ is infinite.*

Furthermore, we prove the solvability of $FP(\mathcal{F})$ whenever the size of $\mathcal{F}$ coincides with the order and the domination number of its graphs.

Theorem 1.4. *Let $\mathcal{F}$ be a family of graphs, each of which has order $\aleph$. $FP(\mathcal{F})$ has a solution whenever the following two conditions hold:*

(1) $|\mathcal{F}| = \aleph$, *and*

(2) *the domination number of each graph in $\mathcal{F}$ is $\aleph$.*

When $\mathcal{F}$ contains only copies of a given graph F satisfying condition (1) of Theorem 1.2 (i.e., F is locally finite), then $\mathcal{F}$ satisfies both conditions (1) and (2) of Theorem 1.4. Therefore, Theorem 1.4 can be seen as a generalization of Theorem 1.2, even though it does not provide any information on the automorphisms of a solution to FP.

Note that if we just require that the domination number of each graph of $\mathcal{F}$ is $\aleph$, there may exist factorizations with fewer factors than $\aleph$; this means that the two conditions in Theorem 1.4 are independent. Indeed, the Rado graph R has no finite dominating set and Corollary 2.4 shows that for every $n \geq 2$ there exists a factorization of $K_{\mathbb{N}}$ into n copies of R. We point out that Theorem 1.4 constructs instead factorizations of $K_{\mathbb{N}}$ into infinite copies of R.

The paper is organized as follows. In Sections 2 and 3, we prove the main results of this paper, Theorems 1.3 and 1.4. In Section 4, we deal with F-factorizations of $K_{\mathbb{N}}$ when F belongs to a special class of graphs with finite domination number (and hence not satisfying condition (2) of Theorem 1.4): the countable k-stars (briefly, S_k), that is, the vertex disjoint union of k countable stars. We prove that $FP(S_k)$ has a solution whenever $k > 3$, and there is no solution for $k \in \{1, 2\}$. This shows that there are families $\mathcal{F}$ of graphs for which $FP(\mathcal{F})$ is not solvable. We leave open the problem when $k = 3$. In the last section, inspired by [9], we provide a sufficient condition for a decomposition $\mathcal{F}$ of $K_\aleph$ to be resolvable (i.e., the graphs of $\mathcal{F}$ can be partitioned into factors of $K_\aleph$).

2 Factorizing the Rado graph

In this section, we prove Theorem 1.3. Also, since the Rado graph R is self-complementary, that is, $K_{\mathbb{N}} \setminus R$ is isomorphic to R, we obtain as a corollary the countable version of Theorem 1.4.

We start by recalling an important characterization of the Rado graph (see, for example, [5]).

Theorem 2.1. *A countable graph is isomorphic to the Rado graph if and only if it satisfies the following property:*

$\star$ *for every disjoint finite sets of vertices U and W, there exists a vertex z adjacent to all the vertices of U and non-adjacent to all the vertices of V.*

Property $\star$ is usually referred to as the *existentially closed* property. Therefore, Theorem 2.1 states that, up to isomorphism, there is exactly one existentially closed countable graph: the Rado graph.

Now we slightly generalize the construction of the Rado graph given in the introduction.

Definition 2.2. Given a set $I \subset \{0, \dots, q-1\}$, with $1 \leq |I| < q$, we denote by R^q_I the following graph: $V(R^q_I) = \mathbb{N}$, and $\{x, y\}$, with $x < y$, is an edge of R^q_I whenever the x-th digit of y in the base q expansion of y belongs to I.

Cleraly, when $q = 2$ and $I = \{1\}$ we obtain our initial description of the Rado graph (i.e. $R = R^2_{\{1\}}$).

Proposition 2.3. *Every graph R^q_I is isomorphic to the Rado graph.*

Proof. By Theorem 2.1, it is enough to show R^q_I is existentially closed. We assume, without loss of generality, that $0 \in I$ while $1 \notin I$, and let U and V be two disjoint finite subsets of $\mathbb{N}$. Then there are infinitely many positive integers whose base q expansion has 0 in each position $u \in U$ and 1 in each position $v \in V$. Denoting by z one of these integers larger than $\max(U \cup V)$, we have that z is adjacent to all the vertices of U but to none in V. □

Note that $K_{\mathbb{N}} = \bigcup_{i=0}^{q-1} R^q_{\{i\}}$ and $R^q_{\{0,\ldots,q-2\}} = \bigcup_{i=0}^{q-2} R^q_i$. Considering that the $R^q_{\{i\}}$s are pairwise edge-disjoint and isomorphic to the Rado graph, by taking $n = q-1$ we obtain the following.

Corollary 2.4. *For every positive integer n, the graphs R and $K_{\mathbb{N}}$ can be factorized into n and $n+1$ copies of R, respectively.*

The following result is crucial to prove Theorem 1.3. It strengthens a result given in [5] and allows us to suitably embed in the Rado graph R any countable graph with infinite domination number.

Proposition 2.5. *Let F be a countable graph with no finite dominating set. For every edge $e \in E(R)$, there exists an embedding σ_e of F in R such that:*

(1) *$\sigma_e(F)$ is a spanning subgraph of R containing the edge e;*

(2) *$R \setminus \sigma_e(F)$ is isomorphic to R.*

Proof. By Proposition 2.3, the graphs $R^3_{\{0,1\}}$, $R^3_{\{0\}}$ and $R^3_{\{1\}}$ are isomoprhic to R. Therefore, we can take $R = R^3_{\{0,1\}}$.

Let e be an edge of $R = R^3_{\{0\}} \cup R^3_{\{1\}}$. We can assume without loss of generality that e lies in $R^3_{\{0\}}$. In [5, Proposition 8], it is shown that there exists an embedding σ_e of F into the Rado graph $R^3_{\{0\}} \subset R$ satisfying condition (1). It is then left to prove that condition (2) holds. By Theorem 2.1, this is equivalent to saying that $R \setminus \sigma_e(F)$ satisfies $\star$.

Let U and V be two finite disjoint subsets of $\mathbb{N}$. Clearly, there are infinitely many positive integers whose base 3 expansion has 1 in each position $u \in U$ and 2 in each position $v \in V$. Let z be one of these integers larger than $\max(U \cup V)$. Since $R \setminus \sigma_e(F)$ contains $R^3_{\{1\}}$ and it is edge-disjoint from $R^3_{\{2\}}$, it follows that z is adjacent in $R \setminus \sigma_e(F)$ to all the vertices of U and is non-adjacent to all the vertices of V. This means that $R \setminus \sigma_e(F)$ is existentially closed. □

We are now ready to prove Theorem 1.3, whose statement is recalled here, for clarity.

Theorem 1.3. *Let $\mathcal{F}$ be a countable family of countable graphs. Then, $FP(\mathcal{F}, R)$ has a solution if and only if the domination number of each graph of $\mathcal{F}$ is infinite.*

Proof. Since the Rado graph has no finite dominating set, the same holds for its spanning subgraphs. Hence, each graph of $\mathcal{F}$ must have infinite domination number. Under this assumption, we are going to show that $FP(\mathcal{F}, R)$ has a solution.

By definition of Rado graph, it is easy to see that $|E(R)| = \aleph_0$, which is also the cardinality of $\mathcal{F}$. Let $E(R) = \{e_1, \ldots, e_n, \ldots\}$ and $\mathcal{F} = \{F_1, \ldots, F_n, \ldots\}$. By recursively applying Proposition 2.5, we obtain a sequence of isomorphisms $\sigma_{e_i} : F_i \to \Gamma_i$ satisfying for each $i \in \mathbb{N}$ the following properties:

- Γ_i is a spanning subgraph of R;
- $R \setminus (\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_{i-1})$ is isomorphic to R and contains Γ_i;
- e_i lies in $\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_i$.

It follows that the Γ_is are pairwise edge-disjoint factors of R which partition $E(R)$. Therefore, $\{\Gamma_i : i \in \mathbb{N}\}$ is a solution to $FP(\mathcal{F}, R)$. □

The proof of Theorem 1.3 allows us to construct solutions to $FP(\mathcal{F}, R)$ even when the cardinality of $\mathcal{F}$ is finite, provided that $\mathcal{F}$ contains a copy of the Rado graph. In other words, we have the following.

Corollary 2.6. *Let $\mathcal{F}$ be a finite family of countable graphs such that*

(1) *$\mathcal{F}$ contains at least one graph isomorphic to the Rado graph;*

(2) *the domination number of each graph in $\mathcal{F}$ is infinite.*

Then, $FP(\mathcal{F}, R)$ has a solution.

Recalling that R is self complementary, the countable version of Theorem 1.4 can be easily obtained as a corollary to Theorem 1.3.

Corollary 2.7. *Let $\mathcal{F}$ be a countable family of countable graphs. $FP(\mathcal{F})$ has a solution whenever the domination number of each graph in $\mathcal{F}$ is infinite.*

Proof. Recall that $R^2_{\{0\}}$ and $R^2_{\{1\}}$ are copies of R which together factorize $K_{\mathbb{N}}$. Therefore, it is enough to partition $\mathcal{F}$ into two countable families $\mathcal{F}_1$ and $\mathcal{F}_2$, and then apply Theorem 1.3 to get a solution $\mathcal{G}_i$ to $FP(\mathcal{F}_i, R^2_{\{i\}})$, for $i = 0, 1$. Clearly, $\mathcal{G}_1 \cup \mathcal{G}_2$ provides a solution to $FP(\mathcal{F})$. □

The natural generalization of property $\star$ to a generic cardinality $\aleph$ is the following one:

$\star_\aleph$ for every disjoint sets of vertices U and W whose cardinality is smaller than $\aleph$, there exists a vertex z adjacent to all the vertices of U and non-adjacent to all the vertices of V.

Then, using the transfinite induction (see Theorem 3.5 below), one could also prove the following generalization of Proposition 2.1:

Proposition 2.8. *Any two graphs of order $\aleph$ that satisfy property $\star_\aleph$ are pairwise isomorphic.*

Therefore, we can refer to any graph of order $\aleph$ and satisfying property $\star_\aleph$ as the $\aleph$-Rado graph $R_\aleph$. Its existence is guaranteed under the Generalized Continuum Hypothesis (GCH) which states that if $\aleph' \prec \aleph$ then $2^{\aleph'} \preceq \aleph$. Under GCH, one can see that the set $\mathcal{S}$ of all q-ary sequences of length $\prec \aleph$ has size $\aleph$. Indeed, for every $\aleph' \prec \aleph$, the set of all q-ary sequences of length $\aleph'$ has cardinality $2^{\aleph'}$, and by GCH we have that $2^{\aleph'} \preceq \aleph$; therefore, $|\mathcal{S}|$ has size at most $\aleph$. Clearly, $\aleph' \prec 2^{\aleph'} \preceq |\mathcal{S}|$ for every $\aleph' \prec \aleph$. It then follows that $|\mathcal{S}| = \aleph$.

This means that the construction of the countable Rado graph (Definition 2.2) based on representing every natural number with a finite q-ary sequence (its base q expansion) can be generalized to any order.

By assuming that GCH holds, we will prove as a corollary to Theorem 1.4 the following generalization of Theorem 1.3.

Theorem 2.9. *Let $\mathcal{F}$ be a family of graphs of order $\aleph$ and assume that $|\mathcal{F}| = \aleph$. Then $FP(\mathcal{F}, R_\aleph)$ has a solution if and only if the domination number of each graph in $\mathcal{F}$ is $\aleph$.*

3 Factorizing infinite complete graphs

We say that a graph or a set of vertices is $\aleph$-small (resp. $\aleph$-bounded) if their order or cardinality is smaller than $\aleph$ (resp. smaller than or equal to $\aleph$). Given two graphs F and Λ of order $\aleph$, we denote by $\Sigma_\aleph(F, \Lambda)$ the set of all graph embeddings between an induced $\aleph$-small subgraph of F and a subgraph of Λ. A partial order on $\Sigma_\aleph(F, \Lambda)$ can be easily defined as follows: if $\sigma\colon G \to \Gamma$ and $\sigma'\colon G' \to \Gamma'$ are embeddings of $\Sigma_\aleph(F, \Lambda)$, we say that $\sigma \leq \sigma'$ whenever σ' is an extension of σ, namely, G and Γ are subgraphs of G' and Γ', respectively, and $\sigma'|_G = \sigma$ (where $\sigma'|_G$ is the restriction of σ' to G).

Lemma 3.1. *Let F be a graph of order $\aleph$ and with no $\aleph$-small dominating set. Also, let Θ be an $\aleph$-small subgraph of $K_\aleph$, and let $\sigma \in \Sigma_\aleph(F, K_\aleph \setminus \Theta)$.*

(1) *If $v \in V(F)$, then there is an embedding $\sigma'\colon G' \to \Gamma'$ in $\Sigma_\aleph(F, K_\aleph \setminus \Theta)$ such that*

$$|V(G')| \leq |V(G)| + 1, \;\; \sigma \leq \sigma' \;\; \textit{and} \;\; v \in V(G');$$

(2) *If $x \in V(K_\aleph)$, then there is an embedding $\sigma''\colon G'' \to \Gamma''$ in $\Sigma_\aleph(F, K_\aleph \setminus \Theta)$ such that*

$$|V(G'')| \leq |V(G)| + 1, \;\; \sigma \leq \sigma'' \;\; \textit{and} \;\; x \in V(\Gamma'').$$

Proof. Let $\sigma\colon G \to \Gamma$ be an embedding in $\Sigma_\aleph(F, K_\mathbb{N} \setminus \Theta)$, and let $v \in V(F)$ and $x \in V(K_\aleph)$. Clearly, when $v \in V(G)$ or $x \in V(\Gamma)$, we can take $\sigma' = \sigma$ or $\sigma'' = \sigma$, respectively. Therefore, we can assume $v \notin V(G)$ and $x \notin V(\Gamma)$.

1. Let G' be the subgraph of F induced by v and $V(G)$. Since $V(\Theta)$ is $\aleph$-small, we can choose $a \in V(K_\aleph) \setminus V(\Theta)$ and let $\sigma'\colon V(G) \cup \{v\} \to V(\Gamma) \cup \{a\}$ be the extension of σ such that $\sigma'(v) = a$. Setting $\Gamma' = \sigma'(G')$, we have that σ' is the required embedding of $\Sigma_\aleph(F, K_\aleph \setminus \Theta)$.

2. Since F has no $\aleph$-small dominating set, $V(G)$ (which is an $\aleph$-small set) cannot be a dominating set for F. Hence, there is a vertex $a \in V(F)$ that is not adjacent to any of the vertices of G. We denote by G'' (resp., Γ'') the graph obtained by adding a to G (resp., x to Γ) as an isolated vertex. Clearly, G'' is an induced subgraph of F; also, Γ'' and Θ have no edge in common, since $E(\Gamma'') = E(\Gamma)$. Therefore, the extension $\sigma''\colon G'' \to \Gamma''$ of σ such that $\sigma''(a) = x$ is the required embedding of $\Sigma_\aleph(F, K_\aleph \setminus \Theta)$. □

From now on, we will work within the Zermelo-Frankel axiomatic system with the Axiom of Choice in the form of the Well-Ordering Theorem. We recall the definition of a well-order.

Definition 3.2. A well-order $\prec$ on a set X is a total order on X with the property that every non-empty subset of X has a least element.

The following theorem is equivalent to the Axiom of Choice.

Theorem 3.3 (Well-Ordering)**.** *Every set X admits a well-order $\prec$.*

Given an element $x \in X$, we define the section $X_{\prec x}$ associated to it:

$$X_{\prec x} = \{y \in X : y \prec x\}.$$

Corollary 3.4. *Every set X admits a well-order $\prec$ such that the cardinality of any section is smaller than $|X|$.*

Proof. Let us consider a well-order $\prec$ on X. Let x be the smallest element such that $X_{\prec x}$ has the same cardinality as X. The set $Y = X_{\prec x}$ is such that all its sections with respect to the order $\prec$ have smaller cardinality. Since Y instead has the same cardinality as X, the order $\prec$ on Y induces an order $\prec'$ on X with the required property. □

We recall now that well-orderings allow proofs by induction.

Theorem 3.5 (Transfinite induction)**.** *Let X be a set with a well-order $\prec$ and let P_x denote a property for each $x \in X$. Set $0 = \min X$ and assume that:*

- *P_0 is true, and*
- *for every $x \in X$, if P_y holds for every $y \in X_{\prec x}$, then P_x holds.*

Then P_x is true for every $x \in X$.

We are now ready to prove Theorem 1.4. The idea behind the proof can be better understood by restricting our attention to the countable case, $\aleph = \mathbb{N}$. To solve $FP(\{F_\alpha : \alpha \in \mathbb{N}\})$, we first order the edges of $K_{\mathbb{N}}$: $\{e_0, e_1, \ldots, e_\gamma, \ldots\}$. Then, we define embeddings $\sigma_\alpha^\beta : G_\alpha^\beta \to \Gamma_\alpha^\beta$ where G_α^β is an induced subgraph of F_α, and Γ_α^β is a subgraph of $K_{\mathbb{N}}$. These embeddings are obtained by recursively applying Lemma 3.1 which adds, at each step, a vertex to G_α^β and a vertex to Γ_α^β and makes sure that the vertex β belongs to both these graphs (this procedure can be seen as a variation of Cantor's "back-and-forth" method). We also make sure that, for every γ, the graphs $\Gamma_0^\gamma, \Gamma_1^\gamma, \ldots, \Gamma_\gamma^\gamma$ are pairwise edge-disjoint and contain between them the edge e_γ. The solution to $FP(\{F_\alpha : \alpha \in \mathbb{N}\})$ will be represented by $\mathcal{G} = \{\Gamma_\alpha : \alpha \in \mathbb{N}\}$ where $\Gamma_\alpha = \bigcup_\beta \Gamma_\alpha^\beta$.

Theorem 1.4 *Let $\mathcal{F}$ be a family of graphs, each of which has order $\aleph$. $FP(\mathcal{F})$ has a solution whenever the following two conditions hold:*

1. *$|\mathcal{F}| = \aleph$, and*
2. *the domination number of each graph in $\mathcal{F}$ is $\aleph$.*

Proof. Let $\mathcal{F} = \{F_\alpha : \alpha \in \mathcal{A}\}$. We consider a well-order $\prec$ on $\mathcal{A}$ satisfying Corollary 3.4. Since by assumption $|V(F_\alpha)| = |\mathcal{A}| = \aleph$, for every $\alpha \in \mathcal{A}$, we can take $V(F_\alpha) = V(K_\aleph) = \mathcal{A}$ and index the edges of $K_\aleph$ over $\mathcal{A}$: $E(K_\aleph) = \{e_\alpha : \alpha \in \mathcal{A}\}$.

To prove the assertion, we construct a chain of families $(\mathcal{E}_\gamma)_{\gamma\in\mathcal{A}}$, where

$$\mathcal{E}_\gamma := \{\sigma_\alpha^\beta : G_\alpha^\beta \to \Gamma_\alpha^\beta \mid \sigma_\alpha^\beta \in \Sigma_\aleph(F_\alpha, K_\aleph), (\alpha, \beta) \in \mathcal{A}_{\preceq\gamma} \times \mathcal{A}_{\preceq\gamma}\},$$

which satisfy the ascending property, that is, $\mathcal{E}_{\gamma'} \subseteq \mathcal{E}_\gamma$ if $\gamma' \preceq \gamma$, and the following three conditions:

(1_γ) for every $(\alpha, \beta) \in \mathcal{A}_{\preceq\gamma} \times \mathcal{A}_{\preceq\gamma}$ and $\beta' \prec \beta$ we have that $\sigma_\alpha^{\beta'} \leq \sigma_\alpha^\beta$ and $\beta \in V(G_\alpha^\beta) \cap V(\Gamma_\alpha^\beta)$;

(2_γ) for every $\beta \in \mathcal{A}_{\preceq\gamma}$, the graphs $\Gamma_\alpha^\beta : \alpha \preceq \beta$ are pairwise edge-disjoint, and the edge e_β belongs to their union;

(3_γ) for every $\alpha, \beta \in \mathcal{A}_{\preceq\gamma}$, the graph Γ_α^β is either finite or $|\mathcal{A}_{\preceq\gamma}|$-bounded.

The desired factorization of $K_\aleph$ is then $\mathcal{G} = \{\Gamma_\alpha \colon \alpha \in \mathcal{A}\}$, where $\Gamma_\alpha = \bigcup_{\beta\in\mathcal{A}} \Gamma_\alpha^\beta$ for every $\alpha \in \mathcal{A}$. Indeed, property (1_γ) guarantees that each Γ_α is a factor of $K_\aleph$ isomorphic to F_α. Also, property (2_γ) ensures that the Γ_αs are pairwise edge-disjoint and between them contain all the edges of $K_\aleph$.

We proceed by transfinite induction on γ.

BASE CASE. Let $0 = \min \mathcal{A}$, choose an edge $e \in E(F_0)$ and let $\sigma \in \Sigma_\aleph(F_0, K_\aleph)$ be the embedding that maps e to e_0. By Lemma 3.1, there exists an embedding $\sigma_0^0 : G_0^0 \to \Gamma_0^0$ in $\Sigma_\aleph(F_0, K_\aleph)$ such that Γ_0^0 is a finite graph and

$$\sigma \leq \sigma_0^0 \text{ and } 0 \in V(G_0^0) \cap V(\Gamma_0^0).$$

Clearly, $\mathcal{E}_0 := \{\sigma_0^0\}$ satisfies properties (1_0), (2_0) and (3_0).

TRANSFINITE INDUCTIVE STEP. We assume that, for any $\gamma' \prec \gamma$, there is a family $\mathcal{E}_{\gamma'}$ satisfying properties ($1_{\gamma'}$), ($2_{\gamma'}$) and ($3_{\gamma'}$), and prove that it can be extended to a family $\mathcal{E}_\gamma$ that satisfies properties (1_γ), (2_γ) and (3_γ). Clearly it is enough to provide the maps σ_α^β where either $\alpha = \gamma$ or $\beta = \gamma$.

We start by constructing the maps σ_α^γ for every $\alpha \prec \gamma$. We proceed by transfinite induction on α.

- Base case. Set $\Theta_0 := \bigcup_{\alpha,\beta\prec\gamma} \Gamma_\alpha^\beta$ and note that, by property ($3_{\gamma'}$), Θ_0 is $\aleph$-small. We also set $\sigma_0^{\prec\gamma} \colon \bigcup_{\beta\prec\gamma} G_0^\beta \to \bigcup_{\beta\prec\gamma} \Gamma_0^\beta$ to be the map of $\Sigma_\aleph(F_0, K_\aleph \setminus \Theta_0)$ whose restriction to G_0^β is σ_0^β. We note that property ($3_{\gamma'}$) guarantees that the order of $\bigcup_{\beta\prec\gamma} G_0^\beta$ is either finite or $|\mathcal{A}_{\preceq\gamma}|$-bounded, hence $\aleph$-small.

  Therefore, we can apply Lemma 3.1 (with $\sigma = \sigma_0^{\prec\gamma}$) to obtain the map $\sigma_0^\gamma \colon G_0^\gamma \to \Gamma_0^\gamma$ in $\Sigma_\aleph(F_0, K_\aleph \setminus \Theta_0)$ such that $|V(\Gamma_0^\gamma)| \leq |V(\bigcup_{\beta\prec\gamma} \Gamma_0^\beta)| + 2$ and, for every $\gamma' \prec \gamma$,

  $$\sigma_0^{\gamma'} \leq \sigma_0^\gamma \text{ and } \gamma \in V(G_0^\gamma) \cap V(\Gamma_0^\gamma).$$

- Inductive step. Assume we have defined the maps $\sigma^\gamma_{\alpha'}$ for every $\alpha' \prec \alpha$, and set

$$\Theta_\alpha := \bigcup_{\alpha' \prec \alpha} \Gamma^\gamma_{\alpha'} \cup \bigcup_{\alpha \prec \alpha' \prec \gamma, \beta \prec \gamma} \Gamma^\beta_{\alpha'}.$$

  As before, by Lemma 3.1 there exists $\sigma^\gamma_\alpha : G^\gamma_\alpha \to \Gamma^\gamma_\alpha$ in $\Sigma_\aleph(F_\alpha, K_\aleph \setminus \Theta_\alpha)$ such that $|V(\Gamma^\gamma_\alpha)| \leq |V(\bigcup_{\beta \prec \gamma} \Gamma^\beta_\alpha)| + 2$ and, for every $\gamma' \prec \gamma$,

$$\sigma^{\gamma'}_\alpha \leq \sigma^\gamma_\alpha \text{ and } \gamma \in V(G^\gamma_\alpha) \cap V(\Gamma^\gamma_\alpha).$$

Finally, we define the maps σ^β_γ when $\beta \preceq \gamma$. We set $\Theta := \bigcup_{\alpha \prec \gamma} \Gamma^\gamma_\alpha$ and proceed by transfinite induction on β.

- Base case. If $e_\gamma \in \Theta$, let σ be the empty map of $\Sigma_\aleph(F_\gamma, K_\aleph \setminus \Theta)$. Otherwise, chose an edge $e \in E(F_\gamma)$, and let $\sigma \in \Sigma_\aleph(F_\gamma, K_\aleph \setminus \Theta)$ be the embedding that maps e to e_γ. By Lemma 3.1, there exists $\sigma^0_\gamma\colon G^0_\gamma \to \Gamma^0_\gamma$ in $\Sigma_\aleph(F_\gamma, K_\aleph \setminus \Theta)$ such that Γ^0_γ is a finite graph and

$$\sigma \leq \sigma^0_\gamma \text{ and } 0 \in V(G^0_\gamma) \cap V(\Gamma^0_\gamma).$$

- Inductive step. Assume we have defined the maps $\sigma^{\beta'}_\gamma$ for any $\beta' \prec \beta$. Again by Lemma 3.1, there exists $\sigma^\beta_\gamma\colon G^\beta_\gamma \to \Gamma^\beta_\gamma$ in $\Sigma_\aleph(F_\gamma, K_\aleph \setminus \Theta)$ such that $|V(\Gamma^\beta_\gamma)| \leq |V(\bigcup_{\beta' \prec \beta} \Gamma^{\beta'}_\gamma)| + 2$ and, for any $\beta' \prec \beta$,

$$\sigma^{\beta'}_\gamma \leq \sigma^\beta_\gamma \text{ and } \beta \in V(G^\beta_\gamma) \cap V(\Gamma^\beta_\gamma).$$

It follows from the construction that the family

$$\mathcal{E}_\gamma := \{\sigma^\beta_\alpha\colon G^\alpha_\beta \to \Gamma^\beta_\alpha \mid \sigma^\beta_\alpha \in \Sigma_\aleph(F_\alpha, K_\aleph), \alpha, \beta \preceq \gamma\}$$

satisfies properties (1_γ), (2_γ) and (3_γ). □

Assuming that GCH holds, we obtain Theorem 2.9 as a corollary to Theorem 1.4.

Theorem 2.9. *Let $\mathcal{F}$ be a family of graphs of order $\aleph$ and assume that $|\mathcal{F}| = \aleph$. Then $FP(\mathcal{F}, R_\aleph)$ has a solution if and only if the domination number of each graph in $\mathcal{F}$ is $\aleph$.*

Proof. By property $\star_\aleph$, one can easily check that $R_\aleph$ is self-complementary, that is $K_\aleph \setminus R_\aleph$ is isomorphic to $R_\aleph$, and the domination number of the $\aleph$-Rado graph is $\aleph$. Therefore, the domination number of each graph of $\mathcal{F}$ must be $\aleph$.

To prove sufficiency, note that $\mathcal{F}' := \mathcal{F} \cup \{R_\aleph\}$ satisfies the hypothesis of Theorem 1.4. Therefore $FP(\mathcal{F}')$ admits a solution. This means that $\mathcal{F}$ factorizes $K_\aleph \setminus R_\aleph \simeq R_\aleph$. □

4 The factorization problem for k-stars

Theorem 1.4 does not provide solutions to $FP(F)$ whenever the graph F has a dominating set of cardinality less than its order. In particular, if F is countable with a finite dominating set, then the existence of a solution to $FP(F)$ is an open problem. In this section, we consider a special class of such graphs, the *k-stars* S_k. More precisely,

- the star S_1, which we also call a 1-star, is the graph with vertex-set $\mathbb{N}$ whose edges are of the form $\{0, i\}$ for every $i \in \mathbb{N} \setminus \{0\}$;
- the k-star S_k is the vertex-disjoint union of k stars.

Note that S_k contains exactly k vertices of infinite degree, which we call *centers* and form a finite dominating set of S_k.

In the following, we show that $FP(S_k)$ has no solution whenever $k \in \{1, 2\}$, while it admits a solution for every $k > 3$. Unfortunately, we leave open the problem for 3-stars.

4.1 The case $k \in \{1, 2\}$

Proposition 4.1. *$FP(S_1)$ has no solution.*

Proof. Assume for a contradiction that there is a factorization $\mathcal{G}$ of $K_{\mathbb{N}}$ into 1-stars. Choose any star $\Gamma \in \mathcal{G}$ and let g denote its center. Note that all the edges of $K_{\mathbb{N}}$ incident with g belong Γ. By recalling that $\mathcal{G}$ is a factorization of $K_{\mathbb{N}}$ (and that a 1-star has no isolated vertices), it follows that g cannot be a vertex in any other star of $\mathcal{G}$. Therefore, every star of $\mathcal{G} \setminus \{\Gamma\}$ is not spanning, contradicting the assumption. □

With essentially the same proof, one obtains the following.

Remark 4.2. Let F be the vertex-disjoint union of S_1 with a finite set of isolated vertices. Then $FP(F)$ has no solution.

To prove the non-existence of a solution to $FP(S_2)$ it will be useful the following lemma.

Lemma 4.3. *If $\mathcal{G}$ is a factorization of $K_{\mathbb{N}}$ into k-stars, then there is at most one vertex of $K_{\mathbb{N}}$ that is never a center in any k-star of $\mathcal{G}$. It follows that $|\mathcal{G}| = |\mathbb{N}|$.*

Proof. It is enough to notice that every pair $\{a, b\}$ of vertices of $K_{\mathbb{N}}$ is the edge of some 2-star Γ of $\mathcal{G}$; hence, either a or b is a center of Γ. □

Proposition 4.4. *$FP(S_2)$ has no solution.*

Proof. Assume for a contradiction that there is a factorization $\mathcal{G}$ of $K_{\mathbb{N}}$ into 2-stars. For every $\Gamma \in \mathcal{G}$, letting c be a center of Γ, we denote by $\Gamma(c)$ the set of vertices adjacent with c in Γ (i.e., the neighborhood of c in Γ).

Choose any 2-star $\Gamma \in \mathcal{G}$ and let a and b denote its centers. Also, let Γ' be the 2-star of $\mathcal{G} \setminus \{\Gamma\}$ containing the edge $\{a, b\}$. Without loss of generality, we can assume that a is a center of Γ'. Finally, by Lemma 4.3, we can choose $x \in \Gamma'(a) \setminus \{b\}$ such that there exists a 2-star $\Gamma'' \in \mathcal{G}$ having x as one of its centers.

Since Γ is a factor of $K_{\mathbb{N}}$, it follows that $x \in \Gamma(b)$. In other words, $\Gamma \cup \Gamma'$ contains the edges $\{x, a\}$ and $\{x, b\}$. Therefore, $a, b \notin \Gamma''(x)$. Since Γ'' is a factor of $K_{\mathbb{N}}$ and $\{a, b\}$ is an edge of Γ, it follows $a, b \in \Gamma''(y)$, where y is the other center of Γ''. In other words, $\{y, a\}$ and $\{y, b\}$ belong to Γ'', hence y cannot lie in Γ, contradicting the fact that Γ is a factor. □

4.2 The case $k \geq 4$

In this section we prove the solvability of $FP(S_k)$ whenever $k \geq 4$. For our constructions we need to introduce the following notation.

Let $\mathbb{D}$ be an integral domain and set $V = \mathbb{D} \times \{0, 1, \ldots, h\}$, for $h \geq 0$. For the sake of brevity, we will denote each pair $(a, i) \in V$ by a_i. Given a graph Γ with vertices in V, for every $a, b \in \mathbb{D}$ we denote by $a\Gamma + b$ the graph obtained by replacing each vertex x_i of Γ with $(ax + b)_i$; further, if $\{x_i, y_i\}$ is an edge of Γ, then $\{(ax + b)_i, (ay + b)_i\}$ is an edge of $a\Gamma + b$. Also, we denote by $Orb_{\mathbb{D}}(\Gamma) = \{\Gamma + d : d \in \mathbb{D}\}$ the $\mathbb{D}$-orbit of Γ, that is, the set of all translates of Γ by the elements of $\mathbb{D}$.

Proposition 4.5. *For every $k \geq 4$, there exists a k-star Γ with vertex set $V = \mathbb{Z} \times \{0, 1\}$ such that $Orb_{\mathbb{Z}}(\Gamma)$ is a factorization of K_V into k-stars.*

Proof. We first deal with the case $k = 4$. Set $\Gamma = \bigcup_{i=1}^{4} \Gamma_i$, where each Γ_i is the 1-star with vertices in $V = \mathbb{Z} \times \{0, 1\}$ and center x_i defined as follows (see Figure 1):

- $x_1 = 0_0$ and $\Gamma_1(x_1) = \{i_0 : i \geq 1\}$;
- $x_2 = -1_1$ and $\Gamma_2(x_2) = \{i_1 : i \geq 0\} \cup \{-1_0\}$;
- $x_3 = -2_0$ and $\Gamma_2(x_3) = \{i_1 : i \leq -3\}$;
- $x_4 = -2_1$ and $\Gamma_4(x_4) = \{i_0 : i \leq -3\}$.

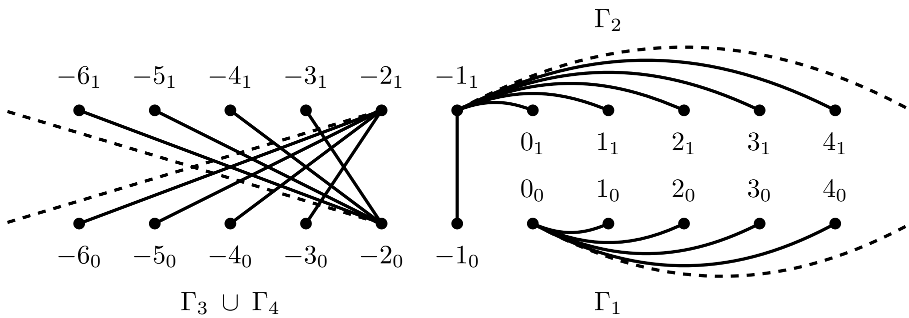

Figure 1: The graph Γ when $k = 4$.

We claim that $\mathcal{G} := Orb_{\mathbb{Z}}(\Gamma)$ is a factorization of K_V into 4-stars. Denote by $K_{U,W}$ the complete bipartite graph whose parts are $U = \mathbb{Z} \times \{0\}$ and $W = \mathbb{Z} \times \{1\}$, and consider the 1-factor $I = \{\{i_0, i_1\} : i \in \mathbb{Z}\}$ of $K_{U,W}$. Clearly, K_V decomposed into K_U, $K_W \cup I$ and $K_{U,W} \setminus I$. One can check that

- $Orb_{\mathbb{Z}}(\Gamma_1)$ decomposes K_U,
- $Orb_{\mathbb{Z}}(\Gamma_2)$ decomposes $K_W \cup I$, and
- $Orb_{\mathbb{Z}}(\Gamma_3 \cup \Gamma_4)$ decomposes $K_{U,W} \setminus I$.

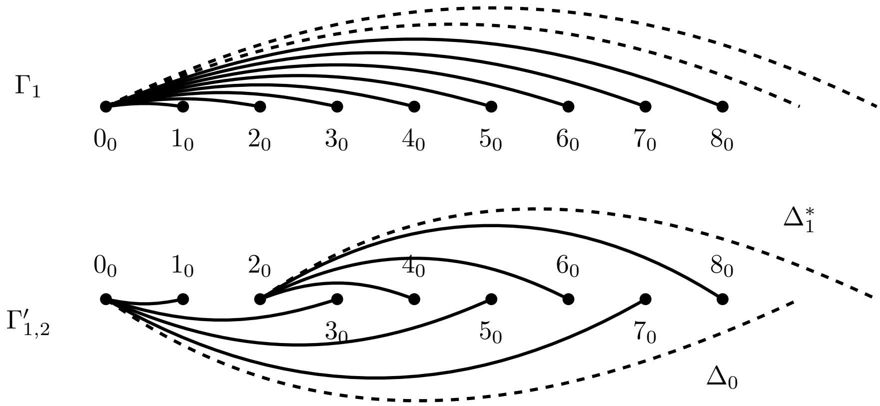

Figure 2: Replacing Γ_1 with $\Gamma'_{1,2}$ produces Γ when $k = 5$.

Hence, $\mathcal{G}$ is a decomposition of K_V. Considering that the Γ_is are pairwise vertex-disjoint and their vertex-sets partition V, we have that Γ and each of its translates (under the action of $\mathbb{Z}$) are factors of K_V isomorphic to a 4-star. Therefore, $\mathcal{G}$ is a factorization of K_V into 4-stars.

To deal with the case $k \geq 5$, it is enough to replace the component Γ_1 of Γ with a $(k-3)$-star Γ'_1 satisfying the following conditions:

$$V(\Gamma'_1) = V(\Gamma_1), \quad \text{and} \tag{4.1}$$

$$Orb_{\mathbb{Z}}(\Gamma'_1) \text{ decomposes } K_U. \tag{4.2}$$

Indeed, letting $\Gamma' = (\Gamma \setminus \Gamma_1) \,\cup\, \Gamma'_1$, by condition (4.1) we have that Γ' is a k-star with vertex-set V. Recalling that $Orb_{\mathbb{Z}}(\Gamma_1)$ decomposes K_U, by condition (4.2) it follows that $Orb_{\mathbb{Z}}(\Gamma')$ and $Orb_{\mathbb{Z}}(\Gamma)$ decompose the same graph, that is, K_V. Hence, $Orb_{\mathbb{Z}}(\Gamma')$ is a factorization of K_V into k-stars.

Let $k = h + 3$ with $h \geq 2$. It is left to construct an h-star $\Gamma'_{1,h}$ satisfying conditions (4.1) and (4.2), for every $h \geq 2$. For sake of clarity, in the rest of the proof we identify $U = \mathbb{Z} \times \{0\}$ with $\mathbb{Z}$. Therefore, Γ_1 is the 1-star centered in 0 with $\Gamma_1(0) = \{i : i \geq 1\}$.

Let Δ_j and Δ_j^* be the 1-stars centered in $c_j = 2(2^j - 1)$ such that

$$\Delta_j(c_j) = \{c_j + i : 0 < i \equiv 2^j \pmod{2^{j+1}}\},$$
$$\Delta_j^*(c_j) = \{c_j + i : 0 < i \equiv 0 \pmod{2^j}\},$$

for $j \geq 0$, and set $\Gamma'_{1,h} = \Delta_0 \,\cup\, \Delta_1 \,\cup \ldots \cup\, \Delta_{h-2} \,\cup\, \Delta_{h-1}^*$ for $h \geq 2$. It is not difficult to check that $\{\Delta_j - c_j : 0 \leq j \leq h-2\} \,\cup\, \{\Delta_{h-1}^* - c_{h-1}\}$ decomposes Γ_1. Therefore, $Orb_{\mathbb{Z}}(\Gamma'_{1,h})$ and $Orb_{\mathbb{Z}}(\Gamma_1)$ decompose the same graph, that is, K_U. Hence, $\Gamma'_{1,h}$ satisfies condition (4.2).

We show that $\Gamma'_{1,h}$ is an h-star satisfying condition (4.1) by induction on h. If $h = 2$, then $V(\Delta_0) = \{0, 1, 3, 5, \ldots\}$ and $V(\Delta_1^*) = \{2, 4, 6, \ldots\}$. Therefore, $\Gamma'_{1,2} = \Delta_0 \,\cup\, \Delta_1^*$ is a 2-star with the same vertex-set as Γ_1. Now assume that $\Gamma'_{1,h}$ is an h-star satisfying condition (4.1) for some $h \geq 2$. Recalling the definition of $\Gamma'_{1,h}$ and $\Gamma'_{1,h+1}$, and considering that the vertex-sets of Δ_{h-1} and Δ^*_h partition $V(\Delta^*_{h-1})$, we have that $\Gamma'_{1,h+1}$ is an $(h+1)$-star with the same vertex-set as $\Gamma'_{1,h}$, that is, $V(\Gamma_1)$, and this concludes the proof. □

Propositions 4.1, 4.4 and 4.5 leave open $FP(S_k)$ only when $k = 3$. In this case, an approach similar to Theorem 4.5 cannot work, as shown in the following.

Proposition 4.6. *There is no* 3*-star* Γ *with vertex-set* $V = \mathbb{Z} \times \{0, 1, \dots, k\}$ *such that the* $\mathbb{Z}$*-orbit of* Γ *is an* S_3*-factorization of* K_V*.*

Proof. Assume for a contradiction that there exists a 3-star Γ with vertex-set $V = \mathbb{Z} \times \{0, 1, \dots, k\}$ such that $\mathcal{G} = Orb_{\mathbb{Z}}(\Gamma)$ is a factorization of K_V.

We first notice that Γ must have at least a center in $\mathbb{Z} \times \{i\}$, for every $i \in \{0, 1, \dots, k\}$. Indeed, if Γ has no center in $\mathbb{Z} \times \{i\}$ for some $i \in \{0, 1, \dots, k\}$, then no edge of $K_{\mathbb{Z}\times\{i\}}$ can be covered by $\mathcal{G}$. Since Γ has 3 centers, it follows that $k \leq 2$. Note that if $k = 2$, the centers of Γ must be x_0, y_1, z_2 for some $x, y, z \in \mathbb{Z}$, but in this case the edge $\{x_0, y_1\}$ cannot lie in any translate of Γ. Therefore $k \leq 1$.

If $k = 1$, without loss of generality we can assume that the centers of Γ are 0_0, x_1 and y_1 with $x \neq y$. Since the edge $\{0_0, x_1\}$ does not belong to Γ, it lies in some of its translates, say $\Gamma + z$ with $z \neq 0$. This is equivalent to saying that $\{(-z)_0, (x-z)_1\} \in \Gamma$. It follows that $x - z = y$, hence $\{(y-x)_0, y_1\} \in \Gamma$. Similarly, we can show that $\{(x-y)_0, x_1\} \in \Gamma$. It follows that Γ cannot contain the edges $\{0_0, (x-y)_0\}$ and $\{0_0, (y-x)_0\}$. This implies that no edge of the form $\{w_0, (x-y+w)_0\}$ lies in any translate of Γ, contradicting again the assumption that $\mathcal{G}$ is a factorization of K_V. Therefore $k = 0$.

Let $V = \mathbb{Z}$ and denote by $\Delta\Gamma$ the multiset of all differences $y - x$ between any two adjcent vertices x and y of Γ, with $x < y$:

$$\Delta\Gamma = \{y - x : \{x, y\} \in E(\Gamma),\ x < y\}.$$

It is not difficult to see that $\mathcal{G} = Orb_{\mathbb{Z}}(\Gamma)$ is a factorization of $K_{\mathbb{Z}}$ if and only if $\Delta\Gamma = \mathbb{N} \setminus \{0\}$. Denoting by $\Gamma + i$ the translate of Γ obtained by replacing each vertex $x \in V(\Gamma)$ with $x + i$, one can easily see that $\Delta(\Gamma + i) = \Delta\Gamma$ for every $i \in \mathbb{Z}$. Therefore, up to a translation, we can assume that the centers of Γ are $0, x, n$ with $0 < x < n$. Now, for every $i \geq n$, denote by Γ_i the induced subgraph of Γ with vertex-set $\{0, 1, \dots, i\}$. Also, let Γ^* be the induced subgraph of Γ on the vertices $\{-3, -2, -1, 0, x, n\}$. Clearly, $|\Delta\Gamma^*| = 3$, $|\Delta\Gamma_i| = i - 2$ and $\Delta\Gamma_i \subset \{1, 2, \dots, i\}$. Also, since the multiset $\Delta\Gamma$ contains all positive integers with no repetition, it follows that $\Delta\Gamma^*$ and $\Delta\Gamma_i$ are disjoint, hence $\Delta\Gamma_i \subset \{1, 2, \dots, i\} \setminus \Delta\Gamma^*$ for every $i \geq n$. Then, for $i = max(\Delta\Gamma^*)$, we obtain the following contradiction: $i - 2 = |\Delta\Gamma_i| \leq |\{1, 2, \dots, i\} \setminus \Delta\Gamma^*| = i - 3$.

□

5 The resolvability problem

Theorem 1.4 allows us to construct decompositions of $K_{\aleph}$ into $\aleph$ graphs of specified type. More precisely, we have the following.

Corollary 5.1. *Let* $\mathcal{F} = \{F_\alpha : \alpha \in \mathcal{A}\}$ *be an infinite family of (non-empty)* $\aleph$*-bounded graphs, where* $\aleph = |\mathcal{A}|$*. Then there exists a decomposition* $\mathcal{G} = \{\Gamma_\alpha : \alpha \in \mathcal{A}\}$ *of* $K_{\aleph}$ *such that each* Γ_α *is isomorphic to* F_α*.*

Furthermore, for any $\beta \in \mathcal{A}$ such that the domination number of F_β is less than $\aleph$, we have that $|V(K_\aleph) \setminus V(\Gamma_\beta)| = \aleph$. Otherwise, for every $0 \preceq \aleph' \preceq \aleph$, the decomposition $\mathcal{G}$ can be constructed so that $|V(K_\aleph) \setminus V(\Gamma_\beta)| = \aleph'$.

Proof. For every $\alpha \in \mathcal{A}$, set $\aleph_\alpha = \aleph$ if the domination number of F_α is less than $\aleph$; otherwise, let $0 \preceq \aleph_\alpha \preceq \aleph$. By adding to each graph F_α a set of $\aleph_\alpha$ isolated vertices we obtain a graph F'_α whose order and domination number are $\aleph$. Since the assumptions of Theorem 1.4 are satisfied, there exists a factorization $\mathcal{G}' = \{\Gamma'_\alpha : \alpha \in \mathcal{A}\}$ of $K_\aleph$ such that each Γ'_α is isomorphic to F'_α. By replacing Γ'_α with the isomorphic copy of F_α, we obtain the desired decomposition $\mathcal{G}$. □

Inspired by [9], we ask under which conditions a decomposition $\mathcal{G}$ of $K_\aleph$ is resolvable, namely, its graphs can be partitioned into factors of $K_\aleph$, also called *resolution classes*. It follows that a resolvable decomposition $\mathcal{G}$ of $K_\aleph$ must satisfy the following two conditions:

N1. if $\Gamma \in \mathcal{G}$ is not a factor of $K_\aleph$, then $|V(K_\aleph) \setminus V(\Gamma)| \geq \min\{|\Gamma| : \Gamma \in \mathcal{G}\}$;

N2.
$$\mathcal{G}(z) \subseteq \mathcal{G}(x) \cup \mathcal{G}(y) \quad \Rightarrow \quad \mathcal{G}(z) \supseteq \mathcal{G}(x) \cap \mathcal{G}(y),$$
where $\mathcal{G}(v) = \{\Gamma \in \mathcal{G} : v \in V(\Gamma)\}$ is the set of all graphs of $\mathcal{G}$ passing through v.

In the following, we easily construct decompositions of $K_\aleph$ that do not satisfy the above conditions, and therefore they are non-resolvable.

Example 5.2. Let $\mathcal{F} = \{F_\alpha : \alpha \in \mathcal{A}\}$ be an infinite family of (non-empty) $\aleph$-bounded graphs, where $\aleph = |\mathcal{A}|$. Also, assume that the domination number of at least one of its graphs, say F_β, is $\aleph$. Then, by applying Corollary 5.1 with $\aleph' \prec \min\{|\Gamma_\alpha| \colon \alpha \in \mathcal{A}\}$, we construct a decomposition that does not satisfy condition N1.

For instance, if $\aleph = |\mathbb{N}|$, each F_α is a countable locally finite graph (hence, its domination number is $\aleph$) and $\aleph' = 1$ for every $\beta \in \mathbb{N}$, then we construct a decomposition $\mathcal{G} = \{G_\beta \colon \beta \in \mathbb{N}\}$ of $K_\mathbb{N}$ into connected regular graphs where $V(G_\beta) = \mathbb{N} \setminus \{x_\beta\}$ for some $x_\beta \in \mathbb{N}$. Clearly, no graph of $\mathcal{G}$ is a factor of $K_\mathbb{N}$, and any two graphs of $\mathcal{G}$ have common vertices. Therefore, $\mathcal{G}$ is not resolvable.

Example 5.3. Let $\mathcal{G}$ be any decomposition of the infinite complete graph K_V (for example, one of those constructed by Corollary 5.1). Let y and z be vertices not belonging to K_V and set $W = V \cup \{y, z\}$. We can easily extend $\mathcal{G}$ to a non-resolvable decomposition $\mathcal{G}'$ of K_W in the following way.

Choose $x \in V$ and let $\mathcal{C}$ be the following family of paths of length 1 or 2:

$$\mathcal{C} = \{[y, v, z] : v \in V \setminus \{x\}\} \cup \{[x, z, y], [x, y]\}.$$

Clearly, $\mathcal{C}$ decomposes $K_W \setminus K_V$, hence $\mathcal{G}' = \mathcal{G} \cup \mathcal{C}$ is a decomposition of K_W. Also, x, y and z do not satisfy condition N2, since $\mathcal{G}'(z) \subseteq \mathcal{G}'(x) \cup \mathcal{G}'(y)$, while $[x, y]$ belongs to $\mathcal{G}'(x) \cap \mathcal{G}'(y)$, but not to $\mathcal{G}'(z)$. Therefore, $\mathcal{G}'$ is non-resolvable. Indeed, any resolution class of $\mathcal{G}'$ could cover the vertex z only with graphs passing through x or y. This means that the graph $[x, y]$ cannot belong to any resolution class of $\mathcal{G}'$.

The following result provides sufficient conditions for a decomposition $\mathcal{G}$ to be resolvable.

Theorem 5.4. *Let $\mathcal{G}$ be a decomposition of the infinite complete graph $K_{\aleph}$ satisfying the following properties for some $\aleph' \prec \aleph$:*

R1. *each graph in $\mathcal{G}$ is $\aleph'$-bounded;*

R2. $|\mathcal{G}(x) \cap \mathcal{G}(y)| \preceq \aleph'$ *for every distinct $x, y \in V(K_{\aleph})$.*

Then $\mathcal{G}$ is resolvable.

Proof. Let $\mathcal{G} = \{G_\alpha : \alpha \in \mathcal{A}\}$. We consider a well-order $\prec$ on $\mathcal{A}$ satisfying Corollary 3.4. Since the graphs of $\mathcal{G}$ are $\aleph'$-bounded, we have that $|\mathcal{A}| = \aleph$ and we can assume $V(K_{\aleph}) = \mathcal{A}$. Here we need to construct an ascending chain $(\mathcal{G}_\gamma)_{\gamma\in\mathcal{A}}$ of families $\mathcal{G}_\gamma := \{\Gamma^\gamma_\alpha : \alpha \in \mathcal{A}_{\preceq\gamma}\}$ (where $\Gamma^{\gamma'}_\alpha$ is a subgraph of Γ^γ_α whenever $\gamma' \preceq \gamma$) that satisfy the following proprieties:

(1_γ) each Γ^γ_α is a vertex-disjoint union of graphs of $\mathcal{G}$;

(2_γ) for every $\alpha \in \mathcal{A}_{\preceq\gamma}$, $\gamma \in V(\Gamma^\gamma_\alpha)$;

(3_γ) G_γ is contained in exactly one Γ^γ_α where $\alpha \in \mathcal{A}_{\preceq\gamma}$;

(4_γ) for every $\alpha \in \mathcal{A}_{\preceq\gamma}$, Γ^γ_α is either a finite graph or $(\aleph' \cdot |\mathcal{A}_{\preceq\gamma}|)$-bounded.

The desired resolution of $K_{\aleph}$ is then $\mathcal{R} = \{\Gamma_\alpha : \alpha \in \mathcal{A}\}$, where $\Gamma_\alpha = \bigcup_{\gamma\in\mathcal{A}} \Gamma^\gamma_\alpha$ for every $\alpha \in \mathcal{A}$. Indeed, due to properties (1_γ) and (2_γ), each Γ_α is a resolution class of $\mathcal{G}$ and, by property (3_γ), $\mathcal{R}$ is a partition of $\mathcal{G}$ into resolution classes.

We proceed by transfinite induction on γ.

BASE CASE. Let $0 = \min X$. By condition R2, if 0 is not a vertex of G_0, $|\mathcal{G}(0) \cap \mathcal{G}(x)| \preceq \aleph'$ for any $x \in V(G_0)$. Since, due to condition R1, $|\mathcal{G}(0)| = \aleph$, there exists $G \in \mathcal{G}(0)$ disjoint from G_0. Therefore we can define $\mathcal{G}_0 = \{\Gamma^0_0\}$ where Γ^0_0 is either $G_0 \cup G$ or, if 0 belongs to $V(G_0)$, G_0.

TRANSFINITE INDUCTIVE STEP. For every $\gamma' \prec \gamma$, we assume there is a family $\mathcal{G}_{\gamma'}$ satisfying $(i_{\gamma'})$ for $1 \leq i \leq 4$. We show that $\mathcal{G}_{\gamma'}$ can be extended to a family $\mathcal{G}_\gamma$ that satisfies the same properties, (i_γ) for $1 \leq i \leq 4$.

We are going to define, recursively, the graphs Γ^γ_α whenever $\alpha \preceq \gamma$. First, we consider the case $\alpha \prec \gamma$. We start by setting $\Gamma^{\prec\gamma}_\alpha := \bigcup_{\gamma'\prec\gamma} \Gamma^{\gamma'}_\alpha$. Note that property ($4_{\gamma'}$) guarantees that $\Gamma^{\prec\gamma}_\alpha$ is either finite or $|\Gamma^{\prec\gamma}_\alpha| \preceq \aleph' \cdot |\mathcal{A}_{\preceq\gamma}|$; hence, $\Gamma^{\prec\gamma}_\alpha$ is $\aleph$-small.

- Base case. If $\gamma \in V(\Gamma^{\prec\gamma}_0)$, set $\Gamma^\gamma_0 = \Gamma^{\prec\gamma}_0$.

  If $\gamma \notin V(\Gamma^{\prec\gamma}_0)$, by condition R2 we have $|\mathcal{G}(\gamma) \cap \mathcal{G}(x)| \preceq \aleph'$ for every $x \in V(\Gamma^{\prec\gamma}_0)$. Since $\Gamma^{\prec\gamma}_0$ is $\aleph$-small, this means that the family of graphs of $\mathcal{G}(\gamma)$ that intersect $V(\Gamma^{\prec\gamma}_0)$ is $\aleph$-small.

  Moreover, any $\Gamma^{\prec\gamma}_\alpha$ is either finite or $(\aleph' \cdot |\mathcal{A}_{\preceq\gamma}|)$-bounded (note that $\aleph' \cdot |\mathcal{A}_{\preceq\gamma}| \prec \aleph$, since $|\mathcal{A}_{\preceq\gamma}| \prec \aleph$). Hence, the set of graphs in $\mathcal{G}(\gamma)$ that are contained in some $\Gamma^{\prec\gamma}_\alpha$ is $\aleph$-small.

  Finally, by condition R1, we have that $|\mathcal{G}(\gamma)| = \aleph$. Therefore, there exists a graph $G \in \mathcal{G}(\gamma)$ that is not contained in any $\Gamma^{\prec\gamma}_\alpha$ and such that $V(G) \cap V(\Gamma^{\prec\gamma}_0) = \emptyset$. Then, we set $\Gamma^\gamma_0 = \Gamma^{\prec\gamma}_0 \cup G$.

- Recursive step. Let $\alpha \prec \gamma$. If $\gamma \in V(\Gamma_\alpha^{\prec\gamma})$, set $\Gamma_\alpha^\gamma = \Gamma_\alpha^{\prec\gamma}$. Otherwise, by proceeding as in the previous case, we obtain the existence of a graph $G \in \mathcal{G}(\gamma)$ that is not in any $\Gamma_{\alpha'}^{\prec\gamma}$ or any $\Gamma_{\alpha''}^{\gamma}$ (where $\alpha' \prec \gamma$ and $\alpha'' \prec \alpha$), and such that $V(G) \cap V(\Gamma_\alpha^{\prec\gamma}) = \emptyset$. In this case, we set $\Gamma_\alpha^\gamma = \Gamma_\alpha^{\prec\gamma} \cup G$.

It is left to define Γ_γ^γ. We proceed by constructing, recursively, an ascending chain of graphs Γ_γ^α, for $\alpha \in \mathcal{A}_{\preceq\gamma}$, that are either finite or $(\aleph' \cdot |\mathcal{A}_{\preceq\gamma}|)$-bounded.

- Base case. Let us first suppose that G_γ is not contained in any $\Gamma_{\alpha'}^\gamma$ (where $\alpha' \prec \gamma$). Again, by conditions R1 and R2, there exists $G \in \mathcal{G}(0)$ that is also not contained in any $\Gamma_{\alpha'}^\gamma$ such that G is either G_γ or is disjoint from G_γ. We set Γ_γ^0 to be $G_\gamma \cup G$. Otherwise, we set Γ_γ^0 to be any graph G in $\mathcal{G}(0)$ that is not contained in any $\Gamma_{\alpha'}^\gamma$.

- Recursive step. Let us suppose that $\alpha \neq 0$ and that we have defined $\Gamma_\gamma^{\alpha'}$ for every $\alpha' \prec \alpha$. Here we set $\Gamma_\gamma^{\prec\alpha}$ to be $\bigcup_{\alpha' \prec \alpha} \Gamma_\gamma^{\alpha'}$. Note that, by construction, $\Gamma_\gamma^{\prec\alpha}$ is either a finite graph or $|\Gamma_\gamma^{\prec\alpha}| \preceq \aleph' \cdot |\mathcal{A}_{\preceq\gamma}|$. If α belongs to $V(\Gamma_\gamma^{\prec\alpha})$, we set Γ_γ^α to be $\Gamma_\gamma^{\prec\alpha}$. Otherwise, proceeding as in the previous case, we obtain that there exists $G \in \mathcal{G}(\alpha)$ disjoint from $\Gamma_\gamma^{\prec\alpha}$ that does not belong to any of the $\Gamma_{\alpha'}^\gamma$. Now we set Γ_γ^α to be $G \cup \Gamma_\gamma^{\prec\alpha}$.

Then the family $\mathcal{G}_\gamma = \{\Gamma_\alpha^\gamma : \alpha \in \mathcal{A}_{\preceq\gamma}\}$ satisfies the properties (1_γ), (2_γ), (3_γ) and (4_γ) by construction. □

Remark 5.5. A cardinal $\aleph$ is said to be *regular* if any $\aleph$-small union of $\aleph$-small sets (resp. graphs) is still an $\aleph$-small set (resp. graph) otherwise it is said to be *singular*. It is easy to see that, for regular cardinals, conditions R1 and R2 of Theorem 5.4 can be relaxed to:

R1'. each graph in $\mathcal{G}$ is $\aleph$-small;

R2'. $|\mathcal{G}(x) \cap \mathcal{G}(y)| \prec \aleph$ for every distinct $x, y \in V(K_\aleph)$.

However, if $\aleph$ is a singular cardinal, then conditions R1' and R2' are no longer sufficient. Indeed, we can construct a decomposition $\mathcal{G}$ of $K_\aleph$ into $\aleph$-small graphs such that

a. $|\mathcal{G}|$ is $\aleph$-small,

b. $\mathcal{G}$ satisfies conditions R1' and R2',

c. there are two (possibly isolated) vertices x and y belonging to every graphs of $\mathcal{G}$, that is, $\mathcal{G} = \mathcal{G}(x) \cap \mathcal{G}(y)$.

Then, choosing any vertex z such that $\mathcal{G}(z) \neq \mathcal{G}$, we have that

$$\mathcal{G}(z) \subseteq \mathcal{G}(x) \cup \mathcal{G}(y) = \mathcal{G} \quad \text{but} \quad \mathcal{G}(z) \not\supseteq \mathcal{G}(x) \cap \mathcal{G}(y) = \mathcal{G}.$$

This means that condition N2 does not hold, therefore the decomposition $\mathcal{G}$ is not resolvable.

We conclude by showing that there is always a resolution for an 'almost' 2-design with blocks that are $\aleph'$-bounded for some $\aleph' \prec \aleph$, that is, a decomposition of $K_\aleph$ whose graphs are almost all $\aleph'$-bounded complete graphs. This extends some results on the resolvability of 2-designs given in [9].

Proposition 5.6. *Let $\mathcal{G}$ be a decomposition of the infinite complete graph $K_\aleph$ into $\aleph'$-bounded graphs for some $\aleph' \prec \aleph$, where $\aleph'$ is not necessarily infinite. If the subset of $\mathcal{G}$ consisting of all non-complete graphs is $\aleph'$-bounded, then $\mathcal{G}$ has a resolution.*

Proof. By assumption, condition R1 of Theorem 5.4 holds. To prove that $\mathcal{G}$ satisfies condition R2 for some $\aleph'' \prec \aleph$, we assume for a contradiction the existence of vertices x and y such that $|\mathcal{G}(x) \cap \mathcal{G}(y)| \succ \aleph'' := (\aleph' + 1)$. It follows that there are at least two complete graphs in $\mathcal{G}(x) \cap \mathcal{G}(y)$, meaning that the edge $\{x, y\}$ is covered more than once by graphs in $\mathcal{G}$, and this is a contradiction. The assertion follows from Theorem 5.4. □

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Redundantly globally rigid braced triangulations*

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Abstract

By mapping the vertices of a graph G to points in $\mathbb{R}^3$, and its edges to the corresponding line segments, we obtain a three-dimensional realization of G. A realization of G is said to be globally rigid if its edge lengths uniquely determine the realization, up to congruence. The graph G is called globally rigid if every generic three-dimensional realization of G is globally rigid.

We consider global rigidity properties of braced triangulations, which are graphs obtained from maximal planar graphs by adding extra edges, called bracing edges. We show that for every even integer $n \geq 8$ there exist braced triangulations with $3n - 4$ edges which remain globally rigid if an arbitrary edge is deleted from the graph. The bound is best possible. This result gives an affirmative answer to a recent conjecture. We also discuss the connections between our results and a related more general conjecture, due to S. Tanigawa and the third author.

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1 Introduction

A d-dimensional *framework* (or geometric graph) is a pair (G, p), where G is a simple graph and $p\colon V(G) \to \mathbb{R}^d$ is a map. We also call (G, p) a *realization* of G in $\mathbb{R}^d$. The length of an edge uv in the framework is defined to be the distance between the points $p(u)$ and $p(v)$. The framework is said to be *rigid* in $\mathbb{R}^d$ if every continuous motion of its vertices in $\mathbb{R}^d$ that preserves all edge lengths preserves all pairwise distances. It is *globally rigid* in $\mathbb{R}^d$ if the edge lengths uniquely determine all pairwise distances. A realization (G, p) is *generic* if the set of the $d|V(G)|$ coordinates of the vertices is algebraically independent over the rationals. It is known that for generic frameworks rigidity and global rigidity in $\mathbb{R}^d$ depend only on the graph of the framework, for every $d \geq 1$. So we may call a graph G *rigid* (resp. *globally rigid*) in $\mathbb{R}^d$ if every (or equivalently, if some) generic d-dimensional realization of G is rigid (resp. globally rigid). The characterization of rigid and globally rigid graphs is known for $d = 1, 2$. For $d \geq 3$ these are major open problems. We refer the reader to [8, 10] for more details on the theory of rigid and globally rigid frameworks and graphs.

Rigid and globally rigid graphs occur in several applications, including sensor network localization [4], molecular conformation [3], formation control [13], and statics [9]. In some applications it is desirable to have a graph which remains rigid or globally rigid even if some vertices or edges are removed. In this paper we study graphs G for which $G - e$ is globally rigid in $\mathbb{R}^d$ for each edge e of G. They are called *redundantly globally rigid* in $\mathbb{R}^d$. In the rest of the paper we focus on the three-dimensional case, i.e. $d = 3$, and the following two conjectures concerning redundant global rigidity.

A *triangulation* $T = (V, E)$ is a maximal planar graph on at least three vertices. A *braced triangulation* $G = (V, E \cup B)$ is a graph obtained from a triangulation $T = (V, E)$ by adding a set B of new edges, called the *bracing edges*. If $|B| = 1$ (resp. $|B| = 2$) then we say that G is a *uni-braced* (resp. *doubly braced*) triangulation. The characterization of globally rigid braced triangulations in $\mathbb{R}^3$ is known, see Theorem 2.6 below. A conjectured sufficient condition for redundant global rigidity is as follows.

Conjecture 1.1 ([7]). *Every 5-connected braced triangulation $G = (V, E \cup B)$ with $|B| \geq 2$ is redundantly globally rigid in $\mathbb{R}^3$.*

A related extremal problem is to determine the smallest number of edges in a redundantly globally rigid graph in $\mathbb{R}^3$ on n vertices, as a function of n, for all (sufficiently large) n. By a theorem of B. Hendrickson [3] every globally rigid graph G in $\mathbb{R}^d$ on $n \geq d + 2$ vertices remains rigid in $\mathbb{R}^d$ after removing any edge of G. It is well-known that a rigid graph in $\mathbb{R}^3$ on $n \geq 3$ vertices has at least $3n - 6$ edges. These facts imply that $3n - 4$ is a lower bound for the extremal value, and $n \geq 6$ must hold. It was conjectured in [6] that this lower bound is tight.

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Conjecture 1.2 ([6]). *For every integer k there exists a redundantly globally rigid graph G in $\mathbb{R}^3$ on $n \geq k$ vertices with $3n-4$ edges.*

The truth of Conjecture 1.1, combined with the fact that there exist arbitrarily large 5-connected triangulations, would imply Conjecture 1.2. We remark that W. Whiteley [12] conjectured that every 5-connected doubly braced triangulation G remains rigid in $\mathbb{R}^3$ after removing any pair of its edges. The truth of Conjecture 1.1, together with Hendrickson's theorem, would imply an affirmative answer to his conjecture.

In the rest of the paper – after introducing the results from rigidity theory that we shall use – we consider doubly braced triangulations in which both bracing edges are dihedral (i.e. they connect non-adjacent vertices that belong to edge sharing faces). We shall prove sufficient conditions that guarantee that a specific edge can be removed from such a triangulation while preserving global rigidity.

Based on these results we can analyse special families of such triangulations which will lead to the proof of (a stronger form of) Conjecture 1.2. We shall prove that for every even integer $n \geq 8$ there exist redundantly globally rigid graphs in $\mathbb{R}^3$ on n vertices with $3n-4$ edges[1]. In the last section we prove necessary conditions for the redundant global rigidity of braced triangulations and formulate a couple of conjectures.

2 Rigid and globally rigid graphs

We shall use the following results in order to verify the rigidity or global rigidity of a graph. Let $G = (V, E)$ be a graph. For a vertex $v \in V$ let $N_G(v)$ (resp. $d_G(v)$) denote the set (resp. the number) of neighbours of v in G. For a set $X \subseteq V$ the graph obtained from G by adding a complete graph on vertex set X (that is, by adding new edges connecting the vertex pairs $x, y \in X$ which are not adjacent in G) is denoted by $G + K(X)$.

Theorem 2.1 ([11]). *Let $G = (V, E)$ be a graph and $v \in V$ with $d_G(v) \geq d+1$, for some $d \geq 1$. If $G - v$ is rigid and $G - v + K(N_G(v))$ is globally rigid in $\mathbb{R}^d$ then G is globally rigid in $\mathbb{R}^d$.*

A bracing edge uv in a braced triangulation G is called *dihedral* if it connects two non-adjacent vertices u, v of two edge sharing triangles on vertices uab and vab, respectively, of the triangulation.

A block and hole graph is obtained from the graph of an (embedded) plane triangulation by removing the interiors of some discs, defined by their boundary cycles, and then rigidifying the vertex sets of some of these cycles by adding new edges. This operation creates some holes and blocks. We shall only consider special block and hole graphs. By removing a single edge or a vertex of degree five from an (embedded) triangulation, we may create a face whose boundary is a 4-cycle or a 5-cycle, respectively. We shall say that such a cycle is a 4-*hole* or 5-*hole* in (some planar embedding of) the graph. The addition of a dihedral bracing edge creates a K_4 subgraph, which can be viewed as a 4-*block* that rigidifies the cycle $aubv$, provided the two edge sharing triangles uab, vab are both faces in the embedding. Since we shall only consider 5-connected triangulations, these triangles will always be faces (in any embedding) and the resulting 4-block will be uniquely defined. For simplicity we shall call a braced triangulation with dihedral bracing edges and a removed edge or degree-five vertex a *block and hole graph*. See [2, 12] for a more general

[1]We can extend our result to odd values of n by using different techniques. We do not discuss this extension in this paper.

definition and results on rigid block and hole graphs in three-space. We need the following corollaries of their results.

Theorem 2.2 ([12]). *Let G be a 4-connected block and hole graph which has a single 4-hole and a single 4-block. Then G is rigid in $\mathbb{R}^3$.*

Theorem 2.3 ([2]). *Let G' be a 5-connected block and hole graph with two 4-blocks and let $G = G' - v$, where v is a vertex of degree five in G' which is disjoint from the blocks. Then G is rigid in $\mathbb{R}^3$.*

Let G be a graph and let uv, vw be a pair of incident edges in G. Let E^v_{uw} be the set of the remaining edges incident with v and let $E^v_{uw} = F \cup F'$ be a bipartition of E^v_{uw}. The (3-dimensional) *vertex splitting* operation (at v, on edges uv, vw) adds a new vertex v' to the graph, adds the new edges uv', $v'w$, vv', and then replaces every edge xv in F' by an edge xv'. The edges in F stay incident to v. See Figure 1. The vertex splitting is said to be *non-trivial* if F and F' are both non-empty.

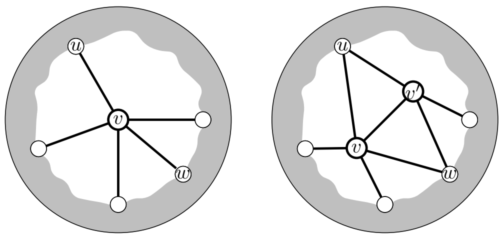

Figure 1: A non-trivial vertex splitting operation on edges uv, vw.

An important conjecture in rigidity theory is that non-trivial vertex splitting preserves global rigidity in $\mathbb{R}^d$, for all $d \geq 1$, see [1]. The next result verifies a special case.

Theorem 2.4 ([7]). *A graph is globally rigid in $\mathbb{R}^3$ if it can be obtained from K_5 by a sequence of non-trivial vertex splitting operations.*

This theorem can be used in the analysis of globally rigid braced triangulations, due to the following combinatorial result.

Theorem 2.5 ([7]). *Every 4-connected uni-braced triangulation can be obtained from K_5 by a sequence of non-trivial vertex splitting operations.*

Thus 4-connected uni-braced triangulations are globally rigid. A complete characterization, with no bounds on the number of bracing edges, is the following.

Theorem 2.6 ([7]). *A braced triangulation $G = (V, E \cup B)$ with $|V| \geq 5$ is globally rigid in $\mathbb{R}^3$ if and only if G is 4-connected and $|B| \geq 1$.*

The inverse operation of vertex splitting is the contraction of an edge uv for which u and v have exactly two common neighbours. This operation takes a triangulation to a smaller triangulation. We shall also use the fact that an edge contraction decreases the vertex connectivity of a graph by at most one.

3 Redundant edges in braced triangulations

In this section we fix the dimension $d = 3$. Every 5-connected braced triangulation with at least one bracing edge is globally rigid by Theorem 2.6. We shall describe several situations in which the removal of an edge from a 5-connected braced triangulation preserves global rigidity. The first lemma is an immediate corollary of Theorem 2.6.

Lemma 3.1. *Let $G = (V, E \cup B)$ be a 5-connected braced triangulation with $|B| \geq 2$. Then $G - e$ is globally rigid for every $e \in B$.*

In the rest of this section we shall assume that G is a 5-connected graph obtained from an (embedded) triangulation by adding exactly two dihedral bracing edges that create two 4-blocks, with at most two vertices in common.

Lemma 3.2. *Let $G = (V, E \cup B)$ be a 5-connected doubly braced triangulation with two 4-blocks. Suppose that $e = uv \in E$ is an edge with $d_G(v) = 5$ and v is disjoint from the 4-blocks. Then $G - e$ is globally rigid.*

Proof. We shall prove that v satisfies the conditions of Theorem 2.1 in graph $G - e$. The inequality $d_{G-e}(v) \geq 4$ is clearly satisfied. Since v is disjoint from the 4-blocks of G, the graph $(G - e) - v$ (= $G - v$) is a block and hole graph with one 5-hole and two 4-blocks. The 5-connectivity of G and Theorem 2.3 imply that $(G - e) - v$ is rigid. Next consider the graph $H = (G - e) - v + K(N_{G-e}(v))$. By 5-connectivity the four neighbours of v in $G - e$ induce three edges in $G - e$. Thus three new edges are added to $G - e$ to obtain H. Notice that H is a braced triangulation: two new edges can be used to triangulate the graph obtained from $T = (V, E)$ by removing v, while the third one becomes a bracing edge. See Figure 2.

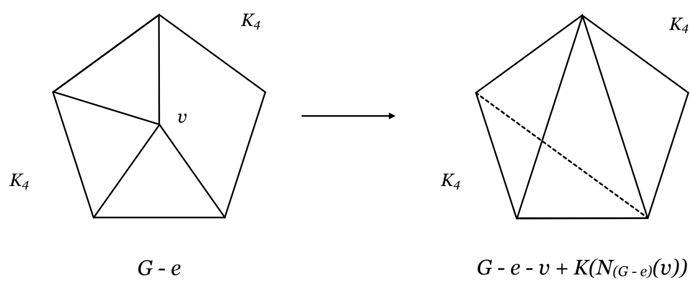

Figure 2: The neighbourhood of v in $G - e$ and the edges they induce in H. The dashed edge is a bracing edge.

Since G is 5-connected, $(G - e) - v = G - v$ is 4-connected. This implies that H is 4-connected. Hence H is globally rigid by Theorem 2.6. The lemma now follows from Theorem 2.1, applied to $G - e$ and v. □

Lemma 3.3. *Let $G = (V, E \cup B)$ be a 5-connected doubly braced triangulation with two 4-blocks. Suppose that $e = uv \in E$ is an edge with $d_G(v) = 5$ and v belongs to exactly one of the 4-blocks. Then $G - e$ is globally rigid.*

Proof. Suppose that the 4-blocks are C_1 and C_2, and v is part of C_1, say. Then the deletion of v from $G - e$ creates a block and hole graph with a 4-block (namely, C_2) and a 4-hole. Note that if v is not incident with the bracing edge f of C_1 then f becomes an edge of the underlying (almost) triangulation of $(G-e)-v$. Since G is 5-connected, $(G-e)-v = G-v$ is 4-connected. Thus $(G - e) - v$ is rigid by Theorem 2.2. Furthermore, it follows that $G-v+K(N_{G-e}(v))$ is a 4-connected braced triangulation with two bracing edges. Hence it is globally rigid by Theorem 2.6.

The Lemma now follows from Theorem 2.1, applied to $G - e$ and v. □

Lemma 3.4. *Let $G = (V, E \cup B)$ be a 5-connected doubly braced triangulation with two 4-blocks C_1, C_2 and let $v \in V(C_1) - V(C_2)$. Suppose that $vw \in E \cap E(C_1)$ for which there is a triangular face uvw of $T = (V, E)$ with $u \notin V(C_1)$. Let $e = uv$. Then $G - e$ is globally rigid.*

Proof. We show that $G - e$ can be obtained from K_5 by a sequence of non-trivial vertex splitting operations. Observe that $G - e$ has a 4-hole and two 4-blocks in which v and w have exactly two common neighbours (the two other vertices of C_1) by 5-connectivity. Let H be the graph obtained from $G-e$ by contracting the edge vw. It is easy to see that H is a 4-connected uni-braced triangulation. Thus H (and hence also $G-e$) can be obtained from K_5 by a sequence of non-trivial vertex splitting operations by Theorem 2.5. The Lemma now follows from Theorem 2.4. □

The last lemma of this section is concerned with the case when the two 4-blocks share two vertices.

Lemma 3.5. *Let $G = (V, E \cup B)$ be a 5-connected doubly braced triangulation with two 4-blocks C_1, C_2 with $V(C_1) \cap V(C_2) = \{a, b\}$, where $V(C_1) = \{a, b, c, d\}$, and the dihedral bracing edge in C_1 is ad. Then $G - ab$ and $G - ac$ are globally rigid. Furthermore, if v is a vertex which is disjoint from the blocks and $cv, av \in E$ then $G - av$ is globally rigid.*

Proof. We have $V(C_1) - V(C_2) = \{c, d\}$. Let us consider the removal of edge $e = ab$. Observe that in $G-e$ the vertices c and a have exactly two common neighbours. Moreover, the graph H obtained from $G - e$ by contracting the edge ca is a 4-connected uni-braced triangulation. Thus H (and hence also $G - e$) can be obtained from K_5 by a sequence of non-trivial vertex splitting operations by Theorem 2.5. Thus $G - ab$ is globally rigid by Theorem 2.4.

The proof for edge ac is similar. In this case we delete the edge ac, contract the edge cd, and apply the same argument. Finally, to show that $G - av$ is globally rigid, we use a similar proof again in which we delete av and then contract ac. □

4 Two families of graphs

In this section, we define two infinite families of redundantly globally rigid doubly braced triangulations in $\mathbb{R}^3$.

Definition 4.1 (Belted bipyramid)**.** For every $n \geq 3$, an n-gonal *belted bipyramid*, denoted by $G(n)$, is a graph on $2n{+}2$ vertices that is constructed as follows. Take two n-gonal pyramids with poles N and S, respectively, and label the vertices on the base of one pyramid 1 to n and on that of the other 1' to n' consecutively. Insert edges between the corresponding pairs of vertices (i.e. between 1 and 1', 2 and 2', and so on) and insert an edge between k and $(k+1)$' for every $1 \leq k \leq (n-1)$. Finally, insert an edge between n and 1'. See Figure 3.

It is easy to see that $G(n)$ is a triangulation. Let $G(n,k)$ denote the graph obtained by inserting the edges $1n'$ and $k(k{-}1)'$ to $G(n)$. Then $G(n,k)$ is a doubly braced triangulation with two dihedral bracing edges. See Figure 3.

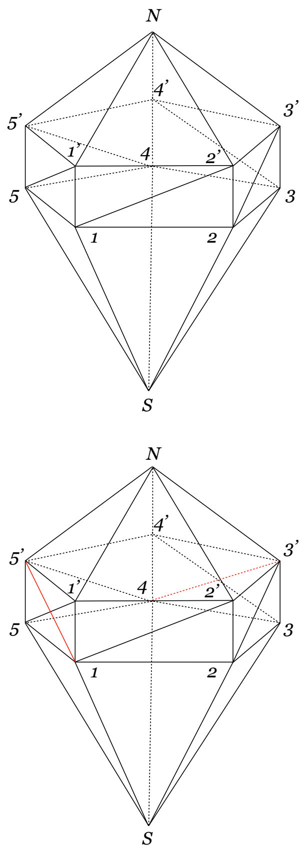

Figure 3: The graphs $G(5)$ and $G(5,4)$.

Lemma 4.2. *For every $n \geq 5$, $G(n)$ (and hence, $G(n,k)$ for every $2 \leq k \leq n$) is 5-connected.*

Proof. By using the structure and the symmetry of $G(n)$ it is not hard to check that it is 5-connected. A simple argument is as follows: consider the base cycle C of one of the pyramids on vertex set $(1, 2, ..., n)$. It is easy to verify that for every $v \in V - V(C)$ there exist 5 paths from v to $V(C)$ that are vertex-disjoint, apart from v. Furthermore, for every $u, v \in V(C)$ there exist 5 u-v-paths that are vertex-disjoint apart from u, v. Since $|V(C)| \geq 5$, this implies that $G(n)$ cannot have a vertex separator of size less than 5. □

Theorem 4.3. *For every $n \geq 5$ and $2 \leq k \leq n$ the graph $G(n, k)$ is redundantly globally rigid in $\mathbb{R}^3$.*

Proof. Theorem 2.6 implies that $G(n, k)$ is globally rigid in $\mathbb{R}^3$. It remains to show that the removal of any edge preserves global rigidity. First suppose that $3 \leq k \leq n - 1$, in which case the two 4-blocks are disjoint.

Each bracing edge is redundant by Lemma 3.1. Note that each vertex has degree five in $G(n, k)$, except for the two poles (when $n \geq 6$) and the end-vertices of the bracing edges. Thus we can use Lemmas 3.2 and 3.3 to show that most of the edges are redundant. The edges that do not satisfy the conditions of at least one of these two lemmas are the edges from the poles to the end-vertices of the bracing edges and, possibly, an edge that connects the end-vertices of different bracing edges. These edges are redundant by Lemma 3.4. So every edge is redundant and the graph is redundantly globally rigid, as required.

We can also show that $G(n, 2)$ and $G(n, n)$ are redundantly globally rigid by a similar argument. In these two special cases the two 4-blocks share two vertices, so we also need Lemma 3.5 in order to handle some of the edges incident with the intersection of the blocks. □

A slightly different construction is the following.

Definition 4.4 (Flat belted bipyramid)**.** For every $n \geq 4$, an n-gonal *flat belted bipyramid*, denoted by $F(n)$, is a graph on $2n$ vertices that is constructed as follows. Take $G(n)$ and delete its two poles. Retaining the vertex labels described in Definition 1, for every $1 \leq k \leq n$, insert an edge between vertex 3 and vertex k (unless 3 is already adjacent to k). Then, for every $1 \leq k \leq n$, insert an edge between vertex $2'$ and vertex k' (unless 2' is already adjacent to k'). See Figure 4.

It is easy to see that $F(n)$ is a triangulation. Let $H(n)$ be the graph obtained from $F(n)$ by inserting edges $1'2$ and $3'4$. See Figure 4. Thus $H(n)$ is a doubly braced triangulation with two dihedral bracing edges that create two disjoint 4-blocks. Although $F(n)$ is not 5-connected, a proof strategy similar to that of Lemma 4.2 can be used to show that $H(n)$ is 5-connected.

Lemma 4.5. *For every $n \geq 4$ the graph $H(n)$ is 5-connected.*

In fact we can show that $H(4)$ is the smallest 5-connected doubly braced triangulation[2].

Theorem 4.6. *$H(n)$ is redundantly globally rigid in $\mathbb{R}^3$ for $n \geq 4$.*

[2]The minimum degree condition implies that the number of vertices is at least eight, and equality holds only if the graph is 5-regular. Thus the complement of the graph is isomorphic to one of the following: (i) the disjoint union of a three-cycle and a five-cycle, (ii) the disjoint union of two four-cycles, (iii) a cycle on eight vertices. In the first two cases a simple analysis shows that the graph cannot be made planar by removing at most two edges. In the third case the graph is $H(4)$.

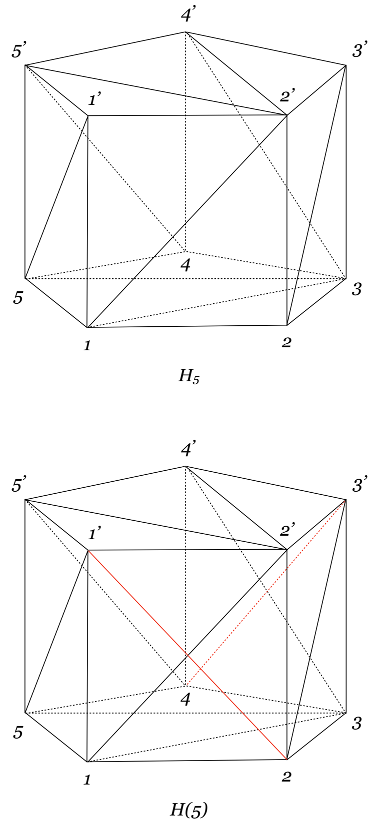

Figure 4: The graphs $F(5)$ and $H(5)$.

Proof. Theorem 2.6 implies that $H(n)$ is globally rigid in $\mathbb{R}^3$. It remains to show that the removal of any edge preserves global rigidity. The rest of the proof is similar to that of Theorem 4.3, using the lemmas of the previous section. Note that in the case of $H(n)$ the two 4-blocks are disjoint. □

The results of this section provide an affirmative answer to Conjecture 1.2.

Theorem 4.7. *For every even integer $n \geq 8$ there exist redundantly globally rigid graphs in $\mathbb{R}^3$ on n vertices with $3n - 4$ edges.*

A simple degree count shows that there are no such graphs for $n \leq 7$.

As we noted earlier, redundantly globally rigid graphs are "doubly redundantly rigid", that is, they remain rigid after the removal of any pair of edges. Thus the graphs defined in this section are also examples of doubly redundantly rigid graphs with the smallest number of edges for every even $n \geq 8$. They are different from the ones constructed in [6], and easier to analyse.

5 Concluding remarks and conjectures

A natural question is whether the 5-connectivity condition in Conjecture 1.1 can be weakened. The next example shows that 5-connectivity is not necessary.

Example 5.1. Consider the graph G in Figure 5. It is a 4-connected (but not 5-connected) doubly braced triangulation, and hence it is globally rigid by Theorem 2.6. We sketch a proof which shows that $G - e$ is globally rigid for every edge e. By the symmetry of G we have four cases to consider: the deleted edge e is

(i) a cross edge in the top K_4,

(ii) a side in the top K_4,

(iii) an edge from the K_4 to the 4-cycle of the 4-separator,

(iv) an edge of the 4-cycle of the separator.

In case (i) $G - e$ is a 4-connected braced triangulation. In cases (ii) and (iii) we can apply (the proof of) Lemma 3.3 by noting that its proof works here by using the specific structure of G (rather than 5-connectivity). In case (iv) we perform two contractions and obtain a 4-connected uni-braced triangulation as follows. Suppose, by symmetry, that $e = cd$. Then first contract an edge between c and the top K_4. Next contract one of the edges from c to the remainder of the top K_4. By contracting the appropriate edge we obtain a 4-connected uni-braced triangulation. Then global rigidity follows by Theorem 2.4.

This leads us to the next question: is it possible to obtain a complete characterization of redundantly globally rigid braced triangulations, at least in some special cases (say, for doubly braced triangulations with two dihedral bracing edges)?

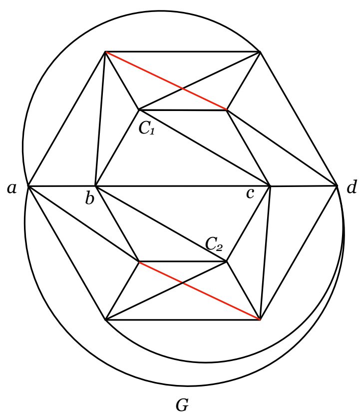

Figure 5: A redundantly globally rigid doubly braced triangulation G with a 4-separator $S = \{a, b, c, d\}$.

In this section we prove some necessary conditions and then formulate a conjecture. A *k-separator* S in a connected graph $G = (V, E)$ is a set of vertices with $|S| = k$ for which $G - S$ is disconnected. For some $X \subseteq V$ we use $G[X]$ to denote the subgraph of

G induced by vertex set X. It is known that for a minimal separator S in a triangulation G we have $|S| \geq 3$, the graph $G - S$ has exactly two connected components, and $G[S]$ is a cycle (see e.g. [7, Section 5]). For a separator S and connected component C of $G - S$ we say that $G[C \cup S]$ is an *extended component* of S in G.

Lemma 5.2. *Let $G = (V, E \cup B)$ be a redundantly globally rigid braced triangulation and let S be a 4-separator in G. Suppose that S is a minimal separator in the underlying triangulation (V, T). Then for every component C of $G - S$ there exists a bracing edge incident with C.*

Proof. Let $T = (V, E)$. Since S is a minimal separator in T, the graph $T - S$ (and hence also $G - S$) has exactly two connected components C, D. For a contradiction suppose that there is no bracing edge incident with C. Since $T[S]$ induces a 4-cycle the graph K obtained from the extended component $G[C \cup S]$ of S by adding the edges that connect those vertex pairs of S which are not adjacent in G, is a 4-connected uni-braced triangulation in which S induces a K_4. Let e be an edge of K incident with C. Then $K - e$ is a minimally rigid graph on at least five vertices. By Hendrickson's theorem $K - e$ is not globally rigid. The fact that $G - e$ can be obtained from $K - e$ by merging $K - e$ and the other extended component $G[D \cup S]$ along a complete graph (and, possibly, by deleting edges) implies that $G - e$ is not globally rigid. This contradiction completes the proof. □

The proof shows that the lemma holds even if redundantly globally rigid is weakened to doubly redundantly rigid in the condition. If the underlying triangulation T is 4-connected, then every 4-separator of G is obviously a minimal separator in T, so the conditions of Lemma 5.2 are satisfied.

Let us consider the case when T is not 4-connected and G is doubly braced. Then for every 3-separator S of T, and corresponding components C, D of $T - S$, both bracing edges must connect C and D (for otherwise S is a 3-separator in $G - e$ for some bracing edge e, contradicting redundant global rigidity). Call a component C arising by the removal of a 3-separator of T a 3-*separator component* of T. It is not hard to see that this implies that T has exactly two minimal 3-separator components C_1 and C_2, both bracing edges connect C_1 and C_2, and that T can be made 4-connected by adding a single edge (from C_1 to C_2). We believe that in this rather special case G is redundantly globally rigid. Otherwise, when T is 4-connected, the necessary condition of Lemma 5.2, together with Hendrickson's connectivity condition, might be sufficient.

Conjecture 5.3. *Let $G = (V, E \cup B)$ be a doubly braced triangulation. Then G is redundantly globally rigid in $\mathbb{R}^3$ if and only if*

(i) *$G - e$ is 4-connected for all $e \in E \cup B$, and*

  (a) *either $T = (V, E)$ has a 3-separator, or*

  (b) *for every 4-separator S of G and component C of $G - S$ there is a bracing edge incident with C.*

Note that if G is doubly braced and the bracing edges induce two disjoint 4-blocks then T must be 4-connected. Thus in this case the conjecture can be simplified.

We close this section by noting that an interesting related open problem is to characterize globally rigid block and hole graphs with a single block (with no constraints on the size of the block and the number of holes - see [2] for the definition). It is possible that the global rigidity of these graphs can be characterized by Hendrickson's necessary conditions.

Conjecture 5.4. *A block and hole graph with a single block is globally rigid in* $\mathbb{R}^3$ *if and only if it is* 4*-connected and redundantly rigid in* $\mathbb{R}^3$ *.*

A characterization of redundantly rigid block and hole graphs with a single block can be obtained from a recent result in [5].

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Intersecting families of graphs of functions over a finite field*

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Abstract

Let U be a set of polynomials of degree at most k over $\mathbb{F}_q$, the finite field of q elements. Assume that U is an intersecting family, that is, the graphs of any two of the polynomials in U share a common point. Adriaensen proved that the size of U is at most q^k with equality if and only if U is the set of all polynomials of degree at most k passing through a common point. In this manuscript, using a different, polynomial approach, we prove a stability version of this result, that is, the same conclusion holds if $|U| > q^k - q^{k-1}$. We prove a stronger result when $k = 2$.

For our purposes, we also prove the following results. If the set of directions determined by the graph of f is contained in an additive subgroup of $\mathbb{F}_q$, then the graph of f is a line. If the set of directions determined by at least $q - \sqrt{q}/2$ affine points is contained in the set of squares/non-squares plus the common point of either the vertical or the horizontal lines, then up to an affinity the point set is contained in the graph of some polynomial of the form αx^{p^k}.

Keywords: Direction problem, Erdős-Ko-Rado, finite field, polynomial.

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1 Introduction

In 1961, Erdős et al. [6] proved that if F is a k-uniform intersecting family of subsets of an n-element set X, then $|F| \leq \binom{n-1}{k-1}$ when $2k \leq n$. Furthermore, they proved that if $2k+1 \leq n$, then equality holds if and only if F is the family of all subsets containing a fixed element $x \in X$. There are several versions of the Erdős-Ko-Rado theorem. For a survey of this type of results, see [7, 13] or [5].

In this manuscript, we investigate an Erdős-Ko-Rado type problem for graphs of functions over a finite field. The idea of this work comes from the recent manuscript [1] by Adriaensen, where the author studies intersecting families of ovoidal circle geometries and, as a consequence, of graphs of functions over a finite field.

Definition 1.1. If f is an $\mathbb{F}_q \to \mathbb{F}_q$ function, then the graph of f is the affine q-set:

$$U_f = \{(x, f(x)) : x \in \mathbb{F}_q\}.$$

The set of directions determined by (the graph of) f is

$$D_f = \left\{ \frac{f(x) - f(y)}{x - y} : x, y \in \mathbb{F}_q,\ x \neq y \right\}.$$

Definition 1.2. For a family of polynomials, U, we say that U is t-intersecting if for any two polynomials $f_1, f_2 \in U$, the graphs of f_1 and f_2 share at least t points, that is,

$$|\{(x, f_1(x)) : x \in \mathbb{F}_q\} \cap \{(x, f_2(x)) : x \in \mathbb{F}_q\}| \geq t.$$

Instead of 1-intersecting, we will also use the term "intersecting".

Note that if U is a t-intersecting family of polynomials of degree at most k, then also

$$|\{(x, f_1(x)) : x \in \mathbb{F}_q\} \cap \{(x, f_2(x)) : x \in \mathbb{F}_q\}| \leq k$$

holds for any pairs $f_1, f_2 \in U$, since $(x, f_1(x)) = (x, f_2(x))$ implies that x is a root of $f_1 - f_2$ which has degree at most k.

In this note, we improve a result due to Adriaensen [1, Theorem 6.2] by using different techniques. Adriaensen's proof goes through association schemes and circle geometries, our proof does not use these, as we rely on two classical results (Results 2.3 and 2.8) about polynomials over finite fields.

To be more precise, our main results are the following theorems.

Theorem 1.3. *Let U be a set of intersecting polynomials of degree $k \leq 2$ over $\mathbb{F}_q$. Assume that $q \geq 53$, when q is odd and $q \geq 8$ when q is even. If*

$$|U| > q^2 - \frac{q\sqrt{q}}{4} + \frac{cq}{8} + \frac{\sqrt{q}}{8},$$

where $c = 1$ for q even and $c = 3$ for q odd, then the graphs of the functions in U share a common point.

We will prove Theorem 1.3 separately for q odd (Theorem 3.11) and for q even (Theorem 3.15). For $k > 2$, the proof can be finished by induction as in [1, page 33]. We obtain the following result.

Theorem 1.4. *If U is a set of more than $q^k - q^{k-1}$ intersecting polynomials over $\mathbb{F}_q$, $q \geq 53$ when q is odd and $q \geq 8$ when q is even, and of degree at most k, $k > 1$, then there exist $\alpha, \beta \in \mathbb{F}_q$ such that $g(\alpha) = \beta$ for all $g \in U$. Furthermore, U can be uniquely extended to a family of q^k intersecting polynomials over $\mathbb{F}_q$ and of degree at most k.*

While finalizing our manuscript, a stronger stability version of the above mentioned result for $k = 2$ was published by Adriaensen; see [2]. Our proof is different, based on polynomials and hence might be of independent interest.

2 Preliminaries

Throughout this paper, $q = p^n$ for some prime p and a positive integer n. The algebraic closure of the finite field $\mathbb{F}_q$ will be denoted by $\overline{\mathbb{F}}_q$.

The absolute trace function is defined as $\mathrm{Tr}_{q/p}\colon \mathbb{F}_q \to \mathbb{F}_p$, $\mathrm{Tr}_{q/p}(x) = x + x^p + \ldots + x^{p^{n-1}}$. Recall that for q even and $c \neq 0$, $a + bx + cx^2 \in \mathbb{F}_q[x]$ has a root in $\mathbb{F}_q$ if and only if $b = 0$, or $b \neq 0$ and

$$\mathrm{Tr}_{q/2}\left(\frac{ac}{b^2}\right) = 0. \tag{2.1}$$

When q is a square, we will also use the notation $\mathrm{N}\colon \mathbb{F}_q \to \mathbb{F}_{\sqrt{q}}$, $x \mapsto x^{\sqrt{q}+1}$, which is the norm of x over $\mathbb{F}_{\sqrt{q}}$.

We will frequently need the following result of Ball, Blokhuis, Brouwer, Storme, Szőnyi and Ball.

Result 2.1 (Part of [3, 4])**.** Let f be an $\mathbb{F}_q \to \mathbb{F}_q$ function such that $|D_f| \leq (q+1)/2$. Then U_f is affinely equivalent to the graph of a linearised polynomial, that is, a polynomial of the form $\sum_{i=0}^{n-1} a_i x^{p^i} \in \mathbb{F}_q[x]$.

Theorem 2.2. *Let U denote a proper $\mathbb{F}_p$-subspace of $\mathbb{F}_q$, $q = p^n > 2$, p prime and consider a function $\sigma\colon \mathbb{F}_q \to \mathbb{F}_q$. If the set of directions*

$$D_\sigma = \left\{ \frac{\sigma(x) - \sigma(y)}{x - y} : x, y \in \mathbb{F}_q,\ x \neq y \right\}$$

is contained in U, then $\sigma(x) = ax + b$ for some $a, b \in \mathbb{F}_q$.

Proof. First note that U is contained in an $(n-1)$-dimensional $\mathbb{F}_p$-subspace V and hence $|D_\sigma| \leq p^{n-1}$. Then by Result 2.1, $\sigma(x) = \alpha + g(x)$, where $\alpha \in \mathbb{F}_q$ and $g(x) = \sum_{i=0}^{n-1} b_i x^{p^i} \in \mathbb{F}_q[x]$, thus

$$D_\sigma = \left\{ \frac{g(x)}{x} : x \in \mathbb{F}_q \setminus \{0\} \right\}.$$

It is well-known that $\beta^{q/p} \prod_{\gamma \in V}(x - \gamma) = \mathrm{Tr}_{q/p}(\beta x)$ for some $\beta \in \mathbb{F}_q \setminus \{0\}$. Next define $f(x) := \beta g(x)$. Then $D_\sigma \subseteq V$ implies

$$\mathrm{Tr}_{q/p}\left(\frac{f(x)}{x}\right) = 0$$

for each $x \in \mathbb{F}_q \setminus \{0\}$. To prove the assertion, it is enough to prove that $f(x)$ is linear. With $f(x) = \sum_{i=0}^{n-1} a_i x^{p^i} \in \mathbb{F}_q[x]$,

$$\mathrm{Tr}_{q/p}\left(\frac{f(x)}{x}\right) = \mathrm{Tr}_{q/p}\left(\sum_{i=0}^{n-1} a_i x^{p^i-1}\right) = \sum_{j=0}^{n-1}\sum_{i=0}^{n-1} a_i^{p^j} x^{p^{i+j}-p^j},$$

and because of our assumption, this polynomial vanishes at every element of $\mathbb{F}_q \setminus \{0\}$.

The $p > 2$ case:

If we multiply this polynomial by $x^{1+p+p^2+\ldots+p^{n-1}}$, then we obtain

$$\sum_{j=0}^{n-1}\sum_{i=0}^{n-1} a_i^{p^j} x^{1+p+\ldots+p^{n-1}+p^{i+j}-p^j}$$

and this polynomial vanishes at every element of $\mathbb{F}_q$. As a function, this polynomial remains the same if we consider it modulo $x^{p^n} - x$, so it is the same function as the polynomial we obtain when we replace the exponents p^{i+j} with p^{i+j-n} for each $i + j \geq n$. Denote this new polynomial with $\tilde{f}$. The fact that we multiplied f by $x^{1+p+p^2+\ldots+p^{n-1}}$ ensures that the exponents of $\tilde{f}$ are larger than 0 and smaller than q. We claim that each monomial has different degree in $\tilde{f}$. It is clear that $\tilde{f}$ is the sum of at most n^2 monomials and the set of degrees of these monomials is contained in the set

$$A := \{1 + p + p^2 + \ldots + p^{n-1} + p^c - p^d : c, d \in \{0, 1, \ldots, n-1\}\}.$$

Assume that for some $c_1, c_2 \in \{0, 1, \ldots, n-1\}$, $d_1, d_2 \in \{0, 1, \ldots, n-1\}$ and $(c_1, d_1) \neq (c_2, d_2)$

$$1 + p + p^2 + \ldots + p^{n-1} + p^{c_1} - p^{d_1} = 1 + p + p^2 + \ldots + p^{n-1} + p^{c_2} - p^{d_2},$$

or equivalently $p^{c_1} + p^{d_2} = p^{c_2} + p^{d_1}$. Since the base p-digits of an integer are uniquely determined, this implies $\{c_1, d_2\} = \{c_2, d_1\}$, so either $c_1 = c_2$ and $d_1 = d_2$, or $c_1 = d_1$ and $c_2 = d_2$. We conclude that two distinct monomials of $\tilde{f}$ have the same degree d if any only if $d = 1 + p + \ldots + p^{n-1}$, that is, when in

$$1 + p + \ldots + p^{n-1} + p^{i+j} - p^j$$

we have $i = 0$.

Note that the degree of $\tilde{f}$ is at most

$$m := 1 + p + \ldots + p^{n-1} + p^{n-1} - 1.$$

Since $p > 2$, m is clearly smaller than q, but $\tilde{f}$ has q roots (the elements of $\mathbb{F}_q$). Thus it is the zero polynomial, all of its coefficients are zero. The coefficients of $\tilde{f}$ are the p^j-powers of $a_1, \ldots, a_{n-1}$ and $\mathrm{Tr}_{q/p}(a_0)$. So $f(x) = a_0 x$.

The $p = 2$ case:

Note that when $i \neq 0$, then

$$2^{i+j} - 2^j = 2^j + 2^{j+1} + \ldots + 2^{i+j-1}.$$

If $2^{i+j}-2^j \geq 2^n = q$ then write this number as

$$2^{i+j}-2^j = 2^j+2^{j+1}+\ldots+2^{i+j-1} = 2^j+2^{j+1}+\ldots+2^{n-1}+2^n(1+\ldots+2^{i+j-1-n}).$$

Clearly, as $\mathbb{F}_q \to \mathbb{F}_q$ functions,

$$x \mapsto x^{2^{i+j}-2^j}$$

are the same as

$$x \mapsto x^{2^j+2^{j+1}+\ldots+2^{n-1}} x^{1+\ldots+2^{i+j-1-n}} = x^{1+\ldots+2^{i+j-1-n}+2^j+2^{j+1}+\ldots+2^{n-1}}.$$

When $2^{i+j}-2^j \geq 2^n = q$, then substitute these exponents in

$$\mathrm{Tr}_{q/2}\left(\frac{f(x)}{x}\right) = \sum_{j=0}^{n-1}\sum_{i=0}^{n-1} a_i^{2^j} x^{2^{i+j}-2^j}$$

with the exponents

$$B_{ij} := 1+\ldots+2^{i+j-1-n}+2^j+2^{j+1}+\ldots+2^{n-1} < q,$$

and denote this new polynomial with $\tilde{f}$. Note that in this case $i+j-1-n < j-1$. When $2^{i+j}-2^j < q$, $i \neq 0$, then define

$$A_{ij} := 2^{i+j}-2^j = 2^j+2^{j+1}+\ldots+2^{i+j-1} < q.$$

If $i=0$ then put $A_{0j}=0$. Since base 2-digits of an integer are uniquely determined, we have the following: (1) If $i_1 \neq 0$, then $A_{i_1j_1} = A_{i_2j_2}$ iff $(i_1,j_1)=(i_2,j_2)$, (2) $A_{0j_1} = A_{0j_2}$ for any pair (j_1,j_2), (3) $B_{i_1j_1} = B_{i_2j_2}$ iff $(i_1,j_1)=(i_2,j_2)$, finally (4) $B_{i_1j_1} = A_{i_2j_2}$ iff $i_1+j_1-n=j_1$ and $j_2=0$ and $i_2=n$ (otherwise $B_{i_1j_1}$ in base 2 has the form $11\ldots110\ldots011\ldots11$, while $A_{i_2j_2}$ has the form $11\ldots1100\ldots00$, a contradiction); but $i_1,i_2<n$. It follows that the only exponent which appears in more than one monomial is the 0. The degree of $\tilde{f}$ is at most $q-2$ (obtained in $x^{A_{(n-1)1}}$) and it has $q-1$ roots (the elements of $\mathbb{F}_q \setminus \{0\}$), so it is the zero polynomial. Hence all of its coefficients are zero. These coefficients are the 2^j-powers of $a_1,\ldots,a_{n-1}$ and $\mathrm{Tr}_{q/2}(a_0)$. It follows that $f(x)=a_0x$. □

We will need the following two results regarding functions over finite fields.

Result 2.3 ([9, Theorem 5.41], Weil's bound)**.** Let ψ be a multiplicative character of $\mathbb{F}_q$ of order $m>1$ and let $f \in \mathbb{F}_q[x]$ be a monic polynomial of positive degree that is not an m-th power of a polynomial. Let d be the number of distinct roots of f in $\overline{\mathbb{F}}_q$. Then for every $a \in \mathbb{F}_q$ we have

$$\left|\sum_{c\in\mathbb{F}_q}\psi(af(c))\right| \leq (d-1)\sqrt{q}.$$

We will also consider polynomials of degree $\sqrt{q}+1$ admitting square values for almost every element of $\mathbb{F}_q$. In this case, the inequality above seems to be useless. In Lemma 2.6, we show a way how to derive information from Weil's bound also in this case. When $m=d=2$ then the following, stronger result holds which can be easily proved by counting $\mathbb{F}_q$-rational points of a conic of $\mathrm{PG}(2,q)$:

Result 2.4 ([11, Exercise 5.32])**.** Let q be an odd prime power, $f(x) = ax^2+bx+c \in \mathbb{F}_q[x]$ with $a \neq 0$, and let ψ denote the quadratic character $\mathbb{F}_q \to \{-1, 1, 0\}$. Then

$$\sum_{x \in \mathbb{F}_q} \psi(ax^2 + bx + c)$$

equals $-\psi(a)$ if $b^2 - 4ac \neq 0$ and $(q-1)\psi(a)$ if $b^2 - 4ac = 0$.

To use Result 2.3, we will need the following.

Lemma 2.5. *Put $f(x) = ax^{p^k+1} + dx^{p^k} + bx + c \in \mathbb{F}_q[x]$, $k \neq 0$. If q is odd and $f(x) = g(x)^2$, then $d^{p^k}a = ba^{p^k}$ and $d^{p^k+1}a = ca^{p^k+1}$, or $a = b = d = 0$.*

Proof. If $a = 0$, then $b = d = 0$ otherwise the degree of f was odd. Assume $a \neq 0$ and suppose $f(x) = g(x)^2$. Then the roots of f have multiplicities at least 2 and hence they are also roots of $f'(x) = ax^{p^k} + b = (a^{p^{-k}}x + b^{p^{-k}})^{p^k}$. It follows that $f(x)$ has a unique root, $-(b/a)^{p^{-k}}$, so

$$f(x) = a(x+\gamma)^{p^k+1} = a(x^{p^k} + \gamma^{p^k})(x+\gamma) = ax^{p^k+1} + \gamma a x^{p^k} + a\gamma^{p^k}x + \gamma^{p^k+1}a,$$

with $\gamma = (b/a)^{p^{-k}}$. It follows that f has the listed properties. □

Lemma 2.6. *If for some odd, square $q > 9$ there is a subset D of $\mathbb{F}_q$ of size larger than $q - \sqrt{q}/2 + 1/2$ such that the $\mathbb{F}_q \to \mathbb{F}_q$ function $x \mapsto \ell(x) := ax^{\sqrt{q}+1} + dx^{\sqrt{q}} + bx + c$, $a \neq 0$, has the property that $\ell(x)$ is a square of $\mathbb{F}_q$ for each $x \in D$, then $a^{\sqrt{q}}b = d^{\sqrt{q}}a$.*

Proof. Suppose $a^{\sqrt{q}}b \neq d^{\sqrt{q}}a$. Then the value set of ℓ clearly does not change if we replace x with $g(y) = ((b/a)^{\sqrt{q}} - (d/a))y - d/a$, since g is a permutation polynomial. Also, $C := g^{-1}(D)$, will have the properties that $|C| > q - \sqrt{q}/2 + 1/2$ and for each $y \in C$,

$$f(y) := \ell(g(y))$$

is a square of $\mathbb{F}_q$. One can easily verify $f(y) = \ell(g(y)) = \beta y^{\sqrt{q}+1} + \beta y + \alpha$, where

$$\beta = \frac{(a^{\sqrt{q}}b - ad^{\sqrt{q}})^{\sqrt{q}+1}}{a^{2\sqrt{q}+1}}$$

and $\alpha = c - bd/a$. We will show that this is not possible.

Since the norm $x \mapsto \mathrm{N}(x)$ takes $(\sqrt{q} - 1)$ distinct non-zero values in $\mathbb{F}_q$, and $(\sqrt{q} - 1) - (\sqrt{q}/2 - 1/2) \geq 2$ (here we use $q > 9$ square), we may take $t_1, t_2 \in \mathbb{F}_q \setminus \{0\}$ such that $\mathrm{N}(t_1) \neq \mathrm{N}(t_2)$ and $\mathrm{N}(t_1), \mathrm{N}(t_2) \notin \{\mathrm{N}(d) : d \in \mathbb{F}_q \setminus C\}$. We show that if $f(x)$ is a square for each $x \in C$ then also the polyomials

$$f(t_1 y^{\sqrt{q}-1}) = \mathrm{N}(t_1)\beta y^{q-1} + \beta t_1 y^{\sqrt{q}-1} + \alpha \in \mathbb{F}_q[y],$$

$$f(t_2 y^{\sqrt{q}-1}) = \mathrm{N}(t_2)\beta y^{q-1} + \beta t_2 y^{\sqrt{q}-1} + \alpha \in \mathbb{F}_q[y],$$

have only square values for each $y \in C$. Indeed, this follows from the fact that $\mathrm{N}(t_i y^{\sqrt{q}-1}) \notin \{\mathrm{N}(d) : d \in \mathbb{F}_q \setminus C\}$ and hence $t_i y^{\sqrt{q}-1} \in C$ for $i = 1, 2$.

Then the polynomials

$$G_1(y) := \mathrm{N}(t_1)\beta + \beta t_1 y^{\sqrt{q}-1} + \alpha \in \mathbb{F}_q[y],$$

$$G_2(y) := \mathrm{N}(t_2)\beta + \beta t_2 y^{\sqrt{q}-1} + \alpha \in \mathbb{F}_q[y],$$

take only square values on the non-zero elements of C. Denote by ψ the multiplicative character of $\mathbb{F}_q$ of order two. The polynomial G_i has at most $\sqrt{q}-1$ roots, and $\psi(G_i(x)) = 1$ for every element x of $C \setminus \{0\}$ if x is not a root of G_i. Define ε to be $\psi(G_i(0))$ if $0 \in C$ and to be 0 otherwise. Then $|C \setminus \{0\}| - (\sqrt{q}-1) + \varepsilon \leq \sum_{x\in C} \psi(G_i(x))$. On the other hand, $-(q - |C|) \leq \sum_{x\in \mathbb{F}_q \setminus C} \psi(G_i(x))$, and hence

$$2|C| - q - \sqrt{q} - 1 \leq \left| \sum_{y\in\mathbb{F}_q} \psi(G_i(y)) \right|.$$

Since

$$(\sqrt{q}-2)\sqrt{q} < 2|C| - q - \sqrt{q} - 1,$$

by Result 2.3 (with $m = 2$) this can only happen if $G_i = g_i^2$ for some polynomials g_i, $i = 1, 2$. Then the roots of G_i (in the algebraic closure of $\mathbb{F}_q$) are multiple roots of G_i and hence also roots of $\gcd(G_i, G_i')$. The only root of G_i' is 0, thus $G_i(0) = 0$ and hence $\mathrm{N}(t_i)\beta + \alpha = 0$. Since $a^{\sqrt{q}}b \neq d^{\sqrt{q}}a$, we have $\beta \neq 0$. It follows that $\mathrm{N}(t_i) = -\alpha/\beta$ for $i = 1, 2$, a contradiction because of the choice of t_1 and t_2. □

The next example shows that $\ell(x) = ax^{\sqrt{q}+1} + dx^{\sqrt{q}} + bx + c$ can have only square values if $a^{\sqrt{q}}b = d^{\sqrt{q}}a$ holds.

Example 2.7. For $t, r \in \mathbb{F}_q$, the polynomial

$$f(x) = r^{\sqrt{q}+1}x^{\sqrt{q}+1} + r^{\sqrt{q}}tx^{\sqrt{q}} + rt^{\sqrt{q}}x + t^{\sqrt{q}+1} = (t+rx)^{\sqrt{q}+1}$$

has only square values in $\mathbb{F}_q$.

We will need a generalisation of the following result by Göloğlu and McGuire.

Result 2.8 ([8, Theorem 1.2])**.** Let q be odd and consider a non zero polynomial $L(x) = \sum_{i=0}^{n-1} a_i x^{p^i} \in \mathbb{F}_q[x]$. Denote by $\square_q$ the set of non-zero squares in $\mathbb{F}_q$. Then

$$\mathrm{Im}\left(\frac{L(x)}{x}\right) \subseteq \square_q \cup \{0\}$$

if and only if $L(x) = ax^{p^d}$ for some $a \in \square_q$ and $0 \leq d \leq n$.

Definition 2.9. If U is a point set of $\mathrm{AG}(2, q)$, then the set of directions defined by U is

$$D_U = \left\{ \left(\frac{a-b}{c-d}\right) : (a,b),(c,d) \in U, (a,b) \neq (c,d) \right\}.$$

(If the denominator is zero then $\left(\frac{a-b}{0}\right) = (\infty)$, the ideal point of vertical lines.)

In the proof of Theorem 2.13 the following result of Szőnyi will be crucial.

Result 2.10 ([10, Theorem 4 and Proposition 6])**.** Let U be a point set of $\mathrm{AG}(2,q)$ of size at least $q-\sqrt{q}/2$ and let D_U be the set of directions determined by U.

1. If U determines less than $(q+1)/2$ directions, then U can be extended to a q-set determining the same set of directions as U.

2. If U determines exactly $(q+1)/2$ directions, one of them is (∞) and there is no point $P \in \mathrm{AG}(2,q) \setminus U$ such that $U \cup \{P\}$ determines the same set of directions as U, then the $(q+1)/2$-set
$$\{d \in \mathbb{F}_q,\ (d) \notin D_U\}$$
is the set of Y coordinates of the points of an irreducible conic $\mathcal{C}$ of $\mathrm{AG}(2,q)$ and the direction (0) is an internal point of $\mathcal{C}$.

Remark 2.11. By [12, Remark 3.3] a blocking set of size at most $2q$ contains a unique minimal blocking set. Let U denote an affine point set of size at least $q-\sqrt{q}/2$ such that U determines less than $(q+1)/2$ directions. Assume that $\mathcal{P}$ and $\mathcal{P}'$ are two affine point sets of size $q-|U|$ which extend U to a q-set determining the same set of directions as U. Then $\mathcal{B} := U \cup \mathcal{P} \cup \mathcal{P}' \cup D_U$ is a blocking set of size at most $\lfloor q+\sqrt{q}/2+(q+1)/2 \rfloor \le 2q$ and hence $\mathcal{B}$ contains a unique minimal blocking set. But both $U \cup \mathcal{P} \cup D_U$ and $U \cup \mathcal{P}' \cup D_U$ are minimal blocking sets and this proves $\mathcal{P} = \mathcal{P}'$, that is, the unicity of the extension of U in Result 2.10.

Lemma 2.12. *Let $\mathcal{S}$ denote the set of non-zero squares or non-squares in* $\mathrm{GF}(q)$*. If the set of Y coordinates of the points of an irreducible conic $\mathcal{C}$ of* $\mathrm{AG}(2,q)$*, $q \ge 53$ odd, is contained in $\mathcal{S} \cup \{0\}$ then $\mathcal{C}$ is a parabola with equation*
$$Y = \alpha(a'X + b'Y + c')^2,$$
where $\alpha \in \mathcal{S}$.

Proof. Note that horizontal translations of $\mathcal{C}$ does not affect the properties that we are examining, so after substituting X by $X-\beta$ for a suitable $\beta \in \mathbb{F}_q$ we may assume that $(0,0)$ is not a point of $\mathcal{C}$ and hence the equation of the conic is
$$aX^2 + bXY + cY^2 + dX + eY + 1 = 0.$$
The direction (0) cannot be a point of the projective extension of $\mathcal{C}$ since otherwise we would get at least $q-1 > (q+1)/2$ different Y coordinates. It follows that there are at most 2 horizontal lines meeting $\mathcal{C}$ in 1 point and at least $(q-3)/2$ horizontal lines meeting $\mathcal{C}$ in 2 points. Fix some $\alpha \in \mathcal{S}$. At least $(q-5)/2$ horizontal lines meet $\mathcal{C}$ in 2 points $(A_i, \alpha B_i^2)$ and $(A_i', \alpha B_i^2)$ with $B_i \neq 0$; and $\mathcal{C}$ has at most 2 points on the X axis. Next define the quartic $\mathcal{Q}$ (which might as well be of smaller degree if $c = 0$):
$$aX^2 + \alpha bXY^2 + \alpha^2 cY^4 + dX + \alpha eY^2 + 1 = 0.$$
Points of $\mathcal{C}$ on the X axis are points of $\mathcal{Q}$ as well, and if $(A_i, \alpha B_i^2)$, $(A_i', \alpha B_i^2)$, $B_i \neq 0$, were two points of $\mathcal{C}$ then $(A_i, \pm B_i)$, $(A_i', \pm B_i)$ are 4 points of $\mathcal{Q}$. It follows that
$$\mathcal{A} = \{(x,y), (x,-y) : (x, \alpha y^2) \in \mathcal{C}\}$$

is a subset of the point set of $\mathcal{Q}$ of size at least $2+2(q-5)=2q-8$. Note that $2q-8 > q+1+6\sqrt{q}$ (here we use $q \geq 53$). It follows instantly from the Hasse-Weil bound that $\mathcal{Q}$ cannot be an irreducible cubic or an irreducible quartic.

First we show that $\mathcal{Q}$ cannot be a quartic curve which is the product of an irreducible cubic and a line. Vertical and horizontal lines meet $\mathcal{A}$ in at most 2 points and hence if such a line would be a factor of $\mathcal{Q}$ then the remaining at least $2q-10$ points of $\mathcal{A}$ should lie on the cubic, a contradiction again by the Hasse-Weil bound now applied to cubic curves. Similarly, if $Y = mX + n$ was a factor of $\mathcal{Q}$, for some $m \neq 0$, then

$$aX^2 + \alpha bX(mX+n)^2 + \alpha^2 c(mX+n)^4 + dX + \alpha e(mX+n)^2 + 1$$

was the zero polynomial. The coefficient of X^4 is $\alpha^2 cm^4$, so c has to be zero but then $\mathcal{Q}$ is not a quartic curve, a contradiction.

From now on we may assume that

$$aX^2 + \alpha bXY^2 + \alpha^2 cY^4 + dX + \alpha eY^2 + 1 = F \cdot G,$$

where F and G are of degree at most 2. Put

$$F = (a_1Y^2 + b_1X + c_1Y + 1 + d_1X^2 + e_1XY),$$
$$G = (a_2Y^2 + b_2X + c_2Y + 1 + d_2X^2 + e_2XY).$$

In $F \cdot G$ the coefficient of Y is $c_1 + c_2$, while it is 0 in the equation of $\mathcal{Q}$, so clearly $c_2 = -c_1$ and we will use this from now on. The coefficient of X^4 is d_1d_2 and it has to be zero, so from now on we may assume $d_1 = 0$. Then the coefficient of X^3 is d_2b_1 and it has to be zero.

In this paragraph assume $\boxed{d_2 \neq 0}$, then $b_1 = 0$ and the coefficient of X^3Y is d_2e_1 so $e_1 = 0$. The coefficient of YX^2 is c_1d_2, so $c_1 = 0$. Then the coefficient of XY is e_2, so $e_2 = 0$. The coefficient of X^2Y^2 is d_2a_1, so $a_1 = 0$. We arrived to the conclusion that the equation of $\mathcal{Q}$ is $a_2Y^2 + b_2X + 1 + d_2X^2$. It follows that the equation of $\mathcal{C}$ is $Y = -\alpha(b_2X+1+d_2X^2)/a_2$. Then by Result 2.4, $-(b_2X+1+d_2X^2)/a_2 = (a'X+b')^2$ for some $a', b' \in \mathbb{F}_q$ and this finishes the proof of the $d_2 \neq 0$ case.

Now assume $\boxed{d_2 = 0}$ (recall also $d_1 = 0$). Then the coefficient of X^2Y^2 is e_1e_2. We may assume $e_1 = 0$. The coefficient of X^2Y is e_2b_1. First assume $\underline{e_2 \neq 0}$, so $b_1 = 0$. Then the coefficient of Y^3X is a_1e_2, so $a_1 = 0$. Then the coefficient of Y^3 is c_1a_2. We cannot have $c_1 = 0$ since then the coefficient of XY would be $e_2 \neq 0$, so $a_2 = 0$. The coefficient of XY is $c_1b_2 + e_2 = 0$ and hence the equation of $\mathcal{Q}$ is: $(1+c_1Y)(1-c_1Y)(1+b_2X)$, a contradiction since vertical and horizontal lines contain at most 2 points of $\mathcal{A}$. So $\underline{e_2 = 0}$ and from now on we may assume that $\mathcal{Q}$ has equation

$$(a_1Y^2 + b_1X + c_1Y + 1)(a_2Y^2 + b_2X - c_1Y + 1) = 0.$$

The fact that Y^3 and XY should have zero coefficient yields $a_1 = a_2$ and $b_1 = b_2$, or $c_1 = 0$. In the former case the equation of $\mathcal{Q}$ is

$$(a_1Y^2 + b_1X + c_1Y + 1)(a_1Y^2 + b_1X - c_1Y + 1) = 0,$$

so $\mathcal{C}$ had equation

$$(\alpha^{-1}a_1Y + b_1X + 1)^2 - \alpha^{-1}c_1^2Y = 0,$$

which proves the assertion.

In the latter case the equation of $\mathcal{Q}$ is

$$(a_1Y^2 + b_1X + 1)(a_2Y^2 + b_2X + 1) = 0,$$

so $\mathcal{C}$ had equation

$$(a_1Y/\alpha + b_1X + 1)(a_2Y/\alpha + b_2X + 1) = 0,$$

a contradiction since $\mathcal{C}$ was irreducible. □

The following can be considered as a generalisation of Result 2.8.

Theorem 2.13. *Let U denote a point set of $\mathrm{AG}(2,q)$, $q \geq 53$ odd, of size at least $q - \sqrt{q}/2$. Let $\mathcal{S}$ denote the set of non-zero squares or non-squares in $\mathbb{F}_q$ and let (d) denote one of the directions (0) or (∞). If D_U is contained in $\{(s) : s \in \mathcal{S}\} \cup \{(d)\}$, then U is affinely equivalent to a subset of the graph of a function of the form*

$$f(x) = \alpha x^{p^k},$$

where $\alpha \in \mathcal{S}$.

Proof. If $|D_U| < (q+1)/2$ then by Result 2.10 U can be extended to a q-set U' determining the same set of directions. According to Result 2.1 U' is affinely equivalent to the graph of a linearised polynomial f. Then Result 2.8 shows that f has the requested form.

Now assume $|D_U| = (q+1)/2$. If $(d) = (0)$, then apply the affinity $\varphi\colon (x : y : z) \mapsto (y : x : z)$. Clearly, $D_{U^\varphi} = (D_U)^\varphi$ and U can be extended if and only if U^φ can be extended. We have $(0)^\varphi = (\infty)$ and if $m \neq 0$ then $(m)^\varphi = (1/m)$, so

$$\{(s)^\varphi : s \in S\} = \{(s) : s \in S\}.$$

By Result 2.10, if U^φ (or U, if $(d) = (\infty)$) cannot be extended, then the set of non-zero squares or non-squares together with the zero equals the set of Y coordinates of an irreducible affine conic $\mathcal{C}$ and (0) is an internal point of $\mathcal{C}$. Then the line at infinity is not a tangent to $\mathcal{C}$, thus $\mathcal{C}$ is not a parabola (and not a hyperbola because then the size of the set of Y coordinates would be $(q-1)/2$; but we don't need this) but this is not possible because of the Lemma 2.12. It follows that U can be extended to a q-set determining the same set of directions as U and the proof can be finished as in the previous paragraph. □

3 On intersecting families of graphs of functions

Our first aim is to prove Theorem 1.3 which we will do separately in the odd and even case.

Lemma 3.1. *If U is a set of t-intersecting polynomials of degree at most k over $\mathbb{F}_q$, then the $(k-t+1)$-ple of coefficients of the monomials $x^t, \dots, x^k$ in elements of U are distinct elements of $\mathbb{F}_q^{k-t+1}$.*

Proof. If the coefficients of $x^t, \dots, x^k$ coincide in $f_1, f_2 \in U$, then $f_1 - f_2$ would have degree at most $t-1$, and hence at most $t-1$ roots, thus the graphs of f_1 and f_2 would share at most $t-1$ points, a contradiction. □

Next, we report below Lemma 6.1 from [1] with an alternative proof.

Lemma 3.2 ([1, Lemma 6.1])**.** *Assume $t \le k < q$. Let U be a set of polynomials of degree at most k over $\mathbb{F}_q$.*

(1) *If for any $f, g \in U$ there exist $x_1, \dots, x_t \in \mathbb{F}_q$ such that $f(x_i) = g(x_i)$ for $i = 1, \dots, t$, then $|U| \le q^{k-t+1}$.*

(2) *If for any $f, g \in U$ there are no $x_1, \dots, x_t \in \mathbb{F}_q$ such that $f(x_i) = g(x_i)$ for $i = 1, \dots, t$, then $|U| \le q^t$.*

Proof. Proof of Part (1): It is a direct consequence of Lemma 3.1.

Proof of Part (2): Take any t distinct field elements, say, $x_1, \dots, x_t$. For any polynomial f over $\mathbb{F}_q$, $(f(x_1), \dots, f(x_t))$ can take at most q^t distinct values of $\mathbb{F}_q^t$ and hence if $|U| > q^t$ then there will be at least 2 polynomials in U which have the same values on the set $\{x_1, \dots, x_t\}$. □

Lemma 3.3. *Let U be a set of intersecting polynomials of degree at most 2 over $\mathbb{F}_q$. Assume that there are more than $\lfloor (q+1)/2 \rfloor$ polynomials h_i in U, so that their x^2 coefficients are c, for some fixed $c \in \mathbb{F}_q$ and suppose also that there exist values α and β so that $h_i(\alpha) = \beta$. Then for every polynomial $f \in U$, whose coefficient in x^2 is not c, $f(\alpha) = \beta$.*

Proof. First assume that $\alpha = 0$. Then the constant term in the polynomials h_i is always β and we want to show that for any polynomial $f \in U$, if the coefficient of x^2 is not c, the constant term must be β. Assume to the contrary, that there is a polynomial $g \in U$, whose constant term is not β. Consider the polynomials:

$$\{g - h_i\}.$$

Since g and h_i are intersecting, $g - h_i$ must have a root in $\mathbb{F}_q$. Also, by the assumptions of the lemma, $(g - h_i)(x) = dx^2 + vx + w$, where $d \neq 0$ and $w \neq 0$ are fixed. We claim that there are at most $\lfloor (q+1)/2 \rfloor$ such polynomials, hence a contradiction. Indeed, if $(g - h_i)(x)$ has a root in $\mathbb{F}_q$, then it can be written as $d(x - u)(x - \frac{w}{du})$. So to bound the number of possible polynomials, we have to bound the number of different $(u, \frac{w}{du})$ pairs, where the order does not matter.

First assume q to be odd. If w/d is not a square, then we get $(q-1)/2$ such pairs. If it is a square, then we see $2 + (q-3)/2$ pairs, which is $(q+1)/2$.

Now, assume q to be even. In this case the number of different pairs $(u, \frac{w}{du})$ is $(q-2)/2 + 1$, that is, $q/2$.

Finally, if $\alpha \neq 0$, then instead of the polynomials f in U, consider the polynomials $\bar{f}(x) := f(x + \alpha)$. This new family is clearly an intersecting family, the $\bar{h}_i$ polynomials will still have the same leading coefficients and $\bar{h}_i(0) = \beta$, so we are in the previous case. □

Lemma 3.4. *Let q be even and U be a set of intersecting polynomials of degree at most 2 over $\mathbb{F}_q$. Assume that $|U| > \frac{q^2+q}{2}$ and assume also that H is a subset of U with more than $\frac{q^2}{2}$ polynomials h_i, so that there exist values α and β for which $h_i(\alpha) = \beta$. Then for every polynomial $f \in U$, $f(\alpha) = \beta$.*

Proof. By the pigeon hole principle, there exists a value c such that there are more than $\frac{q}{2}$ polynomials in H with x^2 coefficient c. Let U^c denote the polynomials in U with x^2 coefficient c. Then Lemma 3.3 implies that for any polynomial $f \in (U \setminus U^c)$, $f(\alpha) = \beta$. Note that $|U \setminus U^c| > \frac{q^2+q}{2} - q$. Again by the pigeon hole principle, there exists a value $c' \neq c$ so that there are more than $\frac{q^2-q}{2(q-1)} = \frac{q}{2}$ polynomials in $(U \setminus U^c)$ with x^2 coefficient c'. Lemma 3.3 yields that for any polynomial $g \in U^c$, $g(\alpha) = \beta$. □

The next result can be proved in exactly the same way as Lemma 3.4.

Lemma 3.5. *Let q be odd and U be a set of intersecting polynomials of degree at most 2 over $\mathbb{F}_q$. Assume that $|U| > \frac{q^2+2q-1}{2}$ and suppose also that H is a subset of U with more than $\frac{q^2+q}{2}$ polynomials h_i, so that there exist values α and β for which $h_i(\alpha) = \beta$. Then for every polynomial $f \in U$, $f(\alpha) = \beta$.* □

Lemma 3.6. *Let U be a set of intersecting polynomials of degree at most $k > 1$ over $\mathbb{F}_q$. Assume that there are more than $(q-1)q^{k-2}$ polynomials h_i in U, so that their x^k coefficients are c, for some fixed $c \in \mathbb{F}_q$ and suppose also that there exist values α and β so that $h_i(\alpha) = \beta$. Then for every polynomial $f \in U$, whose coefficient of x^k is not c, it holds that $f(\alpha) = \beta$.*

Proof. First assume that $\alpha = 0$. Then the constant term in the polynomials h_i is always β and we want to show that for any polynomial $f \in U$, if the coefficient of x^k is not c, the constant term must be β. Assume to the contrary, that there is a polynomial $g \in U$, whose constant term is not β. Consider the polynomials:

$$\{g - h_i\}.$$

Since g and h_i are intersecting, $g - h_i$ must have a root in $\mathbb{F}_q$. Also, by the assumptions of the lemma, $(g - h_i)(x) = dx^k + v_1x^{k-1} + v_2x^{k-2} + \ldots + v_{k-1}x + w$, where $d \neq 0$ and $w \neq 0$ are fixed. We claim that there are at most $(q-1)q^{k-2}$ such polynomials, hence a contradiction. Indeed, such polynomials can be written in the form $(x-u)(dx^{k-1} + \ldots - w/u)$. Note that $u \neq 0$, because $w \neq 0$, hence u can take $q-1$ values. The second term is a polynomial of degree $k-1$, its coefficient in x^{k-1} and its constant term are fixed, so there are at most q^{k-2} different such polynomials.

As before, if $\alpha \neq 0$, then instead of the polynomials f in U, consider the polynomials $\bar{f}(x) := f(x+\alpha)$. This new family is clearly an intersecting family, the $\bar{h}_i$ polynomials will still have the same leading coefficients and $\bar{h}_i(0) = \beta$, so we are in the previous case. □

3.1 Intersecting families of polynomials of degree at most 2, over $\mathbb{F}_q$, q odd

According to Lemma 3.1, the members of an intersecting family of polynomials of degree at most 2 are of the form $f(b,c) + bx + cx^2$ for some function f. More precisely:

Definition 3.7. Suppose that U is a set of intersecting polynomials. Put $D = \{(b,c) \in \mathbb{F}_q \times \mathbb{F}_q : a + bx + cx^2 \in U\}$ and define $f\colon D \to \mathbb{F}_q$ as $f(b,c) = a$, where $a \in \mathbb{F}_q$ is the unique field element such that $a + bx + cx^2 \in U$.

Lemma 3.8. *Let U be a set of intersecting polynomials of degree at most 2 and for $b \in \mathbb{F}_q$, $q \geq 53$ odd, define*

$$C_b := \{c \in \mathbb{F}_q : f(b,c) + bx + cx^2 \in U\}$$

and

$$dom_o := \{b \in \mathbb{F}_q : |C_b| > q - \sqrt{q}/2 + 1/2\}.$$

There exist functions $s, t\colon dom_o \to \mathbb{F}_q$ *and* $h\colon dom_o \to \{0, 1, \ldots, n-1\}$ *(where* $q = p^n$*) such that for every* $c \in C_b$

$$f(b, c) = s(b)c^{p^{h(b)}} + t(b),$$

and $-s(b)$ *is square in* $\mathbb{F}_q$.

Proof. If $f(b, c) + bx + cx^2$ and $f(d, e) + dx + ex^2$ are members of U, then the difference of the two polynomials must have a root in $\mathbb{F}_q$ and hence

$$F(b, c, d, e) := (b-d)^2 - 4(f(b, c) - f(d, e))(c-e)$$

is a square. For $c \in C_b$ put $f_b(c)$ for $f(b, c)$.

For each $b \in \mathbb{F}_q$ and $C, E \in C_b$, $C \neq E$, consider $F(b, C, b, E) = -4(f_b(C) - f_b(E))(C-E)$, which has to be a square, or, equivalently, after dividing by $(C-E)^2$,

$$-\frac{f_b(C) - f_b(E)}{C - E}$$

is in $\square_q \cup \{0\}$ for each $C, E \in C_b$.

If $b \in dom_o$, then by Theorem 2.13, f_b can be uniquely extended to a function $\tilde{f}_b\colon \mathbb{F}_q \to \mathbb{F}_q$ determining the same set of directions as f_b and for each $c \in \mathbb{F}_q$ $\tilde{f}_b(c) = s(b)c^{p^{h(b)}} + t(b)$ for some $dom_o \to \mathbb{F}_q$ functions s, t such that $-s(b)$ is a square, and a function $h\colon dom_o \to \{0, 1, \ldots, n-1\}$. Then for $c \in C_b$ we have $f_b(c) = s(b)c^{p^{h(b)}} + t(b)$. □

Lemma 3.9. *If* $q \geq 53$ *is odd,* U *is a set of intersecting polynomials of degree at most* 2 *such that* $|dom_o| > 1$*, then for* $b, d \in dom_o$ *and* $c \in C_b$*,* $e \in C_d$ *recall that*

$$\begin{aligned} F(b, c, d, e) &= (b-d)^2 - 4(f(b, c) - f(d, e))(c-e) \\ &= (b-d)^2 - 4(s(b)c^{p^{h(b)}} + t(b) - s(d)e^{p^{h(d)}} - t(d))(c-e). \end{aligned}$$

For $b \in dom_o$*, one of the following holds*

1. $s(b) = s(d) = 0$ *and* $t(d) = t(b)$ *for each* $d \in dom_o$*,*

2. $s(b) = s(d) \neq 0$*,* $h(d) = h(b) = 0$ *and* $(t(b) - t(d))^2 = -s(b)(b-d)^2$ *for each* $d \in dom_o$*,*

3. $s(b) = s(d) \neq 0$*,* $h(d) = h(b) = n/2$ *and* $t(b) = t(d)$ *for each* $d \in dom_o$.

Proof. For $b, d \in dom_o$ and $c \in C_b$, $e \in C_d$ recall that $F(b, c, d, e)$ is a square in $\mathbb{F}_q$.

Define the function $G_{b,d,e}\colon \mathbb{F}_q \to \mathbb{F}_q$, as

$$c \mapsto (b-d)^2 - 4(s(b)c^{p^{h(b)}} + t(b) - s(d)e^{p^{h(d)}} - t(d))(c-e) =$$

$$-4s(b)c^{p^{h(b)}+1} + 4es(b)c^{p^{h(b)}} - 4(t(b) - s(d)e^{p^{h(d)}} - t(d))c +$$

$$(b-d)^2 + 4e(t(b) - s(d)e^{p^{h(d)}} - t(d)).$$

First assume $0 < h(b) < n/2$ for some $b \in dom_o$.

Denote by ψ the quadratic character of $\mathbb{F}_q$ and apply Result 2.3 to the function $G_{b,d,e}$. Then we have

$$q - \sqrt{q} - p^{h(b)} < -(q - |C_b|) + (|C_b| - p^{h(b)} - 1) \leq \sum_{c \in \mathbb{F}_q} \psi(G_{b,d,e}(c)),$$

as $G_{b,d,e}$ is a polynomial of degree $p^{h(b)} + 1$ in c, so the number of its roots is at most $p^{h(b)} + 1$.

Thus, we cannot have

$$\left| \sum_{c \in \mathbb{F}_q} \psi(G_{b,d,e}(c)) \right| \leq p^{h(b)} \sqrt{q}.$$

It follows that $G_{b,d,e}$ is the square of a polynomial in c. And hence by Lemma 2.5, one of the following holds:

(i) $s(b) = 0$ and $t(b) - s(d)e^{p^{h(d)}} - t(d) = 0$ for each $d \in dom_o$, $e \in C_d$. If we fix d as well and let e run through C_d then we obtain $s(d) = 0$ and $t(b) = t(d)$, for each $d \in dom_o$.

(ii) $s(b) \neq 0$ and

$$(4es(b))^{p^{h(b)}}(-4s(b)) = -4(t(b) - s(d)e^{p^{h(d)}} - t(d))(-4s(b))^{p^{h(b)}}, \qquad (3.1)$$

$$(4es(b))^{p^{h(b)}+1}(-4s(b)) = ((b-d)^2 + 4e(t(b) - s(d)e^{p^{h(d)}} - t(d)))(-4s(b))^{p^{h(b)}+1}. \qquad (3.2)$$

Then (3.1) yields $s(d)e^{p^{h(d)}} - s(b)e^{p^{h(b)}} = t(b) - t(d)$, for each $d \in dom_o$, $e \in C_d$. Fix d as well and let e run through C_d. Put K for the dimension over $\mathbb{F}_p$ of the kernel of the $\mathbb{F}_p$-linear $\mathbb{F}_q \to \mathbb{F}_q$ function $e \mapsto s(d)e^{p^{h(d)}} - s(b)e^{p^{h(b)}}$. Then

$$\frac{|C_d|}{p^K} \leq \left| \left\{ s(d)e^{p^{h(d)}} - s(b)e^{p^{h(b)}} : e \in C_d \right\} \right| = |\{t(b) - t(d)\}| = 1,$$

thus $K = n$ ($q = p^n$) and hence $s(d) = s(b)$, $h(d) = h(b)$ and $t(d) = t(b)$ for each $d \in dom_o$.

Then (3.2) reads as $0 = (b - d)^2$ for each $d \in dom_o$, a contradiction since $|dom_o| > 1$.

We proved that $0 < h(b) < n/2$ implies $s(d) = 0$ and $t(b) = t(d)$ for each $d \in dom_o$.

Next assume $n/2 < h(b) < n$ for some $b \in dom_o$.

Apply Result 2.3 to the function $c \mapsto (G_{b,d,e}(c))^{p^{n-h(b)}} \pmod{c^q - c}$ and continue as above. It turns out that $s(d) = 0$ and $t(b) = t(d)$ for each $d \in dom_o$ also in this case.

Now assume $h(b) = n/2$ for some $b \in dom_o$.

If $s(b) = 0$, then

$$-4(t(b) - s(d)e^{p^{h(d)}} - t(d))c + (b-d)^2 + 4e(t(b) - s(d)e^{p^{h(d)}} - t(d))$$

is a square for each $d \in dom_o$ and $c \in C_b$, $e \in C_d$. If we consider d and e fixed as well, then it follows that as a function of c it has to be a constant, so $t(b) - s(d)e^{p^{h(d)}} - t(d) = 0$ for each $e \in C_d$ and hence $s(d) = 0$ and $t(b) = t(d)$ for each $d \in dom_o$.

If $s(b) \neq 0$, then Lemma 2.6 applied to $G_{b,d,e}$ gives (3.1) and hence, as before, $s(d) = s(b)$, $t(d) = t(b)$ and $h(d) = h(b)$ for each $d \in dom_o$.

Finally, consider the case when $h(b) = 0$ for some $b \in dom_o$.

Then again from Result 2.3, one obtains $G_{b,d,e}(c) = (b-d)^2 - 4(s(b)c + t(b) - s(d)e^{p^{h(d)}} - t(d))(c-e) = (\alpha + \beta c)^2$, for some $\alpha, \beta \in \mathbb{F}_q$.

If $s(b) = 0$, that is, when $G_{b,d,e}$ is a constant, then $t(b) - s(d)e^{p^{h(d)}} - t(d) = 0$ for each $d \in dom_o$, $e \in C_d$, so $s(d) = 0$ and $t(b) = t(d)$ for each $d \in dom_o$.

If $s(b) \neq 0$, that is, when $G_{b,d,e}$ is of degree two, then the discriminant of $G_{b,d,e}$ has to be zero, i.e.

$$s(b)(b-d)^2 + (s(b)e - s(d)e^{p^{h(d)}} + t(b) - t(d))^2 = 0.$$

For $d \in dom_o$ let ε_d be an element of $\mathbb{F}_q$ for which $\varepsilon_d^2 = -s(b)(b-d)^2$. Consider $d \in dom_o$ fixed as well, then for $e \in C_d$:

$$s(b)e - s(d)e^{p^{h(d)}} \in \{\varepsilon_d + t(d) - t(b), -\varepsilon_d + t(d) - t(b)\}.$$

Put K for the dimension over $\mathbb{F}_p$ of the kernel of the $\mathbb{F}_p$-linear $\mathbb{F}_q \to \mathbb{F}_q$ function $e \mapsto s(b)e - s(d)e^{p^{h(b)}}$. Then

$$\frac{|C_d|}{p^K} \leq \left|\left\{s(b)e - s(d)e^{p^{h(d)}} : e \in C_d\right\}\right| \leq 2,$$

which is a contradiction, unless $K = n$. It follows that $s(b)e - s(d)e^{p^{h(b)}} = 0$ for each $e \in \mathbb{F}_q$, so $h(d) = 0$, $s(d) = s(b)$ and $t(d) - t(b)$ is one of ε_d and $-\varepsilon_d$. □

Lemma 3.10. *If $q \geq 53$ is odd, U is a set of intersecting polynomials of degree at most 2 such that $|dom_o| > (q+1)/2$, then there exist $\alpha, \beta \in \mathbb{F}_q$ such that $g(\alpha) = \beta$ for all $g \in U$ with $g = f(b,c) + bx + cx^2$ where $b \in dom_o$.*

Proof. According to Lemma 3.9, we consider the following two cases.

Suppose that there exists some $b' \in dom_o$ such that $s(b') = 0$.

Then $s(d) = 0$ and $t(d) = t(b)$ for each $d \in dom_o$. Put T for $t(b)$. It follows that for $b \in dom_o$ and $c \in C_b$ the polynomials $f(b,c) + bx + cx^2 \in U$ have the shape $T + bx + cx^2 \in U$ and hence $(0, T)$ is a common point of their graphs.

Suppose that $s(b) \neq 0$ for each $b \in dom_o$.

Then $s(b) = s(d)$ and $h(d) = h(b)$ for each $d \in dom_o$. We will denote these values by S and h, respectively. Note that $h = 0$, or $h = n/2$.

When $h = 0$.
Then $(t(b) - t(d))^2 = -S(b-d)^2$ for each $b, d \in dom_o$, and so

$$\left\{ \frac{t(b) - t(d)}{b - d} : b, d \in dom_o,\ b \neq d \right\} \subseteq \{s, -s\},$$

where $s^2 = -S$.

It follows that the point set $\{(b, t(b)) : b \in dom_o\}$ determines at most two directions. But there is no point set determining exactly two directions, thus t determines a unique direction, i.e.

$$t(d) = \gamma d + T, \text{ for } d \in dom_o,$$

where γ is a constant satisfying $\gamma^2 = s^2 = -S$. Then for $b \in dom_o$ and $c \in C_b$ the polynomials $f(b, c) + bx + cx^2 \in U$ have the shape $Sc + \gamma b + T + bx + cx^2$, so $(-\gamma, T)$ is the common point of their graphs.

When $h = n/2$.
Then also $t(b) = t(d)$ for each $d \in dom_o$. Then

$$(b - d)^2 - 4S(c - e)^{\sqrt{q}+1}$$

has to be a square for each $b, d \in dom_o$ and $c \in C_b$, $e \in C_d$.

Fix b, c, d. Note that for $k \in \mathbb{F}_{\sqrt{q}} \setminus \{0\}$ there are $\sqrt{q} + 1$ elements x in $\mathbb{F}_q$ such that $x^{\sqrt{q}+1} = k$. Since e runs through more than $q - \sqrt{q}/2 + 1/2$ values, we have

$$\mathbb{F}_{\sqrt{q}} \setminus \{0\} \subseteq \{(c - e)^{\sqrt{q}+1} : e \in C_d\}$$

so

$$(b - d)^2 - Sk = b^2 - 2db + d^2 - Sk \tag{3.3}$$

has to be a square in $\mathbb{F}_q$ for each $b, d \in dom_o$, $k \in \mathbb{F}_{\sqrt{q}} \setminus \{0\}$. As a polynomial in b, the discriminant of (3.3) is

$$4d^2 - 4(d^2 - Sk) = Sk.$$

Recall $S = s(b) \neq 0$, so this discriminant cannot be zero. By Result 2.4, for fixed $d \in dom_o$ and $k \in \mathbb{F}_{\sqrt{q}} \setminus \{0\}$ and for the character ψ of order 2,

$$\sum_{b \in \mathbb{F}_q} \psi(b^2 - 2db + d^2 - Sk) = -\psi(1) = -1.$$

On the other hand, a lower bound for this sum is

$$|dom_o| - 2 - |\mathbb{F}_q \setminus dom_o|,$$

which is at least $2|dom_o| - q - 2$, a contradiction when $|dom_o| > (q + 1)/2$. □

Next, we prove Theorem 1.3 when q is odd.

Theorem 3.11. *If $q \geq 53$ is odd and U is a set of $q^2 - \varepsilon$, $\varepsilon < \frac{q\sqrt{q}}{4} - \frac{3q}{8} - \frac{\sqrt{q}}{8}$, intersecting polynomials of degree at most 2 over $\mathbb{F}_q$, then the graphs of the functions in U share a common point.*

Proof. Let dom_o denote the set of values as before and let us call a polynomial $f(b,c) + bx + cx^2 \in U$ good if $b \in dom_o$. According to the previous lemma, if $|dom_o| > (q+1)/2$ the graphs of the good polynomials share a common point. If there are more than $\frac{q^2+q}{2}$ good polynomials then Lemma 3.5 finishes the proof.

Clearly,

$$|U| \leq |dom_o|q + (q - |dom_o|)(q - \sqrt{q}/2 + 1/2) = q^2 - (q - |dom_o|)(\sqrt{q}/2 - 1/2).$$

Assume to the contrary that the number of good polynomials is at most $\frac{q^2+q}{2}$, then $|dom_o| < \frac{q^2+q}{2(q-\sqrt{q}/2+1/2)} < \frac{q}{2} + \frac{\sqrt{q}}{4} + \frac{1}{2}$. Hence:

$$|U| \leq q^2 - \left(\frac{q\sqrt{q}}{4} - \frac{3q}{8} - \frac{\sqrt{q}}{8} + \frac{1}{4}\right),$$

which is a contradiction. □

3.2 Intersecting families of polynomials of degree at most 2, over $\mathbb{F}_q$, q even

Lemma 3.12. *Let U be a set of intersecting polynomials of degree at most 2 and for $t \in \mathbb{F}_q$, $q > 2$ even, define*

$$B_t := \{b \in \mathbb{F}_q : f(b, b+t) + bx + (b+t)x^2 \in U\}$$

and

$$dom_e := \{t \in \mathbb{F}_q : |B_t| \geq q - \sqrt{q}/2\}.$$

There exist functions $A, B\colon dom_e \to \mathbb{F}_q$ such that for every $b \in B_t$

$$f(b, b+t) = A(t)b + B(t),$$

and $A(t) \in \ker \mathrm{Tr}_{q/2}$.

Proof. Consider $F(x) = f(b,c) + bx + cx^2$ and $G(x) = f(d,e) + dx + ex^2$. Then the graphs of F and G share a common point if and only if $F - G$ has a root in $\mathbb{F}_q$, that is, $b = d$ or

$$H(b,c,d,e) := \mathrm{Tr}_{q/2}\left(\frac{(c+e)(f(b,c) + f(d,e))}{(b+d)^2}\right) = 0.$$

Then for each $t \in \mathbb{F}_q$, $b, d \in B_t$, $b \neq d$,

$$H(b, b+t, d, d+t) = \mathrm{Tr}_{q/2}\left(\frac{(b+d)(f(b,b+t) + f(d,d+t))}{(b+d)^2}\right) = 0.$$

Simplifying by $b + d$ yields

$$\mathrm{Tr}_{q/2}\left(\frac{f(b,b+t) + f(d,d+t)}{b+d}\right) = 0.$$

Define $R_t\colon B_t \to \mathbb{F}_q$ as $R_t(x) = f(x, x+t)$. For each $x, y \in B_t$, $x \neq y$, it holds that

$$\mathrm{Tr}_{q/2}\left(\frac{R_t(x) + R_t(y)}{x+y}\right) = 0.$$

In particular, the set of directions determined by the graph of R_t is contained in $\ker \mathrm{Tr}_{q/2}$, and hence it has size at most $q/2$.

From now on assume $t \in dom_e$ and hence $|B_t| \geq q - \sqrt{q}/2$. By results of Szőnyi [10], there exists a unique extension $\tilde{R}_t \colon \mathbb{F}_q \to \mathbb{F}_q$ of R_t such that the set of directions determined by $\tilde{R}_t$ is the same as the set of directions determined by the point set $\{(x, R_t(x)) : x \in B_t\} \subseteq \mathrm{AG}(2,q)$. So the set of directions determined by $\tilde{R}_t$ is contained in $\ker \mathrm{Tr}_{q/2}$ and hence by Theorem 2.2 there exist $A(t), B(t) \in \mathbb{F}_q$ such that $\tilde{R}_t(x) = A(t)x + B(t)$ with $\mathrm{Tr}_{q/2}(A(t)) = 0$. It follows that for $b \in B_t$ we have

$$R_t(b) = f(b, b+t) = A(t)b + B(t).$$

□

Lemma 3.13. *Let U be a set of intersecting polynomials of degree at most 2 and define B_t, dom_e and the functions A and B as in the previous lemma. Then there exist $\alpha, \beta \in \mathbb{F}_q$, $q \geq 8$, such that $A(t) = \alpha^{q/2} + \alpha$ and $B(t) = \alpha t + \beta$ for each $t \in dom_e$.*

Proof. If $|dom_e| = 1$, then the assertion is trivial, so assume $|dom_e| \geq 2$ and take any $s, t \in dom_e$. Fix some $b \in B_s$. Then for each $d \in B_t \setminus \{b\}$,

$$H(b, b+s, d, d+t) = \mathrm{Tr}_{q/2}\left(\frac{(b+s+d+t)(f(b,b+s)+f(d,d+t))}{(b+d)^2}\right) = 0,$$

that is,

$$\mathrm{Tr}_{q/2}\left(\frac{(b+s+d+t)(f(b,b+s)+A(t)d+B(t))}{(b+d)^2}\right) = 0,$$

i.e.,

$$\mathrm{Tr}_{q/2}\Bigg(A(t) + \frac{f(b,b+s)+B(t)+A(t)b+A(t)(s+t)}{b+d} + \\ (s+t)\frac{f(b,b+s)+B(t)+A(t)b}{(b+d)^2}\Bigg) = 0.$$

Applying $\mathrm{Tr}_{q/2}(A(t)) = 0$ and $\mathrm{Tr}_{q/2}(z) = \mathrm{Tr}_{q/2}(z^2)$ for each $z \in \mathbb{F}_q$, we obtain for each $d \in B_t \setminus \{b\}$, $d \neq b$

$$\mathrm{Tr}_{q/2}\Bigg(\frac{f^2(b,b+s)+B^2(t)+A^2(t)b^2+A^2(t)(s+t)^2+}{(b+d)^2} \\ \frac{(s+t)(f(b,b+s)+B(t)+A(t)b)}{(b+d)^2}\Bigg) = 0.$$

The numerator does not depend on d, while the denominator ranges over a subset of $\mathbb{F}_q^*$ of size $|B_t \setminus \{b\}| > \deg \mathrm{Tr}_{q/2} = q/2$ and hence this is possible only if

$$f^2(b,b+s)+B^2(t)+A^2(t)b^2+A^2(t)(s+t)^2+(s+t)(f(b,b+s)+B(t)+A(t)b) = 0.$$

Since $f(b, b+s) = A(s)b + B(s)$, this is equivalent to

$$(A(s)b+B(s))^2+B^2(t)+A^2(t)b^2+A^2(t)(s+t)^2+ \\ (s+t)(A(s)b+B(s)+B(t)+A(t)b) = 0,$$

that is,

$$b^2(A^2(s)+A^2(t))+b(A(s)+A(t))(s+t)+ \\ (B(s)+B(t))(B(s)+B(t)+s+t)+A^2(t)(t+s)^2=0.$$

Since this holds for every $b \in B_s$, and $|B_s|>2$, it follows that as a polynomial of b, this is the zero polynomial, so $A(s)=A(t)$ and

$$(B(s)+B(t))(B(s)+B(t)+s+t)+A^2(t)(t+s)^2=0. \tag{3.4}$$

Since $\mathrm{Tr}_{q/2}(A(t))=0$, and $A(t)$ is a constant function, this proves the existence of $\alpha' \in \mathbb{F}_q$ such that $A(x)=\alpha'^{q/2}+\alpha'$ for each $x \in dom_e$. If $|dom_e|=2$, then clearly B is linear, so assume $|dom_e| \geq 3$ and take some $t' \in dom_e \setminus \{s,t\}$. The same arguments show

$$(B(s)+B(t'))(B(s)+B(t')+s+t')+A^2(t')(t'+s)^2=0. \tag{3.5}$$

Summing up (3.4) and (3.5) we obtain

$$B(s)(t+t')+s(B(t)+B(t'))+B^2(t)+B^2(t')+B(t)t+B(t')t'+(\alpha'^2+\alpha')(t+t')^2=0,$$

so for $x \in dom_e \setminus \{t,t'\}$ it holds that

$$B(x)=x\frac{B(t)+B(t')}{t+t'}+\frac{B^2(t)+B^2(t')+B(t)t+B(t')t'}{t+t'}+(\alpha'^2+\alpha')(t+t'),$$

and from (3.4) (with $s=t'$) one obtains the same for $x \in \{t,t'\}$, so B is linear. Put $B(x)=\gamma x+\beta$, then from (3.4)

$$\gamma(s+t)(\gamma(s+t)+(s+t))=(\alpha'^2+\alpha')(s+t)^2,$$

so $\gamma^2+\gamma=\alpha'^2+\alpha'$ which proves $\gamma=\alpha'$ or $\gamma=\alpha'+1$. Now, if $\gamma=\alpha'$ then we set $\alpha:=\alpha'$ whereas if $\gamma=\alpha'+1$ we set $\alpha:=\alpha'+1$. Since $\alpha'^{q/2}+\alpha'=(\alpha'+1)^{q/2}+\alpha'+1$, our lemma follows. □

Corollary 3.14. *If $g(x)=f(b,c)+bx+cx^2 \in U$ and $b+c \in dom_e$, then $g(\alpha^{q/2})=\beta$.*

Proof. Put $t=b+c$. Then $t \in dom_e$ and hence

$$f(b,c)=f(b,b+t)=A(t)b+B(t)=\alpha^{q/2}b+\alpha b+\alpha t+\beta,$$

hence

$$g(\alpha^{q/2})=\alpha^{q/2}b+\alpha b+\alpha t+\beta+b\alpha^{q/2}+(b+t)\alpha=\beta.$$ □

Finally, we prove Theorem 1.3 for q even.

Theorem 3.15. *If $q \geq 8$ is even and U is a set of $q^2-\varepsilon$, $\varepsilon<\frac{q\sqrt{q}}{4}-\frac{q}{8}-\frac{\sqrt{q}}{8}$, intersecting polynomials of degree at most 2 over $\mathbb{F}_q$, then the graphs of the functions in U share a common point.*

Proof. Let dom_e denote the set of values as before and let us call a polynomial $f(b,c) + bx + cx^2 \in U$ good if $b + c \in dom_e$. According to the previous corollary, the graphs of the good polynomials share a common point. If there are more than $\frac{q^2}{2}$ good polynomials then Lemma 3.4 finishes the proof.

Clearly,

$$|U| \leq |dom_e|q + (q - |dom_e|)(q - \sqrt{q}/2) = q^2 - (q - |dom_e|)\sqrt{q}/2.$$

Assume to the contrary that the number of good polynomials is at most $\frac{q^2}{2}$, then $|dom_e| \leq \frac{q^2}{2(q-\sqrt{q}/2)} < \frac{q}{2} + \frac{\sqrt{q}}{4} + \frac{1}{4}$. Hence:

$$|U| < q^2 - \left(\frac{q\sqrt{q}}{4} - \frac{q}{8} - \frac{\sqrt{q}}{8}\right),$$

which is a contradiction. □

3.3 Intersecting families of polynomials of degree at most $k > 2$

Theorem 1.4. *If U is a set of more than $q^k - q^{k-1}$ intersecting polynomials over $\mathbb{F}_q$, $q \geq 53$ when q is odd and $q \geq 8$ when q is even, and of degree at most k, $k > 1$, then there exist $\alpha, \beta \in \mathbb{F}_q$ such that $g(\alpha) = \beta$ for all $g \in U$. Furthermore, U can be uniquely extended to a family of q^k intersecting polynomials of degree at most k over $\mathbb{F}_q$.*

Proof. Let U be a set of more than $q^k - q^{k-1}$ intersecting polynomials over $\mathbb{F}_q$ and of degree at most k, $k > 1$. First we show that there exist $\alpha, \beta \in \mathbb{F}_q$ such that $g(\alpha) = \beta$ for all $g \in U$. We prove this by induction.

For $k = 2$, this is true by Theorem 1.3. Now assume that it is true for $k - 1$ and we want to prove it for k. By the pigeon hole principle there must be a value c, such that there are more than $q^{k-1} - q^{k-2}$ polynomials h_i in U whose coefficient in x^k is c. Now consider the family of polynomials in the form of $\{h_i - cx^k\}$. Clearly, this is an intersecting family of polynomials of degree at most $k - 1$. So by the induction hypothesis, there are values α and β so that for every i, $(h_i - cx^k)(\alpha) = \beta$ and hence of course $h_i(\alpha) = \beta + c\alpha^k$ and so Lemma 3.6 finishes the proof of the first part.

Next, we will prove that U can be uniquely extended to a family of q^k intersecting polynomials of degree at most k over $\mathbb{F}_q$. Hence, let $\mathcal{F}$ and $\mathcal{F}'$ be two intersecting families of size q^k, both of them containing U. Then, there exist $\alpha, \alpha', \beta, \beta' \in \mathbb{F}_q$ such that $g(\alpha) = \beta$ for all $g \in \mathcal{F}$ and $g(\alpha') = \beta'$ for all $g \in \mathcal{F}'$. The polynomials in U are in $\mathcal{F} \cap \mathcal{F}'$, a contradiction unless $(\alpha, \beta) = (\alpha', \beta')$, since there are at most $q^{k-1} < |U|$ polynomials of degree at most k, whose graph contains two distinct, fixed points. Theorem 1.4 follows. □

4 Large intersecting families whose graphs do not share a common point

The following construction was drawn to our attention in a talk by Sam Adriaensen. Note that it shows the sharpness of the lower bound on $|U|$ in Lemma 3.4.

Example 4.1 (Hilton-Milner type)**.** Pick a point $P := (\alpha, \beta)$ and a line $e := \{(x, vx+w) : x \in \mathbb{F}_q\}$ in $\mathrm{AG}(2, q)$, so that $\beta \neq v\alpha + w$. Let U' be the set of those polynomials over $\mathbb{F}_q$, which are of the form $h(x) = cx^2 + bx + a$ and for which $h(\alpha) = \beta$ and there exist values α' and β' so that $h(\alpha') = \beta'$ and $\beta' = v\alpha' + w$. The set $U = U' \cup \{e\}$ is a set of intersecting polynomials of degree at most 2 over $\mathbb{F}_q$. The size of U is $\frac{q^2+q}{2}$ and clearly there exist no values $s, t \in \mathbb{F}_q$ so that for every polynomial $f \in U$, $f(s) = t$.

Proof. Clearly, we may assume that $P = (0, 1)$ and $e = \{(x, 0) : x \in \mathbb{F}_q\}$. Then $a = 1$ for the polynomials in U'. Pick a point $R := (u, 0)$ from e. The number of polynomials g in U', so that $g(u) = 0$ is 0 if $u = 0$, q otherwise. Hence if we count the polynomials of U' corresponding to R when R runs on the points of e, we see $q(q-1)$ polynomials. But most of the polynomials in U' will correspond to two different points R and R' of e. Actually, only the polynomials which are of the form $bx + 1$ ($b \in \mathbb{F}_q^*$) and polynomials of the form $c^{-2}(x + c)^2$ ($c \in \mathbb{F}_q^*$) in U' will correspond to exactly one point in e. Hence $|U'| = \frac{q(q-1)-2(q-1)}{2} + 2(q-1) = \frac{q^2+q}{2} - 1$ and so $|U| = \frac{q^2+q}{2}$. □

Example 4.2. Let q be odd. There is a family $\mathcal{M}$ of intersecting polynomials of degree at most 2 such that $|\mathcal{M}| = \frac{q^2-q+1}{2}$ and there exists $f \in \mathcal{M}$ with the property that $|U_f \cap U_g| = 1$ for each $g \in \mathcal{M}$, $g \neq f$.

Proof. Choose a polynomial $f(x) = Ax^2 + Bx + C$ and let $\square_q$ be the set of non-zero squares in $\mathbb{F}_q$. Let

$$\mathcal{P} = \left\{ a_i x^2 + b_i x + C - \frac{(B - b_i)^2}{4(A - a_i)} : A - a_i \in \square_q,\ b_i \in \mathbb{F}_q \right\}.$$

Note that $|\mathcal{P}| = q(q-1)/2$.

If (a_i, b_i) and (a_j, b_j) correspond to two elements of $\mathcal{P}$ then the graphs of the corresponding polynomials meet each other if and only if

$$(b_i - b_j)^2 - 4(a_i - a_j)\left(\frac{(B - b_j)^2}{4(A - a_j)} - \frac{(B - b_i)^2}{4(A - a_i)}\right) =$$

$$\frac{(a_i B - a_j B - A b_i + a_j b_i + A b_j - a_i b_j)^2}{(A - a_i)(A - a_j)}$$

is a (possibly zero) square in $\mathbb{F}_q$. This certainly holds since both $(A - a_i)$ and $(A - a_j)$ are squares. Hence $\mathcal{P}$ is an intersecting family. Finally, we prove that $\mathcal{M} = \mathcal{P} \cup \{f\}$ is also an intersecting family. We will do this by proving that for each $g \in \mathcal{P}$, U_g meets U_f in a unique point. So assume $g(x) = ax^2 + bx + C - \frac{(B-b)^2}{4(A-a)}$. It is easy to see that the discriminant of $f - g$ is zero and hence the result follows. □

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Saturated 2-plane drawings with few edges

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Abstract

A drawing of a graph is k-plane if every edge contains at most k crossings. A k-plane drawing is saturated if we cannot add any edge so that the drawing remains k-plane. It is well-known that saturated 0-plane drawings, that is, maximal plane graphs, of n vertices have exactly $3n-6$ edges. For $k > 0$, the number of edges of saturated n-vertex k-plane graphs can take many different values. In this note, we establish some bounds on the minimum number of edges of saturated 2-plane graphs under various conditions.

Keywords: Saturated drawing, 2*-planar, graphs, discharging.*

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1 Preliminaries

In a drawing of a graph in the plane, vertices are represented by points, edges are represented by curves connecting the points, which correspond to adjacent vertices. The points (curves) are also called vertices (edges). We assume that an edge does not go through any vertex, and three edges do not cross at the same point. A graph together with its drawing is a *topological graph*. A drawing or a topological graph is *simple* if any two edges have at most one point in common, that is either a common endpoint or a crossing. In particular, there is no self-crossing. In this paper, we assume the underlying graph has neither loops nor multiple edges.

For any $k \geq 0$, a topological graph is *k-plane* if each edge contains at most k crossings. A graph G is *k-planar* if it has a *k-plane* drawing in the plane.

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There are several versions of these concepts, see e.g. [4]. The most studied one is when we consider only simple drawings. A graph G is *simple k-planar* if it has a *simple k-plane* drawing in the plane.

A simple k-plane drawing is *saturated* if no edge can be added so that the obtained drawing is also simple k-plane. The 0-planar graphs are the well-known planar graphs. A plane graph of n vertices has at most $3n - 6$ edges. If it has exactly $3n - 6$ edges, then it is a triangulation of the plane. If it has fewer edges, then we can add some edges so that it becomes a triangulation with $3n - 6$ edges. That is, saturated plane graphs have $3n - 6$ edges.

Pach and Tóth [6] proved the maximum number of edges of an n-vertex (simple) 1-planar graph is $4n - 8$. Brandenburg et al. [3] noticed that saturated simple 1-plane graphs can have much fewer edges, namely $\frac{45}{17}n + O(1) \approx 2.647n$. Barát and Tóth [2] proved that a saturated simple 1-plane graph has at least $\frac{20n}{9} - O(1) \approx 2.22n$ edges.

For any k, n, let $s_k(n)$ be the minimum number of edges of a saturated n-vertex simple k-plane drawing. With these notations, $\frac{45n}{17} + O(1) \geq s_1(n) \geq \frac{20}{9}n - O(1)$. For $k > 1$, the best bounds known for $s_k(n)$ are shown by Auer et al [1] and by Klute and Parada [5]. Interestingly for $k \geq 5$ the bounds are very close.

In this note, we concentrate on 2-planar graphs on n vertices. Pach and Tóth [6] showed the maximum number of edges of a (simple) 2-planar graph is $5n - 10$. Auer et al [1] and Klute and Parada [5] proved that $\frac{4n}{3} + O(1) \geq s_2(n) \geq \frac{n}{2} - O(1)$. We improve the lower bound.

Theorem 1.1. *For any $n > 0$, $s_2(n) \geq n - 1$.*

A drawing is *l-simple* if any two edges have at most l points in common. By definition a simple drawing is the same as a 1-simple drawing. Let $s_k^l(n)$ be the minimum number of edges of a saturated n-vertex l-simple k-plane drawing. In [5] it is shown that $\frac{4n}{5} + O(1) \geq s_2^2(n) \geq \frac{n}{2} - O(1)$ and $\frac{2n}{3} + O(1) \geq s_2^3(n) \geq \frac{n}{2} - O(1)$. We make the following improvements:

Theorem 1.2. (i) *$s_2^2(3) = 3$, and $\lfloor 3n/4 \rfloor \geq s_2^2(n) \geq \lfloor 2n/3 \rfloor$ for $n \neq 3$,*

(ii) *$s_2^3(3) = 3$, and $s_2^3(n) = \lfloor 2n/3 \rfloor$ for $n \neq 3$.*

The saturation problem for k-planar graphs has many different settings, we can allow self-crossings, parallel edges, or we can consider non-extendable *abstract* graphs. See [4] for many recent results and a survey.

2 Proofs

Definition 2.1. Let G be a topological graph and u a vertex of degree 1. For short, u is called a *leaf* of G. Let v be the only neighbor of u. The pair (u, uv) is called a *flag*. If there is no crossing on uv, then (u, uv) is an *empty flag*.

Definition 2.2. Let G be an l-simple 2-plane topological graph. If an edge contains two crossings, then its piece between the two crossings is a *middle segment*. The edges of G divide the plane into cells. A cell C is *special* if it is bounded only by middle segments and isolated vertices. Equivalently, C is *special*, if there is no vertex on its boundary, apart from isolated vertices. An edge that bounds a special cell is also *special*.

Let G be a saturated l-simple 2-plane topological graph, where $1 \leq l \leq 3$. Suppose a cell C contains an isolated vertex v. Since G is saturated, C must be a special cell and there is no other isolated vertex in C. Now suppose C is an empty special cell. Each boundary edge contains two crossings. Therefore, if we put an isolated vertex in C, then the topological graph remains saturated. So if we want to prove a lower bound on the number of edges, we can assume without loss of generality that each special cell contains an isolated vertex.

Claim 2.3. *A special edge can bound at most one special cell.*

Proof. Suppose uv is a special edge and let pq be its middle segment. If uv bounds more than one special cell, then there is a special cell on both sides of pq, C_1 and C_2 say. Let p be a crossing of the edges uv and xy. There is no crossing on xy between p and one of the endpoints, x say. Therefore, one of the cells C_1 and C_2 has x on its boundary, a contradiction. □

Proof of Theorem 1.1. Suppose G is a saturated simple 2-plane topological graph of n vertices and e edges. We assume that each special cell contains an isolated vertex.

Claim 2.4. *All flags are empty in G.*

Proof. Let (u, uv) be a flag. Suppose to the contrary there is at least one crossing on uv. Let p be the crossing on uv closest to u, with edge xy. Since it is a 2-plane drawing, there is no crossing on xy between p and one of the endpoints, x say. In this case, we can connect u to x along up and px. Since the drawing was saturated, u and x are adjacent in G, and $x \neq v$, that contradicts to $d(u) = 1$. □

Remove all empty flags from G. Observe the resulting topological graph G' is also saturated. If we can add an edge to G', then we could have added the same edge to G.

Suppose to the contrary that G' contains a flag (v, vw). Since G' is saturated, the flag is empty by Claim 2.4. In G, vertex v had degree at least 2, so v had some other neighbors, $u_1, \ldots, u_m$ say, in clockwise order. The flags $(u_i, u_i v)$ were all empty. However, u_1 can be connected to w, which is a contradiction. Therefore, there are no flags in G'. On the other hand, the graph G' may contain isolated vertices. Let n' and e' denote the number of vertices and edges of G'. Since $n - n' = e - e'$, it suffices to show that $e' \geq n' - 1$. If there are no isolated vertices in G', then $e' \geq n'$ is immediate.

We assign weight 1 to each edge. If G' has no edge, then it has one vertex and we are done. We discharge the weights to the vertices so that each vertex gets weight at least 1. If uv is not a special edge, then it gives weight $1/2$ to both endpoints u and v. Suppose now that uv is a special edge. It bounds the special cell C containing the isolated vertex x. If $d(u) = 2$, then uv gives weight $1/2$ to u, if $d(u) \geq 3$, then it gives weight $1/3$ to u. We similarly distribute the weight to vertex v. We give the remaining weight of uv to x.

We show that each vertex gets weight at least 1. This holds immediately for all vertices of positive degree. We have to show the statement only for isolated vertices. Let x be an isolated vertex in a special cell C bounded by $e_1, e_2, \ldots, e_m$ in clockwise direction. Let $e_i = u_i v_i$ such that the oriented curve $\overrightarrow{u_i v_i}$ has C on its right. See Figure 1 for $m = 5$. Let p_i be the crossing of e_i and e_{i+1}. Indices are understood modulo m. In general, it may happen that some of the points in $\{ u_i, v_i \mid i = 1, \ldots, m \}$ coincide. For each vertex u_i or v_i of degree at least 3, the corresponding boundary edge of C has a remainder charge

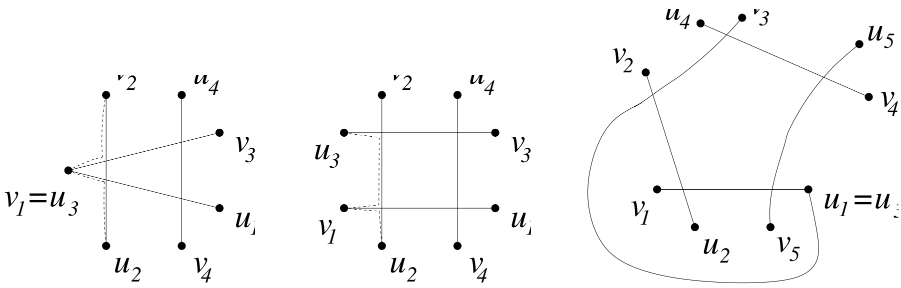

Figure 1: Case 1, $d(v_1) \geq 4$, Case 2, $d(v_1) \geq 3$ and Case 2, $u_1 = u_3$.

at least $1/6$. We have to prove that (with multiplicity) at least 6 of the vertices u_i, v_i have degree at least 3. Consider vertex v_i.

Case 1: $v_i = u_{i+2}$. The vertex $v_i = u_{i+2}$ can be connected to u_{i+1} along the segments $v_i p_i$ and $p_i u_{i+1}$, that are crossing-free segments of the corresponding edges. Similarly, $v_i = u_{i+2}$ can be connected to v_{i+1} along $v_i p_{i+1}$ and $p_{i+1} v_{i+1}$. Since the drawing was simple and saturated, u_i, u_{i+1}, v_{i+1}, v_{i+2} are all different and they are already connected to $v_i = u_{i+2}$, so it has degree at least 4.

Case 2: $v_i \neq u_{i+2}$. The vertex v_i can be connected to u_{i+1} as before, and to u_{i+2} along $v_i p_i$, $p_i p_{i+1}$ and $p_{i+1} u_{i+2}$. Since the drawing was saturated, v_i is already adjacent to u_i, u_{i+1}, u_{i+2}. Unless $u_i = u_{i+2}$, vertex v_i has degree at least 3. Note that $u_{i+1} \neq u_i$ and $u_{i+1} \neq u_{i+2}$, since the drawing was 1-simple.

We can argue analogously for u_i. We conclude that v_i has degree 2 only if $u_i = u_{i+2}$, and u_i has degree 2 only if $v_i = v_{i-2}$.

Recall that m is the number of bounding edges of the special cell C. For $m = 3$, it is impossible that $u_i = u_{i+2}$ or $v_i = v_{i-2}$, therefore, for $i = 1, 2, 3$ all six vertices u_i, v_i have degree at least 3.

Let $m > 3$, and suppose v_1 has degree 2, consequently $u_1 = u_3$. In this case, we prove that u_m, u_1, u_2, u_3, v_m, v_2 all have degree at least 3.

We show it for u_2, the argument is the same for the other vertices. Let γ be the closed curve formed by the segments $u_1 p_1$, $p_1 p_2$ and $p_2 u_3$. (We have $u_1 = u_3$.) Suppose $d(u_2) = 2$. By the previous observations, $v_m = v_2$. However, v_m and v_2 lie on different sides of γ, therefore they cannot coincide. Therefore, there are always at least six vertices u_i, v_i, with multiplicity, which have degree at least 3, so the isolated vertex x gets weight at least 1. This concludes the proof. □

We recall that $s_2^3(n)$ denotes the minimum number of edges of a saturated n-vertex 3-simple 2-plane drawing.

Proof of Theorem 1.2. We start with the upper bounds. Let

$$f(n) = \begin{cases} 3 & \text{if } n = 3 \\ \lfloor 3n/4 \rfloor & \text{otherwise.} \end{cases}$$

First we construct a saturated 2-plane, 2-simple topological graph with n vertices and $f(n)$ edges, for every n. Let $k \geq 3$. A *k-propeller* is isomorphic to a star with k edges as an

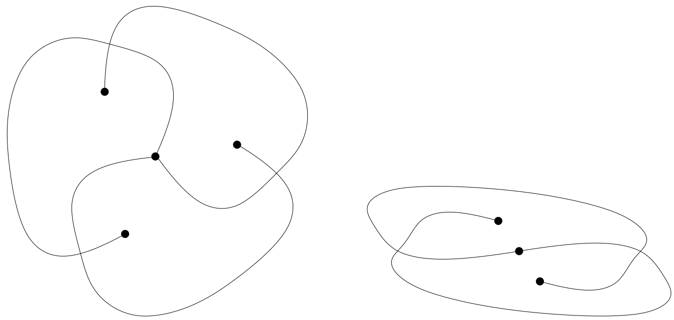

Figure 2: A 3-propeller and a 2-propeller.

abstract graph, drawn as in Figure 2. Clearly it is a saturated 2-plane, 2-simple topological graph with $k+1$ vertices, k edges and the unbounded cell is special.

For $n = 1, 2, 3$, a complete graph of n vertices satisfies the statement. For $n \geq 4$, $n \equiv 0 \bmod 4$, consider $n/4$ disjoint 3-propellers such that each of them is in the unbounded cell of the others. For $n \geq 4$, $n \equiv 1, 2, 3 \bmod 4$, replace one of the propellers by an isolated vertex, a K_2, and a 4-propeller, respectively. This implies the upper bound in (i), that is, $s_2^2(n) \leq f(n)$.

Now we construct a saturated 2-plane, 3-simple topological graph with n vertices and $\lfloor 2n/3 \rfloor$ edges, for every n. A 2-*propeller* is isomorphic to a path of 2 edges as an abstract graph, drawn as in Figure 2. Clearly it is a saturated 2-plane, 3-simple topological graph with 3 vertices, 2 edges and the unbounded cell is special.

For $n \equiv 0 \bmod 3$, take $n/3$ disjoint 2-propellers such that each of them is in the unbounded cell of the others. For $n \equiv 1, 2 \bmod 3$, add an isolated vertex or an independent edge. This implies the upper bound in (ii), $s_2^3(n) \leq \lfloor 2n/3 \rfloor$.

We prove by induction on n that $s_2^2(n) \geq \lfloor 2n/3 \rfloor$ and $s_2^3(n) \geq \lfloor 2n/3 \rfloor$. It is trivial for $n \leq 4$. Let $n > 4$ and assume that $s_2^2(m), s_2^3(m) \geq \lfloor 2m/3 \rfloor$ for every $m < n$. Let G be a saturated 2-plane, 2-simple or 3-simple drawing with n vertices and e edges. We may assume again that every special cell contains an isolated vertex.

Suppose that (u, uv) is an empty flag. We remove u from G. Analogous to the proof of Theorem 1.1, the obtained topological graph is saturated, it has $n-1$ vertices and $e-1$ edges. By the induction hypothesis, $e - 1 \geq \lfloor 2(n-1)/3 \rfloor$, which implies that $e \geq \lfloor 2n/3 \rfloor$. Therefore, we assume for the rest of the proof that G does not contain empty flags.

Claim 2.5. *If (u, uv) is a flag, then either $d(v) \geq 3$ or u and v are included in a 2-propeller.*

Proof. Since G does not contain empty flags, there is a crossing on uv. Let p be the crossing on uv closest to u, with edge xy. There is no crossing on xy between p and one of the endpoints, x say, and $x \neq u$ by the assumptions. We can connect u to x along the segments up and px. Since the drawing was saturated, u and x are adjacent in G. Since u has degree 1, $x = v$. This implies $d(v) \geq 2$. We exclude parallel edges, so $y \neq u$.

Suppose $d(v) = 2$. There is a crossing on the segment py of vy, otherwise we could connect u to y along the segments up and py contradicting the degree assumption on u. Let q be the crossing of vy and ab. There is no crossing on ab between q and one of the endpoints, a say. If a and u are on the same side of edge vy (that is, the directed edges $\overrightarrow{ab}$

and $\overrightarrow{ub}$ cross the directed edge $\overrightarrow{vy}$ from the same side), then we can connect u to a along the segments up, pq, qa. Therefore $a = v$, so either $d(v) \geq 3$, or $b = u$, and edges uv and vy form a 2-propeller. Note that this case is possible only if G is 3-simple.

So we may assume that a is on the other side. If $a = v$, then $d(v) \geq 3$, so we also assume that $a \neq v$. Consider now the edge uv. If there was no crossing on the segment pv of uv, then we can connect u to a along up, the segment pv of yv, the segment vp of uv, pq, and qa. Therefore, there is a crossing on the segment pv of uv. Let r be this crossing of uv with edge cd, and we can assume there is no crossing on the segment cr. (Here, c or d might coincide with a.) If c and y are on the same side of uv (that is, the directed edges $\overrightarrow{vy}$ and $\overrightarrow{dc}$ cross the directed edge $\overrightarrow{vu}$ from the same side), then we can connect u to c along up, px, xr, rc, which means that $c = v$, so $d(v) \geq 3$. If c and y are on opposite sides of uv, then we can connect c to v, so they are already connected. Therefore, $c = y$. However, we assumed that $\overrightarrow{vy}$ and $\overrightarrow{dc}$ cross the directed edge $\overrightarrow{vu}$ from the opposite sides, so there is another crossing of uv and vy. If G is 2-simple, this is impossible and we are done. If G is 3-simple, then this crossing can only be r, so $c = y$ and $d = x$. Now the edges uv and vy form a 2-propeller. □

In a graph G, a connected component with at least two vertices is an *essential component*. If G has only one essential component, then G is *essentially connected.*

Claim 2.6. *We can assume without loss of generality that G is essentially connected.*

Proof. Suppose to the contrary G has at least two essential components. We define a partial order on the essential components of G: $G_i \prec G_j$ if and only if G_i lies in a bounded cell of G_j. Let G_1 be a minimal element with respect to $\prec$ and let G_2 be the union of all other essential components. There is a cell C of G, which is bounded by both G_1 and G_2. Let C correspond to cell C_1 of G_1 and cell C_2 of G_2. By the definition of G_1, C_1 is the unbounded cell of G_1. Since G is saturated, at least one of C_1 or C_2 is a special cell, otherwise G_1 and G_2 can be connected.

For $i = 1, 2$, let H_i be the topological graph G_i together with an isolated vertex in every special cell. Let n_i denote the number of vertices and e_i the number of edges in H_i. We notice $e = e_1 + e_2$ and $n = n_1 + n_2 - 1$ if exactly one of C_1 and C_2 is a special cell. Also $n = n_1 + n_2 - 1$ if both of them are special cells, since we can add 1 isolated vertex instead of 2. By the induction hypothesis, we have $e_i \geq \lfloor 2n_i/3 \rfloor$, so $e \geq \lfloor 2n_1/3 \rfloor + \lfloor 2n_2/3 \rfloor$, and it is easy to check, that for any $n_1, n_2 \geq 2$, $\lfloor 2n_1/3 \rfloor + \lfloor 2n_2/3 \rfloor \geq \lfloor 2(n_1 + n_2 - 1)/3 \rfloor$. Therefore, $e \geq \lfloor 2n_1/3 \rfloor + \lfloor 2n_2/3 \rfloor \geq \lfloor 2(n_1 + n_2 - 1)/3 \rfloor = \lfloor 2n/3 \rfloor$. So, if G is not essentially connected, then we reduce the problem and proceed by induction. □

Assume the 3-simple 2-plane drawing G has a flag (u, uv). If $d(v) = 1$, then G is isomorphic to K_2 and the theorem holds. If $d(v) = 2$, then G contains a 2-propeller u, v, w by Claim 2.5. Since G is essentially connected, but there is an isolated vertex in every special cell, there is an isolated vertex x in the special cell of the 2-propeller. Therefore, if $d(v) = 2$ and $d(w) = 1$, then G is isomorphic to a 2-propeller plus an isolated vertex and we are done. If $d(v) = 2$ and $d(w) > 1$, then remove vertices u, v, x. We removed 3 vertices and 2 edges, so we can use induction.

In the rest of the proof, we assume that every leaf of G is adjacent to a vertex of degree at least 3, and there is no 2-propeller subgraph in G. We give weight $3/2$ to every edge. We discharge the weights to the vertices and show that either every vertex gets weight at least 1, or we can prove the lower bound on the number of edges by induction.

Let uv be an edge. Vertex u gets $1/d(u)$ weight and v gets $1/d(v)$ weight from uv. Every edge has a non-negative remaining charge.

If uv is a special edge, then it is easy to verify that uv bounds only one special cell, and the special cell contains an isolated vertex by the assumption, just like in the proof of Claim 2.3. In this case, edge uv gives the remaining charge to this isolated vertex. After the discharging step, any vertex x with $d(x) > 0$ gets charge at least 1.

Now let x be an isolated vertex, its special cell being C. We distinguish several cases.

Case 1: The special cell C has two sides. Let u_1v_1 and u_2v_2 be the bounding edges. They cross twice, in p and q say, so there are no further crossings on u_1v_1 and u_2v_2. The four endpoints are either distinct, or two of them u_1 and u_2 might coincide, if G was 3-simple. Suppose the order of crossings on the edges is u_ipqv_i, for $i = 1, 2$. If the vertices u_1 and u_2 are distinct, then they can be connected along u_1p and pu_2. Therefore, u_1 and u_2 are either adjacent or coincide in G. Similarly, v_1 and v_2 are also adjacent. Therefore, all four endpoints have degree at least 2, and both u_1v_1 and u_2v_2 give at most charge $1/2$ to its endpoints. Their remaining charges are at least $1/2$, so x gets at least charge 1.

For the rest of the proof, suppose C is bounded by $e_1, e_2, \ldots, e_m$ in clockwise direction, $e_i = u_iv_i$ such that $\overrightarrow{u_iv_i}$ has C on its right.

Case 2: $m = 3$. If none of the bounding edges is a flag, then we are done since each of those edges give weight at least $1/2$ to x. Suppose that u_1 is a leaf. We can connect u_1 to v_2 along segments of the edges u_1v_1 and u_2v_2. Since u_1 is a leaf and the drawing was saturated, u_1 and v_2 are adjacent, consequently $v_1 = v_2$. Similarly, we can connect u_1 to v_3, so $v_1 = v_2 = v_3$.

If u_2 is not a leaf, then u_1v_1 and u_3v_3 both give at least $1/6$ to x, and u_2v_2 gives at least $2/3$, so we have charge at least 1 for x. The same applies if u_3 is not a leaf. So assume u_1, u_2 and u_3 are all leaves. If there are no other edges in G, then we can see from the crossing pattern that G is a 3-propeller and an isolated vertex. That is, $n = 5$ and $e = 3$ and the required inequality holds.

Suppose there are further edges. By Claim 2.6, G is essentially connected. Since u_1, u_2, u_3 are leaves, v_1 is a cut vertex. Let $H_1 = G \setminus \{x, u_1, u_2, u_3\}$. The induced subgraph H_1 has $n - 4$ vertices and $e - 3$ edges, and it is saturated. Therefore, by the induction hypothesis, $e - 3 \geq f(n - 4)$. Notice that $f(n) \leq f(n - 4) + 3$, consequently $e \geq f(n)$.

Case 3: $m > 3$. Each edge gives at least $1/6$ charge to x by Claim 2.5. If an edge is not a flag, then it gives at least $1/2$ charge to x. If there is at least one non-flag bounding edge, we are done. Suppose that each edge u_iv_i is a flag (that is, $d(u_i)$ or $d(v_i)$ is 1). We may also assume that u_1 is a leaf. Now, as in the previous case, we can argue that $v_3 = v_2 = v_1$. It implies u_2 and u_3 are leaves, and by the same argument, $v_5 = v_4 = v_3 = v_2 = v_1$. We can continue and finally we obtain that all v_i are identical and all u_i are leaves. So the vertices u_i, v_i $1 \leq i \leq m$ form a star, and they have the same crossing pattern as an m-propeller. Therefore, u_i, v_i $1 \leq i \leq m$ span an m-propeller. We can finish this case exactly as Case 2. If there are no further edges in G, then the graph is an m-propeller and an isolated vertex. That is, $n = m + 2$ and $e = m$ and the inequality holds. If there are further edges, then v_1 is a cut vertex, and we can apply induction. This concludes the proof of Theorem 1.2. □

Remarks

- We have established lower and upper bounds on the number of edges of a saturated, k-simple, 2-plane drawing of a graph. As we mentioned in the introduction, this problem has many modifications, generalizations. Probably the most natural modification is that instead of graphs already drawn, we consider saturated *abstract* graphs. A graph G is saturated l-simple k-planar, if it has an l-simple k-planar drawing but adding any edge, the resulting graph does not have such a drawing. Let $t_k^l(n)$ be the minimum number of edges of a saturated l-simple k-planar graph of n vertices. By definition, $s_k^l(n) \leq t_k^l(n)$. We are not aware of any case when the best lower bound on $t_k^l(n)$ is better than for $s_k^l(n)$. On the other hand, it seems to be much harder to establish an upper bound construction for $t_k^l(n)$ than for $s_k^l(n)$. In fact, we know nontrivial upper bounds only in two cases, $t_1^1(n) \leq 2.64n + O(1)$ [3] and $t_2^1(n) \leq 2.63n + O(1)$ [1], the latter without a full proof.

  It is known that a k-planar graph has at most $c\sqrt{k}n$ edges [6], so $t_k^l(n) \leq c\sqrt{k}n$, for some $c > 0$.

  Problem 1. Prove that for every $c > 0$, $t_k^l(n) \leq c\sqrt{k}n$ if k, l, n are large enough.

- For any n and k, the best known upper and lower bounds on s_k^l decrease or stay the same as we increase l. This would suggest that $s_k^l \leq s_k^{l-1}$ for any n, k, l, or at least if n is large enough, however, we cannot prove it.

  Problem 2. Is it true, that for any k and l, and n large enough, $s_k^l \leq s_k^{l-1}$?

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Classification of thin regular map representations of hypermaps*

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Abstract

There are two well known maps representations of hypermaps, namely the Walsh and the Vince map representations, being dual of each other. They correspond to normal subgroups of index two of a free product $\Gamma = (C_2 \times C_2) * C_2$ which decompose as "elementary" free product $C_2 * C_2 * C_2$. However Γ has three normal subgroups that decompose as "elementary" free product $C_2 * C_2 * C_2$, the third of these sbgroups giving the less known petrie-path map representation. By relaxing the "elementary" free product condition to free product of rank 3, and under the extra condition "words of smaller length" on the generators, we prove that the number of map representations of hypermaps increases to 15 (up to a restrictedly dual), all of which described in this paper.

Keywords: Map representation, hypermaps, maps, regularity, restricted regularity, orientably regular.

Math. Subj. Class. (2020): 05C10, 05C25, 05C65, 05E18, 20F65

1 Introduction

Using maps to describe hypermaps is not new. The well-known Walsh [8] bipartite map representation uses a bipartite map $\mathcal{M}$ to describe a hypermap $\mathcal{H}$ by interpreting the two monochromatic vertices of the map as hypervertices and hyperedges (respectively), and the faces of $\mathcal{M}$ as the hyperfaces of $\mathcal{H}$. The Vince 2-face bipartite map [7], a dual of a bipartite map, also describes a hypermap by assigning the two monochromatic faces to hyperedges and hyperfaces respectively, and vertices to hypervertices. These are two, out

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of three, Θ-marked map representations realised by an index 2 normal subgroups Θ of the free product

$$\Gamma = \Delta(\infty, 2, \infty) = \langle R_0, R_1, R_2 \mid R_0^2, R_1^2, R_2^2, (R_0R_2)^2 \rangle = C_\infty * (C_2 \times C_2)\,,$$

which are isomorphic to $\Delta = \Delta(\infty, \infty, \infty) = \langle S_0, S_1, S_2 \mid S_0^2, S_1^2, S_2^2 \rangle$ (see [1] and section 3). They are namely, $\Gamma_{2.4} = \langle R_1^{R_0}, R_1, R_2 \rangle$ and $\Gamma_{2.1} = \langle R_0, R_1, R_1^{R_2} \rangle$. The third subgroup of Γ of index 2 isomorphic to Δ is $\Gamma_{2.5} = \langle R_1^{R_0}, R_1, R_0R_2 \rangle$ (see Subsection 4.2). This induces the third less known representation, succinctly described in [2], given by $\Gamma_{2.5}$-marked maps. In this representation Petrie-path-bipartite maps represent hypermaps by assigning the two monochromatic Petrie polygons (closed zig-zag paths turning alternately left and right) to hypervertices and hyperedges, and faces to hyperfaces.

More generally, a *regular* representation of hypermaps by maps is given by an epimorphism ρ from a finite index *normal* subgroup Θ of Γ to Δ.

This paper is inspired by the work of Lynne James on map representation of topological categories (see [5]) and is organised as follow: In Section 2 we give an introduction to the theory of hypermaps and maps focusing on restrictedly marked hypermaps and maps, a theory developed in [1]. In particular, we focus on Θ-marked maps for normal subgroups Θ of finite index in Γ. Section 3 is devoted to define the notion of clean and thin Θ-marked representation of a hypermap by a map. As we will focus on Θ-marked representations for rank 3 normal subgroups Θ of Γ, in Section 4 we derive a rank formula and classify the rank 3 normal subgroups of Γ. The rank formula is derived using presentations for NEC groups (see [3]). Last section is devoted to thin representations (given in Table 2) and its geometric description by means of an example.

In what follows by "representation" we always mean "regular representation". Note that we use right notation, that is, we denote by xf the image of x by the function f.

2 Preliminaries

Hypermaps are 4-tuples $\mathcal{H} = (F; r_0, r_1, r_2)$ where F is a finite set and r_0, r_1, r_2 are involutory permutations of F ($r_i^2 = 1$) generating a transitive group on F. The elements of F are called *flags* and the transitive group $Mon(\mathcal{H}) = \langle r_0, r_1, r_2 \rangle$ is the *monodromy group* of $\mathcal{H}$. The orbits of the action of the subgroups of $Mon(\mathcal{H})$ generated by $\{r_0, r_1, r_2\} \setminus \{r_i\}$ for $i = 0, 1, 2$ are respectively the *hypervertices*, *hyperedges* and *hyperfaces* of the hypermap $\mathcal{H}$, called respectively 0-*cells*, 1-*cells* and 2-*cells* of $\mathcal{H}$. The *valency* of an $i-$cell is the length of the orbit of one of its flags by r_jr_k where $\{i, j, k\} = \{0, 1, 2\}$. If, for some positive integers k, ℓ, m all hypervertices have valency k, all hyperedges have valency ℓ and all hyperfaces have valency m, then we say that $\mathcal{H}$ is a *uniform* hypermap (of type (k, ℓ, m)). In this case, $(k, \ell, m) = (|r_1r_2|, |r_2r_0|, |r_0r_1|)$, where $|g|$ denotes the order of g. If r_0, r_1 and r_2 have no fixed point then we say that $\mathcal{H}$ has *no boundary*. Thus, a uniform hypermap $\mathcal{H} = (F; r_0, r_1, r_2)$ without boundary has $V = \frac{|F|}{2|r_1r_2|}$ hypervertices, $E = \frac{|F|}{2|r_2r_0|}$ hyperedges and $F = \frac{|F|}{2|r_0r_1|}$ hyperfaces.

A *morphism* or *covering* from the hypermap $\mathcal{H}_1 = (E; r_0, r_1, r_2)$ to the hypermap $\mathcal{H}_2 = (F; s_0, s_1, s_2)$ is a function $\phi\colon E \to F$ satisfying

$$xr_i\phi = x\phi s_i,$$

for any $x \in E$ and any $i \in \{0, 1, 2\}$. We say that the hypermap $\mathcal{H}_1$ *covers* the hypermap $\mathcal{H}_2$ if there is a covering from $\mathcal{H}_1$ to $\mathcal{H}_2$. It is straightforward to see that any covering is onto and uniquely determined by the image of a flag. Injective coverings are therefore called *isomorphisms*. An *automorphism* of $\mathcal{H}$ is an isomorphism from $\mathcal{H}$ to itself. We will denote by $\mathrm{Aut}(\mathcal{H})$ the set of automorphisms of $\mathcal{H}$, which is obviously a group under composition.

Topologically, a hypermap $\mathcal{H}$ can be seen as a triangulation of a compact surface $\mathcal{S}$ with vertices labelled $0, 1$ and 2 such that each triangle (a flag of $\mathcal{H}$) has labels $0, 1$ and 2 assigned to its vertices; the vertices labelled $0, 1$ and 2 are respectively the hypervertices, hyperedges and hyperfaces. For each $x \in F$ the two triangles x and xr_i share the common edge e opposite to the vertices labelled i if $x \neq xr_i$; if $x = xr_i$, then the edge e is on the boundary of $\mathcal{S}$ and so $\mathcal{S}$ is a bordered surface. This triangulation is a topological map representation of hypermaps whose dual is the James topological map representation of hypermaps [4]; here the faces are labelled 0 (grey faces), 1 (dotted faces) and 2 (white faces) (see Figure 3). The hypermap $\mathcal{H}$ has (no) boundary if and only if $\mathcal{S}$ has (no) boundary. The *characteristic* of $\mathcal{H}$ is the Euler characteristic of $\mathcal{S}$. In particular, if $\mathcal{H} = (F; r_0, r_1, r_2)$ is a uniform hypermap without boundary, then the Euler characteristic of $\mathcal{H}$ is

$$\chi(\mathcal{H}) = \frac{|F|}{2}\left(\frac{1}{|r_1 r_2|} + \frac{1}{|r_2 r_0|} + \frac{1}{|r_0 r_1|} - 1\right).$$

Alternatively, a hypermap is a cellular embedding of a hypergraph in a compact connected surface.

The monodromy group $Mon(\mathcal{H})$ of a hypermap $\mathcal{H}$ is a quotient of the triangle group Δ. Hence we have an epimorphism $\pi\colon \Delta \to Mon(\mathcal{H})$ and an action

$$F \times \Delta \to F,\ (x, d) \mapsto x(d\pi)$$

of Δ on the set F of flags of $\mathcal{H}$. The stabiliser H of a flag under this action is a subgroup of Δ called a *hypermap subgroup* of $\mathcal{H}$. As the action of Δ is transitive, hypermap subgroups of $\mathcal{H}$ are conjugate. The hypermap $\mathcal{H}$ is then isomorphic to the hypermap $(\Delta/H; H_\Delta R_0, H_\Delta R_1, H_\Delta R_2)$, where Δ/H denotes the set of right cosets of a hypermap subgroup H of $\mathcal{H}$ in Δ, H_Δ is the normal core of H in Δ and $(Hd)H_\Delta R_i = HdR_i$ for any $d \in \Delta$ and any $i \in \{0, 1, 2\}$ (see, for instance [1]).

Let Θ be a normal subgroup of finite index n in Δ and let $\mathcal{H}$ be a hypermap with hypermap subgroup H. Then Θ acts (as a subgroup of Δ) on the set $F = \Delta/H$ of flags of $\mathcal{H}$ partitioning it into at most n orbits, called Θ-*orbits*; in fact, suppose that H is not a subgroup of Θ and let $b \in H \setminus \Theta$. Then $Hb = H$ and $b\Theta \neq \Theta$. Therefore the Θ-orbit $\{Hbt : t \in \Theta\}$ is equal to the Θ-orbit $\{Ht : t \in \Theta\}$, forcing the number of Θ-orbits being at most n. The number of Θ-orbits is n if and only if $H < \Theta$; in this case we say that $\mathcal{H}$ is Θ-*conservative*. A Θ-conservative hypermap $\mathcal{H}$ is Θ-*regular* if the group $Aut^{\Theta}(\mathcal{H})$ of automorphisms preserving Θ-orbits acts transitively on each Θ-orbit, or equivalently, if H is normal in Θ. However, if H is normal in Δ, then $\mathcal{H}$ is a *regular* hypermap, that is, Δ-regular. We shall say that a hypermap $\mathcal{H}$ is *restrictedly-marked* if it is Θ-conservative for some normal subgroup Θ of finite index in Δ. Ought to emphasise that not every hypermap is restrictedly-marked (see [1] for examples).

A hypermap $(F; r_0, r_1, r_2)$ satisfying $(r_0 r_2)^2 = 1$ is called a *map*. The hypervertices, hyperedges and hyperfaces of a map are called *vertices*, *edges* and *faces*, since topologically a map is a cellular embedding of a graph on a compact surface. The monodromy group of a map $\mathcal{M}$ is then a quotient of the "right" triangle group Γ. This group acts on the set of flags of $\mathcal{M}$ via the canonical projection $\pi\colon \Gamma \to Mon(\mathcal{M})$ sending R_i to r_i. The stabiliser of a flag under this action will be called a *map subgroup* of $\mathcal{M}$. Keeping the same notation as already used for hypermaps, we have that a map $\mathcal{M}$ is then isomorphic to the map $(\Gamma/M; M_\Gamma R_0, M_\Gamma R_1, M_\Gamma R_2)$, where M is a map subgroup of $\mathcal{M}$. The theory of restrictedly-marked maps unfolds in the same way as the theory of restrictedly-marked hypermaps by taking finite index normal subgroups Θ of Γ instead of Δ. The group Γ is a free product of $C_2 = \langle R_1 \rangle$ with $D_2 = \langle R_0, R_2 \rangle$ and by the Kurosh's Subgroup Theorem, any normal subgroup Θ of Γ freely decomposes uniquely (up to a permutation of factors) in a (indecomposable) free product (see [6] page 243 and 245)

$$C_2 * \cdots * C_2 * D_2 * \cdots * D_2 * C_\infty * \cdots * C_\infty =$$
$$\langle A_1 \rangle * \cdots * \langle A_s \rangle * \langle B_1, C_1 \rangle * \langle B_t, C_t \rangle * \langle Z_1 \rangle * \cdots * \langle Z_u \rangle$$

for a certain numbers s, t and u of factors $\langle A_i \rangle = C_2$, $\langle B_j, C_j \rangle = D_2$ and $\langle Z_u \rangle = C_\infty$ respectively, whereas s, t or u may be zero. Let $m = s + 2t + u = rank(\Theta)$ and let

$$\{A_1, \ldots, A_s, B_1, \ldots, B_t, C_1, \ldots, C_t, Z_1, \ldots, Z_u\} = \{X_1, \ldots, X_m\}.$$

Then a Θ-conservative map $\mathcal{M}$ with map subgroup M can be represented by the Θ-*marked map*

$$\mathcal{Q} = (\Omega; x_1, \ldots, x_m),$$

where $\Omega = \Theta/M$ is the set of right cosets of M in Θ and $x_1, \ldots, x_m$ are permutations of Ω generating a group G acting transitively on Ω such that the function

$$X_1 \mapsto x_1\,, \ldots\,, X_m \mapsto x_m$$

extends to an epimorphism from Θ to G.

Any Θ-regular map $\mathcal{M}$ covers the regular map

$$\mathcal{T}_\Theta = (\Gamma/\Theta; \Theta R_0, \Theta R_1, \Theta R_2),$$

called the Θ-*trivial map*. As $\mathcal{T}_\Theta$ is a regular map, we have that

- any two vertices of $\mathcal{T}_\Theta$ have same valency, say k,
- any two edges of $\mathcal{T}_\Theta$ have same valency, say $l \in \{1, 2\}$,
- any two faces of $\mathcal{T}_\Theta$ have same valency, say m.

The triple (k, l, m) is called *the type* of the regular map $\mathcal{T}_\Theta$. As $\mathcal{M}$ is Θ-regular and covers $\mathcal{T}_\Theta$, we also have that:

- the vertices of $\mathcal{M}$ covering a vertex v of $\mathcal{T}_\Theta$ also have same valency, say k_v (which is a multiple of k),
- the faces of $\mathcal{M}$ covering a face f of $\mathcal{T}_\Theta$ also have same valency, say m_f (which is a multiple of m).

Denoting by V, E and F the sets of vertices, edges and faces of $\mathcal{T}_\Theta$ and assuming that $\mathcal{M}$ has no boundary, then this together with Euler formula gives that the characteristic of $\mathcal{M}$ is

$$\chi(\mathcal{M}) = \frac{|\Theta : M|}{2} \left(\sum_{v \in V} \mu_v \frac{k}{k_v} + \sum_{e \in E} \mu_e \frac{l}{2} + \sum_{f \in F} \mu_f \frac{m}{m_f} - |\Gamma : \Theta| \right), \qquad (2.1)$$

where $\mu_v = 1$ or 2 according as the vertex v is on the boundary or not and similarly for μ_e and μ_f. For details we refer the reader to [1].

3 Thin map representations of hypermaps

Let Θ be a finite index normal subgroup of Γ of rank 3 and let $\{X_1, X_2, X_3\}$ be a set of generators of Θ. The pair $R = (\Theta, \{X_1, X_2, X_3\})$ will be called a Θ-*marked representation (of hypermaps by maps)* if the function

$$X_1 \mapsto S_0\,,\ X_2 \mapsto S_1\,,\ X_3 \mapsto S_2$$

extends to an epimorphism ρ from Θ onto Δ. We call ρ the *canonical epimorphism* of the representation R. Two representations $(\Theta_1, \{X_1, X_2, X_3\})$ and $(\Theta_2, \{Y_1, Y_2, Y_3\})$ are to be considered equal if $\Theta_1 = \Theta_2 = \Theta$ and their canonical epimorphisms $\rho_1, \rho_2 \colon \Theta \to \Delta$ are such that $\rho_1 = \iota\rho_2$ for some inner automorphism ι of Θ. For example, since S_0, S_1, S_2 are involutions, inverting one or more generators of R give the same representation.

Given a hypermap $\mathcal{H}$ with hypermap subgroup H, setting $\Omega = \{(H\rho^{-1})t : t \in \Theta\}$ and

$$x_i \colon \Omega \to \Omega,\ (H\rho^{-1})t \mapsto (H\rho^{-1})tX_i, \qquad i = 1, 2, 3$$

we get a Θ-marked map $(\Omega; x_1, x_2, x_3)$ called a Θ-*marked map representation of* $\mathcal{H}$.

Remark 3.1. In fact, denoting by N the normal core of $H\rho^{-1}$ in Θ, the group $G = \langle x_1, x_2, x_3\rangle$ is isomorphic to Θ/N by an isomorphism φ mapping x_i to NX_i for any $i \in \{1, 2, 3\}$. Hence $\pi\varphi^{-1}$, where $\pi \colon \Theta \to \Theta/N$ is the canonical epimorphism, is an epimorphism from Θ to G extending the function $X_1 \mapsto x_1, X_2 \mapsto x_2, X_3 \mapsto x_2$. Remark also that ρ induces a bijection $\widetilde{\rho}$ from Ω to $\{Hd : d \in \Delta\}$ which sends $(H\rho^{-1})t$ to $H(t\rho)$ and satisfies $x_i\,\widetilde{\rho} = \widetilde{\rho}\,r_{i-1}$, where r_{i-1} maps Hd to HdR_{i-1} for any $i \in \{1, 2, 3\}$. Thus, we say that $\widetilde{\rho}$ is an *isomorphism* from the Θ-marked map representation $(\Omega; x_1, x_2, x_3)$ of $\mathcal{H}$ to $\mathcal{H}$.

A (Θ-marked) representation $R = (\Theta, (X_1, X_2, X_3))$ will be called *clean* if Θ is the free product of the cyclic groups $\langle X_1\rangle, \langle X_2\rangle, \langle X_3\rangle$, in which case we write

$$\Theta = \langle X_1\rangle * \langle X_2\rangle * \langle X_3\rangle\,.$$

A clean representation is called *thin* if the sum of the lengths of its generators (as words in the free group over $\{R_0, R_1, R_2\}$) is minimal.

The number of rank 3 normal subgroups Θ of Γ is finite, but there are infinitely many clean representations given by all possible sets $\{X_1, X_2, X_3\}$ such that $\Theta = \langle X_1\rangle * \langle X_2\rangle * \langle X_3\rangle$. On the other hand the number of thin representations is finite (see Sections 4 and 5).

4 The rank 3 normal subgroups of Γ

4.1 Rank computation (see also [3])

In order to compute the rank of a normal subgroup Θ of finite index in Γ, we remark that Γ acts as a group of isometries on the hyperbolic plane $\mathbf{H}$, regarding its generators R_0, R_1, R_2 as the reflections on the geodesics given in Figure 1 in the Poincaré disk model.

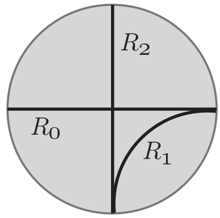


Figure 1: The generators of Γ as hyperbolic reflections.

The action of Θ on $\mathbf{H}$ gives rise to a quotient orbifold $\mathbf{H}/\Theta$ which is a punctured surface (with or without boundary) punctured at the vertices and at the face centers of the regular map $\mathcal{M} = (\Gamma/\Theta; \Theta R_0, \Theta R_1, \Theta R_2)$ with underlying surface $\mathbf{S}$. If $\Theta R_0 = \Theta R_2$, then the covering $\mathbf{H} \to \mathbf{H}/\Theta$ is also branched at the edge centers of $\mathcal{M}$. The group Θ, being the fundamental group of $\mathbf{H}/\Theta$, has a presentation P with $p + 2 - \chi$ generators $X_1, \ldots, X_p, Y_1, \ldots, Y_{2-\chi}$, where p is the total number of punctures and branching points of $\mathbf{H}/\Theta$ and χ is the characteristic of $\mathcal{S}$. The presentation P has a relator $S = X_1 \cdots X_p \cdot \prod_{i=1}^{k} Y_i \cdot W(Y_{k+1}, \ldots, Y_{2-\chi})$, where

$$\langle Y_1, \ldots, Y_{2-\chi} \mid W(Y_{k+1}, \ldots, Y_{2-\chi}) \rangle$$

is a presentation of the fundamental group of the surface $\mathbf{S}$ with k boundary components (setting $\prod_{i=1}^{k} Y_i = 1$ if $k = 0$), and eventually e relators $X_1^2, \ldots, X_e^2$ if $\Theta R_0 = \Theta R_2$, where e is the number of edges of $\mathcal{M}$.
Hence $rank(\Theta) = p + 2 - \chi - 1 = p + 1 - \chi$. More precisely:

- If $\mathbf{S}$ has no boundary ($k = 0$) and is non-orientable, then $2 - \chi$ is the genus g of $\mathbf{S}$, $W(Y_1, \ldots, Y_{2-\chi}) = W(Y_1, \ldots, Y_g) = \prod_{i=1}^{g} Y_i^2$ and therefore

$$S = X_1 \cdots X_p \cdot \prod_{i=1}^{g} Y_i^2 \,.$$

- If $\mathbf{S}$ has no boundary and is orientable, then $2 - \chi$ is even and the genus g of $\mathbf{S}$ is $\frac{2-\chi}{2}$. Replacing $(Y_1, \ldots, Y_{2-\chi})$ by $(A_1, B_1, \ldots, A_g, B_g)$ we have

$$S = X_1 \cdots X_p \cdot \prod_{i=1}^{g} [A_i, B_i] \,.$$

  In the particular case of $\chi = 2$ (sphere) the word $W(Y_1, \ldots, Y_{2-\chi})$ is empty and therefore

$$S = X_1 \cdots X_p.$$

- If $\mathbf{S}$ has boundary, then $\{R_0, R_1, R_2\} \cap \Theta \neq \emptyset$. Thus, any triangle of the triangulation of $\mathbf{S}$ given by the flags of $\mathcal{M}$ has at least an edge on the boundary, since Θ is normal in Γ. This shows that **S** is a closed disk, that is, a bordered surface on a sphere with only one boundary component ($k = 1$). Hence $\chi = 2 - k = 1$ and therefore, setting $Y = Y_1$ we have that $\prod_{i=1}^{k} Y_1 = Y$, $W(Y_{k+1}, \dots, Y_{2-\chi})$ is the empty word and $\langle Y \rangle \cong C_\infty$ is the fundamental group of **S**. Hence

$$S = X_1 \cdots X_p \cdot Y\,.$$

In particular, $rank(\Theta) = p$ in this case.

The next proposition relates the rank of Θ with its index n in Γ for $n > 4$. Relating rank with all indices will give a clumsy formula which does not give more information about the index bound for fixed rank.

Proposition 4.1. If Θ is a normal subgroup of finite index $n > 4$ in Γ, then n is even and

$$rank(\Theta) = \begin{cases} 1+n & \text{if } \mathbf{H}/\Theta \text{ has boundary and branching points;} \\ 1+\frac{n}{2} & \text{if } \mathbf{H}/\Theta \text{ has boundary and no branching point;} \\ & \text{or } \mathbf{H}/\Theta \text{ has no boundary but has branching points;} \\ 1+\frac{n}{4} & \text{if } \mathbf{H}/\Theta \text{ has no boundary and no branching point.} \\ & \text{(in this case } n \text{ is a multiple of } 4). \end{cases}$$

Proof. Using the above notations and remarks we have the following:
If **S** has boundary, then $\Theta R_1 \neq \Theta$ since $|\Gamma/\Theta| = n > 4$. Hence $\Gamma/\Theta = \langle \Theta R_1, \Theta R_j \rangle$ for some $j \in \{0, 2\}$, that is, Γ/Θ is dihedral of even order n. The total number of vertices and faces of the map $\mathcal{M} = (\Gamma/\Theta; \Theta R_0, \Theta R_1, \Theta R_2)$ is then $1 + \frac{n}{2}$. This gives

$$rank(\Theta) = p = \begin{cases} 1+\frac{n}{2} & \text{if } \mathbf{H}/\Theta \text{ has no branching points,} \\ 1+n & \text{if } \mathbf{H}/\Theta \text{ has branching points,} \end{cases}$$

since in the case when $\mathbf{H}/\Theta$ has branching points, $\mathcal{M}$ has $\frac{n}{2}$ edges.

If **S** has no boundary, then n is a multiple of 4 and from Euler formula we have that

$$\chi = \begin{cases} p-\frac{n}{4} & \text{if } \mathbf{H}/\Theta \text{ has no branching points,} \\ p-\frac{n}{2} & \text{if } \mathbf{H}/\Theta \text{ has branching points.} \end{cases}$$

Therefore

$$rank(\Theta) = p+1-\chi = \begin{cases} 1+\frac{n}{4} & \text{if } \mathbf{H}/\Theta \text{ has no branching points,} \\ 1+\frac{n}{2} & \text{if } \mathbf{H}/\Theta \text{ has branching points.} \end{cases}$$ □

Corollary 4.2. *If $rank(\Theta) = 3$, then the index n is 2, 4 or 8 and Γ/Θ is isomorphic to C_2, $C_2 \times C_2$, $C_2 \times C_2 \times C_2$ or D_4.*

Proof. Proposition 4.1 guaranties that $n \in \{2, 4, 8\}$ if $rank(\Theta) = 3$. The groups of order $2, 4$ and 8 not listed in the statement are not generated by involutions. □

4.2 The rank 3 normal subgroups of Γ

(1) $n = 2$: As mentioned in the introduction, there are seven epimorphisms from Γ to C_2 having kernels $\Gamma_{2.1}, \dots, \Gamma_{2.7}$. Only three of them have rank 3, as it is easily checked by applying the Reidemeister-Schreier rewriting process. In this way, one gets that the rank 3 kernels $\Theta = \langle X\rangle * \langle Y\rangle * \langle Z\rangle$ are

$$\Gamma_{2.1} = \langle R_0\rangle * \langle R_1\rangle * \langle R_1^{R_2}\rangle\,, \quad \Gamma_{2.4} = \langle R_1\rangle * \langle R_2\rangle * \langle R_1^{R_0}\rangle \quad \text{and}$$
$$\Gamma_{2.5} = \langle R_1\rangle * \langle R_0R_2\rangle * \langle R_1^{R_0}\rangle\,.$$

These three groups are isomorphic to the free product $C_2 * C_2 * C_2$ and therefore isomorphic to Δ. The remaining four epimorphisms have kernels

$$\Gamma_{2.2} = \langle R_0, R_2\rangle * \langle R_0^{R_1}, R_2^{R_1}\rangle\,, \quad \Gamma_{2.3} = \langle R_0\rangle * \langle R_1R_2\rangle\,, \quad \Gamma_{2.6} = \langle R_2\rangle * \langle R_0R_1\rangle \quad \text{and}$$
$$\Gamma_{2.7} = \langle R_0R_2\rangle * \langle R_1R_2\rangle\,.$$

The group $\Gamma_{2.2}$ has rank 4 and is isomorphic to the free product $D_2 * D_2$, while the other three groups $\Gamma_{2.3}$, $\Gamma_{2.6}$ and $\Gamma_{2.7}$ have rank 2 and are all isomorphic to $C_2 * C_\infty$.

(2) $n = 4$: Up to an automorphism of $G = C_2 \times C_2$ there are seven epimorphisms from Γ to G with kernels $\Gamma_{4.1}, \dots, \Gamma_{4.7}$. One can check that three of them have rank 3, namely

$$\Gamma_{4.1} = \langle R_0\rangle * \langle R_0^{R_1}\rangle * \langle (R_1R_2)^2\rangle\,, \quad \Gamma_{4.4} = \langle R_2\rangle * \langle R_2^{R_1}\rangle * \langle (R_0R_1)^2\rangle \quad \text{and}$$
$$\Gamma_{4.5} = \langle R_0R_2\rangle * \langle (R_0R_1)^2\rangle * \langle (R_0R_2)^{R_1}\rangle\,.$$

These groups are all isomorphic to the free product $C_2 * C_2 * C_\infty$ so that Δ is an epimorphic image of each of them.

Remark 4.3. $\Gamma_{4.1} = \Gamma_{2.3} \cap \Gamma_{2.2} = \Gamma_{2.3} \cap \Gamma_{2.1} = \Gamma_{2.2} \cap \Gamma_{2.1} = \Gamma_{2.3} \cap \Gamma_{2.2} \cap \Gamma_{2.1}\,,$
$\Gamma_{4.4} = \Gamma_{2.4} \cap \Gamma_{2.2} = \Gamma_{2.4} \cap \Gamma_{2.6} = \Gamma_{2.2} \cap \Gamma_{2.6} = \Gamma_{2.4} \cap \Gamma_{2.2} \cap \Gamma_{2.6}\,,$
$\Gamma_{4.5} = \Gamma_{2.7} \cap \Gamma_{2.5} = \Gamma_{2.7} \cap \Gamma_{2.2} = \Gamma_{2.5} \cap \Gamma_{2.2} = \Gamma_{2.7} \cap \Gamma_{2.5} \cap \Gamma_{2.2}\,.$

(3) $n = 8$, $G = D_4$: Up to an automorphism of G there are six epimorphism from Γ to G with kernels $\Gamma_{8.1}, \dots, \Gamma_{8.6}$. Three of them have rank 3 and are all free groups, namely

$$\Gamma_{8.4} = \langle R_0R_1R_2R_1\rangle * \langle R_1R_0R_1R_2\rangle * \langle (R_0R_1)^2R_0R_2\rangle\,,$$
$$\Gamma_{8.5} = \langle (R_1R_2)^2\rangle * \langle R_2(R_1R_0)^2\rangle * \langle R_2(R_0R_1)^2\rangle \quad \text{and}$$
$$\Gamma_{8.6} = \langle (R_0R_1)^2\rangle * \langle R_0(R_1R_2)^2\rangle * \langle R_0(R_2R_1)^2\rangle\,.$$

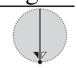
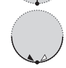
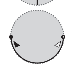
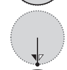
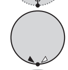
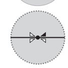
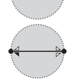
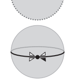
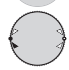
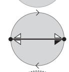
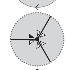
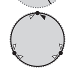
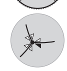
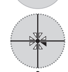
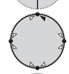
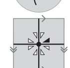
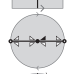
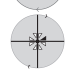
Remark 4.4. $\Gamma_{4.5}$ is the unique normal subgroup of index 4 containing $\Gamma_{8.4}$, while $\Gamma_{4.4}$ is the unique normal subgroup of index 4 containing $\Gamma_{8.5}$ and $\Gamma_{4.1}$ is the unique normal subgroup of index 4 containing $\Gamma_{8.6}$.

(4) $n = 8$, $G = C_2 \times C_2 \times C_2$: Up to an automorphism of G there is only one epimorphism from Γ to G with kernel isomorphic to the rank 3 free group $C_\infty * C_\infty * C_\infty$, namely

$$\Gamma_{8.7} = \langle (R_0R_1)^2\rangle * \langle (R_1R_2)^2\rangle * \langle R_0(R_1R_2)^2R_0\rangle\,.$$

Remark 4.5. $\Gamma_{8.7} = \Gamma_{4.i} \cap \Gamma_{4.j}$ for any distinct $i, j \in \{1, \dots, 7\}$.

The following table gives a overall description of Θ and the Θ-trivial map for each normal subgroup Θ of Γ of index $2, 4, 6$ and 8.

Θ	index	rank	Free-Product dec.	Type of $\mathcal{T}_\Theta$	surface	χ	fig
$\Gamma_{2.1}$	2	3	$C_2 * C_2 * C_2$	(2,2,1)	border	1	
$\Gamma_{2.2}$	2	4	$D_2 * D_2$	(2,1,2)	border	1	
$\Gamma_{2.3}$	2	2	$C_2 * C_\infty$	(1,2,2)	border	1	
$\Gamma_{2.4}$	2	3	$C_2 * C_2 * C_2$	(1,2,2)	border	1	
$\Gamma_{2.5}$	2	3	$C_2 * C_2 * C_2$	(2,1,2)	border	1	
$\Gamma_{2.6}$	2	2	$C_2 * C_\infty$	(2,2,1)	border	1	
$\Gamma_{2.7}$	2	2	$C_2 * C_\infty$	(1,1,1)	orient.	2	
$\Gamma_{4.1}$	4	3	$C_2 * C_2 * C_\infty$	(2,2,2)	border	1	
$\Gamma_{4.2}$	4	4	$C_2 * C_2 * C_2 * C_2$	(2,2,2)	border	1	
$\Gamma_{4.3}$	4	2	$C_\infty * C_\infty$	(2,2,1)	orient.	2	
$\Gamma_{4.4}$	4	3	$C_2 * C_2 * C_\infty$	(2,2,2)	border	1	
$\Gamma_{4.5}$	4	3	$C_2 * C_2 * C_\infty$	(2,1,2)	orient.	2	
$\Gamma_{4.6}$	4	2	$C_\infty * C_\infty$	(1,2,2)	orient.	2	
$\Gamma_{4.7}$	4	2	$C_\infty * C_\infty$	(2,2,2)	nonori.	1	
$\Gamma_{6.1}$	6	4	$C_2 * C_2 * C_2 * C_\infty$	(3,2,2)	border	1	
$\Gamma_{6.2}$	6	4	$C_2 * C_2 * C_2 * C_\infty$	(2,2,3)	border	1	
$\Gamma_{6.3}$	6	4	$C_2 * C_2 * C_2 * C_\infty$	(3,1,3)	orient.	2	
$\Gamma_{8.1}$	8	5	$C_2 * C_2 * C_2 * C_2 * C_\infty$	(4,2,2)	border	1	
$\Gamma_{8.2}$	8	5	$C_2 * C_2 * C_2 * C_2 * C_\infty$	(2,2,4)	border	1	
$\Gamma_{8.3}$	8	5	$C_2 * C_2 * C_2 * C_\infty * C_\infty$	(4,1,4)	orient.	2	
$\Gamma_{8.4}$	8	3	$C_\infty * C_\infty * C_\infty$	(4,2,4)	orient.	0	
$\Gamma_{8.5}$	8	3	$C_\infty * C_\infty * C_\infty$	(2,2,4)	nonori.	1	
$\Gamma_{8.6}$	8	3	$C_\infty * C_\infty * C_\infty$	(4,2,2)	nonori.	1	

$\Gamma_{8.7}$	8	3	$C_\infty * C_\infty * C_\infty$	(2,2,2)	orient.	2	

Table 1: Normal subgroups of indices $2, 4, 6$ and 8 in $\Gamma = C_2 * D_2$.

5 Description of the thin map representations

In the previous section we computed all rank 3 normal subgroups $\Theta = \Gamma_{i.j}$ together with a set of generators $\{X_1, X_2, X_3\}$ such that $\Gamma_{i.j} = \langle X_1 \rangle * \langle X_2 \rangle * \langle X_3 \rangle$ is a thin representation $Ri.j$. Since some $\Gamma_{i.j}$ gives rise to more than one thin representation, we label the corresponding representations by $Ri.ja$, $Ri.jb$, etc. Note that the generators of a thin representation can be read out as fundamental group generators (written as words on $\{R_0, R_1, R_2\}$) from the respective trivial map (Section 4). The classification is done up to a restrictedly dual, that is, the generators of a Θ-marked representation are computed up to the usual map dual if its restriction to Θ is an automorphism of Θ (see also Remark below). The following table gives all the thin Θ-marked representations. Generators of $\Gamma_{i.j}$ which are involutions will be denoted by A, B, C and those which are not will be denoted by X, Y, Z.

Remark 5.1. The assignments

$$\begin{aligned} &R_0 \mapsto R_2\,, && R_1 \mapsto R_1\,, && R_2 \mapsto R_0 \quad \text{and} \\ &R_0 \mapsto R_0R_2\,, && R_1 \mapsto R_1\,, && R_2 \mapsto R_2 \end{aligned}$$

extend to automorphisms of Γ and give rise to the map dualities D (the usual map duality) and P (the Petrie duality). Together they generate the outer automorphism group $Out(\Gamma) = \langle D, P \rangle \cong S_3$. The following diagram graphically pictures the action of $Out(\Gamma)$ on the set of rank 3 normal subgroups of Γ, where lines and dash lines represent the action of D and P, respectively. Note that D, or P, fixes some Θ and therefore for those Θ's it is a Θ-

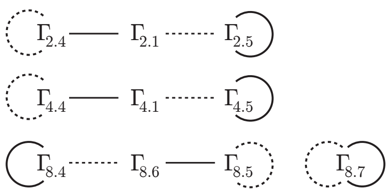


Figure 2: The actions of D and P on the Θ's.

restrictedly duality. The Petrie duality is not a thin-preserving duality except in the case of $\Gamma_{8.7}$; here $R8.7a$ and $R8.7b$ are Petrie duals of each other. The duality D fixes $\Gamma_{2.5}$, $\Gamma_{4.5}$, $\Gamma_{8.4}$ and $\Gamma_{8.7}$. These give rise to the restrictedly-dual representations given in the following Table, but not listed in Table 2.

To illustrate each thin representation, we exhibit the Θ-marked map representation of the toroidal regular hypermap $\mathcal{H}$ pictured in Figure 3 using the James hypermap representation [4], where hypervertices, hyperedges and hyperfaces of $\mathcal{H}$ are represented by simply connected regions colored grey, dotted and white, respectively, and flags are the numbered points. We note that Lynne James hypermap representation is actually the $\Gamma_{6.1}$-marked map representation sending $(R_2R_1)^3$ to 1. Here $\Gamma_{6.1}$ is the normal subgroup of index 6 of

Γ isomorphic to $C_2 * C_2 * C_2 * C_\infty$ generated by R_0, $R_0^{R_1}$, $R_0^{R_1 R_2}$ and $(R_2R_1)^3$ (given in Table 1). This representation is not listed in Table 2 because this restrictedly marked representation is not thin (it is not even clean).

From Figure 3, $\mathcal{H} = (F; r_0, r_1, r_2)$ with $F = \{1, \ldots, 6\}$ and, up to permutation (coloring),

$$r_0 = (1,4)(2,5)(3,6)\,, \quad r_1 = (1,2)(3,4)(5,6)\,, \quad r_2 = (1,6)(2,3)(4,5)\,.$$

The hypermap $\mathcal{H}$ has one hypervertex, one hyperedge and one hyperface all of valency 3. The monodromy group of $\mathcal{H}$ is $G = \langle r_0, r_1, r_2\rangle \cong S_3$. The Euler characteristic of a map representation of $\mathcal{H}$ is given by (2.1) taking into account the Θ-trivial map given in Table 2 and using the isomorphism $\widetilde{\rho}$ given in 3.1.

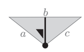
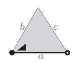
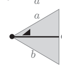
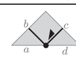
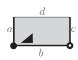
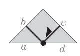
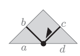
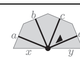
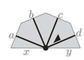
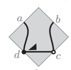
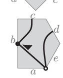
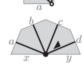
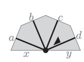
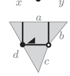
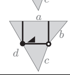
#	Rep.	Generators			Epim.	Θ-slice
1	$R2.1$	$A{=}R_0$	$B{=}R_1$	$C{=}R_2R_1R_2$	$A{\to}r_0$ $B{\to}r_1$ $C{\to}r_2$	
2	$R2.4$	$A{=}R_0R_1R_0$	$B{=}R_1$	$C{=}R_2$	$A{\to}r_0$ $B{\to}r_1$ $C{\to}r_2$	
3	$R2.5$	$A{=}R_0R_1R_0$	$B{=}R_1$	$C{=}R_0R_2$	$A{\to}r_0$ $B{\to}r_1$ $C{\to}r_2$	
4	$R4.1$	$A{=}R_0$	$B{=}R_1R_0R_1$	$X{=}R_1R_2R_1R_2$	$A{\to}r_0$ $B{\to}r_1$ $X{\to}r_2$	
5	$R4.4$	$A{=}R_2$	$B{=}R_1R_2R_1$	$Z{=}R_0R_1R_0R_1$	$A{\to}r_2$ $B{\to}r_1$ $Z{\to}r_0$	
6	$R4.5a$	$A{=}R_0R_2$	$B{=}R_1R_0R_2R_1$	$Z{=}R_0R_1R_0R_1$	$A{\to}r_0$ $B{\to}r_2$ $Z{\to}r_1$	
7	$R4.5b$	$A{=}R_0R_2$	$B{=}R_1R_0R_2R_1$	$X{=}R_0R_1R_2R_1$	$A{\to}r_0$ $B{\to}r_2$ $X{\to}r_1$	
8	$R8.4a$	$X{=}R_0R_1R_2R_1$	$Y{=}R_1R_0R_1R_2$	$Z{=}R_0R_1R_0R_1R_0R_2$	$X{\to}r_0$ $Y{\to}r_1$ $Z{\to}r_2$	
9	$R8.4b$	$X{=}R_0R_1R_2R_1$	$Y{=}R_1R_0R_1R_2$	$Z{=}R_0R_2R_1R_0R_2R_1$	$X{\to}r_0$ $Y{\to}r_1$ $Z{\to}r_2$	
10	$R8.5a$	$X{=}R_1R_2R_1R_2$	$Y{=}R_0R_1R_0R_1R_2$	$Z{=}R_0R_2R_1R_0R_1$	$X{\to}r_0$ $Y{\to}r_1$ $Z{\to}r_2$	
11	$R8.5b$	$X{=}R_1R_2R_1R_2$	$Y{=}R_0R_1R_0R_2R_1$	$Z{=}R_0R_2R_1R_0R_1$	$X{\to}r_0$ $Y{\to}r_1$ $Z{\to}r_2$	
12	$R8.6a$	$X{=}R_0R_1R_2R_1R_2$	$Y{=}R_0R_1R_0R_1$	$Z{=}R_0R_2R_1R_2R_1$	$X{\to}r_1$ $Y{\to}r_0$ $Z{\to}r_2$	
13	$R8.6b$	$X{=}R_0R_2R_1R_2R_1$	$Y{=}R_0R_1R_0R_1$	$Z{=}R_1R_0R_2R_1R_2$	$X{\to}r_1$ $Y{\to}r_0$ $Z{\to}r_2$	
14	$R8.7a$	$X{=}R_1R_2R_1R_2$	$Y{=}R_0R_1R_2R_1R_2R_0$	$Z{=}R_0R_1R_0R_1$	$X{\to}r_1$ $Y{\to}r_2$ $Z{\to}r_0$	
15	$R8.7b$	$X{=}R_1R_2R_1R_2$	$Y{=}R_0R_2R_1R_2R_0R_1$	$Z{=}R_0R_1R_0R_1$	$X{\to}r_1$ $Y{\to}r_2$ $Z{\to}r_0$	

Table 2: The 15 thin representations.

Θ-dual of Rep.	Generators		
$R2.5$	$A = R_2R_1R_2$	$B = R_1$	$C = R_0R_2$
$R4.5a$	$A = R_0R_2$	$B = R_1R_0R_2R_1$	$Z = R_2R_1R_2R_1$
$R4.5b$	$A = R_0R_2$	$B = R_1R_0R_2R_1$	$Z = R_2R_1R_0R_1$
$R8.4a$	$X = R_2R_1R_0R_1$	$Y = R_1R_2R_1R_0$	$Z = R_2R_1R_2R_1R_0R_2$
$R8.7a$	$X = R_1R_0R_1R_0$	$Y = R_0R_2R_1R_2R_0R_1$	$Z = R_2R_1R_2R_1$

Table 3: The dual representations.

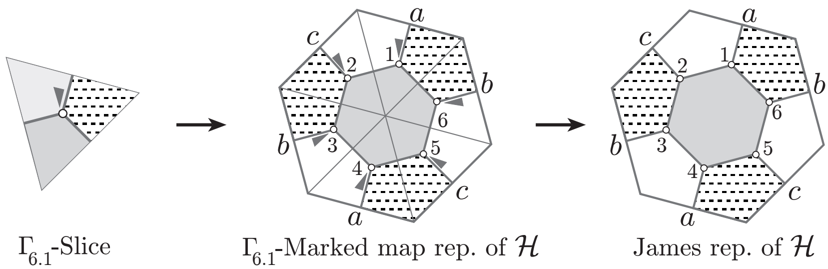


Figure 3: The toroidal regular hypermap $\mathcal{H}$.

As an example, we give a detailed construction of the thin representation $R4.1$ of $\mathcal{H}$ following the generic description given in [1]:
The words R_0, $R_0^{R_1}$ and $(R_1R_2)^2$, in this order, generate the subgroup $\Theta = \Gamma_{4.1}$ as a free product $C_2 * C_2 * C_\infty$ (Table 2). A rooted Θ-slice can be obtained from a Schreier transversal of Θ in Γ, or alternatively by a cut-opening of the trivial Θ-map (see Table 1). The rooted Θ-slice we are taking here is the one given by the Schreier transversal $\{1, R_1, R_2, R_1R_2\}$. Another Schreier transversal may lead to a different rooted Θ-

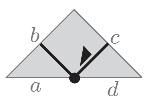
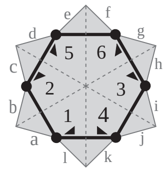


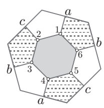
Figure 4: The rooted $\Gamma_{4.1}$-slice.

slice, and a choice of another flag as root corresponds to take another Schreier transversal, and both will lead to “similar” Θ-marked maps, in the sense that the underlying map is the same. The Θ-marked map representation of $\mathcal{H}$ is obtained by the isomorphism $\rho\colon \Theta/H\rho^{-1} \to \Delta/H$ given by $R_0 \mapsto r_0$, $R_0^{R_1} \mapsto r_1$ and $(R_1R_2)^2 \mapsto r_2$. So we have $R_0 = (1,4)(2,5)(3,6)$, $R_0^{R_1} = (1,2)(3,4)(5,6)$ and $(R_1R_2)^2 = (1,6)(2,3)(4,5)$. Now we take 6 rooted Θ-slices labelled $1, 2, 3, 4, 5$ and 6 and join them through their sides a, b, c and d accordingly to the action of the words R_0, $R_0^{R_1}$ and $(R_1R_2)^2$ on the root flag of the slices. In this way, the word R_0 joins the slices 1 and 4, 2 and 5, and 3 and 6, by their sides labelled c, while $R_0^{R_1}$ joins the slices 1 and $2, 3$ and 4, and 5 and 6, by their sides b. This leaves to an incomplete picture:



Now $(R_1R_2)^2$, which is an involution, says that the slices 1 and $6, 2$ and 3, and 4 and 5, are joined together through their sides a and d, that is, in the picture above we have the following equality between labels: $g = a$ and $f = l$, $h = b$ and $i = c$, and $d = j$ and $k = e$. This lead to the final picture of $R4.1$ in Table 5.

In the following tables we illustrate the fifteen map representations Rep of the toroidal regular hypermap $\mathcal{H}$, we display the general Euler's characteristic formula for the map representation Rep of any hypermap, the actual Euler's characteristic of $\mathrm{Rep}(\mathcal{H})$ and the orientability (up to restricted dual) of $\mathrm{Rep}(\mathcal{H})$ - and when possible we record their overall orientability behaviour in parenthesis.

$\mathcal{H} =$  Euler characteristic of $\mathcal{H} = 0$ oriented

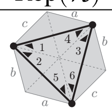
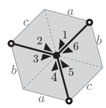
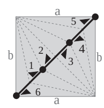
Rep	Rep($\mathcal{H}$)	Euler characteristic of Rep($\mathcal{H}$)	orient.
$R2.1$		$\lvert G\rvert\left(\frac{1}{2\lvert AB\rvert}+\frac{1}{2\lvert BC\rvert}+\frac{1}{2\lvert CA\rvert}-\frac{1}{2}\right)=0$	yes
$R2.4$		$\lvert G\rvert\left(\frac{1}{2\lvert AB\rvert}+\frac{1}{2\lvert BC\rvert}+\frac{1}{2\lvert CA\rvert}-\frac{1}{2}\right)=0$	yes
$R2.5$		$\lvert G\rvert\left(\frac{1}{2\lvert BCAC\rvert}+\frac{1}{2\lvert BA\rvert}-\frac{1}{2}\right)=1$	no

Table 4: The Θ-marked map representations of $\mathcal{H}$ for $|\Gamma : \Theta| = 2$.

We discuss now orientability in more details. The first two thin representations $R2.1$ and $R2.4$ (Vince and Walsh representations) are the unique orientation-preserving representations, that is, if they are orientable they represent orientable hypermaps and if they are nonorientable they represent nonorientable hypermaps. However, the maps coming out from the representations $R4.5a$, $R4.5b$, $R8.4a$, $R8.4b$, $R8.7a$ and $R8.7b$ are always orientable, since the Θ-trivial maps for $\Theta \in \{\Gamma_{4.5}, \Gamma_{8.4}, \Gamma_{8.7}\}$ are orientable. This poses the question: when they represent non-orientable hypermaps? The same question hang over the other representations with an additional hitch, both orientable and non-orientable maps can represent orientable and nonorientable hypermaps. This means that for these representations we no longer have the clue given by $R2.1$ and $R2.4$, and for this reason we need to make a local teste. In general, a Θ-marked map representation $\mathcal{M} = (\Omega; x_1, x_2, x_3)$

is a representation of an orientable hypermap if and only if x_1x_2 and x_2x_3 act on the set of Θ-slices with two orbits (Θ-orbits). As a hypermap $\mathcal{H}$ is orientable if and only if $\mathcal{H}$ covers the orientably-trivial hypermap $\mathcal{T}^+_{\mathcal{H}}$ (Figure 5), a thin map representation Rep($\mathcal{H}$) of $\mathcal{H}$ represents an orientable hypermap if and only if Rep($\mathcal{H}$) covers the corresponding representation Rep($\mathcal{T}^+_{\mathcal{H}}$) of the orientably-trivial hypermap, call this representation *RoriT-map*. In the cases of $R2.1$ and $R2.4$, the RoriT-map is spherical and so for any hypermap

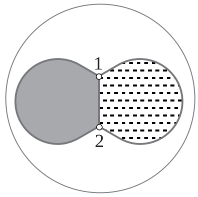
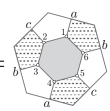


Figure 5: The orientably-trivial hypermap $\mathcal{T}^+_{\mathcal{H}}$.

$\mathcal{H}$ the representations $R2.1(\mathcal{H})$ and $R2.4(\mathcal{H})$ are orientable if and only if $\mathcal{H}$ is orientable. For the other cases, and specially the cases in which the representation R is always orientable ($R4.5a$, $R4.5b$, $R8.4a$, $R8.4b$, $R8.7a$, $R8.7b$), the representation $\mathcal{M} = R(\mathcal{H})$ is an orientable hypermap $\mathcal{H}$ if $\mathcal{M}$ covers the respective RoriT-map. Any RoriT-map has two flags, so a Θ-marked map representation $\mathcal{M} = (\Omega; x_1, x_2, x_3)$ is a representation of an orientable hypermap if and only if the triple (x_1, x_2, x_3) induce a two blocks system on the set of Θ-slices Ω (the two Θ-orbits) such that each x_i permutes the two blocks exactly as the permutation $\overline{x}_i$ of the two flags of the RoriT-map does. That is, x_i exchanges the two blocks if and only if $\overline{x}_i$ exchanges the two flags.

$\mathcal{H} =$  Euler characteristic of $\mathcal{H} = 0$ oriented

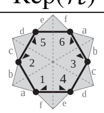
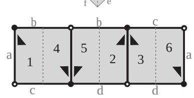
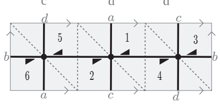
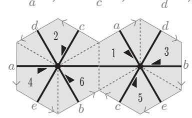
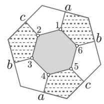
Rep	Rep($\mathcal{H}$)	Euler characteristic of Rep($\mathcal{H}$)	orient.
$R4.1$		$\lvert G\rvert \left(\frac{1}{2\lvert AB\rvert} + \frac{1}{2\lvert AXBX\rvert} - \frac{1}{2}\right) = 1$	no
$R4.4$		$\lvert G\rvert \left(\frac{1}{2\lvert BA\rvert} + \frac{1}{2\lvert AZBZ\rvert} - \frac{1}{2}\right) = 1$	no
$R4.5a$		$\lvert G\rvert \left(\frac{1}{\lvert AZB\rvert} - \frac{1}{2}\right) = 0$	yes (always)
$R4.5b$		$\lvert G\rvert \left(\frac{1}{\lvert AX\rvert} + \frac{1}{\lvert XB\rvert} - 1\right) = -2$	yes (always)

Table 5: The Θ-marked map representations of $\mathcal{H}$ for $\lvert\Gamma : \Theta\rvert = 4$.

$\mathcal{H} =$  Euler characteristic of $\mathcal{H} = 0$ oriented

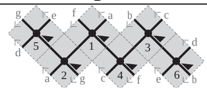
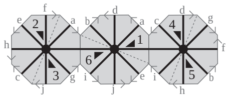
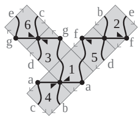
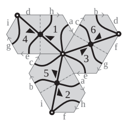
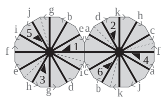
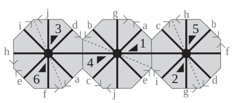
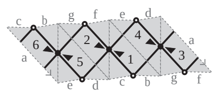
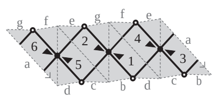
Rep	Rep($\mathcal{H}$)	Euler's charac. form. on Rep($\mathcal{H}$)	orient.
$R8.4a$		$\vert G\vert\left(\frac{1}{\vert XZXY\vert}+\frac{1}{\vert YZ\vert}-2\right)=-4$	yes (always)
$R8.4b$		$\vert G\vert\left(\frac{1}{\vert XZY\vert}+\frac{1}{\vert ZXY\vert}-2\right)=-6$	yes (always)
$R8.5a$		$\vert G\vert\left(\frac{1}{\vert YXZ\vert}+\frac{1}{\vert ZY\vert}-\frac{3}{2}\right)=-4$	no
$R8.5b$		$\vert G\vert\left(\frac{1}{\vert YZ\vert}+\frac{1}{\vert YXZ\vert}-\frac{3}{2}\right)=-4$	no
$R8.6a$		$\vert G\vert\left(\frac{1}{\vert XZ\vert}+\frac{1}{\vert ZYX\vert}-\frac{3}{2}\right)=-4$	no
$R8.6b$		$\vert G\vert\left(\frac{1}{\vert XYZ\vert}+\frac{1}{\vert ZX\vert}-\frac{3}{2}\right)=-4$	no
$R8.7a$		$\vert G\vert\left(\frac{1}{\vert YZX\vert}-\frac{1}{2}\right)=0$	yes (always)
$R8.7b$		$\vert G\vert\left(\frac{1}{\vert XY\vert}+\frac{1}{\vert YZ\vert}-1\right)=-2$	yes (always)

Table 6: The Θ-marked map representations of $\mathcal{H}$ for $|\Gamma : \Theta| = 8$.

Take for example the two map representations (always orientable) given by $R4.5a$ and $R4.5b$ on the non-orientable hypermap $\mathcal{H}$ pictured in Figure 6 left, a non-regular, but uniform of type $(3,3,3)$, the 6 flags hypermap with monodromy group generated by

$$r_0 = (1,5)(2,4)(3,6)\,, \quad r_1 = (1,2)(3,4)(5,6)\,, \quad r_2 = (1,6)(2,3)(4,5)\,.$$

It is simple to see that $R4.5a(\mathcal{H})$ represents a non-orientable hypermap because by having two vertices of valency 2 the map $R4.5a(\mathcal{H})$ does not cover the uniform (regular) toroidal map $R4.5a(\mathcal{T}^+_{\mathcal{H}})$ of type $\{4,4\}$.

For the case $R4.5b(\mathcal{H})$, the argument is not so simple as before because this map is uniform of type $\{6,6\}$ and the trivial oriented map $R4.5b(\mathcal{T}^+_{\mathcal{H}})$ is also uniform of type $\{2,2\}$. However, the word $AXB = R_0R_2R_0R_1R_0R_1$ fix the root flag 1 in the map $R4.5b(\mathcal{H})$, but does not fix any flag on the RoriT-map $R4.5b(\mathcal{T}^+_{\mathcal{H}})$.

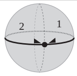
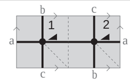
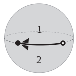
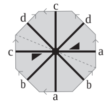
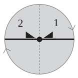
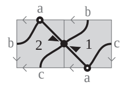
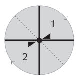
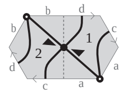
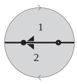
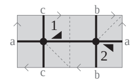
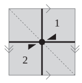
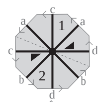
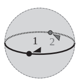
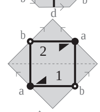
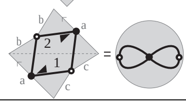
Rep	Rep($\mathcal{T}^+_{\mathcal{H}}$)	Rep	Rep($\mathcal{T}^+_{\mathcal{H}}$)
$R2.1$		$R8.4a$	
$R2.4$		$R8.4b$	
$R2.5$		$R8.5a$	
$R4.1$		$R8.5b$	
$R4.4$		$R8.6a$	
$R4.5a$		$R8.6b$	
$R4.5b$		$R8.7a$	
		$R8.7b$	

Table 7: The 15 RoriT-maps (thin representations of the orientably-trivial hypermap).

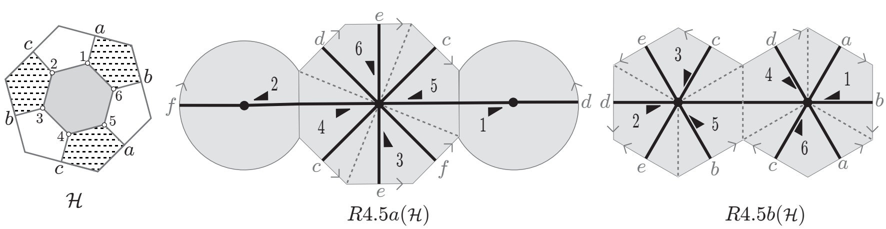


Figure 6: A non-orientable hypermap $\mathcal{H}$ and its representations $R4.5a(\mathcal{H})$ and $R4.5b(\mathcal{H})$.

Alternatively, following the block system argument described above, painting by red and blue the possible two blocks, we have for $R4.5a(\mathcal{T}^+_{\mathcal{H}})$

$$1(red) \xrightarrow{A} 5(blue) \xrightarrow{Z} 6(red) \xrightarrow{B} 1(blue)$$

and for $R4.5b(\mathcal{T}^+_{\mathcal{H}})$

$$1(red) \xrightarrow{A} 5(blue) \xrightarrow{X} 6(red) \xrightarrow{B} 1(blue)$$

In both cases such two block system does not exist.

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The covering lemma and q-analogues of extremal set theory problems

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Abstract

We prove a general lemma (inspired by a lemma of Holroyd and Talbot) about the connection of the largest cardinalities (or weight) of structures satisfying some hereditary property and substructures satisfying the same hereditary property. We use it to show how results concerning forbidden subposet problems in the Boolean poset imply analogous results in the poset of subspaces of a finite vector space. We also study generalized forbidden subposet problems in the poset of subspaces.

Keywords: Subspace lattice, forbidden subposet, covering, profile polytope.

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1 Introduction

One of the most basic questions in extremal finite set theory is the following. Given a property of families of subsets of an n-element set set, what is the cardinality of the largest family satisfying it? Sperner [29] showed that for the property that no member of the family contains another member (in other words: the family is an *antichain*), the answer is $\binom{n}{\lfloor n/2 \rfloor}$. This cardinality is realized by the family of all the $\lfloor n/2 \rfloor$-element subsets.

Our underlying set is $[n] := \{1, 2, \ldots, n\}$. We denote the family of all its subsets by $2^{[n]}$. This family together with the containment relation forms the *Boolean lattice* and is denoted by $\mathcal{B}_n$. The family of all i-element subsets of $[n]$ is called *level* i and is denoted by $\binom{[n]}{i}$. Let $\Sigma(n, k)$ denote the cardinality of the largest k levels (i.e. the middle k levels) of $\mathcal{B}_n$. More precisely, $\Sigma(n, k) = \sum_{i=1}^{k} \binom{n}{\lfloor \frac{n-k}{2} \rfloor + i}$.

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To generalize Sperner's theorem, Katona and Tarján [24] initiated the study of properties given by forbidding inclusion patterns. More precisely, let P be a finite poset. We say that a family $\mathcal{F} \subset 2^{[n]}$ *(weakly) contains* P if there is an order-preserving injection $f\colon P \to \mathcal{F}$, i.e., an injection such that if $x <_P y$, then $f(x) \subset f(y)$. Otherwise $\mathcal{F}$ is P-free. Let $La(n, P)$ denote the size of the largest P-free family $\mathcal{F}$ of $\mathcal{B}_n$. We say that a poset is a *chain* if its members pairwise contain each other The chain of k elements is said to have *size* k and is denoted by P_k. A chain in $\mathcal{B}_n$ is called a *full chain* if it has $n + 1$ members (thus one from each level).

Let us denote by $e(P)$ the largest integer m such that for any n, any family $\mathcal{F}$ of $\mathcal{B}_n$ that consists of m *consecutive* levels is P-free. Every result in this area suggests that the following might hold.

Conjecture 1.1. *For any integer* n *and poset* P*, we have* $La(n, P) = (1+o(1))\Sigma(n, e(P)) = (e(P) + o(1))\binom{n}{\lfloor n/2 \rfloor}$.

This conjecture was first stated by Griggs and Lu [19] and by Bukh [2], although it was already widely believed in the extremal finite set theory community. For a survey on forbidden subposet problems see [18].

Another basic type of extremal finite set theory problems is related to intersection patterns. We say that a family $\mathcal{F}$ is *intersecting* if any two members of it share at least one element. Erdős, Ko and Rado [6] proved that if $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting and $n \geq 2k$, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. For a treatment of several kinds of extremal finite set theory questions, see [16].

A variant of the basic question arises when we are given a weight function (in addition to a property) and we want to determine the largest weight of a family satisfying the property. The most usual version is when the weight of a family is the sum of the weights of its members, and the weight of a subset of $[n]$ depends only on its size. For example the celebrated LYM inequality [25, 26, 31] states that for any antichain $\mathcal{F} \subset 2^{[n]}$, we have $\sum_{F \in \mathcal{F}} 1/\binom{n}{|F|} \leq 1$.

A method to handle together all the weights of the above kind was introduced by P. L. Erdős, Frankl and Katona [7]. The *profile vector* of a family $\mathcal{F}$ is $\underline{p}(\mathcal{F}) = (f_0, \ldots, f_n)$, where $f_i = |\mathcal{F} \cap \binom{[n]}{i}|$. The weight vector corresponding to a weight function is $\underline{w} = (w_0, \ldots, w_n)$, where w_i is the weight of an i-element set. Then the weight of $\mathcal{F}$ is the scalar product of the profile vector and the weight vector. For a property T and a positive integer n, there is a set of profile vectors in the $(n + 1)$-dimensional Euclidean space corresponding to the families with property T. It is well-known that the scalar product is maximized at one of the extreme points of the convex hull of the set of profile vectors, which is called the *profile polytope*. The extreme points of the profile polytopes have been since determined for several properties of families, see [5, 10] for most of them.

We say that a property T of families is *hereditary* if for any family $\mathcal{F}$ with property T, every subfamily of $\mathcal{F}$ has property T. It is easy to see that a property is hereditary if and only if it can be defined by some forbidden substructures, like all the properties considered above. We remark that in the case of hereditary properties, we can assume that all the coordinates of weight functions are non-negative, as we could simply delete the sets of negative weights anyway. Regarding the extreme points, it means that we can obtain all the extreme points by changing to zero some coordinates of those extreme points that maximize the non-negative weight functions.

Forbidden subposet problems can be studied in any poset, and intersection problems can also be studied in structures other than the Boolean poset. A structure where both have been studied is the lattice of subspaces. Let q be a prime power, $\mathbb{F}_q$ be a field of order q and $\mathbb{F}_q^n$ be a vector space of dimension n over $\mathbb{F}_q$. Let $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\dots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\dots(q-1)}$ be the Gaussian (q-nomial) coefficient. It is well-known that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the number of k-dimensional subspaces in $\mathbb{F}_q^n$. Let us denote by $\mathcal{L}_n(q)$ the lattice of subspaces with the containment relation. We also say that the k-dimensional subspaces form level k. The family of all k-dimensional subspaces is called *level k* of $\mathcal{L}_n(q)$.

We are going to consider analogues of extremal finite set theory questions, where i-element subsets of $[n]$ are replaced by i-dimensional subspaces of $\mathbb{F}_q^n$. We say that two subspaces *intersect* if their intersection is more than just the zero vector, i.e. there is a 1-dimensional subspace contained in both. Hsieh [22] proved an analogue of the Erdős-Ko-Rado theorem by showing that an intersecting family of k-dimensional subspaces has cardinality at most $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$, provided $n > 2k$. Greene and Kleitman [17] extended it to the case $n = 2k$. The analogue of Sperner's theorem is also well-known (see [5]). Profile polytopes were studied in this setting in [15].

Recently, other forbidden subposet problems have been examined in $\mathcal{L}_n(q)$ [27, 28]. Let $La_q(n, P)$ denote the largest number of members of a P-free family in $\mathcal{L}_n(q)$. Analogously to the Boolean case, we can define $e_q(P)$ to be the largest integer such that the union of the middle $e_q(P)$ levels of $\mathcal{L}_n(q)$ does not contain P for any n, and let $\Sigma_q(n, k) = \sum_{i=1}^{k} \begin{bmatrix} n \\ \lfloor \frac{n-k}{2} \rfloor + i \end{bmatrix}_q$. One might formulate the following.

Conjecture 1.2. *For any integer n and poset P, we have $La_q(n, P) = (1 + o(1))$ $\Sigma_q(n, e_q(P))$.*

Observe that for several posets we have $e_q(P) = e(P)$. Rather than proving results analogous to those known in the Boolean case, the focus of the papers mentioned above is to prove "stronger" results. For example, the *diamond* poset D_2 has four elements with relations $a < b < d$ and $a < c < d$. It is unknown if Conjecture 1.1 holds for this poset. The best upper bound is $La(n, D_2) \leq (2.20711 + o(1))\binom{n}{\lfloor n/2 \rfloor}$ [20]. Sarkis, Shahriari and students [27] obtained, for the analogous question in the lattice of subspaces, the upper bound $La_q(n, D_2) \leq (2 + 1/q)\begin{bmatrix} n \\ \lfloor n/2 \rfloor \end{bmatrix}_q$.

Let $\vee$ be the poset on three elements with relations $a < b$ and $a < c$, and $\wedge$ be the poset on three elements with relations $a < c$ and $b < c$. Katona and Tarján [24] determined $La(n, \{\vee, \wedge\})$, where we forbid $\vee$ and $\wedge$ at the same time. The solution is $\binom{n}{n/2}$ if n is even, but slightly more than $\binom{n}{\lfloor n/2 \rfloor}$ if n is odd. Shahriari and Yu [28] showed that in $\mathcal{L}_n(q)$ we have $La_q(n, \{\vee, \wedge\}) = \begin{bmatrix} n \\ \lfloor n/2 \rfloor \end{bmatrix}_q$ for every prime power q and $n \geq 2$. They also studied the case we forbid a *broom* $\wedge_u$ and a *fork* $\vee_v$ at the same time, where $\wedge_u$ has $u + 1$ elements $a_1, \dots, a_u, b$ and the relations $a_i < b$ for any $i \leq u$, while $\vee_v$ has $v + 1$ elements $a, b_1, \dots, b_v$ and the relations $a < b_i$ for every $i \leq v$.

The *butterfly* poset B has four elements and the relations $a < c$, $a < d$, $b < c$ and $b < d$. De Bonis, Katona and Swanepoel [4] proved $La(n, B) = \Sigma(n, 2)$. Shahriari and Yu [28] proved $La_q(n, B) = \Sigma_q(n, 2)$.

In this paper we state a simple lemma (Lemma 2.1), that generalizes the so-called *permutation method* and explore its consequences. It can be applied to other structures, and in particular for the subspaces it implies the following theorem.

Theorem 1.3. *Let T be a hereditary property. If any family of $\mathcal{B}_n$ with property T has at most $\Sigma(n,k)$ members, then any family in $\mathcal{L}_n(q)$ with property T has at most $\Sigma_q(n,k)$ members.*

This means that the result of De Bonis, Katona and Swanepoel [4] about butterflies implies the result of Shahriari and Yu [28]. Note that Shahriari and Yu also determined the extremal families. They also consider the poset Y_k on $k+2$ elements, defined by the following relations: $c_k < c_{k-1} < \cdots < c_1 < a$ and $c_1 < b$. Let Y_k' defined in the same way but all the relations are reversed. Shahriari and Yu [28] conjectured that $La_q(n,\{Y_k,Y_k'\}) = \Sigma_q(n,k)$; this follows from a result in [13], using Theorem 1.3. We remark that Xiao and Tompkins [30] independently also found the connection between $La(n,P)$ and $La_q(n,P)$ and used it to prove the conjecture of Shahriari and Yu [28].

The asymptotic version of Theorem 1.3 is also true, giving the following result.

Theorem 1.4. *Let T be a hereditary property. If any family of $\mathcal{B}_n$ satisfying T has at most $(1+o(1))\Sigma(n,k)$ members, then any family in $\mathcal{L}_n(q)$ with property T has at most $(1+o(1))\Sigma_q(n,k)$ members.*

Corollary 1.5. *If* Conjecture 1.1 *holds for P and $e_q(P) = e(P)$, then* Conjecture 1.2 *also holds for P.*

To state the Covering Lemma (Lemma 2.1), we need some preparation, hence we postpone it to Section 2. We also describe how it relates to several known proofs. In Section 3 we prove Theorems 1.3 and 1.4. In Section 4 we examine how the Covering Lemma can be modified to apply in the study of profile polytopes and related topics, and we initiate the study of generalized forbidden subposet problems in $\mathcal{L}_n(q)$.

2 The main lemma

Our lemma is motivated by a lemma by Holroyd and Talbot [21]. We say that a family of subsets of S is a *t-covering* family of S if every element of S is contained in exactly t sets of the family. Given a partition of S into $S_0 \cup S_1 \cup \ldots \cup S_n$ and a vector $\underline{t} = (t_0, t_1, \ldots, t_n)$, we say that a family of subsets of S is a *$\underline{t}$-covering* family of S if for each $0 \le i \le n$, every element of S_i is contained in exactly t_i sets of the family.

In our applications, S will be $2^{[n]}$ or the family of subspaces of $\mathbb{F}_q^n$, and S_i will be level i. Holroyd and Talbot [21] considered coverings of subfamilies $\mathcal{F}$ of one level $\binom{[n]}{i}$. Their lemma states that if $\mathcal{F} \subset \binom{[n]}{i}$, Γ is a t-covering family of subfamilies of $\mathcal{F}$, and an element x has the property that the largest intersecting family in every $\mathcal{G} \in \Gamma$ is $\{G \in \mathcal{G} : x \in G\}$, then the largest intersecting family in $\mathcal{F}$ is $\{F \in \mathcal{F} : x \in F\}$. Our main contribution is the simple observation that we can extend their method to other forbidden configurations and more levels.

For a weight vector $\underline{w} = (w_0, \ldots, w_n)$ and a set $F \subset S$, let $\underline{w}(F) = \sum_{i=0}^{n} w_i |F \cap S_i|$. Let $\underline{w/t} = (w_0/t_0, \ldots, w_n/t_n)$. We will always assume that every coordinate of every weight vector is non-negative. A version of the lemma below has already appeared in my master's thesis [9].

Lemma 2.1 (Covering Lemma)**.** *Let T be a hereditary property of subsets of S and Γ be a $\underline{t}$-covering family of S. Assume that there exists a real number x such that for every $G \in \Gamma$, every subset G' of G with property T has $\underline{w/t}(G') \le x$. Then $\underline{w}(F) \le |\Gamma| x$ for every $F \subset S$ with property T.*

Proof. Let F be a set with property T.

Observe that we have $t_i|F \cap S_i| = \sum_{G\in\Gamma}|G \cap F \cap S_i|$, as every element of $F \cap S_i$ is counted t_i times on both sides. Thus we have

$$\underline{w}(F) = \sum_{i=0}^{n} w_i|F \cap S_i| = \sum_{i=0}^{n} \frac{w_i}{t_i} t_i|F \cap S_i| = \sum_{i=0}^{n} \frac{w_i}{t_i} \sum_{G\in\Gamma}|G \cap F \cap S_i|$$
$$= \sum_{G\in\Gamma}\sum_{i=0}^{n} \frac{w_i}{t_i}|G \cap F \cap S_i| = \sum_{G\in\Gamma} \underline{w/t}(G \cap F) \leq \sum_{G\in\Gamma} x = |\Gamma|x. \qquad \square$$

Let us describe how one can use this lemma in extremal finite set theory. Let $S = 2^{[n]}$ and $S_i = \binom{[n]}{i}$. Then the subsets of S are families of $\mathcal{B}_n$, and we will denote them by $\mathcal{F}$ and $\mathcal{G}$ instead of F and G.

The prime examples of covering families where the above lemma is useful are given by the *permutation method*. Given a permutation $\alpha : [n] \to [n]$, and a set $F \subset [n]$, let $\alpha(F) = \{\alpha(i) : i \in F\}$. Similarly, for a family $\mathcal{F}$ of $\mathcal{B}_n$, let $\alpha(\mathcal{F}) = \{\alpha(F) : F \in \mathcal{F}\}$.

Let $\mathcal{G}_0$ be a family of $\mathcal{B}_n$ that has at least one i-element set for every $0 \leq i \leq n$, and let Γ consist of $\alpha(\mathcal{G}_0)$ for all permutations α. Let $g_i = |\mathcal{G}_0 \cap \binom{[n]}{i}| > 0$ and $t_i = g_i i!\,(n-i)!$, then Γ is a $\underline{t}$-covering of $\mathcal{B}_n$.

The simplest example is when $\mathcal{G}_0$ is a full chain. Consider a Sperner family $\mathcal{F}$ of $\mathcal{B}_n$ and let $\underline{w} = \underline{t}$. Then

$$\sum_{F\in\mathcal{F}}|F|!\,(n-|F|)! = \underline{w}(\mathcal{F}) = \sum_{\mathcal{G}\in\Gamma}\sum_{H\in\mathcal{G}\cap\mathcal{F}} \underline{w/t}(H) \leq \sum_{\mathcal{G}\in\Gamma} 1 = |\Gamma| = n!\,.$$

Dividing by $n!$ we obtain the already mentioned LYM-inequality. Another example is when $\mathcal{G}_0$ is the family of intervals in a cyclic ordering of $[n]$, resulting in the cycle method [23].

Any family $\mathcal{G}_0$ of $\mathcal{B}_n$ can be used to give upper bounds on problems in extremal finite set theory, but these bounds are unlikely to be sharp. For that, $\mathcal{G}_0$ has to be very symmetric in a sense. We need that for *every* permutation α, the largest subfamily of $\alpha(\mathcal{G}_0)$ with property T has the same size. Other examples for families $\mathcal{G}_0$ that sometimes give sharp bounds are the chain-pairs [10] and double chains [3].

Let us return to Lemma 2.1 and examine a very special case. Assume that $S_{i_1} \cup \ldots \cup S_{i_k}$ has property T and for every $G \in \Gamma$, $\underline{w/t}(G') = x$ for $G' = G \cap (S_{i_1} \cup \ldots \cup S_{i_k})$ (in the case of the permutation method, it means that the union of k full levels has property T, and the weight inside $\alpha(\mathcal{G}_0)$ is maximized by those k levels). This implies that we have equality in Lemma 2.1.

Now assume that we conjecture that $\underline{w}(\mathcal{F})$ is maximized by a family that is the union of k full levels (among families with property T). Let $\mathcal{H}_0$ be the intersection of those k levels with $\mathcal{G}_0$, then $\mathcal{H}_0$ has property T. If $\mathcal{H}_0$ happens to have the largest weight $\underline{w/t}$ among subfamilies of $\mathcal{G}_0$ with property T, then it proves the conjecture (here we use the simple observation that $\alpha(\mathcal{H}_0)$ would maximize $\underline{w/t}$ among subfamilies of $\alpha(\mathcal{G}_0)$). Thus our goal would be to find $\mathcal{G}_0$ with this property.

For example, in the case of antichains, it is a natural idea to consider a full chain as $\mathcal{G}_0$. Indeed, for every weight, the maximum will be given by a family that consists of one member, which is a full level on the chain. Moreover, it is one of the levels with the largest weight, thus we can choose the same level all the time. This implies that for every weight function, the maximum in the Boolean poset is also given by a full level, giving us not only Sperner's theorem and the LYM inequality, but all the extreme points of the profile polytope, reproving a result in [7]. Moreover, we say that a family is *k-Sperner* if it is P_{k+1}-free. The above argument works for k-Sperner families as well, since on any chain, for any weight, the maximum is given by k full levels. This, again, gives the extreme points of the profile polytope as well, reproving a result in [8].

Observe that we do not need to have full levels in our conjecture to obtain an exact result without further computations. Assume that in our conjecture, for every i, the extremal family $\mathcal{H}$ contains $\gamma_i\binom{n}{i}$ sets from level i, and $\mathcal{H}$ contains a γ_i fraction of the intersection of $\alpha(\mathcal{G}_0)$ and level i. Then the same argument works. For example consider intersecting families on level k, and use the cycle method [23]. We choose a cyclic ordering of the elements of $[n]$ and let $\mathcal{G}_0$ be the family of k-intervals, i.e. k-sets of consecutive elements. There are n such k-sets, and k of them contain a fixed element x. Let $\mathcal{H}$ be the family of k-sets containing x, and $\mathcal{H}_0$ be its intersection with $\mathcal{G}_0$. It is not hard to see that $\mathcal{H}_0$ is the largest intersecting subfamily of $\mathcal{G}_0$ (provided $k \le n/2$). Thus, for every α we have that $\mathcal{H}$ contains a k/n fraction of the members of $\alpha(\mathcal{G}_0)$. As $\mathcal{H}$ contains a k/n fraction of all the sets, we are done.

To finish this section, let us remark that we are mostly interested in the case where every $w_i = 1$. For that $w_i/t_i = 1/(g_i i!\,(n-i)!\,) = \binom{n}{i}/(n!\,g_i)$. In the case where $\mathcal{G}_0$ is a full chain, every g_i is the same. In the case where $\mathcal{G}_0$ is the family of intervals on the cycle, almost every g_i is the same (with the exception of g_0 and g_n). As multiplying with the same number does not change the extremal families, we can consider maximizing the weight function with $w'_i = \binom{n}{i}$ instead (assuming we can deal with the empty set and the full set some other way). If, on the other hand we can deal with the case of constant weight on the chain or the cycle for a property T, and the optimal family consists of the middle levels, then we obtain a LYM-type inequality for subfamilies of $2^{[n]}$ with property T, see for example the case of butterfly-free families in [4].

3 Subspaces

Let us turn our attention to q-analogues. Similarly to the Boolean case and the permutation method, it will again simplify our tasks if all $\mathcal{G} \in \Gamma$ are isomorphic. Moreover, we would prefer to use $\mathcal{G}$ where proving extremal results is either easy or has already been done. Therefore, we will use a subfamily $\mathcal{G}$ of $\mathcal{L}_n(q)$ that is isomorphic to $\mathcal{B}_n$. Choose an arbitrary basis $B = \{v_1, \dots, v_n\}$ of $\mathbb{F}_q^n$, and let $\mathcal{G}_B$ be the family of those subspaces that are generated by a set of these vectors. Obviously the function that maps $H \subset [n]$ to the subspace $\langle v_x : x \in H\rangle$ keeps inclusion and intersection properties. Let Γ be the union over all bases B of the families $\mathcal{G}_B$.

There are $f(q,n) = (q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{n-1})/n!$ ways to choose a basis, as we pick the vectors one by one, and we obtain a basis $n!$ ways. Hence $f(q,n)$ is the cardinality of Γ, which is a $\underline{t}$-covering of $\mathcal{L}_n(q)$ with $t_i = \frac{(q^i-1)\cdots(q^i-q^{i-1})(q^n-q^i)\cdots(q^n-q^{n-1})}{i!(n-i)!}$. Indeed, to count how many times an i-dimensional subspace is covered, we have to pick a basis of the i-dimensional subspace first, and then extend it to a basis of $\mathbb{F}_q^n$. We counted every $\mathcal{G} \in \Gamma$ exactly $i!\,(n-i)!$ times, as we picked the basis in an ordered way. Observe that we have $t_0 > t_1 > \cdots > t_{\lfloor n/2\rfloor} = t_{\lceil n/2\rceil} < t_{\lceil n/2\rceil+1} < \cdots < t_n$.

Now we are ready to prove Theorem 1.3, which states that if every family $\mathcal{F}$ of $\mathcal{B}_n$ satisfying a hereditary property T has cardinality at most $\Sigma(n,k)$, then families in $\mathcal{L}_n(q)$

with property T have cardinality at most $\Sigma_q(n,k)$. We note that the actual calculation could be omitted by the arguments presented in Section 2. We include it here for the sake of completeness.

Proof of Theorem 1.3. Let $\mathcal{F}$ be a family in $\mathcal{L}_n(q)$ satisfying T. Consider the $\underline{t}$-covering family Γ defined above and let $w_i = t_i$. Then every $\mathcal{G} \in \Gamma$ is isomorphic to $\mathcal{B}_n$, thus by our assumption, the largest weight $\underline{w/t}$, i.e. the largest cardinality of a subfamily $\mathcal{G}' \subset \mathcal{G}$ satisfying T is $\Sigma(n,k)$. This implies $\underline{w}(\mathcal{F}) \leq |\Gamma|\Sigma(n,k)$. Now we will maximize $|\mathcal{F}|$ among families that satisfy the above inequality, without requiring property T. To do this, we need to pick subspaces with the smallest weight, i.e. from the middle levels. We claim that we can pick exactly the k full middle levels, i.e. $\underline{w}(\mathcal{F}_0) = |\Gamma|\Sigma(n,k)$ for the family $\mathcal{F}_0$ consisting of k middle levels. (Note that if $n+k$ is even, we have two options for $\mathcal{F}_0$). This will finish the proof, because more than $\Sigma_q(n,k)$ subspaces would have larger weight than $|\Gamma|\Sigma(n,k)$.

We have

$$\begin{aligned}
\underline{w}(\mathcal{F}_0) &= \sum_{i=\lfloor\frac{n-k}{2}\rfloor+1}^{\lfloor\frac{n-k}{2}\rfloor+k} w_i \begin{bmatrix} n \\ i \end{bmatrix}_q \\
&= \sum_{i=\lfloor\frac{n-k}{2}\rfloor+1}^{\lfloor\frac{n-k}{2}\rfloor+k} \frac{(q^i-1)\cdots(q^i-q^{i-1})(q^n-q^i)\cdots(q^n-q^{n-1})}{i!\,(n-i)!} \begin{bmatrix} n \\ i \end{bmatrix}_q \\
&= \sum_{i=\lfloor\frac{n-k}{2}\rfloor+1}^{\lfloor\frac{n-k}{2}\rfloor+k} \frac{(q^i-1)\cdots(q^i-q^{i-1})(q^n-q^i)\cdots(q^n-q^{n-1})}{i!\,(n-i)!} \\
&\qquad\qquad \frac{(q^n-1)\dots(q^n-q^{n-1})}{(q^i-1)\dots(q^i-q^{i-1})(q^n-q^i)\dots(q^n-q^{n-1})} \\
&= \sum_{i=\lfloor\frac{n-k}{2}\rfloor+1}^{\lfloor\frac{n-k}{2}\rfloor+k} \frac{f(q,n)n!}{i!\,(n-i)!} = \sum_{i=\lfloor\frac{n-k}{2}\rfloor+1}^{\lfloor\frac{n-k}{2}\rfloor+k} |\Gamma|\binom{n}{i} = |\Gamma|\Sigma(n,k).
\end{aligned}$$

Another way to see that $\underline{w}(\mathcal{F}_0) = |\Gamma|\Sigma(n,k)$ is by observing that the left hand side counts the number of pairs (S,B), where S is an i-dimensional subspace and B is a basis for S (organized by subspaces), while the right hand side counts the same thing, but organized by the basis. □

Note that there are several statements similar to Theorems 1.3 and 1.4 that we could prove. We chose to state this one because it immediately gives the exact value of $La_q(n,B)$. Observe that the Boolean result actually gives a weighted result in the case of subspaces, that is stronger than Theorem 1.3. In the case of the butterfly poset, one can prove an even stronger result. If $\mathcal{F}$ is a butterfly-free family of $\mathcal{B}_n$, then we have the LYM-type inequality $\sum_{F\in\mathcal{F}} 1/\binom{n}{|F|} \leq 2$ by [4]. This and the same calculation as in the proof of Theorem 1.3 imply that for a butterfly-free family $\mathcal{G}$ in $\mathcal{L}_n(q)$, we have $\sum_{G\in\mathcal{G}} 1/\begin{bmatrix} n \\ \dim(G)\end{bmatrix} \leq 2$.

Let us prove now Theorem 1.4, which is the asymptotic version of Theorem 1.3.

Proof of Theorem 1.4. We follow the proof of Theorem 1.3. Using its notation, we obtain $\underline{w}(\mathcal{F}) \leq (1+o(1))|\Gamma|\Sigma(n,k)$. Again, to maximize $|\mathcal{F}|$ among those families satisfying the above inequality, we need to pick subspaces with the smallest weight, i.e. from the middle levels. This time we claim that we can pick the subspaces in $\mathcal{F}_0$, and $o(|\mathcal{F}_0|)$ additional subspaces. This will finish the proof similarly to the proof of Theorem 1.3.

We have proved $\underline{w}(\mathcal{F}_0) = |\Gamma|\Sigma(n,k)$, thus the remaining subspaces have total weight $o(|\Gamma|\Sigma(n,k)) = o(\underline{w}(\mathcal{F}_0))$. As each of those has weight not smaller than any weight of a subspace in $\mathcal{F}_0$, more than $\varepsilon|\mathcal{F}_0|$ of them would have weight more than $\varepsilon\underline{w}(\mathcal{F}_0)$, a contradiction that finishes the proof. □

4 Profile polytopes, chain profile polytopes, generalized forbidden subposet problems

In the previous sections we considered arbitrary weights. This means our method can potentially determine the extreme points of the profile polytope for a hereditary property T. If every extreme point in the Boolean case is the union of full levels, and the corresponding union of full levels has property T in the case of subspaces, then this is the situation. Unfortunately, we are only aware of one particular property with this situation. For k-Sperner families, the Boolean result was proved in [8]. We note that instead of using the substructure isomorphic to $\mathcal{B}_n$ with Lemma 2.1, one could use a simpler substructure: a full chain with Lemma 2.1, to obtain the same result, i.e. to determine the extreme points. Moreover, it also easily follows from the LYM-inequality, which is known to hold in $\mathcal{L}_n(q)$. In fact, one can analogously define the profile vectors and polytopes for any graded poset and show for a large class of posets (those with the so-called Sperner property) that the extreme points of k-Sperner families are the profiles of the unions of at most k full levels.

Gerbner and Patkós [14] introduced l-chain profile vectors. Given a family $\mathcal{F}$ of $\mathcal{B}_n$, its l-chain profile vector is an element of the $\binom{n+1}{l}$-dimensional Euclidean space. A coordinate corresponds to a set $\{i_1, \dots, i_l\}$ with $i_1 < i_2 < \dots < i_l$. The value of that coordinate is the number of chains of size l in $\mathcal{F}$ with one element from level i_j for every $1 \le j \le l$. They determined the extreme points of the l-chain profile polytopes of intersecting families and of k-Sperner families of $\mathcal{B}_n$.

They mentioned in [15], after determining the extreme points of the profile polytope of intersecting families in $\mathcal{L}_n(q)$, that with the same method, one can determine the extreme points of the l-chain profile polytope of intersecting families in $\mathcal{L}_n(q)$ as well. Here we show that similarly, the extreme points of the l-chain profile polytope of k-Sperner families in $\mathcal{L}_n(q)$ can be determined. We will state a modified version of Lemma 2.1 that counts copies of a poset Q in a family instead of counting the members of that family.

Let Q be an arbitrary poset with elements $a_1, \dots, a_l$. Consider the $r = (n+1)^l$ functions that map every a_j to an S_i. Let us fix an ordering of these functions and let β_i be the ith of them. We will consider ordered l-sets, i.e. l-sequences $(s_1, \dots, s_l)$ of the base set S. For each $1 \le i \le r$, let $\mathcal{S}_i$ be an arbitrary family of l-sequences with $s_j \in \beta_i(a_j)$ for every $1 \le j \le l$. In the applications, where S_i is a level, we will let $\mathcal{S}_i$ consist of those l-sequences, where the elements form a copy of Q. In particular, if for an embedding β_i and for some j, j' with $a_j < a_{j'}$ we have that $\beta_i(a_j)$ is a higher level than $\beta_i(a_{j'})$, then $\mathcal{S}_i$ is empty. Let us consider only those $r' \le r$ functions β_i, where $\mathcal{S}_i$ is not empty. We can assume without loss of generality that these functions are $\beta_1, \dots, \beta_{r'}$.

Let $\underline{t} = (t_1, \dots, t_{r'})$ be a vector. We say that a family Γ of subsets of S is an $(l, \underline{t})$-*covering* if for each $1 \le i \le r'$, and each l-sequence in $\mathcal{S}_i$, there are exactly t_i members of Γ containing all the elements of that l-sequence (i.e. a particular copy of Q). Let us consider a weight vector $\underline{w} = (w_1, \dots, w_{r'})$. For a set $F \subset S$, let f_i denote the number of l-sets in $\mathcal{S}_i$ with every element in F. Let $\underline{w}(F) = \sum_{i=1}^{r'} w_i f_i$. Let $\underline{w/t} = (w_1/t_1, \dots, w_{r'}/t_{r'})$. We will assume that every weight is non-negative (as T is hereditary, elements of S with negative weight could simply be deleted anyway from any subset of S with property T).

Lemma 4.1. *Let T be a hereditary property of subsets of S and Γ be an $(l, \underline{t})$-covering family of S. Assume that there exists a real number x such that for every $G \in \Gamma$, every subset G' of G with property T has $\underline{w/t}(G') \leq x$. Then $\underline{w}(F) \leq |\Gamma| x$ for every $F \subset S$ with property T.*

Proof. Observe that we have $t_i f_i = \sum_{G \in \Gamma} h_i$, where h_i denotes the number of l-sequences in $\mathcal{S}_i$ with each element of it in $F \cap G$. Indeed, the l-sequences in $\mathcal{S}_i$ with each element in F are counted t_i times on both sides. Thus we have

$$\begin{aligned}\underline{w}(F) &= \sum_{i=1}^{r'} w_i f_i = \sum_{i=1}^{r'} \frac{w_i}{t_i} t_i f_i = \sum_{i=1}^{r'} \frac{w_i}{t_i} \sum_{G \in \Gamma} h_i = \sum_{G \in \Gamma} \sum_{i=1}^{r'} \frac{w_i}{t_i} h_i \\ &= \sum_{G \in \Gamma} \underline{w/t}(G \cap F) \leq \sum_{G \in \Gamma} x = |\Gamma| x. \qquad \square\end{aligned}$$

We have equality here if for every $G \in \Gamma$, there is a $G' \subset G$ satisfying T with $\underline{w/t}(G') = x$, and $G' = G \cap F$. This holds in the following situation. Let T be the k-Sperner property, S be $\mathcal{L}_n(q)$ with the usual partition into levels, and $\mathcal{S}_i$ be the set of those l-sets that form a chain. Let Γ consist of copies of the Boolean poset, as described in Section 3 (note that we could use instead the chains given by a basis and its ordering). Let us assume levels $j_1, \dots, j_k$ have the maximum weight $\underline{w/t}$ in the Boolean poset, and let F consist of the subspaces on levels $j_1, \dots, j_k$. Then by the above, F has the largest weight $\underline{w}(F) = |\Gamma| x$ among k-Sperner families. We obtained that for every non-negative weight the union of k levels has the largest weight, which implies the following result.

Corollary 4.2. *The extreme points of the l-chain profile polytope of k-Sperner families of subspaces of $\mathbb{F}_q^n$ are the unions of at most k levels.*

We mentioned the l-chain polytopes here because the above result gives the first instance of a generalized forbidden subposet problem in $\mathcal{L}_n(q)$. The generalized forbidden subposet problem seeks to find $La(n, P, Q)$, the largest number of copies of the poset Q in a P-free subfamily of $\mathcal{B}_n$. Its study was initiated by Gerbner, Keszegh and Patkós [11], analogously to the graph case [1] that has recently attracted a lot of attention. Further results on $La(n, P, P_l)$ can be found in [12].

We propose to study generalized forbidden subposet problems in $\mathcal{L}_n(q)$. Let $La_q(n, P, Q)$ denote the largest number of copies of the poset Q in a P-free family in $\mathcal{L}_n(q)$. Corollary 4.2 implies that $La_q(n, P_k, P_l)$ is given by k full levels (it is not hard to see that the best way to choose the k levels $i_1, \dots, i_k$ is when the values $i_1, i_2 - i_1, i_3 - i_2, \dots, i_k - i_{k-1}, n - i_k$ differ by at most one). For other pairs of posets, a weighted version in the Boolean case could give bounds on $La_q(n, P, Q)$.

Let us mention that even though Lemma 4.1 immediately implied a generalized forbidden subposet result in $\mathcal{L}_n(q)$, it may be the only particular problem where we can use it to obtain a sharp bound. Lemma 4.1 requires studying a weighted version of a generalized forbidden subposet problem in the smaller structure G, similarly to Lemma 2.1. Observe that in the case of counting the members of a family of $\mathcal{B}_n$, we had the useful property that w_i/t_i is the largest in the middle, exactly where the (conjectured) extremal families are. Therefore, an unweighted result on the cycle gave a weighted result of $\mathcal{B}_n$ that implied the unweighted result. And similarly, an unweighted result of $\mathcal{B}_n$ immediately implied the analogous bound in $\mathcal{L}_n(q)$. However, this is not the case with the more complicated weight functions and more diverse extremal families that we deal with in generalized forbidden subposet problems.

To finish the paper, we present some simple results for $La_q(n, P, Q)$. They are unrelated to the earlier parts of the paper, but we would like to present some results concerning this function, since we initiate the study of this topic in this paper. Let the *generalized diamond* poset D_r have $r + 2$ elements $a, b_1, \ldots, b_r, c$ and relations $a < b_i < c$ for $1 \leq i \leq r$.

Proposition 4.3. (i) $La_q(n, \vee, \wedge_r) = La_q(n, \wedge, \vee_r) = \binom{\left[{n \atop \lfloor n/2 \rfloor}\right]_q}{r}$.

(ii) $La_q(n, B, D_r) = \binom{\left[{n \atop \lfloor n/2 \rfloor}\right]_q}{r}$.

(iii) $La_q(n, P_3, \wedge_r) = \max_{0 \leq k \leq n} \left[{n \atop k}\right]_q \binom{\left[{k \atop \lfloor k/2 \rfloor}\right]_q}{r}$.

The Boolean analogues of the above statements were proved in [11], and the proofs of them also work in our case. We include them for the sake of completeness. We will use the *canonical partition* of k-Sperner families $\mathcal{F}$; it is a partition of $\mathcal{F}$ into k antichains $\mathcal{F}_1, \ldots, \mathcal{F}_k$, where $\mathcal{F}_i$ is the set of minimal elements of $\mathcal{F} \setminus \cup_{j=1}^{i-1} \mathcal{F}_j$.

Proof. The lower bounds for (i) and (ii) are given by the families consisting of all the $\lfloor n/2 \rfloor$-dimensional subspaces together with the zero-dimensional and/or the n-dimensional subspace. For (iii) consider all the k-dimensional and $\lfloor k/2 \rfloor$-dimensional subspaces for every k.

For the upper bound in (i), the first equality is trivial by symmetry. Let us consider now the canonical partition $\mathcal{F}_1 \cup \mathcal{F}_2$ of a $\vee$-free family $\mathcal{F}$ in $\mathcal{L}_n(q)$. Observe that every copy of $\wedge_r$ consists of a member of $\mathcal{F}_2$, and r members of $\mathcal{F}_1$ contained in it. Every member of $\mathcal{F}_1$ is contained in at most one member of $\mathcal{F}_2$ by the $\vee$-free property, thus for every set of r members of $\mathcal{F}_1$, at most one member of $\mathcal{F}_2$ forms a copy of $\wedge_r$ with them. This implies $La_q(n, \vee, \wedge_r) \leq \binom{|\mathcal{F}_1|}{r}$. As $\mathcal{F}_1$ is an antichain, it has at most $\left[{n \atop \lfloor n/2 \rfloor}\right]_q$ members, finishing the proof of (i).

To prove the upper bound in (ii), let $\mathcal{F}$ be a B-free family in $\mathcal{L}_n(q)$ and $\mathcal{M} = \{M \in \mathcal{F} : \exists F', F'' \in \mathcal{F}$ such that $F' \subset M \subset F''\}$. As $\mathcal{F}$ is P_4-free, $\mathcal{M}$ is an antichain. Observe that for an $M \in \mathcal{M}$ there is exactly one $F' \in \mathcal{F}$ with $F' \subset M$ and there is exactly one $F'' \in \mathcal{F}$ with $M \subset F''$. Thus, for every r-tuple from $\mathcal{M}$ there is at most one copy of D_r in $\mathcal{F}$, and there are at most $\binom{\left[{n \atop \lfloor n/2 \rfloor}\right]_q}{r}$ such r-tuples.

To prove the upper bound in (iii), let $\mathcal{F}$ be a P_3-free family in $\mathcal{L}_n(q)$ and consider its canonical partition $\mathcal{F}_1 \cup \mathcal{F}_2$. Every copy of $\wedge_r$ consists of a member of $\mathcal{F}_2$ and r members of $\mathcal{F}_1$. For a member F of $\mathcal{F}_2$ with dimension k, we have to pick r subspaces of it that are in $\mathcal{F}_1$. Those members of $\mathcal{F}_1$ that can be picked form an antichain of subspaces of a k-dimensional space, thus there are at most $\left[{i \atop \lfloor k/2 \rfloor}\right]_q$ of them, and there are $\left[{n \atop k}\right]_q \binom{\left[{k \atop \lfloor k/2 \rfloor}\right]_q}{r}$ ways to pick r of them. It means that a k-dimensional member of $\mathcal{F}_2$ is in at most $w(k) := \binom{\left[{k \atop \lfloor k/2 \rfloor}\right]_q}{r}$ copies of $\wedge_r$. Hence the total number of copies of $\wedge_r$ is at most the total weight of $\mathcal{F}_2$, i.e. $w(\mathcal{F}_2)$. As $\mathcal{F}_2$ is an antichain, this is maximized by a level (for a number of reasons mentioned earlier, for example Corollary 4.2 implies this). The weight of level k is $\begin{bmatrix} n \\ k \end{bmatrix}_q \binom{\begin{bmatrix} k \\ \lfloor k/2 \rfloor \end{bmatrix}_q}{r}$, finishing the proof. □

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On the beta distribution, the nonlinear Fourier transform and a combinatorial problem

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Abstract

The paper describes some probabilistic and combinatorial aspects of the nonlinear Fourier transform associated with the AKNS-ZS problems. We show that the volumes of a family of polytopes that appear in a power expansion of the nonlinear Fourier transform are distributed according to the beta probability distribution. We establish this result by studying an Euler-type discretization of the nonlinear Fourier transform. This approach leads to the combinatorial problem of finding the number of alternating ordered partitions of an integer into a fixed number of distinct parts. We find the explicit formula for these numbers and show that they are essentially distributed according to a novel discretization of the beta distribution for a suitable choice of the shape parameters. We also find the generating functions of the numbers of alternating sums. These functions are expressed in terms of the our discrete nonlinear Fourier transform.

Keywords: Beta distribution, nonlinear Fourier transform, discretisation.

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1 Introduction

As announced in the title, this paper investigates relations between three topics from different parts of mathematics: probability distributions, combinatorics and the theory of nonlinear partial differential equations, more concretely, the nonlinear Fourier transform. Despite the apparent heterogeneity of the topics, the relations between them are rather natural.

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The construction and the study of various versions of the nonlinear Fourier transform stem from the theory of integrable nonlinear partial differential equations. The most famous examples of such equations include the Korteweg-de Vries, nonlinear Schrödinger, and sine-Gordon equations, Heisenberg ferromagnet model, Toda lattices and many others. The role of the nonlinear Fourier transform in the theory of integrable equations is roughly analogous to the role of the linear Fourier transform and, more generally, the Sturm-Liouville expansions in the theory of linear partial differential equations.

The transformation $\mathcal{F}$ used in this text can be thought of as a non-linearization of the usual Fourier transformation. Let $u\colon [0,1] \to \mathbb{R}$ be a function. The nonlinear Fourier transform $\mathcal{F}$ of u that we shall consider in this paper is of the form

$$\mathcal{F}[u](n) = I + \begin{pmatrix} 0 & F[u](n) \\ -F[\overline{u}](-n) & 0 \end{pmatrix} + \sum_{d=2}^{\infty} A_d[u](n),$$

where F is the linear Fourier transform (Fourier series) and $u \mapsto A_d[u]$ are the suitable matrix-valued nonlinear operators.

The beta distribution is one of the oldest and most important probability distributions with a broad spectrum of applications in different areas of probability and statistics, particularly in Bayesian statistical inference. It has been recently mentioned in virtually every book on machine learning and related topics. The beta distribution $\mathrm{Beta}(x; a, b)$ with shape parameters a and b is given by the probability density function

$$p_\beta(x; a, b) = \frac{1}{B(a+1, b+1)} x^a (1-x)^b, \quad x \in [0,1].$$

In this paper, we shall establish a link between the nonlinear Fourier transform and the beta distribution. Let $u_c(x) \equiv u$ be a constant function. The transformation $\mathcal{F}$ is related to a two-parameter family of polytopes $\widehat{D}_d(\lambda)$, where $d \in \mathbb{N}$ and $\lambda \in [0,1]$, given by

$$\widehat{D}_d(\lambda) = \{(x_1, x_2, \ldots, x_d);\ \ 1 \geq x_1 \geq x_2 \ldots \geq x_d \geq 0,\ \ \sum_{i=1}^{d} (-1)^{i-1} x_i = \lambda\}$$

and their projections $D_d(\lambda)$ in the hyperplane $\{(x_1, x_2, \ldots, x_{(d-1)}, 0)\} \subset \mathbb{R}^d$. For the nonlinear Fourier transform $\mathcal{F}[u_c](n)$ of the constant function $u_c \equiv u$ on $[0,1]$, we have

$$\mathcal{F}[u_c](n) = I + \sum_{d=1}^{\infty} u^d \int_0^1 \mathrm{Vol}(D_d(\lambda)) \begin{pmatrix} e^{-2\pi i \lambda n} & 0 \\ 0 & e^{2\pi i \lambda n} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \, d\lambda.$$

This formula is proved in Proposition 2.1 on page 113. We shall see that for every fixed d_0, the volumes of the family $\{D_{d_0}(\lambda); \lambda \in [0,1]\}$ are given by the formula for the probability density function of the beta distribution. Theorem 4.3 on page 122 gives the formula

$$\mathrm{Vol}(D_d(\lambda)) = \frac{1}{d!} \begin{cases} p_\beta(\lambda; \frac{d}{2}, \frac{d}{2}+1); & d \text{ even} \\ p_\beta(\lambda; \frac{d+1}{2}, \frac{d+1}{2}); & d \text{ odd.} \end{cases} \tag{1.1}$$

The probabilistic contents of the above formula will be described below.

The statement and the proof of Theorem 4.3 are obtained by considering a suitable discretization $\mathcal{F}_N$ of the nonlinear Fourier transform $\mathcal{F}$. In the expression for $\mathcal{F}_N[u_c]$, the role of the volumes of the polytopes $D_d(\lambda)$ is assumed by the numbers

$$AQ_N(L,d) = \sharp\{(l_1, l_2, \ldots, l_d) \in \mathbb{N};\ \ l_1 - l_2 + l_3 - \ldots + (-1)^{(d-1)} l_d = L\},$$

where $N - 1 \geq l_1 > l_2 > \ldots > l_d \geq 0$. So, $AQ_N(L,d)$ is the number of ordered alternating partitions of L into d distinct parts not grater than $N-1$.

The central result of the paper is the explicit formula for the numbers $AQ_N(L,d)$. It is given in Theorem 3.3 on page 116. We show that

$$AQ_N(L,d) = \begin{cases} \binom{L-1}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L}{\lfloor \frac{d}{2} \rfloor}\,; & d \ \text{even} \\ \binom{L}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L-1}{\lfloor \frac{d}{2} \rfloor}\,; & d \ \text{odd.} \end{cases} \tag{1.2}$$

The relationship between the numbers $AQ_N(L,d)$ and the nonlinear Fourier transform is best described by the fact that the generating functions for the numbers $AQ_N(L,d)$ are in a natural way expressed in terms of the discrete nonlinear Fourier transform $\mathcal{F}_N$. This is proved in Proposition 3.2 on page 115. We actually get separate generating functions for odd and for even values of d. Understanding the structure of the numbers $AQ_N(L,d)$ was important in the construction of the inverse of $\mathcal{F}_N$ in our recent paper [14].

Results (1.1) and (1.2) can be recast into probabilistic terms. Let our sample space consist of all strictly decreasing d-tuples of integers

$$\Delta_d^D(N) = \{(l_1, l_2, \ldots, l_d);\ \ N-1 \geq l_1 > l_2 > \ldots, l_d \geq 0)\},$$

Let all the samples $(l_1, l_2, \ldots, l_d)$ be equally probable and let

$$X_{AS}[N,d] \colon \Delta_d^D(N) \to \mathbb{N}$$

be the random variable which assigns to a randomly chosen point in $\Delta_d^D(N)$ the alternating sum,

$$X_{AS}[N,d](l_1, l_2, \ldots, l_d) = l_1 - l_2 + l_3 - \ldots + (-1)^{(d-1)} l_d.$$

We want to compute the probability $P(X_{AS}[N,d] = L)$ of the event that a randomly chosen d-tuple has the alternating sum equal to L. We shall show that

$$AS[N,d](L) = P(X_{AS}[N,d] = L) = \begin{cases} \dfrac{\binom{L-1}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L}{\lfloor \frac{d}{2} \rfloor}}{\binom{N}{d}}\,; & d \ \text{even} \\ \dfrac{\binom{L}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L-1}{\lfloor \frac{d}{2} \rfloor}}{\binom{N}{d}}\,; & d \ \text{odd.} \end{cases} \tag{1.3}$$

The random variable X_{AS} is distributed according to the probability mass function $AS[N,d]$ defined by the right hand side of the above formula.

The question arises: Does this distribution have a sensible limit as N goes to infinity? One possibility is to proceed as follows. Let $\lambda \in [0,1]$ be arbitrary. Let us choose a sequence $\{L_N\}_{N \in \mathbb{N}}$ such that $L_N < N$ and $\lim_{N\to\infty} L_N/N = \lambda$. We shall see that

$$\lim_{N\to\infty} P(X_{AS}[N,d] = L_N)N = \begin{cases} p_\beta(\lambda; \frac{d}{2}, \frac{d}{2}+1)\,; & d \ \text{even} \\ p_\beta(\lambda; \frac{d+1}{2}, \frac{d+1}{2})\,; & d \ \text{odd,} \end{cases} \tag{1.4}$$

where $p_\beta(\lambda; a, b)$ is the beta distribution with shape parameters a and b.

Let our sample space now be the ordered simplex $\Delta_d \subset \mathbb{R}^d$ of the dimension d, given by

$$\Delta_d = \{(x_1, x_2, \ldots, x_d);\ \ 1 \geq x_1 \geq x_2 \ldots \geq x_d \geq 0\},$$

and let all the samples $(x_1, x_2, \ldots, x_d)$ be equally probable. This means that we assigned on Δ_d the uniform distribution $v\colon \Delta_d \to \mathbb{R}$ given by $v(x_1, x_2, \ldots, x_d) \equiv d!$. Let the random variable $X_{as}[d]$ defined on Δ_d be given by

$$X_{as}[d](x_1, x_2, \ldots, x_d) = x_1 - x_2 + x_3 - \ldots + (-1)^{(d-1)} x_d.$$

Formula (1.4) shows that the cumulative probability distribution

$$F_{as}[d]\colon [0,1] \longrightarrow \mathbb{R}, \quad \lambda \to F_{as}[d](\lambda)$$

of the random variable X_{as} is given by

$$F_{as}[d](\lambda) = P(X_{as}[d] \leq \lambda) = \begin{cases} \int_0^\lambda p_\beta(\mu; \frac{d}{2}, \frac{d}{2}+1)\, d\mu\,; & d \text{ even} \\ \int_0^\lambda p_\beta(\mu; \frac{d+1}{2}, \frac{d+1}{2})\, d\mu\,; & d \text{ odd.} \end{cases}$$

This result can be recast in geometric terms. Taking into account that the d-dimensional volume of the simplex Δ_d is equal to $\frac{1}{d!}$, we see from the above that the $(d-1)$-dimensional volume of the polytope $D_d(\lambda)$ is indeed given by formula (1.1) explained in Theorem 4.3.

The equality (1.4) suggests a natural generalisation of the probability mass function of $X_{AS}[N, d]$. It can be defined by

$$P_N(L; a, b) = \frac{\binom{L-1}{a}\binom{N-L}{b}}{\binom{N}{a+b+1}},$$

where $L \in \{1, 2, \ldots, N\}$ and are integers such that $a + b < N$. In Proposition 4.2 we show that

$$\lim_{N\to\infty} P_N(\frac{L_N}{N}, a, b)N = p_\beta(\lambda; a, b) = \frac{1}{\beta(a+1, b+1)}\ \lambda^a(\lambda-1)^b.$$

So, the probability mass function $P_N(a, b)$ is a natural discretization of the continuous beta distribution for arbitrary values of shape parameters. But, at the moment, a convincing combinatorial or geometric description of $P_N(a, b)$ remains a task for the future.

Above, we have described a way how the beta distribution emerges as an appropriate limit from a discrete and finite probability distribution. This result is reminiscent to the relation between the Pólya-Eggenberger urn and the beta distribution. Pólya-Eggenberger urn is an urn model with replacement and is tenable - one can draw the balls from the urn infinitely many times. The limit of the quotient $\frac{W_n}{n}$, where W_n is the number of white balls drawn in n draws, is given by

$$\lim_{n\to\infty} P(\frac{W_n}{n} < \lambda) = \int_0^\lambda p_\beta(\mu; \frac{W_0}{s}, \frac{B_0}{s})\, d\mu,$$

where W_0 and B_0 are the initial numbers of white and black balls in the urn and s is the number of the balls added to the urn after drawing and returning a ball of the same colour.

Let $s = 1$. After a finite number n of draws, the probability of $W_n = w$ and $B_n = b$ is equal to

$$P(W_n = w, B_n = b) = \frac{\binom{w-1}{W_0-1}\binom{b-1}{B_0-1}}{\binom{n+\tau_0-1}{\tau_0-1}}, \tag{1.5}$$

where $\tau_0 = W_0 + B_0$. The proofs can be found in the comprehensive treatment of Pólya urn models [9] by H. M. Mahmoud. The values in the (1.5) are related by $w + b = W_0 + B_0 + n = \tau_0 + n$. Our formula (1.3) could therefore be tentatively understood as an outcome of Pólya-Eggenberger process after roughly $N + d$ steps. But the number of steps in constructing an alternating sum $l_1 - l_2 + \ldots + (-1)^{d-1} l_d$ is d. That d is indeed the correct number of steps in our process will become even clearer in the proofs of Theorem 3.3 and Corollary 3.4. These proofs are different from the usual proof of formula (1.5). While the number of steps in the limit of the Pólya-Eggenberger process is infinite, the number of steps in our process remains d even after performing the limit. This comes naturally from the source of our construction which is the nonlinear Fourier transform. The core of our limiting construction is the replacement of the alternating sum of integers: first, by alternating sums of rational numbers and then, in the limit, by the alternating sum of real numbers. This also leads to the geometric expression of our results in terms of the volumes of the polytopes $D_d(\lambda)$, mentioned above.

There are other discretization of the beta distribution with useful properties. One possible approach was studied by A. Punzo in [11]. But, as far as the author is aware, this discretization does not come from some combinatorial source and is given by a very different formula.

The plan of the paper is the following. In section 2, we recall the AKNS-ZS type of the nonlinear Fourier transform and prepare the necessary formulas. We establish the connection between $\mathcal{F}$ and the family of polytopes $\{D_d(\lambda)\}$. We also introduce the discretization $\mathcal{F}_N$ of $\mathcal{F}$. In section 3, we describe and prove the main facts about our central combinatorial problem: the evaluation of the numbers $AQ_N(L, d)$, and the derivation of the generating functions of the numbers $AQ_N(L, d)$ in terms of $\mathcal{F}_N$. In section 4, we prove proposition 4.2 and theorem 4.3 stated above. Section 5 contains graphs illustrating the relation between the beta distribution and our discrete approximation. We conclude the paper by mentioning some problems for further research.

2 Nonlinear Fourier transforms and its discretization

We review the definition of the nonlinear Fourier transform $\mathcal{F}$ which first appeared in the work of M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur in [1] and [2] and more or less simultaneously in the work of V. Zakharov and A. Shabat in [19]. They studied and solved a certain class of integrable partial differential equations which are now called AKNS-ZS equations. The acronym is also used to denote the nonlinear Fourier transform which figures in the AKNS-ZS theory. In this section, we shall also introduce the Euler-type discretization $\mathcal{F}_N$ of $\mathcal{F}$.

2.1 Nonlinear Fourier transform of AKNS-ZS type

We shall consider the nonlinear Fourier transform $\mathcal{F}$ which appears in the study of the *periodic* AKNS-ZS problems. To every well-behaved function $u(x)\colon [0, 1] \to \mathbb{C}$ it assigns the doubly infinite sequence $\{\mathcal{F}[u](n)\}_{n\in\mathbb{Z}}$ of $SU(2)$ matrices, given by $\mathcal{F}[u](n) =$

$(-1)^n\Phi(x=1,n)$, where $\Phi(x,n)$ is the solution of the linear initial value problem

$$\Phi_x(x,n) = L(x,n)\cdot\Phi(x,n), \quad \Phi(0,n) = I. \tag{2.1}$$

The coefficient matrix $L(x,n)$ is given by

$$L(x,n) = \begin{pmatrix} \pi i\, n & u(x) \\ -\overline{u(x)} & -\pi i\, n \end{pmatrix}.$$

As we mentioned in the Introduction, we will see that $\mathcal{F}$ is of the form

$$\mathcal{F}[u](n) = I + \begin{pmatrix} 0 & F[u](n) \\ -F[\overline{u}](-n] & 0 \end{pmatrix} + \sum_{d=2}^{\infty} A_d[u](n).$$

The amount of literature on various aspects of the inverse scattering method is truly vast, so we shall only mention a few works in which the Fourier analysis aspect is more pronounced. The foundational work was done by Gardner, Greene, Kruskal and Miura in [8] and [7]. The transform, used in this paper was first constructed by Ablowitz, Kaup, Newell and Segur, in [1, 2], and simultaneously by Zakharov and Shabat in [19]. Nonlinear Fourier transforms of functions, defined on $\mathbb{R}$ and $\mathbb{R}^+$, were studied by I. Gelfan'd, A. Fokas and B. Pelloni in [5, 6, 10], and in their other works. A version of transformation, closely related to the one studied in this paper is described by T. Tao and C. Thiele in [15]. Some aspects of the transformation, defined above, were studied in my papers [12, 13] and [14].

Definition of $\mathcal{F}$, given above is the one that is usually found in the texts which study the integrable ANKS-ZS equations. We shall rather represent $\mathcal{F}$ in a different gauge. Let $G(x,n) = \operatorname{diag}(e^{-\pi inx}, e^{\pi inx})$ be the (diagonal) matrix of our gauge transformation. In the new gauge Φ is replaced by $\Phi^G = G\cdot\Phi$ and Φ^G is the solution of the initial-value problem

$$\Phi^G_x(x,n) = L^G(x,z)\cdot\Phi^G(x,n), \quad \Phi^G(0,n) = I. \tag{2.2}$$

The transformed coefficient matrix is then $L^G(x,n) = G_x\cdot G^{-1}(x,n) + G(x,n)\cdot L(x,n)\cdot G^{-1}(x,n)$. Its explicit expression is

$$L^G(x,n) = \begin{pmatrix} 0 & e^{-2\pi inx}u(x) \\ -e^{2\pi inx}\overline{u(x)} & 0 \end{pmatrix}. \tag{2.3}$$

In the new gauge we set $\mathcal{F}^G[u](n) = \Phi^G(x=1,n)$. Since $n\in\mathbb{Z}$, the equation $\Phi^G(1,n) = G(1,n)\cdot\Phi(1,n)$ gives $\mathcal{F}[u](n) = \mathcal{F}^G[u](n)$. The solution to the problem (2.2) can be given in the form of the Dyson series.

$$\Phi^G(x,n) = I + \sum_{d=1}^{\infty}\int_{\Delta_d(x)} L^G(x_1,n)\cdot L^G(x_2,n)\cdots L^G(x_d,n)\,d\vec{x}, \tag{2.4}$$

where $\Delta_d(x)$ is the ordered simplex of dimension d with the edge length equal to x,

$$\Delta_d(x) = \{(x_1,x_2,\ldots,x_d)\in\mathbb{R}^d; x\ge x_1\ge x_2\ge\ldots\ge x_d\ge 0\}.$$

Let us denote

$$E(x,n) = \begin{pmatrix} e^{\pi ixn} & 0 \\ 0 & e^{-\pi ixn} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.5}$$

and let $u(x)$ be real valued. Then we have $L^G(x,n) = u(x)\,E(-2x,n)\cdot J$. Matrices $E(x,n)$ and J do not commute. Instead, they obey the relation

$$E(x,n)\cdot J = J\cdot E(-x,n). \tag{2.6}$$

Recall that $\widehat{D}_d(\lambda)$ denotes the polytope given by

$$\widehat{D}_d(\lambda) = \{(x_1,x_2,\dots x_d)\in\Delta_d(1);\ \sum_{j=1}^{d}(-1)^{j-1}x_j = \lambda\},$$

and $D_d(\lambda)$ is its projection on the hyperplane $x_d = 0$. These are the polytopes, mentioned in the introduction. Denote

$$\mathcal{U}(x_1,x_2,\dots,x_{d-1};\lambda) = u(x_1)\cdots u(x_{d-1})u((-1)^{d-1}(\lambda - \sum_{j=1}^{d-1}(-1)^{j-1}x_j)),$$

and let $d_\lambda\vec{x}$ be the measure on $\widehat{D}_d(\lambda)\subset\mathbb{R}^d$, inherited from the Euclidean measure on $\mathbb{R}^d$. Using (2.6) in the Dyson series and evaluating at $x = 1$ gives

$$\mathcal{F}[u](n) = I + \sum_{d=1}^{\infty}\int_{\Delta_d(1)} u(x_1)\,u(x_2)\cdots u(x_d)\,E\Big(-2(\sum_{j=1}^{d}(-1)^{j-1}x_j),n\Big)\cdot J^d\,d\vec{x}$$

which, upon setting $x_1 - x_2 + \ldots + (-1)^{d-1}x_d = \lambda$, can be rewritten as

$$\begin{aligned}\mathcal{F}[u](n) &= I + \sum_{d=1}^{\infty}\int_0^1 E(-2\lambda,n)\,\Big(\int_{\widehat{D}_d(\lambda)} u(x_1)\,u(x_2)\cdots u(x_d)\,d_\lambda\vec{x}\Big)\cdot J^d\frac{1}{\sqrt{d}}\,d\lambda\\ &= I + \sum_{d=1}^{\infty}\int_0^1 E(-2\lambda,n)\,\Big(\int_{D_d(\lambda)}\mathcal{U}(x_1,\dots,x_{d-1};\lambda)\,dx_1\cdots dx_{d-1}\Big)J^d\,d\lambda,\end{aligned}$$

Inserting the constant function $u_c(x)\equiv u$ we immediately get the following proposition.

Proposition 2.1. *In the case where $u_c(x)\equiv u$ is a constant function, we get*

$$\mathcal{F}[u_c](n) = I + \sum_{d=1}^{\infty}u^d\int_0^1 \mathrm{Vol}(D_d(\lambda))\,E(-2\lambda,n)\cdot J^d\,d\lambda. \tag{2.7}$$

□

2.2 Euler-type discretization of $\mathcal{F}$

Many authors studied various discretizations of transformations similar to $\mathcal{F}$, but usually acting on the functions defined on $\mathbb{R}$ or $\mathbb{R}^+$, see e.g. [16, 17, 18]. Important are the discretizations that preserve the integrability of the AKNS-ZS systems. These are constructed in well known works of M. Ablowitz and J. Ladik and also L. Faddeev and A. Yu Volkov, see [3, 4]. A discrete nonlinear Fourier transform, similar to the one studied below, was considered by Tao and Thiele in [15]. In the author's paper [14] an algorithm for evaluating the inverse of the nonlinear Fourier transform, defined below, is constructed. (In [14] a nonlinear Fourier transform of distributions of the form $u(x) = \sum_{n=1}^{N} u_n\,\delta_{x_n}(x)$ is also constructed, together with its inverse.)

We have obtained the nonlinear Fourier transform from an initial value problem for a particular first-order linear differential equation. An obvious approach to construct a discretization is to replace the differential equation with a suitable difference equation. Let $\vec{u} = (u_0, u_1, \ldots, u_{N-1}) \in \mathbb{R}^N$ be a vector which plays a role of a function of a discrete variable. Let the L-matrix be given by

$$L_N(k,n) = \begin{pmatrix} 0 & e^{-2\pi i \frac{kn}{N}} u_k \\ -e^{2\pi i \frac{kn}{N}} u_k & 0 \end{pmatrix}.$$

Definition 2.2. Let $k, n \in \{0, 1, \ldots, N-1\}$. Discrete nonlinear Fourier transform $\mathcal{F}_N[\vec{u}]$ of $\vec{u}$ is defined by $\mathcal{F}_N[\vec{u}](n) = \Phi_N(k = N-1, n)$, where Φ_N is the solution of the difference initial value problem

$$\frac{\Phi_N(k+1,n) - \Phi_N(k,n)}{\frac{1}{N}} = L_N(k,n) \cdot \Phi_N(k,n), \quad \Phi_N(0,n) = I.$$

Solving the above initial value problem and evaluating at $k = N-1$ gives

$$\mathcal{F}_N[\vec{u}](n) = \prod_{k=N-1}^{0} \Big(I + \frac{1}{N} L_N(k,n) \Big),$$

and this can be expanded into

$$\mathcal{F}_N[\vec{u}](n) = I + \sum_{d=1}^{N} \frac{1}{N^d} \sum_{N-1 \geq l_1 > l_2 > \ldots > l_d \geq 0} L_N(l_1,n) \cdot L_N(l_2,n) \cdots L_N(l_d,n). \quad (2.8)$$

This expression is a discrete analogue of Dyson's expansion (2.4).

Let us introduce the notation

$$E_\delta(l,n) = E(l, \frac{n}{N}) = \begin{pmatrix} e^{\pi i l \frac{n}{N}} & 0 \\ 0 & e^{-\pi i l \frac{n}{N}} \end{pmatrix} \quad l, n \in \{0, 1, \ldots, N-1\}$$

where E is given by (2.5), and the subscript δ refers to the use in the discretized context. The coefficient matrix L_N can be written in the form

$$L_N(l,n) = u_l \, E_\delta(-2l, n) \cdot J,$$

with J defined in (2.5). By means of relation (2.6), we can collect all the copies of J in (2.8) on the right. Let $\vec{u}_c = (u, \ldots, u)$ be a constant vector. We get

$$\mathcal{F}_N[\vec{u}_c](n) = I + \sum_{d=1}^{N} (\frac{u}{N})^d \sum_{N-1 \geq l_1 > l_2 > \ldots > l_d \geq 0} E_\delta\Big(-2(l_1 - l_2 + \ldots + (-1)^{d-1} l_d), n\Big) \cdot J^d.$$

If we denote $L = l_1 - l_2 + \ldots + (-1)^{d-1} l_d$, we can finally write

$$\mathcal{F}_N[\vec{u}_c](n) = I + \sum_{d=1}^{N-1} (\frac{u}{N})^d \sum_{L=0}^{N-1} E_\delta(-2L, n) \sum_{(l_1,\ldots,l_d) \in \widehat{D}_{d,N}^{disc}(L)} J^d, \quad (2.9)$$

where

$$\widehat{D}_{d,N}^{disc}(L) = \{(l_1, \ldots, l_d) \in \mathbb{N}^d;\ N-1 \geq l_1 > \ldots > l_d \geq 0,\ \sum_{j=1}^{d} (-1)^{j-1} l_j = L\}. \quad (2.10)$$

3 Ordered alternating partitions with distinct parts

In this section we introduce the central combinatorial object of the paper, namely the numbers $AQ_N(L,d)$. We establish the connection between the family $\{AQ_N(L,d)\}$ and the discrete nonlinear Fourier transform $\mathcal{F}_N$. The transformation $\mathcal{F}_N$ yields the generating functions for $\{AQ_N(L,d)\}$ separately for even and odd values of d. The main result of the section is the statement and proof of the explicit formula for the numbers $AQ_N(L,d)$ and the evaluation of the probability distribution of these numbers.

Definition 3.1. Let

$$\Delta^D_{N,d} = \{(l_1, l_2, \ldots, l_d) \in \mathbb{N}^d; N-1 \geq l_1 > l_2 > \ldots > l_d \geq 0\}$$

be the discrete ordered simplex. Denote by $AQ_N(L,d)$ the numbers which count the *ordered alternating partitions of $L \in \mathbb{N}$ into d distinct parts not greater than $N-1$,*

$$AQ_N(L,d) = \sharp\{(l_1, l_2, \ldots, l_d) \in \Delta^D_{N,d};\ \ l_1 - l_2 + l_3 - \ldots (-1)^{d-1} l_d = L\}. \tag{3.1}$$

In other words, $AQ_N(L,d)$ is the number of solutions of the equation

$$l_1 - l_2 + l_3 - \ldots + (-1)^{d-1} l_d = L$$

where $(l_1, l_2, \ldots, l_d)$ is an element of the simplex $\Delta^D_{N,d}$.

The next proposition shows that $\mathcal{F}_N[u_c](n)$ can, roughly speaking, be understood as the discrete *linear* Fourier transform of the generating polynomial of the finite sequence $\{AQ_N(L,d)\}_{d=1}^N$.

Let us denote by $\mathcal{F}_{ev}[u_c](n)$ the upper left entry and by $\mathcal{F}_{odd}[u_c](n)$ the upper right entry of the 2×2 matrix $\mathcal{F}_N[u_c](n)$.

Proposition 3.2. *The power series expansion of $\mathcal{F}_N[u_c]$ around $u = 0$ is given by*

$$\mathcal{F}_N[u_c](n) = I + \sum_{d=1}^{N} (\frac{u}{N})^d \sum_{L=0}^{N-1} AQ_N(L,d)\, E_\delta(-2L, n) \cdot J^d. \tag{3.2}$$

For every $L \in \{0, 1, \ldots, N-1\}$, the generating polynomials of the numbers

$$\{AQ_N(L, 2k)\}_{k=1,\ldots,\lfloor \frac{N}{2} \rfloor} \quad \text{and} \quad \{AQ_N(L, 2k-1)\}_{k=1,\ldots,\lfloor \frac{N+1}{2} \rfloor}$$

are given by the equations

$$\sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} (-1)^k (\frac{u}{N})^{2k} AQ_N(L, 2k) = \sum_{n=0}^{N-1} e^{2\pi i \frac{Ln}{N}} \cdot \mathcal{F}_{ev}[u_c](n) \tag{3.3}$$

$$\sum_{k=1}^{\lfloor \frac{N+1}{2} \rfloor} (-1)^{k+1} (\frac{u}{N})^{2k-1} AQ_N(L, 2k-1) = \sum_{n=0}^{N-1} e^{-2\pi i \frac{Ln}{N}} \cdot \mathcal{F}_{odd}[u_c](n). \tag{3.4}$$

Proof. Recall formula (2.9)

$$\mathcal{F}_N[\vec{u}_c](n) = I + \sum_{d=1}^{N-1} (\frac{u}{N})^d \sum_{L=0}^{N-1} E_\delta(-2L, n) \sum_{(l_1,\ldots,l_d)\in \widehat{D}_{d,N}^{disc}(L)} J^d.$$

The last sum in the formula yields the constant matrix J^d multiplied by the integer $\sharp\widehat{D}_{d,N}^{disc}(L)$. The number $AQ_N(L,d)$ is by its definition the number of elements in $\widehat{D}_{d,N}^{disc}(L)$, so we have

$$\sum_{(l_1,\ldots,l_d)\in \widehat{D}_{d,N}^{disc}(L)} J^d = AQ_N(L,d)\cdot J^d.$$

Let us now take into account

$$J^{2k} = (-1)^k \cdot I = \begin{pmatrix} (-1)^k & 0 \\ 0 & (-1)^k \end{pmatrix}$$

and

$$J^{2k-1} = (-1)^{k+1}\cdot J = \begin{pmatrix} 0 & (-1)^{2k-1} \\ -(-1)^{2k-1} & 0 \end{pmatrix},$$

and consider the diagonal and anti-diagonal parts of $\mathcal{F}_N$ separately. From 3.2, we get two equations, one for each parity of k:

$$\begin{aligned}\mathcal{F}_{ev}[u_c](n) &= \sum_{k=1}^{\lfloor \frac{N}{2}\rfloor} (-1)^k (\frac{u}{N})^{2k} \sum_{L=0}^{N-1} e^{-2\pi i \frac{Ln}{N}} \cdot AQ_N(L,2k) \\ \mathcal{F}_{odd}[u_c](n) &= \sum_{k=1}^{\lfloor \frac{N+1}{2}\rfloor} (-1)^{k+1} (\frac{u}{N})^{2k-1} \sum_{L=0}^{N-1} e^{2\pi i \frac{Ln}{N}} AQ_N(L,2k-1).\end{aligned}$$

Now, we perform the inverse discrete linear Fourier transforms on both of the above equations and get the expressions (3.3) and (3.4). ☐

We now state and prove the explicit formula for the function $AQ_N(L,d)$.

Theorem 3.3. *For any $N\in\mathbb{N}$, $d\le N$ and $L\in\{0,\ldots,N-1\}$, we have*

$$AQ_N(L,d) = \begin{cases} \binom{L-1}{\lfloor \frac{d-1}{2}\rfloor}\binom{N-L}{\lfloor \frac{d}{2}\rfloor}; & d \text{ even} \\ \binom{L}{\lfloor \frac{d-1}{2}\rfloor}\binom{N-L-1}{\lfloor \frac{d}{2}\rfloor}; & d \text{ odd.}\end{cases} \tag{3.5}$$

Above we use the definition of the binomial symbol for which $\binom{a}{b} = 0$ for negative a.

Proof. Let us define

$$\widehat{AQ}_N(L,d) = \sharp\{(l_1,\ldots,l_d); N\ge l_1 > \ldots > l_d \ge 1, \text{ and } \sum_{j=1}^{d}(-1)^{j-1} l_j = L\}.$$

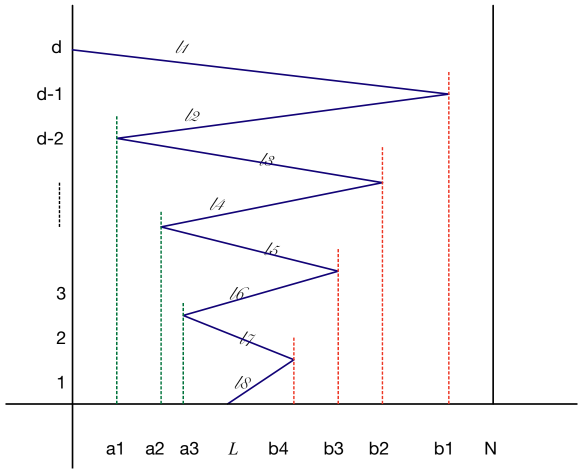


Figure 1: Zigzag path interpretation of an element of $\widehat{AQ}_N(L,d)$ with $d=8$.

We claim that for $\widehat{AQ}_N(L,d)$ we have

$$\widehat{AQ}_N(L,d) = \binom{L-1}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L}{\lfloor \frac{d}{2} \rfloor}. \tag{3.6}$$

Suppose that $d=2k$ is even. Let us consider the partial sums of the alternating sum

$$\widehat{AQ}_N(L,d) = (l_1 - l_2) + (l_3 - l_4) + \ldots + (l_{d-1} - l_d) = L, \tag{3.7}$$

namely:

$$\begin{aligned}
a_1 &= (l_1 - l_2)\\
a_2 &= (l_1 - l_2) + (l_3 - l_4)\\
\vdots & \qquad \vdots\\
a_{k-1} &= (l_1 - l_2) + (l_3 - l_4) + (l_5 - l_6) + \ldots + (l_{d-3} - l_{d-2}).
\end{aligned}$$

Let us also introduce the integers b_m, given by

$$\begin{aligned}
b_1 &= (N - l_1)\\
b_2 &= (N - l_1) + (l_2 - l_3)\\
\vdots & \qquad \vdots\\
b_k &= (N - l_1) + (l_2 - l_3) + (l_4 - l_5) + \ldots + (l_{d-2} - l_{d-1})
\end{aligned}$$

From the above definitions we see that

$$\begin{aligned}
l_1 &= N - b_1\\
l_{2m} &= (N - b_m) - a_m\\
l_{2m-1} &= (N - b_m) - a_{m-1}.
\end{aligned}$$

We shall now turn the situation around. Let

$$1 \leq \alpha_1 < \alpha_2 < \ldots, < \alpha_{k-1} \leq L - 1 \tag{3.8}$$

be an arbitrary ordered subset of $\{1, 2, 3 \ldots, L-1\}$ and let

$$0 \leq \beta_1 < \beta_2 < \ldots < \beta_k \leq N - L - 1 \tag{3.9}$$

be an arbitrary ordered subset of $\{0, 1, 2, \ldots, N-L-1\}$. Let us define

$$\begin{aligned} \lambda_1 &= N - \beta_1 \\ \lambda_{2m} &= (N - \beta_m) - \alpha_m, \qquad m = 1, 2, \ldots, k-1 \\ \lambda_{2m-1} &= (N - \beta_m) - \alpha_{m-1}, \quad m = 2, 3, \ldots, k \end{aligned}$$

From (3.8) and (3.9) we see that

$$N \geq \lambda_1 > \lambda_2 > \lambda_3 > \ldots > \lambda_{d-1} > 1$$

and

$$\lambda_1 - \lambda_2 + \lambda_3 - \ldots + \lambda_{d-1} \geq L + 1.$$

Therefore there exists precisely one λ_d such that

$$(\lambda_1 - \lambda_2 + \lambda_3 - \ldots + \lambda_{d-1}) - \lambda_d = L$$

From the construction we also see that $\lambda_d < \lambda_{d-1}$.

We have shown that for every choice of a pair (3.8) and (3.9) of subsets of

$$\{1, 2, 3 \ldots, L-1\} \quad \text{and} \quad \{0, 1, 2, \ldots, N-L-1\},$$

respectively, there exists precisely one solution $\{\lambda_1, \lambda_2, \ldots, \lambda_d\}$ of the equation (3.7). Since the number of such pairs is equal to

$$\binom{L-1}{k-1}\binom{N-L}{k} = \binom{L-1}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L}{\lfloor \frac{d}{2} \rfloor},$$

our proposition is proved for even d. The proof for odd d is only a slight variation of the above and we shall omit it.

Proof by induction. Our formula can be proved by induction on N. For $N = 2$, formula (3.6) can be checked by hand. If we divide the alternating sums from $\widehat{AQ}_N(L, d)$ into those, for which $l_1 = N$ and those for which $l_1 < N$, we get the recursion relation

$$\widehat{AQ}_N(L, d) = \widehat{AQ}_{N-1}(L, d) + \widehat{AQ}_{N-1}(N-L, d-1).$$

By the induction hypothesis, the above equation becomes

$$\begin{aligned} \widehat{AQ}_N(L, d) &= \binom{L-1}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L-1}{\lfloor \frac{d}{2} \rfloor} + \binom{N-L-1}{\lfloor \frac{d-2}{2} \rfloor}\binom{L-1}{\lfloor \frac{d-1}{2} \rfloor} \\ &= \binom{L-1}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L}{\lfloor \frac{d}{2} \rfloor}, \end{aligned}$$

and this proves (3.6). The second equality above comes from the recurrence relation of the Pascal triangle.

Finally, we observe that

$$AQ_N(L,d) = \widehat{AQ}_N(L,d), \text{ for } d \text{ even}, \quad \text{and} \quad AQ_N(L,d) = \widehat{AQ}_N(L{+}1,d), \text{ for } d \text{ odd}.$$

These relations, together with formula (3.6), prove the proposition. □

Two of the central results of this paper are corollaries of the above theorem.

Corollary 3.4. *Let the random variable*

$$X_{AS}[N,d] \colon \Delta_d^D(N) \longrightarrow \mathbb{R}$$

defined on the discrete ordered simplex

$$\Delta_d^D(N) = \{(l_1, l_2, \ldots, l_d);\ \ N-1 \geq l_1 > l_2 > \ldots > l_d \geq 0\}$$

be given by

$$X_{AS}[N,d](l_1, l_2, \ldots, l_d) = l_1 - l_2 + l_3 - \ldots + (-1)^{(d-1)} l_d.$$

Then its probability mass function is

$$P(X_{AS}[N,d] = L) = \begin{cases} \dfrac{\binom{L-1}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L}{\lfloor \frac{d}{2} \rfloor}}{\binom{N}{d}}; & d \text{ even} \\ \dfrac{\binom{L}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L-1}{\lfloor \frac{d}{2} \rfloor}}{\binom{N}{d}}; & d \text{ odd.} \end{cases}$$

Proof. The number of the favourable events is given by Theorem 3.3, proved above. To evaluate the number of all outcomes it helps to consider Figure 1. We see that the number of all outcomes is equal to the number of the subsets which are composed of all the integer points a_i, all the points b_i and the point L. These are precisely all the subsets with d elements in the set $\{1, 2, \ldots, N\}$. Their number is of course $\binom{N}{d}$. This proves our corollary. □

Inserting the formula (3.5) in the expressions (3.3) and (3.4) yields the following corollary:

Corollary 3.5. *The power series of* $\mathcal{F}_N[u_c](n)$ *around* $u = 0$ *is given by*

$$\begin{aligned}\mathcal{F}_N[u_c](n) = I \quad &+ \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} (-1)^k (\frac{u}{N})^{2k} \sum_{L=0}^{N-1} \binom{L-1}{k-1}\binom{N-L}{k} \cdot E_\delta(-2L, n) \\ &+ \sum_{k=0}^{\lfloor \frac{N+1}{2} \rfloor} (-1)^k (\frac{u}{N})^{2k+1} \sum_{L=0}^{N-1} \binom{L}{k}\binom{N-L-1}{k} \cdot E_\delta(-2L, n) \cdot J.\end{aligned}$$

4 Beta distribution and polytopes $D_d(\lambda)$

In this section we prove our second theorem which is the expression of the volumes of polytopes $D_d(\lambda)$ in terms of the probability density function of the beta distribution. We obtain this result by taking a suitable limit of the probability mass functions of the random variables $X_{AS}[N, d]$.

4.1 Discretization of beta distribution

The subset $\widehat{D}^{disc}_{d,N}(L)$, given by (2.10) of the discrete ordered simplex

$$\Delta^{disc}_{d,N} = \{(l_1, l_2, \ldots, l_d) \in (\mathbb{N} \cup \{0\})^d; N - 1 \geq l_1 > l_2 > \ldots > l_d \geq 0\}$$

with the edge of size N is given by one equation. The size $AQ_N(L, d)$ of $\widehat{D}^{disc}_{d,N}(L)$ is therefore of the order N^{d-1}.

Lemma 4.1. *Let λ be a real number in $[0, 1]$ and let $\{L_N\}_{N\in\mathbb{N}}$ be a sequence of positive integers such that $L_N < N$ and $\lim_{N\to\infty} \frac{L_N}{N} = \lambda$. Then we have*

$$\lim_{N\to\infty} \frac{AQ_N(L_N, d)}{\binom{N}{d}} N = \begin{cases} p_\beta(\lambda; \frac{d}{2}, \frac{d}{2} + 1)\,; & d \text{ even} \\ p_\beta(\lambda; \frac{d+1}{2}, \frac{d+1}{2})\,; & d \text{ odd.} \end{cases} \tag{4.1}$$

where

$$p_\beta(\lambda; a, b) = \frac{1}{B(a + 1, b + 1)} \lambda^a (1 - \lambda)^b$$

is the probability density function of the beta distribution $\mathrm{Beta}(\lambda; a, b)$.

Proof. We shall prove the formula only for even d. The proof for odd d is essentially the same. Consider first the numerator of the quotient under the limit. For $d = 2m$, formula (3.5) gives

$$AQ_N(L_N, d) = \binom{L_N - 1}{m - 1}\binom{N - L_N}{m}.$$

This expression can be expanded into

$$AQ_N(L_N, d) = \frac{1}{(m-1)!\, m!} \prod_{k=0}^{m-2} ((L_N - 1) - k) \prod_{k=0}^{m-1} ((N - L_N) - k). \tag{4.2}$$

Consider the first product above. It is a polynomial of degree $m - 1$ in the variable $(L_N - 1) = (N\frac{L_N}{N} - 1)$. Expanding this polynomial gives

$$(N\frac{L_N}{N} - 1)^{m-1} + \sum_{k=1}^{m-2} (N\frac{L_N}{N} - 1)^k \cdot n(k) = (N\frac{L_N}{N} - 1)^{m-1} + \mathcal{O}(N\frac{L_N}{N})^{m-2}$$

For large N we can replace $\frac{L_N}{N}$ by λ. Taking into account also the second product, (4.2) gives

$$\begin{aligned} AQ_N(L_N, d) &= \frac{1}{(m-1)!\, m!}\Big((N\frac{L_N}{N} - 1)^{m-1} + \mathcal{R}_1\Big)\Big((N - N\frac{L_N}{N})^m + \mathcal{R}_2\Big) \\ &= \frac{1}{(m-1)!\, m!}\Big(N^{m+m-1}(\frac{L_N}{N})^{m-1}(1 - \frac{L_N}{N})^m + \mathcal{R}_3\Big) \end{aligned} \tag{4.3}$$

where

$$\mathcal{R}_1 = \mathcal{R}_3 = \mathcal{O}(\frac{1}{(N\lambda)^{m-2}}) \quad \text{and} \quad \mathcal{R}_2 = \mathcal{O}(\frac{1}{(N\lambda)^{m-1}}).$$

For the denominator $\binom{N}{d}$ we have

$$\binom{N}{d} = \frac{1}{d!}\Big(N(N-1)\cdots(N-(d-1))\Big) = \frac{N^d}{d!} + \mathcal{O}(\frac{1}{N^{(d-2)}}). \tag{4.4}$$

Because $d-1 = m + (m-1)$ and $\lim_{N\to\infty} \frac{L_N}{N} = \lambda$ formulas (4.3) and (4.4) yield

$$\lim_{N\to\infty} \frac{AQ_N(L_N, d)}{\binom{N}{d}} N = \frac{d!}{(m-1)!m!}\ \lambda^{m-1}\lambda^m.$$

The definition of the Euler beta function for positive integers gives $\frac{d!}{(m-1)!\,m!} = \frac{1}{B(m,m+1)}$, and this proves formula (4.1) for even d. □

The above calculation suggests the definition of a discrete version $\mathrm{Beta}_N(a,b)$ of beta distribution for arbitrary choice of the shape parameters. Let a, b and N be integers. Let the probability mass function of $\mathrm{Beta}_N(a,b)$ be defined by

$$P_N(L; a, b) = \frac{\binom{L-1}{a}\binom{N-L}{b}}{\binom{N}{a+b+1}}$$

for $L \in \{1, 2, \ldots, N\}$.

Proposition 4.2. *Let λ be an arbitrary real number in the unit interval $[0,1]$ and let $\{L_N\}_{N\in\mathbb{N}}$ be a sequence, such that $L_N < N$ and $\lim_{N\to\infty} \frac{L_N}{N} = \lambda$. Then*

$$\lim_{N\to\infty} P_N(\frac{L_N}{N}, a, b)N = p_\beta(\lambda; a, b) = \frac{1}{\beta(a+1, b+1)}\ \lambda^a(\lambda-1)^b$$

Proof. The proof is an obvious adaptation of the proof of Lemma 4.1. We only have to replace the particular values $m-1$ and m of the shape parameters by an arbitrary pair a and b of positive integers. Then the same calculations as those performed in the proof of Lemma 4.1 yield the proof of the proposition. □

4.2 Volumes of polytopes $D_d(\lambda)$

Recall formula (2.7):

$$\mathcal{F}[u_c](n) = I + \sum_{d=1}^{\infty} u^d \int_0^1 \mathrm{Vol}(D_d(\lambda)) \begin{pmatrix} e^{-2\pi i\lambda n} & 0 \\ 0 & -e^{2\pi i\lambda n} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^d d\lambda.$$

We have the following theorem.

Theorem 4.3. *For every dimension* d*, the volumes of polytopes* $D_d(\lambda)$ *are essentially distributed according to the beta distribution with the shape parameters* $(\frac{d}{2}, \frac{d}{2}+1)$*, if* d *is even, and* $(\frac{d+1}{2}, \frac{d+1}{2})$*, if* d *is odd. More concretely, we have the following expression:*

$$\mathrm{Vol}(D_d(\lambda)) = \frac{1}{d!}\begin{cases} \frac{1}{B(\frac{d}{2},\frac{d}{2}+1)}\,\lambda^{\frac{d}{2}-1}(1-\lambda)^{\frac{d}{2}} & = p_\beta(\lambda;\frac{d}{2},\frac{d}{2}+1);\ d \text{ even} \\ \frac{1}{B(\frac{d+1}{2},\frac{d+1}{2})}\,\lambda^{\frac{d-1}{2}}(1-\lambda)^{\frac{d-1}{2}} & = p_\beta(\lambda;\frac{d+1}{2},\frac{d+1}{2});\ d \text{ odd}, \end{cases} \tag{4.5}$$

where $p_\beta(\lambda; a, b)$ *is the probability density function of the distribution with shape parameters* a *and* b*.*

Proof. Recall the set $\widehat{D}^{disc}_{d,N}(L)$, given by (2.10). Rescaling it by the factor $1/N$ gives the set

$$\widehat{D}^{disc}_d(\frac{L}{N}) = \{(\frac{l_1}{N}, \frac{l_2}{N}\dots, \frac{l_d}{N}); \frac{N-1}{N} \geq \frac{l_1}{N} > \dots > \frac{l_d}{N} \geq 0, \sum_{j=1}^{d}(-1)^{j-1}l_j = L\}$$

which contains the same number of points as $\widehat{D}^{disc}_{d,N}(L)$, but lies in the polytope $\widehat{D}_d(\frac{L}{N})$. Let $D^{disc}_d(\frac{L}{N})$ denote the orthogonal projection of $\widehat{D}^{disc}_d(\frac{L}{N})$ on the hyperplane

$$\{(x_1, \dots, x_{d-1}, 0)\} \subset \mathbb{R}^d.$$

The number $\sharp D^{disc}_d(\frac{L}{N})$ of points in $D^{disc}_d(\frac{L}{N})$ is clearly equal to the number of points in $\widehat{D}^{disc}_d(\frac{L}{N})$. The value $\frac{1}{N^{d-1}}\sharp D^{disc}_d(\frac{L}{N})$ is approximately equal to the volume $\mathrm{Vol}(D_d(\frac{L}{N}))$ of the projection $D_d(\frac{L}{N})$ of $\widehat{D}_d(\frac{L}{N})$ on the hyperplane $x_d = 0$ in $\mathbb{R}^d$. So, on the one hand, the number $\sharp D^{disc}_d(\frac{L}{N})$ is equal to $AQ_N(L, d)$, while on the other, the value $\frac{1}{N^{d-1}}\sharp D^{disc}_d(\frac{L}{N})$ is an approximation of $\mathrm{Vol}(D_d(L))$. Let now $\{\lambda_N\}_{N\in\mathbb{N}}$ be a sequence of rationals $\frac{L_N}{N}$ converging to $\lambda \in [0, 1]$. We have

$$\lim_{N\to\infty} \frac{1}{N^{d-1}}\,AQ_N(NL_N, d) = \mathrm{Vol}(D_d(\lambda)).$$

In the proof of Lemma 4.1 we have seen that

$$\binom{N}{d} = \frac{N^d}{d!} + \mathcal{O}(\frac{1}{N^{(d-2)}}),$$

so

$$\frac{\binom{N}{d}}{N} = \frac{N^{(d-1)}}{d!} + \mathcal{O}(\frac{1}{N^{(d-3)}}).$$

Therefore

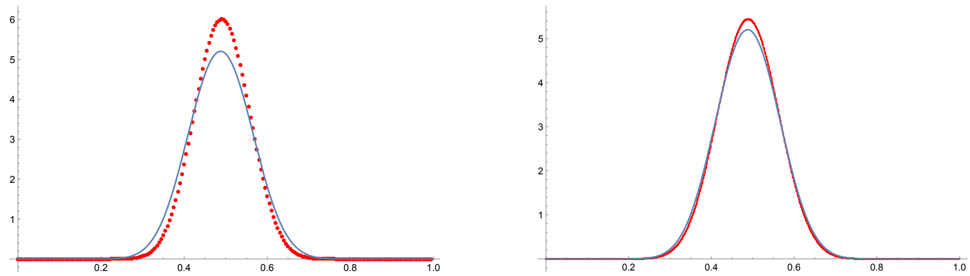
$$\lim_{N\to\infty} \frac{1}{N^{d-1}}\,AQ_N(NL_N, d) = \lim_{N\to\infty} \frac{AQ_N(L_N, d)}{\binom{N}{d}} N.$$

This equality, together with Lemma 4.1, proves our theorem. □

5 Quantitative comparisons

In this section we shall investigate by experimental means the comparison between the probability density function of $\mathrm{Beta}(l; a, b)$ distribution and its approximations, given by the probability mass functions $P_N(l; a, b)$. For the sake of brevity we shall concentrate on the shape parameters $(a, b) = (m - 1, m)$ which appear in connection with the non-linear Fourier transform. It is now clear that absolute value of the difference $p_\beta(l; a, b) - P_N(l; a, b)$ decreases for every $l = \frac{L_N}{N}$ as N increases. But the quality of the approximation also depends crucially on the choice of the shape parameters. We shall see that, roughly speaking, the value $|p_\beta(l; a, b) - P_N(l; a, b)|$ at a fixed N, increases with increasing of $a + b$. Explicit formula for this difference can be deduced from formulas (4.1) and (4.2), but it is quite complicated. The images will provide a better illustration of the relations between $p_\beta(l; a, b)$ and $P_N(l; a, b)$.

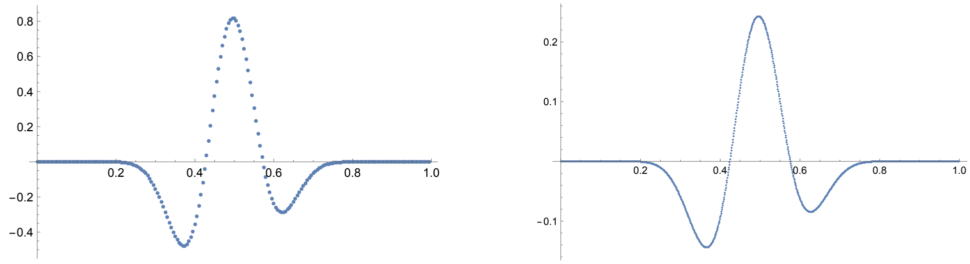
The two images in Figure 2 show the comparison between $p_\beta(l; 21, 22)$ and $P_N(l; 21, 22)$ for $N = 200$ and $N = 1000$.



(a) $N = 200$, $a = 21$, $b = 22$ (b) $N = 1000$, $a = 21$, $b = 22$

Figure 2: Comparison of graphs.

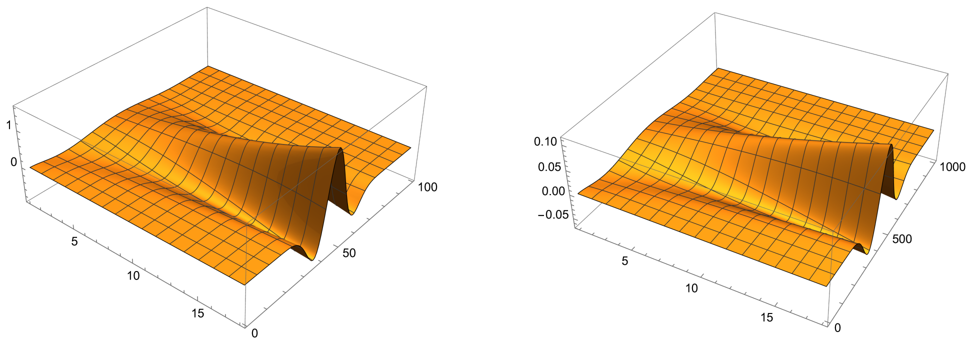
Figure 3 shows that for any choice of the shape parameters the difference $P_N(l; a, b) - p_\beta(l; a, b))$ has three local extrema. For the cases, related to the number of alternating partitions of integers where $a = b - 1$ or $a = b$, the maximum is located roughly at the center of the interval $[0, 1]$.



(a) $N = 200$, $a = 21$, $b = 22$ (b) $N = 1000$, $a = 21$, $b = 22$

Figure 3: The shape of the difference.

The two images in Figure 4 illustrate the dependence of the difference $p_\beta(l;a,b) - P_N(l;a,b)$ on the size of the shape parameters. Again we consider $(a,b) = (a, a+1)$. We see, that at a fixed N the difference increases with increasing of the shape parameter a.



(a) $N = 100$, $a \in [3, 20]$ (b) $N = 1000$, $a \in [3, 20]$

Figure 4: Dependence of the difference on the size of the shape parameter.

Even if the shape parameters a and b are very different, the corresponding graphs are of similar shapes to the above. The only difference is that, in case where the shape parameters a and b are significantly different the peaks of the graphs are shifted away from the center. This is clear from the following fact. Suppose that a is considerably larger than b. Then the left zero of limit function $p_\beta(l;a,b) = \frac{1}{B(a+1,b+1)} l^a (1-l)^b$ is of higher degree than the right one. The function is therefore flatter and closer to zero in the vicinity of 0 and the peak of the graph is pushed towards the right. Qualitatively the shape of the difference does not change.

6 Conclusions and outlook

In the paper we arrived at the construction of a discrete probability distribution with probability mass function $P_N(l;a,b)$ which converges to the probability density function $p_\beta(l;a,b)$ as $N \to \infty$. The result is precisely stated in Proposition 4.2. Crucial in the construction is the connection of $P_N(l;a,a)$ and $P_N(l;a-1,a)$ to the following combinatorial problem: find the number $AQ_N(L,d)$ of alternating ordered partitions of the positive integer $L < N$ into d distinct parts, not greater than $N-1$. The number $AQ_N(L,d)$ can also be represented by the number of the zig-zag paths, drawn in Figure (1). This combinatorial problem naturally appeared in the context of the discretization $\mathcal{F}_N$ of the nonlinear Fourier transform $\mathcal{F}$, described in Section 2. The essential connection between the numbers $AQ_N(L,d)$ and $\mathcal{F}_N$ is given by Proposition 3.2 where we show that the inverse linear Fourier transform of the entries of $\mathcal{F}_N$ yields the generating polynomials of the numbers $AQ_N(L,d)$.

The formula for distribution $P_N(l;a,b)$ can also be interpreted as the distribution describing the Pólya-Eggenberger urn, but this interpretation is different from ours. We have the connection of $P_N(l;a,b)$ to the combinatorial problem and the nonlinear Fourier transform only for the shape parameters of the form $(a,b) = (a,a)$ or $(a,b) = (a-1,a)$. The natural question arises: can we find a combinatorial problem whose relation with $P_N(l;a,b)$ for an arbitrary choice of a and b would be analogous to the relation between $P_N(l; a-1, a)$ and $P_N(l; a, a)$ and the problem of alternating ordered partitions $AQ_N(L, d)$? Does there exist a meaningful generalisation $\mathcal{F}_{a,b}$ of the nonlinear Fourier transform $\mathcal{F}$, whose relation with $p_\beta(x; a, b)$ would be analogous to the relation between $\mathcal{F}$ and $p_\beta(x; a, a)$ and $p_\beta(x; a-1, a)$, described in theorem 4.3. These are the natural problems for further investigation, based on this paper. Finding answers to these questions would importantly improve understanding the nonlinear Fourier transform and its structure.

In this paper, we considered the nonlinear Fourier transform $\mathcal{F}[u]$ evaluated on the simplest of functions, namely, the constant function $u \equiv c$. An obvious direction of further research is to try to extend the approach used in this paper, to the context, where $\mathcal{F}[u]$ is evaluated on some more interesting class of functions u.

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Products of subgroups, subnormality, and relative orders of elements

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Abstract

Let G be a group. We give an explicit description of the set of elements $x \in G$ such that $x^{|G:H|} \in H$ for every subgroup of finite index $H \leqslant G$. This is related to the following problem: given two subgroups H and K, with H of finite index, when does $|HK : H|$ divide $|G : H|$?

Keywords: Relative order, product of subgroups, subnormal subgroup.

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1 Introduction

Let G be an arbitrary group, and let us write $H \leqslant_f G$ to say that H is a subgroup of G of finite index. Let $x \in G$ and $H \leqslant_f G$. If H is a normal subgroup of G, then it is easy to see that $x^{|G:H|} \in H$. The same is not true in general: fixed $H \leqslant_f G$, the set $\{x \in G : x^{|G:H|} \in H\}$ may not even be closed under multiplication (take $G = \mathrm{Sym}(3)$ and $H = \langle (1\ 2) \rangle$). The goal of this paper is to understand this phenomenom and its implications. As far as we can see, this has not been dealt with before in the literature.

Definition 1.1. Let $x \in G$ and $H \leqslant G$. The *relative order* of x with respect to H is

$$o_H(x) := |\langle x \rangle : \langle x \rangle \cap H|.$$

The following result is proved in Section 2.

Lemma 1.2. *Let $n \geq 1$. Then $x^n \in H$ if and only if $o_H(x)$ is finite and divides n.*

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Given $H, K \leqslant G$, $|HK : H|$ is the cardinality of the set of all cosets of H which are intersected by K (we refer to Section 2 for more details). Since $o_H(x) = |H\langle x\rangle : H|$, we obtain

Corollary 1.3. *$x^{|G:H|} \in H$ if and only if $|H\langle x\rangle : H|$ divides $|G : H|$.*

If $H, K \leqslant_f G$, then $|HK : H|$ divides $|G : H|$ if and only if $|HK : K|$ divides $|G : K|$. If G is finite, both are equivalent to $|HK|$ dividing $|G|$. In Section 3, we prove the following two results:

Proposition 1.4. *Let $H \lhd \lhd G$. Then $|HK : K|$ divides $|G : K|$ for every $K \leqslant_f G$.*

Theorem 1.5. *Let $H \leqslant_f G$. Then $H \lhd \lhd G$ if and only if $|HK : H|$ divides $|G : H|$ for every $K \leqslant G$.*

The converse of Proposition 1.4 is not true in general (see Example 5.11). In particular, some attention is needed with subgroups of infinite index. During the preparation of this manuscript, the author found out that the finite version of Theorem 1.5 already appeared in [5, Theorem 2]. In Section 4, we study the following class of subgroups

Definition 1.6. A subgroup $H \leqslant_f G$ is *exponential* if $x^{|G:H|} \in H$ for every $x \in G$.

This is a generalization of subnormality, and we prove that it is equivalent to normality in some cases, namely for the Hall subgroups of a finite group and for the maximal subgroups of a solvable group. From the dual point of view, in Section 5 we study the set

$$S(G) := \{x \in G : x^{|G:H|} \in H \text{ for every } H \leqslant_f G\}.$$

At first glance $S(G)$ is quite elusive, and indeed working directly with the definition is not easy. Using the results of Section 3, we give an elementary proof of the next theorem. Given $N \lhd G$, let $F_N(G)$ be the preimage of $F(G/N)$, where $F(G)$ denotes the Fitting subgroup of G.

Theorem 1.7. *If G is any group, then $S(G) = \cap_{N \lhd_f G} F_N(G)$.*

In particular, $S(G) = F(G)$ when G is finite (Proposition 5.1). Of course, Theorem 1.7 implies that $S(G)$ is closed under multiplication, a fact which is not immediately clear from the definition.

2 Preliminaries

We start with the proof of the key Lemma 1.2.

Proof of Lemma 1.2. Let $ord_H(x) := \min\{n \geq 1 : x^n \in H\}$. We first notice that $o_H(x) = ord_H(x)$. Indeed, from the definitions we have $o_H(x) = o_{H\cap\langle x\rangle}(x)$ and $ord_H(x) = ord_{H\cap\langle x\rangle}(x)$. The fact that $o_{H\cap\langle x\rangle}(x) = ord_{H\cap\langle x\rangle}(x)$ is a simple exercise. Now, the "if" part of the statement is trivial. On the other hand, if $x^n \in H$ for some $n \geq 1$, then clearly $ord_H(x) < \infty$. Let $n = q \cdot ord_H(x) + r$ with $r, q \geq 0$ and $r < ord_H(x)$. Since H is a subgroup, the fact that $x^n = x^{q\cdot ord_H(x)} x^r \in H$ implies that $x^r \in H$, which in turn means $r = 0$. □

The bulk of this paper is about finite groups. We summarize here the basic tools and notation that are used with regard to general non-finite groups. Let G be an arbitrary group and $H, K \leqslant G$. If H and K have finite index, then so has $H \cap K$, and $|G : H \cap K| = |G : H||H : H \cap K|$. As we have said in the introduction, we write $|HK : H|$ for the cardinality of the set of all cosets of H which are intersected by K. This is not accidental, because the product set $HK = \{hk : h \in H, k \in K\}$ is a union of cosets of H. It is not relevant to distinguish between left-cosets and right-cosets, since $k \in Hx$ if and only if $k^{-1} \in x^{-1}H$. We also observe that $|HK : H| = |K : H \cap K| = |KH : H|$.
The *finite residual* $R(G)$ is the intersection of the subgroups of G of finite index. If $R(G) = 1$, then G is said to be *residually finite*. It is easy to check that $G/R(G)$ is always residually finite. Finally, the *Fitting subgroup* $F(G)$ is defined as the subgroup generated by the nilpotent normal subgroups, and coincides with the set of the elements $x \in G$ such that the normal closure $\langle x\rangle^G$ is nilpotent [1]. In general, this is a stronger condition than $\langle x\rangle$ being subnormal in G. If G is finite, then $F(G)$ itself is nilpotent, i.e. it is the largest nilpotent normal subgroup.

3 Products of subgroups

The proof of Proposition 1.4 follows immediately from the following

Lemma 3.1. *Let $H \lhd M \leqslant G$, and let $K \leqslant_f G$. Then $|HK : K|$ divides $|MK : K|$.*

Proof. We have to prove that the ratio

$$\frac{|MK : K|}{|HK : K|} = \frac{|M : M \cap K|}{|H : H \cap K|}$$

is an integer. Now $H \lhd M$ implies that $H(M \cap K)$ is a subgroup of M, and so we can write

$$\begin{aligned} |M : M \cap K| &= |M : H(M \cap K)||H(M \cap K) : M \cap K| \\ &= |M : H(M \cap K)||H : H \cap K|. \end{aligned}$$

In particular, the original ratio equals $|M : H(M \cap K)|$. □

We continue with the easiest direction of Theorem 1.5.

Lemma 3.2. *Let $H \leqslant_f M \leqslant_f G$, and let $K \leqslant G$. Then $\frac{|HK:H|}{|MK:M|} = |M \cap K : H \cap K|$.*

Proof. We have

$$\begin{aligned} \frac{|HK : H|}{|MK : M|} &= \frac{|K : H \cap K|}{|K : M \cap K|} \\ &= \frac{|K : M \cap K||M \cap K : H \cap K|}{|K : M \cap K|} \\ &= |M \cap K : H \cap K|. \end{aligned}$$

□

We prove the claim of Theorem 1.5 by induction on the subnormal defect of H, so let $H \lhd_f M \lhd\lhd_f G$, and $K \leqslant G$. Using Lemma 3.2, we have

$$\frac{|G : H|}{|HK : H|} = \frac{|G : M||M : H|}{|MK : M||M \cap K : H \cap K|}.$$

By induction, it is sufficient to prove that $\frac{|M:H|}{|M\cap K:H\cap K|}$ is an integer. Now $H \lhd M$ implies that $H(M \cap K)$ is a subgroup of M, and so we can write

$$\begin{aligned} |M : H| &= |M : H(M \cap K)||H(M \cap K) : H| \\ &= |M : H(M \cap K)||M \cap K : H \cap K|. \end{aligned}$$

This concludes the proof of the "only if" part.

3.1 The Kegel-Wielandt-Kleidman theorem, revisited

Definition 3.3. Let G be a finite group, $H \leqslant G$, and let p be a prime. Then H is p-*subnormal* in G if $H \cap P$ is a p-Sylow of H for every p-Sylow P of G.

We characterize p-subnormality with the following

Lemma 3.4. *A subgroup H is p-subnormal if and only if $|HP|$ divides $|G|$ for every p-Sylow $P \leqslant G$.*

Proof. We have that $H \cap P$ is a p-Sylow of H if and only if $|H : H \cap P| = |HP : P|$ is not divisible by p. Since $|H : H \cap P|$ is a divisor of $|G|$, the last condition is equivalent to $|HP : P|$ dividing $|G : P|$, i.e. $|HP| \mid |G|$. □

The famous Kegel-Wielandt conjecture [3, 7], proved by Kleidman [4] using the classification of the finite simple groups, says that $H \lhd \lhd G$ whenever H is p-subnormal for every p.

Theorem 3.5 (Kegel-Wielandt conjecture)**.** *If $|HP|$ divides $|G|$ for every Sylow subgroup $P \leqslant G$, then $H \lhd \lhd G$.*

See [2] for some consequences of p-subnormality for a single p. The "if" part of Theorem 1.5 follows easily. Let $H \leqslant_f G$, and assume that $|HK : H|$ divides $|G : H|$ for every $K \leqslant G$. Let $N \lhd_f G$ be the normal core of H, and let $N \leqslant K \leqslant G$ be any intermediate subgroup. Working with G/N and K/N, Theorem 3.5 gives $H/N \lhd \lhd G/N$, i.e. $H \lhd \lhd G$.

We point out that Kegel [3] did not use the classification to prove Theorem 3.5 when H is solvable. We give a very short proof in the case where H is nilpotent, which is enough for the characterization of $S(G)$ we will present in Section 5.

Lemma 3.6 (Kegel-Wielandt for nilpotent subgroups)**.** *Let $H \leqslant G$ be a nilpotent subgroup of the finite group G. If $|HP|$ divides $|G|$ for every Sylow subgroup $P \leqslant G$, then $H \lhd \lhd G$.*

Proof. Suppose that H is not subnormal, and in particular $H \nleqslant F(G)$. So there exists a p-element x such that $x \in H \setminus F(G)$. Since $x \notin O_p(G)$, there exists a p-Sylow P of G such that $x \notin P$. By hypothesis $H \cap P$ is a p-Sylow of H and, since H is nilpotent, $H \cap P$ contains all the p-elements of H. This contradicts the fact that $x \notin P$. □

Levy [5] proves the same result when H is a p-subgroup of G. Another consequence of Theorem 1.5 is that p-subnormality for every p implies that $|HK|$ divides $|G|$ for every $K \leqslant G$. We provide an elementary proof of this fact.

Lemma 3.7. *Let G be a finite group and $H \leqslant G$. If $|HP|$ divides $|G|$ for every Sylow $P \leqslant G$, then $|HK|$ divides $|G|$ for every $K \leqslant G$.*

Proof. Let $K \leqslant G$. We have to show that $|HK : K| = |H : H \cap K|$ divides $|G : K|$. Let p^α be a prime power that divides $|H : H \cap K|$. Since p^α is arbitrary, it is sufficient to prove that $p^\alpha \mid |G : K|$. Let $P_0 \leqslant K$ be a p-Sylow of K, and let $P \leqslant G$ be a p-Sylow of G such that $P \cap K = P_0$. Of course, $p^\alpha \mid |H : H \cap P_0|$. By hypothesis $|H : H \cap P| = |HP : P|$ divides $|G : P|$, and so is not divisible by p. Therefore, $p^\alpha \mid |H \cap P : H \cap P_0|$. Now $|H \cap P : H \cap P_0| = |(H \cap P)P_0 : P_0|$, and this divides $|P : P_0|$ because P is a p-group. So $p^\alpha \mid |P : P_0|$, and then of course $p^\alpha \mid |G : P_0|$. Since $p \nmid |K : P_0|$, we obtain $p^\alpha \mid |G : K|$ as desired. □

4 Exponential subgroups

We write $H \leqslant_{exp} G$ if $x^{|G:H|} \in H$ for all $x \in G$. We observe immediately that exponentiality is preserved by quotients.

Lemma 4.1. *Let $N \lhd G$, and $N \leqslant H \leqslant G$. Then $H \leqslant_{exp} G$ if and only if $H/N \leqslant_{exp} G/N$.*

Proof. Let $x \in G$ and $H \leqslant_{exp} G$. Then $(Nx)^{|G/N:H/N|} = Nx^{|G:H|} \in H/N$ and so $H/N \leqslant_{exp} G/N$. If $H/N \leqslant_{exp} G/N$, then $Nx^{|G:H|} = (Nx)^{|G/N:H/N|} \in H$, and so $x^{|G:H|} \in H$. □

Since exponential subgroups have finite index, we can apply Lemma 4.1 with the normal core, and work with a finite group. Let G be a finite group and $H \leqslant G$. From Corollary 1.3 and Theorem 1.5, we have

- $H \lhd \lhd G$ if and only if $|HK|$ divides $|G|$ for every $K \leqslant G$;
- $H \leqslant_{exp} G$ if and only if $|HC|$ divides $|G|$ for every cyclic $C \leqslant G$.

We stress that $H \leqslant_{exp} G$ whenever $|G : H|$ is a multiple of the exponent $\exp(G)$.

Remark 4.2. Every finite group of order other than a prime has a non-trivial exponential subgroup: if $\exp(G) < |G|$, then it is sufficient to take any subgroup whose order divides $|G|/\exp(G)$. Otherwise, all the Sylow subgroups of G are cyclic, and it is well known that G is solvable. In particular, G has a non-trivial normal subgroup, which is certainly exponential.
We notice a difference with the stronger condition that HK is a subgroup for every K i.e. H is a *permutable* subgroup. Indeed, it is easy to prove that if HC is a subgroup for every cyclic $C \leqslant G$, then HK is a subgroup for every $K \leqslant G$.

For every $n \geq 1$, let $G^n := \langle\{x^n : x \in G\}\rangle$. The exponential subgroups of G of index n are in correspondence with the subgroups of G/G^n of index n. Since G^n is characteristic, the property of being exponential is preserved by automorphisms. Moreover, we have the following

Lemma 4.3. *Let $H \leqslant G$ have a trivial characteristic core. Then $H \leqslant_{exp} G$ if and only if $|G : H|$ is a multiple of the exponent of G.*

Proof. Let $n = |G : H|$. By the exponentiality of H we have $G^n \leqslant H$. Since G^n is a characteristic subgroup of G contained in H, we obtain $G^n = 1$. But this means exactly that n is a multiple of $\exp(G)$. The converse is trivial. □

In general, there exist non-subnormal exponential subgroups whose index is not a multiple of the exponent. A simple example is $G = C_4 \times \mathrm{Sym}(3)$ and $H \cong C_2 \times C_2$. The following corollaries of Lemma 4.3 are obtained with the same strategy.

Corollary 4.4. *Let $H \leqslant G$ be a Hall subgroup. If $H \leqslant_{exp} G$, then $H \lhd G$.*

Proof. Suppose that H is not normal, and let $N \lhd G$ be the normal core of H. Since H/N is a Hall subgroup of G/N, by induction and Lemma 4.1, we can assume that H is core-free. Now $\exp(G)$ captures every prime dividing $|G|$, and so the contradiction is given by Lemma 4.3. □

Corollary 4.5. *Let $M \leqslant G$ be a maximal subgroup of the solvable group G. If $M \leqslant_{exp} G$, then $M \lhd G$.*

Proof. Suppose that M is not normal, and let $N \lhd G$ be the normal core of M. Since M/N is a maximal subgroup of G/N, by induction and Lemma 4.1, we can assume that M is core-free. Now $|G : M| = q^\alpha$ for some prime power q^α. If G is a q-group we are done. Otherwise, the contradiction is given by Lemma 4.3. □

We cannot drop the hypothesis of solvability in Corollary 4.5: the alternating group $G = \mathrm{Alt}(10)$ has a conjugacy class of maximal subgroups M of size 720. Since $\exp(G) = 2520 = |G : M|$, it appears that M is an exponential maximal subgroup which is not normal.
We conclude this section with the hereditary properties of exponential subgroups.

Lemma 4.6. *The following are true:*

- *If $H \leqslant_{exp} M \leqslant_{exp} G$, then $H \leqslant_{exp} G$;*
- *The intersection of exponential subgroups is exponential.*

Proof. Let $x \in G$. Since $M \leqslant_{exp} G$, we have $m = x^{|G:M|} \in M$. Then $x^{|G:H|} = m^{|M:H|} \in H$. To prove the second statement, it is sufficient to notice that $|G : H \cap K|$ is a multiple of both $|G : H|$ and $|G : K|$. □

Other important properties of the lattice of the subnormal subgroups are not true for exponential subgroups, and the dihedral group $G = D_{12}$ is a good source of counterexamples. Every subgroup of G whose order is 2 is exponential in G, since $\exp(G) = 6$. Let H be any non-central subgroup of order 2. Now

- The subgroup $H_1 = \langle H, Z(G)\rangle \cong C_2 \times C_2$ provides a counterexample to the statement that two exponential subgroups generate an exponential subgroup: choosing any involution $x \in G \setminus H_1$ we get $x^{|G:H_1|} = x \notin H_1$.
- The subgroup H_2 which satisfies $H < H_2 \cong \mathrm{Sym}(3)$ provides a counterexample to the statement that the intersection of an exponential subgroup of G with any subgroup of G is exponential in that subgroup: choosing any involution $x \in H_2 \setminus H$, we get that H is not exponential in H_2 although it is exponential in G.

5 The set $S(G)$

Let us recall the definition of $S(G)$ given in the introduction:

$$S(G) := \{x \in G : x^{|G:H|} \in H \text{ for every } H \leqslant_f G\}.$$

From Corollary 1.3, we have

$$S(G) = \{x \in G : |H\langle x\rangle : H| \text{ divides } |G:H| \text{ for every } H \leqslant_f G\}.$$

The results of Section 3 allow to settle the finite case easily:

Proposition 5.1. *If G is finite, then* $S(G) = F(G)$.

Proof. Let $x \in G$. Then $x \in S(G)$ if and only if $|H\langle x\rangle|$ divides $|G|$ for every $H \leqslant G$. From Proposition 1.4 and Lemma 3.6, this is equivalent to $\langle x\rangle \lhd\lhd G$, i.e. $x \in F(G)$. □

5.1 A top-down approach

Let G be an arbitrary group and let $R(G) = \cap_{H \leqslant_f G} H$ be its finite residual. The condition in the definition of $S(G)$ is empty on $R(G)$, and so $R(G) \subseteq S(G)$. In fact, $S(G)$ is the preimage of $S(G/R(G))$ under the projection $G \twoheadrightarrow G/R(G)$.

Lemma 5.2. *Let* $N \lhd G$. *Then* $S(G/N) = \{Nx : x^{|G:H|} \in H$ *for every* $N \leqslant H \leqslant_f G\}$. *In particular,* $S(G/R(G)) = S(G)/R(G)$.

Proof. Let $x \in G$ and $N \leqslant H \leqslant_f G$. The equality $(Nx)^{|G:H|} = Nx^{|G:H|}$ implies that $Nx \in H/N$ if and only if $x^{|G:H|} \in H$, and the first part follows because H is arbitrary. The second part follows because $R(G)$ contains all the subgroups of G of finite index. □

As a consequence of Lemma 5.2, we can assume that G is residually finite. Given $N \lhd G$, let $F_N(G)$ be the preimage of $F(G/N)$.

Proof of Theorem 1.7. We have to prove that $S(G) = \cap_{N \lhd_f G} F_N(G)$. Let $x \in S(G)$ and $N \lhd_f G$. From Lemma 5.2 and Proposition 5.1 we have $Nx \in S(G/N) = F(G/N)$, i.e. $x \in F_N(G)$.
On the other hand, let $x \in \cap_{N \lhd_f G} F_N(G)$ and $H \leqslant_f G$. If $N \lhd_f G$ is the normal core of H, then in particular $x \in F_N(G)$. From Proposition 5.1 we have

$$Nx \in \frac{F_N(G)}{N} = F(G/N) = S(G/N),$$

and so Lemma 5.2 provides $x^{|G:H|} \in H$. The proof follows because H is arbitrary. □

The following observation deletes a bunch of terms from $\cap_{N \lhd_f G} F_N(G)$.

Lemma 5.3. *Let G be a finite group and* $N \lhd G$. *Then* $F(G) \leqslant F_N(G)$.

Proof. We have that $NF(G)/N \cong F(G)/(N \cap F(G))$ is a nilpotent normal subgroup of G/N. Then $NF(G)/N \leqslant F(G/N) = F_N(G)/N$, and so $NF(G) \leqslant F_N(G)$. □

Corollary 5.4. *If* $N, K \lhd_f G$ *and* $K \leqslant N$, *then* $F_K(G) \leqslant F_N(G)$.

As a particular case of Theorem 1.7, we have

Proposition 5.5. *Let G be a group. The following are equivalent:*

(A) $G = S(G)$*;*

(B) *every subgroup of finite index of G is exponential;*

(C) *every finite quotient of G is nilpotent;*

(D) *every subgroup of finite index of G is subnormal.*

Proof. This follows easily from Theorem 1.7. □

We say that a group G is S-free if $S(G) = 1$.

Lemma 5.6. *Let G be a group which is residually S-free. Then $S(G) = 1$.*

Proof. Let $1 \neq x \in G$. By definition, there exists $N \lhd G$ such that $x \notin N$ and $S(G/N) = 1$. In particular $Nx \notin S(G/N)$, and so from Lemma 5.2 we obtain $x \notin S(G)$. Since x is arbitrary, it follows that $S(G) = 1$. □

Corollary 5.7. *If F is a finitely generated free group, then $S(F) = 1$.*

5.2 Baer groups and S-groups

Following a different approach, now we study $S(G)$ starting from the subgroups of G. This will provide a counterexample to the converse of Proposition 1.4.
Let $B(G) := \{x \in G : \langle x \rangle \lhd \lhd G\}$ be the *Baer radical* of G. It is clear that $B(G)$ is a characteristic subgroup. Moreover, $B(G)$ coincides with $F(G)$ if G is finite, but it can be much larger in general (see [1, Example 85]). A group which equals its Baer radical is called a *Baer group*. The same argument in the proof of Proposition 5.1 shows that $B(G) \subseteq S(G)$. We say that a group is an *S-group* if it satisfies the equivalent conditions of Proposition 5.5. It is easy to see that the class of S-groups is closed by subgroups of finite index and quotients. Of course, every Baer group is an S-group.

Proposition 5.8 (Theorem 73 in [1])**.** *A group is a Baer group if and only if every its finitely generated subgroup is subnormal and nilpotent. In particular, every finitely generated Baer group is nilpotent.*

By Propositions 5.5 and 5.8, every finitely generated non-nilpotent p-group is an S-group which is not Baer. The next theorem of Wilson [8] provides many groups with *trivial* Baer radical. We recall that an infinite group is just-infinite if every its proper quotient is finite.

Theorem 5.9 (Theorem 2 in [8])**.** *Let G be a just-infinite group. If $B(G) \neq 1$, then $B(G)$ is a free abelian group of finite rank, which coincides with its own centralizer in G.*

Lemma 5.10. *Let G be a just-infinite p-group. Then $S(G) = G$, but $B(G) = 1$.*

Proof. The fact that $G = S(G)$ follows from Proposition 5.5 and the fact that finite p-groups are nilpotent. If $B(G) \neq 1$, then $B(G)$ is a free abelian group by Theorem 5.9, which contraddicts that G is a p-group. □

Example 5.11 (No converse to Proposition 1.4)**.** Let G be a just-infinite p-group, and let $K \leqslant G$ be any nilpotent subgroup. Since every subgroup of finite index of G is subnormal, from Theorem 1.5 we have that $|HK : H|$ divides $|G : H|$ for every $H \leqslant_f G$. On the other hand, K is not subnormal in G, because $B(G) = 1$.

Finally, it is worth to mention the following theorem of Robinson [6]. Given a group property $\mathcal{P}$, a group is hyper-$\mathcal{P}$ if every its non-trivial homomorphic image has some non-trivial normal subgroup with the property $\mathcal{P}$.

Theorem 5.12 (Theorem 1 in [6])**.** *Let G be a finitely generated hyperabelian or hyperfinite group. If G is an S-group, then G is nilpotent.*

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Coincident-point rigidity in normed planes

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Abstract

A bar-joint framework (G, p) is the combination of a graph G and a map p assigning positions, in some space, to the vertices of G. The framework is rigid if every edge-length-preserving continuous motion of the vertices arises from an isometry of the space. We will analyse rigidity when the space is a (non-Euclidean) normed plane and two designated vertices are mapped to the same position. This non-genericity assumption leads us to a count matroid first introduced by Jackson, Kaszanitsky and the third author. We show that independence in this matroid is equivalent to independence as a suitably regular bar-joint framework in a non-Euclidean normed plane with two coincident points; this characterises when a regular non-Euclidean normed plane coincident-point framework is rigid and allows us to deduce a delete-contract characterisation. We then apply this result to show that an important construction operation (generalised vertex splitting) preserves the stronger property of global rigidity in non-Euclidean normed planes and use this to construct rich families of globally rigid graphs when the non-Euclidean normed plane is analytic.

Keywords: Bar-joint framework, global rigidity, non-Euclidean framework, count matroid, recursive construction, normed spaces, analytic norm.

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1 Introduction

A bar-joint *framework* (G, p) is the combination of a graph $G = (V, E)$ and a map $p\colon V \to \mathbb{R}^d$ assigning positions to the vertices of G (and hence lengths to the edges). Note that in this article graphs are taken to be finite and simple. Intuitively, the framework is *rigid* if every edge-length-preserving continuous motion of the vertices arises from an isometry of $\mathbb{R}^d$. More strongly, (G, p) is *globally rigid* if every framework in $\mathbb{R}^d$, on the same graph, with the same edge lengths actually has the same distance between every pair of vertices.

The rigidity and global rigidity of bar-joint frameworks in Euclidean spaces has been intensely studied in recent years (e.g. [2, 3, 13, 16, 21, 23]) and has a rich history going as far back as classical work of Euler and Cauchy on Euclidean polyhedra. In the last decade, work on rigidity has been generalised to various non-Euclidean normed spaces (e.g. [6, 9, 10, 19, 20]). All of these results concern characterising the combinatorial nature of the 'generic' behaviour of frameworks. This article extends this to frameworks with two points lying in the same location. The difficulty that already arises in this context shows how necessary the genericity assumption in those papers really was. Frameworks with coincident points have been considered in the Euclidean context [12, 14] and applied to global rigidity there [4], as well as for frameworks on surfaces [17].

Beyond the natural extension towards non-generic frameworks (and thus nearer to being of potential use in applications), we are motivated by the study of global rigidity in non-Euclidean normed planes. The first and third author recently instigated research in this direction [10] proving global rigidity for an infinite class of graphs in non-Euclidean analytic normed planes. In this paper we use our analysis of frameworks with two coincident points to improve this result by creating a substantially richer class of globally rigid graphs.

We now give a short outline of what follows. After introducing the necessary background on the theory of rigid frameworks in normed planes, coincident-point frameworks and the relevant notion of graph sparsity, in Section 2, the majority of the paper is contained in Section 3. Here we provide a detailed geometric analysis of the effect of certain graph operations on the rigidity of a coincident-point framework in a normed plane. In Section 4 we combine these geometric results with combinatorial results of [17] to establish a purely combinatorial characterisation of independence in the 'coincident-point normed plane rigidity matroid' and we deduce from this a delete-contract characterisation of coincident-point rigidity in any strictly convex non-Euclidean normed plane. By delete-contract characterisation we mean that we characterise the coincident-point rigidity of a graph G in terms of the rigidity of two graphs related to G; the graphs obtained from G by deleting the edge between the coincident vertices and the graph obtained by contracting the two coincident vertices. In Section 5 we provide our other main results; these concern global rigidity. We deduce from our delete-contract characterisation that another graph operation preserves global rigidity, and we use this result alongside the results of [10] to establish global rigidity in the special case of non-Euclidean analytic normed planes for a rich family of graphs.

We conclude the introduction with a brief comparison with the more familiar Euclidean case to give context for the reader. Both our characterisation of independence in the coincident-point normed plane rigidity matroid and our delete-contract characterisation are precise analogues of results obtained by Fekete, Jordán and Kaszanitzky for the Euclidean case [12]. Furthermore, in the Euclidean case generic global rigidity in the plane is completely characterised [16]. Our results provide a key step towards establishing an analogue of that result in non-Euclidean analytic normed planes. It is worth noting though that the results of [12] came later than the Euclidean plane characterisation and, to our knowledge, have not been used to provide an alternative proof of the global rigidity characterisation in the Euclidean plane. The non-Euclidean normed plane case requires both subtly different combinatorics and geometry which motivated our deployment of this technique. We would expect that our application to global rigidity through 'generalised vertex splitting' (defined in Section 5) could be adapted to the Euclidean case.

2 Rigidity and uv-coincident frameworks in normed spaces

2.1 Rigidity in normed spaces

Let X be a real finite-dimensional normed space with norm $\|\cdot\|$. We define a *support functional* of $z \in X$ to be a linear functional $f\colon X \to \mathbb{R}$ such that $f(z) = \|z\|^2$ and $\sup_{\|x\|=1} f(x) = \|z\|$. It follows from the Hahn-Banach theorem that every point has a support functional and every linear functional of X is the support functional of a point in X. A non-zero point in X is said to be *smooth* if it has exactly one support functional, and we shall denote the unique support functional of a smooth point z by φ_z. We say X is *smooth* if every non-zero point in X is smooth, and *strictly convex* if every linear functional of X is the support functional of at most one, and hence exactly one, point in X.[1] We note that for normed planes (2-dimensional normed spaces), strict convexity is equivalent to the property that any two linearly independent smooth points have linearly independent support functionals.

Now let (G, p) be a framework in X; that is the combination of a graph $G = (V, E)$ and a map $p\colon V \to X$ (called a *placement* of G). A *finite flex* of (G, p) is a continuous path $\alpha\colon [0, 1] \to X^V$ where $\alpha(0) = p$ and $\|\alpha_x(t) - \alpha_y(t)\| = \|p_x - p_y\|$ for each edge $xy \in E$ and every $t \in [0, 1]$. If every framework $(G, \alpha(t))$ is congruent to (G, p), i.e. there exists an isometry $f_t\colon X \to X$ so that $\alpha_x(t) = f_t(p_x)$ for every $x \in V$, then we say α is *trivial*. We now define (G, p) to be *(continuously) rigid* if every finite flex of (G, p) is trivial.

Since determining whether a framework is rigid is computationally challenging [1], we follow the literature and linearise the problem. First, let (G, p) be a *well-positioned* framework, i.e. the point $p_x - p_y$ is smooth for each edge $xy \in E$. An *infinitesimal flex* of (G, p) is a map $u\colon V \to X$ where $\varphi_{p_x - p_y}(u_x - u_y) = 0$ for all $xy \in E$. An infinitesimal flex is *trivial* if there exists a linear map $T\colon X \to X$ and a point $z_0 \in X$ so that $u_x = T(p_x) + z_0$ for every vertex $x \in V$, and the map T is tangent to the linear isometry group of X at the identity map. Importantly, when X has finitely many linear isometries – for example, when X is a non-Euclidean normed plane [25, page 83] – the only trivial infinitesimal flexes are those that stem from translations, i.e., infinitesimal flexes $u = (u_x)_{x \in V}$ where there exists $z \in X$ such that $u_x = z$ for all $x \in V$. We now say that a well-positioned framework (G, p) is *infinitesimally rigid* if every infinitesimal flex of (G, p) is trivial.

For a d-dimensional normed space X, a well-positioned framework (G, p) in X, and a fixed basis $b_1, \ldots, b_d$ of X, we can define the *rigidity matrix* to be the $|E| \times d|V|$ matrix

[1] Here we have opted to use a more relevant – but still equivalent – definition for strict convexity. The more conventional definition for the property is as follows: a normed space is said to be strictly convex if $\|tx + (1 - t)y\| < 1$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and each $0 < t < 1$. To see the equivalence, note that if $\sup_{\|z\|=1} f(z) = 1$ and $\|x\| = \|y\| = 1$, then $f(x) = f(y) = 1$ if and only if $\|tx + (1-t)y\| = 1$ for all $0 < t < 1$: this latter fact stems from the inequality $tf(x) + (1-t)f(y) \leq f(tx + (1-t)y) \leq \|tx + (1-t)y\| \leq 1$.

$R(G,p)$, where for every $e \in E$, $x \in V$ and $i \in \{1,\dots,d\}$ we have

$$R(G,p)_{e,(x,i)} = \begin{cases} \varphi_{p_x - p_y}(b_i) & \text{if } e = xy, \\ 0 & \text{otherwise.} \end{cases}$$

The choice of basis used to define $R(G,p)$ can be arbitrary as we are only interested in the sets of linearly independent rows of the matrix. We say a well-positioned framework is *independent* if $\operatorname{rank} R(G,p) = |E|$, *minimally (infinitesimally) rigid* if it is both independent and infinitesimally rigid, and *regular* if $\operatorname{rank} R(G,p) \geq \operatorname{rank} R(G,q)$ for all other well-positioned frameworks (G,q). It is immediate that all independent and/or infinitesimally rigid frameworks are regular. Given k is the dimension of the linear space of trivial infinitesimal flexes of (G,p), it can be shown that so long as the affine span of the set $\{p_x : x \in V\}$ is X, the framework (G,p) will be infinitesimally rigid if and only if $\operatorname{rank} R(G,p) = d|V| - k$; see [6, Proposition 3.13]. Consequently any well-positioned framework where the affine span of the set $\{p_x : x \in V\}$ is X, is minimally rigid if and only if $|E| = \operatorname{rank} R(G,p) = d|V| - k$.

We can link infinitesimal rigidity to rigidity with the following result.

Theorem 2.1. *Let (G,p) be a well-positioned framework in a normed space X.*

(i) [7, Theorem 3.7] *If (G,p) is infinitesimally rigid, then it is rigid.*

(ii) [6, Theorem 1.1 & Lemma 4.4] *If (G,p) is regular and rigid, and the set of smooth points in X is open, then (G,p) is infinitesimally rigid.*

We shall make use of the following perturbation result throughout the paper. It will be convenient to refer to properties of placements rather than frameworks. To this end we say that a placement p of G has property P if the framework (G,p) has property P.

Lemma 2.2 ([6, Lemmas 4.1 and 4.4])**.** *For any graph G and any normed space X, the set of well-positioned placements of G in X is a conull (i.e. the complement of a set with Lebesgue measure zero) subset of X^V, and the set of regular placements of G in X is a non-empty open subset of the set of well-positioned placements.*

We say that a graph is *rigid* (respectively, *independent*, *minimally rigid*) if it has an infinitesimally rigid (respectively, independent, minimally rigid) placement.

Whether a graph $G = (V,E)$ is rigid/independent in a normed plane can be determined by simple sparsity counting conditions. For $\emptyset \neq U \subseteq V$, $i_G(U)$ will denote the number of edges in the subgraph, $G[U]$, of G induced by U. For non-negative integers k, ℓ, we say G is *(k,ℓ)-sparse* if $i_G(U) \leq k|U| - \ell$ for every $\emptyset \neq U \subseteq V$ with $|U| \geq k$; if G is (k,ℓ)-sparse and $|E| = k|V| - \ell$, then we say G is *(k,ℓ)-tight*.

Theorem 2.3 ([5])**.** *A graph G is minimally rigid in a non-Euclidean normed plane X if and only if G is $(2,2)$-tight.*

For a family $\mathcal{S} = \{S_1, S_2, \dots, S_k\}$ of subsets $S_i \subseteq V$, $1 \leq i \leq k$, we say that $\mathcal{S}$ is a *cover* of $F \subseteq E$ if $F \subseteq \{xy : \{x,y\} \subseteq S_i$ for some $1 \leq i \leq k\}$. We can combine Theorem 2.3 with [17, Section 3.1] (which simply applies a classical result of Edmonds [11] on matroids induced by submodular functions) to obtain the following result.

Corollary 2.4. *Let (G, p) be a well-positioned framework in a non-Euclidean normed plane X. Let $\mathcal{S}$ be the set of all covers $\mathcal{X} := \{X_1, \ldots, X_k\}$. Given $s\colon \mathbb{N} \to \{0, 1\}$ is the map with $s(x) = 1$ if $x = 2$ and $s(x) = 0$ otherwise, we have*

$$\operatorname{rank} R(G, p) \leq \min_{\mathcal{X} \in \mathcal{S}} \sum_{i=1}^{k} (2|X_i| - 2 - s(|X_i|)),$$

with equality if and only if (G, p) is regular. Moreover it suffices to minimise over all covers $\mathcal{Y} := \{Y_1, \ldots, Y_k\}$ of the edge set E where $|Y_i| \geq 2$ for each i and $|Y_i \cap Y_j| \leq 1$ for all $i \neq j$, with equality only if $\min\{|Y_i|, |Y_j|\} = 2$.

2.2 uv-coincident rigidity and uv-sparse graphs

Let $G = (V, E)$ be a graph with vertices $u, v \in V$, and let X be a normed space. A framework (G, p) in X is *uv-coincident* if $p_u = p_v$; if the framework $(G - uv, p)$ is well-positioned, then we say that (G, p) is a *well-positioned* uv-coincident framework. Since $p_u = p_v$, we consider $G - uv$ so as to maintain smoothness of the support functionals associated with the framework; otherwise, no uv-coincident framework with uv as an edge would be well-positioned.

A well-positioned uv-coincident framework (G, p) is *infinitesimally rigid* if $(G - uv, p)$ is infinitesimally rigid in X. Given the linear space $X^V/uv := \{q \in X^V : q_u = q_v\}$, we say that a well-positioned uv-coincident framework (G, p) is *regular* if $\operatorname{rank} R(G - uv, p) \geq \operatorname{rank} R(G - uv, q)$ for all $q \in X^V/uv$, and *independent* if $uv \notin E$ and (G, p) is independent in X. A well-positioned uv-coincident framework (G, p) is *minimally (infinitesimally) rigid* if it is both infinitesimally rigid and independent. We say a graph G is *uv-rigid* (respectively, *uv-independent*, *minimally uv-rigid*) if there exists a uv-coincident framework (G, p) that is infinitesimally rigid (respectively, independent, minimally rigid).

By applying the same methods used to prove Lemma 2.2, we can obtain the natural analogue for uv-coincident frameworks. The two main observations for proving the result are: (i) the set of smooth points of a normed space form a conull subset (i.e. the complement of a set with Lebesgue measure zero) of X and (ii) the map $p \mapsto R(G - uv, p)$ is lower semi-continuous.

Lemma 2.5. *For any graph G and any normed space X, the set of well-positioned uv-coincident placements of G in X is a conull subset of X^V/uv, and the set of regular uv-coincident placements of G in X is a non-empty open subset of the set of well-positioned uv-coincident placements.*

As will be shown in Section 4, uv-rigidity in non-Euclidean normed planes is closely related to the following sparsity property of graphs.

Let $G = (V, E)$ be a graph and let u, v be two distinct vertices of G. Let $\mathcal{X} = \{X_1, X_2, ..., X_k\}$ be a family with $X_i \subseteq V$, $1 \leq i \leq k$. We say that $\mathcal{X}$ is *uv-compatible* if $u, v \in X_i$ and $|X_i| \geq 3$ hold for all $1 \leq i \leq k$. We define the *value* of non-empty subsets of V and of uv-compatible families, denoted $\operatorname{val}(\cdot)$, as follows. For $\emptyset \neq U \subseteq V$, we let

$$\operatorname{val}(U) = 2|U| - t_U,$$

where $t_U = 4$ if $U = \{u, v\}$, $t_U = 3$ if $U \neq \{u, v\}$ and $|U| \in \{2, 3\}$, and $t_U = 2$

otherwise. For a uv-compatible family $\mathcal{X} = \{X_1, X_2, \dots, X_k\}$ we let

$$\mathrm{val}(\mathcal{X}) = \left(\sum_{i=1}^{k} \mathrm{val}(X_i) \right) - 2(k-1) = 2 + \sum_{i=1}^{k} (2|X_i| - t_{X_i} - 2).$$

Note that if $\mathcal{X} = \{U\}$ is a uv-compatible family containing only one set then the two definitions agree, i.e. $\mathrm{val}(\mathcal{X}) = \mathrm{val}(U)$ holds.

We say that G is *uv-sparse* if for all $U \subseteq V$ with $|U| \geq 2$ we have $i_G(U) \leq \mathrm{val}(U)$ and for all uv-compatible families $\mathcal{X}$ we have $i_G(\mathcal{X}) := \left| \bigcup_{i=1}^{k} E(G[X_i]) \right| \leq \mathrm{val}(\mathcal{X})$. A graph G is *uv-tight* if it is uv-sparse and $|E| = 2|V| - 2$. Note that if G is uv-sparse then $uv \notin E$. It was shown in [17] that if $G = (V, E)$ is a graph and $u, v \in V$ are distinct vertices of G then $\mathcal{I} = \{F : F \subseteq E$ and (V, F) is uv-sparse$\}$ is the family of independent sets of a matroid $\mathcal{M}_{uv}$ on E.

It is straightforward to construct $(2,2)$-sparse graphs which are not uv-sparse. Perhaps the simplest way is to notice that the complete bipartite graph $K_{2,3}$, with the part of size two comprising of u and v, is clearly $(2,2)$-sparse but fails to be uv-sparse. To see this let v_1, v_2, v_3 be the vertices in the part of size three and consider the uv-compatible family $\mathcal{X} = \{X_1, X_2, X_3\}$ where $X_1 = \{u, v, v_1\}$, $X_2 = \{u, v, v_2\}$ and $X_3 = \{u, v, v_3\}$. Then $i_G(\mathcal{X}) = 2+2+2 = 6$ and $\mathrm{val}(\mathcal{X}) = (2 \cdot 3 - 3) + (2 \cdot 3 - 3) + (2 \cdot 3 - 3) - 2(3-1) = 5$.

3 Recursive operations

Let $G = (V, E)$ be a graph. The 0-*extension* operation (on a pair of distinct vertices $a, b \in V$) adds a new vertex z and two edges za, zb to G. The *1-extension* operation (on edge $ab \in E$ and vertex $c \in V \setminus \{a, b\}$) deletes the edge ab, adds a new vertex z and edges za, zb, zc. The *vertex-to-H move* adds a copy of a $(2,2)$-tight graph H with $V(H) \cap V = \{w\}$, along with an arbitrary replacement of each edge xw by an edge of the form xy with $y \in V(H)$. A *vertex-to-4-cycle move* takes a vertex w with neighbours $v_1, v_2, \dots, v_k$ for any $k \geq 2$, splits w into two new vertices w, w' with $w' \notin V$, adds edges $wv_1, w'v_1, wv_2, w'v_2$ and then arbitrarily replaces edges xw with edges of the form xy where $x \in \{v_3, \dots, v_k\}$ and $y \in \{w, w'\}$. All $(2,2)$-tight graphs can be constructed from a single vertex by a sequence of 0- and 1-extensions, vertex-to-4-cycle and vertex-to-K_4 operations; see [22, Theorem 3.1] for more details. The operations we use are illustrated in Figures 1 and 2.

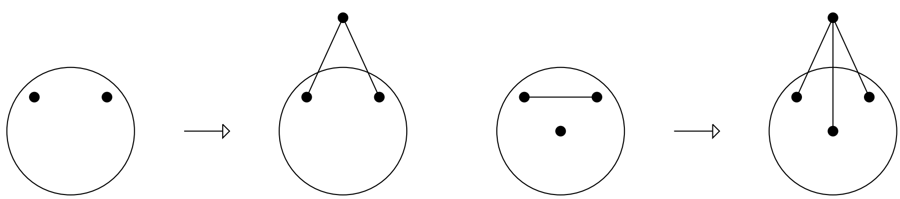

Figure 1: 0-extension and 1-extension.

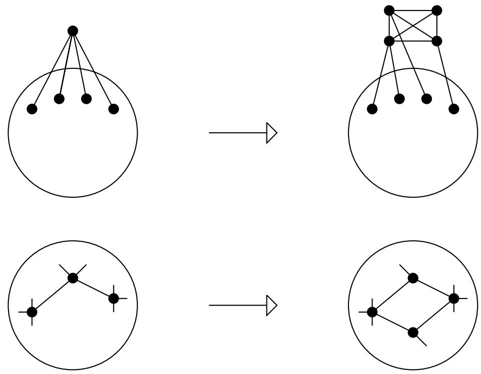

Figure 2: The vertex-to-H (with H being the complete graph on 4 vertices) and vertex-to-4-cycle operations.

We shall need the following specialized versions. First, suppose that $|V \cap \{u, v\}| = 1$. The *0-extension that adds* u (respectively, *0-extension that adds* v) operation is a 0-extension where $z = u$ and $v \in V \setminus \{a, b\}$ (respectively, with $z = v$ and $u \in V \setminus \{a, b\}$). The *vertex-to-4-cycle move that adds* u (respectively, *vertex-to-4-cycle move that adds* v) is a vertex-to-4-cycle move where $w = v$ and $\{w, w'\} = \{u, v\}$ (respectively, $w = u$ and $\{w, w'\} = \{u, v\}$). The *vertex-to-H move that adds* u (respectively, *vertex-to-H move that adds* v) is a vertex-to-H move where $w = v$ and $u \in V(H) \setminus V$ (respectively, $w = u$ and $v \in V(H) \setminus V$), and the graph H is uv-tight.

Now suppose $u, v \in V$ are two distinct vertices. The *uv-0-extension* operation is a 0-extension on a pair a, b with $\{a, b\} \neq \{u, v\}$. The *uv-1-extension* operation is a 1-extension on some edge ab and vertex c for which $\{u, v\}$ is not a subset of $\{a, b, c\}$. The *uv-vertex-to-4-cycle* and *uv-vertex-to-H* moves are simply any vertex-to-4-cycle and vertex-to-H moves applied to a graph containing both u and v.

We can immediately obtain the following result using the proof technique of [5, Lemmas 5.1 and 5.2]. In particular, since the uv-0- and uv-1-extensions are local operations that relate to at most one of u and v, their coincidence does not have any effect on the proofs presented in [5].

Lemma 3.1. *Let G be a graph that contains both u and v, and let G' be formed from G by either a uv-0-extension or a uv-1-extension. If G is uv-independent in a normed plane X, then G' is uv-independent in X.*

The next lemma shows 0-extensions that add either u or v preserve independence. It should be noted that our proof technique requires strict convexity.

Lemma 3.2. *Let $G = (V, E)$ be a graph that contains u but not v, and let X be a strictly convex normed plane. Suppose G' is formed from G by a 0-extension that adds v. Then G' is uv-independent if and only if G is independent.*

Proof. We note that as G' contains G as a subgraph, if G' is uv-independent then G will be independent. Suppose there is an independent placement p of G in X. By applying translations, we may suppose that $p_u = 0$. Let v_1, v_2 be the two neighbours of v in G'. We may also assume that p_{v_1} and p_{v_2} are linearly independent and smooth; indeed if this was not true, we could apply Lemma 2.2 to (G, p) to find a placement of G where it is true. Define p' to be the well-positioned placement of G' with $p'_x = p_x$ for all $x \in V$ and $p'_v = p_u$. From our choice of placement of G', we see that

$$R(G', p') = \left[\begin{array}{c|c} R(G,p) & \mathbf{0}_{|E|\times 2} \\ \hline A & -\varphi_{p_{v_1}} \\ B & -\varphi_{p_{v_2}} \end{array}\right]$$

for some $1 \times 2|V|$ matrices A and B. Hence (G', p') is independent if and only if $\varphi_{p_{v_1}}, \varphi_{p_{v_2}}$ are linearly independent. Since p_{v_1}, p_{v_2} are linearly independent and X is strictly convex, the pair $\varphi_{p_{v_1}}, \varphi_{p_{v_2}}$ are linearly independent as required. □

For the vertex-to-4-cycle move we will use the technique of [17, Lemma 11] to show that a vertex-to-4-cycle move which creates two coincident vertices preserves independence. Similarly to the previous result, we will require that the normed plane in question is strictly convex.

Lemma 3.3. *Let $G = (V, E)$ and $G' = (V', E')$ be graphs and let X be a strictly convex normed plane.*

(i) *If G is independent in X and G' is formed from G by a vertex-to-4-cycle move that adds either u or v, then G' is uv-independent in X.*

(ii) *If G is uv-independent in X and G' is formed from G by a uv-vertex-to-4-cycle move, then G' is uv-independent in X.*

Proof. Suppose that G is uv-independent (respectively, independent). Using Lemma 2.5 (respectively, Lemma 2.2), choose a uv-independent (respectively, independent) placement p of G in X so that p_w, p_{v_1}, p_{v_2} are not collinear. By applying translations to p, we shall assume that $p_w = 0$. Now define p' to be the placement of G' with $p'_x = p_x$ for all $x \in V$ and $p'_{w'} = p_w$. The pair (G', p') form a well-positioned uv-coincident framework due to our choice of p'. Since X is strictly convex, the pair $\varphi_{p_{v_1}}, \varphi_{p_{v_2}}$ are linearly independent. Define G'' to be the graph formed from G' by replacing each edge $w'v_i$ for $3 \leq i \leq k$ with the edge wv_i. Then

$$R(G'', p') = \left[\begin{array}{c|c} R(G,p) & \mathbf{0}_{|E|\times 2} \\ \hline A & \varphi_{p'_{w'} - p'_{v_1}} \\ B & \varphi_{p'_{w'} - p'_{v_2}} \end{array}\right] = \left[\begin{array}{c|c} R(G,p) & \mathbf{0}_{|E|\times 2} \\ \hline A & -\varphi_{p_{v_1}} \\ B & -\varphi_{p_{v_2}} \end{array}\right],$$

for some $1 \times 2|V|$ matrices A and B. Since p_{v_1}, p_{v_2} are linearly independent and X is strictly convex, the pair $\varphi_{p_{v_1}}, \varphi_{p_{v_2}}$ are linearly independent. Hence $R(G'', p')$ has linearly independent rows. To prove that G' is uv-independent in X we will describe a series of rank-preserving row operations that will form $R(G', p')$ from $R(G'', p')$.

As $\varphi_{p_{v_1}}$ and $\varphi_{p_{v_2}}$ are linearly independent, there exist for each $3 \leq i \leq k$ a unique pair of values α_i and β_i such that

$$\alpha_i \varphi_{p_{v_1}} + \beta_i \varphi_{p_{v_2}} = \varphi_{p_{v_i}} = \varphi_{p'_{v_i} - p'_z},$$

where $z \in \{w, w'\}$ is chosen so that $v_i z \in E'$. For $1 \leq i \leq k$, let (wv_i) denote the row of $R(G'', p')$ corresponding to the edge wv_i, and similarly let $(w'v_1)$ and $(w'v_2)$ denote the rows of $R(G'', p')$ corresponding to edges $w'v_1$ and $w'v_2$ respectively. For $v_i \in N_{G'}(w')$, let $[w'v_i]$ denote the row of $R(G', p')$ corresponding to the edge $w'v_i$. Now, for all $v_i \in N_{G'}(w') \setminus \{v_1, v_2\}$, we have

$$[w'v_i] = (wv_i) - \alpha_i(wv_1) - \beta_i(wv_2) + \alpha_i(w'v_1) + \beta_i(w'v_2).$$

These row operations, when applied $R(G'', p')$, preserve linear independence and form the matrix $R(G', p')$. Therefore the rows of $R(G', p')$ are linearly independent. □

We now prove that vertex-to-H operations that add either u or v and uv-vertex-to-H operations will preserve uv-independence.

Lemma 3.4. *Let $G = (V, E)$ and $G' = (V', E')$ be graphs and let X be any non-Euclidean normed plane.*

(i) *Suppose G is independent in X and G' is formed from G by a vertex-to-H move that adds either u or v. If H is minimally uv-rigid in X, then G' is uv-independent in X.*

(ii) *Suppose G is uv-independent in X and G' is formed from G by a uv-vertex-to-H move. If H is minimally rigid in X, then G' is uv-independent in X.*

Proof. If (i) holds, let (H, q) be a minimally rigid uv-coincident framework in X and (G, p) be an independent framework in X, while if (ii) holds, let (H, q) be a minimally rigid framework in X and (G, p) be an independent uv-coincident framework in X. By applying translations we may assume $q_w = p_w = 0$. For any matrix A with columns corresponding to a vertex subset of $V \cup V(H)$, define A_w to be the matrix where we delete all columns corresponding to the vertex w. Given the fixed basis $b_1, b_2 \in X$ used to define our rigidity matrices in X, we define the matrix

$$M := \left[\begin{array}{c|c} R(H, q)_w & \mathbf{0}_{|E(H)| \times (2|V|-2)} \\ \hline A & R(G, p)_w \end{array}\right]$$

where A is the $|E| \times (2|V(H)| - 2)$ matrix with entries

$$A_{e,(y,i)} = \begin{cases} \varphi_{p_y - p_w}(b_i) & \text{if } e = xw, \\ 0 & \text{otherwise.} \end{cases}$$

By our choices of p and q, the matrix M has linearly independent rows.

For each $n \in \mathbb{N}$, choose a well-positioned uv-coincident framework (G', p^n) where $p^n_x = q_x/n$ for each $x \in V(H)$ and $\|p^n_x - p_x\| < 1/n$ for each $x \in V$ (this framework can be seen to exist from Lemma 2.5). Define M_n to be the matrix formed from multiplying each row of $R(G', p^n)_w$ corresponding to an edge of H by n. As the map $x \mapsto \varphi_x$ is continuous on the set of smooth points of X (see [24, Theorem 25.5]), the sequence of matrices $(M_n)_{n \in \mathbb{N}}$ will converge to M. Hence for sufficiently large $N \in \mathbb{N}$, the matrix M_{n_0} (and hence $R(G', p^{n_0})_w$) will have linearly independent rows. By setting $p' = p^{n_0}$, we obtain our desired independent uv-coincident framework (G', p'). □

4 Characterising coincident-point independence

With the geometric results of the previous section in hand, we can use the combinatorics of [17] to prove the difficult sufficiency direction of our main result on coincident frameworks. We begin with the following result which can be extracted from the proof of [17, Theorem 4].

Proposition 4.1 ([17]). *Any uv-tight graph on at least five vertices can be constructed from either a $(2,2)$-tight graph with at least four vertices that contains exactly one of u and v, or from the graph consisting of two copies of K_4 intersecting in a single vertex $x \notin \{u,v\}$ where u and v are in different copies of K_4 (see* Figure 3*), by a sequence of 0-extensions that add u or v, vertex-to-4-cycle and vertex-to-H moves that add u or v, uv-0- and uv-1-extensions, and uv-vertex-to-4-cycle and uv-vertex-to-H moves.*

Sketch of proof. Since the proof of [17, Theorem 4] is long and technical we provide a sketch of the proposition to orient the interested reader into how it can be extracted from that theorem. It is easy to see that every graph generated as described in the statement is uv-tight. For the converse, firstly [17, Theorem 4] is stated for independence in $\mathcal{M}_{uv}$, i.e. for uv-sparse graphs, but we can extend to a base E of $\mathcal{M}_{uv}$ which induces a graph $G=(V,E)$ that, since $|V|\geq 5$, necessarily has $2|V|-2$ edges and hence is uv-tight.

Suppose G has a vertex, w, of degree 2. If $w \in \{u,v\}$ then $G-w$ is $(2,2)$-tight. If $w \notin \{u,v\}$ then an easy argument shows that $G-w$ is uv-tight.

Therefore we may assume that the minimum degree is exactly 3, however it is much harder to reduce degree 3 vertices. [17, Theorem 4] deals firstly with three straightforward special cases. Firstly, if there is a 4-cycle in G containing u and v then uv is not an edge of this 4-cycle and we see that G is obtained from a $(2,2)$-tight graph by a vertex-to-4-cycle operation that split u into u and v. Secondly, if G contains a uv-tight subgraph H such that $V(H) \subsetneq V$ then we may assume H is a maximal such subgraph (that is there is no vertex in $V \setminus V(H)$ with more than one neighbour in $V(H)$). Then G/H (the graph obtained from G by contracting all vertices of H to a single vertex) is $(2,2)$-tight and G is obtained from G/H by a vertex-to-H move that expands u into a subgraph H that contains u and v. Thirdly, if G contains a degree 3 vertex contained in a subgraph of G isomorphic to K_4 and there is a vertex $x \in V \setminus V(H)$ such that $|V(H) \cap N(x)| = 2$ (and since we may assume the second special case does not occur $\{u,v\} \not\subset V(H) \cup \{x\}$), then we may apply a uv-vertex-to-H move to a uv-tight graph $G/(H \cup \{x\})$ to obtain G.

The proof is then completed by applying the arguments in Cases 5 and 6 of [17, Theorem 4], which use the fact we do not have the special structures we just dealt with, to analyse all possibilities for reducing a vertex of degree 3. Note that it is still not true that 1-extensions and uv-1-extensions suffice, however it is true that using precisely the operations listed in the proposition is sufficient. □

We will also require the following lemmas.

Lemma 4.2. *Let $G=(V,E)$ be a graph with at most 4 vertices that contains both u and v, and let X be a strictly convex non-Euclidean normed plane. Then G is uv-sparse if and only if it is uv-independent in X.*

Proof. The only graphs on 4 or fewer vertices that are not uv-sparse are those which contain the edge uv, and if G contains the edge uv then it is not uv-independent. Suppose $uv \notin E$. We note that G must be a subgraph of $K_4 - uv$, so it is sufficient to consider the

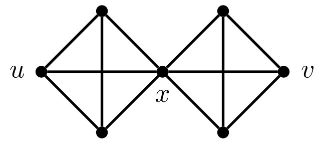

Figure 3: A uv-tight graph that is one of the base graphs of the construction described in Proposition 4.1.

case $G = K_4 - uv$. As G can be formed from $G - u$ by a 0-extension that adds u, G is uv-independent by Theorem 2.3 and Lemma 3.2. □

Lemma 4.3. *Let $G = (V, E)$ be the graph consisting of two copies of K_4 intersecting in a single vertex $x \notin \{u, v\}$, where u and v are in different copies of K_4 (see* Figure 3*). Then G is minimally uv-rigid in any non-Euclidean normed plane X.*

Proof. Let $V_u = \{x, u, a_u, b_u\}$ and $V_v = \{x, v, a_v, b_v\}$ be the vertex sets of the two copies of K_4 in G. As can be seen in Figure 3, $V_u \cap V_v = \{x\}$. By Theorem 2.3, there exists a placement $p^u : V_u \to X$ so that the framework (K_{V_u}, p^u), where K_{V_u} is the complete graph with vertex set V_u, is minimally rigid in X. Define the placement $p : V \to X$ by setting $p_{a_v} = p^u_{a_u}$, $p_{b_v} = p^u_{b_u}$, $p_v = p^u_u$, and $p_y = p^u_y$ for all $y \in V_u$. We now note that (G, p) is a minimally rigid uv-coincident framework; this follows from the fact that joining two minimally rigid frameworks in a non-Euclidean normed plane produces a minimally rigid framework, since the trivial infinitesimal flexes of a non-Euclidean normed plane correspond only to translations. Hence G is minimally uv-rigid as required. □

Theorem 4.4. *A graph is uv-independent in a strictly convex non-Euclidean normed plane X if and only if it is uv-sparse.*

Proof. First suppose G is uv-independent in X. Let G/uv denote the graph obtained from G by contracting the vertex pair u, v into a new vertex which we denote as z[2]. Let (G, p) be a regular (and hence independent) uv-coincident framework in X. We obtain a framework $(G/uv, p^{uv})$ in X by putting $p^{uv}_z = p_u = p_v$ and $p^{uv}_x = p_x$ for all $x \in V \setminus \{u, v\}$. For any $U \subseteq V$, the (possibly uv-coincident) induced subframework $(G[U], p|_U)$ is independent. Hence, if $\{u, v\} \nsubseteq U$, then $i_G(U) \leq \mathrm{val}(U)$ by Theorem 2.3. Since the case when $U = \{u, v\}$ is trivial, it now remains to show that $i_G(\mathcal{X}) \leq \mathrm{val}(\mathcal{X})$ for all uv-compatible families $\mathcal{X}$ in G. (Note that the case when $U \subseteq V$ and $\{u, v\} \subseteq U$ will be included by taking $\mathcal{X} = \{U\}$.)

Let $\mathcal{X} = \{X_1, \ldots, X_k\}$ be a uv-compatible family and consider the subgraph $H = (U, F)$ of G, where $U = \bigcup_{i=1}^k X_i$ and $F = \bigcup_{i=1}^k E(G[X_i])$. By contracting the vertex pair u, v in H, we obtain the graph H/uv. Define q to be the restriction of p to the vertex set U and q^{uv} to be the restriction of p^{uv} to the vertex set $U - \{u, v\} + z$. We have $\mathcal{X}_{uv} = \{X_1/uv, \ldots, X_k/uv\}$ is a cover of $E(H/uv)$ where X_i/uv denotes the set that we

[2]For us, a contraction will always be the more general vertex-contraction (which does not require u and v be adjacent) not the stricter edge-contraction (which does require u and v be adjacent).

get from X_i by identifying u and v. By Corollary 2.4, we have

$$\begin{aligned}\operatorname{rank} R(H/uv, q^{uv}) &\leq \sum_{i=1}^{k}(2|X_i/uv| - 2 - s(|X_i/uv|)\\ &= \sum_{i=1}^{k}(2|X_i| - 2 - t_{X_i})\\ &= \operatorname{val}(\mathcal{X}) - 2.\end{aligned}$$

Every vector μ^{uv} in the kernel of $R(H/uv, q^{uv})$ determines a unique vector μ in the kernel of $R(H, q)$ with $\mu_u = \mu_v = \mu_z^{uv}$ and $\mu_x = \mu_x^{uv}$ for all for all $x \in U \setminus \{u, v\}$. Hence $\dim\ker R(H, q) \geq \dim\ker R(H/uv, q^{uv})$. The rigidity matrix $R(H, q)$ has linearly independent rows since $R(G, p)$ has linearly independent rows, hence we have

$$i_G(\mathcal{X}) = \operatorname{rank} R(H, q) \leq \operatorname{rank} R(H/uv, q^{uv}) + 2 \leq \operatorname{val}(\mathcal{X}).$$

Thus G is uv-sparse.

We prove the sufficiency by induction on $|V|$. Suppose that G is uv-sparse. If $|V| \leq 4$, then G is uv-independent in X by Lemma 4.2. So we may suppose that $|V| \geq 5$. By adding additional edges, if necessary, we may assume G is uv-tight[3]. By Proposition 4.1, G can be constructed from either a $(2, 2)$-tight graph containing exactly one of u and v, or the graph pictured in Figure 3, by the operations defined in Section 3. Furthermore, as X is strictly convex, the corresponding geometric operations preserve minimal rigidity in X (see Section 3). The result now follows from Theorem 2.3 (i.e., every $(2, 2)$-tight graph is independent in X) and Lemma 4.3. □

We next use this result to prove the following delete-contract characterisation of uv-rigidity.

Theorem 4.5. *Let G be a graph with distinct vertices u, v, and let X be a strictly convex non-Euclidean normed plane. Then G is uv-rigid in X if and only if $G - uv$ and G/uv are both rigid in X.*

Proof. Suppose that G is uv-rigid. It is immediate from the definition that $G - uv$ must be rigid. Choose a regular uv-coincident placement p of G, and define p^{uv} to be the placement of G/uv where $p_x^{uv} = p_x$ for all $x \in V - \{u, v\}$ and (given that z is the vertex obtained from u and v during the contraction) $p_z^{uv} = p_u = p_v$. Given an infinitesimal flex μ^{uv} of $(G/uv, p^{uv})$, we can form an infinitesimal flex μ of (G, p) by setting $\mu_x = \mu_x^{uv}$ for all $x \in V - \{u, v\}$ and $\mu_u = \mu_v = \mu_z^{uv}$. Since (G, p) is infinitesimally rigid as a uv-coincident framework, we must have that $\mu = (\lambda)_{x\in V}$ (and hence $\mu^{uv} = (\lambda)_{x\in V(G/uv)}$) for some vector $\lambda \in X$. Thus $(G/uv, p^{uv})$ is infinitesimally rigid and G/uv is rigid.

The converse follows from Theorem 4.4 as in the proof of [17, Theorem 1]. □

We conjecture that the last two results apply in arbitrary non-Euclidean normed planes.

Conjecture 4.6. *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Then G is uv-independent in a non-Euclidean normed plane X if and only if G is uv-sparse.*

[3] Recall that uv-sparse graphs are the independent sets of a matroid and, when $|V| \geq 5$, the bases of this matroid have rank $2|V| - 2$.

Indeed extending the proof of Theorem 4.4 to the non-convex case requires only improvements to Lemmas 3.2 and 3.3. For the first of these, the issue is that 0-extensions that add v require us to precisely place v on top of the placement of u. However in the not strictly convex case, the proof of [5, Lemmas 5.1] requires one to choose the position of v carefully so that the support functionals of the edges incident to v guarantee linear independence. For the latter case, both the vertex-to-4-cycle move that adds v and the uv-vertex-to-4-cycle move have similar complications that would need to be resolved.

5 Global rigidity in analytic normed planes

A framework (G, p) in a normed space X is said to be *globally rigid* if every other framework (G, q) in X with $\|p_v - p_w\| = \|q_v - q_w\|$ for every edge $vw \in E$ is congruent to (G, p). A graph is then said to be *globally rigid* in X if the set

$$\{p \in X^V : (G, p) \text{ is globally rigid}\}$$

has a non-empty interior as a subset of the linear space X^V with the product topology inherited from X. It can be quickly seen that any globally rigid framework/graph will also be rigid.

Although much is known about global rigidity in Euclidean spaces, very little is known about the property for non-Euclidean normed spaces. The results that are known are only for *analytic normed spaces*, i.e., normed spaces where the norm restricted to the non-zero points is a real analytic function. As well as being strictly convex ([10, Lemma 3.1]), analytic normed spaces have many useful properties, including the following.

Lemma 5.1. *Let G be a graph with distinct vertices u, v and let X be a non-Euclidean analytic normed space.*

(i) *The set of all $p \in X^V$ where (G, p) is a regular framework is an open conull subset of X^V.*

(ii) *The set of all $p \in X^V/uv$ where (G, p) is a regular uv-coincident framework is an open conull subset of X^V/uv.*

Proof. If $\dim X = 1$ then the result follows immediately from noticing that all well-positioned frameworks and uv-coincident frameworks are regular. Suppose $\dim X \geq 2$. It was shown in [10, Proposition 3.2] that the set of well-positioned but non-regular placements of G are exactly the zero set of a non-constant analytic function defined on the connected open conull set of well-positioned placements. This gives (i). For (ii) we can use the same technique to show that the set of well-positioned but non-regular uv-coincident placements of G are exactly the zero set of a non-constant analytic function defined on the connected open conull set of well-positioned uv-coincident placements. The result now holds as the zero set of a non-constant analytic function with connected domain is always a closed null subset (see [10, Proposition 2.3]). □

Importantly, we can define a large class of globally rigid graphs in any non-Euclidean analytic normed plane.

Proposition 5.2 ([10])**.** *Let X be a non-Euclidean analytic normed plane. Then the graphs $K_5 - e$ and H, depicted in* Figure 4*, are globally rigid in X. Moreover any graph obtained from either of these by a sequence of degree* 3 *vertex additions (i.e., add a vertex and join it to three other vertices) and edge additions is globally rigid.*

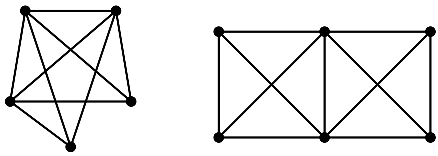

Figure 4: The graphs K_5^- (left) and H (right).

We next increase this class of graphs with the following construction operation introduced in [18]. A *generalised vertex split*, is defined as follows. Choose $z \in V$ and a partition N_u, N_v of the neighbours of z. Next, delete z from G and add two new vertices u, v joined to N_u, N_v, respectively. Finally add two new edges uv, uw for some $w \in V \setminus N_u$. See Figure 5 for an illustration of the operation.

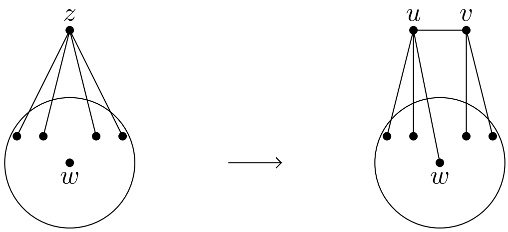

Figure 5: Generalised vertex split.

As the name suggests, this operation generalises the usual vertex splitting operation, see [26], which is the special case when w is chosen to be a neighbour of v. Note also that the special case when u has degree 3 (and $v = z$) is the well known 1-extension operation. Previously it was not known whether the 1-extension operation or a suitably restricted version of the vertex splitting operation preserves global rigidity in any non-Euclidean normed plane X.

As an application of our main result we will deduce that global rigidity can, under certain conditions, be preserved for generalised vertex splits. We will first need the following result which can be seen to follow from adapting the methods in [10, Section 3.2] to allow frameworks with zero-length edges[4].

Lemma 5.3. *Let (G, p) be a uv-coincident framework in a smooth normed space X with finitely many linear isometries. If (G, p) is globally rigid and infinitesimally rigid, then there exists an open neighbourhood $U \subset X^V$ of p where for each $q \in U$ the framework (G, q) is globally rigid.*

Theorem 5.4. *Let G be a globally rigid graph in a non-Euclidean analytic normed plane X. Let G' be a generalised vertex split of G at the vertex z with new vertices u, v and suppose that $G' - uv$ is rigid in X. Then G' is globally rigid in X.*

[4]Although it is a prerequisite in [10, Section 3.2] that the frameworks are well-positioned, the proof technique only requires that the squared edge-length map is differentiable. Since the map $x \mapsto \|x\|^2$ is always differentiable at the point 0, we can refine the result so that it holds for frameworks with zero-length edges.

Proof. Since $G'/uv = G$ is globally rigid in X it is also rigid in X by Theorem 2.1. As $G' - uv$ is also rigid in X, Theorem 4.5 implies that G' is uv-rigid in X. Hence by Lemma 5.1, we may choose an infinitesimally and globally rigid framework (G, p) so that if we define (G', p') to be the uv-coincident framework with $p'_x = p_x$ for all $x \in V$ and $p'_u = p'_v = p_z$, then (G', p') will be infinitesimally rigid also. Furthermore, (G', p') will also be globally rigid as (G, p) is globally rigid. We can now use Lemma 5.3 to deduce that (G', q) is globally rigid in X for all q sufficiently close to p'. Hence G' is globally rigid in X also. □

We can now improve upon Proposition 5.2. Here a graph $G = (V, E)$ is *redundantly rigid* in X if $G - e$ is rigid in X for any edge $e \in E$.

Corollary 5.5. *Let G be a graph obtained from K_5^- or H by a sequence of generalised vertex splits that preserve redundant rigidity, edge additions and degree at least* 3 *vertex additions. Then G is globally rigid in any non-Euclidean analytic normed plane.*

Proof. Follows immediately from Proposition 5.2 and Theorem 5.4. □

Since minimally rigid graphs in X have $2|V| - 2$ edges by Theorem 2.3, it is natural to expect that if $G = (V, E)$ is globally rigid then $|E| \geq 2|V| - 1$. The graphs K_5^- and H both achieve equality, but the inequality is strict for every graph in the infinite family obtained from these as in Proposition 5.2. To illustrate the power of Corollary 5.5 we note that we now have infinitely many globally rigid graphs for which equality holds and that this still holds if we restrict generalised vertex splitting to just one of vertex splitting or 1-extension. Two examples are depicted in Figure 6. The graph on the left is obtained from H by a vertex split and the graph on the right is obtained from H by a 1-extension. Both are globally rigid in X by Corollary 5.5.

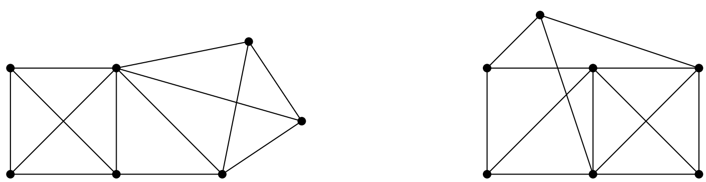

Figure 6: Examples of globally rigid graphs.

6 Concluding remarks

1. Following submission of this article we were able to improve upon Corollary 5.5. Specifically in [8], using the results of this article in a crucial way, we obtained a complete combinatorial description of graphs that are globally rigid in any non-Euclidean analytic normed plane. It turns out that we needed just 1 additional operation to those used in Corollary 5.5: this operation deletes an edge xy and adds two new vertices z, w and 5 new edges xz, xw, yz, yw, zw. In different language, the characterisation of [8] shows that a graph is globally rigid in any non-Euclidean analytic normed plane if and only if it is 2-connected and redundantly rigid (which means that it is still rigid after deleting any edge).

2. Theorem 4.4 and Theorem 4.5 provide a detailed combinatorial understanding of coincident point rigidity for frameworks in strictly convex non-Euclidean normed planes. As noted in the introduction, similar results exist for the Euclidean plane [12] and for frameworks supported on a cylinder in $\mathbb{R}^3$ [17]. Given the applicability of coincident point rigidity to analysing global rigidity (e.g. [4]) it would be interesting to develop analogues of Theorem 4.4 and Theorem 4.5 in other natural settings in rigidity theory. It may also be interesting to explore rigidity for frameworks with larger (or multiple) sets of coincident points. This line of investigation has begun in the case of the Euclidean plane [15].

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