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Finite simple groups on triple systems*

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Abstract

Let $\mathcal{D}$ be a triple system, and let G be a finite simple group. In this paper we almost determine all possibilities of $\mathcal{D}$ admitting G as its flag-transitive automorphism group.

Keywords: Triple system, flag-transitivity, finite simple group.

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1 Introduction

A 2-(v, k, λ) design is a pair $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ where $\mathcal{P}$ is a set of v points and $\mathcal{B}$ is a collection of b k-subsets (blocks) of $\mathcal{P}$ with the property that every 2-subset of $\mathcal{P}$ occurs in λ blocks of $\mathcal{B}$. If no blocks are identical, then $\mathcal{D}$ is called simple.

An *automorphism* of a design $\mathcal{D}$ is a permutation of $\mathcal{P}$ which leaves $\mathcal{B}$ invariant. The full automorphism group of $\mathcal{D}$, denoted by $\mathrm{Aut}(\mathcal{D})$, is the group consisting of all automorphisms of $\mathcal{D}$. A $flag$ of $\mathcal{D}$ is a point-block pair (α, B) such that $\alpha \in B$. For $G \leq \mathrm{Aut}(\mathcal{D})$, G or $\mathcal{D}$ is called *flag-transitive* if G acts transitively on the set of flags, and *point-primitive* if G acts primitively on $\mathcal{P}$. A set of blocks of $\mathcal{D}$ is called a set of *base blocks* with respect to an automorphism group G of $\mathcal{D}$ if it contains exactly one block from each G-orbit on the block set. In particular, if G is a flag-transitive automorphism group of $\mathcal{D}$, then any block B is a base block of $\mathcal{D}$.

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In this paper, we focus on simple 2-$(v, 3, \lambda)$ designs also known as simple triple systems, which can be denoted by $TS(v, \lambda)$. One possibility is to take all possible 3-subsets of $\mathcal{P}$ however such designs are called *complete* and will be ignored. A triple system is a *Steiner triple system*, or $STS(v)$, when $\lambda = 1$.

Let r be the number of the blocks through a given point. For a $TS(v, \lambda)$, it is well known that a necessary and sufficient condition for the existence of a $TS(v, \lambda)$ is $v \neq 2$ and $\lambda \equiv 0 \pmod{(v-2, 6)}$, and

$$3b = vr; \tag{1.1}$$

$$r = \frac{\lambda(v-1)}{2}; \tag{1.2}$$

$$b = \frac{\lambda v(v-1)}{6}; \tag{1.3}$$

$$b \geq v. \tag{1.4}$$

A 2-$(v, k, 1)$ design is also called a finite linear space. A classic result is that of Higman and McLaughlin [8] who proved that for a finite linear space, flag-transitivity implies point-primitivity. Then Buekenhout, Delandtsheer and Doyen in [1] proved that if G acts flag-transitively on a linear space, then G is of affine or almost simple type. In 1990, the six-person team [2] classified all flag-transitive linear spaces apart from those with an one-dimensional affine automorphism group.

For 2-$(v, k, 1)$ designs with small values of k, one of the first classifications was for Steiner triple systems in [4], which considered what happens when the action was block-transitive but not 2-transitive on points. It is described in [11] what happens when the action on points is 2-transitive. This result depends on the classification of all finite simple groups and is subsumed into the general results proved by Kantor in [10].

Let G be a flag-transitive automorphism group of a $TS(v, \lambda)$. It is shown in [6, 2.3.7(c), (e)] that G is point-primitive. Moreover, we can easily prove that G is 2-homogeneous (see Lemma 2.2 below). This result makes it possible to classify all flag-transitive triple systems using the classification of the finite 2-transitive permutation groups. Our main purpose is to give a classification of all triple systems admitting a simple flag-transitive automorphism group.

We now state the main result of this paper:

Theorem 1.1. *Let $\mathcal{D}$ be a triple system, and let G be a finite simple group. If G acts flag-transitively on $\mathcal{D}$, then one of the following* LINES *of* Table 1 *holds.*

Remark 1.2.

- All but the triple systems listed in LINES 20 and 21 exist.
- If $G = PSU(3, q)$ with $q = 5$, then there are only two flag-transitive triple systems corresponding to LINES 19 and 20.
- The existence of triple systems with $3 \nmid q$ and $q \neq 5$ corresponding to LINES 20 and 21 is in doubt.

Table 1: G and corresponding triple systems.

LINE	G	$\mathcal{D}$	Notes
1	A_7	$TS(15,1)$	
2		$TS(15,12)$	
3	$PSL(2,11)$	$TS(11,3)$	
4		$TS(11,6)$	
5	HS	$TS(176,12)$	
6		$TS(176,72)$	
7		$TS(176,90)$	
8	Co_3	$TS(276,112)$	
9		$TS(276,162)$	
10	$PSp(2d,2)$	$TS(2^{d-1}(2^d+1),2^{2d-2})$	$d\geq 3$
11		$TS(2^{d-1}(2^d+1),2(2^{d-1}-1)(2^{d-2}+1))$	
12	$PSp(2d,2)$	$TS(2^{d-1}(2^d-1),2^{2d-2})$	$d\geq 3$
13		$TS(2^{d-1}(2^d-1),2(2^{d-1}+1)(2^{d-2}-1))$	
14	$PSL(d,q)$	$TS(\frac{q^d-1}{q-1},q-1)$	$d\geq 3$
15		$TS(\frac{q^d-1}{q-1},\frac{q^d-1}{q-1}-q-1)$	
16	$PSL(2,q)$	$TS(q+1,\frac{q-1}{2})$	$q\equiv 1(\bmod 4)$
17	$Ree(q)$	$TS(q^3+1,2(q-1))$	$q=3^{2e+1}>3$
18		$TS(q^3+1,q-1)$	
19	$PSU(3,q)$	$TS(q^3+1,q-1)$	$q\geq 3$
20		$TS(q^3+1,\frac{q^2-1}{(3,q+1)})$	
21		$TS(q^3+1,\frac{2(q^2-1)}{(3,q+1)})$	

2 Useful lemmas

The notation and terminology used is standard and can be found in [5, 6] for design theory and in [7, 9] for group theory. In particular, if G is a permutation group on a set Ω, and $\{\alpha,\beta\}\subseteq\Delta\subseteq\Omega$, then G_α denotes the stabilizer of a point α in G, and $G_{\alpha\beta}$ denotes the pointwise stabilizer of two points α and β in G, and G_Δ denotes the setwise stabilizer of Δ in G.

The following result about flag-transitive 2-designs is well-known.

Lemma 2.1. *Let $\mathcal{D}=(\mathcal{P},\mathcal{B})$ be a 2-(v,k,λ) design, and let G be an automorphism group of $\mathcal{D}$. For any $\alpha\in\mathcal{P}$ and $B\in\mathcal{B}$, G is flag-transitive if and only if G is point-transitive and G_α is transitive on the pencil $P(\alpha)$ (the set of blocks through α), if and only if G is block-transitive and G_B is transitive on the points of B.*

Lemma 2.2. *Let $\mathcal{D}=(\mathcal{P},\mathcal{B})$ be a triple system, and let G be a flag-transitive automorphism group of $\mathcal{D}$. If G is a simple group, then G acts 2-transitively on $\mathcal{P}$.*

Proof. Let $\{\alpha,\beta\}$ and $\{\gamma,\delta\}$ be arbitrary two unordered pairs of $\mathcal{P}$. By the definition of a triple system, there are two points ε and θ such that $B_1=\{\alpha,\beta,\varepsilon\}$ and $B_2=\{\gamma,\delta,\theta\}$ are two blocks of $\mathcal{D}$. The flag-transitivity of G implies that there is a $g\in G$ such that

$$(\varepsilon,B_1)^g=(\varepsilon^g,B_1^g)=(\theta,B_2),$$

and so $\{\alpha,\beta\}^g=\{\gamma,\delta\}$. Thus G is 2-homogeneous. If G is a simple group, then G acts 2-transitively on $\mathcal{P}$ by [7, Theorem 9.4B]. □

Lemma 2.3. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a triple system, and let $G \leq \mathrm{Aut}(\mathcal{D})$ be a 2-transitive group on $\mathcal{P}$. Then the following conditions are equivalent:*

(i) *G acts flag-transitively on $\mathcal{D}$.*

(ii) *If $B = \{\alpha, \beta, \gamma\} \in \mathcal{B}$, then $\{\{\alpha, \beta, \gamma_i\} \mid \gamma_i \in \gamma^{G_{\{\alpha,\beta\}}}\}$ is the set of all blocks through points α and β.*

Proof. (i) $\Rightarrow$ (ii): Let $B(\alpha, \beta) = \{B_1, B_2, \dots, B_\lambda\}$ be the set of blocks through points α and β, where $B_i = \{\alpha, \beta, \gamma_i\}$, $\gamma_i \in \mathcal{P} \setminus \{\alpha, \beta\}$. Clearly, $B(\alpha, \beta)^{G_{\{\alpha,\beta\}}} = B(\alpha, \beta)$. If G acts flag-transitively on $\mathcal{D}$, then for any two flags (γ_i, B_i) and (γ_j, B_j), there is a $g \in G$ such that $(\gamma_i, B_i)^g = (\gamma_j, B_j)$, so $\gamma_i^g = \gamma_j$ and $g \in G_{\{\alpha,\beta\}}$. Thus $G_{\{\alpha,\beta\}}$ acts transitively on $B(\alpha, \beta)$ and hence $\{\gamma_1, \dots, \gamma_\lambda\} = \gamma_i^{G_{\{\alpha,\beta\}}}$.

(ii) $\Rightarrow$ (i): Let (γ, B) and (ϵ, C) be two flags of $\mathcal{D}$ with $B = \{\alpha, \beta, \gamma\}$, $C = \{\delta, \eta, \epsilon\}$. By the 2-transitivity of G, there exists $g_1 \in G$ such that $\{\alpha, \beta\}^{g_1} = \{\delta, \eta\}$, thus $B^{g_1} = \{\delta, \eta, \gamma^{g_1}\}$ is a block containing δ and η. Since $\{\{\delta, \eta, \epsilon_i\} \mid \epsilon_i \in \epsilon^{G_{\{\delta,\eta\}}}\}$ is the set of all blocks through δ and η, there exists $g_2 \in G_{\{\delta,\eta\}}$ such that $\gamma^{g_1 g_2} = \epsilon$, and then $(\gamma, B)^{g_1 g_2} = (\epsilon, C)$. Therefore, G acts flag-transitively on $\mathcal{D}$. □

Corollary 2.4. *Let G be a 2-transitive group on a point set $\mathcal{P}$ with $|\mathcal{P}| = v$, and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be all sizes of orbits of $G_{\alpha\beta}$ on $\mathcal{P} \setminus \{\alpha, \beta\}$. If $\lambda_i \neq \lambda_j$ for $i \neq j$, then there exist k different flag-transitive $TS(v, \lambda_i)$.*

Proof. Without loss of generality, let $\Delta = \gamma^{G_{\alpha\beta}}$ with $|\Delta| = \lambda_1$, where $\gamma \in \mathcal{P} \setminus \{\alpha, \beta\}$. Since $G_{\alpha\beta} \trianglelefteq G_{\{\alpha,\beta\}}$, the group $G_{\alpha\beta}$ acts $\frac{1}{2}$-transitively on $\gamma^{G_{\{\alpha,\beta\}}}$, that is, $G_{\alpha\beta}$-orbits on $\gamma^{G_{\{\alpha,\beta\}}}$ have the same length. The uniqueness of the $G_{\alpha\beta}$-orbit with size λ_1 implies that $\gamma^{G_{\alpha\beta}} = \gamma^{G_{\{\alpha,\beta\}}}$. Thus $G_{\{\alpha,\beta\}}$ has a unique orbit with size λ_1. Let $B = \{\alpha, \beta, \gamma\}$ and $\mathcal{B} = B^G$. We shall prove below that $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a $TS(v, \lambda_1)$ admitting G as its flag-transitive automorphism group.

Since G is 2-transitive, for any pair $\{\delta, \eta\}$, there exists $g \in G$ such that $\{\alpha, \beta\}^g = \{\delta, \eta\}$. So $G_{\{\delta,\eta\}}$ has a unique orbit $\Delta^g = (\gamma^g)^{G_{\{\delta,\eta\}}}$ with $|\Delta^g| = |\Delta| = \lambda_1$. Let $B(\delta, \eta)$ be the set of elements of $\mathcal{B}$ containing δ, η with $|B(\delta, \eta)| = \lambda$. It is easy to see that $\Lambda = \{\{\delta, \eta, \epsilon\} \mid \epsilon \in \Delta^g\} \subseteq \mathcal{B}$, so we have $\lambda \geq \lambda_1$. On the other hand, for $C = \{\delta, \eta, \theta\} \in B(\delta, \eta)$, there exists $h \in G$ such that $C = B^h$. As $|\gamma^{G_{\alpha\beta}}| = |\alpha^{G_{\gamma\beta}}| = |\beta^{G_{\alpha\gamma}}| = \lambda_1$, we may assume that $\theta = \gamma^h$. Then $|\theta^{G_{\{\delta,\eta\}}}| = |\gamma^{hG_{\{\delta,\eta\}}}| = |\gamma^{G_{\{\alpha,\beta\}}h}| = |\Delta^h| = \lambda_1$, it implies $\lambda_1 \geq \lambda$. Thus, $\lambda = \lambda_1$ and $B(\delta, \eta) = \Lambda$. Hence $\mathcal{D}$ is a $TS(v, \lambda_1)$, and G is a flag-transitive automorphism group of $\mathcal{D}$ by Lemma 2.3(ii). □

Lemma 2.5. *Let G be a 2-transitive group on a point set $\mathcal{P}$ with $|\mathcal{P}| = v$, and let $\Delta = \{\alpha, \beta, \gamma\}$ be a 3-subset of $\mathcal{P}$. If $G_{\alpha\beta}$ is a cyclic group of order λ and $|\gamma^{G_{\alpha\beta}}| = \lambda$, then*

(i) *$\mathcal{D} = (\mathcal{P}, \Delta^G)$ is a flag-transitive $TS(v, \lambda)$ if and only if $G_\Delta^\Delta \cong S_3$, or*

(ii) *$\mathcal{D} = (\mathcal{P}, \Delta^G)$ is a flag-transitive $TS(v, 2\lambda)$ if and only if $G_\Delta^\Delta \cong \mathbb{Z}_3$.*

Proof. Here we only prove case (i), and case (ii) can be proved by same procedure. Since $G_{\alpha\beta}$ is a cyclic group for any points α and β, we have that $G_\Delta = G_\Delta^\Delta$. Let $\mathcal{D} = (\mathcal{P}, \Delta^G)$. If $\mathcal{D}$ is a flag-transitive $TS(v, \lambda)$, then using Lemma 2.1 and Equation (1.3), we have that

$$b = \frac{\lambda v(v-1)}{6} = |\Delta^G| = [G : G_\Delta] = [G : G_{\alpha\beta}][G_{\alpha\beta} : G_\Delta].$$

By 2-transitivity of G and $|G_{\alpha\beta}| = \lambda$, we obtain $|G_\Delta| = 6$. The flag-transitivity of G implies that G_Δ acts transitively on the points of Δ by Lemma 2.1. Thus $G_\Delta \cong S_3$.

If $G_\Delta \cong S_3$, then $G_{\{\alpha,\beta\}\gamma} \cong \mathbb{Z}_2$ and $|\Delta^G| = [G : G_{\alpha\beta}][G_{\alpha\beta} : G_\Delta] = \frac{\lambda v(v-1)}{6}$. Thus, $\mathcal{D}$ is a $TS(v, \lambda)$ as G acts 2-transitively on $\mathcal{P}$. Clearly,

$$|\gamma^{G_{\{\alpha,\beta\}}}| = [G_{\{\alpha,\beta\}} : G_{\{\alpha,\beta\}\gamma}] = \lambda,$$

where $G_{\{\alpha,\beta\}\gamma} = G_{\{\alpha,\beta\}} \cap G_\gamma$. Therefore, G acts flag-transitively on $\mathcal{D}$ by Corollary 2.4. □

Lemma 2.6. *Let $G = Ree(q)$ act 2-transitively on Ω, where $|\Omega| = q^3+1$ and $q = 3^{2e+1} > 3$. Then there exist subsets Δ, Σ of size 3 such that*

$$G_\Delta^\Delta = \mathbb{Z}_3, \;\; G_\Sigma^\Sigma = S_3.$$

Proof. Let Q be a Sylow 3-subgroup of G. Then $|Q| = q^3$, and there exists $\alpha \in \Omega$ such that Q is regular on $\Omega \setminus \{\alpha\}$. Thus each subgroup of Q is semiregular on $\Omega \setminus \{\alpha\}$. Let $x, y \in Q$ such $|x| = |y| = 3$, $x \notin \mathbf{Z}(Q)$ and $y \in \mathbf{Z}(Q)$, where the centre $\mathbf{Z}(Q)$ is elementary abelian of order q.

Let Δ be an orbit of $\langle x\rangle$. Then $|\Delta| = 3$ and $G_\Delta^\Delta = \mathbb{Z}_3$ or S_3. Further, since x is not conjugate to x^{-1} in G (reference [12]), we have $G_\Delta^\Delta \cong \langle x\rangle \cong \mathbb{Z}_3$.

Consider y acting on $\Omega \setminus \{\alpha\}$. Since y is in the centre $\mathbf{Z}(Q)$, there is an involution $z \in G_\alpha$ such that $y^z = y^{-1}$, and the subgroup $H = \langle y, z\rangle \cong S_3$. Since $\langle y\rangle$ is semiregular on $\Omega \setminus \{\alpha\}$, the set $\Omega \setminus \{\alpha\}$ is divided into $\frac{1}{3}q^3$ orbits of $\langle y\rangle$:

$$\Delta_1, \Delta_2, \ldots, \Delta_m,$$

where $m = \frac{1}{3}q^3$ is odd. Since each H-orbit Σ contains a $\langle y\rangle$-orbit, the cardinality $|\Sigma| = 3$ or 6. As the number $\frac{1}{3}q^3$ of $\langle y\rangle$-orbits is odd, it follows that there is at least one H-orbit Σ on $\Omega \setminus \{\alpha\}$ has length 3. Therefore, $G_\Sigma^\Sigma = H_\Sigma^\Sigma = S_3$ with $|\Sigma| = 3$. □

3 Proof of Theorem 1.1

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a $TS(v, \lambda)$, and let G be a simple group acting flag-transitively on $\mathcal{D}$. Then G acts 2-transitively on $\mathcal{P}$ by Lemma 2.2. Since we neglect the case $\mathcal{D}$ is complete, we may assume that G is not 3-homogeneous group on $\mathcal{P}$. Thus, all such groups are known and we can find a classification in [3] and we have that G must be one of the following Table 2.

We will prove Theorem 1.1 by analyzing the 11 cases in Table 2 one by one.

Proof of Theorem 1.1. Let α and β be two points of $\mathcal{P}$. For Cases 1 – 7, we have the following facts by the proof of [10, Theorem 1]:

If $G = A_7$ and $v = 15$, then $G_{\alpha\beta}$ has orbit-lengths 1 and 12 on $\mathcal{P} \setminus \{\alpha, \beta\}$.

If $G = PSL(2, 11)$ and $v = 11$, then $G_{\alpha\beta}$ has orbit-lengths 3 and 6 on $\mathcal{P} \setminus \{\alpha, \beta\}$.

If $G = HS$ and $v = 176$, then $G_{\alpha\beta}$ has orbit-lengths 12, 72 and 90 on $\mathcal{P} \setminus \{\alpha, \beta\}$.

If $G = Co_3$ and $v = 276$, then $G_{\alpha\beta}$ has orbit-lengths 112 and 162 on $\mathcal{P} \setminus \{\alpha, \beta\}$.

If $G = PSp(2d, 2)$ and $v = 2^{2d-1} + 2^{d-1}$, then $G_{\alpha\beta}$ has orbit-lengths $2(2^{d-1} - 1)(2^{d-2} + 1)$ and 2^{2d-2} on $\mathcal{P} \setminus \{\alpha, \beta\}$.

Table 2: 2-transitive, not 3-homogeneous simple groups.

Case	Group	Degree	Notes
1	A_7	15	
2	$PSL(2,11)$	11	
3	HS	176	
4	Co_3	276	
5	$PSp(2d,2)$	$2^{2d-1}+2^{d-1}$	$d \geq 3$
6	$PSp(2d,2)$	$2^{2d-1}-2^{d-1}$	$d \geq 3$
7	$PSL(d,q)$	$(q^d-1)/(q-1)$	$d \geq 3$
8	$PSL(2,q)$	$q+1$	$q \equiv 1 \pmod 4$
9	$Suz(q)$	q^2+1	$q = 2^{2e+1} > 2$
10	$Ree(q)$	q^3+1	$q = 3^{2e+1} > 3$
11	$PSU(3,q)$	q^3+1	$q \geq 3$

If $G = PSp(2d,2)$ and $v = 2^{2d-1} - 2^{d-1}$, then $G_{\alpha\beta}$ has orbit-lengths $2(2^{d-1}+1)(2^{d-2}-1)$ and 2^{2d-2} on $\mathcal{P} \setminus \{\alpha,\beta\}$.

If $G = PSL(d,q)$ with $d \geq 3$ and $v = \frac{q^d-1}{q-1}$, $G_{\alpha\beta}$ has orbit-lengths $q-1$ and $\frac{q^d-1}{q-1} - q - 1$ on $\mathcal{P} \setminus \{\alpha,\beta\}$.

It follows from Corollary 2.4 that $\mathcal{D}$ is one of triple systems corresponding LINES 1-15 in Table 1.

Case 8: $G = PSL(2,q)$ with $q \equiv 1 \pmod 4$ and $v = q+1$. In this case, there are exactly two G-orbits on 3-subsets of $q+1$ points with size $\frac{q(q^2-1)}{12}$. Also, $G_{\alpha\beta} \cong Z_{\frac{q-1}{2}}$ has two orbits with length $\frac{q-1}{2}$ on $\mathcal{P} \setminus \{\alpha,\beta\}$, denoted by Γ_1 and Γ_2. Suppose that $\Gamma_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_{\frac{q-1}{2}}\}$, $\Gamma_2 = \{\beta_1, \beta_2, \ldots, \beta_{\frac{q-1}{2}}\}$. For $i \in \{1,2\}$, let $\mathcal{D}_i = (\mathcal{P}, \Delta_i^G)$ where $\Delta_i = \{\alpha, \beta, \gamma_i\}$ and $\gamma_i \in \Gamma_i$. It is easy to calculate that $|G_{\Delta_i}| = 6$, and hence $G_{\Delta_i} \cong S_3$. By Lemma 2.5(i), both $\mathcal{D}_1$ and $\mathcal{D}_2$ are $TS(q+1, \frac{q-1}{2})$. Let

$$g = (\alpha,\beta)(\alpha_1,\beta_1)\cdots(\alpha_{\frac{q-1}{2}}, \beta_{\frac{q-1}{2}}).$$

Clearly, g is an isomorphism from $\mathcal{D}_1$ to $\mathcal{D}_2$, that is $\mathcal{D}_1 \cong \mathcal{D}_2$. Thus, $\mathcal{D}$ is a $TS(q+1, \frac{q-1}{2})$.

Case 9: $G = Sz(q)$ and $v = q^2+1$. Since G acts flag-transitively on $\mathcal{D}$, then $3 \mid |G|$ by Lemma 2.1. But this contracts the fact that $3 \nmid |G|$ (see [9, Theorem 3.6]). Therefore, there is no triple system admitting $Sz(q)$ as its flag-transitive automorphism group.

Case 10: $G = Ree(q)$ and $v = q^3+1$ with $q = 3^{2e+1} > 3$. From Lemmas 2.5 and 2.6, we have that $\mathcal{D}$ is one of triple systems corresponding LINES 17 and 18 in Table 1.

Case 11: $G = PSU(3,q)$ and $v = q^3+1$. Since $G_{\alpha\beta} \cong Z_{\frac{q^2-1}{(3,q+1)}}$ has a unique orbit O with size $q-1$ and $q(3,q+1)$ orbits with size $\frac{q^2-1}{(3,q+1)}$. Similar to proof of Lemma 2.4, we can prove that there exists a unique $TS(q^3+1, q-1)$ admitting G as its flag-transitive automorphism group.

If $q = 3^e \geq 3$, there exist subsets Δ, Σ of size 3 such that $G_\Delta^\Delta = \mathbb{Z}_3$, $G_\Sigma^\Sigma = S_3$ by the same proof as Lemma 2.6. In this case, $\mathcal{D}$ is one of triple systems corresponding LINES 20 and 21 in Table 1 from Lemma 2.5.

If $q = 5$ then $\mathcal{D}$ can only be a flag-transitive $TS(126, 8)$ in addition to $TS(126, 4)$ by a simple calculation. This means that there is no flag-transitive $TS(126, 16)$ in this case.

Unfortunately, we don't know whether Lemma 2.6 holds when $3 \nmid q$. Thus the existence of $TS(q^3 + 1, \frac{q^2-1}{(3,q+1)})$ (or $TS(q^3 + 1, \frac{2(q^2-1)}{(3,q+1)})$) with $3 \nmid q$ and $q \neq 5$ is in doubt.

This completes the proof of Theorem 1.1. □

Conjecture 3.1. *Let $\mathcal{D}$ be a triple system $TS(q^3 + 1, \lambda)$, and let $G = PSU(3, q)$ act flag-transitively on $\mathcal{D}$ with $3 \nmid q$ and $q \neq 5$. If $\lambda \neq q - 1$ then one of following holds:*

(i) *If q is even, then $\lambda = \frac{2(q^2-1)}{(3,q+1)}$.*

(ii) *If q is odd, then $\lambda = \frac{q^2-1}{(3,q+1)}$ or $\frac{2(q^2-1)}{(3,q+1)}$.*

In fact, using MAGMA, we have already proved that the conjecture holds when $q \leq 100$.

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