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Contents

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The edge-transitive polytopes that are not vertex-transitive*

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Abstract

In 3-dimensional Euclidean space there exist two exceptional polyhedra, the *rhombic dodecahedron* and the *rhombic triacontahedron*, the only known polytopes (besides polygons) that are edge-transitive without being vertex-transitive. We show that these polyhedra do not have higher-dimensional analogues, that is, that in dimension $d \geq 4$, edge-transitivity of convex polytopes implies vertex-transitivity.

More generally, we give a classification of all convex polytopes which at the same time have all edges of the same length, an edge in-sphere and a bipartite edge-graph. We show that any such polytope in dimension $d \geq 4$ is vertex-transitive.

Keywords: Convex polytopes, symmetry of polytopes, vertex-transitive, edge-transitive.

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1 Introduction

A d-dimensional (convex) polytope $P \subset \mathbb{R}^d$ is the convex hull of finitely many points. P is said to be *vertex-transitive* resp. *edge-transitive* if its (orthogonal) symmetry group $\mathrm{Aut}(P) \subset \mathrm{O}(\mathbb{R}^d)$ acts transitively on its vertices resp. edges. For a general overview over the state of the art regarding symmetries in convex and abstract polytopes we refer to [9].

It has long been known that there are exactly *nine* edge-transitive polyhedra in $\mathbb{R}^3$ (see *e.g.* [6]). These are the five Platonic solids (tetrahedron, cube, octahedron, icosahedron and dodecahedron) together with the cuboctahedron, the icosidodecahedron, and their duals, the *rhombic dodecahedron* and the *rhombic triacontahedron* (depicted in this order):

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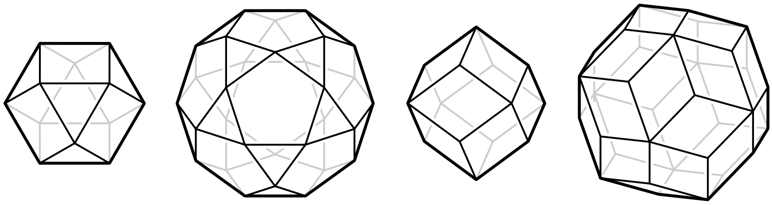


Little is known about analogous questions in higher dimensions. Branko Grünbaum writes in "Convex Polytopes" [5, page 413]

> No serious consideration seems to have been given to polytopes in dimension $d \geq 4$ about which transitivity of the symmetry group is assumed only for faces of suitably low dimensions, [...].

Even though families of higher-dimensional edge-transitive polytopes have been studied, to the best of our knowledge, no classification of these has been achieved so far. Equally striking, all the known examples of such polytopes in dimension at least four are simultaneously *vertex-transitive*. In dimension up to three, certain polygons (see Figure 1), as well as the rhombic dodecahedron and rhombic triacontahedron are edge- but *not* vertex-

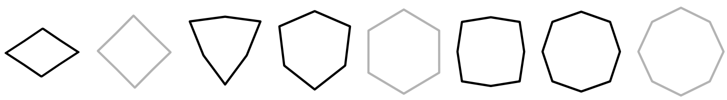


Figure 1: Some examples of edge-transitive $2n$-gons with $2n \in \{4, 6, 8\}$ (the same works for all n). The polygons depicted with black boundary are not vertex-transitive.

transitive. No higher dimensional example of this kind has been found. In this paper we prove that this is not for lack of trying:

Theorem 1.1. *In dimension $d \geq 4$, edge-transitivity of convex polytopes implies vertex-transitivity.*

As immediate consequence, we obtain the classification of all polytopes that are edge-but not vertex-transitive. The list is quite short:

Corollary 1.2. *If $P \subset \mathbb{R}^d, d \geq 2$ is edge- but not vertex-transitive, then P is one of the following:*

(i) *a non-regular $2k$-gon (see* Figure 1*),*

(ii) *the rhombic dodecahedron, or*

(iii) *the rhombic triacontahedron.*

Theorem 1.1 is proven by embedding the class of edge- but not vertex-transitive polytopes in a larger class of polytopes, defined by geometric regularities instead of symmetry. In Theorem 2.4 we show that a polytope $P \subset \mathbb{R}^d$ which is edge- but not vertex-transitive must have all of the following properties:

(i) all edges are of the same length,

(ii) it has a bipartite edge-graph $G_P = (V_1 \mathbin{\dot\cup} V_2, E)$, and

(iii) there are radii $r_1 \leq r_2$, so that $\|v\| = r_i$ for all $v \in V_i$.

We compile this into a definition: a polytope that has these three properties shall be called *bipartite* (*cf.* Definition 2.1). The edge- but not vertex-transitive polytopes then form a subclass of the bipartite polytopes, but the class of bipartite polytopes is much better behaved. For example, faces of bipartite polytopes are bipartite (Proposition 2.5), something which is not true for edge/vertex-transitive polytopes[1]. Our quest is then to classify all bipartite polytopes. The surprising result: already being bipartite is very restrictive:

Theorem 1.3. *If $P \subset \mathbb{R}^d, d \geq 2$ is bipartite, then P is one of the following:*

(i) *an edge-transitive $2k$-gon (see* Figure 1*),*

(ii) *the rhombic dodecahedron,*

(iii) *the rhombic triacontahedron, or*

(iv) *a Γ-permutahedron for some finite reflection group $\Gamma \subset \mathrm{O}(\mathbb{R}^d)$ (see* Definition 2.10*; some 3-dimensional examples are shown in* Figure 2*).*

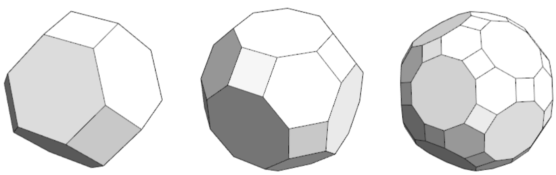


Figure 2: From left to right: the A_3-, B_3 and H_3-permutahedron.

The Γ-permutahedra are vertex-transitive, and all the other entries in the list are of dimension $d \leq 3$. This immediately implies Theorem 1.1.

Remarkably, despite the definition of bipartite polytope being purely geometric, all bipartite polytopes are highly symmetric, that is, at least vertex- or facet-transitive, and sometimes even edge-transitive.

Overview

In Section 2 we introduce the central notion of *bipartite polytope* and prove its most relevant properties: that being bipartite generalizes being edge- but not vertex-transitive, and that all faces of bipartite polytopes are again bipartite. We then investigate certain subclasses of bipartite polytopes: bipartite polygons and inscribed bipartite polytopes. We prove that the latter coincide with the Γ-permutahedra, a class of vertex-transitive polytopes. It therefore remains to classify the non-inscribed cases, the so-called *strictly* bipartite polytopes. We

[1] For example, consider a vertex-transitive but not uniform antiprism. Its faces are non-regular triangles, which are thus not vertex-transitive. Alternatively, consider the (n, n)-duoprism, $n \neq 4$, that is, the cartesian product of a regular n-gon with itself. This polytope is edge-transitive, but its facets are n-gonal prisms (the cartesian product of a regular n-gon with an edge), which are not edge-transitive.

show that the classification of these reduces to the classification of bipartite *polyhedra*, *i.e.,* the case $d = 3$.

From Section 3 on the investigation is focused on the class of strictly bipartite polyhedra. We successively determine restrictions on the structure of such, *e.g.* the degrees of their vertices and the shapes of their faces. This quite elaborate process uses many classical geometric results and techniques, including spherical polyhedra, the classification of rhombic isohedra and the realization of graphs as edge-graphs of polyhedra. As a result, we can exclude all but two cases, namely, the rhombic dodecahedron, and the rhombic triacontahedron. Additionally, we shall find a remarkable near-miss, that is, a polyhedron which fails to be bipartite only by a tiny (but quantifiable) amount.

2 Bipartite polytopes

From this section on let $P \subset \mathbb{R}^d, d \geq 2$ denote a d-dimensional polytope of full dimension (*i.e.,* P is not contained in a proper affine subspace). By $\mathcal{F}(P)$ we denote the face lattice of P, and by $\mathcal{F}_\delta(P) \subset \mathcal{F}(P)$ the subset of δ-dimensional faces.

Definition 2.1. P is called *bipartite*, if

(i) all its edges are of the same length ℓ,

(ii) its edge-graph is bipartite, which we write as $G_P = (V_1 \cup V_2, E)$, and

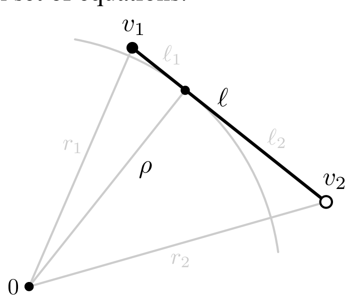
(iii) there are radii $r_1 \leq r_2$ so that $\|v\| = r_i$ for all $v \in V_i$.

If $r_1 < r_2$, then P is called *strictly bipartite*. A vertex $v \in V_i$ is called an *i-vertex*. The numbers r_1, r_2 and ℓ are called the *parameters* of a bipartite polytope.

Remark 2.2. Since P is full-dimensional by convention, Definition 2.1 only defines *full-dimensional* bipartite polytopes.

To extend this notion to not necessarily full-dimensional polytopes, we shall call a polytope *bipartite* even if it is just bipartite as a subset of its affine hull where we made an appropriate choice of origin in the affine hull (note that whether a polytope is bipartite depends on its placement relative to the origin and that there is at most one such placement if the polytope is full-dimensional). This comes in handy when we discuss faces of bipartite polytopes.

Remark 2.3. An alternative definition of bipartite polytope would replace (iii) by the condition that P has an *edge in-sphere*, that is, a sphere that touches each edge of P in a single point (this definition was used in the abstract). The configuration depicted below (an edge of P connecting two vertices $v_1 \in V_1$ and $v_2 \in V_2$) shows how any one of the four quantities r_1, r_2, ℓ and ρ (the radius of the edge in-sphere) is determined from the other three by solving the given set of equations:



$$\begin{aligned} \rho^2 + \ell_1^2 &= r_1^2 \\ \rho^2 + \ell_2^2 &= r_2^2 \\ \ell_1 + \ell_2 &= \ell \end{aligned}$$

There is a subtlety: for the edge in-sphere to actually touch the edge (rather than only its affine hull outside of the edge) it is necessary that the perpendicular projection of the origin onto the edge ends up inside the edge (equivalently, that the triangle $\mathrm{conv}\{0, v_1, v_2\}$ is acute at v_1 and v_2). One might regard this as intuitively clear since we are working with convex polytopes, but this will also follows formally as part of our proof of Proposition 3.7 (as we shall mention there in a footnote).

This alternative characterization of bipartite polytopes via edge in-spheres will become relevant towards the end of the classification (in Section 3.9). Still, for the larger part of our investigation, Definition 2.1(iii) is the more convenient version to work with.

2.1 General obsevations

Proposition 2.4. *If P is edge- but not vertex-transitive, then P is bipartite.*

This is a geometric analogue to the well known fact that every edge- but not vertex-transitive *graph* is bipartite. A proof of the graph version can be found in [4]. The following proof can be seen as a geometric analogue:

Proof of Proposition 2.4. Clearly, all edges of P are of the same length.

Fix some edge $e \in \mathcal{F}_1(P)$ with end vertices $v_1, v_2 \in \mathcal{F}_0(P)$. Let V_i be the orbit of v_i under $\mathrm{Aut}(P)$. We prove that $V_1 \cup V_2 = \mathcal{F}_0(P)$, $V_1 \cap V_2 = \varnothing$ and that the edge graph G_P is bipartite with partition $V_1 \mathbin{\dot\cup} V_2$.

Let $v \in \mathcal{F}_0(P)$ be some vertex and $\tilde{e} \in \mathcal{F}_1(P)$ an incident edge. By edge-transitivity, there is a symmetry $T \in \mathrm{Aut}(P)$ that maps $\tilde{e}$ onto e, and therefore maps v onto v_i for some $i \in \{1, 2\}$. Thus, v is in the orbit V_i. This holds for all vertices of P, and therefore $V_1 \cup V_2 = \mathcal{F}_0(P)$.

The orbits of v_1 and v_2 must either be identical or disjoint. Since $V_1 \cup V_2 = \mathcal{F}_0(P)$, from $V_1 = V_2$ it would follow $V_1 = \mathcal{F}_0(P)$, stating that P has a single orbit of vertices. But since P is *not* vertex-transitive, this cannot be. Thus, $V_1 \cap V_2 = \varnothing$, and therefore $V_1 \mathbin{\dot\cup} V_2 = \mathcal{F}_0(P)$.

Let $\tilde{e} \in \mathcal{F}_1(P)$ be an edge with end vertices $\tilde{v}_1$ and $\tilde{v}_2$. By edge-transitivity, $\tilde{e}$ can be mapped onto e by some symmetry $T \in \mathrm{Aut}(P)$. Equivalently $\{T\tilde{v}_1, T\tilde{v}_2\} = \{v_1, v_2\}$. Since v_1 and v_2 belong to different orbits under $\mathrm{Aut}(P)$, so do $\tilde{v}_1$ and $\tilde{v}_2$. Hence $\tilde{e}$ has one end vertex in V_1 and one end vertex in V_2. This holds for all edges, and thus, G_P is bipartite with partition $V_1 \mathbin{\dot\cup} V_2$.

It remains to determine the radii $r_1 \le r_2$. Set $r_i := \|v_i\|$ (assuming w.l.o.g. that $\|v_1\| \le \|v_2\|$). Then for every $v \in V_i$ there is a symmetry $T \in \mathrm{Aut}(P) \subset \mathrm{O}(\mathbb{R}^d)$ so that $Tv_i = v$, and thus

$$\|v\| = \|Tv_i\| = \|v_i\| = r_i. \qquad \square$$

Bipartite polytopes are more comfortable to work with than edge- but not vertex-transitive polytopes because their faces are again bipartite polytopes (in the sense as explained in Remark 2.2). Later, this will enable us to reduce the problem to an investigation in lower dimensions.

Proposition 2.5. *Let $\sigma \in \mathcal{F}(P)$ be a face of P. Then it holds*

(i) *if P is bipartite, so is σ.*

(ii) *if P is strictly bipartite, then so is σ, and $v \in \mathcal{F}_0(\sigma) \subseteq \mathcal{F}_0(P)$ is an i-vertex in P if and only if it is an i-vertex in σ.*

(iii) *if $r_1 \le r_2$ are the radii of P and $\rho_1 \le \rho_2$ are the radii of σ, then there holds*

$$h^2 + \rho_i^2 = r_i^2,$$

where h is the height of σ, that is, the distance of $\operatorname{aff}(\sigma)$ from the origin.

Proof. Properties clearly inherited by σ are that all edges are of the same length and that the edge graph is bipartite. It remains to show the existence of the radii $\rho_1 \le \rho_2$ compatible with the bipartition of the edge-graph of σ.

Let $c \in \operatorname{aff}(\sigma)$ be the orthogonal projection of 0 onto $\operatorname{aff}(\sigma)$. Then $\|c\| = h$, the height of σ as defined in (iii). For any vertex $v \in \mathcal{F}_0(\sigma)$ which is an i-vertex in P, the triangle $\Delta := \operatorname{conv}\{0, c, v\}$ has a right angle at c. Set $\rho_i := \|v - c\|$ and observe

$$\rho_i^2 := \|v - c\|^2 = \|v\|^2 - \|c\|^2 = r_i^2 - h^2. \tag{$*$}$$

In particular, the value ρ_i does only depend on i. In other words, σ is a bipartite polytope when considered as a subset of its affine hull, where the origin is chosen to be c (*cf.* Remark 2.2). This proves (i), and $(*)$ is equivalent to the equation in (iii). From $(*)$ also follows $r_1 < r_2 \Leftrightarrow \rho_1 < \rho_2$, which proves (ii). □

The following observation will be of use later on.

Observation 2.6. Given two adjacent vertices $v_1, v_2 \in \mathcal{F}_0(P)$ with $v_i \in V_i$, and if P has parameters r_1, r_2 and ℓ, then

$$\ell^2 = \|v_1 - v_2\|^2 = \|v_1\|^2 + \|v_2\|^2 - 2\langle v_1, v_2\rangle = r_1^2 + r_2^2 - 2r_1r_2 \cos\measuredangle(v_1, v_2),$$

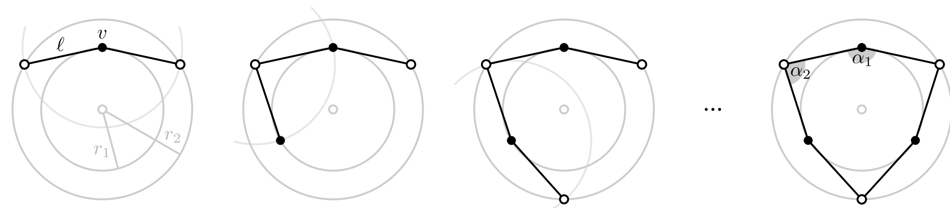
This can be rearranged for $\cos\measuredangle(v_1, v_2)$. While the exact value of this expression is not of relevance to us, this shows that this angle is determined by the parameters and does not depend on the choice of the adjacent vertices v_1 and v_2.

2.2 Bipartite polygons

The easiest to describe (and to explicitly construct) are the bipartite *polygons*.

Foremost, the edge-graph is bipartite, and thus, a bipartite polygon must be a $2k$-gon for some $k \ge 2$. One can show that the bipartite polygons are exactly the edge-transitive $2k$-gons (*cf.* Figure 1), and that such one is *strictly* bipartite if and only if it is *not* vertex-transitive (or equivalently, not regular). We will not make use of these symmetry properties of bipartite polygons.

The parameters r_1, r_2 and ℓ uniquely determine a bipartite polygon, as can be seen by explicit construction:



One starts with an arbitrary 1-vertex $v \in \mathbb{R}^2$ placed on the circle $\mathbf{S}_{r_1}(0)$. Its neighboring vertices are then uniquely determined as the intersections $\mathbf{S}_{r_2}(0) \cap \mathbf{S}_\ell(v)$. The procedure is repeated with the new vertices until the edge cycle closes (which only happens if the parameters are chosen appropriately).

The procedure also makes clear that the interior angle $\alpha_i \in (0, \pi)$ at an i-vertex only depends on i, but not on the chosen vertex $v \in V_i$.

Corollary 2.7. *A bipartite polygon $P \subset \mathbb{R}^2$ is a $2k$-gon with alternating interior angles $\alpha_1, \alpha_2 \in (0, \pi)$ (α_i being the interior angle at an i-vertex), and its shape is uniquely determined by its parameters (up to congruence).*

The exact values for the interior angles are not of relevance. Instead, we only need the following properties:

Proposition 2.8. *The interior angles $\alpha_1, \alpha_2 \in (0, \pi)$ satisfy*

$$\alpha_1 + \alpha_2 = 2\alpha^k_{\text{reg}} \quad \textit{and} \quad \alpha_2 \le \alpha^k_{\text{reg}} \le \alpha_1, \tag{2.1}$$

where $\alpha^k_{\text{reg}} := (1 - 1/k)\pi$ is the interior angle of a regular $2k$-gon, and the inequalities are satisfied with equality if and only $r_1 = r_2$.

Proof. The sum of interior angles of a $2k$-gon is $2(k-1)\pi$, and thus $k\alpha_1 + k\alpha_2 = 2(k-1)\pi$, which, after division by k, yields the first part of (2.1).

For two adjacent vertices $v_1, v_2 \in \mathcal{F}_0(P)$ (where $v_i \in V_i$), consider the triangle $\Delta := \operatorname{conv}\{0, v_1, v_2\}$ whose edge lengths are r_1, r_2 and ℓ, and whose interior angles at v_1 resp. v_2 are $\alpha_1/2$ resp. $\alpha_2/2$. From $r_1 \le r_2$ (resp. $r_1 < r_2$) and the law of sine follows $\alpha_1 \ge \alpha_2$ (resp. $\alpha_1 > \alpha_2$). With $\alpha_1 + \alpha_2 = 2\alpha^k_{\text{reg}}$ this yields the second part of (2.1). $\square$

Observation 2.9. For later use (in Corollary 3.18), consider Proposition 2.8 with $2k = 4$. In this case we find,

$$\alpha_2 \le \frac{\pi}{2} \le \alpha_1,$$

that is, α_1 is never acute, and α_2 is never obtuse.

2.3 The case $r_1 = r_2$

We classify the inscribed bipartite polytopes, that is, those with coinciding radii $r_1 = r_2$. This case is made especially easy by a classification result from [10]. We need the following definition:

Definition 2.10. Let $\Gamma \subset \mathrm{O}(\mathbb{R}^d)$ be a finite reflection group and $v \in \mathbb{R}^d$ a *generic* point *w.r.t.* Γ (*i.e.,* v is not fixed by a non-identity element of Γ). The orbit polytope

$$\operatorname{Orb}(\Gamma, v) := \operatorname{conv}\{Tv \mid T \in \Gamma\} \subset \mathbb{R}^d$$

is called a Γ-*permutahedron.*

The relevant result then reads

Theorem 2.11 (Corollary 4.6. in [10])**.** *If P has only centrally symmetric 2-dimensional faces (that is, it is a zonotope), has all vertices on a common sphere and all edges of the same length, then P is a Γ-permutahedron.*

This provides a classification of bipartite polytopes with $r_1 = r_2$.

Theorem 2.12. *If $P \subset \mathbb{R}^d$ is bipartite with $r_1 = r_2$, then it is a Γ-permutahedron.*

Proof. If $r_1 = r_2$, then all vertices are on a common sphere (that is, P is inscribed). By definition, all edges are of the same length. Both statements then also hold for the faces of P, in particular, the 2-dimensional faces. An inscribed polygon with a unique edge length is necessarily regular. With Corollary 2.7 the 2-faces are then regular $2k$-gons, therefore centrally symmetric.

Summarizing, P is inscribed, has all edges of the same length, and all 2-dimensional faces of P are centrally symmetric. By Theorem 2.11, P is a Γ-permutahedron. □

Γ-permutahedra are vertex-transitive by definition, hence do not provide examples of edge- but not vertex-transitive polytopes.

2.4 Strictly bipartite polytopes

It remains to classify the *strictly* bipartite polytopes. This problem is divided into two independent cases: dimension $d = 3$, and dimension $d \geq 4$. The detailed study of the case $d = 3$ (which turns out to be the actual hard work) is postponed until Section 3, the result of which is the following theorem:

Theorem 2.13. *If $P \subset \mathbb{R}^3$ is strictly bipartite, then P is the rhombic dodecahedron or the rhombic triacontahedron.*

Presupposing Theorem 2.13, the case $d \geq 4$ is done quickly.

Theorem 2.14. *There are no strictly bipartite polytopes in dimension $d \geq 4$.*

Proof. It suffices to show that there are no strictly bipartite polytopes in dimension $d = 4$, as any higher-dimensional example has a strictly bipartite 4-face (by Proposition 2.5).

Let $P \subset \mathbb{R}^4$ be a strictly bipartite 4-polytope. Let $e \in \mathcal{F}_1(P)$ be an edge of P. Then there are $s \geq 3$ cells (aka. 3-faces) $\sigma_1, ..., \sigma_s \in \mathcal{F}_3(P)$ incident to e, each of which is again strictly bipartite (by Proposition 2.5). By Theorem 2.13 each σ_i is a rhombic dodecahedron or rhombic triacontahedron.

The dihedral angle of the rhombic dodecahedron resp. triacontahedron is $120°$ resp. $144°$ at every edge [3]. However, the dihedral angles meeting at e must sum up to less than 2π. With the given dihedral angles this is impossible. □

3 Strictly bipartite polyhedra

In this section we derive the classification of strictly bipartite polyhedra. The main goal is to show that there are only two: the rhombic dodecahedron and the rhombic triacontahedron.

From this section on, let $P \subset \mathbb{R}^3$ denote a fixed *strictly bipartite polyhedron* with radii $r_1 < r_2$ and edge length ℓ. The 2-faces of P will be shortly referred to as just *faces* of P. Since they are bipartite, they are necessarily $2k$-gons.

Definition 3.1. We use the following terminology:

(i) a face of P is of *type* $2k$ (or called a $2k$-*face*) if it is a $2k$-gonal polygon.

(ii) an edge of P is of *type* $(2k_1, 2k_2)$ (or called a $(2k_1, 2k_2)$-*edge*) if the two incident faces are of type $2k_1$ and $2k_2$ respectively.

(iii) a vertex of P is of *type* $(2k_1, ..., 2k_s)$ (or called a $(2k_1, ..., 2k_s)$-*vertex*) if its incident faces can be enumerated as $\sigma_1, ..., \sigma_s$ so that σ_i is a $2k_i$-face (note, the order of the numbers does not matter).

We write $\tau(v)$ for the type of a vertex $v \in \mathcal{F}_0(P)$.

3.1 General observations

In a given bipartite polyhedron, the type of a vertex, edge or face already determines much of its metric properties. We prove this for faces:

Proposition 3.2. *For some face $\sigma \in \mathcal{F}_2(P)$, any of the following properties of σ determines the other two:*

(i) *its type $2k$,*

(ii) *its interior angles $\alpha_1 > \alpha_2$.*

(iii) *its height h (that is, the distance of* $\operatorname{aff}(\sigma)$ *from the origin).*

Corollary 3.3. *Any two faces of P of the same height, or the same type, or the same interior angles, are congruent.*

Proof of Proposition 3.2. Fix a face $\sigma \in \mathcal{F}_2(P)$.

Suppose that the height h of σ is known. By Proposition 2.5, a face of P of height h is bipartite with radii $\rho_i^2 := r_i^2 - h^2$ and edge length ℓ. By Corollary 2.7, these parameters then uniquely determine the shape of σ, which includes its type and its interior angles. This shows (iii) $\implies$ (i), (ii).

Suppose now that we know the interior angles $\alpha_1 > \alpha_2$ of σ (it actually suffices to know one of these, say α_1). Fix a 1-vertex $v \in V_1$ of σ and let $w_1, w_2 \in V_2$ be its two adjacent 2-vertices in σ. Consider the simplex $S := \operatorname{conv}\{0, v, w_1, w_2\}$. The length of each edge of S is already determined, either by the parameters alone, or by additionally using the known interior angles via

$$\begin{aligned}\|w_1 - w_2\|^2 &= \|w_1 - v\|^2 + \|w_2 - v\|^2 - 2\langle w_1 - v, w_2 - v\rangle \\ &= 2\ell^2(1 - \cos \underbrace{\measuredangle(w_1 - v, w_2 - v)}_{\alpha_1}).\end{aligned}$$

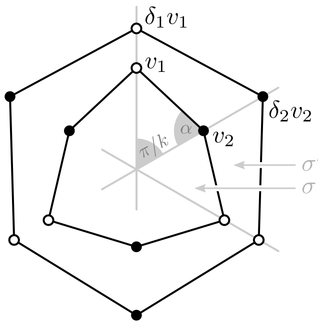
Thus, the shape of S is determined. In particular, this determines the height of the face $\operatorname{conv}\{v, w_1, w_2\} \subset S$ over the vertex $0 \in S$. Since $\operatorname{aff}\{v, w_1, w_2\} = \operatorname{aff}(\sigma)$, this determines the height of σ in P. This proves (ii) $\implies$ (iii).

Finally, suppose that the type $2k$ is known. We then want to show that the height h is uniquely determined.[2] For the sake of contradiction, suppose that the type $2k$ does *not*

[2] The reader motivated to prove this himself should know the following: it is indeed possible to write down a polynomial in h of degree four whose coefficients involve only r_1, r_2, ℓ and $\cos(\pi/k)$, and whose zeroes include all possible heights of any $2k$-face of P. However, it turns out to be quite tricky to work out which zeroes correspond to feasible solutions. For certain values of the coefficients there are multiple positive solutions for h, some of which correspond to *non-convex* $2k$-faces. There seems to be no easy way to tell them apart.

uniquely determine the height of the face. Then there is another $2k$-face $\sigma' \in \mathcal{F}_2(P)$ of some height $h' \neq h$. W.l.o.g. assume $h > h'$.

Visualize both faces embedded in $\mathbb{R}^2$, on top of each other and centered at the origin as shown in the figure below:



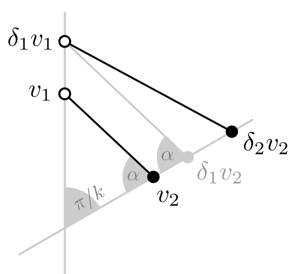
The vertices in both polygons are equally spaced by an angle of π/k (*cf.* Observation 2.6) and we can therefore assume that the vertex v_i of σ (resp. v_i' of σ') is a positive multiple of $(\sin(i\pi/k), \cos(i\pi/k)) \in \mathbb{R}^2$ for $i \in \{1, ..., 2k\}$. There are then factors $\delta_i \in \mathbb{R}_+$ with $v_i' = \delta v_i$.

The norms of vectors v_1, v_2, $\delta_1 v_1$ and $\delta_2 v_2$ are the radii of the bipartite polygons σ and σ'. With Proposition 2.5(iii) from $h > h'$ follows $\|v_1\| < \|\delta_1 v_1\|$ and $\|v_2\| < \|\delta_2 v_2\|$, and thus, $(*)$ $\delta_1, \delta_2 > 1$. W.l.o.g. assume $\delta_1 \leq \delta_2$.

Since both faces have edge length ℓ, we have $\|v_1 - v_2\| = \|\delta_1 v_1 - \delta_2 v_2\| = \ell$. Our goal is to derive the following contradiction:

$$\ell = \|v_1 - v_2\| \overset{(*)}{<} \delta_1 \|v_1 - v_2\| = \|\delta_1 v_1 - \delta_1 v_2\| \overset{(**)}{<} \|\delta_1 v_1 - \delta_2 v_2\| = \ell,$$

To prove $(**)$, consider the triangle Δ with vertices $\delta_1 v_1$, $\delta_2 v_2$ and $\delta_1 v_2$:



Since σ is convex, the angle α is smaller than $90°$. It follows that the interior angle of Δ at $\delta_1 v_2$ is obtuse (here we are using $\delta_1 \leq \delta_2$). Hence, by the sine law, the edge of Δ opposite to $\delta_1 v_2$ is the longest, which translates to $(**)$. $\square$

As a consequence of Proposition 3.2, the interior angles of a face of P do only depend on the type of the face (and the parameters), and so we can introduce the notion of *the* interior angle $\alpha_i^k \in (0, \pi)$ of a $2k$-face at an i-vertex. Furthermore, set $\epsilon_k := (\alpha_1^k - \alpha_2^k)/2\pi$. By Proposition 2.8 we have $\epsilon_k > 0$ and

$$\alpha_1^k = \Big(1 - \frac{1}{k} + \epsilon_k\Big)\pi, \qquad \alpha_2^k = \Big(1 - \frac{1}{k} - \epsilon_k\Big)\pi.$$

Definition 3.4. If $\tau = (2k_1, ..., 2k_s)$ is the type of a vertex, then define

$$K(\tau) := \sum_{i=1}^{s} \frac{1}{k_i}, \qquad E(\tau) := \sum_{i=1}^{s} \epsilon_{k_i}.$$

Both quantities are strictly positive.

Proposition 3.5. *Let $v \in \mathcal{F}_0(P)$ be a vertex of type $\tau = (2k_1, ..., 2k_s)$.*

(i) *If $v \in V_1$, then $E(\tau) < K(\tau) - 1$ and $s = 3$.*

(ii) *If $v \in V_2$, then $E(\tau) > s - 2 - K(\tau)$.*

Proof. Let $\sigma_1, ..., \sigma_s \in \mathcal{F}_2(P)$ be the faces incident to v, so that σ_j is a $2k_j$-face. The interior angle of σ_j at v is $\alpha_i^{k_j}$, and the sum of these must be smaller than 2π. In formulas

$$2\pi > \sum_{j=1}^{s} \alpha_i^{k_j} = \sum_{j=1}^{s} \Big(1 - \frac{1}{k_j} \pm \epsilon_{k_j}\Big)\pi = (s - K(\tau) \pm E(\tau))\pi,$$

where $\pm$ is the plus sign for $i = 1$, and the minus sign for $i = 2$. Rearranging for $E(v)$ yields $(*)$ $\mp E(\tau) > s - 2 - K(\tau)$. If $i = 2$, this proves (ii). If $i = 1$, note that from the implication $k_j \geq 2 \implies K(\tau) \leq s/2$ follows

$$s \overset{(*)}{<} -E(\tau) + K(\tau) + 2 \leq 0 + \frac{s}{2} + 2 \qquad \implies \qquad s < 4.$$

The minimum degree of a vertex in a polyhedron is at least three, hence $s = 3$, and $(*)$ becomes (i). □

This allows us to exclude all but a manageable list of types for 1-vertices. Note that a vertex $v \in V_1$ has a type of some form $(2k_1, 2k_2, 2k_3)$.

Corollary 3.6. *For a* 1*-vertex $v \in V_1$ of type τ holds $K(\tau) > 1 + E(\tau) > 1$. One checks that this leaves exactly the options in* Table 1.

τ	$K(\tau)$	Γ
$(4,4,\ 4)$	$3/2$	$I_1 \oplus I_1 \oplus I_1$
$(4,4,\ 6)$	$4/3$	$I_1 \oplus I_2(3)$
$(4,4,\ 8)$	$5/4$	$I_1 \oplus I_2(4)$
$\vdots$	$\vdots$	$\vdots$
$(4,4,2k)$	$1 + 1/k$	$I_1 \oplus I_2(k)$
$(4,6,\ 6)$	$7/6$	$A_3 = D_3$
$(4,6,\ 8)$	$13/12$	B_3
$(4,6,10)$	$31/30$	H_3

Table 1: Possible types of 1-vertices, their K-values and the Γ of the Γ-permutahedron in which all vertices have this type.

The types in Table 1 are called the *possible types* of 1-vertices. Each of the possible types is realizable in the sense that there exists a bipartite polyhedron in which all 1-vertices have this type. Examples are provided by the Γ-permutahedra (the Γ of that Γ-permutahedron is listed in the right column of Table 1). These are not *strictly* bipartite though.

The convenient thing about Γ-permutahedra is that all their vertices are of the same type. We cannot assume this for general strictly bipartite polyhedra, not even for all 1-vertices.

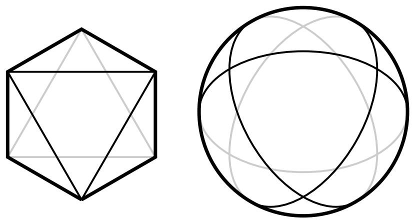
3.2 Spherical polyhedra

The purpose of this section is to define a second notion of interior angle for each face. These angles can be defined in several equivalent ways, one of which is via spherical polyhedra.

A *spherical polyhedron* is an embedding of a planar graph into the unit sphere, so that all edges are embedded as great circle arcs, and all regions are convex[3]. If $0 \in \operatorname{int}(P)$, we can associate to P a spherical polyhedron P^S by applying central projection

$$\mathbb{R}^3 \setminus \{0\} \to \mathbf{S}_1(0), \quad x \mapsto \frac{x}{\|x\|}$$

to all its vertices and edges (this process is visualized below).



The vertices, edges and faces of P have spherical counterparts in P^S obtained as projections onto the unit sphere. Those will be denoted with a superscript "S". For example, if $e \in \mathcal{F}_1(P)$ is an edge of P, then e^S denotes the corresponding "spherical edge", which is a great circle arc obtained as the projection of e onto the sphere.

We still need to justify that the spherical polyhedron of P is well-defined, by proving that P contains the origin:

Proposition 3.7. $0 \in \operatorname{int}(P)$.

Proof. The proof proceeds in several steps.

Step 1: Fix a 1-vertex $v \in V_1$ with neighbors $w_1, w_2, w_3 \in V_2$, and let $u_i := w_i - v$ be the direction of the edge $\operatorname{conv}\{v, w_i\}$ emanating from v. Let $\sigma_{ij} \in \mathcal{F}_2(P)$ denote the $2k$-face containing v, w_i and w_j. The interior angle of σ_{ij} at v is then $\measuredangle(u_i, u_j)$, which by Proposition 2.8 and $k \geq 2$ satisfies

$$\measuredangle(u_i, u_j) > \Big(1 - \frac{1}{k}\Big)\pi \geq \frac{\pi}{2} \quad \implies \quad \langle u_i, u_j \rangle < 0.$$

Step 2: Besides v, the polyhedron P contains another 1-vertex $v' \in V_1$. It then holds $v' \in v + \operatorname{cone}\{u_1, u_2, u_3\}$, which means that there are non-negative coefficients

[3]Convexity on the sphere means that the shortest great circle arc connecting any two points in the region is also contained in the region.

$a_1, a_2, a_3 \geq 0$, at least one positive, so that $v + a_1 u_1 + a_2 u_2 + a_3 u_3 = v'$. Rearranging and applying $\langle v, \cdot \rangle$ yields

$$\begin{aligned} a_1\langle v, u_1\rangle + a_2\langle v, u_2\rangle + a_3\langle v, u_3\rangle &= \langle v, v'\rangle - \langle v, v\rangle \\ &= r_1^2 \cos \measuredangle(v, v') - r_1^2 < 0. \end{aligned} \tag{$*$}$$

The value $\langle v, u_i\rangle$ is independent of i (see Observation 2.6). Since there is at least one positive coefficient a_i, from $(*)$ follows $\langle v, u_i\rangle < 0$.[4]

Step 3: By the previous steps, $\{v, u_1, u_2, u_3\}$ is a set of four vectors with pair-wise negative inner product. The convex hull of such an arrangement in 3-dimensional Euclidean space does necessarily contain the origin in its interior, or equivalently, there are positive coefficients $a_0, ..., a_3 > 0$ with $a_0 v + a_1 u_1 + a_2 u_2 + a_3 u_3 = 0$ (for a proof, see Proposition A.1). In other words: $0 \in v + \mathrm{int}(\mathrm{cone}\{u_1, u_2, u_3\})$.

Step 4: If $H(\sigma)$ denotes the half-space associated with the face $\sigma \in \mathcal{F}_2(P)$, then

$$0 \in v + \mathrm{int}(\mathrm{cone}\{u_1, u_2, u_3\}) = \bigcap_{\sigma \sim v} \mathrm{int}(H(\sigma)).$$

Thus, $0 \in \mathrm{int}(H(\sigma))$ for all faces σ incident to v. But since every face is incident to a 1-vertex, we obtain $0 \in \mathrm{int}(H(\sigma))$ for all $\sigma \in \mathcal{F}_2(P)$, and thus $0 \in \mathrm{int}(P)$ as well. □

The main reason for introducing spherical polyhedra is that we can talk about the *spherical interior angles* of their faces.

Let $\sigma \in \mathcal{F}_2(P)$ be a face, and $v \in \mathcal{F}_0(\sigma)$ one of its vertices. Let $\alpha(\sigma, v)$ denote the interior angle of σ at v, and $\beta(\sigma, v)$ the spherical interior angle of σ^S at v^S. It only needs a straight-forward computation (involving some spherical geometry) to establish a direct relation between these angles: *e.g.* if v is a 1-vertex, then

$$\sin^2(\ell^S) \cdot (1 - \cos\beta(\sigma, v)) = \Big(\frac{\ell}{r_2}\Big)^2 \cdot (1 - \cos\alpha(\sigma, v)),$$

where ℓ^S denotes the arc-length of an edge of P^S (indeed, all edges are of the same length). An equivalent formula exists for 2-vertices. The details of the computation are not of relevance, but can be found in Appendix A.2.

The core message is that the value of $\alpha(\sigma, v)$ uniquely determines the value of $\beta(\sigma, v)$ and vice versa. In particular, since the value of $\alpha(\sigma, v) = \alpha_i^k$ does only depend on the type of the face and the partition class of the vertex, so does $\beta(\sigma, v)$, and it makes sense to introduce the notion β_i^k for the spherical interior angle of a $2k$-gonal spherical face of P^S at (the projection of) an i-vertex. Thus, we have

$$\beta_i^{k_1} = \beta_i^{k_2} \quad \Longleftrightarrow \quad \alpha_i^{k_1} = \alpha_i^{k_2} \quad \overset{3.2}{\Longleftrightarrow} \quad k_1 = k_2, \tag{3.1}$$

where we use Proposition 3.2 for the last equivalence.

Observation 3.8. The spherical interior angles β_i^k have the following properties:

(i) The spherical interior angles surrounding a vertex add up to exactly 2π. That is, for an i-vertex $v \in \mathcal{F}_0(P)$ of type $(2k_1, ..., 2k_s)$ holds

$$\beta_i^{k_1} + \cdots + \beta_i^{k_s} = 2\pi.$$

[4]Note that this provides the formal proof mentioned in Remark 2.3, namely, that the triangle $\mathrm{conv}\{0, v_1, v_2\}$ is acute at v_1 and v_2.

(ii) The sum of interior angles of a spherical polygon always exceed the interior angle sum of a respective flat polygon. That is, it holds

$$k\beta_1^k + k\beta_2^k > 2(k-1)\pi \quad \implies \quad \beta_1^k + \beta_2^k > 2\Big(1 - \frac{1}{k}\Big)\pi.$$

This has some consequences for the strictly bipartite polyhedron P:

Corollary 3.9. *P contains at most two different types of 1-vertices, and if there are two, then one is of the form $(4,4,2k)$, and the other one is of the form $(4,6,2k')$ for distinct $k \neq k'$ and $2k' \in \{6,8,10\}$.*

Proof. Each possible type listed in Table 1 is either of the form $(4,4,2k)$ or of the form $(4,6,2k')$ for some $2k \geq 4$ or $2k' \in \{6,8,10\}$.

If P contains simultaneously 1-vertices of type $(4,4,2k_1)$ and $(4,4,2k_2)$, apply Observation 3.8(i) to see

$$\beta_1^2 + \beta_1^2 + \beta_1^{k_1} \overset{(i)}{=} \beta_1^2 + \beta_1^2 + \beta_1^{k_2} \quad \implies \quad \beta_1^{k_1} = \beta_1^{k_2} \quad \overset{(3.1)}{\implies} \quad k_1 = k_2.$$

If P contains simultaneously 1-vertices of type $(4,6,2k_1')$ and $(4,6,2k_2')$, then

$$\beta_1^2 + \beta_1^3 + \beta_1^{k_1'} \overset{(i)}{=} \beta_1^2 + \beta_1^3 + \beta_1^{k_2'} \quad \implies \quad \beta_1^{k_1'} = \beta_1^{k_2'} \quad \overset{(3.1)}{\implies} \quad k_1' = k_2'.$$

Finally, if P contains simultaneously 1-vertices of type $(4,4,2k)$ and $(4,6,2k')$, then

$$\beta_1^2 + \beta_1^2 + \beta_1^k \overset{(i)}{=} \beta_1^2 + \beta_1^3 + \beta_1^{k'} \quad \implies \quad \beta_1^k - \beta_1^{k'} = \underbrace{\beta_1^3 - \beta_1^2}_{\neq 0 \text{ by (3.1)}} \quad \overset{(3.1)}{\implies} \quad k \neq k'. \qquad \square$$

Since each edge of P is incident to a 1-vertex, we obtain

Observation 3.10. If P has only 1-vertices of types $(4,4,2k)$ and $(4,6,2k')$, then each edge of P is of one of the types

$$\underbrace{(4,4),\ (4,2k)}_{\text{from a } (4,4,2k)\text{-vertex}},\ \underbrace{(4,6),\ (4,2k') \quad \text{or} \quad (6,2k')}_{\text{from a } (4,6,2k')\text{-vertex}}.$$

Corollary 3.11. *The dihedral angle of an edge $e \in \mathcal{F}_1(P)$ of P only depends on its type.*

Proof. Suppose that e is a $(2k_1, 2k_2)$-edge. Then e is incident to a 1-vertex $v \in V_1$ of type $(2k_1, 2k_2, 2k_3)$. By Observation 3.8(i) holds $\beta_1^{k_3} = 2\pi - \beta_1^{k_1} - \beta_1^{k_2}$, which further determines k_3. By Proposition 3.2 we have uniquely determined interior angles $\alpha_1^{k_1}, \alpha_1^{k_2}$ and $\alpha_1^{k_3}$.

It is known that for a simple vertex (that is, a vertex of degree three) the interior angles of the incident faces already determine the dihedral angles at the incident edges (for a proof, see the Appendix, Proposition A.2). Consequently, the dihedral angle at e is already determined. $\square$

The next result shows that Γ-permutahedra are the only bipartite polytopes in which a 1-vertex and a 2-vertex can have the same type.

Corollary 3.12. *P cannot contain a 1-vertex and a 2-vertex of the same type.*

Proof. Let $v \in \mathcal{F}_0(P)$ be a vertex of type $(2k_1, 2k_2, 2k_3)$. The incident edges are of type $(2k_1, 2k_2)$, $(2k_2, 2k_3)$ and $(2k_3, 2k_1)$ respectively. By Corollary 3.11 the dihedral angles of these edges are uniquely determined, and since v is simple (that is, has degree three), the interior angles of the incident faces are also uniquely determined (*cf.* Appendix, Proposition A.2). In particular, we obtain the same angles independent of whether v is a 1-vertex or a 2-vertex.

A 1-vertex is always simple, and thus, a 1-vertex and a 2-vertex of the same type would have the same interior angles at all incident faces, that is, $\alpha_1^k = \alpha_2^k$ for each incident $2k$-face. But this is not possible if P is *strictly* bipartite (by Proposition 2.5(ii) and Proposition 2.8). □

3.3 Adjacent pairs

Given a 1-vertex $v \in V_1$ of type $\tau_1 = (2k_1, 2k_2, 2k_3)$, for any two distinct $i, j \in \{1, 2, 3\}$, v has a neighbor $w \in V_2$ of type $\tau_2 = (2k_i, 2k_j, *, ..., *)$, where $*$ are placeholders for unknown entries. The pair of types

$$(\tau_1, \tau_2) = ((2k_1, 2k_2, 2k_3),\ (2k_i, 2k_j, *, ..., *))$$

is called an *adjacent pair* of P. It is the purpose of this section to show that certain adjacent pairs cannot occur in P. Excluding enough adjacent pairs for fixed τ_1 then proves that the type τ_1 cannot occur as the type of a 1-vertex.

Our main tools for achieving this will be the inequalities established in Proposition 3.5 (i) and (ii), that is

$$E(\tau_1) \overset{\text{(i)}}{<} K(\tau_1) - 1 \quad \text{and} \quad E(\tau_2) \overset{\text{(ii)}}{>} s - 2 - K(\tau_2),$$

where s is the number of elements in τ_2. For a warmup, and as a template for further calculations, we prove that the adjacent pair $(\tau_1, \tau_2) = ((4, 6, 8), (6, 8, 8))$ will not occur in P.

Example 3.13. By Proposition 3.5(i) we have

$$(*) \quad \epsilon_2 + \epsilon_3 + \epsilon_4 = E(\tau_1) \overset{(i)}{<} K(\tau_1) - 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - 1 = \frac{1}{12}.$$

On the other hand, by Proposition 3.5(ii) we have

$$\begin{aligned}(**) \quad \frac{2}{12} = 3 - 2 - \Big(\frac{1}{3} + \frac{1}{4} + \frac{1}{4}\Big) &= s - 2 - K(\tau_2) \\ &\overset{(ii)}{<} E(\tau_2) = \underbrace{\epsilon_3 + \epsilon_4}_{<1/12} + \underbrace{\epsilon_4}_{<1/12} < \frac{2}{12},\end{aligned}$$

which is a contradiction. Hence this adjacent pair cannot occur. Note that we used $(*)$ to upperbound certain sums of ϵ_i in $(**)$.

An adjacent pair excluded by using the inequalities from Proposition 3.5(i) and (ii) as demonstrated in Example 3.13 will be called *infeasible*.

The argument applied in Example 3.13 will be repeated many times for many different adjacent pairs in the upcoming Sections 3.5, 3.4, 3.6, 3.8, and we shall therefore use a tabular form to abbreviate it. After fixing, $\tau_1 = (4, 6, 8)$, the argument to refute the adjacent pair $(\tau_1, \tau_2) = ((4, 6, 8), (6, 8, 8))$ is abbreviated in the first row of the following table:

τ_2	$s - 2 - K(\tau_2)$	$\overset{?}{<}$	$E(\tau_2)$	
$(6, 8, 8)$	$2/12$	$\not<$	$(\epsilon_3 + \epsilon_4) + \epsilon_4$	$< 2/12$
$(6, 8, 6, 6)$	$9/12$	$\not<$	$(\epsilon_3 + \epsilon_4) + \epsilon_3 + \epsilon_3$	$< 3/12$

The second row displays the analogue argument for another example, namely, the pair $((4, 6, 8), (6, 8, 6, 6))$, showing that it is infeasible as well. Both rows will reappear in the table of Section 3.5 where we exclude $(4, 6, 8)$ as a type for 1-vertices entirely. Note that the terms in the column below $E(\tau_2)$ are grouped by parenthesis to indicate which subsums are upper bounded via Proposition 3.5(i). In this example, if there are n groups, then the sum is upper bounded by $n/12$.

The placeholders in an adjacent pair $((2k_1, 2k_2, 2k_3), (2k_i, 2k_j, *, ..., *))$ can, in theory, be replaced by an arbitrary sequence of even numbers, and each such pair has to be refuted separately. The following fact will make this task tractable: write $\tau \subset \tau'$ if τ is a *subtype* of τ', that is, a vertex type that can be obtained from τ' by removing some of its entries. We then can prove

Proposition 3.14. *If (τ_1, τ_2) is an infeasible adjacent pair, then the pair (τ_1, τ_2') is infeasible as well, for every $\tau_2' \supset \tau_2$.*

Proof. Suppose $\tau_2 = (2k_1, ..., 2k_s)$, $\tau_2' = (2k_1, ..., 2k_s, 2k_{s+1}, ..., 2k_{s'}) \supset \tau_2$, and that the pair (τ_1, τ_2') is *not* infeasible. Then τ_2' satisfies Proposition 3.5(ii)

$$\begin{aligned} & E(\tau_2') > s' - 2 - K(\tau_2') \\ \implies \quad & E(\tau_2) > s - 2 - K(\tau_2) + \sum_{i=s+1}^{s'} \overbrace{\Big(1 - \frac{1}{k_i} - \epsilon_{k_i}\Big)}^{\alpha_2^{k_i}/\pi > 0} > s - 2 - K(\tau_2). \end{aligned}$$

But this is exactly the statement that τ_2 satisfies Proposition 3.5(ii) as well, *i.e.,* that the pair (τ_1, τ_2) is also not infeasible. □

By Proposition 3.14 it is sufficient to exclude so-called *minimal infeasible adjacent pairs*, that is, infeasible adjacent pairs (τ_1, τ_2) for which (τ_1, τ_2') is not infeasible for any $\tau_2' \subset \tau_2$.

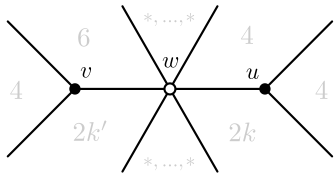
A second potential problem is, that we know little about the values that might replace the placeholders in $\tau_2 = (2k_i, 2k_j, *, ..., *)$. For our immediate goal, dealing with the following special case is sufficient:

Proposition 3.15. *The placeholders in an adjacent pair $((4, 6, 2k'), (6, 2k', *, ..., *))$ can only contain 4, 6 and $2k'$.*

Proof. Suppose that P contains an adjacent pair

$$(\tau_1, \tau_2) = ((4, 6, 2k'), (6, 2k', 2k, *, ..., *))$$

induced by a 1-vertex $v \in V_1$ of type τ_1 with neighbor $w \in V_2$ of type τ_2. Suppose further that $2k \notin \{4, 6, 2k'\}$. The vertex w is then incident to a $2k$-face, and therefore also to a 1-vertex $u \in V_1$ of type $(4, 4, 2k)$ (u cannot be of type $(4, 6, 2k)$ because of $k \neq k'$ and Corollary 3.9). This configuration is depicted below:



Note that w is also incident to a 4-face, and thus $(6, 2k', 2k, 4) \subseteq \tau_2$.

By Proposition 3.5(i) the existence of 1-vertices of type $(4, 4, 2k)$ and $(4, 6, 2k')$ yields inequalities

$$\epsilon_2 + \epsilon_2 + \epsilon_k < \frac{1}{k} \quad \text{and} \quad \epsilon_2 + \epsilon_3 + \epsilon_{k'} < \frac{1}{k'} - \frac{1}{6}. \tag{3.2}$$

Since τ_2 has $\tau := (6, 2k', 2k, 4)$ as a subtype, by Proposition 3.14 it suffices to show that the pair $((4, 6, 2k'), (6, 2k', 2k, 4))$ is infeasible. This follows via Proposition 3.5(ii):

$$\begin{aligned} \frac{7}{6} - \frac{1}{k} - \frac{1}{k'} &= 4 - 2 - K(\tau) \\ &\overset{(ii)}{<} E(\tau) = \underbrace{\epsilon_2 + \epsilon_3 + \epsilon_{k'}}_{<1/k'-1/6} + \underbrace{\epsilon_k}_{<1/k} \overset{(3.2)}{<} \frac{1}{k} + \frac{1}{k'} - \frac{1}{6}, \end{aligned}$$

which rearranges to $1/k + 1/k' > 2/3$. Recalling $2k' \in \{6, 8, 10\} \implies k' \geq 3$ (from Corollary 3.9) and $2k \notin \{4, 6, 2k'\} \implies k \geq 4$ shows that this is not possible. $\square$

3.4 The case $\tau_1 = (4, 6, 10)$

If P contains a 1-vertex of type $(4, 6, 10)$, then it contains an adjacent pair of the form

$$(\tau_1, \tau_2) = ((4, 6, 10), (6, 10, *, ..., *)).$$

We proceed as demonstrated in Example 3.13. Proposition 3.5(i) yields $\epsilon_2 + \epsilon_3 + \epsilon_5 < 1/30$. By Proposition 3.15 the placeholders can only take on values 4, 6 or 10. The following table lists the minimally infeasible adjacent pairs and proves their infeasibility.

τ_2	$s - 2 - K(\tau_2)$	$\overset{?}{<}$	$E(\tau_2)$	
$(6, 10, 6)$	$4/30$	$\not<$	$(\epsilon_3 + \epsilon_5) + \epsilon_3$	$< 2/30$
$(6, 10, 10)$	$8/30$	$\not<$	$(\epsilon_3 + \epsilon_5) + \epsilon_5$	$< 2/30$
$(6, 10, 4, 4)$	$14/30$	$\not<$	$(\epsilon_2 + \epsilon_3 + \epsilon_5) + \epsilon_2$	$< 2/30$

By Proposition 3.14 we conclude: the placeholder in $\tau_2 = (6, 10, *, ..., *)$ can contain no 6 or 10, and at most one 4. This leaves us with the option $\tau_2 = (4, 6, 10)$, which is the same as τ_1 and therefore not possible by Corollary 3.12. Therefore, P cannot contain a 1-vertex of type $(4, 6, 10)$.

3.5 The case $\tau_1 = (4, 6, 8)$

If P contains a 1-vertex of type $(4, 6, 8)$, then it also contains an adjacent pair of the form

$$(\tau_1, \tau_2) = ((4, 6, 8), (6, 8, *, ..., *)).$$

We proceed as demonstrated in Example 3.13. Proposition 3.5(i) yields $\epsilon_2 + \epsilon_3 + \epsilon_4 < 1/12$. By Proposition 3.15 the placeholders can only take on values 4, 6 or 8. The following table lists the minimally infeasible adjacent pairs and proves their infeasibility.

τ_2	$s-2-K(\tau_2)$	$\overset{?}{<}$	$E(\tau_2)$	
$(6,8,8)$	$2/12$	$\not<$	$(\epsilon_3+\epsilon_4)+\epsilon_3$	$< 2/12$
$(6,8,4,4)$	$5/12$	$\not<$	$(\epsilon_2+\epsilon_3+\epsilon_4)+\epsilon_2$	$< 2/12$
$(6,8,4,6)$	$7/12$	$\not<$	$(\epsilon_2+\epsilon_3+\epsilon_4)+\epsilon_3$	$< 2/12$
$(6,8,6,6)$	$9/12$	$\not<$	$(\epsilon_2+\epsilon_3+\epsilon_4)+\epsilon_3+\epsilon_3$	$< 3/12$

By Proposition 3.14 we conclude: the placeholder in $\tau_2 = (6,8,*,...,*)$ can contain no 8, and at most one 4 or 6, but not both at the same time.

This leaves us with the options $\tau_2 = (4,6,8)$ and $\tau_2 = (6,6,8)$. In the first case, $\tau_1 = \tau_2$ which not possible by Corollary 3.12. In the second case, there would be two adjacent 6-faces, but P does not contain $(6,6)$-edges by Observation 3.10 with $2k' = 8$. Therefore, P cannot contain a 1-vertex of type $(4,6,8)$.

3.6 The case $\tau_1 = (4,6,6)$

If P contains a 1-vertex of type $(4,6,6)$, then it also contains an adjacent pair of the form

$$(\tau_1,\tau_2) = ((4,6,6),(6,6,*,...,*)).$$

We proceed as demonstrated in Example 3.13. Proposition 3.5(i) yields $\epsilon_2+\epsilon_3+\epsilon_3 < 1/6$. By Proposition 3.15 the placeholders can only take on values 4 or 6. The following table lists the minimally infeasible adjacent pairs and proves their infeasibility.

τ_2	$s-2-K(\tau_2)$	$\overset{?}{<}$	$E(\tau_2)$	
$(6,6,4,4)$	$2/6$	$\not<$	$(\epsilon_2+\epsilon_3+\epsilon_3)+\epsilon_2$	$< 2/6$
$(6,6,6,4)$	$3/6$	$\not<$	$(\epsilon_2+\epsilon_3+\epsilon_3)+\epsilon_3$	$< 2/6$
$(6,6,6,6)$	$4/6$	$\not<$	$(\epsilon_3+\epsilon_3)+(\epsilon_3+\epsilon_3)$	$< 2/6$

By Proposition 3.14 we conclude: the placeholder in $\tau_2 = (6,6,*,...,*)$ can contain at most one 4 or 6, but not both at the same time.

This leaves us with the options $\tau_2 = (4,6,6)$ and $\tau_2 = (6,6,6)$. In the first case we have $\tau_1 = \tau_2$, which is not possible by Corollary 3.12. Excluding $(6,6,6)$ needs more work: fix a 6-gon $\sigma \in \mathcal{F}_2(P)$. Each edge of σ is either of type $(4,6)$ or of type $(6,6)$ (by Observation 3.10). Each 1-vertex of σ (which must be of type $(4,6,6)$) is then incident to exactly one of these $(6,6)$-edges of σ. Thus, there are exactly *three* $(6,6)$-edges incident to σ (see Figure 3). On the other hand, each 2-vertex of σ is incident to an even number of $(6,6)$-edges of σ (since if a 2-vertex is incident to at least one $(6,6)$-edge, then we have previously shown that its type must be $(6,6,6)$, implying another incident $(6,6)$-edge). Therefore the number of $(6,6)$-edges incident to σ must be *even* (see Figure 3), in contradiction to the previously obtained number three of such edges.

Consequently, P cannot contain a 1-vertex of type $(4,6,6)$.

Observation 3.16. It is a consequence of Sections 3.6, 3.5, 3.4 that P cannot have a 1-vertex of a type $(4,6,2k')$ for a $2k' \in \{6,8,10\}$. By Corollary 3.9 this means that *all* 1-vertices of P are of the same type $\tau_1 = (4,4,2k)$ for some fixed $2k \geq 4$.

It is worth to distinguish the case $(4,4,4)$ from the cases $(4,4,2k)$ with $2k \geq 6$.

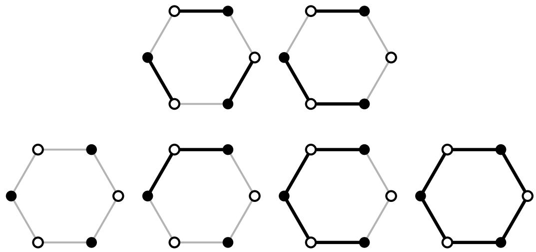


Figure 3: Possible distributions of $(4,6)$-edges (gray) and $(6,6)$-edges (thick) around a 6-gon as discussed in Section 3.6. The top row shows configurations compatible with the conditions set by 1-vertices (black), and the bottom row shows the configurations compatible with the conditions set by the 2-vertices (white).

3.7 The case $\tau_1 = (4,4,4)$

In this case, all 2-faces are 4-gons, and all 4-gons are congruent by Proposition 3.2. A 4-gon with all edges of the same length is known as a *rhombus*, and the polyhedra with congruent rhombic faces are known as *rhombic isohedra* (from german *Rhombenisoeder*). These have a known classification:

Theorem 3.17 (S. Bilinksi, 1960 [2])**.** *If P is a polyhedron with congruent rhombic faces, then P is one of the following:*

(i) *a member of the infinite family of rhombic hexahedra, i.e., P can be obtained from a cube by stretching or squeezing it along a long diagonal,*

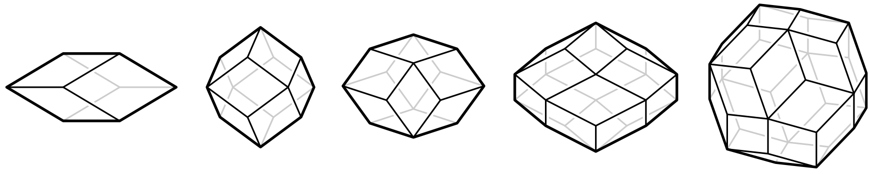
(ii) *the rhombic dodecahedron,*

(iii) *the Bilinski dodecahedron,*

(iv) *the rhombic icosahedron, or*

(v) *the rhombic triacontahedron.*

The figure below depicts these polyhedra in the given order (from left to right; including only one instance from the family (i)):



The rhombic dodecahedron and triacontahedron are known edge- but not vertex-transitive polytopes. We show that the others are not even strictly bipartite.

Corollary 3.18. *If P is strictly bipartite with all 1-vertices of type $(4,4,4)$, then P is one of the following:*

(i) *the rhombic dodecahedron,*

(ii) *the rhombic triacontahedron.*

Proof. The listed ones are edge-transitive but not vertex-transitive. Also they are not inscribed. By Proposition 2.4 they are therefore *strictly* bipartite.

We then have to exclude the other polyhedra listed in Theorem 3.17. The rhombic hexahedra include the cube, which is inscribed, hence not strictly bipartite. In all the other cases, there exist vertices where acute and obtuse angles meet (see the figure). So this vertex cannot be assigned to either V_1 or V_2 (*cf.* Observation 2.9), and the polyhedron cannot be bipartite. □

These are the only strictly bipartite polyhedra we will find, and both are edge-transitive without being vertex-transitive.

3.8 The case $\tau_1 = (4, 4, 2k), 2k \geq 6$

If P contains a 1-vertex of type $(4, 4, 2k)$ with $2k \geq 6$, then it also has an adjacent pair of the form

$$(\tau_1, \tau_2) = ((4, 4, 2k), (4, 2k, *, ..., *)).$$

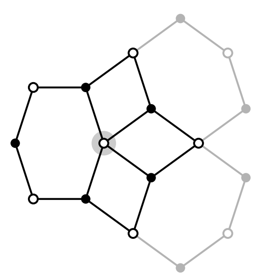
We proceed as demonstrated in Example 3.13. Proposition 3.5(i) yields $\epsilon_2 + \epsilon_2 + \epsilon_k < 1/k$. Since $(4, 4, 2k)$ is the only type of 1-vertex of P, there are only 4-faces and $2k$-faces and the placeholders can only take on the values 4 and $2k$ (note that we do *not* use Proposition 3.15 for this). The following table lists some inequalities derived for infeasible pairs:

τ_2	$s - 2 - K(\tau_2)$	$\overset{?}{<}$	$E(\tau_2)$	
$(4, 2k, 4, 4, 4)$	$1 - 1/k$	$<$	$(\epsilon_2 + \epsilon_2 + \epsilon_k) + (\epsilon_2 + \epsilon_2)$	$< 2/k$
$(4, 2k, 4, 4, 2k)$	$3/2 - 2/k$	$<$	$(\epsilon_2 + \epsilon_2 + \epsilon_k) + (\epsilon_2 + \epsilon_k)$	$< 2/k$

One checks that these inequalities are not satisfied for $2k \geq 6$. Proposition 3.14 then states that the placeholders can contain at most two 4-s, and if exactly two, then nothing else. Moreover, τ_2 must contain at least as many 4-s as it contains $2k$-s, as otherwise we would find two adjacent $2k$-faces while P cannot contain a $(2k, 2k)$-edge by Observation 3.10. We are therefore left with the following options for τ_2:

$$(4, 4, 2k), \ (4, 4, 4, 2k) \quad \text{and} \quad (4, 2k, 4, 2k).$$

The case $\tau_2 = (4, 4, 2k)$ is impossible by Corollary 3.12. We show that $\tau_2 = (4, 4, 4, 2k)$ is also not possible: consider the local neighborhood of a $(4, 4, 4, 2k)$-vertex (the highlighted vertex) in the following figure:



Since the 1-vertices (black dots) are of type $(4, 4, 6)$, this configuration forces on us the existence of the two gray 6-faces. These two faces intersect in a 2-vertex, which is then incident to two $2k$-faces and must be of type $(4, 2k, 4, 2k)$. But we can show that the types $(4, 4, 4, 2k)$ and $(4, 2k, 4, 2k)$ are incompatible by Observation 3.8(i):

$$\beta_2^2 + \beta_2^2 + \beta_2^2 + \beta_2^k \overset{(i)}{=} \beta_2^2 + \beta_2^k + \beta_2^2 + \beta_2^k \implies \beta_2^2 = \beta_2^k \overset{(3.1)}{\implies} 4 = 2k \geq 6.$$

Thus, $(4, 4, 4, 2k)$ cannot occur.

We conclude that every 2-vertex incident to a $2k$-face must be of type $(4, 2k, 4, 2k)$. Consider then the following table:

τ_2	$s - 2 - K(\tau_2)$	$\overset{?}{<}$	$E(\tau_2)$	
$(4, 2k, 4, 2k)$	$1 - 2/k$	$<$	$(\epsilon_2 + \epsilon_2 + \epsilon_k) + \epsilon_2$	$< 2/k$

The established inequality yields $2k \leq 6$, and hence $2k = 6$. We found that then all 1-vertices must be of type $(4, 4, 6)$, and all 2-vertices incident to a 6-face must be of type $(4, 6, 4, 6)$.

3.9 The case $\tau_1 = (4, 4, 6)$

At this point we can now assume that all 1-vertices of P are of type $(4, 4, 6)$ and that each 2-vertex of P that is incident to a 6-face is of type $(4, 6, 4, 6)$. In particular, P contains a 2-vertex $w \in V_2$ of this type. Since there is no $(6, 6)$-edge in P, the two 6-faces incident to w cannot be adjacent. In other words, the faces around w must occur alternatingly of type 4 and type 6, which is the reason for writing the type $(4, 6, 4, 6)$ with alternating entries.

On the other hand, P contains $(4, 4)$-edges, and none of these is incident to a $(4, 6, 4, 6)$-vertex surrounded by alternating faces. Thus, there must be further 2-vertices of a type other than $(4, 6, 4, 6)$, necessarily *not* incident to any 6-face. These must then be of type

$$(4^r) := (\underbrace{4, ..., 4}_{r}), \quad \text{for some } r \geq 3.$$

Proposition 3.19. $r = 5$.

Proof. If there is a (4^r)-vertex, Observation 3.8(i) yields $\beta_2^2 = 2\pi/r$. Analogously, from the existence of a $(4, 6, 4, 6)$-vertex follows

$$2\beta_2^2 + 2\beta_2^3 \overset{(i)}{=} 2\pi \quad \implies \quad \beta_2^3 = \frac{2\pi - 2\beta_2^2}{2} = \Big(1 - \frac{2}{r}\Big)\pi.$$

Recall $k\beta_1^k + k\beta_2^k > 2\pi(k-1)$ from Observation 3.8(ii). Together with the previously established values for β_2^2 and β_2^3, this yields

$$\begin{aligned} \beta_1^2 &> \frac{2\pi(2-1) - 2\beta_2^2}{2} = \Big(1 - \frac{2}{r}\Big)\pi, \quad \text{and} \\ \beta_1^3 &> \frac{2\pi(3-1) - 3\beta_2^3}{3} = \Big(\frac{1}{3} + \frac{2}{r}\Big)\pi. \end{aligned} \tag{3.3}$$

Since the 1-vertices are of type $(4, 4, 6)$, Observation 3.8(i) yields

$$2\pi \overset{(i)}{=} 2\beta_1^2 + \beta_1^3 \overset{(3.3)}{>} 2\Big(1 - \frac{2}{r}\Big)\pi + \Big(\frac{1}{3} + \frac{2}{r}\Big)\pi = \Big(\frac{7}{3} - \frac{2}{r}\Big)\pi.$$

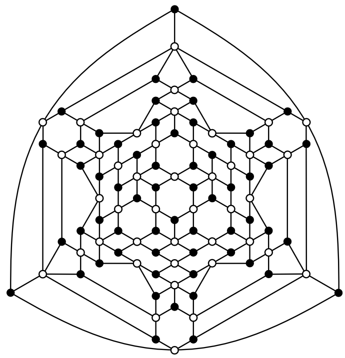


Figure 4: The edge-graph of the final candidate polyhedron.

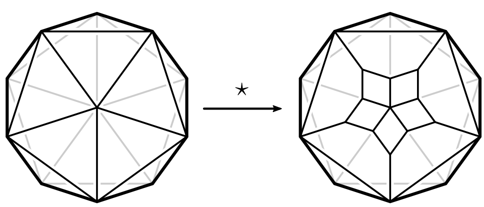
And one checks that this rearranges to $r < 6$.

This leaves us with the options $r \in \{3, 4, 5\}$. If $r = 4$, then $\beta_2^3 = \pi/2 = \beta_2^2$, which is impossible by Equation (3.1). And if $r = 3$, then (3.3) yields $\beta_1^3 > \pi$, which is also impossible for a convex face of a spherical polyhedron. We are left with $r = 5$. $\square$

To summarize: P is a strictly bipartite polyhedron in which all 1-vertices are of type $(4, 4, 6)$, and all 2-vertices are of types $(4, 6, 4, 6)$ or (4^5), and both types actually occur in P. This information turns out to be sufficient to uniquely determine the edge-graph of P, which is shown in Figure 4.

This graph can be constructed by starting with a hexagon in the center with vertices of alternating colors (indicating the partition classes). One then successively adds further faces (according to the structural properties determined above), layer by layer. This process involves no choice and thus the result is unique.

As mentioned in Remark 2.3, a bipartite polyhedron has an edge in-sphere. Thus, P is a polyhedral realization of the graph in Figure 4 with an edge in-sphere. It is known that any two such realizations are related by a projective transformation [8]. One representative $Q \subset \mathbb{R}^3$ from this class (which we do not yet claim to coincide with P) can be constructed by applying the following operation $\star$ to each vertex of the regular icosahedron:

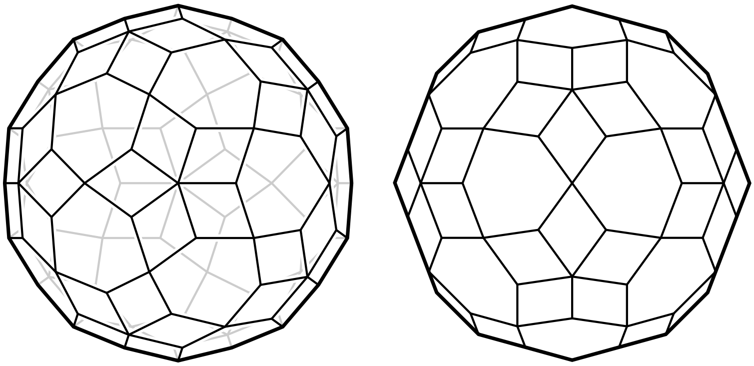


The operation is performed in such a way, so that

- the five new "outer" vertices of the new 4-gons are positioned in the centers of edges of the icosahedron.

- the edges of each new 4-gon are tangent to a common sphere centered at the center of the icosahedron

The resulting polyhedron Q looks as follows:



One can verify that Q has indeed the desired edge-graph.

It is clear from the construction that Q has an edge in-sphere, and any two of its 4-gonal or 6-gonal faces are congruent (as we would expect from a bipartite polyhedron). Like-wise, P has an edge in-sphere and the same edge-graph. Hence, P must be a projective transformation of Q. However, any projective transformation that is not just a re-orientation or a uniform rescaling will inevitably destroy the property of congruent faces. In conclusion, we can assume that P is identical to Q (up to scale and orientation).

It remains to check whether Q is indeed a bipartite polyhedron. For this, recall that any two of the following properties imply the third (*cf.* Remark 2.3):

(i) Q has an edge in-sphere.

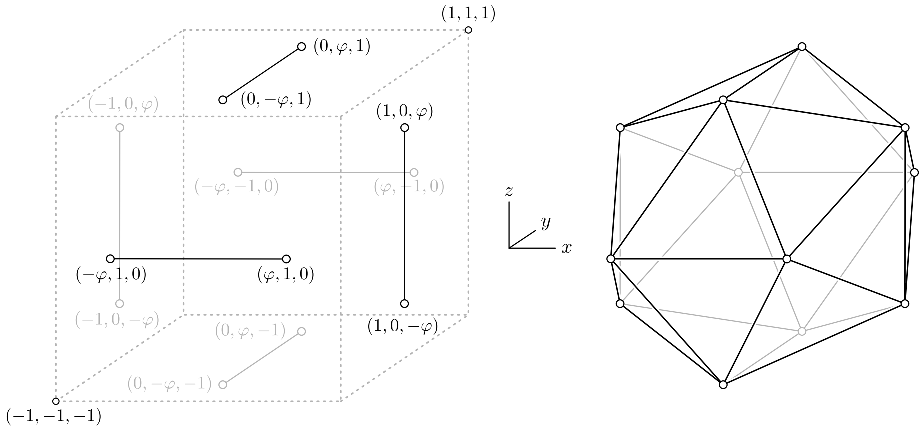
(ii) Q has all edges of the same length.

(iii) for each vertex $v \in \mathcal{F}_0(Q)$, the distance $\|v\|$ only depends on the partition class of the vertex.

Now, Q satisfies (i) by construction, and it would need to satisfy both (ii) and (iii) in order to be bipartite. The figure certainly suggests that all edges of Q are of the same length. However, as we shall show now, Q cannot satisfy both (ii) and (iii) at the same time, and thus, can satisfy neither. In particular, the edges must have a tiny difference in length that cannot be spotted visually, making Q into a remarkable near-miss (we will quantify this below).

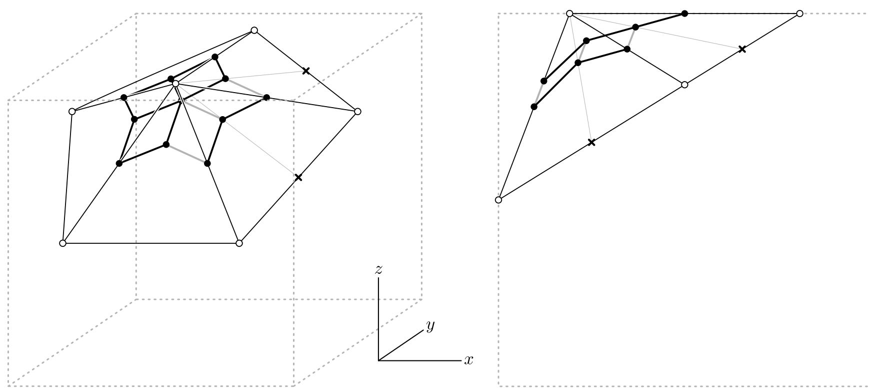
For what follows, let us assume that (ii) holds, that is, that all edges of Q are of the same length, in particular, that all 4-gons are rhombuses. Our goal is to show that $\|v\|$ depends on the type of the vertex $v \in V_2$ (not only its partition class), establishing that (iii) does not hold.

For this, start from the following well-known construction of the regular icosahedron from the cube of edge-length 2 centered at the origin.

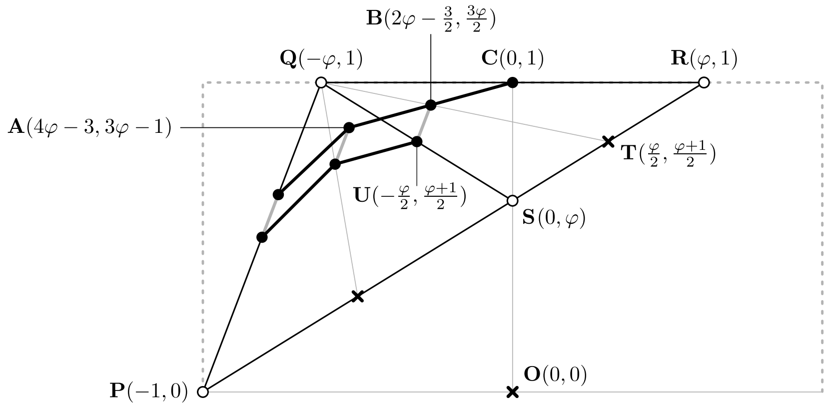


The construction is as follows: insert a line segment in the center of each face of the cube as shown in the left image. Each line segment is of length 2φ, where $\varphi \approx 0.61803$ is the positive solution of $\varphi^2 = 1 - \varphi$ (one of the numbers commonly knows as the *golden ratio*). The convex hull of these line segments gives the icosahedron with edge length 2φ.

It is now sufficient to consider a single vertex of the icosahedron together with its incident faces. The image below shows this vertex after we applied $\star$.



The image on the right is the orthogonal projection of the configuration on the left onto the yz-plane. This projection makes it especially easy to give 2D-coordinates for several important points:



The points $\mathbf{A}$ and $\mathbf{C}$ are 2-vertices of Q of type (4^5) and $(4,6,4,6)$ respectively. Both points and the origin $\mathbf{O}$ are contained in the yz-plane onto which we projected. Consequently, distances between these points are preserved during the projection, and assuming that Q is bipartite, we would expect to find $|\overline{\mathbf{OA}}| = |\overline{\mathbf{OC}}| = r_2$. We shall see that this is not the case, by explicitly computing the coordinates of $\mathbf{A}$ and $\mathbf{C}$ in the new coordinate system (y,z).

By construction, $\mathbf{C} = (0,1)$ and $|\overline{\mathbf{OC}}| = 1$. Other points with easily determined coordinates are $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S}$, $\mathbf{T}$ (the midpoint of $\mathbf{R}$ and $\mathbf{S}$) and $\mathbf{U}$ (the midpoint of $\mathbf{Q}$ and $\mathbf{S}$).

By construction, the point $\mathbf{B}$ lies on the line segment $\overline{\mathbf{QT}}$. The parallel projection of a rhombus is a (potentially degenerated) parallelogram, and thus, opposite edges in the projection are still parallel. Hence, the gray edges in the figure are parallel. For that reason, the segment $\overline{\mathbf{UB}}$ is parallel to $\overline{\mathbf{PQ}}$. This information suffices to determine the coordinates of $\mathbf{B}$, which is now the intersection of $\overline{\mathbf{QT}}$ with the parallel of $\overline{\mathbf{PQ}}$ through $\mathbf{U}$. The coordinates are given in the figure.

The rhombus containing the vertices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ degenerated to a line. Its fourth vertex is also located at $\mathbf{B}$. Therefore, the segments $\overline{\mathbf{CB}}$ and $\overline{\mathbf{BA}}$ are translates of each other. Since the point $\mathbf{B}$ and the segment $\overline{\mathbf{CB}}$ are known, this allows the computation of the coordinates of $\mathbf{A}$ as given in the figure.

We can finally compute $|\overline{\mathbf{OA}}|$. For this, recall $(*)\ \varphi^{2n} = F_{2n-2} - \varphi F_{2n-1}$, where F_n denotes the n-th *Fibonacci number* with initial conditions $F_0 = F_1 = 1$. Then

$$\begin{aligned}
|\overline{\mathbf{OA}}|^2 &= (4\varphi-3)^2 + (3\varphi-1)^2 \\
&= 25\varphi^2 - 30\varphi + 10 \\
&\overset{(*)}{=} 25(1-\varphi) - 30\varphi + 10 \\
&= 35 - 55\varphi \\
&= 1 + (34 - 55\varphi) \overset{(*)}{=} 1 + \varphi^{10} > 1,
\end{aligned}$$

and thus, Q cannot be bipartite. Remarkably, we find that

$$|\overline{\mathbf{OA}}| = \sqrt{1+\varphi^{10}} \approx 1.00405707$$

is only about 0.4% larger than $|\overline{\mathbf{OC}}| = 1$, and so while Q is not bipartite, it is a remarkable near-miss.

Since P was assumed to be bipartite, but was also shown to be identical to Q, we reached a contradiction, which finally proves Theorem 2.13, and the goal of the paper is achieved.

4 Conclusions and open questions

In this paper we have shown that any edge-transitive (convex) polytope in four or more dimensions is necessarily vertex-transitive. We have done this by classifying all polytopes which simultaneously have all edges of the same length, an edge in-sphere and a bipartite edge graph (which we named *bipartite* polytopes).

The obstructions we derived for being edge-transitive without being vertex-transitive have been primarily geometric and less a matter of symmetry (a detailed investigation of the Euclidean symmetry groups was not necessary, but it might be interesting to view the problem from this perspective). We suspect that dropping convexity or considering combinatorial symmetries instead of geometric ones will quickly lead to examples of "just edge-transitive structures". For example, it is easy to find embeddings of graphs into $\mathbb{R}^d$ with these properties.

Slightly stronger than being simultaneously vertex- and edge-transitive is being transitive on *arcs*, that is, on incident vertex-edge pairs. This additional degree of symmetry allows an edge to be not only mapped onto any other edge, but also onto itself with reversed orientation. While there are graphs that are vertex- and edge-transitive without being arc-transitive (the so-called *half-transitive* graphs, see [7]), we believe it is unlikely that this distinction is necessary for convex polytopes.

Question 4.1. Is there a polytope $P \subset \mathbb{R}^d$ that is edge-transitive and vertex-transitive, but not arc-transitive?

In a different direction, the questions of this paper naturally generalize to faces of higher dimensions. In general, the interactions between transitivities of faces of different dimensions have been little investigated. For example, already the following question seems to be open:

Question 4.2. For fixed $k \in \{2, ..., d-3\}$, are there convex d-polytopes for arbitrarily large $d \in \mathbb{N}$ that are transitive on k-dimensional faces without being transitive on either vertices or facets?

Of course, any such question could be attacked by attempting to classify the k-face-transitive (convex) polytopes for some $k \in \{1, ..., d-2\}$. It seems to be unclear for which k this problem is tractable (for comparison, $k = 0$ is intractable, see [1]), and it appears that there are no techniques applicable to all (or many) k at the same time.

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A

A.1 Geometry

Proposition A.1. *Given a set $x_0, ..., x_d \in \mathbb{R}^d \setminus \{0\}$ of $d+1$ vectors with pair-wise negative inner product, then there are* positive *coefficients $\alpha_0, ..., \alpha_d > 0$ with*

$$\alpha_0 x_0 + \cdots + \alpha_d x_d = 0.$$

Proof. We proceed by induction. The induction base $d = 1$ which is trivially true.

Now suppose $d \geq 2$, and, W.l.o.g. assume $\|x_0\| = 1$. Let π_0 be the orthogonal projection onto $x_0^\perp$, that is, $\pi_0(u) := u - x_0\langle x_0, u\rangle$. In particular, for $i \neq j$ and $i, j > 0$

$$\langle \pi_0(x_i), \pi_0(x_j)\rangle = \underbrace{\langle x_i, x_j\rangle}_{<0} - \underbrace{\langle x_0, x_i\rangle}_{<0} \underbrace{\langle x_0, x_j\rangle}_{<0} < 0.$$

Then $\{\pi(x_1), ..., \pi_0(x_d)\}$ is a set of d vectors in $x_0^\perp \cong \mathbb{R}^{d-1}$ with pair-wise negative inner product. By induction assumption there are positive coefficients $\alpha_1, ..., \alpha_d > 0$ so that $\alpha_1\pi_0(x_1) + \cdots + \alpha_d\pi_0(x_d) = 0$.

Set $\alpha_0 := -\langle x_0, \alpha_1 x_1 + \cdots + \alpha_d x_d\rangle > 0$. We claim that $x := x_0\alpha_0 + \cdots + \alpha_d x_d = 0$. Since $\mathbb{R}^d = \operatorname{span}\{x_0\} \oplus x_0^\perp$, it suffices to check that $\langle x_0, x\rangle = 0$ as well as $\pi_0(x) = 0$. This follows:

$$\begin{aligned}
\langle x_0, x\rangle &= \alpha_0 \underbrace{\langle x_0, x_0\rangle}_{=1} + \underbrace{\langle x_0, \alpha_1 x_1 + \cdots + \alpha_d x_d\rangle}_{=-\alpha_0} = 0, \\
\pi_0(x) &= \alpha_0 \underbrace{\pi_0(x_0)}_{=0} + \underbrace{\alpha_1\pi_0(x_1) + \cdots + \alpha_d\pi_0(x_d)}_{=0} = 0.
\end{aligned}$$

□

Proposition A.2. *Let $P \subset \mathbb{R}^3$ be a polyhedron with $v \in \mathcal{F}_0(P)$ a vertex of degree three. The interior angles of the faces incident to v determine the dihedral angles at the edges incident to v and vice versa.*

Proof. For $w_1, w_2, w_3 \in \mathcal{F}_0(P)$ the neighbors of v, let $u_i := w_i - v$ denote the direction of the edge e_i from v to w_i. Let σ_{ij} be the face that contains v, w_i and w_j. Then $\measuredangle(u_i, u_j)$ is the interior angle of σ_{ij} at v.

The set $\{u_1, u_2, u_3\}$ is uniquely determined (up to some orthogonal transformation) by the angles $\measuredangle(u_i, u_j)$. Furthermore, since P is convex, $\{u_1, u_2, u_3\}$ forms a basis of $\mathbb{R}^3$, and this uniquely determines the *dual basis* $\{n_{12}, n_{23}, n_{31}\}$ for which $\langle n_{ij}, u_i\rangle = \langle n_{ij}, u_j\rangle = 0$. In other words, n_{ij} is a normal vector to σ_{ij}. The dihedral angle at the edge e_j is then $\pi - \measuredangle(n_{ij}, n_{jk})$, hence uniquely determined. The other direction is analogous, via constructing $\{u_1, u_2, u_3\}$ as the dual basis to the set of normal vectors. □

A.2 Computations

The edge lengths in a spherical polyhedron are measured as angles between its end vertices. Consider adjacent vertices $v_1^S, v_2^S \in \mathcal{F}_0(P^S)$, then the incident edge has (arc-)length $\ell^S := \measuredangle(v_1^S, v_2^S) = \measuredangle(v_1, v_2)$.

It follows from Observation 2.6 that these angles are completely determined by the parameters, hence the same for all edges of P^S.

Proposition A.3. *For a face $\sigma \in \mathcal{F}_2(P)$ and a vertex $v \in \mathcal{F}_0(\sigma)$, there is a direct relationship between the value of $\alpha(\sigma, v)$ and the value of $\beta(\sigma, v)$.*

Proof. Let $w_1, w_2 \in V_2$ be the neighbors of v in the $2k$-face σ, and set $u_i := w_i - v$. Then $\measuredangle(u_1, u_2) = \alpha(\sigma, v)$. W.l.o.g. assume that v is a 1-vertex (the argument is equivalent for a 2-vertex).

For convenience, we introduce the notation $\chi(\theta) := 1 - \cos(\theta)$. We find that

$$\begin{aligned}(*) \quad 2\ell^2 \cdot \chi(\alpha(\sigma, v)) &= \ell^2 + \ell^2 - 2\ell^2 \cos(\measuredangle(u_1, u_2)) \\ &= \|u_1\|^2 + \|u_2\|^2 - 2\langle u_1, u_2\rangle \\ &= \|u_1 - u_2\|^2 = \|w_1 - w_2\|^2 \\ &= \|w_1\|^2 + \|w_2\|^2 - 2\langle w_1, w_2\rangle \\ &= r_2^2 + r_2^2 - r_2^2 \cos\measuredangle(w_1, w_2) = 2r_2^2 \cdot \chi(\measuredangle(w_1, w_2)).\end{aligned}$$

The side lengths of the spherical triangle $w_1^S v^S w_2^S$ are $\measuredangle(w_1, w_2), \ell^S$ and ℓ^S. By the spherical law of cosine[5] we obtain

$$\begin{aligned}\cos\measuredangle(w_1, w_2) &= \cos(\ell^S)\cos(\ell^S) + \sin(\ell^S)\sin(\ell^S)\cos(\beta(\sigma, v)) \\ &= \cos^2(\ell^S) + \sin^2(\ell^S)(\cos(\beta(\sigma, v)) - 1 + 1) \\ &= [\cos^2(\ell^S) + \sin^2(\ell^S)] + \sin^2(\ell^S)(\cos(\beta(\sigma, v)) - 1) \\ &= 1 - \sin^2(\ell^s) \cdot \chi(\beta(\sigma, v))\end{aligned}$$

$$\Longrightarrow \quad \sin^2(\ell^S) \cdot \chi(\beta(\sigma, v)) = \chi(\measuredangle(w_1, w_2)) \overset{(*)}{=} \Big(\frac{\ell}{r_2}\Big)^2 \cdot \chi(\alpha(\sigma, v)). \qquad \square$$

[5] $\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(\gamma)$, where a, b and c are the side lengths (arc-lengths) of a spherical triangle, and γ is the interior angle opposite to the side of length c.

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Double generalized majorization and diagrammatics*

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Abstract

In this paper we show that the generalized majorization of partitions of integers has a surprising completing-squares property. Together with the previously obtained transitivity-like property, this enables a compelling diagrammatical interpretation. Apart from purely combinatorial interest, the main result has applications in matrix completion problems, and representation theory of quivers.

Keywords: Partitions, majorization, diagrammatics, inequalities.

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1 Introduction

By a partition we mean a finite non-increasing sequence of integers. Let $a_1 \geq \ldots \geq a_s$ be integers, then we can define the corresponding partition $\mathbf{a} = (a_1, \ldots, a_s)$. For a partition

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$\mathbf{a} = (a_1, \dots, a_s)$ we shall assume that $a_i := +\infty$, for $i \leq 0$, and $a_i := -\infty$, for $i > s$. The following notation will be used throughout the paper

$$\mathbf{a} = (a_1, \dots, a_s), \tag{1.1}$$
$$\mathbf{b} = (b_1, \dots, b_k), \tag{1.2}$$
$$\mathbf{c} = (c_1, \dots, c_m), \tag{1.3}$$
$$\mathbf{d} = (d_1, \dots, d_{m+s}), \tag{1.4}$$
$$\mathbf{g} = (g_1, \dots, g_{m+k}), \tag{1.5}$$
$$\mathbf{f} = (f_1, \dots, f_{m+k+s}). \tag{1.6}$$

Arguably, the most famous comparison between two partitions of integers is a classical majorization in Hardy-Littlewood-Polya sense [16]. In this paper we deal with its generalisation given in [2, 9, 10]. More precisely, we compare three partitions of integers in the following way:

Definition 1.1. Let $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{g}$ be partitions (1.2), (1.3) and (1.5), respectively. If

$$c_i \geq g_{i+k}, \qquad i = 1, \dots, m, \tag{1.7}$$
$$\sum_{i=1}^{h_j} g_i - \sum_{i=1}^{h_j - j} c_i \leq \sum_{i=1}^{j} b_i, \qquad j = 1, \dots, k \tag{1.8}$$
$$\sum_{i=1}^{m+k} g_i = \sum_{i=1}^{m} c_i + \sum_{i=1}^{k} b_i, \tag{1.9}$$

where

$$h_j := \min\{i | c_{i-j+1} < g_i\}, \quad j = 1, \dots, k,$$

then we say that $\mathbf{g}$ is *majorized* by $\mathbf{c}$ and $\mathbf{b}$. This type of majorization we call *the generalized majorization*, and we write

$$\mathbf{g} \prec' (\mathbf{c}, \mathbf{b}).$$

The generalized majorization generalizes the classical majorization. Indeed, if $m = 0$, i.e. if the partition $\mathbf{c}$ is empty, the generalized majorization becomes the classical majorization between the partitions $\mathbf{g}$ and $\mathbf{b}$. Many intrinsic, purely combinatorial properties of generalized majorization, including generalizations of some of the well-known properties of the classical majorization, have been obtained in [10, 11, 14]. These results demonstrate rich structure of generalized majorization as an independent combinatorial object.

Apart from purely combinatorial interest, this relationship between three partitions of integers naturally appears in Matrix and Matrix Pencils completion problems [2, 7, 9, 12], as well as in Representation Theory of Quivers [22], and Perturbation Theory [1, 12].

In this paper we go further, and show that generalized majorization, apart from transitivity-like property that has been shown in [10, Theorem 8], also has certain completing-squares property. This novel property of generalized majorization is motivated by the study of two problems given below, that naturally appear both from matrix pencils completions, and representation theory of quivers point of view.

The first problem has appeared in [9, 11] and turned out to be very challenging and the key point in solving many perturbation and completion problems of Matrix Pencils, see e.g. [6, 7, 9, 12, 13].

Problem 1.2 (Double general majorization problem)**.** Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$, and $\mathbf{g}$ be partitions (1.1), (1.2), (1.4) and (1.5). Find necessary and sufficient conditions for the existence of a partition $\mathbf{f} = (f_1, \dots, f_{m+k+s})$, such that

$$\mathbf{f} \prec' (\mathbf{d}, \mathbf{b}) \quad \text{and} \quad \mathbf{f} \prec' (\mathbf{g}, \mathbf{a}). \tag{1.10}$$

We note that in the case of classical majorization there always exists a minimal partition of a given sum, i.e. for any two partitions of the same length and total sum, there exists a partition that is majorized by both of them. However, here the problem is much more complicated, and involved. A complete solution to Problem 1.2 was obtained in [11, 14]:

Theorem 1.3 ([14, Theorem 3])**.** *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{d}$*, and* $\mathbf{g}$ *be partitions* (1.1), (1.2), (1.4) *and* (1.5)*. There exists a partition* $\mathbf{f} = (f_1, \dots, f_{m+k+s})$*, such that*

$$\mathbf{f} \prec' (\mathbf{d}, \mathbf{b}) \quad \textit{and} \quad \mathbf{f} \prec' (\mathbf{g}, \mathbf{a})$$

if and only if

$$\sum_{i=1}^{m+s} d_i + \sum_{i=1}^{k} b_i = \sum_{i=1}^{m+k} g_i + \sum_{i=1}^{s} a_i$$

and the condition $\bar{\Omega}(\mathbf{g}, \mathbf{d}, \mathbf{b}, \mathbf{a})$ *holds.*

The explicit form of the condition $\bar{\Omega}(\mathbf{g}, \mathbf{d}, \mathbf{b}, \mathbf{a})$ is given in [11, 14], and consists of inequalities between the elements of the partitions $\mathbf{g}, \mathbf{d}, \mathbf{b}, \mathbf{a}$. These involve very technical explicit definition of certain sets S and Δ, and we dismiss it here. We refer the interested reader to [11, 14] for all details and properties on these sets, and for the explicit form of $\bar{\Omega}$.

The second problem has showed its importance when studying bounded rank one perturbations of matrix pencil [12]. Also, it naturally appears in the study of the possible Kronecker invariants of a partially prescribed Matrix Pencil, see e.g. [13, 17]. Apart of the case $k = s = 1$ which has been solved in [12], the following problem is still open:

Problem 1.4 (Pseudo double majorization problem)**.** Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$, and $\mathbf{g}$ be partitions (1.1), (1.2), (1.4) and (1.5). Find necessary and sufficient conditions for the existence of a partition $\mathbf{c} = (c_1, \dots, c_m)$, such that

$$\mathbf{g} \prec' (\mathbf{c}, \mathbf{b}) \quad \text{and} \quad \mathbf{d} \prec' (\mathbf{c}, \mathbf{a}). \tag{1.11}$$

The goal of the paper is to prove the relationship between the double majorization Problem 1.2 and pseudo double majorization Problem 1.4. In Theorem 3.2, as the main result of the paper, we prove that Problem 1.4 implies Problem 1.2. That is, we prove that for four partitions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$, and $\mathbf{g}$ as in (1.1), (1.2), (1.4) and (1.5), the existence of a partition $\mathbf{c}$ satisfying (1.11) implies the existence of a partition $\mathbf{f}$ satisfying (1.10). In addition, we explicitly construct such partition $\mathbf{f}$. This is a surprising, and nontrivial property of the generalized majorization. Also, in Section 4 we give a counterexample that the converse does not hold.

This purely combinatorial result has several interpretations. First, let us introduce some diagrammatics into the story, and denote general majorization by an arrow, i.e. let us denote

$$\mathbf{g} \prec' (\mathbf{c}, \mathbf{b})$$

by

$$\begin{tikzcd} \mathbf{g} \\ \mathbf{c} \arrow[u, "\mathbf{b}"'] \end{tikzcd}$$

Now, as a direct corollary to our main result we obtain the following commutative diamond-like diagram:

$$\begin{tikzcd} & \mathbf{f} & \\ \mathbf{g} \arrow[ur, dashed, "\mathbf{a}"] & & \mathbf{d} \arrow[ul, dashed, "\mathbf{b}"'] \\ & \mathbf{c} \arrow[ul, "\mathbf{b}"] \arrow[ur, "\mathbf{a}"'] & \end{tikzcd}$$

In other words, the lower half of the square (represented by full lines) can always be completed to a full square. More details on diagrammatics are given in Section 4.2.

In addition, the above completion up to a commutative diagram is related to various classical Linear Algebra problems. First of all, both Problems 1.2 and 1.4 naturally appear as cornerstones in solving the classical General Matrix Pencils Completion Problem [17]. In particular, a solution to Problem 1.2 is a key result in obtaining a full description of the possible Kronecker invariants of a quasi-regular matrix pencil with a prescribed subpencil in [13]. For similar contributions and importance of Problems 1.2 and 1.4 in matrix pencils completion problems see [6, 7, 9]. The close relationship between Problems 1.2 and 1.4 obtained in this paper, should have a significant impact in obtaining a complete solution of the General Matrix Pencils Completion Problem. Similar applications are expected in the study of representation of Kronecker quivers, [22].

Another area of applications of results on generalized majorizations is in Bounded Rank Perturbation problems [3–5, 18–21]. In the case when partitions $\mathbf{a}$ and $\mathbf{b}$ are both of length one (i.e. when $s = k = 1$), Problems 1.2 and 1.4 have been addressed and solved separately in [12], and were crucial in solving the rank one perturbation problem for matrix pencils. It is expected that the main result of this paper should lead to a solution of the arbitrary rank perturbation problem in the future. Some steps in this direction have already been done in [8]. Indeed, in [8] we have studied and resolved the classical bounded rank perturbation problem for quasi-regular matrix pencils (pencils with full normal rank). For all details on matrix pencils see [15]. This is a very general result in low rank perturbation theory, and has been open for a long time. The milestone in its solution is the main result of the paper – Theorem 3.2. It allows to choose a special, preferred form of the low rank matrix pencil that performs the perturbation. We expect more impact of Theorem 3.2 in the study of bounded rank perturbations of different classes of matrix pencils in the future.

2 Partitions and generalized majorization

For any two partitions $\mathbf{a} = (a_1, \dots, a_s)$ and $\mathbf{b} = (b_1, \dots, b_k)$ by $\mathbf{a} \cup \mathbf{b}$ we mean a partition obtained as a non-increasing ordering of $\{a_1, \dots, a_s, b_1, \dots, b_k\}$. If $a > b$ are nonnegative integers, then we assume $\sum_{i=a}^{b} a_i := 0$.

Now we shall list some of the basic properties of the auxiliary numbers, h_j, that appear in the definition of the generalized majorization. Below we use the notation from Definition 1.1.

Since $h_j = \min\{i | c_{i-j+1} < g_i\}$, for $j = 1, \dots, k$, we have

$$m + k + 1 > h_k > \cdots > h_2 > h_1 > 0, \tag{2.1}$$

and so in particular

$$h_j \geq j, \quad j = 1, \dots, m + k. \tag{2.2}$$

Also from the definition of h_j we have

$$c_{i-j+1} \geq g_i, \quad \text{for} \quad i < h_j, \quad \text{for any} \quad j = 1, \dots, k. \tag{2.3}$$

We notice that in Definition 1.1, if (1.9) is satisfied, then (1.8) is equivalent to the following:

$$\sum_{i=h_j+1}^{m+k} g_i \geq \sum_{i=h_j-j+1}^{m} c_i + \sum_{i=j+1}^{k} b_i, \quad j = 1, \dots, k. \tag{2.4}$$

The generalized majorization implies *weak majorization* given by the following definition:

Definition 2.1. If partitions $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{g}$ from (1.2), (1.3), and (1.5), respectively, satisfy conditions (1.7), (2.4) and

$$\sum_{i=1}^{m+k} g_i \geq \sum_{i=1}^{m} c_i + \sum_{i=1}^{k} b_i,$$

then we say that $\mathbf{g}$ is *weakly majorized* by $\mathbf{c}$ and $\mathbf{b}$, and we write

$$\mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}).$$

Lemma 2.2 ([7, Theorem 2.5]). *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$, *and* $\mathbf{g}$ *from* (1.1), (1.2), (1.4)*, and* (1.5)*, respectively, satisfy*

$$\sum_{i=1}^{m+s} d_i + \sum_{i=1}^{k} b_i = \sum_{i=1}^{m+k} g_i + \sum_{i=1}^{s} a_i.$$

If there exists a partition $\bar{\mathbf{f}} = (\bar{f}_1, \dots, \bar{f}_{m+k+s})$ *such that*

$$\bar{\mathbf{f}} \prec'' (\mathbf{d}, \mathbf{b}) \quad \textit{and} \quad \bar{\mathbf{f}} \prec'' (\mathbf{g}, \mathbf{a}), \tag{2.5}$$

then there exists a partition $\mathbf{f} = (f_1, \dots, f_{m+k+s})$ *such that*

$$\mathbf{f} \prec' (\mathbf{d}, \mathbf{b}) \quad \textit{and} \quad \mathbf{f} \prec' (\mathbf{g}, \mathbf{a}). \tag{2.6}$$

Moreover, if the partition $\bar{\mathbf{f}}$ *satisfying* (2.5) *consists of nonnegative integers, and*

$$\sum_{i=1}^{m+s} d_i + \sum_{i=1}^{k} b_i \geq 0,$$

then there exists a partition $\mathbf{f}$ *consisting of nonnegative integers satisfying* (2.6).

We also cite the result from [10] which shows the transitivity property of generalized majorization. More on this topic is given in Section 4.

Theorem 2.3 ([10])**.** *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{d}$ *and* $\mathbf{f}$ *be partitions* (1.1), (1.2), (1.4) *and* (1.6)*, respectively. If*

$$\mathbf{f} \prec' (\mathbf{d}, \mathbf{b}) \quad \textit{and} \quad \mathbf{d} \prec' (\mathbf{c}, \mathbf{a}),$$

then

$$\mathbf{f} \prec' (\mathbf{c}, \mathbf{a} \cup \mathbf{b}).$$

3 Main result

We start this section by giving one auxiliary result:

Lemma 3.1. *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{d}$ *and* $\mathbf{g}$ *be the partitions* (1.1), (1.2), (1.4) *and* (1.5)*, respectively. Let* $\mathbf{c} = (c_1, \dots, c_m)$ *be a partition such that*

$$\mathbf{d} \prec' (\mathbf{c}, \mathbf{a}) \quad \textit{and} \quad \mathbf{g} \prec' (\mathbf{c}, \mathbf{b}). \tag{3.1}$$

Let $h_j = \min\{i | c_{i-j+1} < g_i\}$*,* $j = 1, \dots, k$*, and* $\bar{h}_j = \min\{i | c_{i-j+1} < d_i\}$*,* $j = 1, \dots, s$*. Let* $\mathbf{g}' = (g'_1, \dots, g'_m)$ *be a partition obtained from* $\mathbf{g}$ *after removing* $g_{h_1}, \dots, g_{h_k}$*, i.e.*

$$\{g'_1, \dots, g'_m\} = \{g_1, \dots, g_{m+k}\} \setminus \{g_{h_1}, \dots, g_{h_k}\},$$

and let $\mathbf{d}' = (d'_1, \dots, d'_m)$ *be a partition obtained from* $\mathbf{d}$ *after removing* $d_{\bar{h}_1}, \dots, d_{\bar{h}_s}$*, i.e.*

$$\{d'_1, \dots, d'_m\} = \{d_1, \dots, d_{m+s}\} \setminus \{d_{\bar{h}_1}, \dots, d_{\bar{h}_s}\}.$$

Then

$$c_i \geq \max(g'_i, d'_i), \quad i = 1, \dots, m. \tag{3.2}$$

Proof. Fix $i \in \{1, \dots, m\}$. Let $h_0 := 0$, $h_{k+1} := m + k + 1$. Then there exists $j \in \{0, \dots, k\}$ such that

$$h_{j+1} - (j + 1) \geq i > h_j - j.$$

This is true since $h_{u+1} > h_u$, and so $h_{u+1} - (u + 1) \geq h_u - u$, for all $u = 0, \dots, k$, as well as $h_0 - 0 = 0$ and $h_{k+1} - (k + 1) = m$.

Then

$$h_{j+1} > i + j > h_j,$$

and so by the definition of $\mathbf{g}'$ we have

$$g_{i+j} = g'_i.$$

If $j < k$, by (2.3) we have that $c_{l-j} \geq g_l$ for all $l < h_{j+1}$, and so

$$c_i \geq g_{i+j} = g'_i.$$

If $j = k$, by (3.1) and definition of the generalized majorization, we again obtain

$$c_i \geq g_{i+k} = g_{i+j} = g'_i.$$

By replacing the partitions $\mathbf{g}'$ by $\mathbf{d}'$ we shall also obtain

$$c_i \geq d'_i.$$

Altogether we have obtained (3.2), as desired. □

Now we can give the main result of the paper:

Theorem 3.2. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$ *and* $\mathbf{g}$ *be partitions* (1.1), (1.2), (1.4) *and* (1.5), *respectively. If there exists a partition* $\mathbf{c} = (c_1, \dots, c_m)$ *such that*

$$\mathbf{d} \prec' (\mathbf{c}, \mathbf{a}) \quad \textit{and} \quad \mathbf{g} \prec' (\mathbf{c}, \mathbf{b}), \tag{3.3}$$

then there exists a partition $\mathbf{f} = (f_1, \dots, f_{m+k+s})$ *such that*

$$\mathbf{f} \prec' (\mathbf{d}, \mathbf{b}) \quad \textit{and} \quad \mathbf{f} \prec' (\mathbf{g}, \mathbf{a}). \tag{3.4}$$

Proof. By the definition of the generalized majorization (Definition 1.1) and by (2.4), we have that (3.3) is equivalent to:

$$c_i \geq g_{i+k}, \qquad i = 1, \dots, m, \tag{3.5}$$

$$\sum_{i=h_j+1}^{m+k} g_i - \sum_{i=h_j-j+1}^{m} c_i \geq \sum_{i=j+1}^{k} b_i, \qquad j = 1, \dots, k, \tag{3.6}$$

$$\sum_{i=1}^{m+k} g_i = \sum_{i=1}^{m} c_i + \sum_{i=1}^{k} b_i, \tag{3.7}$$

and

$$c_i \geq d_{i+s}, \qquad i = 1, \dots, m, \tag{3.8}$$

$$\sum_{i=\bar{h}_j+1}^{m+s} d_i - \sum_{i=\bar{h}_j-j+1}^{m} c_i \geq \sum_{i=j+1}^{s} a_i, \qquad j = 1, \dots, s, \tag{3.9}$$

$$\sum_{i=1}^{m+s} d_i = \sum_{i=1}^{m} c_i + \sum_{i=1}^{s} a_i, \tag{3.10}$$

where

$$h_j := \min\{i | c_{i-j+1} < g_i\}, \quad j = 1, \dots, k,$$

and

$$\bar{h}_j := \min\{i | c_{i-j+1} < d_i\}, \quad j = 1, \dots, s.$$

Equalities (3.7) and (3.10) together give

$$\sum_{i=1}^{m+s} d_i + \sum_{i=1}^{k} b_i = \sum_{i=1}^{m+k} g_i + \sum_{i=1}^{s} a_i. \tag{3.11}$$

Let us denote by $\mathbf{g}' = (g'_1, \dots, g'_m)$ a partition obtained from $\mathbf{g}$ after removing $\{g_{h_1}, \dots, g_{h_k}\}$. Also, let us denote by $\mathbf{d}' = (d'_1, \dots, d'_m)$, a partition obtained from $\mathbf{d}$ after removing $\{d_{\bar{h}_1}, \dots, d_{\bar{h}_s}\}$. By Lemma 3.1 we have that

$$c_i \geq \max(g'_i, d'_i), \quad i = 1, \dots, m. \tag{3.12}$$

In order to prove the existence of a partition $\mathbf{f} = (f_1, \dots, f_{m+k+s})$ satisfying (3.4), by (3.11) and by Lemma 2.2 it is enough to prove the existence of a partition $\bar{\mathbf{f}} = (\bar{f}_1, \dots, \bar{f}_{m+k+s})$ satisfying

$$\bar{\mathbf{f}} \prec'' (\mathbf{d}, \mathbf{b}) \quad \text{and} \quad \bar{\mathbf{f}} \prec'' (\mathbf{g}, \mathbf{a}). \tag{3.13}$$

We shall define the partition $\bar{\mathbf{f}} = (\bar{f}_1, \dots, \bar{f}_{m+k+s})$ as a non-increasing ordering of integers $\min(g'_1, d'_1), \dots, \min(g'_m, d'_m), g_{h_1}, \dots, g_{h_k}, d_{\bar{h}_1}, \dots, d_{\bar{h}_s}$, i.e.

$$\bar{\mathbf{f}} := \{\min(g'_1, d'_1), \dots, \min(g'_m, d'_m)\} \cup \{g_{h_1}, \dots, g_{h_k}\} \cup \{d_{\bar{h}_1}, \dots, d_{\bar{h}_s}\}.$$

By Definition 2.1, we are left with proving the following:

$$g_i \geq \bar{f}_{i+s}, \qquad i = 1, \dots, m+k, \tag{3.14}$$

$$\sum_{i=l_j+1}^{m+k+s} \bar{f}_i \geq \sum_{i=l_j-j+1}^{m+k} g_i + \sum_{i=j+1}^{s} a_i, \qquad j = 1, \dots, s, \tag{3.15}$$

$$\sum_{i=1}^{m+k+s} \bar{f}_i \geq \sum_{i=1}^{m+k} g_i + \sum_{i=1}^{s} a_i, \tag{3.16}$$

$$d_i \geq \bar{f}_{i+k}, \qquad i = 1, \dots, m+s, \tag{3.17}$$

$$\sum_{i=\bar{l}_j+1}^{m+k+s} \bar{f}_i \geq \sum_{i=\bar{l}_j-j+1}^{m+s} d_i + \sum_{i=j+1}^{k} b_i, \qquad j = 1, \dots, k, \tag{3.18}$$

$$\sum_{i=1}^{m+k+s} \bar{f}_i \geq \sum_{i=1}^{m+s} d_i + \sum_{i=1}^{k} b_i, \tag{3.19}$$

where

$$l_j := \min\{i | g_{i-j+1} < \bar{f}_i\}, \quad j = 1, \dots, s,$$

and

$$\bar{l}_j := \min\{i | d_{i-j+1} < \bar{f}_i\}, \quad j = 1, \dots, k.$$

In fact, we shall prove only (3.14) – (3.16). By replacing the partition $\mathbf{g}$ by $\mathbf{d}$, and the partition $\mathbf{a}$ by $\mathbf{b}$, the formulas (3.17) – (3.19) will follow.

To that end, let us denote by $\bar{\mathbf{f}}' = (\bar{f}'_1, \dots, \bar{f}'_{m+k})$ the following partition:

$$\bar{\mathbf{f}}' := \{\min(g'_1, d'_1), \dots, \min(g'_m, d'_m)\} \cup \{g_{h_1}, \dots, g_{h_k}\}.$$

Then

$$\bar{\mathbf{f}} = \bar{\mathbf{f}}' \cup \{d_{\bar{h}_1}, \dots, d_{\bar{h}_s}\},$$

and so

$$\bar{f}'_i \geq \bar{f}_{i+s}, \quad i = 1, \dots, m + k. \tag{3.20}$$

Since

$$\mathbf{g} = \mathbf{g}' \cup \{g_{h_1}, \dots, g_{h_k}\},$$

we also have

$$g_i \geq \bar{f}'_i, \quad i = 1, \dots, m + k. \tag{3.21}$$

Altogether, (3.20) and (3.21) give (3.14). By the definition of $\bar{\mathbf{f}}$ we have

$$\begin{aligned}
\sum_{i=1}^{m+k+s} \bar{f}_i &= \sum_{i=1}^{m} \min(g'_i, d'_i) + \sum_{i=1}^{k} g_{h_i} + \sum_{i=1}^{s} d_{\bar{h}_i} \\
&= \sum_{i=1}^{m} g'_i + \sum_{i=1}^{m} d'_i - \sum_{i=1}^{m} \max(g'_i, d'_i) + \sum_{i=1}^{k} g_{h_i} + \sum_{i=1}^{s} d_{\bar{h}_i} \\
&= \sum_{i=1}^{m+k} g_i + \sum_{i=1}^{m+s} d_i - \sum_{i=1}^{m} \max(g'_i, d'_i).
\end{aligned}$$

By applying (3.12), we get

$$\sum_{i=1}^{m+k+s} \bar{f}_i \geq \sum_{i=1}^{m+k} g_i + \sum_{i=1}^{m+s} d_i - \sum_{i=1}^{m} c_i,$$

which by (3.10) gives (3.16), as desired.

Hence, we are left with proving (3.15). First, we introduce by convention $h_0 := 0$, and $h_{k+1} := m+k+1$. Now, fix $j \in \{1, \dots, s\}$. Let $u_j \in \{0, \dots, k\}$ and $\alpha_j \in \{0, \dots, m+k\}$ be such that

$$g_{h_{u_j}} \geq d_{\bar{h}_j} > g_{h_{u_j+1}}, \tag{3.22}$$

$$g_{\alpha_j} \geq d_{\bar{h}_j} > g_{\alpha_j+1}. \tag{3.23}$$

Then

$$h_{u_j+1} > \alpha_j \geq h_{u_j}. \tag{3.24}$$

From the definition of h_i we have that $h_i \geq i$, for all $i = 1, \dots, k$, (see (2.2)). This, together with (3.24) gives

$$\alpha_j \geq u_j.$$

Also, by the definition of $\mathbf{g}'$, from (3.22) and (3.23) we obtain that

$$g'_{\alpha_j - u_j} \geq d_{\bar{h}_j} > g'_{\alpha_j - u_j + 1}. \tag{3.25}$$

Moreover, from the definition of $\bar{h}_j$, and from (3.12), we have that

$$d_{\bar{h}_j} > c_{\bar{h}_j - j + 1} \geq g'_{\bar{h}_j - j + 1}.$$

Thus,

$$g'_{\alpha_j - u_j} > g'_{\bar{h}_j - j + 1},$$

and so

$$\alpha_j - u_j \leq \bar{h}_j - j.$$

Hence,

$$\min(\alpha_j - u_j, \bar{h}_j - j) = \alpha_j - u_j. \tag{3.26}$$

Next, we shall prove that

$$l_j = \alpha_j + j, \tag{3.27}$$

and

$$\bar{f}_{l_j} = d_{\bar{h}_j}. \tag{3.28}$$

(Recall that $l_j = \min\{i | g_{i-j+1} < \bar{f}_i\}$). Indeed, we have:

$$g_{h_1} \geq \cdots \geq g_{h_{u_j}} \geq d_{\bar{h}_j} > g_{\alpha_j + 1}, \tag{3.29}$$

$$d_{\bar{h}_1} \geq \cdots \geq d_{\bar{h}_{j-1}} \geq d_{\bar{h}_j} > g_{\alpha_j + 1}, \tag{3.30}$$

$$g'_1 \geq \cdots \geq g'_{\alpha_j - u_j} \geq d_{\bar{h}_j} > g_{\alpha_j + 1}, \tag{3.31}$$

$$d'_1 \geq \cdots \geq d'_{\bar{h}_j - j} \geq d_{\bar{h}_j} > g_{\alpha_j + 1}. \tag{3.32}$$

From the definition of $\bar{\mathbf{f}}$, and by (3.26), we have that there are at least $u_j + j + \min(\alpha_j - u_j, \bar{h}_j - j) = \alpha_j + j$ elements of $\bar{\mathbf{f}}$ that are bigger or equal than $d_{\bar{h}_j}$. Therefore $\bar{f}_{\alpha_j + j} \geq d_{\bar{h}_j} > g_{\alpha_j + 1}$, and so $l_j \leq \alpha_j + j$.

For the other inequality, first suppose that $\bar{f}_{l_j} > d_{\bar{h}_j}$. Then among $\{\bar{f}_1, \ldots, \bar{f}_{l_j}\}$, there would be at most $j - 1$ $d_{\bar{h}_i}$'s, while all other elements would be less than or equal to some of the elements of the partition $\mathbf{g}$. Therefore, we would have that for all $i = 1, \ldots, l_j$, $\bar{f}_i \leq g_{i-(j-1)}$, and so $\bar{f}_{l_j} \leq g_{l_j - j + 1}$, which contradicts the definition of l_j.

Hence $\bar{f}_{l_j} \leq d_{\bar{h}_j}$, and so by (3.23) and the definition of l_j

$$g_{\alpha_j} \geq d_{\bar{h}_j} \geq \bar{f}_{l_j} > g_{l_j - j + 1},$$

and so $l_j \geq \alpha_j + j$. Altogether, this proves (3.27) and (3.28).

In addition, by (3.29) – (3.32), we have also shown that

$$\sum_{i=1}^{l_j} \bar{f}_i = \sum_{i=1}^{\alpha_j + j} \bar{f}_i = \sum_{i=1}^{j} d_{\bar{h}_i} + \sum_{i=1}^{u_j} g_{h_i} + \sum_{i=1}^{\alpha_j - u_j} \min(g'_i, d'_i). \tag{3.33}$$

Now, we have

$$\sum_{i=l_j+1}^{m+k+s} \bar{f}_i = \sum_{i=\alpha_j+j+1}^{m+k+s} \bar{f}_i = \sum_{i=j+1}^{s} d_{\bar{h}_i} + \sum_{i=u_j+1}^{k} g_{h_i} + \sum_{i=\alpha_j-u_j+1}^{m} \min(g_i', d_i')$$
$$= \sum_{i=j+1}^{s} d_{\bar{h}_i} + \sum_{i=u_j+1}^{k} g_{h_i} + \sum_{i=\alpha_j-u_j+1}^{m} g_i' + \sum_{i=\alpha_j-u_j+1}^{m} d_i'$$
$$- \sum_{i=\alpha_j-u_j+1}^{m} \max(g_i', d_i').$$

We note that by (3.22), (3.23) and (3.25) we have

$$\sum_{i=u_j+1}^{k} g_{h_i} + \sum_{i=\alpha_j-u_j+1}^{m} g_i' = \sum_{i=\alpha_j+1}^{m+k} g_i = \sum_{i=l_j-j+1}^{m+k} g_i.$$

Also,

$$\sum_{i=j+1}^{s} d_{\bar{h}_i} + \sum_{i=\alpha_j-u_j+1}^{m} d_i' - \sum_{i=\alpha_j-u_j+1}^{m} \max(g_i', d_i') =$$
$$\sum_{i=j+1}^{s} d_{\bar{h}_i} + \sum_{i=\alpha_j-u_j+1}^{\bar{h}_j-j} d_i' + \sum_{i=\bar{h}_j-j+1}^{m} d_i'$$
$$- \sum_{i=\alpha_j-u_j+1}^{\bar{h}_j-j} \max(g_i', d_i') - \sum_{i=\bar{h}_j-j+1}^{m} \max(g_i', d_i').$$

For all $i \in \{\alpha_j - u_j + 1, \dots, \bar{h}_j - j\}$, by (3.32) and (3.25) we have

$$d_i' \geq d_{\bar{h}_j} > g_i',$$

and so

$$\max(g_i', d_i') = d_i'.$$

We also have

$$\sum_{i=j+1}^{s} d_{\bar{h}_i} + \sum_{i=\bar{h}_j-j+1}^{m} d_i' = \sum_{i=\bar{h}_j+1}^{m+s} d_i.$$

Altogether we have

$$\sum_{i=l_j+1}^{m+k+s} \bar{f}_i = \sum_{i=l_j-j+1}^{m+k} g_i + \sum_{i=\bar{h}_j+1}^{m+s} d_i - \sum_{i=\bar{h}_j-j+1}^{m} \max(g_i', d_i')$$
$$\geq \sum_{i=l_j-j+1}^{m+k} g_i + \sum_{i=\bar{h}_j+1}^{m+s} d_i - \sum_{i=\bar{h}_j-j+1}^{m} c_i,$$

where the last inequality follows from (3.12). Finally by (3.9) we obtain (3.15), as desired. This finishes our proof. □

Remark 3.3. We note that if both $\mathbf{d}$ and $\mathbf{g}$ are partitions consisting of nonnegative integers, such that

$$\sum_{i=1}^{m+s} d_i + \sum_{i=1}^{k} b_i \geq 0,$$

then by Lemma 2.2 the partition $\mathbf{f}$ also consists of nonnegative integers.

In the course of the proof of Theorem 3.2, we have also proved the following result

Corollary 3.4. *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{d}$ *and* $\mathbf{g}$ *be partitions* (1.1), (1.2), (1.4) *and* (1.5)*, respectively. Let* $\mathbf{c} = (c_1, \dots, c_m)$ *be a partition such that*

$$\mathbf{d} \prec'' (\mathbf{c}, \mathbf{a}) \quad \textit{and} \quad \mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}), \tag{3.34}$$

then there exists a partition $\mathbf{f} = (f_1, \dots, f_{m+k+s})$ *such that*

$$\mathbf{f} \prec'' (\mathbf{d}, \mathbf{b}) \quad \textit{and} \quad \mathbf{f} \prec'' (\mathbf{g}, \mathbf{a}). \tag{3.35}$$

Also, by Theorem 2.3 we have

Corollary 3.5. *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{d}$ *and* $\mathbf{g}$ *be partitions* (1.1), (1.2), (1.4) *and* (1.5)*, respectively. If there exists a partition* $\mathbf{c} = (c_1, \dots, c_m)$*, such that*

$$\mathbf{d} \prec' (\mathbf{c}, \mathbf{a}) \quad \textit{and} \quad \mathbf{g} \prec' (\mathbf{c}, \mathbf{b})$$

then there exists a partition $\mathbf{f} = (f_1, \dots, f_{m+k+s})$ *such that*

$$\mathbf{f} \prec' (\mathbf{c}, \mathbf{a} \cup \mathbf{b}).$$

Finally, by combining Theorem 1.3 with the result of Corollary 3.4, we obtain necessary conditions for the pseudo double majorization problem.

Corollary 3.6. *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{d}$ *and* $\mathbf{g}$ *be partitions* (1.1), (1.2), (1.4) *and* (1.5)*, respectively. If there exists a partition* $\mathbf{c} = (c_1, \dots, c_m)$*, such that*

$$\mathbf{d} \prec'' (\mathbf{c}, \mathbf{a}) \quad \textit{and} \quad \mathbf{g} \prec'' (\mathbf{c}, \mathbf{b})$$

then the condition $\bar{\Omega}(\mathbf{g}, \mathbf{d}, \mathbf{b}, \mathbf{a})$ *holds.*

4 Some comments and more on diagrammatics of generalized majorization

4.1 A counter example for the converse of Theorem 3.2

In the following example we show that the converse of Theorem 3.2 does not hold:

Example 4.1. Let us consider the following partitions of integers:

$$\mathbf{d} = (7, 2, 1) \tag{4.1}$$

$$\mathbf{g} = (7, 2, 1) \tag{4.2}$$

$$\mathbf{a} = (3, 1) \tag{4.3}$$

$$\mathbf{b} = (2, 2) \tag{4.4}$$

The partition

$$\mathbf{f} = (4,4,3,2,1) \tag{4.5}$$

satisfies

$$\mathbf{f} \prec' (\mathbf{g},\mathbf{a}) \quad \text{and} \quad \mathbf{f} \prec' (\mathbf{d},\mathbf{b}). \tag{4.6}$$

Indeed, (4.6) is equivalent to

$$\min(g_i, d_i) \geq f_{i+2}, \qquad i = 1,\dots,3, \tag{4.7}$$

$$\sum_{i=l_j+1}^{5} f_i \geq \sum_{i=l_j-j+1}^{3} g_i + \sum_{i=j+1}^{2} a_i, \qquad j = 1,2, \tag{4.8}$$

$$\sum_{i=1}^{5} f_i = \sum_{i=1}^{3} g_i + \sum_{i=1}^{2} a_i = \sum_{i=1}^{3} d_i + \sum_{i=1}^{2} b_i, \tag{4.9}$$

$$\sum_{i=\bar{l}_j+1}^{5} f_i \geq \sum_{i=\bar{l}_j-j+1}^{3} d_i + \sum_{i=j+1}^{2} b_i, \qquad j = 1,2, \tag{4.10}$$

where

$$l_1 = \bar{l}_1 = 2, \quad l_2 = \bar{l}_2 = 3.$$

By (4.1) – (4.5) we directly get that all of (4.7) – (4.10) hold. Hence we have (4.6), as announced.

However, there is no partition $\mathbf{c}$ satisfying

$$\mathbf{g} \prec' (\mathbf{c},\mathbf{b}) \quad \text{and} \quad \mathbf{d} \prec' (\mathbf{c},\mathbf{a}). \tag{4.11}$$

Indeed, by the definition of generalized majorization, we would have that such a partition $\mathbf{c}$ would be of length one, i.e. $\mathbf{c} = (c_1)$ for certain integer c_1, and that

$$c_1 = \sum_{i=1}^{3} g_i - \sum_{i=1}^{2} b_i = \sum_{i=1}^{3} d_i - \sum_{i=1}^{2} a_i = 6$$

Then

$$h_1 = \min\{i | c_i < g_i\} = 2,$$

and hence we would need that

$$\sum_{i=h_1+1}^{3} g_i \geq \sum_{i=h_1-1+1}^{1} c_i + \sum_{i=1+1}^{2} b_i$$

which gives

$$g_3 = 1 \geq b_2 = 2,$$

which is a contradiction. Hence there is no partition $\mathbf{c}$ satisfying (4.11), as announced.

4.2 Diagrammatics

By using diagrammatics introduced in Section 1, Theorem 2.3 implies the following transitivity-like property of the generalized majorization:

$$\begin{tikzcd} \mathbf{f} & & \mathbf{f} \\ \mathbf{g} \arrow[u, "\mathbf{a_2}"'] & \Longrightarrow & \\ \mathbf{d} \arrow[u, "\mathbf{a_1}"'] & & \mathbf{d} \arrow[uu, "\mathbf{a_1} \cup \mathbf{a_2}"'] \end{tikzcd}$$

The main result of the paper, Theorem 3.2, can be described diagrammatically, by stating that every diagram of the form

$$\begin{tikzcd} \mathbf{g} & & \mathbf{d} \\ & \mathbf{c} \arrow[ul, "\mathbf{b}"] \arrow[ur, "\mathbf{a}"'] & \end{tikzcd}$$

can be completed to a square

$$\begin{tikzcd} & \mathbf{f} & \\ \mathbf{g} \arrow[ur, dashed, "\mathbf{a}"] & & \mathbf{d} \arrow[ul, dashed, "\mathbf{b}"'] \\ & \mathbf{c} \arrow[ul, "\mathbf{b}"] \arrow[ur, "\mathbf{a}"'] & \end{tikzcd}$$

The two properties allow various combinations. For example, by combining the result from Theorem 3.2 with the result from Theorem 2.3 we can get the following. Let $\mathbf{c}$, $\mathbf{u}$, $\mathbf{w}$, $\mathbf{g}$, $\mathbf{d}$, $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{b_1}$ and $\mathbf{b_2}$ be partitions such that

$$\mathbf{u} \prec' (\mathbf{g}, \mathbf{a_1}), \quad \mathbf{u} \prec' (\mathbf{c}, \mathbf{b_1}), \quad \mathbf{w} \prec' (\mathbf{c}, \mathbf{a_2}), \quad \mathbf{w} \prec' (\mathbf{d}, \mathbf{b_2}),$$

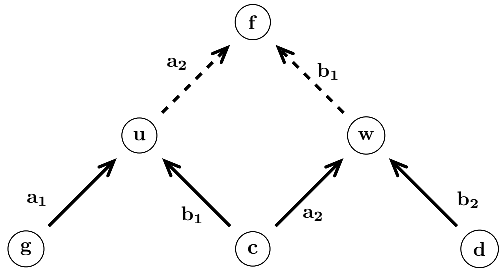
i.e.

$$\begin{tikzcd} & \mathbf{u} & & \mathbf{w} & \\ \mathbf{g} \arrow[ur, "\mathbf{a_1}"] & & \mathbf{c} \arrow[ul, "\mathbf{b_1}"] \arrow[ur, "\mathbf{a_2}"'] & & \mathbf{d} \arrow[ul, "\mathbf{b_2}"'] \end{tikzcd}$$

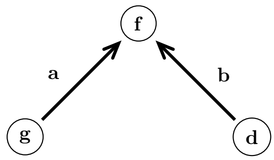
Then by Theorem 3.2 there exists a partition $\mathbf{f}$ such that

$$\mathbf{f} \prec' (\mathbf{u}, \mathbf{a_2}) \quad \text{and} \quad \mathbf{f} \prec' (\mathbf{w}, \mathbf{b_1}).$$

Diagrammatically this gives



Finally, by Theorem 2.3, such $\mathbf{f}$ satisfies



where

$$\mathbf{a} = \mathbf{a_1} \cup \mathbf{a_2}, \quad \mathbf{b} = \mathbf{b_1} \cup \mathbf{b_2}.$$

5 Conclusions

In this paper we study new properties of generalized majorization. The main result of the paper is the proof that the generalized majorization has a completing-squares property. More precisely, we have introduced pseudo double majorization problem for two pairs of partitions (Problem 1.4), and we relate it with double majorization problem (Problem 1.2). In particular, we prove that the existence of a partition $\mathbf{c}$ satisfying (1.11) implies the existence of a partition $\mathbf{f}$ satisfying (1.10). By introducing diagrammatical interpretation of generalized majorization, our main result has an elegant geometric interpretation, which also complements the previous results on transitivity-like property of generalized majorization [10].

Finally, the obtained results are expected to have strong impact in solving the General Matrix Pencil Completion Problem, as well as in solving Bounded Rank Perturbation Problems for matrix pencils.

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Classification of minimal Frobenius hypermaps*

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Abstract

In this paper, we give a classification of orientably regular hypermaps with an automorphism group that is a minimal Frobenius group. A Frobenius group G is called minimal if it has no nontrivial normal subgroup N such that G/N is a Frobenius group. An orientably regular hypermap $\mathcal{H}$ is called a Frobenius hypermap if $\mathrm{Aut}(\mathcal{H})$ acting on the hyperfaces is a Frobenius group. A minimal Frobenius hypermap is a Frobenius hypermap whose automorphism group is a minimal Frobenius group with cyclic point stabilizers. Every Frobenius hypermap covers a minimal Frobenius hypermap. The main theorem of this paper generalizes the main result of Breda D'Azevedo and Fernandes in 2011.

Keywords: Frobenius hypermap, Frobenius group.

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1 Introduction

Let $\mathcal{S}$ be a compact and connected orientable surface. A *topological hypermap* $\mathcal{H}$ on $\mathcal{S}$ is a triple $(\mathcal{S}; V; E)$, where V and E denote closed subsets of $\mathcal{S}$ with the following properties:

(1) $B = V \cap E$ is a finite set. Its elements are called the *brins* of $\mathcal{H}$;

(2) $V \cup E$ is connected;

(3) the components of V (called the *hypervertices*) and of E (called the *hyperedges*), are homeomorphic to closed discs;

(4) the components of the complement $\mathcal{S}\backslash(V \cup E)$ are homeomorphic to open discs, and they are called the *hyperfaces* of $\mathcal{H}$.

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The following Figure 1 shows a topological hypermap on torus with 9 brins, 3 hypervertices (black components), 3 hyperedges (grey components) and 3 hyperfaces (white components).

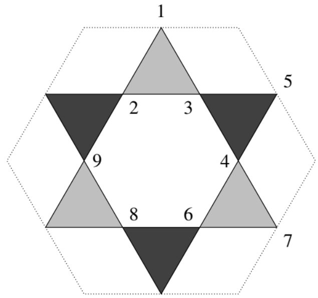

Figure 1: A hypermap on torus.

An important and convenient way to visualize hypermaps was introduced by Walsh in [13]. The *Walsh representation* of a hypermap as a bipartite graph embedding on $\mathcal{S}$ can be described as follows. At the centre of each hypervertex place a white vertex and at the centre of each hyperedge place a black vertex. If a hypervertex intersects a hyperedge then we join the corresponding white vertex and black vertex by an edge. In this way we obtain a bipartite graph. This bipartite graph is said to be the *underlying graph* of $\mathcal{H}$. Figure 2 is the Walsh representation of the hypermap in Figure 1.

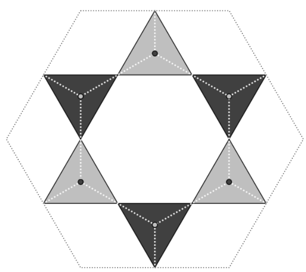

Figure 2: The Walsh representation.

An *algebraic hypermap* is a quadruple $\mathcal{H} = (G, B, \rho_0, \rho_1)$, where G is a finite group which is generated by two elements ρ_0, ρ_1 and acts transitively on a finite set B. By [3], there is a one-to-one correspondence between topological and algebraic hypermaps. The finite group G is the *monodromy group* of $\mathcal{H}$, denoted by $\mathrm{Mon}(\mathcal{H})$. In the Walsh representation, G is a permutation group acting on the set of edges, ρ_0, ρ_1 generate the cyclic permutations of the edges going around the white resp. black vertices in a positive sense, and each cycle of $\rho_0\rho_1$ bounds a hyperface in a negative direction. A permutation α of B is called an *automorphism* of the hypermap $\mathcal{H} = (G, B, \rho_0, \rho_1)$ if it is G-equivariant, i.e. if

$$\alpha(g(b)) = g(\alpha(b))$$

for every $b \in B$ and $g \in G$.

Since $\alpha\rho_0\alpha^{-1} = \rho_0$ and $\alpha\rho_1\alpha^{-1} = \rho_1$, α induces a permutation on the cycles of ρ_0 and ρ_1. So, in the Walsh representation, $\mathrm{Aut}(\mathcal{H})$ induces a subgroup of the automorphism group of the underlying graph of $\mathcal{H}$, and $\mathrm{Aut}(\mathcal{H})$ preserves the hypervertex set and hyperedge set, respectively. A hypermap is called *regular* if G acts regularly on B. In this case, $\mathrm{Aut}(\mathcal{H})$ is isomorphic to G which acts regularly on B as well.

For a regular hypermap $\mathcal{H} = (G, B, \rho_0, \rho_1)$, the set B can be replaced by G, so that $\mathrm{Mon}(\mathcal{H})$ and $\mathrm{Aut}(\mathcal{H})$ can be viewed as the right and left regular multiplications of G, respectively. So, $\mathcal{H}$ can be denoted by a triple $\mathcal{H} = (G; \rho_0, \rho_1)$, where $G = \langle \rho_0, \rho_1 \rangle$. In this way, the hypervertices (resp. hyperedges and hyperfaces) correspond to right cosets of G relative to $\langle \rho_0 \rangle$, (resp. $\langle \rho_1 \rangle$ and $\langle \rho_0\rho_1 \rangle$). In [4], the hypermap $\mathcal{H} = (G; \rho_0, \rho_1)$ is denoted by $(G; a, b)$ where $a = {\rho_1}^{-1}{\rho_0}^{-1}$ and $b = \rho_0$. From now on, we denote a regular hypermap $\mathcal{H}$ by the triple $\mathcal{H} = (G; a, b)$, and then the hyperfaces (resp. hypervertices and hyperedges) correspond to left cosets of G relative to subgroups $\langle a \rangle$ (resp. $\langle b \rangle$ and $\langle ab \rangle$). Let $\mathcal{H} = (G; a, b)$ and $\mathcal{H}' = (G'; a', b')$ be two orientably regular hypermaps. If there is an epimorphism ρ from G to G' such that $a^\rho = a'$ and $b^\rho = b'$, then $\mathcal{H}$ is called a covering of $\mathcal{H}'$ or $\mathcal{H}$ *covers* $\mathcal{H}'$. Given a group G, $(G; a_1, b_1) \cong (G; a_2, b_2)$ if and only if there exists an automorphism σ of G such that $a_1^\sigma = a_2$ and $b_1^\sigma = b_2$.

A (face-)*primer* hypermap is an orientably regular hypermap whose automorphism group induces faithful actions on its hyperfaces, see [4]. The classification of regular hypermaps with given automorphism groups isomorphic to $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$ can be extracted from [12] by Sah. Moreover, Conder, Potočnik and Širáň extended Sah's investigation to reflexible hypermaps, on both orientable and nonorientable surfaces, and provided explicit generating sets for projective linear groups, see [1]. In [2], Conder described all regular hypermaps of genus 2 to 101, and all non-orientable regular hypermaps of genus 3 to 202.

The study of primer hypermaps was initiated by Breda d'Azevedo and Fernandes in 2011. In [4], the authors classified the primer hypermaps with p-hyperfaces for a prime number p, where their automorphism groups are Frobenius groups. Thereafter, they determined all regular hypermaps with p-hyperfaces, see [5]. In [7], Du and Hu classified primer hypermaps with a product of two primes number of hyperfaces. Recently, Du and Yuan characterized primer hypermaps with nilpotent automorphism groups and prime hypervertex valency, see [8].

A Frobenius group is a transitive permutation group G on a set Ω which is not regular on Ω , but has the property that the only element of G which fixes more than one point is the identity element. A Frobenius group G is called *minimal* if it does not have a nontrivial normal subgroup N such that G/N is a Frobenius group. A regular hypermap $\mathcal{H}$ is called a *Frobenius hypermap* if $\mathrm{Aut}(\mathcal{H})$ acting on the hyperfaces is a Frobenius group. Clearly, $\mathcal{H}$ is a primer hypermap. A *minimal Frobenius hypermap* is a Frobenius hypermap whose automorphism group is a minimal Frobenius group with a cyclic point stabilizer. Clearly, every Frobenius hypermap covers a minimal Frobenius hypermap.

This paper has three sections. In the first section, a quick overview of orientably regular hypermaps is given. In Section 2, we introduce minimal Frobenius groups. In the last section, we give a classification of orientably regular minimal Frobenius hypermaps. Furthermore, the main theorem of this paper generalizes the main result of Breda D'Azevedo and Fernandes, see [4].

2 Minimal Frobenius groups

We refer the readers to [10] for standard notation and results in group theory. Set (r, s) to denote the greatest common divisor of two positive integers r and s. We denote the orders of an element x and of a subgroup H of G as $|x|$ and $|H|$, respectively. A semidirect product of a group N by a group H is denoted by $N : H$. Let $\mathbb{Z}_m = \{0, 1, \cdots, m-1\}$ and $\mathbb{Z}_m^* = \{k \mid k \in \mathbb{Z}_m \text{ and } (k, m) = 1\}$.

Let G be a Frobenius group on Ω. A subgroup K of G is called the *Frobenius kernel* if K acts regularly on Ω. Each point stabilizer is called a *Frobenius complement* of K in G. In the following, we give some interesting results about Frobenius groups and primitive groups.

Proposition 2.1 ([6, P86]). *Let G be a Frobenius group on Ω and $\alpha \in \Omega$, K be the Frobenius kernel, and H be a Frobenius complement. Then:*

(i) *K is a normal and regular subgroup of G.*

(ii) *For each odd prime number p, the Sylow p-subgroups of H are cyclic, and the Sylow 2-subgroups are either cyclic or quaternion groups. If G is not solvable, then it has exactly one nonabelian composition factor, namely A_5.*

(iii) *K is a nilpotent group.*

Proposition 2.2 ([6, Corollary 1.5A.]). *Let G be a group acting transitively on a set Ω with at least two points. Then G is primitive if and only if each point stabilizer G_α is a maximal subgroup of G.*

Lemma 2.3. *Assume $G \leqslant \mathrm{Sym}(\Omega)$ has a regular normal subgroup R, where Ω has at least two points. Then G is primitive if and only if no nontrivial subgroup of R is normalized by G_α, for each α.*

Proof. By Proposition 2.2, G is primitive if and only if G_α is a maximal subgroup of G. Because R is a regular normal subgroup of G, $G = G_\alpha R$ and $G_\alpha \cap R = \{1\}$.

We claim that G_α is maximal if and only if no nontrivial subgroup of R is normalized by G_α. Suppose G_α is not maximal, then there exists a proper subgroup K of G such that $G_\alpha < K$. It follows that $K = K \cap G = K \cap G_\alpha R = G_\alpha(K \cap R)$. In this case, $K \cap R$ is a proper subgroup of R which is normalized by G_α. Conversely, suppose that there exists a proper subgroup H, normalized by G_α, of R. Thus $G_\alpha H$ is a proper subgroup of G and so G_α is not maximal. □

Corollary 2.4 follows directly from Lemma 2.3.

Corollary 2.4. *Assume $G \leqslant \mathrm{Sym}(\Omega)$ has a regular normal subgroup R, where Ω has at least two points. If R is abelian, then G is primitive if and only if no nontrivial normal subgroup of G is contained in R.*

Lemma 2.5. *Let K be the Frobenius kernel of a Frobenius group G which acts on a set Ω. If N is a normal subgroup of G, then either $N \leqslant K$ or $K < N$.*

Proof. Assume that N is not a subgroup of K. Set $\alpha \in \Omega$. Since N is a normal subgroup of G, we have $N = (\bigcup_{g \in K} N_\alpha^g) \cup (N \cap K)$ and so N is a subgroup of $N_\alpha K$. Let $|N_\alpha| =$

m, $|K| = n$ and $|N \cap K| = t$. Then, $|N| = n(m-1)+t$. Since $N \leqslant N_\alpha K$ and $N_\alpha \leqslant N$, we get $N = N \cap N_\alpha K = N_\alpha(N \cap K)$. So, $|N| = mt$ which implies $n(m-1)+t = mt$. Note that $m > 1$, then $n = t$. Therefore, $N \cap K = K$ and K is a proper subgroup of N. □

Proposition 2.6 ([11, Lemma 2.3]). *Let K be the Frobenius kernel of a Frobenius group G. If N is a normal subgroup of G and $N < K$, then G/N is a Frobenius group.*

Proposition 2.7 ([11, Corollary 2.6]). *Let $G = KH$ be a Frobenius group, where K is the Frobenius kernel and H is a Frobenius complement. For each $h \in H, h \neq 1$, and for each $k \in K$, the orders of h, kh and hk are equal, that is $|h| = |kh| = |hk|$.*

Based on Lemma 2.5 and Proposition 2.6, we give the following definition of *minimal Frobenius groups*.

Definition 2.8. A Frobenius group G is called *minimal* if it does not have a nontrivial normal subgroup N such that G/N is a Frobenius group.

Lemma 2.9. *If G is a minimal Frobenius group acting on a set Ω with the Frobenius kernel K, then K is an elementary abelian p-group and G is primitive.*

Proof. If G is minimal, then by Proposition 2.6 no nontrivial normal subgroup of G exists in K. Note that K is a nilpotent group. Let P be a Sylow p-group of K, $\Phi(P)$ be the Frattini subgroup of P and L be the p'-Hall group of K. Both $\Phi(P)$ and L are characteristic subgroups of K. So, $L = \Phi(P) = 1$ which implies that K is an elementary abelian p-group.

Because no nontrivial normal subgroup of G is contained in K and K is abelian, it follows that G is primitive by Corollary 2.4. □

Lemma 2.10. *If G is a primitive group acting on a set Ω with non-trivial abelian point stabilizers, then G is a Frobenius group and its Frobenius kernel K is an elementary abelian p-group.*

Proof. It suffices to show that for any two distinct points $\alpha, \beta \in \Omega$, $G_\alpha \cap G_\beta = 1$. Let $J = G_\alpha \cap G_\beta$. Since G is primitive, $G = \langle G_\alpha, G_\beta \rangle$. Note that G_α and G_β are abelian, so J is a normal subgroup of G. Because $\alpha^J = \{\alpha\}$, for any $g \in G$, we have $\alpha^{gJ} = \alpha^{Jg} = \{\alpha^g\}$. That is to say J fixes every point of Ω, so $J = 1$ and G is a Frobenius group. Furthermore, as point stabilizers are maximal, the Frobenius kernel K must be an elementary abelian p-group . □

Corollary 2.11 follows from Lemma 2.9 and 2.10 directly.

Corollary 2.11. *Let G be a permutation group with cyclic point stabilizers. Then, G is a minimal Frobenius group if and only if G is a primitive group.*

For a prime number p and an integer n, an integer m ($m > 1$) is called a *primitive divisor* of $p^n - 1$ if m divides $p^n - 1$, but it does not divide $p^s - 1$ for any $s < n$.

The following Proposition 2.12 can be obtained from some results in [10, Kapitel II: 3.10, 3.11, 7.3].

Proposition 2.12. *For a prime number p and a positive integer n, set $G = GL(n, p)$.*

(i) *The group G contains a cyclic Singer-Zyklus group $S = \langle x\rangle$ of order $p^n - 1$, and $C_G(S) = S$. Moreover, $N_G(S) = S : \langle y\rangle = \langle x, y \mid x^{p^n-1} = y^n = 1, x^y = x^p\rangle$, and $|N_G(S)| = n(p^n - 1)$. Take an element $g \in S$, if $|g|$ is a primitive divisor of $p^n - 1$, then $N_G(\langle g\rangle) = N_G(S), C_G(\langle g\rangle) = S$ and $\langle g\rangle$ is an irreducible subgroup.*

(ii) *Let L be a cyclic irreducible subgroup of G. Then L is conjugate to a subgroup of S, and $|L|$ is a primitive divisor of $p^n - 1$.*

The following lemma generalizes Lemma 3.3 in [9]. The proof is similar to that of Lemma 3.3, so we omit it.

Lemma 2.13. *Let $X = T : \langle x\rangle$ and $Y = T : \langle y\rangle$ be two subgroups of $A = AGL(n, p) = T : G$, where $G = GL(n, p)$, T is the translation subgroup, and x, y are nontrivial elements in G. If σ is an isomorphism from X to Y mapping $\langle x\rangle$ to $\langle y\rangle$, then, there exists an element $u \in G$ such that $\sigma = I(u)|_X$, where $I(u)$ is the inner automorphism of A induced by u. In particular, $u \in N_G(\langle x\rangle)$ if $\langle x\rangle = \langle y\rangle$.*

3 Classification of minimal Frobenius hypermaps

For a prime number p, an integer $n \geqslant 1$ ($n \geqslant 2$ if $p = 2$) and a primitive divisor m of $p^n - 1$, let S be the cyclic Singer-Zyklus group of $GL(n, p)$, $\langle a\rangle$ be a subgroup of S with order m and T be the translation subgroup of $AGL(n, p)$. Define a group M of order mp^n as

$$M = T : \langle a\rangle \leqslant T : S \leqslant AGL(n, p) = T : GL(n, p).$$

By Proposition 2.12, $\langle a\rangle$ is an irreducible subgroup. Hence M is a primitive group, and consequently M is a Frobenius group by Lemma 2.10.

Let F be a minimal Frobenius group acting on a set Ω ($|\Omega| > 2$) with cyclic point stabilizers, and K be its Frobenius kernel. By Lemma 2.9, K is an elementary abelian p-group and F is a primitive group. Set $|K| = p^n$, and then $|\Omega| = p^n$. Take an element $\alpha \in \Omega$ and assume $|F_\alpha| = k$. By Proposition 2.12, k is a primitive divisor of $p^n - 1$, and $GL(n, p)$ has only one conjugacy class of irreducible cyclic subgroups of order k. Hence $AGL(n, p)$ has only one conjugacy class of subgroups isomorphic to F which implies $F \cong M = T : \langle a\rangle$ when $k = m$. These discussions give the following Theorem 3.1.

Theorem 3.1. *Let F be a minimal Frobenius group with cyclic point stabilizers of order m. Then, $F \cong T : \langle a\rangle$, where T is elementary abelian of order p^n for some prime number p and an integer $n \geqslant 1$, m is a primitive divisor of $p^n - 1$ and $|\langle a\rangle| = m$. Clearly, $|F| = mp^n$.*

Lemma 3.2. *Let $M = T : \langle a\rangle$ be the group defined as in the first paragraph of this section. If $\mathcal{H} = (M; R, L)$ is a Frobenius hypermap, then $\mathcal{H}$ is isomorphic to*

$$\mathcal{H}(p, n, m, i, j) = (M; a^i, a^j b),$$

where $1 \neq b \in T$, m is a primitive divisor of $p^n - 1$, $j \in \mathbb{Z}_m$, $i \in \mathbb{Z}_m^*$ and $(i, p) = 1$. Moreover, different parameter pairs (i, j) give non-isomorphic hypermaps with p^n hyperfaces, each of valency m. Furthermore, there are $\frac{m\phi(m)}{n}$ non-isomorphic hypermaps, where ϕ is the Euler's totient function.

Proof. Let $G = GL(n,p)$ and then $M \leqslant AGL(n,p) = T : G$. Since M is a Frobenius group, M has only one conjugacy class of subgroups of order m. So we can assume $R = a^i$ for some $i \in \mathbb{Z}_m^*$. Remember that S is the cyclic Singer-Zyklus group of $GL(n,p)$ and $\langle a \rangle$ is a subgroup of S. So, M is a normal subgroup of $T : S$. Since S fixes a and acts transitively on $T \backslash \{1\}$ by conjugation, we may fix $L = a^j b$, where j is calculated modular m.

If there exists an automorphism σ of M such that $(a^i)^\sigma = a^{i'}$ and $(a^j b)^\sigma = a^{j'} b$, then $b^\sigma = a^\epsilon b$ for some $\epsilon \in \mathbb{Z}_m$. Clearly, the orders of b and $a^\epsilon b$ are equal. While according to Proposition 2.7, the two elements $a^\epsilon b$ and a^ϵ have the same order which is coprime with that of b if $\epsilon \neq 0$ modulo m. So, $b^\sigma = b$. By Lemma 2.13, there exists an element $u \in G$ such that $\sigma = I(u)|_F$, where $u \in N_G(\langle a \rangle)$. According to Proposition 2.12,

$$N_G(\langle a \rangle) = S : \langle y \rangle = \langle x, y \mid x^{p^n - 1} = y^n = 1, x^y = x^p \rangle,$$

where $S = \langle x \rangle$. Because $b^\sigma = b$, it follows that $u = y^t$, where t is calculated modular n. So, $a^\sigma = a^{y^t} = a^{p^t}$. As a result, we may assume $(i,p) = 1$ in $R = a^i$. As a result, we get $\frac{m\phi(m)}{n}$ non-isomorphic hypermaps $(M; a^i, a^j b)$, where ϕ is the Euler's totient function. Clearly, $(M; a^i, a^j b)$ has p^n hyperfaces, each of valency m. □

By Theorem 3.1, the automorphism group of a minimal Frobenius hypermap is isomorphic to $M = T : \langle a \rangle$, where $|T| = p^n$ and $|\langle a \rangle| = m$. Consequently, we give the following classification theorem of minimal Frobenius hypermaps.

Theorem 3.3. *$\mathcal{H}$ is a minimal Frobenius hypermap if and only if $\mathcal{H}$ is isomorphic to*

$$\mathcal{H}(p,n,m,i,j) = (M; a^i, a^j b),$$

where M is a group defined as in the first paragraph of this section, m is a primitive divisor of $p^n - 1$, $j \in \mathbb{Z}_m$, $i \in \mathbb{Z}_m^*$ and $(i,p) = 1$. Moreover, different parameter pairs (i,j) give non-isomorphic hypermaps with p^n hyperfaces, each of valency m. And, there are $\frac{m\phi(m)}{n}$ non-isomorphic minimal Frobenius hypermaps, where ϕ is the Euler's totient function.

According to Corollary 2.11, we have the following Proposition 3.4.

Proposition 3.4. *If $\mathcal{H}$ is a regular hypermap, then $\mathcal{H}$ is a minimal Frobenius hypermap if and only if* $\mathrm{Aut}(\mathcal{H})$ *acts primitively on the hyperfaces.*

The next Proposition 3.5 follows from Lemma 2.5.

Proposition 3.5. *Every Frobenius hypermap covers a minimal Frobenius hypermap.*

The H-sequence of a hypermap $\mathcal{H}$ is a sequence $[|v|, |e|, |f|; V, E, F; |\mathrm{Aut}(\mathcal{H})|]$, where $|v|, |e|, |f|, V, E$ and F stand for the hypervertex valency, hyperedge valency, hyperface valency, number of hypervertices, number of hyperedges and number of hyperfaces of $\mathcal{H}$, respectively.

Corollary 3.6. *The H-sequence of the minimal Frobenius hypermap $\mathcal{H}(p,n,m,i,j) = (M; a^i, a^j b)$ is*

(i) $[p, m, m; mp^{n-1}, p^n, p^n; mp^n]$ *for $j = 0$;*

(ii) $[m, p, m; p^n, mp^{n-1}, p^n; mp^n]$ *for $j = m - i$;*

(iii) $[\frac{m}{(m,j)}, \frac{m}{(m,i+j)}, m; (m,j)p^n, (m,i+j)p^n, p^n; mp^n]$ *for* $j \neq 0$ *and* $j \neq m-i$.

Proof. The sequence is determined by the first three entries, namely $|a^j b|$, $|a^{i+j} b|$ and $|a^i|$. These entries can be easily calculated according to Proposition 2.7. □

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A parametrisation for symmetric designs admitting a flag-transitive, point-primitive automorphism group with a product action*

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Abstract

We study (v, k, λ)-symmetric designs having a flag-transitive, point-primitive automorphism group, with $v = m^2$ and $(k, \lambda) = t > 1$, and prove that if D is such a design with m even admitting a flag-transitive, point-primitive automorphism group G, then either:

(1) D is a design with parameters $\left((2t + s - 1)^2, \frac{2t^2-(2-s)t}{s}, \frac{t^2-t}{s^2}\right)$ with $s \geq 1$ odd, or

(2) G does not have a non-trivial product action.

We observe that the parameters in (1), when $s = 1$, correspond to Menon designs.

We also prove that if D is a (v, k, λ)-symmetric design with a flag-transitive, point-primitive automorphism group of product action type with $v = m^l$ and $l \geq 2$ then the complement of D does not admit a flag-transitive automorphism group.

Keywords: Symmetric-designs, flag-transitivity, primitive groups, automorphism groups of designs.

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1 Introduction

If $D = (P, \mathcal{B})$ is a (v, k, λ)-symmetric design, a *flag* of D is an ordered pair (p, B) such that $p \in P$ is a point of D, $B \in \mathcal{B}$ is a block of D, and $p \in B$. The *order* of D is $n = k - \lambda$.

There are some symmetric designs in which the parameters are related in some special way, such as Hadamard designs in which $v = 4n + 3$, $k = 2n + 1$, and $\lambda = n$ ($n \in \mathbb{Z}^+$), and Menon designs, in which $v = 4t^2$, $k = 2t^2 - t$, and $\lambda = t^2 - t$ for some positive integer t. These last ones will be relevant in the present work.

If $G = \mathrm{Aut}(D)$, then G is *point-transitive* if it is transitive on P (the set of points of D), and it is *flag-transitive* if it is transitive on the set of flags of D. If G is point-transitive, it can either be *point-primitive*, that is, there is no G-invariant non-trivial partition of P, or *point-imprimitive*, which is when there is a non-trivial partition of the points of D invariant under the action of G.

Primitive groups are classified by the O'Nan-Scott Theorem, we will use the classification in [4] by Liebeck, Praeger, and Saxl, with five types, namely affine, almost simple, product, simple diagonal, and twisted wreath.

Buekenhout, Delandtsheer, and Doyen proved in [1] that if a 2-design with $\lambda = 1$ (linear space) admits a point-primitive, flag-transitive automorphism group G, then it must be of affine or almost simple type. O'Reilly-Regueiro proved the same result for symmetric 2-designs with $2 \leq \lambda \leq 4$ in [5, 6]. All designs in this paper will be 2-designs.

In [7], Tian and Zhou extended this result to $\lambda \leq 100$, and conjectured that it holds for all values of λ. Having an upper bound on λ, in [7] they ruled out the simple diagonal, product, and twisted wreath action by finding possible groups and/or sets of parameters of designs and then ruling them out by arithmetic constraints and the use of GAP [2]. Additionally, in [3, 8, 9, 10], Zhou et al. have tackled this issue from different perspectives, and have ruled out the product action for flag-transitive (v, k, λ) symmetric designs in which $\lambda \geq (k, \lambda)^2$, as well as, for those cases in which λ is prime.

We have tried to prove that if D is a (v, k, λ)-symmetric design with $v = m^2$ even and any λ admitting a point-primitve, flag-transitive automorphism group G, then G does not have a product action. In this paper we present our results, namely, a parametrisation for such designs which in some cases correspond to Menon designs.

In 1998, Zieschang proved in [11] that if a (not necessarily symmetric) 2-design in which $(r, \lambda) = 1$ (where r is the number of blocks incident with any given point) admits a flag-transitive group G, then G is of affine or almost simple type. Given this result, in our work we will assume $(k, \lambda) = t > 1$.

2 Product action

We start with a result from [5], which will be useful later.

Corollary 2.1. *If G is a flag-transitive automorphism group of a (v, k, λ)-symmetric design $D = (P, \mathcal{B})$, then k divides $\lambda(v - 1, |G_x|)$ for every point-stabiliser G_x.*

The next lemma gives us an arithmetic condition that will be used throughout this work.

Suppose that the group G has a product action on the set of points P. Then there is a finite set Γ with $|\Gamma| \geq 5$ and a group H acting primitively on Γ, with an almost simple or simple diagonal action, such that

$$P = \Gamma^l \text{ and } G \leq H^l \rtimes S_l = H \mathrm{wr} S_l, \text{ with } l \geq 2.$$

Lemma 2.2. *If G is a point-primitive group acting flag-transitively on a (v, k, λ)-symmetric design $D = (P, \mathcal{B})$, with a product action on P, then k divides $\lambda l(|\Gamma| - 1)$ and $v = |\Gamma|^l \leq \lambda l^2(|\Gamma| - 1)^2$.*

Proof. Take $x \in P = \Gamma^l$. If $x = (\gamma_1, ..., \gamma_l)$, define for $1 \geq j \geq l$ the Cartesian line of the j^{th} parallel class through x to be the set:

$$G_{x,j} = \{(\gamma_1, ..., \gamma_{j-1}, \gamma, \gamma_{j+1}, ..., \gamma_l) | \gamma \in \Gamma\},$$

(So there are l Cartesian lines through x).

Denote $|\Gamma| = m$.

Since G is primitive, G_x is transitive on the l Cartesian lines through x. Denote by Δ the union of those lines (excluding x). Then Δ is a union of orbits of G_x, and so every block through x intersects it in the same number of points. Hence k divides $\lambda l(|\Gamma| - 1)$. Also, $k^2 > \lambda(m^l - 1)$, so $(m^l - 1) < \lambda l^2(m - 1)^2$.

Hence

$$v = m^l \leq \lambda l^2(m - 1)^2. \tag{2.1}$$

□

3 Results

In this section we will only consider $l = 2$, further work may be done for greater values of l.

When $l = 2$, $m = \frac{r^2+2r+4\lambda}{4\lambda - r^2}$ so

$$(m + 1)r^2 + 2r - 4\lambda(m - 1) = 0 \tag{3.1}$$

solving for r we have

$$r = \frac{-2 \pm \sqrt{4 + 16\lambda(m - 1)(m + 1)}}{2(m + 1)} = \frac{-1 \pm \sqrt{1 + 4\lambda(m^2 - 1)}}{m + 1}$$

therefore

$$r = \frac{2(k - 1)}{m + 1}. \tag{3.2}$$

Suppose that $(k, \lambda) = t > 1$ (the case where $(k, \lambda) = 1$ was done by Paul-Hermann Zieschang [11]), so there exist positive integers a and b such that

$$k = at, \quad \lambda = bt. \tag{3.3}$$

Then, by Lemma 2.2 we have

$$k = \frac{2\lambda(m - 1)}{r}, \tag{3.4}$$

and substituting (3.3) in the last one and also in $k(k - 1) = \lambda(v - 1)$ we obtain

$$a = \frac{2b(m - 1)}{r}, \tag{3.5}$$

$$a(at - 1) = b(m^2 - 1). \tag{3.6}$$

From (3.5) we can see that a divides $b(m-1)$. But $(k,\lambda)=t$ so $t=(at,bt)$ implies $(a,b)=1$. Therefore a divides $m-1$, that is, there exists a positive integer s such that $m-1=as$ and substituting in (3.5), we obtain $r=2bs$. Then since $(a,b)=1$, this forces $s=(m-1,\frac{r}{2})$.
We have te following results with respect to the new parameters a and s.

Lemma 3.1. *Let D be a (v,k,λ)-symmetric design with $v=m^2$ admitting a flag-transitive, point-primitive automorphism group with product action. If $k=at$ and $\lambda=bt$ with $t=(k,\lambda)$, then $a\neq 1$.*

Proof. If $a=1$ then $k=t$ and $\lambda=kb$ with $b\geq 1$. This is a contradiction because $k>\lambda$, therefore $a\neq 1$. □

Lemma 3.2. *Let D be a (v,k,λ)-symmetric design with $v=m^2$ admitting a flag-transitive, point-primitive automorphism group with product action. If $k=at$ and $\lambda=bt$ with $t=(k,\lambda)$, then $(a,s)=1$ where s is a positive integer such that $m-1=as$ and $r=2bs$.*

Proof. Note (3.2) can be rewritten as:

$$r+1=k-(m-1)\frac{r}{2}.$$

Using the expressions $k=at$, $\lambda=bt$, $m-1=as$ and $r=2bs$ we obtain $1=a(t-bs^2)-2bs$ and here we can see that $(a,s)=1$. □

The fact that the parameter $s=1$ is a necessary and sufficient condition for Menon designs is seen in the following result:

Lemma 3.3. *Let D be a (v,k,λ)-symmetric design with $v=m^2$ admitting a flag-transitive, point-primitive automorphism group with product action. If $t=(k,\lambda)$ and $s\in\mathbb{Z}^+$ is such that $m-1=as$ and $r=2bs$, then $s=1$ if and only if $v=4t^2$, $k=2t^2-t$, and $\lambda=t^2-t$.*

Proof. Suppose first that $s=1$, so $m-1=a$ which implies $k=(m-1)t$. We also have $\frac{r}{2}=b$, so $\lambda=\frac{r}{2}t$. Now from $m=\frac{r^2+2r+4\lambda}{4\lambda-r^2}$ we obtain

$$m=\frac{b+t+1}{t-b} \tag{3.7}$$

then

$$a=m-1=\frac{2b+1}{t-b}. \tag{3.8}$$

Now if $t=b$ then $\lambda=tb=b^2=\dfrac{r^2}{4}$, and substituting in (3.1) we obtain

$$r^2(m+1)+2r-r^2(m-1)=0,\text{ so } r(r+1)=0.$$

This forces $r=0$ or $r=-1$, which is a contradiction and so $t\neq b$. From (3.8) we have $t-b\geq 1$.

Suppose that $t - b > 1$, and let $x > 1$ be an integer such that $t = b + x$. We will prove that x is an odd number. If $x = 2y$ for some $y \in \mathbb{Z}$, then $t = b + 2y$, and substituting in (3.8) we obtain

$$a = \frac{2b+1}{2y},$$

which is a contradiction since $a \in \mathbb{Z}$, so x is odd. Therefore there exists a positive integer y such that $x = 2y + 1 > 1$ and with this we obtain $t = b + 2y + 1$, substituting in (3.8) results in

$$a = \frac{2b+1}{2y+1}.$$

Using the last expression for a together with $a > b$ we obtain $2b+1 > b(2y+1)$ which results in

$$1 > b(2y-1). \tag{3.9}$$

But we assumed $t - b > 1$ so $x = 2y + 1 > 1$, that is, $2y - 1 > -1$. This together with the expression (3.9) implies that the equation $2y - 1 = 0$ should hold. But that implies $y = \frac{1}{2}$, which is a contradiction since we assumed $y \in \mathbb{Z}$.

From the above we can conclude that $b = t - 1$ and this implies $\lambda = t(t-1)$. Then substituting this expression in (3.8) we have

$$a = 2(t-1) + 1 = 2t - 1,$$

so $k = t(2t-1)$ and $m = a + 1 = 2t$, therefore $v = m^2 = 4t^2$.

Now, suppose that we have a symmetric design with parameters $v = 4t^2$, $k = 2t^2 - t$ and $\lambda = t^2 - t$. Then $a = 2t - 1$ and $b = t - 1$. In addition, we have $m = 2t$ and all of these combined imply $m - 1 = 2t - 1 = a$. But $m - 1 = as$, and so $s = 1$. Hence the result. □

Remark 3.4. When we fix $m - 1$ and we vary $\frac{r}{2}$ we get many possible values for λ that satisfy the equation $m = \frac{r^2+2r+4\lambda}{4\lambda - r^2}$, at this point we observe that if $m - 1$ is a power of an odd prime, then the parameters satisfy the conditions of Menon designs.

Lemma 3.5. *Let D be a (v, k, λ)-symmetric design with $v = m^2$ admitting a flag-transitive, point-primitive automorphism group with product action such that $t = (k, \lambda)$ and $s \in \mathbb{Z}^+$ such that $m - 1 = as$ and $r = 2bs$. If $m - 1 = p^d$ with p an odd prime and $d \in \mathbb{N}$, then $v = 4t^2$, $k = 2t^2 - t$, and $\lambda = t^2 - t$.*

Proof. From $m - 1 = as = p^d$ then we have the following possible cases:

1. $s = p^i$ and $a = p^{d-i}$ for some natural number $i < d$.
   This case is not possible because this would imply that $(a, s) = p^j$ for some natural number j, and this contradicts Lemma 3.2.

2. $a = p^i$ and $s = p^{d-i}$ for some natural number $i < d$.
   This case is not possible because this would imply that $(a, s) = p^j$ for some natural number j, contradicting Lemma 3.2.

3. $s = p^d$ and $a = 1$.
   This is not possible because it contradicts Lemma 3.1.

4. $a = p^d$ and $s = 1$.
   Recall that $s = 1$ (Lemma 3.3), so in this case $v = 4t^2$, $k = 2t^2 - t$ and $\lambda = t^2 - t$ (these are the conditions for Menon designs). With this we have proved the lemma. □

Remark 3.6. We cannot claim the previous result for any odd $m - 1$ because the parameters $(4900, 3267, 2178)$, $(16900, 2752, 448)$ and $(44100, 8019, 1458)$ are counterexamples to that possible generalisation. However we have neither confirmed nor discarded the existence of designs with these parameters. These (and Menon designs) are the only admissible parameters for $v \leq (210)^2$.

Recall the definition of the Cartesian lines from Lemma 2.2. In the case we are studying, when $l = 2$, there are two Cartesian lines through any point in the design, so we have two possibilities. Either:

(i) there exists a point x and a block that contains it such that it intersects only one Cartesian line through x, or

(ii) for any point x in the design, every block that contains it intersects each one of the Cartesian lines through x.

We now study these cases separately. Although there are similarities between both proofs, due to their length and enough differences we present two theorems for clarity.

Theorem 3.7 (**Case** (i)). *Let D be a (v, k, λ)-symmetric design with $v = m^2$ admitting a flag-transitive, point-primitive automorphism group with product action. If there exists a flag (x, A) in the design such that A intersects only one Cartesian line through x then $r + 1$ divides k.*

Proof. Let (x, A) be the flag such that A intersects a Cartesian line through $x := (a_0, b_0)$. Suppose that A intersects the second Cartesian line through x.

First, let us prove that for any element of the block A, A intersects only the second Cartesian line through that point. We have, two subcases: either a point $y \in A$ is in the second Cartesian line through x, or a point $y \in A$ is not in the second Cartesian line through x.

First subcase: we can see that if we take a point $y \in A$ so that it is also in the second Cartesian line through x, then $y = (a_0, \nu)$ for some $\nu \in \Gamma$. In this way, the set of elements in the second Cartesian line through y which are also in A is the same as the intersection of A with the second Cartesian line through x. Also, since by Lemma 2.2 the size of the intersection of A with the second Cartesian line through x is $r + 1$, the size of the set of elements in the second Cartesian line through y which are also in A is $r + 1$, and since the size of the intersection of A with the Cartesian lines through any point is $r + 1$, there are no more elements of any of the two Cartesian lines through y in A. In particular, there are no elements of the first Cartesian line through y in A and so the statement is proved for this subcase.

Second subcase: Now we are going to take $y \in A$ such that it is not in the second Cartesian line through x, in particular $y \neq x$, so if $y := (a_1, b_1)$ then $a_1 \neq a_0$. Let us consider the flag (y, A). Since the group G is flag-transitive, there is a $g \in G$ such that $g(x, A) = (y, A)$, that is, $g(x) = y$, so

$$g(a_0, b_0) = (a_1, b_1), \tag{3.10}$$

this implies $g|_\Gamma(a_0) = a_1$. So, for any $\mu \in \Gamma$ such that $(a_0, \mu) \in A$ we have $g(a_0, \mu) = (a_1, \nu)$ for some $\nu \in \Gamma$. Thus the element $g \in G$ sends every element of the second Cartesian line through x which is also in A to an element of the second Cartesian line through y which is also in A. In this way A intersects only the second Cartesian line through y. This is true for any y which is not in the second Cartesian line through x and by Lemma 2.2 the size of this intersection is $r + 1$ and with this the statement the second subcase is proved.

Let A_0 be the set of points in the second Cartesian line through x which are also in A, including x, the size of this set is $r + 1$. Now let us take an element $x_1 \in A \setminus A_0$. By previous arguments A intersects only the second Cartesian line through x_1, therefore, if A_1 is the set of points in the second Cartesian line through x_1 that are in A including x_1 then the size of this set is also $r + 1$. In the same way as before, we take $x_2 \in A \setminus (A_0 \cup A_1)$ and define the set A_2 as the set of points in the second Cartesian line through x_2 that are in A including x_2 and again its size is $r + 1$.

The process is continued in this way until no more points can be taken in A, thus we get a set of points $x_0, x_1, ..., x_i \in A$ along with a collection of sets $A_0, A_1, ..., A_i$ for some natural number i, such that A_j is the intersection of the second Cartesian line through x_j with A. So, the size of A_j is $r + 1$ for all j. Also, $A = \bigcup_{j=0}^{j=i} A_j$ and by construction if $x_g \neq x_h$ then $A_g \neq A_h$ with $1 \leq g, h \leq i$.

It remains to prove that each pair of sets in this collection is disjoint, that is, if $A_e \neq A_f$ are two sets in the collection that was previously constructed, we must prove that $A_e \cap A_f = \emptyset$ with $1 \leq e, f \leq i$, $e \neq f$. Suppose that there exists an element $p \in A_e \cap A_f$, with $x_e := (a_e, b_e)$ and $x_f := (a_f, b_f)$. Then $p = (a_e, \mu) = (a_f, \nu)$ for some $\mu, \nu \in \Gamma$. We can see that $a_e = a_f$, which implies that x_e is in the second Cartesian line through x_f. This is a contradiction since $A_e \neq A_f$.

From all of the above we can conclude that we obtain a partition of the block A. We know that the size of A is k, and on the other hand $A = \bigcup_{j=0}^{j=i} A_j$.

They are all disjoint and the size of each A_j is $r+1$, so $k = i(r+1)$, and $r+1$ divides k. $\square$

Let (x, A) be a flag such as in Theorem 3.7, that is, A intersects only the second Cartesian line through x. We count the number of flags (y, C) such that $x \in C$ and $y \neq x$ is in the second Cartesian line through x.

The number of these flags is the same as the number of blocks that contain x as well as elements of the second Cartesian line through x, (we denote this number by z), multiplied by the number of elements of the second Cartesian line through x (excluding x) which are in these blocks, that is, r, therefore the number of such flags (y, C) is zr.

On the other hand, x and y are together in λ blocks and there are $m - 1$ points of the second Cartesian line through x, so when we count these flags (y, C) we obtain $\lambda(m - 1)$.

The above implies the equation $zr = \lambda(m - 1)$, but the equation $kr = 2\lambda(m - 1)$ also holds, hence $z = \frac{k}{2}$ and since $z \in \mathbb{N}$, k is even. This means that half of the blocks that contain x intersect with the second Cartesian line through x and the other half intersect with the first Cartesian line through x. This is possible since the previous argument is also valid for the first Cartesian line through x.

In the following theorem we examine Case (ii).

Theorem 3.8 (**Case** (ii))**.** *Let D be a (v, k, λ)-symmetric design with $v = m^2$ admitting a flag-transitive, point-primitive automorphism group with product action. If for every point*

x in the design, every block that contains it intersects with the two Cartesian lines through x, then $\frac{r}{2}+1$ divides k.

Proof. Let $x=(a_0,b_0)$ be an arbitrary point in the design, and let A be a block containing x, then there are r_1 elements of the first Cartesian line through x (excluding x) in A and there are r_2 elements of the second Cartesian line through x (excluding x) in A. The numbers r_1 and r_2 satisfy the equation $r=r_1+r_2$, by Lemma 2.2.

If C is another block containing x, then it must intersect the two Cartesian lines through x. Since G acts transitively on the flags, there is an element $g\in G$ such that $g(x,A)=(x,C)$ and from this we can see that $g(x)=x$, that is, $g|_\Gamma$ fixes a_0 and b_0.

First, we will prove that g sends the elements of the first Cartesian line through x which are also in A to elements of the first Cartesian line through x which are also in C. Let (μ,b_0) be an element of the first Cartesian line through x which is also in A. Then $g(\mu,b_0)=(\nu,b_0)\in C$ for some $\nu\in\Gamma$ since $g|_\Gamma$ fixes b_0. Similarly g sends the elements of the second Cartesian line through x which are also in A to elements of the second Cartesian line through x which are also in C. Let(a_0,μ) be an element of the first Cartesian line through x which is also in A, then $g(a_0,\mu)=(a_0,\nu)\in C$ for some $\nu\in\Gamma$ since $g|_\Gamma$ fixes a_0. Therefore the block C has as many elements of the first Cartesian line through x as A, and as many elements of the second Cartesian line through x as A. The above is true for every block that contains x.

Now let us count the number of flags (y,C) of the design such that $y\neq x$ is an element of the first Cartesian line through x and C is a block containing x. Every block contains r_1 elements of the first Cartesian line through x, when we exclude x, and there are k blocks containing x. All of them intersect the first Cartesian line through x, therefore there are kr_1 flags of this type. On the other hand y and x are together in λ blocks and there are $m-1$ elements of the first Cartesian line through x (excluding x), hence there are $\lambda(m-1)$ flags of this type. This yields the equation $kr_1=\lambda(m-1)$, but from Lemma 2.2 the equation $kr=2\lambda(m-1)$ also holds and we conclude that $r_1=\frac{r}{2}$. However $r_1+r_2=r$, so the intersection of every block containing x with the second Cartesian line through x (excluding x) has $r_2=\frac{r}{2}$ elements.

The above is true for every x, that is, for every point in the design, every block that contains it intersects the first Cartesian line through that point in $\frac{r}{2}$ other points and the same holds for the second Cartesian line through that point (excluding the point itself).

In what follows we will consider A_0 to be the set of points of the second Cartesian line through x which are also in A including x itself. The number of elements in that set is $\frac{r}{2}+1$. Let us consider $x_1\in A\setminus A_0$ so from the previous paragraphs A intersects the second Cartesian line through x_1 in $\frac{r}{2}$ elements, thus if A_1 is the set of points of the second Cartesian line through x_1 which are also in A including x_1 itself, the number of elements in A_1 is $\frac{r}{2}+1$. Now we take an element $x_2\in A\setminus(A_0\cup A_1)$ in the same way as before, and let A_2 be the set of points of the second Cartesian line through x_2 which are also in A including x_2 itself. The number of elements in A_2 is $\frac{r}{2}+1$.

We can continue this process in this way until there are no more elements in A, (everything is finite), so we obtain a collection of points $x_0,x_1,...,x_i\in A$ and a collection of sets $A_0,A_1,...,A_i$ for some natural number i such that for all $j=0,\dots,i$ A_j is the intersection of the second Cartesian line through x_j with A, and A_j has $\frac{r}{2}+1$ elements. By construction, $A=\bigcup_{j=0}^{j=i}A_j$ and the construction implies that if $x_g\neq x_h$ then $A_g\neq A_h$ with $1\leq g,h\leq i$.

It remains to prove that every two sets in this collection are disjoint, that is, we must prove that if $A_e \neq A_f$ then $A_e \cap A_f = \emptyset$ (with $1 \leq e, f \leq i$ and $e \neq f$). Suppose there is an element $p \in A_e \cap A_f$, with $x_e := (a_e, b_e)$ and $x_f := (a_f, b_f)$. Then $p = (a_e, \mu) = (a_f, \nu)$ for some $\mu, \nu \in \Gamma$. We can see that $a_e = a_f$, which implies that x_e is in the second Cartesian line through x_f, a contradiction since $A_e \neq A_f$.

Therefore we have a partition of the block $A = \bigcup_{j=0}^{j=i} A_j$. The size of $\bigcup_{j=0}^{j=i} A_j$ is $i(\frac{r}{2}+1)$ since they are all disjoint, and the size of A is k, therefore $k = i(\frac{r}{2}+1)$ and $\frac{r}{2}+1$ divides k. □

Now we will present some consequences of Theorem 3.8.

Corollary 3.9. *With the same hypotheses of* Theorem 3.8, $\frac{r}{2}+1$ divides m.

Proof. From (3.2) we have

$$k = \frac{r}{2}m + \frac{r}{2} + 1,$$

and there is an integer p such that $k = p(\frac{r}{2}+1)$. Substituting in the previous equation we obtain $(p-1)\left(\frac{r}{2}+1\right) = \frac{r}{2}m$. Since $\left(\frac{r}{2}+1, \frac{r}{2}\right) = 1$, $\frac{r}{2}+1$ necessarily divides m. □

Corollary 3.10. *With the same hypotheses of* Theorem 3.8, $\frac{r}{2}+1$ divides λ.

Proof. There is an integer p such that $k = p(\frac{r}{2}+1)$, and substituting this and (3.2) in $k(k-1) = \lambda(m-1)(m+1)$, we obtain

$$p\frac{r}{2}\left(\frac{r}{2}+1\right) = \lambda(m-1).$$

By Corollary 3.9, $\frac{r}{2}+1$ divides m, so $\left(\frac{r}{2}+1, m-1\right) = 1$ and $\frac{r}{2}+1$ divides λ. □

Since t is the greatest common divisor of k and λ, the following holds:

Corollary 3.11. *With the same hypotheses of Theorem 3.8, $\frac{r}{2}+1$ divides t.*

Proof. Since $\frac{r}{2}+1$ divides k and λ, and also $(k,\lambda) = t$ we conclude $\frac{r}{2}+1$ divides t. □

In the next results, we will introduce a particular case in which we have obtained the parameters of a Menon design, as a consequence of Corollary 3.11. Since $\frac{r}{2}+1$ divides $t > 1$, we will first consider the case in which t is a prime number. The following result is a first approach to our main result.

Lemma 3.12. *Let D be a (v,k,λ)-symmetric design with $(k,\lambda) = t > 1$ a prime number and $v = m^2$, admitting a flag-transitive, point-primitive automorphism group G. If for every point x in the design, every block that contains it intersects the two Cartesian lines through x, then either G does not have a product action or D is a Menon design.*

Proof. From Corollary 3.11, we have $\frac{r}{2}+1$ divides t. Since t is a prime we obtain $\frac{r}{2} = 0$ or $\frac{r}{2}+1 = t$.

If $\frac{r}{2} = 0$ then from (3.2) we have $k-1 = 0$ and this is impossible.

If on the other hand $\frac{r}{2}+1 = t$, then $m = \frac{r^2+2r+4\lambda}{4\lambda - r^2}$ so

$$m = \frac{bs^2+s+t}{t-bs^2} \tag{3.11}$$

which implies $t \geq bs^2$. If $t = bs^2$ then $\lambda = b^2s^2 = \frac{r^2}{4}$. Substituting this in (3.1) we obtain

$$r^2(m+1) + 2r - r^2(m-1) = 0, \text{ therefore } r(r+1) = 0,$$

so $r = 0$ or $r = -1$ which is a contradiction, so $t > bs^2$.

This forces $t = \frac{r}{2} + 1 = bs + 1 > bs^2$, so $1 > bs(s-1)$ and $s = 1$. From Lemma 3.3, $v = 4t^2$, $k = 2t^2 - t$ and $\lambda = t^2 - t$, which are the parameters of a Menon design. □

Remark 3.13. The triples of parameters $(4900, 3267, 2178)$, $(16900, 2752, 448)$, and $(44100, 8019, 1458)$, do not correspond to Menon designs but they satisfy all known necessary arithmetic conditions on the existence of a symmetric design with v even, so we do not prove the conjeture for $v \leq (210)^2$ (we have not tried computational methods).

Lemma 3.14. *Let D be a (v, k, λ)-symmetric design with $(k, \lambda) = t > 1$ and $v = m^2 \leq (210)^2$ with m even admitting a flag-transitive point-primitive automorphism group G, then either G does not have a non-trivial product action or one of the following conditions holds:*

1. *D is a Menon design with parameters $(4t^2, 2t^2 - t, t^2 - t)$, where $t > 1$, or*
2. *D has parameters $(16900, 2752, 448)$.*

Proof. For $m \leq 210$ the admissible parameters that do not satisfy the conditions of Menon designs and in which $k - \lambda$ is a square are $(4900, 3267, 2178)$, $(16900, 2752, 448)$, and $(44100, 8019, 1458)$. For these, $\frac{r}{2} + 1$ is $47, 22$ and 39 respectively, so they do not satisfy Theorem 3.8. Now for those parameters $r + 1$ is $93, 43$ and 78 respectively. The first and third of them do not satisfy Theorem 3.7, but the parameters $(16900, 2752, 448)$ do. Thus, these are the only possible parameters for $m^2 \leq (210)^2$.

In this case, k is even, which is consistent with Theorem 3.7. We also have $s = 3$ so from Lemma 3.3 we know that these parameters cannot correspond to a Menon design. □

Remark 3.15. The triple $(16900, 2752, 448)$ does not correspond to a Menon design since $s = 3$, although it satisfies all the arithmetic conditions for a symmetric design. We make no claim as to whether such a design exists, but perhaps it is not the case that when $l = 2$ only Menon designs are possible (if at all).

The following is our main result, the proof follows Cases (i) and (ii) from Theorems 3.7 and 3.8, that is, either: there is a flag (x, A), such that the block A intersects only one Cartesian line through x (Case (i)), or for every point x, every block that contains it intersects both of the Cartesian lines through x (Case (ii)). The proof based on Case (i) is similar to the proof of the case in which $m - 1$ is the power of an odd prime. In this sense it is a generalisation of this proof, but because of the existence of the parameter s this generalisation was not obtained in an obvious way. For this reason, we need an additional arithmetic condition, which is found in Corollary 3.11.

In the proof based on Case (ii) we also obtain an arithmetic condition for a and so also for k. We believe we do not necessarily obtain parameters for Menon designs for an arbitrary λ when we study symmetric designs admitting a flag-transitive point-primitive automorphism group with product action when $l = 2$. This case also gives us a parametrisation of (v, k, λ) in terms of t and s, and if $s = 1$ then the parameters correspond to Menon designs, that is, our parameterisation is a generalisation of the parameterisation of Menon designs.

Theorem 3.16. *Let D a (v, k, λ)-symmetric design admitting a flag-transitive, point-primitive, automorphism group G with $(k, \lambda) = t > 1$ and $v = m^2$ with m even. Then either:*

(i) *G does not have a non-trivial product action, or*

(ii) *D is a design with parameters $\left((2t+s-1)^2, \frac{2t^2-(2-s)t}{s}, \dfrac{t^2-t}{s^2}\right)$ with $s \geq 1$ odd.*

When $s = 1$ D is a Menon design and if $s > 1$ then t is even.

Proof. From the hypotheses and from Lemma 2.2 there are two possible cases. For any given point, either each block that contains it intersects the two Cartesian lines through it, or there is a point such that a block containing it only intersects one Cartesian line through it. As we have seen, the latter implies that every block intersects only one Cartesian line through each point it contains.

First we will study this last case. Here, Theorem 3.7 is satisfied, so $r + 1$ divides k with $r > 1$ an integer such that $kr = 2\lambda(m-1)$. This implies there is an integer p such that $k = p(r + 1)$. Also $k - 1 = \frac{r}{2}(m + 1)$ holds, so $k = \frac{r}{2}m + \frac{r}{2} + 1$, and therefore

$$m - 1 = (r + 1)(m + 1 - 2p).$$

Then $r + 1$ divides $m - 1$, but $m - 1 = as = x(r + 1)$ where $x := m + 1 - 2p$ and since $r = 2bs$ we have $(r + 1, s) = 1$ so $r + 1$ divides a. Also a divides $r + 1$ which forces $r + 1 = a$, and this implies $k = (r + 1)t$. This all implies $t - bs^2 = 1$, so $b = \frac{t-1}{s^2}$, and we obtain the parameters $\lambda = \frac{t-1}{s^2}t$, $k = \frac{2t+s-2}{s}t$, $v = (2t + s - 1)^2$.

The proof of Theorem 3.7 states that k should be an even number and since $r + 1$ is an odd number then t should be an even number. Since $m = 2t + s - 1$ and m is an even number then s is an odd number.

The triple $(16900, 2752, 448)$ satisfies the conditions we obtained, with $t = 64$ and $s = 3$, so this is not a Menon design.

When $s = 1$ we obtain the parametrisation for Menon designs with t an even number.

Now suppose that for every point, every block that contains it intersects both Cartesian lines through it. Here the hypotheses of Theorem 3.8 hold and from Corollary 3.11, there exists $x \geq 1$ such that

$$t = \left(\frac{r}{2} + 1\right)x = (bs + 1)x. \tag{3.12}$$

From (3.11) we obtain $m = \frac{s(bs+1)+t}{t-bs^2} = \frac{s(bs+1)+(bs+1)x}{t-bs^2}$, that is

$$m = \left(\frac{s + x}{t - bs^2}\right)\left(\frac{r}{2} + 1\right). \tag{3.13}$$

Using (3.12) we obtain

$$t - bs^2 = x + bs(x - s), \tag{3.14}$$

which we divide into the following cases:

1. $x < s$
   From Lemma 3.12, we have $t - bs^2 > 0$, and from (3.14) $x > bs(s-x) > bx(s-x)$. Since $x < s$ then $1 > b(s-x) > 0$, but this cannot be the case since $b(s-x)$ should be an integer.

2. $s < x$
   From (3.13), $s + x \geq t - bs^2$. If $s + x = t - bs^2$ we have from (3.13) that $m = \frac{r}{2} + 1$ so $as = m - 1 = \frac{r}{2} = bs$ and therefore $a = b$, but $(a, b) = 1$ so $a = 1$ this is impossible by Lemma 3.1. Therefore $s + x > t - bs^2$, and from (3.14) we have $s + x > x + bs(x-s)$ so $1 > b(x-s) > 0$ and this is also imposisble since $b(x-s)$ is an integer.

3. $s = x$
   From (3.12), s divides t, and from (3.2) $at - bs(as + 2) = 1$, so $(t, s) = 1$ and therefore $s = 1$. Also from Lemma 3.3, $v = 4t^2$, $k = 2t^2 - t$, and $\lambda = t^2 - t$

This concludes the proof. □

The parameters of Menon designs are not the only ones we can obtain when we assume that the automorphism group of the design has a product action on the points of the design, and the parameters $(16900, 2752, 448)$ are an example of this. However we note that a design with the possible parameters which arise and do not correspond to Menon designs must satisfy that each block only intersects one Cartesian line through each point in that block.

It is not the case that the way in which we consider product action to obtain possible Menon designs does not work because here is a potential counterexample, but rather that with this theorem we give explicit expressions for the parameters v, k, λ, in terms of parameters s, t, and when $s = 1$ they do correspond to Menon designs.

4 One further result

Here we present an additional result for any $l \geq 2$.

Theorem 4.1. *Let D a (v, k, λ)-symmetric design with $v = m^l$ admitting a flag-transitive, point-primitive, automorphism group G with a non-trivial product action. Then the complement of the design is not flag-transitive.*

Proof. Suppose that D' is the complement of the design D, so its parameters are $(v', k', \lambda') = (v, v - k, v - 2k + \lambda)$. If we also assume D' is flag-transitive, then the following equation holds:

$$(v - k)(v - k - 1) = (v - 2k + \lambda)(m - 1)(m^{l-1} + m^{l-2} + ... + 1). \tag{4.1}$$

If D has a point-primitive automorphisms group G, then D' has the same point-primitive automorphisms group G and we can consider the Cartesian lines through a point, since G is transitive on the points of D'. Thus k' divides $\lambda' l(m-1)$, so there is an integer p such that

$$(v - k)p = l(v - 2k + \lambda)(m - 1). \tag{4.2}$$

Substituting this in (4.1) we obtain

$$l(v-k)(v-1-k) = (v-k)p(m^{l-1} + m^{l-2} + ... + 1)$$
$$\text{so} \quad l((m-1)(m^{l-1} + m^{l-2} + ... + 1) - k) = p(m^{l-1} + m^{l-2} + ... + 1),$$

hence

$$lk = q(m^{l-1} + m^{l-2} + ... + 1) \tag{4.3}$$

with $q = l(m-1) - p > 0$.

But for D we know that $k = \dfrac{l\lambda(m-1)}{r}$ and $k(k-1) = \lambda(v-1)$ so

$$k(k-1) = \frac{kr}{l(m-1)}(m^l - 1)$$

and we obtain a generalisation of (3.2):

$$l(k-1) = r(m^{l-1} + m^{l-2} + ... + 1). \tag{4.4}$$

If we substitute (4.3) in (4.4) then

$$q(m^{l-1} + m^{l-2} + ... + 1) - l = r(m^{l-1} + m^{l-2} + ... + 1)$$
$$\text{so} \quad (q-r)(m^{l-1} + m^{l-2} + ... + 1) = l,$$

and therefore $m^{l-1} + m^{l-2} + ... + 1 \leq l$ if $m > 1$ and $l \geq m^{l-1} + m^{l-2} + ... + 1 > l$, which is impossible.

We conclude $m \leq 1$, but this is a contradiction since $m \geq 5$. □

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Mutually orthogonal cycle systems*

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Abstract

An ℓ-cycle system $\mathcal{F}$ of a graph Γ is a set of ℓ-cycles which partition the edge set of Γ. Two such cycle systems $\mathcal{F}$ and $\mathcal{F}'$ are said to be *orthogonal* if no two distinct cycles from $\mathcal{F} \cup \mathcal{F}'$ share more than one edge. Orthogonal cycle systems naturally arise from face 2-colourable polyehdra and in higher genus from Heffter arrays with certain orderings. A set of pairwise orthogonal ℓ-cycle systems of Γ is said to be a set of mutually orthogonal cycle systems of Γ.

Let $\mu(\ell, n)$ (respectively, $\mu'(\ell, n)$) be the maximum integer μ such that there exists a set of μ mutually orthogonal (cyclic) ℓ-cycle systems of the complete graph K_n. We show that if $\ell \geq 4$ is even and $n \equiv 1 \pmod{2\ell}$, then $\mu'(\ell, n)$, and hence $\mu(\ell, n)$, is bounded below by a constant multiple of n/ℓ^2. In contrast, we obtain the following upper bounds: $\mu(\ell, n) \leq n-2$; $\mu(\ell, n) \leq (n-2)(n-3)/(2(\ell-3))$ when $\ell \geq 4$; $\mu(\ell, n) \leq 1$ when $\ell > n/\sqrt{2}$; and $\mu'(\ell, n) \leq n-3$ when $n \geq 4$. We also obtain computational results for small values of n and ℓ.

Keywords: Orthogonal cycle decompositions, cyclic cycle systems, Heffter arrays, completely-reducible, super-simple.

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1 Introduction

We say that a graph Γ *decomposes* into subgraphs $\Gamma_1, \Gamma_2, \dots, \Gamma_t$, if the edge sets of the Γ_i partition the edges of Γ. If $\mathcal{F} = \{\Gamma_i \mid 1 \le i \le t\}$ where $\Gamma_i \cong H$ for each $1 \le i \le t$, then we say that $\mathcal{F}$ is an H-decomposition of Γ. An *ℓ-cycle system* of a graph Γ is a decomposition of Γ into ℓ-cycles. In the case where Γ is the complete graph K_n we say that there is an ℓ-cycle system of *order* n. Necessary and sufficient conditions for the existence of an ℓ-cycle system of order n were given in [1, 26]; see also [6]. Namely, at least one ℓ-cycle system of order $n > 1$ exists if and only if $3 \le \ell \le n$, $n(n-1) \equiv 0 \pmod{2\ell}$ and n is odd.

Two ℓ-cycle systems $\mathcal{F}$ and $\mathcal{F}'$ of the same graph Γ are said to be *orthogonal* if, for all cycles $C \in \mathcal{F}$ and $C' \in \mathcal{F}'$, C and C' share at most one edge. A set of pairwise orthogonal ℓ-cycle systems of Γ is said to be a set of *mutually orthogonal* cycle systems of Γ. In this paper we are interested in the maximum μ such that there exists a set of μ mutually orthogonal ℓ-cycle systems of order n; we denote this value by $\mu(\ell, n)$.

In the array below we exhibit a set of four mutually orthogonal cycle systems of order 9. We have determined computationally that $\mu(4, 9) = 4$; i.e., this set is maximum.

$$\begin{aligned}
&\{(1,2,3,4),(1,3,6,5),(1,6,2,7),(1,8,2,9),(2,4,7,5),(3,5,8,7),(3,8,6,9),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4,5,9,8),(4,6,7,9)\},\\
&\{(1,2,6,8),(1,3,5,7),(1,4,8,5),(1,6,5,9),(2,3,6,4),(2,5,4,9),(2,7,3,8),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3,4,7,9),(6,7,8,9)\},\\
&\{(1,2,8,7),(1,3,4,6),(1,4,9,8),(1,5,3,9),(2,3,8,5),(2,4,5,6),(2,7,5,9),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3,6,9,7),(4,7,6,8)\},\\
&\{(1,2,9,3),(1,4,6,9),(1,5,4,7),(1,6,3,8),(2,3,7,6),(2,4,8,7),(2,5,6,8),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3,4,9,5),(5,7,9,8)\}.
\end{aligned}$$

Orthogonal cycle systems arise from face 2-colourable embeddings of graphs on surfaces, which satisfy two conditions natural to polyhedra and similar phenomena: each pair of faces share at most one edge and each edge belongs to exactly two faces.

Let μK_n be the multigraph in which each edge of K_n is replaced by μ parallel edges. A decomposition $\mathcal{F}$ of μK_n into a subgraph H is said to be *super-simple* if no two copies of H share more than one edge, and *completely-reducible* if $\mathcal{F}$ partitions into μ decompositions of K_n. It follows that a set of μ mutually orthogonal cycle systems of K_n is equivalent to a completely-reducible super-simple decomposition of μK_n into cycles; see [12] for more details.

In the case $\ell = 3$, observe that a pair of ℓ-cycle systems is orthogonal if and only if the cycle systems are disjoint. It is not hard to see that there are at most $n - 2$ pairwise disjoint triple systems of order n; a set of systems which meets this bound is called a *large set* of disjoint Steiner triple systems, or $\mathrm{LTS}(n)$. An $\mathrm{LTS}(7)$ does not exist [13]; however

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in [23, 24], it is shown that an LTS(n) exists if and only if $n > 7$ and $n \equiv 1$ or $3 \pmod 6$, except for a finite list of possible exceptions. The exceptional cases are all solved in [27].

In this paper, we are often interested in *cyclic* cycle systems of the complete graph K_n. Let G be an additive group of order n and suppose K_n has vertex set G. Given a cycle $C = (c_0, c_1, \ldots, c_{\ell-1})$ in K_n, for each element $g \in G$, define the cycle $C + g = (c_0 + g, c_1 + g, \ldots, c_{\ell-1} + g)$. We say that a cycle system $\mathcal{F}$ of K_n is *G-regular* if, for any $C \in \mathcal{F}$ and $g \in G$, we have that $C + g \in \mathcal{F}$. In the case that G is a cyclic group, we refer to a $\mathbb{Z}_n$-regular cycle system as *cyclic*. In a cyclic cycle system $\mathcal{F}$, the *orbit* of the cycle $C \in \mathcal{F}$ is the set of cycles $\{C + g \mid g \in \mathbb{Z}_n\}$; a cyclic cycle system can be completely specified by listing a set of *starter cycles*, that is, a set of representatives for the orbits of the cycles under the action of $\mathbb{Z}_n$.

The existence problem for cyclic cycle systems has attracted much attention. Clearly, in order for a cyclic ℓ-cycle system of odd order n to exist, we must have that $3 \leq \ell \leq n$ and ℓ divides $n(n-1)/2$. However, additional conditions for existence also come into play. There is no cyclic ℓ-cycle system of order n when $(\ell, n) \in \{(3, 9), (15, 15)\}$; $\ell = n = p^m$ for some prime p and integer $m \geq 2$; or $\ell < n < 2\ell$ and $\gcd(\ell, n)$ is a prime power [7, 9]. Buratti [7] has conjectured that a cyclic ℓ-cycle system of order n exists for any other admissible pair (ℓ, n); this conjecture is still open. The existence problem for cyclic cycle systems of the complete graph has been solved in a number of cases, including when $n \equiv 1$ or $\ell \pmod{2\ell}$ [8, 9, 22, 25, 28] (see also [4, 5, 18]), $\ell \leq 32$ [31, 32], ℓ is twice or thrice a prime power [30, 31, 32], or ℓ is even and $m > 2\ell$ [29].

We explore the maximum μ' such that there exists a set of μ' mutually orthogonal *cyclic* ℓ-cycle systems of order n; this value is denoted by $\mu'(\ell, n)$. Pairs of orthogonal cyclic cycle systems of the complete graph arise from Heffter arrays with certain orderings. A *Heffter array* $H(n; k)$ is an $n \times n$ matrix such that each row and column contains k filled cells, each row and column sum is divisible by $2nk + 1$ and either x or $-x$ appears in the array for each integer $1 \leq x \leq nk$. A Heffter array is said to have a *simple ordering* if, for each row and column, the entries may be cyclically ordered so that all partial sums are distinct modulo $2nk + 1$. The following was first shown by Archdeacon [2] as part of a more general result; consult [11] to see this result stated more explicitly.

Theorem 1.1. *If $H(n; k)$ is a Heffter array with a simple ordering, then there exists a pair of orthogonal cyclic decompositions of K_{2nk+1} into k-cycles. In particular, $\mu'(k, 2nk + 1) \geq 2$.*

Thus the following is implied by existing literature on Heffter arrays.

Theorem 1.2 ([3, 11, 14, 17])**.** *Let $n \geq k$. Then $\mu'(k, 2nk + 1) \geq 2$ whenever:*

- $k \in \{3, 5, 7, 9\}$ *and* $nk \equiv 3 \pmod 4$*;*
- $k \equiv 0 \pmod 4$*;*
- $n \equiv 1 \pmod 4$ *and* $k \equiv 3 \pmod 4$*;*
- $n \equiv 0 \pmod 4$ *and* $k \equiv 3 \pmod 4$ *(for large enough n).*

With an extra condition on the orderings of the entries of a Heffter array, these orthogonal cycle systems in turn biembed to yield a face 2-colourable embedding on an orientable surface. Face 2-colourable embeddings on orientable surfaces have been studied for a variety of combinatorial structures [16, 19, 20, 21]. Recently, Costa, Morini, Pasotti and

Pellegrini [15] employed a generalization of Heffter arrays to construct pairs of orthogonal ℓ-cycle systems of the complete multipartite graph in certain cases.

In [12], it is shown that for every graph H and fixed integer $k \geq 1$, for sufficiently large n (satisfying some elementary necessary divisibility conditions), there exists a set of k pairwise orthogonal decompositions of K_n into H (i.e., no two copies of H share more than one edge). Aside from this quite general asymptotic result, to our knowledge, sets of mutually orthogonal ℓ-cycle systems of size greater than 2 have not been studied for $\ell \geq 4$.

In this paper, our focus for cyclic cycle systems is in the case $n \equiv 1 \pmod{2\ell}$, for which it is possible to construct a cyclic ℓ-cycle system with no short orbit. In particular, we will find lower bounds on $\mu(\ell, n)$ by constructing sets of mutually orthogonal cyclic even cycle systems. Specifically, we show that if ℓ is even and $n \equiv 1 \pmod{2\ell}$, then $\mu'(\ell, n)$ is bounded below by a constant multiple of n/ℓ^2, i.e., $\mu'(\ell, n) = \Omega(n/\ell^2)$. Our main result is as follows.

Theorem 1.3. *If $\ell \geq 4$ is even, $n \equiv 1 \pmod{2\ell}$ and $N = (n-1)/(2\ell)$, then*

$$\mu(\ell, n) \geq \mu'(\ell, n) \geq \frac{N}{a\ell + b} - 1,$$

where

$$(a, b) = \begin{cases} (4, -2), & \text{if } \ell \equiv 0 \pmod 4, \\ (24, -18), & \text{if } \ell \equiv 2 \pmod 4. \end{cases}$$

In Section 2, when $\ell = 4$, we improve the bound of Theorem 1.3 to $\mu(\ell, n) \geq \mu'(\ell, n) \geq 4N$ (Lemma 2.1). Section 3 establishes some notation and preliminary results. The general result for $\ell \equiv 0 \pmod 4$ is proved in Section 4 (Theorem 4.3), while the bound for $\ell \equiv 2 \pmod 4$ is proved in Section 5 (Theorem 5.5). In contrast, in Section 6 we establish upper bounds, namely $\mu(\ell, n) \leq n - 2$; $\mu(\ell, n) \leq (n-2)(n-3)/(2(\ell-3))$ for $\ell \geq 4$; $\mu(\ell, n) \leq 1$ for $\ell > \sqrt{n(n-1)/2}$; and $\mu'(\ell, n) \leq n - 3$ for $n \geq 4$. Finally, computational results for small values are given in the appendix.

2 Mutually orthogonal 4-cycle systems

Clearly $n \equiv 1 \pmod 8$ is a necessary condition for a decomposition of K_n into 4-cycles, cyclic or otherwise. Let $[a, b, c, d]_n$ denote the $\mathbb{Z}_n$-orbit of the 4-cycle $(0, a, a+b, a+b+c)$, where $a+b+c+d$ is divisible by n. Observe that $[a, b, c, d]_n = [-d, -c, -b, -a]_n$. Where the context is clear, we write $[a, b, c, d]_n = [a, b, c, d]$. Let $D_n = \{1, 2, \ldots, (n-1)/2\}$; that is, D_n is the set of *differences* in $\mathbb{Z}_n$. We consider $\mathbb{Z}_n$ as the set $\pm D_n \cup \{0\}$.

By observation, the maximum size of a set of mutually orthogonal cyclic 4-cycle systems of K_9 is $\mu'(4, 9) = 2$. Two such systems are $[1, -2, 4, -3]_9$ and $[1, -3, 4, -2]_9$. In the non-cyclic case, an exhaustive computational search indicates that the maximum size of a set of mutually orthogonal 4-cycle systems of K_9 is $\mu(4, 9) = 4$; see the example given in Section 1.

Lemma 2.1. *If $n \equiv 1 \pmod 8$ and $n \geq 17$, then there exists a set of $(n-1)/2$ mutually orthogonal cyclic 4-cycle systems of order n. In particular, $\mu'(4, n) \geq (n-1)/2$.*

Proof. We first describe how to construct a set of $(n-5)/2$ mutually orthogonal cyclic 4-cycle systems; then we add two more by making some adjustments.

Let $N = (n-1)/8$. For each i, j with $1 \leq i < j \leq 2N$, let $C_{i,j}$ and $C'_{i,j}$ be the pair of orbits of 4-cycles:

$$C_{i,j} := \{[2i-1, 2j, -2i, -(2j-1)]\}, \quad C'_{i,j} := \{[2i-1, -(2j-1), -2i, 2j]\}.$$

Next, let $F_1, F_2, \dots F_{2N-1}$ be a set of 1-factors which decompose the complete graph on vertex set $\{1, 2, \dots, 2N\}$.

For each 1-factor F_k, the sets

$$\mathcal{F}_k := \bigcup_{\substack{\{i,j\} \in F_k \\ i<j}} C_{i,j} \quad \text{and} \quad \mathcal{F}'_k := \bigcup_{\substack{\{i,j\} \in F_k \\ i<j}} C'_{i,j}$$

each describe a cyclic decomposition of K_n into 4-cycles. Observe that the set of such decompositions constitutes a mutually orthogonal set of size $4N - 2 = (n-5)/2$.

We next make an adjustment to extend this set. Without loss of generality, let $F_1 = \{\{1,2\}, \{3,4\}, \dots, \{2N-1, 2N\}\}$. Replace $\mathcal{F}_1$ and $\mathcal{F}'_1$ with:

$$\mathcal{F}_* = \{[4i-3, -(4i-2), -(4i-1), 4i] \mid 1 \leq i \leq N\},$$
$$\mathcal{F}'_* = \{[4i-3, 4i, -(4i-1), -(4i-2)] \mid 1 \leq i \leq N\}.$$

Then, we can add another pair of cyclic decompositions, orthogonal to each decomposition in $\{\mathcal{F}_*, \mathcal{F}'_*, \mathcal{F}_2, \dots, \mathcal{F}_{2N-1}, \mathcal{F}'_2, \dots, \mathcal{F}'_{2N-1}\}$, given by:

$$\mathcal{F}_{2N} := \{[1, -3, 4N, -(4N-2)]\} \cup \{[4i+1, -(4i+3), 4i, -(4i-2)] \mid 1 \leq i < N\}$$

and

$$\mathcal{F}'_{2N} := \{[1, -(4N-2), 4N, -3]\} \cup \{[4i+1, -(4i-2), 4i, -(4i+3)] \mid 1 \leq i < N\}.$$

(Note that orthogonality requires $N \geq 2$ at this final step.) □

In the case $n = 17$, we have computationally determined that $\mu'(4, 17) = 10$, which improves on the bound given in Lemma 2.1. A set of ten mutually orthogonal cyclic 4-cycle systems of order 17 is given in the appendix.

We exhibit the methods of the previous proof in the case $n = 25$. We start with a 1-factorization of K_6:

$$F_1 = \{\{1,2\}, \{3,4\}, \{5,6\}\},$$
$$F_2 = \{\{1,3\}, \{2,6\}, \{4,5\}\},$$
$$F_3 = \{\{1,4\}, \{2,5\}, \{3,6\}\},$$
$$F_4 = \{\{1,5\}, \{2,3\}, \{4,6\}\},$$
$$F_5 = \{\{1,6\}, \{2,4\}, \{3,5\}\}.$$

The resulting 12 mutually orthogonal cyclic 4-cycle systems of order 25 are given by:

$$
\begin{aligned}
\mathcal{F}_* &= \{[1,-2,-3,4],[5,-6,-7,8],[9,-10,-11,12]\},\\
\mathcal{F}'_* &= \{[1,4,-3,-2],[5,8,-7,-6],[9,12,-11,-10]\},\\
\mathcal{F}_2 &= \{[1,6,-2,-5],[3,12,-4,-11],[7,10,-8,-9]\},\\
\mathcal{F}'_2 &= \{[1,-5,-2,6],[3,-11,-4,12],[7,-9,-8,10]\},\\
\mathcal{F}_3 &= \{[1,8,-2,-7],[3,10,-4,-9],[5,12,-6,-11]\},\\
\mathcal{F}'_3 &= \{[1,-7,-2,8],[3,-9,-4,10],[5,-11,-6,12]\},\\
\mathcal{F}_4 &= \{[1,10,-2,-9],[3,6,-4,-5],[7,12,-8,-11]\},\\
\mathcal{F}'_4 &= \{[1,-9,-2,10],[3,-5,-4,6],[7,-11,-8,12]\},\\
\mathcal{F}_5 &= \{[1,12,-2,-11],[3,8,-4,-7],[5,10,-6,-9]\},\\
\mathcal{F}'_5 &= \{[1,-11,-2,12],[3,-7,-4,8],[5,-9,-6,10]\},\\
\mathcal{F}_6 &= \{[1,-3,12,-10],[5,-7,4,-2],[9,-11,8,-6]\},\\
\mathcal{F}'_6 &= \{[1,-10,12,-3],[5,-2,4,-7],[9,-6,8,-11]\}.
\end{aligned}
$$

Through computational means we determined that this collection of 12 mutually orthogonal cyclic 4-cycle systems of order 25 is maximal. However, it is not maximum, as we also established computationally that $\mu'(4,25) \geq 17$.

3 Preliminary lemmas for cycle length greater than 4

In this section, we introduce notation and basic results which will be needed later to construct mutually orthogonal cycle systems with even cycle length $\ell \geq 6$.

Henceforth, for any integers a and b with $a \leq b$, $[a,b]$ is the set of integers $\{a, a+1, \ldots, b\}$. For $a, b \in \mathbb{R}$ with $a < b$, we also use the notation (a,b) to denote the set of *integers* strictly between a and b.

Let the vertices of the complete graph K_n be labelled with $[0, n-1]$, where n is odd. Then the *difference* associated with an edge $\{a,b\}$ is defined to be the minimum value in the set $\{|a-b|, n-|a-b|\}$. Let e_1 and e_2 be two edges of differences d and e, respectively. Then we may write $e_1 = \{a, a+d \pmod n\}$ and $e_2 = \{b, b+e \pmod n\}$, where $a, b \in [0, n-1]$ are uniquely determined. We define the *distance* between e_1 and e_2 to be the minimum value in the set $\{|a-b|, n-|a-b|\}$. Given a cycle C with vertices in $\mathbb{Z}_n$, the set ΔC is defined to be the multiset of differences of the edges of C.

The idea is to construct cyclic systems using so-called *balanced* sets of differences. The following definitions and lemma appear in [10].

Definition 3.1. If $D = \{d_1, d_2, \ldots, d_{2k}\}$ is a set of positive integers, with $d_i < d_{i+1}$ for $i \in [1, 2k-1]$, the *alternating difference pattern* of D is the sequence $(s_1, s_2, \ldots, s_k)$ where $s_i = d_{2i} - d_{2i-1}$ for every $i \in [1,k]$. Furthermore, D is said to be *balanced* if there exists an integer $\tau \in [1,k]$ such that $\sum_{i=1}^{\tau} s_i = \sum_{i=\tau+1}^{k} s_i$.

Definition 3.2. Let $D = \{d_1, d_2, \ldots, d_{2k}\}$ be a balanced set of positive integers. Let $\delta_1, \delta_2, \ldots, \delta_{2k}$ be the sequence obtained by reordering the integers in D as follows:

$$
\delta_i = \begin{cases} d_i & \text{if } 1 \leq i \leq 2\tau - 1,\\ d_{i+1} & \text{if } 2\tau \leq i \leq 2k-1,\\ d_{2\tau} & \text{if } i = 2k. \end{cases}
$$

Set $c_0 = 0$ and $c_i = \sum_{h=1}^{i}(-1)^h \delta_h$ for $1 \le i \le 2k-1$. We then define $C(D) := (c_0, c_1, \ldots, c_{2k-1})$.

Lemma 3.3 (Lemma 3.2 of [10])**.** *Let $k \ge 2$. If D is a balanced set of $2k$ positive integers, then $C(D)$ is a $2k$-cycle satisfying $\Delta C(D) = D$ and vertex set $V(C(D)) \subset [-d, d']$, where $d = \max D$ and $d' = \max(D \setminus \{d\})$.*

Corollary 3.4. *Let $k \ge 2$ and $n \equiv 1 \pmod{4k}$. Suppose that the set $[1,(n-1)/2]$ partitions into sets $D_1, D_2, \ldots, D_{(n-1)/(4k)}$, each of which is balanced and of size $2k$. Then cycles $C(D_i)$, $i \in [1,(n-1)/(4k)]$, form a set of starter cycles for a cyclic $2k$-cycle decomposition of K_n; in particular, the set*

$$\{C(D_i) + j \mid i \in [1,(n-1)/4k], j \in [0, n-1]\}$$

is a cyclic decomposition of K_n into $2k$-cycles.

Proof. Let $i \in [1,(n-1)/2k]$. Since $D_i \subset [1,(n-1)/2]$, Lemma 3.3 implies that $V(C(D_i)) \subset [-(n-1)/2, (n-1)/2]$. Thus the vertices of $V(C(D_i))$ are distinct in $\mathbb{Z}_n$. The result follows. □

Our general strategy will be to show that a pair of cyclically generated cycle systems is orthogonal by showing that the sets of differences from any two cycles in different orbits share at most one element. To this end, the following lemma will be used in Sections 4 and 5.

Lemma 3.5. *Let $\delta, N > 0$ and suppose there exist integers d and d' such that $d, d' \in (N/2 - \delta N, N/2 + \delta N)$. If α and α' are integers such that $1 \le \alpha < \alpha' \le (1-2\delta)/4\delta$, then $\alpha d < \alpha' d'$.*

Proof. Note that if $\delta > \frac{1}{10}$, then the result is vacuously true since $(1-2\delta)/4\delta < 2$. So we assume $\delta \le \frac{1}{10}$. For each positive integer s, define

$$I_s = \{si \mid N/2 - \delta N < i < N/2 + \delta N; i \in \mathbb{R}\}.$$

Let $m = \lfloor \frac{1-2\delta}{4\delta} \rfloor$, and let $S = [1, m]$. Observe that $\alpha, \alpha' \in S$. Now, $\delta \le 1/(4m+2)$ implies that:

$$\begin{gathered} m(1+2\delta) \le (m+1)(1-2\delta) \\ \Rightarrow m(N/2 + \delta N) \le (m+1)(N/2 - \delta N). \end{gathered}$$

It follows that for each $s \in S$, every element of I_s is strictly less than every element of I_{s+1}. Since $\alpha d \in I_\alpha$ and $\alpha' d' \in I_{\alpha'}$, it follows that $\alpha d < \alpha' d'$. □

The following variation of Lemma 3.5 will be used in Section 5.

Corollary 3.6. *Let $\delta, N > 0$ and suppose there exist integers d and d' such that $d, d' \in (N/3 - \delta N, N/3 + \delta N)$. If α and α' are integers such that $1 \le \alpha < \alpha' \le (1-3\delta)/6\delta$, then $\alpha d < \alpha' d'$.*

Proof. If m is a positive integer, $m \le (1-3\delta)/6\delta$ implies that

$$m(N/3 + \delta N) \le (m+1)(N/3 - \delta N).$$

The remaining argument is similar to the previous lemma. □

4 Orthogonal sets of $4k$-cycle systems with $k \geq 2$

Our aim in this section is to prove Theorem 4.3. In particular, for each $k \geq 2$ and $n \equiv 1 \pmod{8k}$, we will show that $\mu'(n, 4k) = \Omega(n/k^2)$. That is, we construct a set of mutually orthogonal $4k$-cycle decompositions of K_n of size at least cn/k^2 where c is a constant. In particular, the number of mutually orthogonal decompositions constructed is

$$\left\lceil \frac{n-1}{8k(16k-2)} - 1 \right\rceil;$$

thus we have at least two orthogonal decompositions whenever

$$\frac{n-1}{8k(16k-2)} > 2,$$

or equivalently $n - 1 > 32k(8k - 1)$.

Let N and k be positive integers and let $n = 8kN + 1$. For each integer $d \in (N/2 - N/(16k-2), N/2)$ (there is at least one such integer whenever $N > 16k-2$), we construct a cyclic $4k$-cycle decomposition of K_n which we will denote by $\mathcal{F}(d)$.

The first d starter cycles in $\mathcal{F}(d)$ use the set of differences $[1, 4kd]$. For $i \in [1, d]$, let

$$S_{d,i} = \{i, d + i, 2d + i, \ldots, (4k - 1)d + i\}.$$

Observe that the set $S_{d,i}$ is balanced, with $\tau = k$, for each $i \in [1, d]$.

Henceforth in this section, let $e := N - d$. (In effect, e is a function of d.) Observe that $e \in (N/2, N/2 + N/(16k - 2))$. The remaining e starter cycles in $\mathcal{F}(d)$ use differences $[4kd + 1, 4kN]$. For $i \in [1, e]$, take

$$T_{e,i} = \{4kd + i, 4kd + e + i, 4kd + 2e + i, \ldots, 4kd + (4k - 1)e + i\}.$$

Observe that the set $T_{e,i}$ is balanced for each $i \in [1, e]$, where $\tau = k$. Moreover, since $4kd + 4ke = 4kN$, we have that

$$\left(\bigcup_{i=1}^{d} S_{d,i}\right) \cup \left(\bigcup_{i=1}^{e} T_{e,i}\right) = [1, 4kN],$$

so by Corollary 3.4, the set of cycles

$$\mathcal{F}(d) := \{C(S_{d,i}) \mid i \in [1, d]\} \cup \{C(T_{e,i}) \mid i \in [1, e]\}$$

is a set of starter cycles for a cyclic $4k$-cycle system of order $n = 8kN + 1$.

In order to show that we have constructed an orthogonal set of decompositions, we will make use of the following, which is a direct consequence of Lemma 3.5.

Lemma 4.1. *Suppose $d, d' \in (N/2 - N/(16k - 2), N/2)$ such that $d \neq d'$, and let $e = N - d$ and $e' = N - d'$. Let $\alpha, \alpha' \in [1, 4k - 1]$. Then no two of αd, $\alpha' d'$, αe and $\alpha' e'$ are equal. Moreover, if $\alpha < \alpha'$ then $\alpha d < \alpha' d'$ and $\alpha e < \alpha' e'$.*

Lemma 4.2. *Suppose $d, d' \in (N/2 - N/(16k - 2), N/2)$ such that $d \neq d'$. Then the decompositions $\mathcal{F}(d)$ and $\mathcal{F}(d')$, as defined above, are orthogonal.*

Proof. In what follows, $d \neq d'$, $e = N - d$ and $e' = N - d'$. Observe that $e, e' \in (N/2, N/2 + N/(16k-2))$.

It suffices to show that if C is a cycle from $\mathcal{F}(d)$ and C' is a cycle from $\mathcal{F}(d')$, then C and C' share at most one difference. Equivalently, we will show that:

(i) *For any* $i \in [1, d]$ *and* $i' \in [1, d']$, $|S_{d,i} \cap S_{d',i'}| \leq 1$;

(ii) *For any* $i \in [1, e]$ *and* $i' \in [1, e']$, $|T_{e,i} \cap T_{e',i'}| \leq 1$; and

(iii) *For any* $i \in [1, d]$ *and* $i' \in [1, e']$, $|S_{d,i} \cap T_{e',i'}| \leq 1$.

To show (i), suppose to the contrary that $\{x, y\} \subseteq S_{d,i} \cap S_{d',i'}$ with $x < y$. Thus $y - x = \alpha d = \alpha' d'$ for some $\alpha, \alpha' \in [1, 4k-1]$, contradicting Lemma 4.1. The justification of (ii) is similar. For (iii), if $x, y \in S_{d,i} \cap T_{e',i'}$ with $x < y$, then $y - x = \alpha d$ for some $\alpha \in [1, 4k-1]$ (since $x, y \in S_{d,i}$) and $y - x = \alpha' e'$ for some $\alpha' \in [1, 4k-1]$ (since $x, y \in T_{e',i'}$), so $\alpha d = \alpha' e'$, which again contradicts Lemma 4.1. □

We note that the existence of two distinct integers in $(N/2 - N/(16k-2), N/2)$ is guaranteed when $N > 4(8k-1)$, i.e. $n - 1 > 32k(8k-1)$.

Since $n = 8Nk + 1$, we have the following theorem.

Theorem 4.3. *Let* $k \geq 2$ *and* $n = 8Nk + 1$. *There is a set of mutually orthogonal cyclic* $4k$*-cycle systems of order* n *of size at least*

$$\frac{N}{16k-2} - 1 = \frac{n-1}{8k(16k-2)} - 1.$$

Thus, if $n \equiv 1 \pmod{8k}$,

$$\mu(n, 4k) \geq \mu'(n, 4k) \geq \frac{n-1}{8k(16k-2)} - 1.$$

5 Orthogonal sets of $(4k+2)$-cycles

In this section, we show that for positive integers k and $n \equiv 1 \pmod{8k+4}$, $\mu'(n, 4k+2) = \Omega(n/k^2)$. Specifically, we construct

$$\left\lceil \frac{n-1}{(8k+4)(96k+30)} - 1 \right\rceil$$

mutually orthogonal cyclic $(4k+2)$-cycle decompositions of K_n. Thus we have at least two orthogonal decompositions whenever

$$\frac{n-1}{(8k+4)(96k+30)} > 2,$$

or equivalently $n - 1 > 48(2k+1)(16k+5)$.

Let N and k be positive integers and let $n = 2(4k+2)N+1$. For each $d \equiv N \pmod 2$ with $d \in (N/3 - N/(48k+15), N/3)$ (there is at least one such integer whenever $N > 48k+15$), we form a cyclic $(4k+2)$-cycle decomposition $\mathcal{F}(d)$ of K_n. Let $e = (N-d)/2$, and observe that $N/3 < e < N/3 + N/(2(48k+15)) < N/3 + N/(48k+15)$. Thus $e \in (N/3, N/3 + N/(48k+15))$.

For $i \in [1, d]$, let

$$S_{d,i,1} = \{i, d+i, 2d+i, \dots, (4k-1)d+i\} \text{ and } S_{d,i,2} = \{4kN+4e+i, (4k+2)N-i+1\},$$

and let $S_{d,i} = S_{d,i,1} \cup S_{d,i,2}$.

Now, when constructing the cycles containing differences in $S_{d,i}$, instead of $(4k+2)N - i + 1$, we will use the *negative* of this difference modulo n, that is, the value

$$(8k+4)N + 1 - ((4k+2)N - i + 1) = (4k+2)N + i.$$

We construct a starter cycle $C'(S_{d,i})$ using the set of differences $S_{d,i}$ but in a slightly different way to Lemma 3.3.

$$\begin{aligned} C'(S_{d,i}) =&(0, -i, d, -d-i, \dots, kd, -kd-i, (k+2)d, \\ &-(k+1)d - i, (k+3)d, -(k+2)d - i, \dots, 2kd, -(2k-1)d - i, \\ &(4k+2)N - (2k+1)d, -(2k+1)d - i). \end{aligned}$$

(Note that in the case $k = 1$, $C'(S_{d,i}) = (0, -i, d, -d-i, 4N + e - d, -3d - i)$.)

Lemma 5.1. *Let $i \in [1, d]$. Working modulo n, the ordered sequence $C'(S_{d,i})$ is a $(4k+2)$-cycle with difference set $S_{d,i}$.*

Proof. To see that no vertices are repeated (modulo n) within the sequence $C'(S_{d,i})$, it suffices to observe that:

$$\begin{aligned} &-(4k+2)N < -(2k+1)d - i < -(2k-1)d - i < -(2k-2)d - i < \cdots \\ &< -d - i < -i < 0 < d < 2d < \cdots < kd < (k+2)d < (k+3)d < \cdots < 2kd \\ &< (4k+2)N - (2k+1)d < (4k+2)N. \end{aligned}$$

By inspection, and since $(4k+2)N - (2k+1)d = 4kN + 4e - (2k-1)d$ and $n - ((4k+2)N - i + 1) = (4k+2)N + i$, the set of differences of the edges of the cycle $C'(S_{d,i})$ is $S_{d,i}$. □

Note that

$$\bigcup_{i=1}^{d} S_{d,i} = [1, 4kd] \cup [4kN + 4e + 1, 4kN + 4e + d] \cup [(4k+2)N - d + 1, (4k+2)N];$$

since $4kN + 4e + d = (4k+2)N - d$, we have that

$$\bigcup_{i=1}^{d} S_{d,i} = [1, 4kd] \cup [4kN + 4e + 1, (4k+2)N].$$

For $j, \ell \in [1, e]$, let

$$\begin{aligned}
T_{e,j,1} &= \{4kd + j, 4kd + e + j, \ldots, 4kd + (4k-1)e + j\}, \\
T_{e,j,2} &= \{4kN + j, 4kN + 2e + j\}, \\
U_{e,\ell,1} &= \{4kd + 4ke + \ell, 4kd + (4k+1)e + \ell, \ldots, 4kd + (8k-1)e + \ell\}, \\
U_{e,\ell,2} &= \{4kN + e + \ell, 4kN + 3e + \ell\},
\end{aligned}$$

and set $T_{e,j} = T_{e,j,1} \cup T_{e,j,2}$ and $U_{e,\ell} = U_{e,\ell,1} \cup U_{e,\ell,2}$.

The sets $T_{e,j}$ and $U_{e,\ell}$ are each balanced with $\tau = k + 1$. We have that

$$\begin{aligned}
\left(\bigcup_{j=1}^{e} T_{e,j} \right) \cup \left(\bigcup_{\ell=1}^{e} U_{e,\ell} \right) &= [4kd + 1, 4kd + 8ke] \cup [4kN + 1, 4kN + 4e] \\
&= [4kd + 1, 4kN + 4e],
\end{aligned}$$

since $4kd + 8ke = 4kN$. Observe that for fixed d,

$$\left(\bigcup_{i=1}^{d} S_{d,i} \right) \cup \left(\bigcup_{j=1}^{e} T_{e,j} \right) \cup \left(\bigcup_{\ell=1}^{e} U_{e,\ell} \right) = [1, (4k+2)N],$$

and thus by Corollary 3.4 and Lemma 5.1, the set of cycles

$$\mathcal{F}(d) = \{C'(S_{d,i}) \mid i \in [1, d]\} \cup \{C(T_{e,j}) \mid j \in [1, e]\} \cup \{C(U_{e,\ell}) \mid \ell \in [1, e]\}$$

is a set of starter cycles for a $(4k+2)$-cycle decomposition of K_n.

In order to show that the decompositions $\mathcal{F}(d)$, $d \in (N/3 - N/(48k+15), N/3)$, are orthogonal, we will make use of the following lemma which is directly implied by Corollary 3.6.

Lemma 5.2. *Suppose there exist integers*

$$d, d', e, e' \in \left(\frac{N}{3} - \frac{N}{48k+15}, \frac{N}{3} + \frac{N}{48k+15} \right)$$

such that $d \neq d'$ and $e \neq e'$. Let $\alpha, \alpha' \in [1, 8k+2]$. Then $\alpha d \neq \alpha' d'$ and $\alpha e \neq \alpha' e'$. Moreover, if $\alpha < \alpha'$, then $\alpha d < \alpha' d'$ and $\alpha e < \alpha' e'$.

Lemma 5.3. *Suppose that $\beta d + i = \beta' d' + i'$, where $\beta, \beta' \in [0, 4k-1]$, $i \in [1, d]$, $i' \in [1, d']$ and $d' < d$. Then either $\beta' = \beta$ or $\beta' = \beta + 1$.*

Proof. From Lemma 5.2, $(\beta + 1)d < (\beta + 2)d'$. Now,

$$(\beta - 1)d' + i' \leq \beta d' \leq \beta d < \beta d + i$$

and

$$\beta d + i \leq (\beta + 1)d < (\beta + 2)d' < (\beta + 2)d' + i';$$

hence

$$(\beta - 1)d' + i' < \beta d + i < (\beta + 2)d' + i'. \qquad \square$$

Lemma 5.4. *Let $d \neq d'$ such that $d, d' \equiv N \pmod 2$ and*

$$d, d' \in \left(\frac{N}{3} - \frac{N}{48k+15}, \frac{N}{3} + \frac{N}{48k+15}\right).$$

Let $e = (N-d)/2$ and $e' = (N-d')/2$. Let $i \in [1, d]$, $i' \in [1, d']$, $j, \ell \in [1, e]$ and $j', \ell' \in [1, e']$. Then for each $X \in \{S_{d,i}, T_{e,j}, U_{e,\ell}\}$ and each $Y \in \{S_{d',i'}, T_{e',j'}, U_{e',\ell'}\}$, $|X \cap Y| \leq 1$ with the exception $S_{d,i} \cap S_{d',i} = \{i, (4k+2)N + i\}$.

Proof. Recall from the start of this section that $e, e' \in (N/3, N/3 + N/(48k+15))$. In what follows, we frequently apply Lemma 5.2 to d, d', e and e'. To prove the lemma, it suffices to show the following:

(i) $S_{d,i} \cap S_{d',i} = \{i, (4k+2)N - i + 1\}$ and if $i \neq i'$ then $|S_{d,i} \cap S_{d',i'}| \leq 1$;

(ii) $|T_{e,j} \cap T_{e',j'}| \leq 1$, $|U_{e,\ell} \cap U_{e',\ell'}| \leq 1$ and $|T_{e,j} \cap U_{e',\ell'}| \leq 1$;

(iii) $|S_{d,i} \cap T_{e',j'}| \leq 1$ and $|S_{d,i} \cap U_{e',\ell'}| \leq 1$.

Proof of (i). In this case, we may assume without loss of generality that $d' < d$. We note that

$$\begin{aligned} &4kN + 4e' + i' > 4kN > 4kd \geq (4k-1)d + i \text{ and} \\ &4kN + 4e + i > 4kN > 4kd' \geq (4k-1)d' + i', \end{aligned}$$

so $S_{d,i,1} \cap S_{d',i',2} = S_{d',i',1} \cap S_{d,i,2} = \emptyset$.

Now, supposing that $|S_{d,i,1} \cap S_{d',i',1}| \geq 2$, it follows that for some x, $(x, x + \alpha d) = (x, x + \alpha' d')$ where $\alpha, \alpha' \in [1, 4k-1]$; thus $\alpha d = \alpha' d'$, in contradiction to Lemma 5.2. Next, supposing that $|S_{d,i,2} \cap S_{d',i',2}| \geq 2$, then either

(a) $4kN + 4e + i = 4kN + 4e' + i'$ and $(4k+2)N - i + 1 = (4k+2)N - i' + 1$, or

(b) $4kN + 4e + i = (4k+2)N - i' + 1$ and $4kN + 4e' + i' = (4k+2)N - i + 1$.

In both cases, it is straightforward to check that $e = e'$, a contradiction.

Thus if $|S_{d,i} \cap S_{d',i'}| \geq 2$, it must be that $|S_{d,i,1} \cap S_{d',i',1}| = 1$ and $|S_{d,i,2} \cap S_{d',i',2}| = 1$. If $i = i'$ then $\{i, (4k+2)N - i + 1\} \subseteq S_{d,i} \cap S_{d',i'}$. Moreover, recalling that $S_{d,i,1} \cap S_{d',i',2} = S_{d',i',1} \cap S_{d,i,2} = \emptyset$, it follows that $|S_{d,i} \cap S_{d',i'}| = 2$. Hence if $i = i'$, then $S_{d,i} \cap S_{d',i} = \{i, (4k+2)N - i + 1\}$. We now assume that $i \neq i'$. From Lemma 5.3, $|S_{d,i,1} \cap S_{d',i',1}| = 1$ implies that either

(a) $\beta d + i = \beta d' + i'$, or

(b) $\beta d + i = (\beta + 1)d' + i'$

for some $\beta, \beta' \in [0, 4k-1]$. Now suppose that also $|S_{d,i,2} \cap S_{d',i',2}| = 1$. Since $i \neq i'$, we note that $(4k+2)N - i + 1 \neq (4k+2)N - i' + 1$. Also, it cannot be the case that $4kN + 4e + i = (4k+2)N - i' + 1$, since

$$\begin{aligned} 4kN + 4e + i &= (4k+2)N - 2d + i \leq (4k+2)N - d < (4k+2)N - d' \\ &\leq (4k+2)N - i' < (4k+2)N - i' + 1. \end{aligned}$$

Now suppose that $4kN + 4e + i = 4kN + 4e' + i'$. Then $2d - i = 2d' - i'$. If (a) is true, then $(\beta + 2)d = (\beta + 2)d'$; since $\beta + 2 > 0$, we have $d = d'$, a contradiction.

On the other hand, if (b) is true, then $(\beta+2)d = (\beta+3)d'$, contradicting Lemma 5.2. Thus the only remaining possibility is that $4kN + 4e' + i' = (4k+2)N - i + 1$, so that $i + i' = 2N - 4e' + 1 = 2d' + 1$ is odd. Since d and d' have the same parity, this contradicts (a), so it must be that (b) is true. It follows that

$$(\beta+3)d' - \beta d + 1 = 2i \leq 2d.$$

Thus $(\beta+3)d' \leq (\beta+2)d - 1 < (\beta+2)d$, contradicting Lemma 5.2. $\square$

Proof of (ii). We first note that the largest element in $T_{e,j,1} \cup U_{e,\ell,1}$ is $4kd + (8k-1)e + \ell$, while the smallest element of $T_{e,j,2} \cup U_{e,\ell,2}$ is $4kN + j$. Since

$$4kd + (8k-1)e + \ell \leq 4kd + 8ke = 4kN < 4kN + j,$$

it follows that $T_{e,j,1} \cap T_{e',j',2} = \emptyset$, $U_{e,\ell,1} \cap U_{e',\ell',2} = \emptyset$ and $T_{e,j,1} \cap U_{e',\ell',2} = \emptyset$.

Now, if $|T_{e,j,1} \cap T_{e',j',2}| \geq 2$, $|U_{e,\ell,1} \cap U_{e',\ell',1}| \geq 2$ or $|T_{e,j,1} \cap U_{e',j',1}| \geq 2$, then for some x, $(x, x+\alpha e) = (x, x+\alpha' e')$, where $\alpha, \alpha' \in [1, 8k-1]$. Thus $\alpha e = \alpha' e'$, contradicting Lemma 5.2. If $|T_{e,j,2} \cap T_{e',j',2}| \geq 2$, $|U_{e,\ell,2} \cap U_{e',\ell',2}| \geq 2$ or $|T_{e,\ell,2} \cap U_{e',\ell',2}| \geq 2$, then it follows that $e = e'$, a contradiction.

Thus, if $|T_{e,j} \cap T_{e',j'}| \geq 2$, it must be that $|T_{e,j,1} \cap T_{e',j',1}| = 1$ and $|T_{e,j,2} \cap T_{e',j',2}| = 1$. Since $|T_{e,j,1} \cap T_{e',j',1}| = 1$, we have that for some $\alpha, \alpha' \in [0, 4k-1]$, $4kd + \alpha e + j = 4kd' + \alpha' e' + j'$, which implies that $(8k-\alpha)e - j = (8k-\alpha')e' - j'$. Since $|T_{e,j,2} \cap T_{e',j',2}| = 1$, then $4kN + \beta e + j = 4kN + \beta' e' + j'$ where $\beta, \beta' \in \{0, 2\}$. Hence $(8k - \alpha + \beta)e = (8k - \alpha' + \beta')e'$, which contradicts Lemma 5.2 since $(8k-\alpha+\beta), (8k-\alpha'+\beta') \in [4k+1, 8k+2]$. We conclude that $|T_{e,j} \cap T_{e',j'}| \leq 1$.

In a similar way, the assumption that $|U_{e,\ell,1} \cap U_{e,\ell',1}| = 1$ and $|U_{e,\ell,2} \cap U_{e,\ell',2}| = 1$ leads to a contradiction, as does the assumption that $|T_{e,j,1} \cap U_{e',\ell',1}| = 1$ and $|T_{e,j,2} \cap U_{e',\ell',2}| = 1$. We conclude that $|U_{e,\ell} \cap U_{e',\ell'}| \leq 1$ and $|T_{e,j} \cap U_{e',\ell'}| \leq 1$.

Next, suppose that $|U_{e,\ell,1} \cap U_{e',\ell',1}| = 1$ and $|U_{e,\ell,2} \cap U_{e',\ell',2}| = 1$. Since $|U_{e,\ell,1} \cap U_{e',\ell',1}| = 1$, we have that for some $\alpha, \alpha' \in [4k, 8k-1]$, $4kd + \alpha e + \ell = 4kd' + \alpha' e' + \ell'$, which implies that $(8k-\alpha)e - \ell = (8k-\alpha')e' - \ell$. Since $|U_{e,\ell,2} \cap U_{e',\ell',2}| = 1$, then $4kN + \beta e + \ell = 4kN + \beta' e' + \ell'$ where $\beta, \beta' \in \{1, 3\}$. Hence $(8k - \alpha + \beta)e = (8k - \alpha' + \beta')e'$, which contradicts Lemma 5.2 since $(8k-\alpha+\beta), (8k-\alpha'+\beta') \in [2, 4k+3]$.

Finally, suppose that $|T_{e,j,1} \cap U_{e',\ell',1}| = 1$ and $|T_{e,j,2} \cap U_{e',\ell',2}| = 1$. Since $|T_{e,j,1} \cap U_{e',\ell',1}| = 1$, we have that for some $\alpha \in [0, 4k-1]$, $\alpha' \in [4k, 8k-1]$, $4kd + \alpha e + j = 4kd' + \alpha' e' + \ell'$, which implies that $(8k-\alpha)e - j = (8k-\alpha')e' - \ell'$. Since $|T_{e,j,2} \cap U_{e',\ell',2}| = 1$, then $4kN + \beta e + j = 4kN + \beta' e' + \ell'$, where $\beta \in \{0, 2\}$ and $\beta' \in \{1, 3\}$. Hence $(8k - \alpha + \beta)e = (8k - \alpha' + \beta')e'$, which contradicts Lemma 5.2 since $4k + 1 \leq 8k - \alpha + \beta \leq 8k + 2$ and $2 \leq 8k - \alpha' + \beta' \leq 4k + 3$. $\square$

Proof of (iii). Note that since

$$(4k-1)d + i \leq 4kd < 4kN < 4kN + j' < 4kN + e' + \ell',$$

then $S_{d,i,1} \cap T_{e',j',2} = \emptyset$ and $S_{d,i,1} \cap U_{e',\ell',2} = \emptyset$. Moreover,

$$\begin{aligned} 4kd' + (4k-1)e' + j' \leq 4kd' + 4ke' < 4kd' + (8k-1)e' + \ell' \leq 4kd' + 8ke' = 4kN \\ < 4kN + 4e + i, \end{aligned}$$

and so $S_{d,i,2} \cap T_{e',j',1} = \emptyset$ and $S_{d,i,2} \cap U_{e',\ell',1} = \emptyset$.

By Lemma 5.2,

$$(4k-2)d+i \leq (4k-1)d < 4kd' < 4kd' + j'.$$

It follows that $|S_{d,i,1} \cap T_{e',j',1}| \leq 1$. Also, since $d < N/3 < e'$,

$$(4k-1)d+i \leq 4kd < 4ke' < 4kd' + 4ke' + \ell',$$

and thus $S_{d,i,1} \cap U_{e',\ell',1} = \emptyset$.

Now, using Lemma 5.2, we also have that

$$4kN+e'+\ell' \leq 4kN+2e' < 4kN+2e'+j' \leq 4kN+3e' < 4kN+4e < 4kN+4e+i,$$

and so $S_{d,i,2} \cap T_{e',j',2} = \emptyset$ and $|S_{d,i,2} \cap U_{e',\ell',2}| \leq 1$. It follows that $|S_{d,i} \cap T_{e',j'}| \leq 1$ and $|S_{d,i} \cap U_{e',\ell'}| \leq 1$. □

End of Proof of Lemma 5.4. □

Theorem 5.5. *Let $k \geq 1$ and $n = (8k+4)N+1$. There is a set of mutually orthogonal cyclic $(4k+2)$-cycle systems of order n of size at least*

$$\left\lceil \frac{N}{96k+30} - 1 \right\rceil = \left\lceil \frac{n-1}{(8k+4)(96k+30)} - 1 \right\rceil.$$

Thus, if $n \equiv 1 \pmod{2(4k+2)}$, then

$$\mu(n, 4k+2) \geq \mu'(n, 4k+2) \geq \left\lceil \frac{n-1}{(8k+4)(96k+30)} - 1 \right\rceil.$$

Proof. The number of integers strictly between $N/3 - N/(48k+15)$ and $N/3$ with the same parity as N is at least $\lceil N/(96k+30) - 1 \rceil$. Note that there are at least two distinct integers of the same parity as N in this interval whenever

$$\frac{N}{96k+30} > 2,$$

or equivalently $n-1 > 48(2k+1)(16k+5)$. It thus suffices to show that for distinct integers d and d' with the same parity such that

$$d, d' \in \left(\frac{N}{3} - \frac{N}{48k+15}, \frac{N}{3} \right),$$

the decompositions $\mathcal{F}(d)$ and $\mathcal{F}(d')$ are orthogonal.

In turn, it suffices to deal with the exceptional case from Lemma 5.4. From Lemma 5.1, the edges of differences i and $(4k+2)N - i + 1$ within $C'(S_{d,i})$ are $\{0, -i\}$ and $\{(4k+2)N - (2k+1)d, -(2k+1)d - i\}$, which are at distance $(4k+2)N - (2k+1)d + i$. Similarly, the edges of differences i and $(4k+2)N - i + 1$ within $C'(S_{d',i})$ are $\{0, -i\}$ and $\{(4k+2)N - (2k+1)d', -(2k+1)d' - i\}$, which are at distance $(4k+2)N - (2k+1)d' + i$. If the pairs of edges within cycles generated from the starters $C'(S_{d,i})$ and $C'(S_{d',i})$ coincide, then we must have that $(2k+1)d \equiv (2k+1)d' \pmod{n}$. But n and $2k+1$ are coprime, so $d = d'$. □

6 Concluding remarks

The main results of this paper have been to establish lower bounds on the number of mutually orthogonal cyclic ℓ-cycle systems of order n. For upper bounds on the number of systems (not necessarily cyclic in nature) we have the following lemmata.

Lemma 6.1. *If there exists a set of μ mutually orthogonal ℓ-cycle systems of order n, then $\mu \le n-2$. That is, $\mu(\ell,n) \le n-2$.*

Proof. Consider a vertex w in K_n. The vertex w belongs to precisely $(n-1)(n-2)/2$ paths of length 2 in K_n where w is the center vertex of the path. Moreover, each such path belongs to at most one ℓ-cycle from any set of μ mutually orthogonal ℓ-cycle systems. The number of cycles in one ℓ-cycle system which contain vertex w is equal to $(n-1)/2$. Thus $\mu(n-1)/2 \le (n-1)(n-2)/2$. The result follows. □

Lemma 6.2. *Let $\ell \ge 4$. Then*

$$\mu(\ell,n) \le \frac{(n-2)(n-3)}{2(\ell-3)}.$$

Proof. Suppose there exists a set $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\mu\}$ of mutually orthogonal ℓ-cycle systems of K_n. Consider an edge $\{v,w\}$ in K_n. Then for each $i \in [1,\mu]$, there is an ℓ-cycle $C_i \in \mathcal{F}_i$ containing the edge $\{v,w\}$. Let H be the clique of size $n-2$ in K_n not including vertices v and w. Then the intersection of C_i with H is a path P_i with $\ell-3$ edges. Moreover, orthogonality implies that the paths in the set $\{P_i \mid i \in [1,\mu]\}$ are pairwise edge-disjoint. Thus, $(\ell-3)\mu$ is bounded by the number of edges in H; that is, $(\ell-3)\mu \le (n-2)(n-3)/2$. □

Observe that Lemma 6.2 improves Lemma 6.1 only if $\ell > (n+3)/2$. If $\ell > n/\sqrt{2}$, it is not even possible to find a pair of orthogonal cycle systems, as shown in the following lemma.

Lemma 6.3. *If $2\ell^2 > n(n-1)$ then $\mu(\ell,n) \le 1$.*

Proof. Suppose there exists a pair $\{\mathcal{F}_1, \mathcal{F}_2\}$ of mutually orthogonal ℓ-cycle systems of K_n. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ each contain $n(n-1)/(2\ell)$ cycles of length ℓ. Let C be a cycle in $\mathcal{F}_1$. By the definition of orthogonality, each edge of C intersects a unique cycle in $\mathcal{F}_2$. Thus $\ell \le n(n-1)/(2\ell)$, contradiction. □

When the systems are required to be cyclic, Lemma 6.1 can be slightly improved.

Lemma 6.4. *Let $n \ge 4$. If there exists a set of μ' mutually orthogonal cyclic ℓ-cycle systems of order n, then $\mu' \le n-3$. That is, $\mu'(\ell,n) \le n-3$.*

Proof. Since $\mu'(\ell,n) \le \mu(\ell,n)$, Lemma 6.1 implies that $\mu'(\ell,n) \le n-2$. Suppose, for the sake of contradiction that $\mu'(\ell,n) = n-2$. Thus there exists a set of $n-2$ orthogonal cyclic decompositions of K_n where the vertices are labelled with elements of $\mathbb{Z}_n$. Let $a \in [1,(n-1)/2]$. Suppose that the path $(-a,0,a)$ of length 2 does not occur in a cycle from one of these decompositions. Then the total number of paths of length 2 containing 0 which appear in one of the cycles is less than $(n-1)(n-2)/2$. However, there are $(n-2)(n-1)/2$ cycles containing vertex 0, contradicting the condition of orthogonality.

Let C_a be the cycle containing the path $(-a, 0, a)$ and let $\mathcal{F}$ be the decomposition of K_n containing C_a. Since our decomposition is cyclic, there is also a cycle $C' \in \mathcal{F}$ containing $(0, a, 2a)$; since C' and C_a share an edge we must have $C' = C_a$. Inductively, $C_a = (0, a, 2a, \dots)$. In particular $C_1 = (0, 1, 2, \dots, n-1)$ and thus $\ell = n$. But since $\mu'(\ell, n) = n - 2 \geq 2$ and $n > (n-1)/2$, there is a cycle $C'' \neq C_a$ in a decomposition $\mathcal{F}' \neq \mathcal{F}$ containing a repeated difference $a \in [1, (n-1)/2]$. The cycle C'' shares two edges with C_a, contradicting the condition of orthogonality. □

It is worth noting that for certain congruencies the upper bound in Lemma 6.4 can be made significantly smaller. For example, if $n \equiv 3 \pmod 6$ then $\mu'(3, n) = 1$, because in this case any cyclic decomposition necessarily contains the cycle $(0, n/3, 2n/3)$.

In the appendix we give computational results for $\mu'(\ell, n)$ when ℓ and n are small. As yet we are unaware of any instances for which the bound of Lemma 6.4 is tight, and so we ask if equality ever occurs.

Question 1. For which values of ℓ and n, if any, is $\mu'(\ell, n) = n - 3$?

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Appendix

We computed sets of mutually orthogonal cyclic ℓ-cycle systems of order $n = 2\ell + 1$ for small values of ℓ, and in so doing we empirically determined or bounded $\mu'(\ell, 2\ell + 1)$ in these cases. Recall from Lemma 6.4 that $\mu'(\ell, 2\ell + 1) \leq 2\ell - 2$.

Note that for any cyclic ℓ-cycle system of order $2\ell + 1$, the cycles of the system comprise a single $\mathbb{Z}_{2\ell+1}$-orbit. To find sets of mutually orthogonal cyclic ℓ-cycle systems of order $2\ell + 1$, we first determined the orbit for each possible system and then constructed a graph in which each system is represented as a vertex and adjacency denotes orthogonality. Maximum cliques were then sought. The results for $3 \leq \ell \leq 11$ are summarised in Table 1. For $9 \leq \ell \leq 11$, we found cliques of order 8 but we do not yet know whether larger cliques exist (the computational task becomes increasingly challenging as the number of systems grows).

We now present examples of the $\mathbb{Z}_{2\ell+1}$-orbits for the sets of mutually orthogonal cyclic ℓ-cycle systems of order $2\ell + 1$ that we found. Each orbit is represented by the differences that occur on the edges of its cycles, using notation from Section 2.

$\ell = 3$**,** $n = 7$

$[1, 2, -3]$, $[1, -3, 2]$

$\ell = 4$**,** $n = 9$

$[1, -2, -3, 4]$, $[1, 4, -3, -2]$

$\ell = 5$**,** $n = 11$

$[1, -2, 4, 3, 5]$, $[1, 3, -2, 5, 4]$, $[1, 4, 5, -2, 3]$, $[1, 5, 3, 4, -2]$

ℓ	$n = 2\ell + 1$	No. of Cyclic Systems	$\mu'(\ell, 2\ell + 1)$
3	7	2	2
4	9	6	2
5	11	24	4
6	13	168	5
7	15	1344	8
8	17	11136	8
9	19	128304	≥ 8
10	21	1504248	≥ 8
11	23	19665040	≥ 8

Table 1: Number of mutually orthogonal cyclic ℓ-cycle systems of order $2\ell + 1$.

$\ell = 6$, $n = 13$

$[1, 2, 3, -4, 5, 6]$, $[1, -4, -2, 3, -5, -6]$, $[1, 5, 3, 6, -4, 2]$, $[1, -5, -4, 3, -6, -2]$, $[1, 6, 3, 2, 5, -4]$

$\ell = 7$, $n = 15$

$[1, 2, 6, -4, -7, -3, 5]$, $[1, -2, -3, -5, -4, 7, 6]$, $[1, 3, 4, -2, 6, -5, -7]$, $[1, -3, 4, 2, -5, -6, 7]$, $[1, 5, -3, -7, -4, 6, 2]$, $[1, 6, 7, -4, -5, -3, -2]$, $[1, 7, -6, -5, 2, 4, -3]$, $[1, -7, -5, 6, -2, 4, 3]$

$\ell = 8$, $n = 17$

$[1, 2, 3, 4, -6, -7, -5, 8]$, $[1, -2, -3, 8, 5, -6, 4, -7]$, $[1, 3, 7, -8, -6, 2, -4, 5]$, $[1, -3, -5, 6, 4, -8, 7, -2]$, $[1, 4, 5, 2, -3, 6, -7, -8]$, $[1, 5, -7, 6, -8, -3, 2, 4]$, $[1, -6, -8, 2, 5, -4, 7, 3]$, $[1, -8, -7, 5, -3, 4, 2, 6]$

$\ell = 9$, $n = 19$

$[1, 2, 3, 4, 5, -6, -7, 9, 8]$, $[1, -2, 3, -4, -7, 6, 8, 9, 5]$, $[1, 5, 8, -3, 7, -6, -4, 9, 2]$, $[1, -5, -7, -6, -8, -2, -4, 3, 9]$, $[1, 6, -3, -9, 2, -7, 4, -5, -8]$, $[1, -6, 8, -7, -3, -5, -2, 4, -9]$, $[1, 7, 3, 6, -8, 9, -5, 2, 4]$, $[1, 9, -7, 8, 4, 3, 5, 2, -6]$

$\ell = 10$, $n = 21$

$[1, 2, 3, 4, 5, -7, 6, 9, -10, 8]$, $[1, -2, 3, -4, 5, -6, 8, -9, -7, -10]$, $[1, 3, -2, -5, -10, 9, -6, -8, 4, -7]$, $[1, -6, -10, 7, -3, 9, -5, -2, -8, -4]$, $[1, 7, -9, -8, 5, 6, 4, 3, 2, 10]$, $[1, -8, -5, 3, -6, -9, 7, -10, 2, 4]$, $[1, 10, 3, 6, -7, 5, -8, -2, 4, 9]$, $[1, -10, 3, 5, -4, -9, -2, -7, 8, -6]$

$\ell = 11$, $n = 23$

$[1, 2, 3, 4, 5, 6, 7, 8, 9, -10, 11]$, $[1, -2, 3, -4, 5, -6, 7, -11, -10, 9, 8]$, $[1, 3, 2, -4, -5, -11, -6, 10, 8, -7, 9]$, $[1, -3, 10, 6, 4, 7, 9, 11, 8, -2, -5]$, $[1, -4, 8, 6, 3, 5, -9, -2, 10, -11, -7]$, $[1, 5, -11, -8, -3, 9, -7, 2, 6, -4, 10]$, $[1, 10, -7, -8, 3, -5, 9, 4, 6, -11, -2]$, $[1, -11, 5, -7, 4, 2, -9, -6, 8, 10, 3]$

Below we present examples of mutually orthogonal cyclic 4-cycle systems of orders 17 and 25; these are mentioned in Section 2.

$\ell = 4$**,** $n = 17$

$\{[1, 2, 3, -6], [4, -5, -7, 8]\}$, $\{[1, -2, -3, 4], [5, 8, -7, -6]\}$, $\{[1, -3, -8, -7], [2, 4, 5, 6]\}$,
$\{[1, 4, -7, 2], [3, -5, 8, -6]\}$, $\{[1, -4, 8, -5], [2, 7, -3, -6]\}$, $\{[1, 5, 2, -8], [3, -4, -6, 7]\}$,
$\{[1, -5, -3, 7], [2, -6, 8, -4]\}$, $\{[1, -6, -3, 8], [2, -4, 7, -5]\}$, $\{[1, -7, -8, -3], [2, 6, 5, 4]\}$,
$\{[1, -8, -6, -4], [2, 5, 3, 7]\}$

$\ell = 4$**,** $n = 25$

$\{[1, 2, 3, -6], [4, -5, 12, -11], [7, -8, -9, 10]\}$,
$\{[1, -2, -3, 4], [5, -6, -7, 8], [9, -10, -11, 12]\}$,
$\{[1, 3, 4, -8], [2, 5, 7, 11], [6, -9, -10, -12]\}$, $\{[1, -3, -4, 6], [2, -5, -7, 10], [8, -11, -9, 12]\}$,
$\{[1, 4, 2, -7], [3, -5, -9, 11], [6, -8, -10, 12]\}$, $\{[1, -4, -2, 5], [3, 6, 7, 9], [8, -12, -11, -10]\}$,
$\{[1, 5, 3, -9], [2, -4, 10, -8], [6, 12, -7, -11]\}$, $\{[1, -5, -3, 7], [2, 6, 4, -12], [8, 11, -10, -9]\}$,
$\{[1, 7, -10, 2], [3, -11, -4, 12], [5, 9, -8, -6]\}$, $\{[1, -7, -8, -11], [2, 12, 5, 6], [3, 10, -4, -9]\}$,
$\{[1, 8, 4, 12], [2, -10, -6, -11], [3, -7, 9, -5]\}$, $\{[1, -8, 11, -4], [2, -12, 3, 7], [5, 10, -6, -9]\}$,
$\{[1, -9, -3, 11], [2, -6, -4, 8], [5, 12, -10, -7]\}$,
$\{[1, -10, 11, -2], [3, 8, -5, -6], [4, -7, 12, -9]\}$, $\{[1, -11, -12, -3], [2, 8, 10, 5], [4, -6, 9, -7]\}$,
$\{[1, 12, -3, -10], [2, 7, -5, -4], [6, 11, -9, -8]\}$, $\{[1, -12, 8, 3], [2, 9, -4, -7], [5, 11, -6, -10]\}$

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Some remarks on the square graph of the hypercube

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Abstract

Let $\Gamma = (V, E)$ be a graph. The square graph Γ^2 of the graph Γ is the graph with the vertex set $V(\Gamma^2) = V$ in which two vertices are adjacent if and only if their distance in Γ is at most two. The square graph of the hypercube Q_n has some interesting properties. For instance, it is highly symmetric and panconnected. In this paper, we investigate some algebraic properties of the graph Q_n^2. In particular, we show that the graph Q_n^2 is distance-transitive. We show that the graph Q_n^2 is an imprimitive distance-transitive graph if and only if n is an odd integer. Also, we determine the spectrum of the graph Q_n^2. Finally, we show that when $n > 2$ is an even integer, then Q_n^2 is an automorphic graph, that is, Q_n^2 is a distance-transitive primitive graph which is not a complete or a line graph.

Keywords: Square of a graph, distance-transitive graph, hypercube, automorphism group, Johnson graph, automorphic graph.

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1 Introduction

In this paper, a graph $\Gamma = (V, E)$ is considered as an undirected simple graph where $V = V(\Gamma)$ is the vertex-set and $E = E(\Gamma)$ is the edge-set. For all the terminology and notation not defined here, we follow [1, 3, 5, 6, 9].

Let $\Gamma = (V, E)$ be a graph. The *square graph* Γ^2 of the graph Γ is the (simple) graph with vertex set V in which two vertices are adjacent if and only if their distance in Γ is at most two. It is easy to see that $\mathrm{Aut}(\Gamma) \leq \mathrm{Aut}(\Gamma^2)$, where $\mathrm{Aut}(\Gamma)$ denotes the automorphism group of the graph Γ. Thus, if the graph Γ is a vertex-transitive graph, then Γ^2 is a vertex-transitive graph. A graph Γ of order $n > 2$ is *Hamilton-connected* if for any pair of distinct vertices u and v, there is a Hamilton u-v path, namely, there is a u-v path

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of length $n-1$. It is clear that if a graph Γ is Hamilton-connected then it is Hamiltonian. A graph Γ of order $n > 2$ is panconnected if for every two vertices u and v, there is a u-v path of length l for every integer l with $d(u,v) \leq l \leq n-1$. Note that if a graph Γ is panconnected, then it is Hamilton-connected. It is a well known fact that when a graph Γ is 2-connected, then its square Γ^2 is panconnected [4, 7]. Using this fact, and an algebraic property of Johnson graphs, recently it has been proved that the Johnson graphs are panconnected [10].

Let $n \geq 2$ be an integer. The hypercube Q_n is the graph whose vertex-set is $\{0,1\}^n$, where two n-tuples are adjacent if they differ in precisely one coordinate. This graph has been studied from various aspects by many authors. Some recent works concerning some algebraic aspects of this graph include [14, 17, 24, 28]. It is a well known fact that the graph Q_n is a distance-transitive graph [1, 3], and hence it is edge-transitive. Now, using a well known result due to Watkins [27], it follows that the connectivity of Q_n is maximal, that is, n. Like the hypercube Q_n, its square, namely, the graph Q_n^2 has some interesting properties. For instance, when $n \geq 2$, then Q_n is 2-connected. Now using a known result due to Chartrand and Fleischner [4, 7], it follows that Q_n^2 is a panconnected graph. Also, since Q_n is vertex-transitive, the graph Q_n^2 is vertex-transitive, as well. Hence Q_n^2 is a regular graph and it is easy to check that its valency is $n+\binom{n}{2}=\binom{n+1}{2}$. If $n=2$, then Q_n^2 is the complete graph K_4. When $n=3$, then Q_n^2 is a 6-regular graph with 8 vertices. This graph is isomorphic with a graph known as the *coktail-party* graph $CP(4)$ [1]. It can be shown that when $n=4$, then the graph Q_n^2 is a 10-regular graph with 16 vertices, which is isomorphic to the complement of the graph known as the *Clebsch* graph [9].

In this paper, we determine the automorphism group of the graph Q_n^2. Then we show that Q_n^2 is a distance-transitive graph. This implies that the connectivity of the graph Q_n^2 is maximal, namely, its valency $\binom{n+1}{2}$. Also, we will see that the graph Q_n^2 is an imprimitive distance-transitive graph if and only if n is an odd integer. A graph Γ is called an *automorphic* graph, when it is a distance-transitive primitive graph which is not a complete or a line graph [1]. In the last step of the paper, we show that the graph Q_n^2 is an automorphic graph if and only if n is an even integer.

2 Preliminaries

The graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are called *isomorphic*, if there is a bijection $\alpha \colon V_1 \longrightarrow V_2$ such that $\{a,b\} \in E_1$ if and only if $\{\alpha(a), \alpha(b)\} \in E_2$ for all $a, b \in V_1$. In such a case the bijection α is called an *isomorphism*. An *automorphism* of a graph Γ is an isomorphism of Γ with itself. The set of automorphisms of Γ with the operation of composition of functions is a group called the *automorphism group* of Γ and denoted by $\mathrm{Aut}(\Gamma)$.

The group of all permutations of a set V is denoted by $\mathrm{Sym}(V)$ or just $\mathrm{Sym}(n)$ when $|V| = n$. A *permutation group* G on V is a subgroup of $\mathrm{Sym}(V)$. In this case we say that G *acts* on V. If G acts on V we say that G is *transitive* on V (or G acts *transitively* on V) if given any two elements u and v of V, there is an element β of G such that $\beta(u) = v$. If Γ is a graph with vertex-set V then we can view each automorphism of Γ as a permutation on V and so $\mathrm{Aut}(\Gamma) = G$ is a permutation group on V.

A graph Γ is called *vertex-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. We say that Γ is *edge-transitive* if the group $\mathrm{Aut}(\Gamma)$ acts transitively on the edge set E, namely, for any $\{x,y\}, \{v,w\} \in E(\Gamma)$, there is some π in $\mathrm{Aut}(\Gamma)$, such that $\pi(\{x,y\}) = \{v,w\}$. We say that Γ is *symmetric* (or *arc-transitive* if for all vertices u, v, x, y of Γ such that u and v are adjacent, and also, x and y are adjacent, there is an automorphism π in $\mathrm{Aut}(\Gamma)$ such that $\pi(u) = x$ and $\pi(v) = y$. We say that Γ is *distance-transitive* if for all vertices u, v, x, y of Γ such that $d(u, v) = d(x, y)$, where $d(u, v)$ denotes the distance between the vertices u and v in Γ, there is an automorphism π in $\mathrm{Aut}(\Gamma)$ such that $\pi(u) = x$ and $\pi(v) = y$.

A vertex cut of the graph Γ is a subset U of V such that the subgraph $\Gamma - U$ induced by the set $V - U$ is either trivial or not connected. The *connectivity* $\kappa(\Gamma)$ of a nontrivial connected graph Γ is the minimum cardinality of all vertex cuts of Γ. If we denote by $\delta(\Gamma)$ the minimum degree of Γ, then $\kappa(\Gamma) \leq \delta(\Gamma)$. A graph Γ is called *k-connected* (for $k \in \mathbb{N}$) if $|V(\Gamma)| > k$ and $\Gamma - X$ is connected for every subset $X \subset V(\Gamma)$ with $|X| < k$. It is trivial that if a positive integer m is such that $m \leq \kappa(\Gamma)$, then Γ is an m-connected graph. We have the following fact.

Theorem 2.1 ([27]). *If a connected graph Γ is edge-transitive, then $\kappa(\Gamma) = \delta(\Gamma)$, where $\delta(\Gamma)$ is the minimum degree of vertices of Γ.*

Let $n, k \in \mathbb{N}$ with $k < n$, and let $[n] = \{1, ..., n\}$. The *Johnson graph* $J(n, k)$ is defined as the graph whose vertex set is $V = \{v \mid v \subseteq [n], |v| = k\}$ and two vertices v,w are adjacent if and only if $|v \cap w| = k - 1$. The class of Johnson graphs is a well known class of distance-transitive graphs [3]. It is an easy task to show that the set of mappings $H = \{f_\theta \mid \theta \in \mathrm{Sym}([n])\}$, $f_\theta(\{x_1, ..., x_k\}) = \{\theta(x_1), ..., \theta(x_k)\}$, is a subgroup of $\mathrm{Aut}(J(n, k))$ [9]. It has been shown that $\mathrm{Aut}(J(n, k)) \cong \mathrm{Sym}([n])$ if $n \neq 2k$, and $\mathrm{Aut}(J(n, k)) \cong \mathrm{Sym}([n]) \times \mathbb{Z}_2$, if $n = 2k$, where $\mathbb{Z}_2$ is the cyclic group of order 2 [3, 13, 18].

Although in most situations it is difficult to determine the automorphism group of a graph Γ and how it acts on its vertex and edge sets, there are various papers in the literature, and some of the recent works include [8, 11, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 26, 28].

Let G be any abstract finite group with identity 1, and suppose Ω is a subset of G, with the properties:

(i) $x \in \Omega \Longrightarrow x^{-1} \in \Omega$,

(ii) $1 \notin \Omega$.

The *Cayley graph* $\Gamma = \mathrm{Cay}(G; \Omega)$ is the (simple) graph whose vertex-set and edge-set are defined as follows:

$V(\Gamma) = G$, $E(\Gamma) = \{\{g, h\} \mid g^{-1}h \in \Omega\}$.

It can be shown that the Cayley graph $\Gamma = \mathrm{Cay}(G; \Omega)$ is connected if and only if the set Ω is a generating set in the group G [1].

The group G is called a semidirect product of N by Q, denoted by $G = N \rtimes Q$, if G contains subgroups N and Q such that:

(i) $N \trianglelefteq G$ (N is a normal subgroup of G)

(ii) $NQ = G$; and

(iii) $N \cap Q = 1$.

3 Main results

The hypercube Q_n is the graph whose vertex set is $\{0,1\}^n$, where two n-tuples are adjacent if they differ in precisely one coordinate. It is easy to show that $Q_n = \mathrm{Cay}(\mathbb{Z}_2^n; S)$, where $\mathbb{Z}_2$ is the cyclic group of order 2, and $S = \{e_i \mid 1 \leq i \leq n\}$, where $e_i = (0,...,0,1,0,...,0)$, with 1 at the ith position. It is easy to show that the set $H = \{f_\theta | \theta \in \mathrm{Sym}([n])\}$, $f_\theta(x_1,...,x_n) = (x_{\theta(1)},...,x_{\theta(n)})$ is a subgroup of the group $\mathrm{Aut}(Q_n)$. It is clear that $H \cong \mathrm{Sym}([n])$. We know that in every Cayley graph $\Gamma = \mathrm{Cay}(G; S)$, the group $\mathrm{Aut}(\Gamma)$ contains a subgroup isomorphic with the group G. In fact, if $x \in \mathbb{Z}_2^n$, and we define the mapping $f_x(v) = x + v$, for every $v \in V(Q_n)$, then f_x is an automorphism of the hypercube Q_n. Hence $\mathbb{Z}_2^n$ is (isomorphic with) a subgroup of $\mathrm{Aut}(Q_n)$. It has been proved that $\mathrm{Aut}(Q_n) = \langle \mathbb{Z}_2^n, \mathrm{Sym}([n]) \rangle \cong \mathbb{Z}_2^n \rtimes \mathrm{Sym}([n])$ [14]. It is clear that when Γ is a graph then $\mathrm{Aut}(\Gamma)$ is a subgroup of $\mathrm{Aut}(\Gamma^2)$. Thus we have $\mathrm{Aut}(Q_n) \leq \mathrm{Aut}(Q_n^2)$. In the sequel, we wish to show that the graph Q_n^2 is a distance-transitive graph, and for doing this we need the automorphism group of Q_n^2. When $n = 3$, then Q_n^2 is isomorphic with the coktail-party graph $CP(4)$. The complement of this graph is a disjoint union of 4 copies of K_2. Thus $\mathrm{Aut}(Q_3^2) \cong \mathrm{Sym}([2])\ wr_I\ \mathrm{Sym}([4])$, where $I = \{1,2,3,4\}$ [3, 22] (for an acquaintance with the notion of wreath product of groups see [6]). Now it can be checked that this graph is a distance-transitive graph. Hence, in the sequel we assume that $n > 4$. It is easy to see that for the graph Q_n^2 we have, $Q_n^2 = \mathrm{Cay}(\mathbb{Z}_2^n; T)$, $T = S \cup S_1$, where $S_1 = \{e_i + e_j | \ i,j \in [n], i \neq j\}$. Let $A = \mathrm{Aut}(Q_n^2)$ and A_0 be the stabilizer subgroup of the vertex $v = 0$ in A. Since Q_n^2 is a vertex-transitive graph, then from the orbit-stabilizer theorem we have $|A| = |A_0||V(Q_n^2)| = 2^n|A_0|$. The following lemma determines an upper bound for $|A_0|$.

Lemma 3.1. *Let $n > 4$ and $A = \mathrm{Aut}(Q_n^2)$. Let A_0 be the stabilizer subgroup of the vertex $v = 0$. Then $|A_0| \leq (n+1)!$.*

Proof. Let $\Gamma = Q_n^2$. We know that $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^n; T)$, $T = S \cup S_1$, where $S = \{e_i \mid 1 \leq i \leq n\}$ and $S_1 = \{e_i + e_j | \ i,j \in [n], i \neq j\}$. Let $f \in A_0$. Then $f(T) = T$. Let G be the subgraph of Γ which is induced by the subset T. Let $h = f|_T$ be the restriction of the mapping f to the subset T. It is clear that h is an automorphism of the graph G. It is easy to see that the mapping $\Phi\colon A_0 \to \mathrm{Aut}(G)$, which is defined by the rule $\Phi(g) = g|_T$, is a group homomorphism. Thus we have $\frac{A_0}{\ker(\Phi)} \cong \mathrm{im}(\Phi)$, and hence we have $|A_0| = |\ker(\Phi)||\mathrm{im}(\Phi)|$. Since $im(\Phi)$ is a subgroup of $\mathrm{Aut}(G)$, then we have $|A_0| \leq |\ker(\Phi)||\mathrm{Aut}(G)|$. If we show that $|\mathrm{Aut}(G)| \leq (n+1)!$ and $\ker(\Phi) = \{1\}$, then the lemma is proved. Hence in the rest of the proof we show that:

(i) $|\mathrm{Aut}(G)| \leq (n+1)!$,

(ii) $\ker(\Phi) = \{1\}$.

(i) We give two proofs for proving this claim. The first is more elementary than the second, but we need some parts of it in the proof of (ii). The second is based on the automorphism group of the Johnson graph $J(n,k)$.

Proof 1 *of* (i). Consider the graph G. In $T = V(G)$, consider the subgraphs induced by the subsets $C_0 = S = \{e_i | \ 1 \leq i \leq n\}$, $C_i = \{e_i, e_i + e_j | \ 1 \leq j \leq n, i \neq j\}$, $1 \leq i \leq n$ (we also denote by C_i the subgraph induced by the set C_i). It is clear that C_0 is an n-clique in the graph G. Note that if $e_i + e_r$ and $e_i + e_s$ are two elements of C_i, then we have

$(e_i + e_r) - (e_i + e_s) = e_r + e_s \in T$. Hence each C_i is also an n-clique in the graph G. It can be shown that each C_i, $0 \leq i \leq n$ is a maximal n-clique in G. It is clear that if $i \neq 0$, then $C_0 \cap C_i = \{e_i\}$. Moreover, if $i, j \in \{1, ..., n\}$ and $i \neq j$, then $C_i \cap C_j = \{e_i + e_j\}$. Let M be a maximal n-clique in the graph G. It is not hard to show that $M = C_j$ for some $j \in \{0, 1, ..., n\}$. If a is an automorphism of the graph G, then $a(C_j)$ is a maximal n-clique in the graph G. Hence the natural action of a on the set $X = \{C_0, C_1, ..., C_n\}$ is a permutation on X. Let G_1 be the graph with the vertex set X in which two vertices v and w are adjacent if and only if $v \cap w \neq \emptyset$. Now, it is clear that $G_1 \cong K_{n+1}$, the complete graph on $n + 1$ vertices, and hence $\mathrm{Aut}(G_1) \cong \mathrm{Sym}(X)$. Let $a \in \mathrm{Aut}(G)$ be such that $a(C_j) = C_j$, for each $j \in \{0, 1, ..., n\}$. Noting that $C_0 \cap C_i = \{e_i\}$, $i \neq 0$, we deduce that $a(x) = x$ for every $x \in C_0$. Note that the vertex $e_i + e_j$ is the unique common neighbor of vertices e_i and e_j in the graph G which is not in C_0. This implies that $a(e_i + e_j) = e_i + e_j$. Therefore we have $a(v) = v$ for every $v \in T$. Now it is easy to see that the mapping $\pi\colon \mathrm{Aut}(G) \to \mathrm{Aut}(G_1)$ defined by the rule $\pi(a) = f_a$, where $f_a(C_i) = a(C_i)$ for every $C_i \in X$, is an injection and therefore we have $(n + 1)! \geq |\mathrm{Aut}(G)|$. □

Proof 2 *of* (i). Consider the graph G. We show that this graph is isomorphic with the Johnson graph $J(n + 1, 2)$. We define the mapping

$$f\colon V(G) \to V(J(n+1, 2)),$$

by the rule:

$$f(v) = \begin{cases} \{i, n+1\}, if\ v = e_i \\ \{i, j\}, \quad if\ \ v = e_i + e_j \end{cases}$$

It is clear that f is a bijection. Let $\{v, w\}$ be an edge in the graph G. Then we have three possibilities:
(1) $\{v, w\} = \{e_i, e_j\}$, (2) $\{v, w\} = \{e_i, e_i + e_k\}$, (3) $\{v, w\} = \{e_i + e_k, e_i + e_j\}$.
Now, we have (1) $f(\{v, w\}) = \{\{i, n + 1\}, \{j, n + 1\}\}$, (2) $f(\{v, w\}) = \{\{i, n + 1\}, \{i, k\}\}$, (3) $f(\{v, w\}) = \{\{i, k\}, \{j, k\}\}$. It follows that f is a graph isomorphism. Hence, $\mathrm{Aut}(G) \cong \mathrm{Aut}(J(n+1, 2))$. Since $\mathrm{Aut}(J(n+1, 2)) \cong \mathrm{Sym}([n+1])$ [3, 13, 18], then we have $\mathrm{Aut}(G) \cong \mathrm{Sym}([n + 1])$. □

(ii) we now show that $\ker(\Phi) = \{1\}$. Let $f \in \ker(\Phi)$. Then $f(0) = 0$ and $h = f|_T$ is the identity automorphism of the graph G. Hence $f(x) = x$ for every $x \in T$. Note that when $x \in T$, then $w(x) \in \{1, 2\}$, where $w(x)$ is the weight of x, that is, the number of 1s in the n-tuple x. Let $x \in V(\Gamma)$ and $w(x) = m$. We show by induction on m, that $f(x) = x$. It is clear that when $m = 0, 1, 2$, then the claim is true. Let the claim be true when $w(x) \leq m$, $m \geq 2$. We show that if $w(x) = m + 1$, then $f(x) = x$. Let $y = e_{i_1} + ... + e_{i_m} + e_{i_{m+1}}$ be a vertex of weight $m + 1$. Let $v = y + e_{i_m} + e_{i_{m+1}}$. Since $W(v) = m - 1$, thus $f(v) = v$. Let N be the subgraph of Γ which is induced by the set $N(v)$. Since Γ is vertex-transitive, then $G \cong N$. Also, since $f(v) = v$, then the restriction of f to $N(v)$ is an automorphism of the graph N. In $N(v)$ we define the subsets $M_0 = \{v + e_i | \ 1 \leq i \leq n\}$, $M_i = \{v + e_i, v + e_i + e_j | \ 1 \leq j \leq n, j \neq i\}$, $1 \leq i \leq n$. It can be check that the subgraph induced by each M_i is a maximal n-clique in the graph N. Also, $M_0 \cap M_i = \{v + e_i\}$. Moreover, $v + e_i + e_j$ is the unique common neighbor of the vertices $v + e_i$ and $v + e_j$ in the graph N which is not in M_0. If $x \in M_0$, then $f(x) = x$, because $w(x) \leq m$. This implies that $f(M_i) = M_i$. Now, by an argument similar to what is done in Proof 1, we can see that $f(x) = x$ for every $x \in N(v)$. Since $y \in N(v)$, we have $f(y) = y$. We now conclude that f is the identity automorphism of Γ. Hence we have $\ker(\Phi) = \{1\}$. $\square$

Theorem 3.2. *Let $n > 4$ and $\Gamma = Q_n^2$ be the square of the hypercube Q_n. Then we have* $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_2^n \rtimes \mathrm{Sym}([n+1])$.

Proof. Let A_0 be the stabilizer subgroup of the vertex $v = 0$ in the group $\mathrm{Aut}(\Gamma)$. We know from Lemma 3.1, that $|A_0| \leq (n+1)!$. Let T and $X = \{C_0, ..., C_n\}$ be the sets which are defined in the proof of Lemma 3.1. Note that $\mathbb{Z}_2^n$ is a vector space over the field $\mathbb{Z}_2$ and C_i, $0 \leq i \leq n$, is a basis for this vector space. Let $f_i \colon C_0 \to C_i$ be a bijection. We can linearly extend f_i to an automorphism $e(f_i)$ of the group $\mathbb{Z}_2^n$. It is clear that $e(f_i) \in A_0$. We know that every automorphism of the group $\mathbb{Z}_2^n$ which fixes the set T is an automorphism of the graph Γ. We can see that when $x, y \in C_i$ and $x \neq y$ then $x + y \in T$. Thus we have $e(f_i)(e_r + e_s) = e(f_i)(e_r) + e(f_i)(e_s) \in T$. Hence we have $e(f_i)(T) = T$. Since the number of permutations f_i is $n!$, hence the number of automorphisms of $e(f_i)$ is $n!$. Note that when $i \neq j$, then $e(f_i) \neq e(f_j)$. Now since $0 \leq i \leq n$, then we have at least $(n+1)(n!) = (n+1)!$ distinct automorphisms in the group A_0. Thus by Lemma 3.1, we have $|A_0| = (n+1)!$. We saw, in the proof of Lemma 3.1, that A_0 is isomorphic with a subgroup of $\mathrm{Sym}([n+1])$. Hence we deduce that $A_0 \cong \mathrm{Sym}([n+1])$.

We know, by the orbit-stabilizer theorem, that $|V(\Gamma)||A_0| = |\mathrm{Aut}(\Gamma)|$. Thus we have $|\mathrm{Aut}(\Gamma)| = 2^n[(n+1)!]$. For every $v \in \mathbb{Z}_2^n$, the mapping $f_v(x) = v+x$, for every $x \in \mathbb{Z}_2^n$, is an automorphism of the graph Γ. It is easy to check that $L = \{f_v | \, v \in \mathbb{Z}_2^n\}$ is a subgroup of $\mathrm{Aut}(\Gamma)$ which is isomorphic with $\mathbb{Z}_2^n$. Also it is easy to check that $L \cap A_0 = \{1\}$. Hence we have $|LA_0| = |L||A_0| = 2^n[(n+1)!] = |\mathrm{Aut}(\Gamma)|$. This implies that $\mathrm{Aut}(\Gamma) = LA_0$. Also we can see that for every $v \in \mathbb{Z}_2^n$ and every $a \in A_0$ we have $a^{-1} f_v a = f_{a^{-1}(v)}$. Thus we deduce that L is a normal subgroup of $\mathrm{Aut}(\Gamma)$. We now conclude that

$$\mathrm{Aut}(\Gamma) \cong L \rtimes A_0 \cong \mathbb{Z}_2^n \rtimes \mathrm{Sym}([n+1]). \qquad \square$$

The graph Q_n^2 has some interesting properties. In the next theorem, we show that Q_n^2 is distance-transitive.

Theorem 3.3. *Let $n \geq 4$ be an integer. Then the graph Q_n^2 is a distance-transitive graph.*

Proof. Let v and w be vertices in Q_n^2. It is easy to check that $d(x, y) = \lceil \frac{w(x+y)}{2} \rceil$. Hence we have $d(x, 0) = \lceil \frac{w(x)}{2} \rceil$. Let D be the diameter of Q_n^2. it follows from the first two sentences that $D = \lceil \frac{n}{2} \rceil$. Let A_0 be the stabilizer subgroup the vertex $v = 0$ in $\mathrm{Aut}(Q_n^2)$. Since the graph Q_n^2 is a vertex-transitive graph, it is sufficient to show that the action of A_0 on the set Γ_k is transitive, where Γ_k is the set of vertices at distance k from the vertex $v = 0$. Let x and y be two vertices in Γ_k. There are two possible cases, that is,

(i) $w(x) = w(y)$ or

(ii) $w(x) \neq w(y)$.

(i) Let $w(x) = w(y)$. We know that $w(x) \in \{2k, 2k-1\}$. Without loss of generality, we can assume that $w(x) = 2k$. Let $x = e_{i_1} + ... + e_{i_{2k}}$ and $y = e_{j_1} + ... + e_{j_{2k}}$. There are vertices $e_{x_1}, ..., e_{x_{n-2k}}$ and $e_{y_1}, ..., e_{y_{n-2k}}$ in Q_n^2 such that

$$\{e_{i_1}, ..., e_{i_{2k}}, e_{x_1}, ..., e_{x_{n-2k}}\} = C_0 = \{e_1, e_2, ..., e_n\} = \{e_{j_1}, ..., e_{j_{2k}}, e_{y_1}, ..., e_{y_{n-2k}}\}.$$

Let f be the permutation on the set C_0 which is defined by the rule, $f(e_{i_r}) = e_{j_r}, 1 \leq r \leq 2k$, and $f(e_{x_l}) = e_{y_l}, 1 \leq l \leq n-2k$. We now can see that $e(f)(x) = y$, where $e(f)$ is the linear extension of f to $\mathbb{Z}_2^n$ (see the proof of Theorem 3.2).
(ii) Let $w(x) \neq w(y)$. Without loss of generality we can assume that $w(x) = 2k-1$ and $w(y) = 2k$. Let $x = e_{i_1} + ... + e_{i_{2k-1}}$ and $y = e_{j_1} + ... + e_{j_{2k}}$. Note that $y = (e_{j_1} + e_{j_{2k}}) + (e_{j_2} + e_{j_{2k}}) + ... + (e_{j_{2k-2}} + e_{j_{2k}}) + (e_{j_{2k-1}} + e_{j_{2k}})$. There are vertices $e_{x_1}, ..., e_{x_{n-2k+1}}$ and $e_{y_1} = e_{j_{2k}}, e_{y_2}, ..., e_{y_{n-2k+1}}$ in Q_n^2 such that

$$\begin{aligned}
&\{e_{i_1}, ..., e_{i_{2k-1}}, e_{x_1}, ..., e_{x_{n-2k+1}}\} = C_0, \\
&\{e_{j_1} + e_{j_{2k}}, e_{j_2} + e_{j_{2k}}, ... + e_{j_{2k-2}} + e_{j_{2k}}, e_{j_{2k-1}} + e_{j_{2k}}\} \cup \\
&\qquad\qquad\qquad\{e_{y_1}, e_{y_2} + e_{j_{2k}}, ..., e_{y_{n-2k+1}} + e_{j_{2k}}\} = C_{j_{2k}}
\end{aligned}$$

We now define the bijection g from C_0 to $C_{j_{2k}}$ by the rule $g(e_{i_r}) = e_{j_r} + e_{j_{2k}}$, and $g(e_{x_1}) = e_{y_1}, g(e_{x_i}) = e_{y_i} + e_{j_{2k}}, i \neq 1$. Let $e(g)$ be the linear extension of g to $\mathbb{Z}_2^n$. This yields that $e(g)$ is an automorphism of the graph Q_n^2 such that $e(g)(x) = y$. □

Theorem 3.3 implies many results. For instance, we now can deduce from it the following corollary, which is important in applied graph theory and interconnection networks.

Corollary 3.4. *Let $n \geq 4$ be an integer. Then the connectivity of the graph Q_n^2 is maximal, namely, $n+\binom{n}{2}$ (its valency).*

Proof. By Theorem 3.3 the graph Q_n^2 is distance-transitive, then it is edge-transitive. Thus, it follows from Theorem 2.1, that the connectivity of the graph Q_n^2 is its valency, namely, $n+\binom{n}{2}$. □

A block B, in the action of a group G on a set X, is a subset of X such that $B \cap g(B) \in \{B, \emptyset\}$, for each g in G. If G is transitive on X, then we say that the permutation group (X, G) is primitive if the only blocks are the trivial blocks, that is, those with cardinality $0, 1$ or $|X|$. In the case of an imprimitive permutation group (X, G), the set X is partitioned into a disjoint union of non-trivial blocks, which are permuted by G. We refer to this partition as a block system. A graph Γ is said to be primitive or imprimitive according to the group $\mathrm{Aut}(\Gamma)$ acting on $V(\Gamma)$ has the corresponding property. In the sequel, we need the following definition.

Definition 3.5. A graph $\Gamma = (V, E)$ of diameter D is said to be *antipodal* if for any $u, v, w \in V$ such that $d(u, v) = d(u, w) = D$, then we have $d(v, w) = D$ or $v = w$.

Let $\Gamma_i(x)$ denote the set of vertices of Γ at distance i from the vertex x. Let Γ be a distance-transitive graph. From Definition 3.5 it follows that if $\Gamma_D(x)$ is a singleton set, then the graph Γ is antipodal. It is easy to see that the hypercube Q_n is antipodal, since every vertex u has a unique vertex at maximum distance from it. Note that this graph is at the same time bipartite. We have the following fact [1].

Proposition 3.6. *A distance-transitive graph Γ of diameter D has a block $X = \{u\} \cup \Gamma_D(u)$ if and only if Γ is antipodal, where $\Gamma_D(u)$ is the set of vertices of Γ at distance D from the vertex u.*

Also, we have the following important fact [1].

Theorem 3.7. *An imprimitive distance-transitive graph is either bipartite or antipodal. (Both possibilities can occur in the same graph.)*

We now can state and prove the following fact concerning the square of the hypercube Q_n.

Corollary 3.8. *Let $n \geq 4$ be an integer. Then, the square of the hypercube Q_n, namely, the graph Q_n^2, is an imprimitive distance-transitive graph if and only if n is an odd integer.*

Proof. We know from Theorem 3.3, that the graph $\Gamma = Q_n^2$ is a distance-transitive graph. Let $n = 2k$ be an even integer. If D denotes the diameter of Q_n^2, then $D = k$. Let $C_0 = \{e_1, ..., e_n\}$ be the standard basis of the hypercube Q_n. Let $w = e_1 + e_2 + ... + e_n$ and $B_1 = \{w + e_i \mid 1 \leq i \leq n\}$. Consider the vertex $u = 0$. It is easy to show that $\Gamma_D(u) = \{w\} \cup B_1$. Two vertices w and $w + e_1$ are in $\Gamma_D(u)$, but they are not at distance $k = D$ from each other, since they are adjacent and $k > 1$. Thus, when n is an even integer, then the graph Q_n^2 is not antipodal. Since the girth of Q_n^2 is 3, then this graph is not bipartite. Now, Theorem 3.7 implies that the graph $\Gamma = Q_n^2$ is not imprimitive.

Now assume that $n = 2k + 1$ is an odd integer. It is easy to see that $D = k + 1$ and $\Gamma_D(0) = \{w\}$. Therefore by Proposition 3.6, Γ is antipodal, and hence has the set $\{0, w\}$ as a block. We now conclude that, when n is an odd integer, then Q_n^2 is an imprimitive graph. □

Let $\Gamma = (V, E)$ be a simple connected graph with diameter D. A *distance-regular* graph $\Gamma = (V, E)$, with diameter D, is a regular connected graph of valency k with the following property. There are positive integers

$$b_0 = k, b_1, ..., b_{D-1}; c_1 = 1, c_2, ..., c_D,$$

such that for each pair (u, v) of vertices satisfying $u \in \Gamma_i(v)$, we have

(1) the number of vertices in $\Gamma_{i-1}(v)$ adjacent to u is c_i, $1 \leq i \leq D$.

(2) the number of vertices in $\Gamma_{i+1}(v)$ adjacent to u is b_i, $0 \leq i \leq D - 1$.

The intersection array of Γ is $i(\Gamma) = \{k, b_1, ..., b_{D-1}; 1, c_2, ..., c_d\}$.

It is easy to show that if Γ is a distance-transitive graph, then it is distance-regular [1]. Hence, the hypercube $Q_n,\ n > 2$ is a distance-regular graph. We can verify by an easy argument that the intersection array of Q_n is

$$\{n, n-1, n-2, ..., 1; 1, 2, 3, ..., n\}.$$

In other words, for hypercube Q_n, we have $b_i = n - i$, $c_i = i$, $1 \leq i \leq n - 1$, and $b_0 = n$, $c_n = n$. In the following theorem, we determine the intersection array of the square of the hypercube Q_n [1].

Theorem 3.9. *Let $n > 3$ be an integer and $\Gamma = Q_n^2$ be the square of the hypercube Q_n. Let D denote the diameter of Q_n^2. Then for the intersection array of this graph we have $b_0 = \binom{n+1}{2}$, $b_i = \binom{n-2i+1}{2}$, $c_i = \binom{2i}{2}$, $1 \leq i \leq D - 1$. Also, $c_D = \binom{n+1}{2}$, when n is an odd integer and $c_D = \binom{n}{2}$ when n is an even integer.*

Proof. Since Q_n^2 is a regular graph of valency $\binom{n+1}{2}$, thus we have b_0=$\binom{n+1}{2}$. Let u be a vertex in Q_n^2 at distance i from the vertex $v = 0$. It is easy to check that $w(u) = 2i$ or $w(u) = 2i - 1$. This implies that that the diameter of the graph Q_n^2 is $D = \lceil \frac{n}{2} \rceil$.

Let u be a vertex in Q_n^2 at distance $i \geq 1$ from the vertex $v = 0$, such that $i \neq D$. There are two cases, that is, $w(u) = 2i$, or $w(u) = 2i - 1$. Without lose of generality we can assume that $w(u) = 2i$. Hence u is of the form $u = e_{j_1} + e_{j_2} + ... + e_{j_{2i}}$. Now it is easy to show that if x is a vertex of Q_n^2 adjacent to u and at distance $i - 1$ from the vertex $v = 0$, then x must be of the form $x = u + e_k + e_l$, where $e_k, e_l \in \{e_{j_1}, e_{j_2}, ..., e_{j_{2i}}\}$. It is clear that the number of such xs is equal to $\binom{2i}{2}$. Moreover, If x is a vertex of Q_n^2 adjacent to u and at distance $i + 1$ from the vertex $v = 0$, then x must be of the forms $x = u + e_k$ or $x = u + e_k + e_l$, where $e_k, e_l \in \{e_1, e_2, ..., e_n\} - \{e_{j_1}, e_{j_2}, ..., e_{j_{2i}}\}$. It is clear that the number of such xs is equal to $\binom{n-2i}{1}$+$\binom{n-2i}{2}$=$\binom{n-2i+1}{2}$. We now deduce that when $1 \leq i \leq D - 1$, then c_i=$\binom{2i}{2}$, and b_i=$\binom{n-2i+1}{2}$.

When n is an odd integer, then the vertex $u = e_1 + e_2 + ... + e_n$ is the unique vertex of Q_n^2 at distance D from the vertex $v = 0$. Thus c_D=$\binom{n+1}{2}$, namely, the valency of u. If n is an even integer, then $\Gamma_D(0) = \{u, u + e_i | \ 1 \leq i \leq n\}$ is the set of vertices of $\Gamma = Q_n^2$ at distance D from the vertex $v = 0$. Now, by a similar argument which is done in the first section of the proof, it can be shown that c_D=$\binom{n}{2}$. □

Remark 3.10. There are distance-regular graphs $\Gamma = (V, E)$, with the property that their squares are not distance-regular. For instance, consider the cycle C_n with vertex set $\{0, 1, 2, ..., n - 1\}$. It is well known that C_n is a distance-regular graph of diameter $[\frac{n}{2}]$ with the intersection array:

$$\{2, 1, 1, ..., 1, 1; 1, 1, 1, ..., 1, 2\} \text{ when } n \text{ is an even integer and,}$$
$$\{2, 1, 1, ..., 1, 1; 1, 1, 1, ..., 1, 1\} \text{ when } n \text{ is an odd integer [1].}$$

Now, assume that $n \geq 7$. It can be shown by an easy argument that $\Gamma = C_n^2$ is not a distance-regular graph. To see this fact, let v be a vertex in C_n at distance i from the vertex 0, and $c_i(v) = |\Gamma_{i-1}(0) \cap N(v)|$. It is easy to show that $\Gamma_i(0) = \{2i, -2i, 2i-1, -2i+1\}$, and $c_i(2i) = 1$, but $c_i(2i - 1) = 2$.

Remark 3.11. Let $n, k \in \mathbb{N}$ with $k < n$, and let $[n] = \{1, ..., n\}$. Consider the Johnson graph $J(n, k)$. It is clear that the order of this graph is $\binom{n}{k}$. It is easy to check that $J(n, k) \cong J(n, n - k)$, hence we assume that $1 \leq k \leq \frac{n}{2}$. The class of Johnson graphs is one of the most well known and interesting subclass of distance-regular graphs [3]. It is easy to show that if v and w are vertices in the Johnson graph $J(n, k)$, then $d(v, w) = k - |v \cap w|$. Thus, the diameter of the Johnson graph $J(n, k)$ is k. Note that the graph $J(n, 1)$ is the complete graph K_n and hence it is distance-regular. The diameter of the graph $J(n, 2)$ is 2, hence the diameter of its square is 1. Thus the graph $J^2(n, 2)$ is the complete graph K_m, and hence it is a distance-regular graph (m=$\binom{n}{2}$). We can show that when $k = 3$, then the square of Johnson graph $\Gamma = J(n, k)$ is a distance-regular graph if and only if $n = 6$. For checking this, let $v = \{1, 2, 3\}$. Note that the diameter of the graph Γ^2 is 2 and a vertex w in Γ^2 is at distance 2 from v if and only if $|v \cap w| = 0$. Moreover w is at distance 1 from v if and only if $|v \cap w| \in \{1, 2\}$. Hence $\Gamma_1^2(v) = V(\Gamma) - \{v, v^c\}$ and $\Gamma_2^2(v) = \{v^c\}$, where v^c is the complement of the set v in the set $\{1, 2, ..., 6\}$. Thus $v^c = \{4, 5, 6\}$. Now, it is clear that $b_0(v)$=$\binom{3}{2}\binom{3}{1}$+$\binom{3}{1}\binom{3}{2}$=18. Also, for every $w \in \Gamma_1^2(v)$, $c_1(w) = 1$ and $b_1(w) = 1$, and $c_2(v^c) = |\Gamma_1^2(v)| = 18$. Thus the graph $\Gamma^2 = J^2(6, 3)$ is a distance-regular graph with intersection array $\{18, 1; 1, 18\}$. But, if $n > 6$, then the graph $\Gamma^2 = J^2(n, 3)$ is not distance-regular. In fact if $n > 6$, then for the vertex $v = \{1, 2, 3\}$, each of the vertices $u = \{1, 2, 4\}$ and $w = \{1, 4, 5\}$ is in $\Gamma_1^2(v)$. If $x \in \Gamma_2^2(v)$ is adjacent to u, then $4 \in x$, and hence $x = \{4\} \cup y$, where $y \subset v^c - \{4\}$ and $|y| = 2$. We now can deduce that $b_1(u)$=$\binom{n-4}{2}$. On the other hand, if $x \in \Gamma_2^2(v)$ is adjacent to w, then $4 \in x$ and $5 \notin x$, or $5 \in x$ and $4 \notin x$ or $4, 5 \in x$. Thus, $b_1(w)$=$2\binom{n-5}{2}$+$\binom{n-5}{1}$=$\binom{n-4}{2}$+$\binom{n-5}{2}$. This implies that when $n \geq 7$ then the graph $J^2(n, 3)$ cannot be distance-regular.

By a similar argument we we can show that the graph $J^2(8, 4)$ is distance-regular, but if $n > 8$, then the graph $J^2(n, 4)$ is not distance-regular.

Remark 3.12. Let $\Gamma = (V, E)$ be a graph. Γ is said to be a *strongly regular* graph with parameters (n, k, λ, μ), whenever $|V| = n$, Γ is a regular graph of valency k, every pair of adjacent vertices of Γ have λ common neighbor(s), and every pair of non-adjacent vertices of Γ have μ common neighbor(s). It is clear that the diameter of every strongly regular graph is 2. It is easy to show that if a graph Γ is a distance-regular graph of diameter 2 and order n, with intersection array $(b_0, b_1; c_1, c_2)$, then Γ is a strongly regular graph with parameters $(n, b_0, b_0 - b_1 - 1, c_2)$. We know that the diameter of the graph Q_n^2 is $\lceil \frac{n}{2} \rceil$. Now, it follows from Theorem 3.3, that Q_3^2 is a strongly regular graph with parameter $(8, 6, 4, 6)$. This graph is known as the *coktail-party* graph $CP(4)$ [1]. Also, the graph Q_4^2 is a strongly regular graph with parameter $(16, 10, 6, 6)$. We know that when a graph Γ is a strongly regular graph with parameters (n, k, λ, μ), then its complement is again a strongly regular graph with parameter $(n, n - k - 1, n - 2 - 2k + \mu, n - 2k + \lambda)$ [9]. Hence, the complement of the graph Q_4^2 is a strongly regular graph with parameter $(16, 5, 0, 2)$. This graph is known as the *Clebsch* graph [9] and it is the unique strongly regular graph with parameters $(16, 5, 0, 2)$. Figure 1 displays a version of the Clebsch graph (the complement of the graph Q_4^2) in the plane [9].

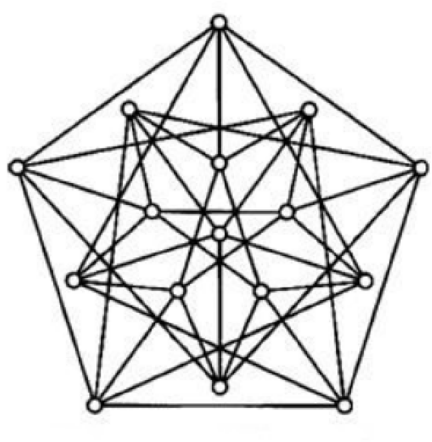

Figure 1: The Clebsch graph.

4 The spectrum of the square of the hypercube

The square of the hypercube Q_n has some further interesting algebraic properties. For obtaining some of those properties, we need the spectrum of this graph. The spectrum of Q_n is known [1], however we are not aware of a paper showing the spectrum of Q_n^2. Here we compute by means of an algebraic and self-contained method the spectrum of Q_n^2.

Let $\Gamma = (V, E)$ be a graph with the vertex set $\{v_1, \cdots, v_n\}$. Then the adjacency matrix of Γ is an $n \times n$ matrix $A = (a_{ij})$, in which columns and rows are labeled by V and a_{ij} is defined as follow:

$$a_{ij} = A(v_i, v_j) = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise.} \end{cases}$$

If $Ax = \lambda x, x \neq 0$, then λ is an eigenvalue of A, and x is an eigenvector of A corresponding to λ [9]. Let $\lambda_1, \cdots, \lambda_r$ be eigenvalues of A with multiplicities $m_1, \cdots, m_r$, respectively. The spectrum of the graph Γ is defined as

$$Spec(\Gamma) = \begin{Bmatrix} \lambda_1, & \lambda_2, & \cdots, & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{Bmatrix}.$$

When we work with graphs there is an additional refinement. We can suppose that an eigenvector is a real function f on the vertices. Then if at any vertex v you sum up the values of f on its neighboring vertices, you should get λ times the values of f at v. Formally,

$$\sum_{w \in N(v)} f(w) = \lambda f(v).$$

Let G be a finite abelian group (written additively) of order $|G|$ with identity element 0=0_G. A character χ of G is a homomorphism from G into the multiplicative group U of complex numbers of absolute value 1, that is, a mapping from G into U with $\chi(g_1 + g_2) = \chi(g_1)\chi(g_2)$ for all $g_1, g_2 \in G$. If G is a finite abelian group, then there are integers $n_1, \cdots, n_k$, such that $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$. Let $S = \{s_1, \cdots, s_n\}$ be a non-empty subset of G such that $0 \notin S$ and $S = -S$. Let $\Gamma = \mathrm{Cay}(G; S)$. Assume $f \colon G \longrightarrow \mathbb{C}^*$ is a character where $\mathbb{C}^*$ is the multiplicative group of the complex numbers. If $\omega_{ij} = e^{\frac{2\pi i j}{n_i}}$, $0 \leq i \leq k$, $1 \leq j \leq n_i$, is an n_ith root of unitary, then f is of the form $f = f_{(\omega_1, \cdots, \omega_k)}$, where $f_{(\omega_1, \cdots, \omega_k)}(x_1, \cdots, x_k) = \omega_1^{x_1}\omega_2^{x_2} \cdots \omega_k^{x_k}$, for each $(x_1, x_2, ..., x_k) \in G$ [12].

If v is a vertex of Γ, then we know that $N(v) = \{v + s_1, \cdots, v + s_n\}$ is the set of vertices that are adjacent to v. We now have

$$\sum_{w \in N(v)} f(w) = \sum_{i=1}^{n} f(v + s_i) = \sum_{i=1}^{n} f(v)f(s_i) = f(v)(\sum_{i=1}^{n} f(s_i)).$$

Therefore, if we let $\lambda = \lambda_f = \sum_{s \in S} f(s)$ then we have $\sum_{w \in N(v)} f(w) = \lambda_f f(v)$, and hence the mapping f is an eigenvector for the Cayley graph Γ with corresponding eigenvalue $\lambda = \lambda_f = \sum_{s \in S} f(s)$.

Theorem 4.1. *Let $n > 3$ be an integer and Q_n^2 be the square of the hypercube Q_n. Then each of the eigenvalues of Q_n^2 is of the form,*

$$\lambda_i = \frac{1}{2}n(n+1) - 2i(n+1) + 2i^2,$$

for $0 \le i \le \lfloor \frac{n+1}{2} \rfloor$. Moreover, the multiplicity of λ_0 is 1, the multiplicity of λ_i is $m(\lambda_i)=\binom{n}{i}+\binom{n}{n+1-i}$, for $1 \le i \le \lfloor \frac{n+1}{2} \rfloor$, when n is an even integer, and $m(\lambda_i)=\binom{n}{i}+\binom{n}{n+1-i}$ for $1 \le i < \lfloor \frac{n+1}{2} \rfloor$, when n is an odd integer, with $m(\lambda_j)=\binom{n}{j}$ for $j = \lfloor \frac{n+1}{2} \rfloor$.

Proof. According to what is stated before this theorem, every eigenvector of the graph $\Gamma = Q_n^2 = \mathrm{Cay}(\mathbb{Z}_2^n; S)$ is of the form $f = f_{(\omega_1,\cdots,\omega_n)}$, where each ω_i, $1 \le i \le n$, is a complex number such that $\omega_i^2 = 1$, namely, $\omega_i \in \{1, -1\}$. We now have

$$\begin{aligned}\lambda_f = \sum_{w\in S} f(w) &= \sum_{i=1}^{n} f(e_i) + \sum_{i,j=1,\ i\neq j}^{n} f(e_i + e_j)\\ &= \sum_{i=1}^{n} f(e_i) + \sum_{i,j=1,\ i\neq j}^{n} f(e_i)f(e_j).\end{aligned}$$

Note that for every vertex $v = (x_1, \ldots, x_n)$, $x_i \in \{0, 1\}$ in Q_n^2, we have

$$f(x_1, \ldots, x_n) = f_{(w_1,\ldots,w_n)}(x_1 \ldots, x_n) = w_1^{x_1} \ldots w_n^{x_n}.$$

Note that in the computing of the value of $w_1^{x_1} \ldots w_n^{x_n}$ we can ignore w_i when $w_i = 1$. Thus, for $e_k = (0, \ldots, 0, 1, 0 \ldots, 0)$, where 1 is the kth entry, we have;

$$\begin{aligned} f(e_k) &= f_{(w_1,\ldots,w_n)}(0, \ldots, 0, 1, 0, \ldots, 0)\\ &= w_1^0 \ldots w_k^1 w_{k+1}^0 \ldots w_n^0 = \begin{cases} -1 & if\ \ w_k = -1\\ 1 & if\ \ w_k = 1\end{cases}\end{aligned}$$

Hence, if in the n-tuple $(w_1, \ldots, w_n)$ the number of $-1s$ is i (and therefore the number of $1s$ is $(n - i)$), then in the sum

$$\sum_{k=1}^{n} f(e_k) = \sum_{k=1}^{n} f_{(w_1,\ldots,w_n)}(0, \ldots, x_k, 0, \ldots, 0), \ \ x_k = 1,$$

the contribution of -1 is i and the contribution of 1 is $n - i$. Therefore, we have

$$\sum_{k=1}^{n} f(e_k) = -i + (n - i) = n - 2i.$$

On the other hand, since

$$(\sum_{k=1}^{n} f(e_k))^2 = \sum_{k=1}^{n} f(e_k)^2 + 2 \sum_{i,j=1,\ i\neq j}^{n} f(e_i)f(e_j),$$

therefore, we have

$$\sum_{i,j=1,\ i\neq j}^{n} f(e_i)f(e_j) = \frac{1}{2}((n-2i)^2 - \sum_{k=1}^{n} f(e_k)^2).$$

Now since $\sum_{k=1}^{n} f(e_k)^2 = n$, thus we have

$$\begin{aligned}\lambda_f &= \sum_{i=1}^{n} f(e_i) + \sum_{i,j=1,\ i\neq j}^{n} f(e_i)f(e_j) = (n-2i) + \frac{1}{2}((n-2i)^2 - n)\\ &= \frac{1}{2}n + \frac{1}{2}n^2 - 2ni + 2i^2 - 2i = \frac{1}{2}n(n+1) - 2i(n+1) + 2i^2.\end{aligned}$$

Note that $f = f_{(w_1,w_2,\ldots,w_n)}$, and the number of sequences $(w_1\ldots,w_n)$ in which i entries are -1 is $\binom{n}{i}$. If we denote λ_f by λ_i, then we deduce that every eigenvalue of the graph Q_n^2 is of the form

$$\lambda_i = \frac{1}{2}n(n+1) - 2i(n+1) + 2i^2,\ 0 \leq i \leq n. \quad (**)$$

Consider the real function $f(x) = \frac{1}{2}n(n+1) - 2x(n+1) + 2x^2$. Then $\lambda_i = f(i)$, $i \in \{0,1,...,n\}$. This function reaches its minimum at $x = \frac{n+1}{2}$. Now by using some calculus, we can see that $f(x) = f(n+1-x)$. Thus, we have $\lambda_i = f(i) = f(n+1-i) = \lambda_{n+1-i}$, $1 \leq i \leq n$. Now it follows that if $n = 2k$, then the multiplicity of λ_i is $\binom{n}{i}+\binom{n}{n+1-i}$, $1 \leq i \leq k$. Note that when $n = 2k+1$, then $n+1-(k+1) = k+1$, thus $\lambda_{n+1-(k+1)} = \lambda_{k+1}$. Hence if $n = 2k+1$, then the multiplicity of λ_i is $\binom{n}{i}+\binom{n}{n+1-i}$, $1 \leq i \leq k$, and the multiplicity of λ_{k+1} is $\binom{n}{k+1}$. Note that since the graph Q_n^2 is a $\binom{n+1}{2}$-regular graph, hence the multiplicity of λ_0=$\binom{n+1}{2}$=$\frac{1}{2}(n+1)n$ is 1. $\square$

Let $\Gamma = (V,E)$ be a graph. The line graph $L(\Gamma)$ of the graph Γ is constructed by taking the edges of Γ as vertices of $L(\Gamma)$, and joining two vertices in $L(\Gamma)$ whenever the corresponding edges in Γ have a common vertex. Note that if $e = \{v,w\}$ is an edge of Γ, then its degree in the graph $L(\Gamma)$ is $\deg(v) + \deg(w) - 2$. Concerning the eigenvalues of the line graphs, we have the following fact [1, 9].

Proposition 4.2. *If λ is an eigenvalue of a line graph $L(\Gamma)$, then $\lambda \geq -2$.*

Therefore, if $\lambda < -2$ is an eigenvalue of a graph graph Γ, then Γ is not a line graph.

A (c,d)-*biregular* graph is a bipartite graph in which each vertex in one part has degree c and each vertex in the other part has degree d [25]. It is known and easy to prove that if the line graph of the graph Γ is regular, then Γ is a regular or a (c,d)-biregular bipartite graph.

Theorem 4.3. *Let $n \geq 4$ be an integer and Q_n^2 be the square of the hypercube Q_n. Then Q_n^2 cannot be a line graph.*

Proof. Let $k = \lfloor \frac{n}{2} \rfloor$. Hence, if n is an even integer, then $n = 2k$ and if n is an odd integer then $n = 2k+1$. It follows from Theorem 4.1, that the smallest eigenvalue of the graph Q_n^2 is λ_k, when n is an even integer and λ_{k+1}, when n is an odd integer. Now consider the eigenvalue λ_k of the graph Q_n^2 in (**) (in the proof of Theorem 4.1). Therefore if n is an even integer, then we have

$$\lambda_k = k(2k+1) - 2k(2k+1) + 2k^2 = k(2k+1-4k-2+2k) = -k.$$

Moreover if $n = 2k + 1$, then we have,

$$\begin{aligned}\lambda_{k+1} &= (2k+1)(k+1) - 2(k+1)(2k+2) + 2(k+1)^2 \\ &= (k+1)(2k+1-4k-4+2k+2) = -k-1.\end{aligned}$$

We now deduce that when $n \geq 5$, then $\lambda_k \leq -3$. Now, it follows from Proposition 4.2, when $n \geq 5$, then the graph Q_n^2 can not be a line graph.

Our argument shows that if λ is an eigenvalue of the graph Q_4^2, then $\lambda \geq -2$, and hence in this way we can not say anything about our claim.

We now show that Q_4^2 is not a line graph. On the contrary, assume that Q_4^2 is a line graph. Thus, there is a graph Δ such that $Q_4^2 = L(\Delta)$. Since Q_4^2 is a regular graph, hence

(i) Δ is a regular graph, or

(ii) Δ is a biregular bipartite graph.

(i) Let $\Delta = (V, E)$ be a t-regular graph of order h. Since Q_4^2 is 10-regular, thus, $L(\Delta) = Q_4^2$ is a $2t-2 = 10$-regular graph, and hence $t = 6$. Therefore we have $16 = |E| = \frac{1}{2}6h = 3h$, which is impossible.

(ii) Let $\Delta = (A \cup B, E)$ be a (c, d)-biregular bipartite graph such that every vertex in A (B) is of degree c (d). Hence we have $16 = |E| = c|A| = d|B|$. Thus c and d must divide 16. On the other hand, if $e = \{a, b\}$ is an edge of Δ, then we must have $\deg(a) + \deg(b) - 2 = 10 = c + d - 2$. Hence we have $c + d = 12$. We now can check that $\{c, d\} = \{4, 8\}$. Without loss of generality, we can assume that $d = 8$ and $c = 4$. Hence we must have $|A| \geq 8$. Now since each vertex in A is of degree $c = 4$, then we must have, $16 = |E| = c|A| = 4|A| \geq 4 \times 8 = 32$, which is impossible.

Our argument shows that the graph Q_4^2 is also not a line graph. $\square$

An *automorphic* graph is a distance-transitive graph whose automorphism group acts primitively on its vertices, and not a complete graph or a line graph.

Automorphic graphs are apparently very rare. For instance, there are exactly three cubic automorphic graphs [1, 2]. It is clear that for $n \geq 3$, the graph Q_n^2 is not a complete graph. We now derive from Corollary 3.8, and Theorem 4.3, the following important result.

Corollary 4.4. *Let $n \geq 4$ be an integer. Then the square of the hypercube Q_n, that is, the graph Q_n^2, is an automorphic graph if and only if n is an even integer.*

5 Conclusion

In this paper, we proved that the square of the distance-transitive graph Q_n, that is, the graph Q_n^2, is again a distance-transitive graph (Theorem 3.3). We showed that there are important classes of distance-transitive graphs (including the cycle C_n, $n \geq 7$), such that their squares are not even distance-regular (and hence are not distance-transitive) (Remark 3.11). Also, we determined the spectrum of the graph Q_n^2 (Theorem 4.1). Moreover, we showed that when $n > 3$ is an even integer, then the graph Q_n^2 is an automorphic graph, that is, a distance-transitive primitive graph which is not a complete or a line graph (Corollary 4.4).

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A new family of additive designs

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Abstract

In this paper we construct a family of 2-(q^n, sp^2, λ) additive designs $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, where q is a power of a prime p and $\mathcal{P}$ is a n-dimensional vector space over $\mathrm{GF}(q)$, and we compute their parameters explicitly. These designs, except for some special cases, had not been considered in the previous literature on additive block designs.

Keywords: Block designs, additive designs.

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1 Additive designs

Point-flat designs $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ of an affine geometry $\mathrm{AG}(d, p)$ over $\mathrm{GF}(p)$, as well as of a projective geometry $\mathrm{PG}(d, 2)$ over $\mathrm{GF}(2)$, are basic examples of 2-(v, k, λ) designs for which, if $\mathcal{P}$ is taken to be $\mathrm{GF}(p^d)$, respectively $\mathrm{GF}(2^{d+1})^* = \mathrm{GF}(2^{d+1}) \setminus \{0\}$, then the blocks have the property that the sum of their points in $\mathcal{P}$ is zero.

As soon as $k > 4$, the family $\mathcal{B}$ of blocks of any of these designs is strictly contained in the family $\mathcal{B}_k$ (respectively, $\mathcal{B}_k^*$) of all the k-subsets of $\mathrm{GF}(p^d)$ (respectively, $\mathrm{GF}(2^{d+1})^*$) whose elements sum up to zero. In [19], and in [13] for the case $p = 2$, it is shown that the incidence structure $\mathcal{D}_k = (\mathcal{P}, \mathcal{B}_k)$ is a 2-(p^d, k, λ) design if and only if $k = mp$ for some integer m, and that, in such a case, the automorphism group of $\mathcal{D}_k$ is the group of invertible affine mappings $\phi(x) = \phi_0(x) + \phi(0)$ over $\mathrm{GF}(p)$, with $\phi_0 \in \mathrm{GL}(d, p)$. In this case, by applying a well-known result of Li and Wan [15] (see also [14, Theorem 2.4] and [20]), one finds that

$$\lambda = \frac{1}{p^d}\binom{p^d - 2}{k - 2} + c_k\,\frac{k-1}{p^d}\binom{p^{d-1} - 1}{m - 1},$$

where $c_k = (-1)^m$ if $p = 2$ and $c_k = 1$ otherwise.

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Moreover, for $p = 2$, the incidence structure $\mathcal{D}_k^* = (\mathrm{GF}(2^{d+1})^*, \mathcal{B}_k^*)$ is a 2-$(2^{d+1} - 1, k, \lambda)$ design for any integer k, and, again, the parameter λ is given by an explicit formula [13, Proposition 2.6], whereas the automorphism group of $\mathcal{D}_k^*$ is the group $\mathrm{GL}(d+1, 2)$ of invertible linear mappings on $\mathrm{GF}(2^{d+1})$ over $\mathrm{GF}(2)$. Among the subdesigns of the latter designs one finds the only known Steiner 2-design over a finite field, found by Braun et al. [2] and revisited in [6], when seen as a 2-$(8191, 7, 1)$ design (note that $8191 = 2^{13} - 1$), as well as the 2-$(2v - 1, 7, 7)$ designs over $\mathrm{GF}(2)$ considered in [4], [21].

More generally, in [8] and [9] a 2-(v, k, λ) design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is said to be *additive* if $\mathcal{P}$ can be embedded in a finite commutative group G in such a way that the sum of the elements in every block is zero. Moreover, it is shown that symmetric and affine resolvable 2-designs are additive and that, for these designs, and for a suitable choice of G, the blocks are *exactly* the (unordered) k-tuples of elements in $\mathcal{P}$ which sum up to zero, so that the automorphism group of $\mathcal{D}$ coincides with the stabilizer of $\mathcal{P}$ in the automorphism group of G. On the contrary, it is shown that the only additive Steiner triple systems are the point-line designs of $\mathrm{AG}(d, 3)$ and $\mathrm{PG}(d, 2)$ (cf. also [11] and [12]).

With a similar construction to that considered in the present paper, in [18] an additive 2-design is provided, for which no embedding can be found in such a way that the blocks are characterized as the k-sets of elements of $\mathcal{P}$ summing up to zero, thereby settling an open question posed in [9].

Interestingly enough, the search for new additive designs occasionally produces new designs which, in addition to being additive, turn out to be also the first known examples of designs with a certain set of parameters. For instance, in [16] an additive 2-$(81, 6, 2)$ design is constructed, which is also the first known example of a simple 2-design (that is, with no repeated blocks) with these parameters, whereas in [17] an additive Steiner 2-$(124, 4, 1)$ design is presented. More generally, some infinite classes of additive Steiner 2-designs are presented in [5] and [3], in the latter case as a notable application of the method of partial differences.

The goal of this paper is to introduce a class of (additive) block designs that are subdesigns of $\mathcal{D} = (\mathrm{GF}(p^d), \mathcal{B}_k)$ and which seem not to have appeared so far in the literature.

2 Some new designs

In [7] we considered the 2-$(n^2, 2n, 2n - 1)$ design obtained by taking the points and the (unordered) pairs of distinct parallel lines of a finite affine plane of order $n > 2$. Similarly, in this paper we consider an incidence structure whose blocks are unions of suitable parallel lines in an affine geometry over $\mathrm{GF}(p)$. We obtain an additive subdesign of the design $\mathcal{D} = (\mathrm{GF}(p^d), \mathcal{B}_k)$ considered here in Section 1, for which we are able to compute the parameters.

Note that one finds, among these designs, the classical point-flat designs $\mathrm{AG}_2(n, 3)$, $n \geq 2$, and $\mathrm{AG}_3(n, 2)$, $n \geq 3$. Interestingly enough, in some special cases the 2-(v, k, λ) designs that we construct have a smaller λ than that of the corresponding point-flat designs of $\mathrm{AG}(d, p)$ with the same parameters v and k.

As usual, we say that m vectors $x_1, x_2, \ldots, x_m$ are *affinely independent* if the $m - 1$ vectors $x_2 - x_1, \ldots, x_m - x_1$ are linearly independent.

Theorem 2.1. *Let q be a power of a prime p, and let $\mathcal{P}$ be a n-dimensional vector space over* $\mathrm{GF}(q)$. *Let m be divisible by p, with $3 \leq m \leq n + 1$, and let $\mathcal{B}$ consist of all* *subsets* $\mathfrak{b}(x_1, x_2, \ldots, x_m)$ *of* $\mathcal{P}$ *of the form*

$$\mathfrak{b}(x_1, x_2, \ldots, x_m) = \{x_j + s(x_1 + x_2 + \cdots + x_m) |\, 1 \le j \le m \text{ and } s \in \mathrm{GF}(p)\},$$

where $x_1, x_2, \ldots, x_m \in \mathcal{P}$ *are affinely independent vectors over* $\mathrm{GF}(q)$*, and* $\mathrm{GF}(p)$ *is the fundamental subfield of* $\mathrm{GF}(q)$*. Then* $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ *is a* $2 - (q^n, mp, \lambda)$ *additive design, with*

$$\lambda = \begin{cases} \frac{(q^n-q)\cdots(q^n-q^{m-2})}{(m-1)!\,p^{m-2}\,(p-1)}\,(mp-1) & \text{if } m > 4, \\ \frac{(q^n-q)(q^n-q^2)}{24} & \text{if } m = 4, \\ \frac{q^n-q}{6} & \text{if } m = 3. \end{cases} \tag{2.1}$$

Proof. Suppose $\mathfrak{b}(y_1, y_2, \ldots, y_m) \in \mathcal{B}$. Since the vectors $y_1, y_2, \ldots, y_m \in \mathcal{P}$ are affinely independent, the sum $(y_2 - y_1) + \cdots + (y_m - y_1)$ is not zero and, since m is divisible by p and $my_1 = 0$, we deduce that $y_1 + y_2 + \cdots + y_m$ is not zero, as well. Since the case $m = 2$ is excluded by hypothesis, the sets $\{y_i + s(y_1 + y_2 + \cdots + y_m) |\, s \in \mathrm{GF}(p)\}$ and $\{y_j + s(y_1 + y_2 + \cdots + y_m) |\, s \in \mathrm{GF}(p)\}$ are disjoint, for $i \neq j$, thus $\mathfrak{b}(y_1, y_2, \ldots, y_m)$ contains exactly mp elements (note that in the excluded case where $m = 2$ the two sets are coincident). Because $m \le n + 1$ and because $G = \mathrm{Aff}(\mathcal{P})$ (the affine group of $\mathcal{P}$ over $\mathrm{GF}(q)$) acts 2-homogeneously on $\mathcal{P}$ and permutes the subsets $\{w_1, w_2, \ldots, w_m\}$ of $\mathcal{P}$ consisting of m affinely independent vectors, the block-set $\mathcal{B}$ may be written as $\mathcal{B} = \mathfrak{b}_0^G$ (the G-orbit of a fixed block $\mathfrak{b}_0 = \mathfrak{b}(x_1, x_2, \ldots, x_m)$), and it follows from [1, Proposition 4.6, page 175] (or from [10, Remark 4.29, page 82]) that $\mathcal{D}$ is a $2 - (v, k, \lambda)$ design with parameters $v = q^n$, $k = mp$ and

$$b = |\mathcal{B}| = \frac{|G|}{|S_{\mathfrak{b}_0}|},$$

where $S_{\mathfrak{b}_0} = \{f \in \mathrm{Aff}(\mathcal{P}) |\, f(\mathfrak{b}_0) = \mathfrak{b}_0\}$ is the setwise stabilizer of the base block $\mathfrak{b}_0$.

Since, for every block $\mathfrak{b} = \mathfrak{b}(y_1, y_2, \ldots, y_m)$ of $\mathcal{D}$,

$$\sum_{y \in \mathfrak{b}} y = \begin{cases} \left(p + m\binom{p}{2}\right)(y_1 + y_2 + \cdots + y_m) & \text{for } p > 2 \\ m(y_1 + y_2 + \cdots + y_m) & \text{for } p = 2 \end{cases},$$

which is the zero vector in either case, the design $\mathcal{D}$ is additive by [8, Proposition 2.7, page 277].

In order to determine the number b of blocks of $\mathcal{D}$, we claim that, if $\mathfrak{b} = \mathfrak{b}(y_1, y_2, \ldots, y_m)$ is any block of the 2-design $\mathcal{D}$ and if we denote by $R_{\mathfrak{b}}$ the number of (unordered) sets $\{z_1, z_2, \ldots, z_m\} \subset \mathfrak{b}$ consisting of affinely independent vectors $z_1, z_2, \ldots, z_m$ having the property that $\mathfrak{b}(z_1, z_2, \ldots, z_m) = \mathfrak{b}(y_1, y_2, \ldots, y_m)$, then $R_{\mathfrak{b}}$ does not depend on $\mathfrak{b}$ and we have

$$R_{\mathfrak{b}} = \begin{cases} p^{m-1}(p-1), & \text{if } m > 4, \\ 56, & \text{if } m = 4, \\ 72, & \text{if } m = 3. \end{cases}$$

Indeed, if $t_1, t_2, \ldots, t_m \in \mathrm{GF}(p)$ are chosen in such a way that $t_1 + t_2 + \cdots + t_m \neq -1 \in \mathrm{GF}(p)$, then the m (distinct) vectors $z_i = y_i + t_i(y_1 + y_2 + \cdots + y_m)$ of $\mathcal{P}$ (belonging to $\mathfrak{b}$) are affinely independent and have the property that $\mathfrak{b}(z_1, z_2, \ldots, z_m) = \mathfrak{b}$. Hence $R_{\mathfrak{b}} \geq p^m - p^{m-1} = p^{m-1}(p-1)$.

On the other hand, since

$$l_j = \{y_j + \tau(y_1 + y_2 + \cdots + y_m) | \tau \in \mathrm{GF}(q)\} \qquad (j = 1, 2, \ldots, m)$$

are m distinct parallel lines of $\mathcal{P}$ such that $\mathfrak{b} \subseteq l_1 \cup l_2 \cup \cdots \cup l_m$, we infer: if $\mathfrak{b}(w_1, w_2, \ldots, w_m) = \mathfrak{b}$ for suitable affinely independent vectors $w_1, w_2, \ldots, w_m \in \mathfrak{b}$, and if $m > 4$, then the block $\mathfrak{b}$ is strictly contained in the affine subspace over $\mathrm{GF}(p)$ through the m given affinely independent points and defines uniquely the direction $y_1 + y_2 + \cdots + y_m$ of the parallel lines, thus the m-set $\{w_1, w_2, \ldots, w_m\}$ meets each of the m lines l_j ($j = 1, 2, \ldots, m$) in just one point (vector), otherwise some of the y_j would not belong to $\mathfrak{b} = \mathfrak{b}(y_1, y_2, \ldots, y_m) = \mathfrak{b}(w_1, w_2, \ldots, w_m)$. Hence there are $c_1, c_2, \ldots, c_m \in \mathrm{GF}(p)$ such that $w_j = y_j + c_j(y_1 + y_2 + \cdots + y_m)$ for $j = 1, 2, \ldots, m$. Therefore we must have $R_{\mathfrak{b}} \leq p^{m-1}(p-1)$, if $m > 4$. Thus we proved that, if $m > 4$, then $R_{\mathfrak{b}} \leq p^{m-1}(p-1) \leq R_{\mathfrak{b}}$, that is, $R_{\mathfrak{b}} = p^{m-1}(p-1)$.

Suppose now $m = 4$. Thus $p = 2$ and the four lines $y_i + \langle y_1 + y_2 + y_3 + y_4\rangle$ (with $i = 1, 2, 3, 4$), whose union is $\mathfrak{b}$, fill a whole 3-dimensional space over $\mathrm{GF}(2)$. Then four vectors (points) $z_1, z_2, z_3, z_4 \in \mathfrak{b}$ have the property that $\mathfrak{b}(z_1, z_2, z_3, z_4) = \mathfrak{b}$ if and only if z_1, z_2, z_3, z_4 are non-coplanar points of (the affine space) $\mathfrak{b}$: choosing 3 points out of the 8, and a further point not in the plane through them, we obtain 4 non-coplanar points, in $\binom{4}{3}$ different ways, hence $R_{\mathfrak{b}} = 4 \times \binom{8}{3}/\binom{4}{3} = 56$, if $m = 4$.

Finally, suppose $m = 3$. Then $p = 3$ and the three lines $y_i + \langle y_1 + y_2 + y_3\rangle$ (with $i = 1, 2, 3$), whose union is $\mathfrak{b}$, are coplanar, hence $\mathfrak{b}$ is a finite affine plane of order 3. Then three vectors (points) $z_1, z_2, z_3 \in \mathfrak{b}$ are affinely independent (and have the property that $\mathfrak{b}(z_1, z_2, z_3) = \mathfrak{b}$) if and only if z_1, z_2, z_3 are non-collinear points of (the affine plane) $\mathfrak{b}$. Therefore $R_{\mathfrak{b}} = \binom{9}{3} - 12 = 72$, if $m = 3$, and the claim is proved.

Since $\frac{q^n(q^n-1)(q^n-q)\cdots(q^n-q^{m-2})}{1\cdot 2\cdot 3\cdots m}$ is the number of all the m-subsets of $\mathcal{P}$ consisting of affinely independent vectors, counting in two ways the number of flags $(W, \mathfrak{b})$, where $W = \{w_1, w_2, \ldots, w_m\}$ is an m-subset of $\mathcal{P}$ consisting of affinely independent vectors and $\mathfrak{b} = \mathfrak{b}(y_1, y_2, \ldots, y_m)$ is a block of $\mathcal{D}$ through W, we obtain by the above argument

$$\begin{cases} \frac{q^n(q^n-1)(q^n-q)\cdots(q^n-q^{m-2})}{1\cdot 2\cdot 3\cdots m} = p^{m-1}(p-1)b \text{ , if } m > 4, \\[2ex] \frac{q^n(q^n-1)(q^n-q)(q^n-q^2)}{24} = 56b \text{ , if } m = 4, \\[2ex] \frac{q^n(q^n-1)(q^n-q)}{6} = 72b \text{ , if } m = 3, \end{cases}$$

and this gives the number b of blocks. The parameter λ follows consequently. □

Remark 2.2. It is worth noting that the cases where $m = 3, 4$ are sensibly different from those where $m > 4$.

Let us first point out that, since the $2-(q^n, mp, \lambda)$ designs $\mathcal{D}$ considered in Theorem 2.1 have $v = q^n$ points, it is natural to ask in what cases such designs arise just as classical point-flat designs $\mathrm{AG}_\mu(n, q)$ of the affine geometries $\mathrm{AG}(n, q)$. It turns out that this is the case only for $\mathrm{AG}_2(n, 3)$, $n \geq 2$, and $\mathrm{AG}_3(n, 2)$, $n \geq 3$. Indeed, the μ-flat through m affinely independent points has $k = q^{m-1}$ points, and this equals $k = mp$ only in the cases where $m = 2$ and $q = 4$ (which is excluded), $m = 3$ and $q = 3$, and $m = 4$ and $q = 2$. The fact that in these two cases the blocks turn out to be affine subspaces has already been pointed out in the above proof.

In all the remaining cases, the designs $\mathcal{D}$ in Theorem 2.1 are not point-flat designs $\mathrm{AG}_\mu(n, q)$. For $q = p^c, m = p^h$, such designs $\mathcal{D}$ are $2-(p^{cn}, p^{h+1}, \lambda)$ designs, hence they have the same parameters v and k as the point-flat designs $\mathrm{AG}_{h+1}(cn, p)$ of the affine geometries $\mathrm{AG}(cn, p)$, thus it is appropriate to compare the value of the parameter λ in (2.1) for $\mathcal{D}$ with the value of λ for $\mathrm{AG}_{h+1}(cn, p)$. As we will now see, for $m = p = 3$, $q = 3^c$ (resp., for $m = 4$, $p = 2$, $q = 2^c$), with $c > 1$, the value of λ in (2.1) is smaller than the corresponding value of λ for the point-plane design $\mathrm{AG}_2(cn, 3)$ (resp., for the point-flat design $\mathrm{AG}_3(cn, 2)$). In either case, the design $\mathcal{D}$ has a $\mathrm{GF}(p)$-structure, but not a $\mathrm{GF}(q)$-structure.

(i) For $m = 3$ and $q = 3^c$, $c > 1$, $\mathcal{D}$ is a $2-\left(3^{cn}, 9, \frac{3^{cn}-3^c}{6}\right)$ design, whereas the point-plane design $\mathrm{AG}_2(cn, 3)$ has a larger $\lambda = \frac{3^{cn}-3}{6}$, whose difference with the parameter λ of $\mathcal{D}$ is $\frac{3^c-3}{6}$, which increases exponentially with c. The smallest example is the case $n = c = 2$: in this case, $\mathcal{D}$ is a $2-(81, 9, 12)$ design, whereas the point-plane design $\mathrm{AG}_2(4, 3)$ is a $2-(81, 9, 13)$ design.

(ii) For $m = 4$ and $q = 2^c$, $c > 1$, $\mathcal{D}$ is a $2-\left(2^{cn}, 8, \frac{(2^{cn}-2^c)(2^{cn}-2^{2c})}{24}\right)$ design, whereas the point-flat design $\mathrm{AG}_3(cn, 2)$ has a larger value of $\lambda = \frac{(2^{cn}-2)(2^{cn}-4)}{24}$.

On the contrary, for $m = p = q > 3$ the parameter λ for $\mathcal{D}$ becomes much larger than that for the point-plane design $\mathrm{AG}_2(n, p)$. For instance, for the smallest case $m = p = q = 5$, $n = 4$, $\mathcal{D}$ is a $2-(625, 25, 372000)$ design, whereas the point-plane design $\mathrm{AG}_2(4, 5)$ is a $2-(625, 25, 31)$ design. And the situation in the cases that do not have a corresponding $\mathrm{AG}_\mu(n, q)$ to be compared with is not different: for $q = 3$, $n = 5$, and $m = 6$, $\mathcal{D}$ is a $2-(243, 18, \lambda)$ design, with $\lambda = 1718496$.

Remark 2.3. *As the affine group* $\mathrm{Aff}(\mathcal{P})$ *has order* $|\mathrm{Aff}(\mathcal{P})| = q^n(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})$ *and* $b = \frac{|\mathrm{Aff}(\mathcal{P})|}{|S_{\mathfrak{b}_0}|}$, *we may conclude that the stabilizer* $S_{\mathfrak{b}_0}$ *is a group of order*

$$|S_{\mathfrak{b}_0}| = \begin{cases} (1\cdot 2\cdot 3\cdots m)p^{m-1}(p-1)(q^n-q^{n-1})(q^n-q^{n-2})\cdots(q^n-q^{m-1}), \text{ if } m > 4, \\ 1344(q^n-q^3)\cdots(q^n-q^{n-1}), \text{ if } m = 4, \\ 432(q^n-q^{n-1})(q^n-q^{n-2})\cdots(q^n-q^2), \text{ if } m = 3. \end{cases}$$

Remark 2.4. The design $\mathcal{D}_k = (\mathrm{GF}(2^n), \mathcal{B}_k)$, considered in [13, Proposition 2.5], is a 3-design for any even k. Similarly, for $p = 2$, the 2-design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ considered in Theorem 2.1 is a 3-design if and only if $q = 2$. Indeed, let $q = 2$, and let $\{P_1, P_2, P_3\}$ and $\{Q_1, Q_2, Q_3\}$ be two 3-subsets of $\mathcal{P}$. Since the group of affinities of $\mathcal{P}$ acts 3-transitively on $\mathcal{P}$, there exists an (invertible) affinity ρ such that $\rho(P_i) = Q_i$, $i = 1, 2, 3$. Moreover, $\rho(\mathfrak{b}(y_1, y_2, \ldots, y_m)) = \mathfrak{b}(\rho(y_1), \rho(y_2), \ldots, \rho(y_m))$ for any subset $\{y_1, y_2, \ldots, y_m\}$ of $\mathcal{P}$ consisting of m affinely independent vectors, hence P_1, P_2, P_3 belong to a block $\mathfrak{b}$ if and only if Q_1, Q_2, Q_3 belong to the block $\rho(\mathfrak{b})$. Therefore $\mathcal{D}$ is a 3-design.

Now let $q \neq 2$. If $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ were a 3-design, then the corresponding derived design at the point 0 would be a 2-design. By definition, every block of the latter design is of the form $\mathfrak{b}(x_1 = 0, x_2, \ldots, x_m) \setminus \{0\} = \{x_j + s(x_1 + x_2 + \cdots + x_m) | 1 \leq j \leq m$ and

$s \in \mathrm{GF}(2)\} \setminus \{0\}$, where $x_2, \ldots, x_m$ are linearly independent vectors over $\mathrm{GF}(q)$, hence one can prove that, for any nonzero x in $\mathcal{P}$, and for any scalar c in $\mathrm{GF}(q) \setminus \mathrm{GF}(2)$, the two vectors x and cx cannot lie in a common block. Therefore $\mathcal{D}$ is not a 3-design for $p = 2$ and $q \neq 2$.

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On metric dimensions of hypercubes

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Abstract

In this note we show two unexpected results concerning the metric, the edge metric and the mixed metric dimensions of hypercube graphs. First, we show that the metric and the edge metric dimensions of Q_d differ by at most one for every integer d. In particular, if d is odd, then the metric and the edge metric dimensions of Q_d are equal. Second, we prove that the metric and the mixed metric dimensions of the hypercube Q_d are equal for every $d \geq 3$. We conclude the paper by conjecturing that all these three types of metric dimensions of Q_d are equal when d is large enough.

Keywords: Edge metric dimension, mixed metric dimension, metric dimension, hypercubes.

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1 Introduction

The metric dimension of connected graphs was introduced about 50 years ago in [6, 22], in connection with modeling navigation systems in networks, although this notion was already known by then for general metric spaces from [1]. Given a connected graph G and two vertices $u, v \in V(G)$, the *distance* $d_G(u, v)$ between these two vertices is the length of a shortest path connecting v and u. The vertices u, v are *distinguished* or *resolved* by a vertex $x \in V(G)$ if $d_G(u, x) \neq d_G(v, x)$. A given set of vertices S is a *metric generator* for the graph G, if every two vertices of G are distinguished by a vertex of S. The cardinality of the smallest possible metric generator for G is the *metric dimension* of G, which is denoted by $\dim(G)$. The terminology of metric generators was introduced in [11], and the previous two works referred to such sets as *resolving sets* and *locating sets*, respectively. We herewith follow the terminology of [11]. A metric generator for G of cardinality $\dim(G)$ is called a *metric basis*. Although the classical metric dimension is an old topic in graph theory, there are still several open problems that remain unsolved. Recent investigations on this concern are [3, 4, 8, 16]. More results and open questions concerning metric dimension and related variants can be found in the recent surveys [15] and [23].

In order to uniquely identify the edges of a graph, by using vertices, the edge metric dimension of connected graphs was introduced in [10] as follows. Let G be a connected graph and let uv be an edge of G such that $u, v \in V(G)$. The *distance* between a vertex $x \in V(G)$ and the edge uv is defined as, $d_G(uv, x) = \min\{d_G(u, x), d_G(v, x)\}$. It is said that two distinct edges $e_1, e_2 \in E(G)$ are *distinguished* or *resolved* by a vertex $v \in V(G)$ if $d_G(e_1, v) \neq d_G(e_2, v)$. A set $S \subset V(G)$ is called an *edge metric generator* for G if and only if for every pair of edges $e_1, e_2 \in E(G)$, there exists an element of S which distinguishes the edges. The cardinality of a smallest possible edge metric generator of a graph is known as the *edge metric dimension*, and is denoted by $\mathrm{edim}(G)$. After the seminal paper [10], a significant number of researches on such parameter have appeared. Among them, some of the most recent ones are [3, 12, 13, 14, 19]. See also the survey [15] for some other contributions. It is natural to consider comparing the metric and edge metric dimensions of graphs. However, as first proved in [10], and continued in [13, 14], both parameters are not in general comparable since there exist connected graphs G for which $\mathrm{edim}(G) < \dim(G)$, $\mathrm{edim}(G) = \dim(G)$ or $\mathrm{edim}(G) > \dim(G)$.

In order to combine the unique identification of vertices and of edges in only one scheme, the mixed metric dimension of graphs was introduced in [9]. For a connected graph G, a vertex $w \in V(G)$ and an edge $uv \in E(G)$ are *distinguished* or *resolved* by a vertex $x \in V(G)$ if $d_G(w, x) \neq d_G(uv, x)$. A set $S \subset V(G)$ is called a *mixed metric generator* for G if and only if for every pair of elements of the graphs (vertices or edges) $e, f \in E(G) \cup V(G)$, there exists a vertex of S which distinguishes them. The cardinality of a smallest possible mixed metric generator of G is known as the *mixed metric dimension* of G, and is denoted by $\mathrm{mdim}(G)$. Some recent studies on mixed metric dimension of graphs are [20, 21]. Clearly, every mixed metric generator must be a metric generator as well as an edge metric generator, and so, $\mathrm{mdim}(G) \geq \max\{\dim(G), \mathrm{edim}(G)\}$, for any connected graph G. Moreover, since $\dim(G)$ and $\mathrm{edim}(G)$ are in general not comparable (see [13, 14] for more information on this fact), several situations relating these three parameters can be found. That is, there are graphs G with $\mathrm{mdim}(G) \gg \max\{\dim(G), \mathrm{edim}(G)\}$, $\mathrm{mdim}(G) = \dim(G) \gg \mathrm{edim}(G)$, $\mathrm{mdim}(G) = \mathrm{edim}(G) \gg \dim(G)$, or $\mathrm{mdim}(G) = \dim(G) = \mathrm{edim}(G)$.

The metric dimension of hypercube graphs has attracted the attention of several researchers from long ago. For instance, the work of Lindström [17] is probably one of the oldest ones, and for some recent ones we suggest the works [7, 18, 24]. Surprisingly, for other related invariants there has been comparatively little research on hypercube graphs, although one can find some interesting recent results on this topic such as those that appeared in [5, 7]. It is our goal to present some results on the close connections that exist among the metric, the edge metric and the mixed metric dimensions of hypercube graphs.

The d-dimensional hypercube, denoted by Q_d, with $d \in \mathbb{N}$, is a graph whose vertices are represented by d-dimensional binary vectors, *i.e.*, $u = (u_1, \ldots, u_2) \in V(Q_d)$ where $u_i \in \{0, 1\}$ for every $i \in \{1, \ldots, d\}$. Two vertices are adjacent in Q_d if their vectors differ in exactly one coordinate. Hypercubes can be also seen as the d times Cartesian product of the graph P_2, that is, $Q_d \cong P_2 \Box P_2 \Box \cdots \Box P_2$, or recursively, $Q_d \cong Q_{d-1} \Box P_2$. The distance between two vertices in Q_d represents the total number of coordinates in which their vectors differ. The hypercube Q_d is bipartite, and has 2^d vertices and $d \cdot 2^{d-1}$ edges. We remark that Q_2 is the cycle C_4 and that Q_4 can be also seen as the torus graphs $C_4 \Box C_4$.

2 Results

Our first contribution is to relate the metric generators with the edge metric generators of bipartite graphs.

Lemma 2.1. *Let G be a connected bipartite graph. Then, every metric generator for G is also an edge metric generator.*

Proof. Let S be an arbitrary metric generator for G. We will show that S is an edge metric generator as well.

Let $e_1 = x_1 y_1$ and $e_2 = x_2 y_2$ be two arbitrary distinct edges of G. Since G is bipartite and e_1, e_2 are distinct, one can w.l.o.g. assume that x_1, x_2 (with $x_1 \neq x_2$) belong to one of the bipartition sets and y_1, y_2 to the other one. Hence the distance between $u = x_1$ and $v = x_2$ is even.

Now, as u and v are distinct, there must be a vertex $s \in S$ that distinguishes them, *i.e.* $d(s, u) \neq d(s, v)$. We may assume that $d(s, u) + 1 \leq d(s, v)$. Since u and v are on even distance, it follows that distances $d(s, u)$ and $d(s, v)$ are of same parity, otherwise we encounter a closed walk of odd length in G, which is not possible in a bipartite graph. This implies $d(s, u) + 2 \leq d(s, v)$, and now we easily derive

$$d(e_1, s) \leq d(u, s) < d(v, s) - 1 \leq d(e_2, s).$$

In particular, $d(e_1, s) < d(e_2, s)$ implies that e_1, e_2 are distinguished by $s \in S$. Since the choice of these two edges was arbitrary, we conclude that S is also an edge metric generator. $\square$

It is then natural to think in the opposite direction with regard to the result above. In particular, we ask if an edge metric generator for a bipartite graph is also a metric generator. In contrast with the result above, achieving this seems to be a challenging task. However, we have at least managed to show a weaker result for an infinite family of bipartite graphs, namely the hypercubes Q_d. That is, when d is odd, every edge metric generator for Q_d is indeed a metric generator, and when d is even, every edge metric generator is “almost” a metric generator.

From now on we denote by α_i the vector of dimension d whose i^{th}-coordinate is 1, and the remaining coordinates are 0. Also, by "$\oplus$" we represent the standard (binary) XOR operation. Notice that, for any vertex $u \in V(Q_d)$, $u \oplus \alpha_i$ means switching the i^{th}-coordinate of u from 0 to 1, or vice versa.

Lemma 2.2. *Let S be an edge metric generator of Q_d. If there exist two distinct vertices u and v not distinguished by S, then they must be antipodal in Q_d and d is even. If d is odd, then S is also a metric generator of Q_d.*

Proof. Suppose that $u = (u_1, u_2, \dots, u_d)$ and $v = (v_1, v_2, \dots, v_d)$ are not antipodal. Then, $u_i = v_i$ for some i. Let Q^0_{d-1} and Q^1_{d-1} be the half-cubes regarding the dimension i. Notice that u and v belongs to a same half-cube, say Q^0_{d-1}. Let e_u and e_v be the edges corresponding to the component i (in Q_d) incident with u and v, respectively. In other words, as $u\oplus\alpha_i$ and $v\oplus\alpha_i$ are the neighbours of u and v in Q^1_{d-1}, we have $e_u = (u, u\oplus\alpha_i)$ and $e_v = (v, v \oplus \alpha_i)$. We claim that the edges e_u and e_v are not distinguished by S. To see this, observe that if $s \in S$ belongs to Q^0_{d-1}, then

$$d(s, e_u) = d(s, u) = d(s, v) = d(s, e_v).$$

Also, if $s \in S$ belongs to Q^1_{d-1}, then

$$d(s, e_u) = d(s, u \oplus \alpha_i) = d(s, v \oplus \alpha_i) = d(s, e_v).$$

We hence derive that the edges e_u and e_v are not distinguished by S, which is a contradiction.

Based on the above arguments we conclude that u and v are antipodal, *i.e.* $d(u, v) = d$. Hence, every vertex x of S satisfies $d(u, x) + d(x, v) = d$. As every vertex $s \in S$ must be equally distanced from u and v, we conclude that $d(u, s) = d(s, v) = d/2$, and consequently, d must be even. This establishes the main claim.

Finally, observe that if d is odd, then no vertex is equally distanced from two antipodal vertices of Q_d, and therefore, S is a metric generator of Q_d. $\square$

Next lemma will ensure that enlarging an edge metric generator of Q_d with one chosen element, we get a metric generator of Q_d.

Lemma 2.3. *Let S be an edge metric generator of Q_d and let s be an arbitrary element of S. Then, $S \cup \{s \oplus \alpha_1\}$ is a metric generator of Q_d.*

Proof. If S is a metric generator of Q_d, then $S \cup \{s \oplus \alpha_1\}$ is so too, and we are done. Thus, we assume that S is not a metric generator of Q_d. Then, by Lemma 2.2, d is even and there must exist antipodal vertices u and v such that $d(u, x) = d(v, x) = d/2$ for every $x \in S$. This will not be the case for $s \oplus \alpha_1$, as $|d(u, s \oplus \alpha_1) - d(v, s \oplus \alpha_1)| = 2$. Therefore, we conclude that $S \cup \{s \oplus \alpha_1\}$ is a metric generator of Q_d. $\square$

Since Q_d is a bipartite graph, the two previous lemmas give us the following consequence.

Theorem 2.4. *Let $d \geq 1$. Then*

$$\operatorname{edim}(Q_d) \leq \dim(Q_d) \leq \operatorname{edim}(Q_d) + 1,$$

with the second inequality being tight only if d is even.

Proof. The lower bound holds by Lemma 2.1. The upper bound and its possible tightness (for more than one case) follows by Lemmas 2.2 and 2.3. $\square$

Notice that the upper bound $\dim(Q_d) \leq \operatorname{edim}(Q_d) + 1$ is indeed tight for the case Q_4, since $4 = \dim(Q_4) = \operatorname{edim}(Q_4) + 1$, as proved in [10].

We now turn our attention to relating the metric dimension with the mixed metric dimension of hypercubes. To this end, we will need the following two results. We must remark that the first of next two lemmas already appeared in [18]. We include its proof for completeness.

Lemma 2.5. *If S is a metric generator (in particular, a metric basis) of Q_d and $s \in S$, then $(S \setminus \{s\}) \cup \{s'\}$ is also a metric generator (in particular, a metric basis) of Q_d, where $s' \in V(Q_d)$ is the antipodal vertex of s.*

Proof. If $s \in S$ distinguishes some pair of vertices x and y of Q_d, then s' distinguishes such pair as well, since $d(x, s') = d - d(x, s)$ and $d(y, s') = d - d(y, s)$. This also means that no metric basis of Q_d contains two antipodal vertices. Thus, if S is a metric generator (or a metric basis) of Q_d, then $S \setminus \{s\} \cup \{s'\}$ is a metric generator (or a metric basis) as well. $\square$

Lemma 2.6. *If S is a metric generator of Q_d, then there is at most one index $i \in \{1, \ldots, d\}$ such that all the vertices from S have the same value at the i^{th} coordinate.*

Proof. Suppose that there exist two different indices i and j such that all vertices from S have the same value at the i^{th} and j^{th} coordinates. First, let us consider the case when there are zeroes at such coordinates. Other cases can be shown by using similar arguments. Now, let $x \in V(Q_d)$ be a vertex having zeroes at all coordinates, except at the i^{th}, and let y be a vertex having zeroes at all positions except at the j^{th}. Then, $d(x, s) = d(y, s)$ for any vertex $s \in S$, a contradiction. $\square$

The mixed metric dimension of hypercubes Q_1 and Q_2 are 2 and 3, respectively. This can be derived from results for paths and cycles from [9]. This gives us that $\dim(Q_d) < \operatorname{mdim}(Q_d)$, for $d \in \{1, 2\}$. For all higher dimensions the mixed metric dimension is equal to the metric dimension as we next show.

Theorem 2.7. *Let $d \geq 3$. Then*

$$\dim(Q_d) = \operatorname{mdim}(Q_d).$$

Proof. First, $\{(1,1,1),(0,1,0),(0,0,1)\}$ and $\{(1,1,1,1),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$ are mixed metric bases for Q_3 and Q_4, respectively. Thus, the equality follows for these cases since $\dim(Q_3) = 3$ and $\dim(Q_4) = 4$. It remains to check the equality for $d \geq 5$.

Let S be a metric basis for Q_d with $d \geq 5$. By Lemma 2.1, S is an edge metric generator of Q_d. In this sense, in Q_d we only need to distinguish those pairs of elements, one of them being a vertex and the other one, an edge. For this, let u be an arbitrary vertex and let $e = xy$ be an arbitrary edge of Q_d.

Suppose first that u is not a vertex of e. As $d(u, x)$ and $d(u, y)$ are of different parity, we may assume that u and x are on even distance. Now, let s_i be a vertex from S that distinguishes u and x. Similarly, as in Lemma 2.1, notice that $d(s_i, u)$ and $d(s_i, x)$ are of the same parity, and as they are different, we have that $|d(s_i, u) - d(s_i, x)| \geq 2$. So, if $d(s_i, u) < d(s_i, x)$, then we derive

$$d(s_i, u) < d(s_i, u) + 1 \leq d(s_i, x) - 1 \leq d(s_i, e),$$

and if $d(s_i, x) < d(s_i, u)$, then we have

$$d(s_i, e) \leq d(s_i, x) < d(s_i, u).$$

Thus, in both cases e and u are distinguished by a vertex from S.

So all the pairs of elements (vertices and edges) considered in the upper part are distinguished by an arbitrary metric basis. To conclude the proof, we need to construct a metric basis of cardinality $|S|$ that will also distinguish incident vertices and edges.

Suppose now that u is an endpoint of e, say $u = x$. To distinguish u and e there needs to be a vertex $s \in S$ which is from the half-cube Q_{d-1} that contains vertex y and does not contain vertex x. To distinguish all such pairs there must be at least one vertex from the mixed metric generator in every half-cube Q_{d-1}. For any index $i \in \{1, \dots, d\}$, there exists a vertex from a mixed metric generator having 0 on the i^{th} coordinate, and a vertex from a mixed metric generator having 1 on the i^{th} coordinate. In other words, a mixed metric basis does not have a column of zeroes or a column of ones at an arbitrary index i (if we arrange all vectors of the mixed metric basis as a matrix with such vectors as the rows of such matrix).

We have started from an arbitrary metric basis S. Since Q_d is a vertex transitive graph, we may assume that the vertex $s_1 = (0, 0, \dots, 0)$ (all coordinates equal to 0) is in S. If S does not contain a column of zeroes, then S is also a mixed metric basis. Otherwise, by Lemma 2.6, there exists only one such column, say at index i_0. By Lemma 2.5, we know that we can replace any of the vertices from the set S with its antipodal vertex and the incurred set $S' = S \setminus \{s\} \cup \{s'\}$ is a metric basis too, since the column at index i_0 (all zeroes) ensures that no two vertices in S are antipodal to each other. Moreover, in view of Lemma 2.1, S is an edge metric generator as well.

There exist at least four different vertices $s_1 = (0, 0, \dots, 0)$, s_2, s_3 and s_4 in the set S, since $\dim(Q_d) \geq 4$, for $d \geq 5$. We construct four sets S'_i in the following way:

$$S'_1 = (S \setminus \{s_1\}) \cup \{s'_1\}, \quad S'_2 = (S \setminus \{s_2, s_3\}) \cup \{s'_2, s'_3\},$$
$$S'_3 = (S \setminus \{s_2\}) \cup \{s'_2\}, \quad S'_4 = (S \setminus \{s_1, s_3\}) \cup \{s'_1, s'_3\},$$

and consider the next situations:
(I): If S'_1 is not a mixed metric generator, then there is a column of ones in S'_1 at some index i_1.
(II): If S'_2 is not a mixed metric generator, then there is a column of zeroes in S'_2 at some index i_2.
(III): If S'_3 is not a mixed metric generator, then there is a column of zeroes in S'_3 at some index i_3.
(IV): If S'_4 is not a mixed metric generator, then there is a column of ones in S'_4 at some index i_4.

Observe that all these indices i_0, i_1, i_2, i_3, and i_4 are different. If none of the four sets S'_i defined above is a mixed metric generator, then the initial set S looks as follows.

$$
\begin{array}{rc|cccc|c}
 & i_0 & i_1 & i_2 & i_3 & i_4 & \dots \\
\hline
s_1: & 0 & 0 & 0 & 0 & 0 & \dots \\
s_2: & 0 & 1 & 1 & 1 & 1 & \dots \\
s_3: & 0 & 1 & 1 & 0 & 0 & \dots \\
\hline
s_4: & 0 & 1 & 0 & 0 & 1 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
s_{|S|}: & 0 & 1 & 0 & 0 & 1 & \dots
\end{array}
$$

We now take a look at the columns i_1, i_2, i_3 and i_4. Let v_1 be a vertex having zeroes at all positions except at i_1 and i_3 and let v_2 be a vertex having zeroes at all positions except at i_2 and i_4. Then, $d(v_1, s) = d(v_2, s)$, for any vertex $s \in S$, a contradiction. Therefore, at least one of the sets S'_i has to be a mixed metric generator, and therefore, the equality $\operatorname{mdim}(Q_d) = \dim(Q_d)$ follows since any mixed metric basis is also a metric basis. □

In view of the asymptotical result for the metric dimension of hypercubes from [2], Theorems 2.4 and 2.7 give us the following consequences.

Corollary 2.8. *Let $d \geq 3$. Then*

$$\dim(Q_d) - 1 \leq \operatorname{edim}(Q_d) \leq \dim(Q_d) = \operatorname{mdim}(Q_d).$$

Corollary 2.9. *Let $d \geq 2$. Then*

$$\operatorname{mdim}(Q_d) \sim \operatorname{edim}(Q_d) \sim \dim(Q_d) \sim \frac{2d}{\log_2 d}.$$

We conclude this short paper with the following conjecture.

Conjecture 2.10. *If d is large enough, then*

$$\operatorname{edim}(Q_d) = \dim(Q_d).$$

As the above conjecture does not hold for $d = 4$, d must be at least 5.

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Complete forcing numbers of graphs*

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Abstract

The complete forcing number of a graph G with a perfect matching is the minimum cardinality of an edge set of G on which the restriction of each perfect matching M is a forcing set of M. This concept can be view as a strengthening of the concept of global forcing number of G. Došlić in 2007 obtained that the global forcing number of a connected graph is at most its cyclomatic number. Motivated from this result, we obtain that the complete forcing number of a graph is no more than 2 times its cyclomatic number and characterize the matching covered graphs whose complete forcing numbers attain this upper bound and minus one, respectively. Besides, we present a method of constructing a complete forcing set of a graph. By using such method, we give closed formulas for the complete forcing numbers of wheels and cylinders.

Keywords: Perfect matching, global forcing number, complete forcing number, cyclomatic number, wheel, cylinder.

Math. Subj. Class. (2020): 05C70, 05C90, 92E10

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *matching* of G is a set of disjoint edges of G. A *perfect matching* M of G is a matching that covers all vertices of G. An edge of G is termed *allowed* if it lies in some perfect matching of G and *forbidden* otherwise. A *forcing set* of M is a subset of M contained in no other perfect matching of G. The *forcing number* of M is the minimum possible cardinality of forcing sets of M. We may refer to a survey [6] on this topic. A subset $S \subseteq E(G) \setminus M$ is called an *anti-forcing set* of M [14] if $G - S$ has a unique perfect matching M. The *anti-forcing number* of M is the smallest cardinality of anti-forcing sets of M.

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Let G be a graph with a perfect matching. Concerning all perfect matchings of G, Vukičević et al. [21, 22] introduced the concept of *global (or total) forcing set*, which is defined as a subset of $E(G)$ on which there are no two distinct perfect matchings coinciding. The minimum possible cardinality of global forcing sets is called the *global forcing number* of G. For more about the global forcing number of a graph, the reader is referred to [3, 7, 20, 27].

As a strengthening of the concept of global forcing set of G, Xu et al. [24] proposed the concept of the *complete forcing set* of G, which is defined as a subset of $E(G)$ on which the restriction of each perfect matching M is a forcing set of M. A complete forcing set with the minimum cardinality is called a *minimum complete forcing set* of G, and its cardinality is called the *complete forcing number* of G, denoted by $cf(G)$. If G has at least two perfect matchings, then $cf(G)$ is larger than the global forcing number of G [24].

A subgraph G_0 of G is said to be *nice* if $G - V(G_0)$ has a perfect matching. Obviously, an even cycle C of G is nice if and only if there is a perfect matching M of G such that $E(C) \cap M$ is a perfect matching of C. We call each of the two perfect matchings of C a *frame* (or a typeset [24]) of C, which was ever used in [1] to obtain a min-max theorem for the Clar problem on 2-connected plane bipartite graphs.

Xu et al. established the following equivalent condition for a subset of edges of a graph to be a complete forcing set.

Theorem 1.1 ([24]). *Let G be a graph with a perfect matching. Then $S \subseteq E(G)$ is a complete forcing set of G if and only if for any nice cycle C of G, the intersection of S and each frame of C is nonempty.*

Let S be a complete forcing set of G. For a perfect matching M of G, from Theorem 1.1, $S \setminus M$ contains at least one edge of every M-alternating cycle of G. By Lemma 2.1 of [14], $S \setminus M$ is an anti-forcing set of M. So a complete forcing set of G both forces and antiforces each perfect matching. Further, Chan et al. [4] obtained that the complete forcing number of a catacondensed hexagonal system is equal to the number of hexagons plus the Clar number and presented a linear-time algorithm for computing it. Besides, some certain explicit formulas for the complete forcing numbers of rectangular polynominoes, polyphenyl systems, spiro hexagonal systems and primitive coronoids have been derived [5, 15, 16, 23]. In recent papers [11, 12], we established a sufficient condition for an edge set of a hexagonal system (HS) to be a complete forcing set in terms of elementary edge-cut cover, which yields a tight upper bound on the complete forcing numbers of HSs. For a normal HS, we gave two lower bounds on its complete forcing number by the number of hexagons and matching numbers of some subgraphs of its inner dual graph, respectively. In addition, we showed that the complete forcing numbers of catacondensed HSs, normal HSs without 2×3 subsystems, parallelogram, regular hexagon- and rectangle-shaped HSs attain one of the two above lower bounds.

Let $c(G) = |E(G)| - |V(G)| + \omega(G)$ denote the *cyclomatic number* of G, where $\omega(G)$ is the number of components of G. In 2007, Došlić [7] obtained that the global forcing number of a connected graph is at most its cyclomatic number and gave a characterization: a connected (bipartite) graph has the global forcing number attaining its cyclomatic number if and only if each cycle is nice (such graphs are called 1-cycle resonant graphs; see [9]). As a corollary, the global forcing number of any catacondensed HS is equal to the number of hexagons.

Motivated by Došlić's result, in this paper we obtain that the complete forcing number of a graph is no more than 2 times its cyclomatic number by presenting a method of constructing a complete forcing set of a graph (see the next section). Moreover, in Section 3, we show that the complete forcing number of a matching covered graph attains the above upper bound if and only if such graph is either K_2 (a complete graph with 2 vertices) or an even cycle. Besides, we characterize the matching covered graphs whose complete forcing numbers are equal to 2 times their cyclomatic numbers minus 1 in terms of ear decomposition. In the last section, we present some lower bounds on the complete forcing numbers of some types of graphs including plane elementary bipartite graphs and cylinders. Combining such methods, we give closed formulas for the complete forcing numbers of wheels and cylinders.

2 An upper bound on complete forcing number

All graphs considered in this paper are simple and all the bipartite graphs are given a proper black and white coloring: any two adjacent vertices receive different colors.

Let G be a graph. Suppose that V' is a nonempty subset of $V(G)$. The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both end-vertices in V' is called the subgraph of G *induced* by V' and is denoted by $G[V']$. The induced subgraph $G[V \setminus V']$ is denoted by $G-V'$. For $E' \subseteq E(G)$, the spanning subgraph $(V(G), E(G) \setminus E')$ is denoted by $G - E'$ [2]. For a nonempty proper subset V' of $V(G)$, the set of all edges of G having exactly one end-vertex in V' is called an *edge cut* of G and denoted by $\partial_G(V')$ (or simply $\partial(V')$).

For $v \in V(G)$ and $e \in E(G)$, for simplicity we use $G-v$, $G-e$ and $\partial(v)$ to represent $G - \{v\}$, $G - \{e\}$ and $\partial(\{v\})$ respectively. Further the cardinality of $\partial_G(v)$ is called the *degree* of v in G and is denoted by $d_G(v)$ (or simply $d(v)$).

In this section, we will present a method of constructing a complete forcing set of G in terms of elementary edge cut, which was introduced in [19, 26] to show the existence of perfect matchings in HS and plays an important role in resonance theory of graphs [13, 25, 28] especially in the computation of Clar number of HSs [10]. Elementary edge cut was previously defined in bipartite graphs, we extend this concept to general graphs as follows.

We call an edge cut D of G an *elementary edge cut* (e-cut for short) if it satisfies the following three conditions:

(1) $\omega(G - D) = \omega(G) + 1$, that is, there are exactly two components O_1 and O_2 of $G - D$ that are different from of all components of G.

(2) At least one of O_1 and O_2 is a bipartite graph.

(3) All edges of D are incident with the vertices of the same color in one bipartite component of O_1 and O_2 (for example, the bold edges of G_1 in Figure 10 form an e-cut of G_1).

A *bridge* of G is an edge cut of G consisting of exactly one edge. A *cut-vertex* of G is a vertex whose deletion increases the number of components. A *block* of G is a maximal connected subgraph of G that has no cut-vertices. Each block with at least 3 vertices is 2-connected. The blocks of a loopless graph are its isolated vertices, bridges, and maximal 2-connected subgraph. A block of G that contains exactly one cut-vertex of G is called an *end-block* of G.

Let D be an e-cut of a graph G with at least two edges. Then we define an *e-cut deletion operation* (simply ED operation) of G in the following steps:

(1) Delete D from G,

(2) Delete the set B consisting of all bridges of $G - D$, and

(3) Delete the isolated vertices of $G - D - B$.

Let G' be the subgraph obtained from G by an ED operation. Then G' has neither isolated vertices nor bridges. If G' is not empty, then each block of G' is 2-connected. Let v' be a non-cut-vertex of a block of G' with at most one cut-vertex of G'. Then $\omega(G' - \partial_{G'}(v')) = \omega(G') + 1$ and v' is a component of $G' - \partial_{G'}(v')$, so $\partial_{G'}(v')$ is an e-cut of G' with at least 2 edges and we can do an ED operation on G'. If we can do l ED operations from G and obtain the following subgraph sequence $G = G_1 \supset G_2 \supset \cdots \supset G_{l+1}$, where G_i is not empty graph and G_{i+1} is obtained by doing an ED operation from G_i for $i = 1, 2, \ldots, l$, then we call this procedure an *e-cut decomposition* from G_1 to G_{l+1}. Let D_i be the deleted e-cut from G_i. Then

$$c(G_i - D_i) = |E(G_i)| - |D_i| - |V(G_i)| + \omega(G_i) + 1 = c(G_i) - (|D_i| - 1).$$

Let B_i be the set of bridges deleted from $G_i - D_i$. Then we have

$$c((G_i - D_i) - B_i) = c(G_i - D_i) = c(G_i) - (|D_i| - 1).$$

Since deleting the isolated vertices from $G_i - D_i - B_i$ keeps its cyclomatic number unchanged,

$$c(G_{i+1}) = c((G_i - D_i) - B_i) = c(G_i) - (|D_i| - 1).$$

So, we have

$$c(G_i) - c(G_{i+1}) = |D_i| - 1. \tag{2.1}$$

Since $|D_i| \geq 2$, Equation (2.1) implies that an ED operation on G_i decrease the cyclomatic number by at least 1.

From the above discussion, we have the following result.

Lemma 2.1. *If a graph G has an e-cut with at least two edges, then there exists an e-cut decomposition from G to empty graph.*

Lemma 2.2. *Let G be a graph without isolated vertices or bridges. If H is a 2-connected induced subgraph of G, then there exists an e-cut decomposition from G to H.*

Proof. If G is not 2-connected, then there is a block B of G such that H is not an induced subgraph of B and B contains at most one cut-vertex of G. Since G has neither isolated vertices nor bridges, B is 2-connected. Let v_1 be a vertex of B that is not a cut-vertex of G. Then $\partial_G(v_1)$ is an e-cut of G with at least two edges. We can use $\partial_G(v_1)$ to do an ED operation on G. If G is 2-connected and $G \neq H$, let v_2 be a vertex of $V(G) \setminus V(H)$. Then $\partial_G(v_2)$ is an e-cut of G with at least two edges. We can use $\partial_G(v_2)$ to do an ED operation on G. In either of the above two cases, we can see that H is still an induced subgraph of the resulting graph. Clearly, we can do ED operations repeatedly like the above until the resulting subgraph is H. $\square$

Lemma 2.3. *Let G be a graph that admits a perfect matching and F be the set of all forbidden edges of G. If there exists an e-cut decomposition from $G - F = G_1$ to G_{l+1} $(l \geq 1)$ such that G_{l+1} is empty graph or each cycle of G_{l+1} is not a nice cycle of G, then $D_1 \cup D_2 \cup \cdots \cup D_l$ is a complete forcing set of G, where D_i $(i = 1, 2, \ldots, l)$ is the e-cut deleted from G_i in the e-cut decomposition. Further, $cf(G) \leq c(G) + l - c(G_{l+1})$.*

Proof. For $i = 1, 2, \ldots, l$, let B_i be the set of bridges deleted from $G_i - D_i$ in the e-cut decomposition. Then $E(G)$ can be partitioned into $F \cup D_1 \cup B_1 \cup D_2 \cup B_2 \cup \cdots \cup D_l \cup B_l \cup E(G_{l+1})$. Since every edge of a nice cycle C is allowed, $E(C) \cap F = \emptyset$.

Claim. For $i = 1, 2, \ldots, l$, if there is a nice cycle C of G that has an edge in $D_i \cup B_i$, then each frame of C intersects $D_1 \cup D_2 \cup \cdots \cup D_i$.

Proof. We shall proceed by induction on i. For $i = 1$, if $E(C) \cap B_1 \neq \emptyset$, then $E(C) \cap D_1 \neq \emptyset$. Choose an edge e_1 in $E(C) \cap D_1$. Since D_1 is an e-cut of G_1, there is a bipartite component O_1 of $G_1 - D_1$ such that all edges of D_1 are incident with the same colored vertices of O_1 (say black). After C passes through e_1 to black end-vertex in O_1, C must pass through another edge of D_1 from black end-vertex in O_1. Let e_2 be the first such edge. Then the path of C_1 between black vertices of e_1 and e_2 in O_1 has even length. This yields that edges e_1 and e_2 in D_1 belong to different frames of C and the claim holds for $i = 1$. Suppose that the claim holds for $i \leq l - 1$. We shall prove it for $i + 1$. If C has an edge in $E(G) \setminus E(G_{i+1})$, that is, C has some edge in $D_1 \cup B_1 \cup D_2 \cup B_2 \cup \cdots \cup D_i \cup B_i$, by the induction hypothesis, the intersection of each frame of C and $D_1 \cup D_2 \cup \cdots \cup D_i$ is nonempty. So we may assume that $E(C) \subseteq E(G_{i+1})$. Since C has an edge in $D_{i+1} \cup B_{i+1}$, similarly we have that each frame of C must have an edge in D_{i+1}. Consequently, the claim holds. □

Since G_{l+1} is empty graph or every cycle of G_{l+1} is not a nice cycle of G, every nice cycle of G contains an edge in $D_1 \cup B_1 \cup D_2 \cup B_2 \cup \cdots \cup D_l \cup B_l$. By the claim, each frame of each nice cycle of G intersects $D_1 \cup D_2 \cup \cdots \cup D_l$. So, by Theorem 1.1, $D_1 \cup D_2 \cup \cdots \cup D_l$ is a complete forcing set of G.

From Equation (2.1), we have

$$c(G_1) - c(G_{l+1}) = \sum_{i=1}^{l} (|D_i| - 1), \tag{2.2}$$

and then

$$\sum_{i=1}^{l} |D_i| = c(G_1) + l - c(G_{l+1}).$$

Since $G_1 = G - F$, $c(G_1) \leq c(G)$ and

$$cf(G) \leq |D_1 \cup D_2 \cup \cdots \cup D_l| = c(G_1) + l - c(G_{l+1}) \leq c(G) + l - c(G_{l+1}). \quad \square$$

From Lemma 2.3, we have the following upper bound on the complete forcing number.

Theorem 2.4. *Let G be a graph that admits a perfect matching. Then $cf(G) \leq 2c(G)$.*

Proof. If G has a unique perfect matching, then $cf(G) = 0$, and the conclusion holds. If G has at least two perfect matchings, let F be the set of all forbidden edges of G. Then each K_2 block of $G-F$ is a component, so there is a 2-connected block B of $G-F$ with at most one cut-vertex of $G-F$. Let v be a vertex of B that is not a cut-vertex of $G-F$. Then $\partial_{G-F}(v)$ is an e-cut of $G-F$ with at least two edges. By Lemma 2.1 there exists an e-cut decomposition from $G-F$ to empty graph: $G_1 \supset G_2 \supset \cdots \supset G_{l+1}$, where $G_1 = G-F$ and $G_{l+1} = \emptyset$. For $i = 1, 2, \ldots, l$, let D_i be the e-cut deleted from G_i in this e-cut decomposition. From Equation (2.2), since $|D_i| \geq 2$ and $c(G_1) \leq c(G)$, we have $l \leq \sum_{i=1}^{l}(|D_i|-1) = c(G_1) \leq c(G)$. Combining with Lemma 2.3, we have $cf(G) \leq c(G) + l - c(G_{l+1}) \leq 2c(G)$. □

3 Some extremal matching covered graphs

A connected graph G is said to be *matching covered* if it has at least two vertices and each edge is allowed. Every matching covered graph with at least four vertices is 2-connected [18].

In this section, we will characterize the matching covered graphs whose complete forcing numbers attain the upper bound given in Theorem 2.4 and minus one, respectively.

Theorem 3.1. *Let G be a matching covered graph. Then $cf(G) = 2c(G)$ if and only if G is either K_2 or an even cycle.*

Proof. The sufficiency is obvious. So we consider the necessity. If $c(G) = 0$, then G is a tree. Since G is matching covered, G can only be K_2. For $c(G) \geq 1$, suppose to the contrary that G is not an even cycle. Then G has a vertex v with degree at least 3. Let $D_1 = \partial(v)$. Then since G is 2-connected, $G - D_1$ has exactly two components and D_1 is an e-cut with $|D_1| \geq 3$. We use D_1 to do an ED operation on $G_1 = G$ and obtain G_2, and then we do ED operations from G_2 repeatedly until the empty graph is obtained. Consequently, we obtain an e-cut decomposition $G = G_1 \supset G_2 \supset \cdots \supset G_{l+1} = \emptyset$ $(l \geq 1)$. For $i = 2, 3, \ldots, l$, let D_i be the e-cut deleted from G_i in this e-cut decomposition. By Equation (2.1), $c(G_2) - c(G_1) = |D_1| - 1 \geq 2$ and $c(G_{i+1}) - c(G_i) = |D_i| - 1 \geq 1$ $(i = 2, 3, \ldots, l)$. Combining with Equation (2.2), we have $l + 1 \leq \sum_{i=1}^{l}(|D_i| - 1) = c(G) - c(G_{l+1})$, and thus $l \leq c(G) - 1$. By Lemma 2.3, $c(G) \leq 2c(G) - 1$, a contradiction. □

Corollary 3.2. *Let G be a graph with a perfect matching. Then $cf(G) = 2c(G)$ if and only if*

(i) *each forbidden edge of G is a bridge, and*

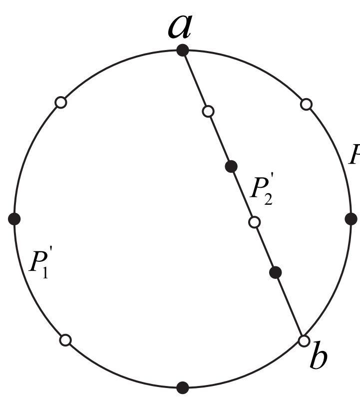
(ii) *each component of the graph obtained by deleting all forbidden edges from G is either K_2 or an even cycle.*

Proof. Let G_0 be the graph obtained from G by deleting all forbidden edges. Since each forbidden edge of G does not appear in any minimum complete forcing set of G, $cf(G) = cf(G_0)$. Let $O_1, O_2, \ldots, O_t$ $(t \geq 1)$ be the components of G_0. By Theorem 2.4 we have

$$cf(G) = cf(G_0) = \sum_{i=1}^{t} cf(O_i) \leq 2\sum_{i=1}^{t} c(O_i) = 2c(G_0) \leq 2c(G).$$

In the above expression, Theorem 3.1 implies that the third equality holds if and only if each O_i is either K_2 or an even cycle, and the fifth equality holds if and only if G_0 and G have the same cyclomatic number, that is, each forbidden edge of G is a bridge. □

A connected graph G is said to be *elementary* if all its allowed edges form a connected subgraph of G. A connected bipartite graph is elementary if and only if each edge is allowed [17]. An elementary bipartite graph has the so-called "bipartite ear decomposition". Let x be an edge. Join the end vertices of x by a path P_1 of odd length (the so-called "first ear"). We proceed inductively to build a sequence of bipartite graphs as follows: If $G_{r-1} = x + P_1 + P_2 + \cdots + P_{r-1}$ has already been constructed, add the r-th ear P_r (a path of odd length) by joining any two vertices in different colors of G_{r-1} such that P_r has no other vertices in common with G_{r-1}. The decomposition $G_r = x + P_1 + P_2 + \cdots + P_r$ will be called an *(bipartite) ear decomposition* of G_r. It is known that a bipartite graph G is elementary if and only if G has a bipartite ear decomposition [17]. We can see that the number r of ears is equal to the cyclomatic number of G.



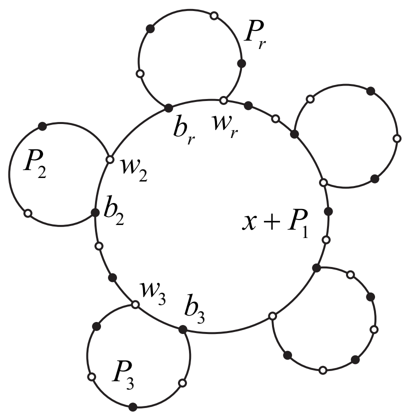


Figure 1: Two examples for graphs G with $cf(G) = 2c(G) - 1$.

Theorem 3.3. *Let G be a matching covered graph. Then $cf(G) = 2c(G) - 1$ if and only if G is a bipartite graph and one of the following holds :*

(i) $c(G) = 2$ *(see* Figure 1(a)*);*

(ii) *G has an ear decomposition $G = x + P_1 + P_2 + \cdots + P_r$ $(r \geq 3)$ such that one frame of $x + P_1$ contains at least $r - 1$ edges $w_2b_2, w_3b_3, \ldots, w_rb_r$ and the two ends of $P_2, P_3, \ldots, P_r$ are the two end-vertices of $w_2b_2, w_3b_3, \ldots, w_rb_r$, respectively (see* Figure 1(b)*).*

Proof. Sufficiency. (i) If $c(G) = 2$, by the ear decomposition of G, G contains two 3-degree vertices, denoted by a and b. Let P_1', P_2', P_3' be the 3 internally disjoint paths from a to b (see Figure 1(a)) and S be a complete forcing set of G. If $|S| \leq 2$, then one of P_1', P_2' and P_3' has no edges in S, say P_1'. We can see that one of the two nice cycles $P_1' \cup P_2'$ and $P_1' \cup P_3'$ has a frame containing no edges of S, which contradicts that S is a complete forcing set by Theorem 1.1. So $cf(G) \geq 3$. Conversely, we can see that $\partial_G(a)$ is a complete forcing set of G, which means that $cf(G) \leq 3$. Consequently, we have $cf(G) = 3 = 2c(G) - 1$.

(ii) For $2 \leq i \leq r$, let $C_i = P_i \cup \{w_i b_i\}$. Then we can see that $C_2, C_3, \ldots, C_r$ are $r-1$ vertex-disjoint nice cycles of G. Let S be a complete forcing set of G. By Theorem 1.1, each frame of C_i $(2 \leq i \leq r)$ has at least one edge of S, so each C_i contains 2 edges of S. Further the frame of the nice cycle $x + P_1$ that does not contain $\{w_2b_2, w_3b_3, \ldots, w_rb_r\}$ has an edge in S. So $|S| \geq 2r-1$. Conversely, let $D_1 = \partial_G(b_2)$ and D_i $(i = 2, 3, \ldots, r-1)$ be any two adjacent edges of C_{i+1}. We use $D_1, D_2, \ldots, D_{r-1}$ to do ED operations from G in turn and obtain empty graph finally. By Lemma 2.3, $D_1 \cup D_2 \cup \cdots \cup D_{r-1}$ is a complete forcing set of G and $cf(G) \leq |D_1 \cup D_2 \cup \cdots \cup D_{r-1}| = 3 + 2(r-2) = 2r-1$. Consequently, we have $cf(G) = 2r-1 = 2c(G)-1$.

Necessity. If $c(G) = 0$ or 1, then G is K_2 or an even cycle. By Theorem 3.1, $cf(G) = 2c(G)$, contradicting $cf(G) = 2c(G)-1$. So $c(G) \geq 2$ and $|V(G)| \geq 4$. Since G is matching covered, G is 2-connected.

Claim 1. For an e-cut decomposition from $G = G_1$ to $G_{l+1} = \emptyset$, if there is an integer k $(1 \leq k \leq l)$ such that $|D_k| \geq 4$ or there are two integers m and n $(1 \leq m < n \leq l)$ such that $|D_m| \geq 3$ and $|D_n| \geq 3$, then $cf(G) \leq 2c(G)-2$.

Proof. If $|D_k| \geq 4$ $(1 \leq k \leq l)$, then since $|D_i| \geq 2$ for $i = 1, 2, \ldots, k-1, k+1, \ldots, l$, $\sum_{i=1}^{l}(|D_i|-1) \geq l+2$. From Equation (2.2), we have $l \leq c(G)-2$, and thus $cf(G) \leq 2c(G)-2$ by Lemma 2.3.

If $|D_m| \geq 3$ and $|D_n| \geq 3$ $(1 \leq m < n \leq l)$. Then since $|D_i| \geq 2$ $(i = 1, 2, \ldots, m-1, m+1, \ldots, n-1, n+1, \ldots, l)$, $\sum_{i=1}^{l}(|D_i|-1) \geq l+2$. From Equation (2.2), we have $l \leq c(G)-2$, and thus $cf(G) \leq 2c(G)-2$ by Lemma 2.3. □

If G has a vertex v_0 with degree at least 4, let $D_1 = \partial(v_0)$. Since G is 2-connected, $G - \partial(v_0)$ has exactly two components and D_1 is an e-cut of G. Then we can give an e-cut decomposition from G to empty graph by taking D_1 as the first e-cut. By Claim 1, $cf(G) \leq 2c(G)-2$, a contradiction. Thus $d_G(v) \leq 3$ for each vertex $v \in V(G)$. In addition, since $c(G) \geq 2$, G has a 3-degree vertex v_1. Since G is 2-connected, $G - v_1$ is connected.

Claim 2. G is a bipartite graph.

Proof. Suppose to the contrary that G is not a bipartite graph. Let v be a vertex of G. Then $G - v$ is not a bipartite graph as well. Otherwise, $G - v$ has a bipartition (W, B) $(|W| < |B|)$. If v is adjacent to a vertex w of W in G, then vw is a forbidden edge of G, which contradicts that G is matching covered. So v can only be adjacent to vertices of B in G, and thus G is a bipartite graph, a contradiction to the supposition. Hence, $G - v_1$ has an odd cycle C_1.

Let $D_1 = \partial(v_1)$. Since G is 2-connected, $G - D_1$ has exactly two components and D_1 is an e-cut of G with $|D_1| = 3$. We obtain G_2 by doing an ED operation on $G_1 = G$ via D_1. Since $G[V(C_1)]$ is 2-connected and $G[V(C_1)]$ is still a subgraph of G_2, from Lemma 2.2, there exists an e-cut decomposition from G_2 to $G[V(C_1)] = G_m$. For $i = 2, 3, \ldots, m-1$, we denote by D_i the deleted e-cut from G_i in this e-cut decomposition. If G_m has a 3-degree vertex v_m, let $D_m = \partial_{G[V(C_1)]}(v_m)$. We can give an e-cut decomposition from G_m to empty graph by taking D_m as the first e-cut. Combining the above two e-cut decompositions, we have an e-cut decomposition from G_1 to empty graph with $|D_1| = |D_m| = 3$. By Claim 1, $cf(G) \leq 2c(G)-2$, a contradiction. If G_m is an odd cycle, by Lemma 2.3, $D_1 \cup D_2 \cup \cdots \cup D_{m-1}$ is a complete forcing set of G and $cf(G) \leq$

$c(G) + (m-1) - c(G_m)$. Since $|D_1| = 3$ and $|D_i| \geq 2$ ($i = 2, 3, \ldots, m-1$), $c(G) - c(G_m) = \sum_{i=1}^{m-1}(|D_i| - 1) \geq m$, so we have $m \leq c(G) - c(G_m) = c(G) - 1$. Hence, $cf(G) \leq c(G) + (m-1) - c(G_m) \leq 2c(G) - 3$, a contradiction. □

Claim 3. Each block of $G - v_1$ is either K_2 or an even cycle.

Proof. Let B be a block of $G - v_1$. Then $d_B(v) \leq 2$ for each $v \in V(B)$. Otherwise, let v' be a vertex of B with $d_B(v') = 3$. By Lemma 2.2, there exists an e-cut decomposition from $G = G_1$ to $B = G_m$ by taking $D_1 = \partial_G(v_1)$ as the first e-cut. Let $D_m = \partial_{G_m}(v')$. Then D_m is an e-cut of B and we can give an ED decomposition from B to empty graph by taking D_m as the first e-cut. Combining with the above two e-cut decompositions, we have an e-cut decomposition from G to empty graph with $|D_1| = |D_m| = 3$. By Claim 1, $cf(G) \leq 2c(G) - 2$, a contradiction. Since G is a bipartite graph by Claim 2, each block of $G - v_1$ is K_2 or an even cycle. □

In the following we may assume that v_1 is a black vertex of G.

Claim 4. If each block of $G - v_1$ is K_2, then $c(G) = 2$ and (i) holds.

Proof. Obviously $G - v_1$ is a tree. If $G - v_1$ has no 3-degree vertices, then it is a path P. Since G is 2-connected, the end-vertices of P are adjacent to v_1 and receive white. Further, since $d_G(v_1) = 3$, v_1 has third white neighbor as an internal vertex of P (see Figure 2(a)). So $c(G) = 2$.

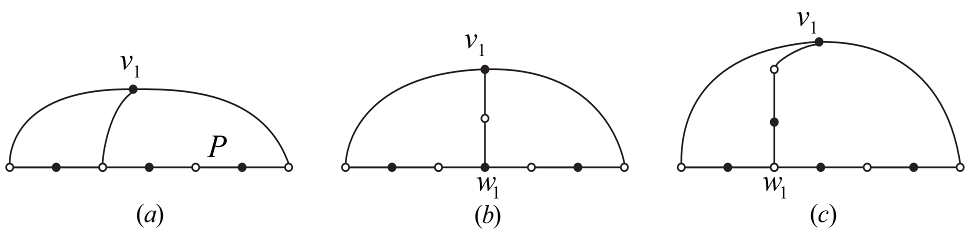


Figure 2: Illustration for Claim 4.

If $G - v_1$ has a 3-degree vertex, then $G - v_1$ has only one 3-degree vertex, denoted by w_1. Otherwise, $G - v_1$ has at least four 1-degree vertices, but just three of them is adjacent to v_1 in G, so G has a 1-degree vertex, a contradiction. Thus $G - v_1$ has three 1-degree vertices which are adjacent to v_1 in G. It follows that w_1 is a white vertex (see Figure 2(c)); Otherwise, G has an odd number of vertices (see Figure 2(b)), a contradiction. So $c(G) = 2$. □

In what follows we suppose that $G - v_1$ has a block that is an even cycle.

Claim 5. Each even cycle block of $G - v_1$ has at most two 3-degree vertices in G.

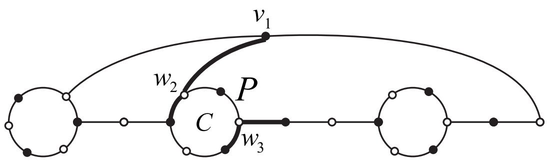


Figure 3: An even cycle block C of $G - v_1$ has exactly three 3-degree vertices of G.

Proof. If $G - v_1$ has an even cycle block that has four 3-degree vertices in G, then it has at least one end-block that has no vertices that are adjacent to v_1 in G. This causes G to have a cut-vertex, which contradicts that G is 2-connected. If there is an even cycle block C of $G - v_1$ that has exactly three 3-degree vertices of G, then two of such three vertices w_2 and w_3 have the same color in G. Let P be a path contained in C with ends w_2 and w_3 (see Figure 3). Then each internal vertex of P is still a 2-degree vertex of G. Further, since G has an e-cut $\partial(V(P))$ of four edges, we can give an e-cut decomposition from G to empty graph by taking $\partial(V(P))$ as the first e-cut. By Claim 1, we have $cf(G) \leq 2c(G) - 2$, a contradiction. □

Claim 6. $G - v_1$ is not 2-connected, and has no vertices contained in three K_2 blocks.

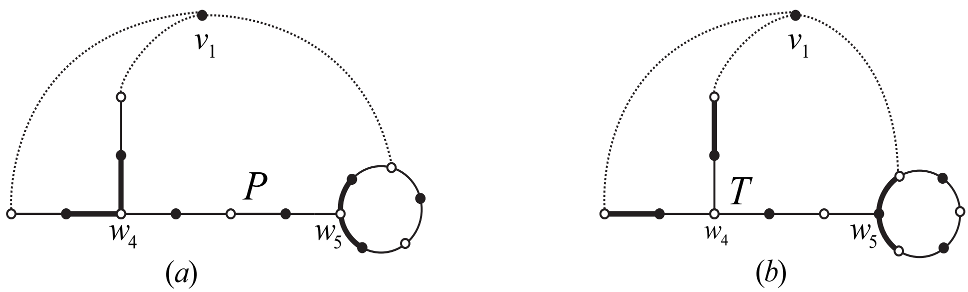


Figure 4: (a) w_4 and w_5 have the same color; (b) w_4 and w_5 have different colors.

Proof. If $G - v_1$ is 2-connected, then by Claim 3, $G - v_1$ is an even cycle and v_1 is adjacent to three vertices of this cycle in G, which contradicts Claim 5. So, $G - v_1$ is not 2-connected.

Suppose to the contrary that $G - v_1$ has a vertex w_4 incident with three K_2 blocks. Then $G - v_1$ has at least 3 end-blocks. Since G is 2-connected and $d_G(v_1) = 3$, $G - v_1$ has exactly three end-blocks. Let P be a shortest path between w_4 and a 3-degree vertex w_5 of $G - v_1$ in an even cycle block so that each internal vertex of P is a 2-degree vertex in G. If w_4 and w_5 have the same color, then $\partial(V(P))$ is an e-cut of G (see Figure 4(a)). There exists an e-cut decomposition from G to empty graph by taking $\partial(V(P))$ as the first e-cut. By Claim 1, we have $cf(G) \leq 2c(G) - 2$, a contradiction. If w_4 and w_5 have different colors, let T be the tree consisting of P and the remaining two K_2 blocks of $G - v_1$ that has an end-vertex w_4. Then $\partial(V(T))$ is an e-cut of G (see Figure 4(b)). Similarly we have $cf(G) \leq 2c(G) - 2$, a contradiction. □

By Claims 3, 5 and 6, $G - v_1$ has exactly two end-blocks which each has a white non-cut-vertex of $G - v_1$ adjacent to v_1 in G, and $G - v_1$ can be constructed as follows: $r - 1$ disjoint paths $P_1', P_2', \ldots, P_{r-1}'$ connect $r-2$ disjoint even cycles $C_1, C_2, \ldots, C_{r-2}$ in turn so that P_i' only connects C_{i-1} and C_i for $i = 2, 3, \ldots, r-2$, where $r \geq 3$, and P_1' and P_{r-1}' connect only C_1 and C_{r-2} respectively (see Figure 5(a)). Let v_2 be the third neighbor of v_1 in $G - v_1$.

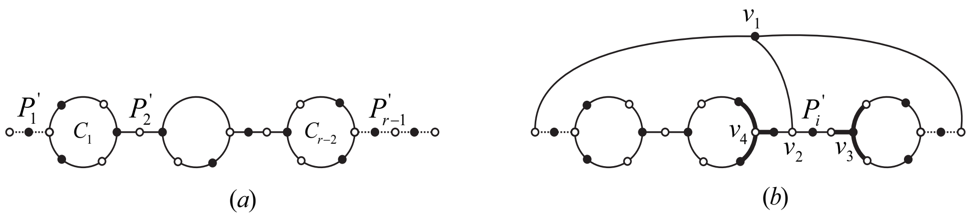


Figure 5: (a) The construction of $G - v_1$; (b) Illustration for Claim 7.

Claim 7. v_2 must be an internal vertex of paths P_1' and P_{r-1}'.

Proof. If v_2 belongs to some even cycle C_k $(1 \leq k \leq r-2)$ in $G - v_1$, then C_k has three 3-degree vertices of G, which contradicts Claim 5. If v_2 is an internal vertex of P_i' $(2 \leq i \leq r-2)$ (see Figure 5(b)), let the ends of P_i' be v_3 and v_4. Then there exists an e-cut decomposition from G to empty graph by taking $\partial(v_3)$ and $\partial(v_4)$ as the first two e-cuts. Since $|\partial(v_3)| = |\partial(v_4)| = 3$, by Claim 1, $cf(G) \leq 2c(G) - 2$, a contradiction. Hence v_2 is an internal vertex of P_1' or P_{r-1}' and the claim holds. □

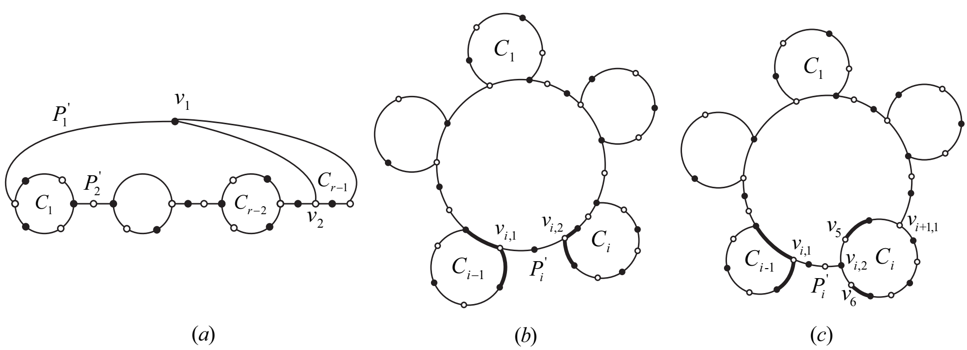


Figure 6: (a) The construction of G; (b) e-cut (bold edges) leaving from P_i'; (c) e-cut (bold edges) leaving from three T.

By Claim 7 we may suppose v_2 is an internal vertex of P_{r-1}' that has length at least 3. Then the subpath of P_{r-1}' between both neighbors of v_1 with two incident edges forms a cycle, denoted by C_{r-1}. Thus G can be constructed from $r-1$ disjoint even cycles $C_1, C_2, \dots, C_{r-1}$ by using $r-1$ disjoint paths to connect them in a cyclic way. More precisely, each P'_i connects vertex $v_{i,1}$ of C_{i-1} and vertex $v_{i,2}$ of C_i, where $i = 1, 2, \dots, r-1$ and $C_0 = C_{r-1}$ (see Figure 6(a)). Note that $P'_2, \dots, P'_{r-2}$ remain unchanged, but P'_1 is lengthened by one edge and P'_{r-1} is shorten. Then $v_{i,1}$ and $v_{i,2}$ have different colors in G. Otherwise, there exists an e-cut decomposition from G to empty graph by taking $\partial(V(P'_i))$ (see Figure 6(b)) as the first e-cut that has four edges. By Claim 1, $cf(G) \leq 2c(G) - 2$, a contradiction. Further, $v_{i,2}$ and $v_{i+1,1}$ have different colors in G, where $v_{r,1} = v_{1,1}$. Otherwise, two edges leaving from even cycle C_i are forbidden edges of G, which contradicts that G is matching covered. Finally we claim that $v_{i,2}$ and $v_{i+1,1}$ are adjacent in C_i. Otherwise, since $v_{i,2}$ and $v_{i+1,1}$ have different colors, two paths between $v_{i,2}$ and $v_{i+1,1}$ in C_i have length at least 3 (see Figure 6(c)). Let v_5 and v_6 be the two neighbors of $v_{i,2}$ in C_i. Then v_5, v_6 and $v_{i,1}$ are all of the same color in G. Let T be the tree induced by $\{v_5, v_6\} \cup V(P'_i)$. Then $\partial(V(T))$ is an e-cut of G of four edges. So there exists an e-cut decomposition from G to empty graph by taking $\partial(V(T))$ as the first e-cut. By Claim 1, we have $cf(G) \leq 2c(G) - 2$, a contradiction.

Let $x + P_1$ be an even cycle formed by the paths P'_i and edges $v_{i,2}v_{i+1,1}$, $i = 1, 2, \dots, r-1$, and let P_{i+1} be the path between $v_{i,2}$ and $v_{i+1,1}$ in C_i of length at least three. Then the edges $v_{i,2}v_{i+1,1}$, $i = 1, 2, \dots, r-1$, are contained in a frame of $x + P_1$ and $G = x + P_1 + P_2 + \cdots + P_r$ is an ear decomposition of G described as in (ii). □

4 Wheels and cylinders

In this section, we first present some lower bounds on the complete forcing numbers of some special types of graphs. We then derive some closed formulas for the complete forcing numbers of wheels and cylinders, respectively. Our main idea is to apply an e-cut decomposition on a given graph to construct a complete forcing set whose cardinality attains a lower bound on the complete forcing number.

Lemma 4.1. *Let G be a graph that admits a perfect matching. If there is a set $\mathcal{C}$ of nice cycles of G such that every edge of G lies in exactly two nice cycles of $\mathcal{C}$, then $cf(G) \geq |\mathcal{C}|$.*

Proof. For a nice cycle C of $\mathcal{C}$, let $T_1(C)$ and $T_2(C)$ be the two frames of C. Let S be a minimum complete forcing set of G. By Theorem 2.1, we have

$$|S \cap T_i(C)| \geq 1, i = 1, 2, \text{ for each nice cycle } C \text{ of } \mathcal{C}.$$

Summing all the above inequalities together, we have

$$2|S| = \sum_{C \in \mathcal{C}} (|S \cap T_1(C)| + |S \cap T_2(C)|) \geq 2|\mathcal{C}|,$$

because each edge of S belongs to exactly two nice cycles of $\mathcal{C}$. Then we have

$$cf(G) = |S| \geq |\mathcal{C}|.$$ □

For a plane elementary bipartite graph G, all facial cycles (including the exterior facial cycle) of G are nice cycles [28]. Since each edge of G lies in exactly two of these facial cycles, by Lemma 4.1, we have

Corollary 4.2. *Let G be a plane elementary bipartite graph with n faces. Then $cf(G) \geq n$.*

This result is a generalization of a lower bound on the complete forcing numbers of normal hexagonal systems (see [11]).

A *wheel* W_n $(n \geq 4)$ is a graph formed by connecting a single vertex (called the *hub*) to all vertices of a cycle (called the *rim*) with $n-1$ vertices. We can check that W_{2n} $(n \geq 2)$ is matching covered by the definition.

Theorem 4.3. *For $n \geq 2$, $cf(W_{2n}) = 2n-1$.*

Proof. We denote by v_0 the hub of W_{2n} and by $v_1, v_2, \ldots, v_{2n-1}$ the vertices in the rim of W_{2n} along one of two directions of it. We can see that the set $\mathcal{C}$ of 4-cycles $\{v_i v_0 v_{i+2} v_{i+1} v_i | i = 1, 2, \ldots, 2n-1\}$ consisting of $2n-1$ nice cycles of W_{2n}, where $v_{2n} = v_1$ and $v_{2n+1} = v_2$. Moreover, each edge of W_{2n} lies in exactly two nice cycles of $\mathcal{C}$. By Lemma 4.1, $cf(W_{2n}) \geq |\mathcal{C}| = 2n-1$. On the other hand, let $D_1 = \partial_{W_{2n}}(v_0)$. Then D_1 is an e-cut of W_{2n} with $2n-1$ edges. We use D_1 to do an e-cut operation on $G_1 = W_{2n}$ and obtain G_2. Since G_2 is an odd cycle, by Lemma 2.3, D_1 is a complete forcing set. So $cf(W_{2n}) \leq |D_1| = 2n-1$. Consequently, $cf(W_{2n}) = 2n-1$. $\square$

The *cartesian product* $G \times H$ of two graphs G and H is a graph with vertex set $V(G) \times V(H)$ specified by putting (u, v) adjacent to (u', v') if and only if (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$. Let $P_m = u_1 u_2 \cdots u_m$ be a path with m vertices. Recently, Chang et al. [5] obtained that $cf(P_m \times P_n) = \lfloor \frac{n}{2} \rfloor (m-1) + \lfloor \frac{m}{2} \rfloor (n-1)$. It is natural to consider the complete forcing numbers of $m \times n$ cylinders. Let $C_n = v_1 v_2 \cdots v_n v_1$ be a cycle with n vertices. An $m \times n$ cylinder $P_m \times C_n$ consists of $m-1$ concentric layers of quadrangles (i.e. each layer is a cyclic chain of n quadrangles), capped on each end by an n-polygon (see G_1 of Figure 7 for an example). If both m and n are odd, then $P_m \times C_n$ has an odd number of vertices and thus has no perfect matchings. So we only consider the complete forcing number of $P_m \times C_n$ with even mn. The operation of inserting a new vertex of degree two on an edge of a graph is called a *subdivision* of the edge.

Lemma 4.4. *If m is even, then*

$$cf(P_m \times C_n) \geq \begin{cases} mn - \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{2mn+m-n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. For $1 \leq i \leq m-1$, let R_i be the subgraph of $P_m \times C_n$ induced by $\{(u_i, v_j), (u_{i+1}, v_j) | \, j = 1, 2, \ldots, n\}$ and $E_{1i} = \{(u_i, v_j)(u_{i+1}, v_j) | j = 1, 2, \ldots, n\}$. For $1 \leq j \leq n$, let L_j be the subgraph of $P_m \times C_n$ induced by $\{(u_i, v_j), (u_i, v_{j+1}) | i = 1, 2, \ldots, m\}$ and $E_{2j} = \{(u_i, v_j)(u_i, v_{j+1}) | i = 1, 2, \ldots, m\}$, where $v_{n+1} = v_1$. Let S be a minimum complete forcing set of $P_m \times C_n$. Since R_i has n nice quadrangles of $P_m \times C_n$ and each quadrangle of R_i has a frame completely contained in E_{1i}, by Theorem 1.1, each quadrangle of R_i has an edge in $S \cap E_{1i}$. If n is even, then $|S \cap E_{1i}| \geq \frac{n}{2}$. And if n is odd, then $|S \cap E_{1i}| \geq \frac{n+1}{2}$. Since L_j has $m-1$ nice quadrangles of $P_m \times C_n$ and each quadrangle of L_j has a frame completely contained in E_{2j}, by Theorem 1.1, each quadrangle of L_j has an edge in $S \cap E_{2j}$. Since m is even, $|S \cap E_{2j}| \geq \frac{m}{2}$. Thus we have if n is even, then $cf(P_m \times C_n) = |S| \geq \sum_{i=1}^{m-1} |S \cap E_{1i}| + \sum_{j=1}^{n} |S \cap E_{2j}| \geq \frac{n(m-1)}{2} + \frac{mn}{2} = mn - \frac{n}{2}$. And if n is odd, then $cf(P_m \times C_n) = |S| \geq \sum_{i=1}^{m-1} |S \cap E_{1i}| + \sum_{j=1}^{n} |S \cap E_{2j}| \geq \frac{(m-1)(n+1)}{2} + \frac{mn}{2} = \frac{2mn+m-n-1}{2}$. $\square$

Lemma 4.5 (Pick's theorem [8])**.** *Let P be a simple polygon constructed on a polyomino such that all the polygon's vertices are polyomino's vertices. Let the number of polyomino's vertices in the interior of P be i and the number of polyomino's vertices on the boundary of P be b. Then the area of P is given by $A = \frac{b}{2} + i - 1$.*

Theorem 4.6.

$$cf(P_m \times C_n) = \begin{cases} mn - n + 2, & \text{if } m \text{ is odd and } n \text{ is even } (m \geq 1, n \geq 4), \\ mn - \frac{n}{2}, & \text{if both } m \text{ and } n \text{ are even } (m \geq 2, n \geq 4), \\ \frac{2mn+m-n-1}{2}, & \text{if } m \text{ is even and } n \text{ is odd } (m \geq 2, n \geq 3). \end{cases}$$

Proof. Since mn is even, we can see that each edge of $P_m \times C_n$ is allowed, so $P_m \times C_n$ is matching covered. To construct a complete forcing set of $P_m \times C_n$, by Lemma 2.3, we can directly apply e-cut decomposition on $P_m \times C_n$.

We divide our proof into the following three cases.

Case 1. m is odd and n is even $(m \geq 1, n \geq 4)$.

If $m = 1$, then $P_m \times C_n$ is an even cycle and $cf(P_m \times C_n) = 2$ by Theorem 3.1, and the conclusion holds. In the following, we suppose that $m \geq 3$.

By Corollary 4.2, $cf(P_m \times C_n) \geq mn - n + 2$. So it suffices to construct a complete forcing set of $P_m \times C_n$ of size $mn - n + 2$.

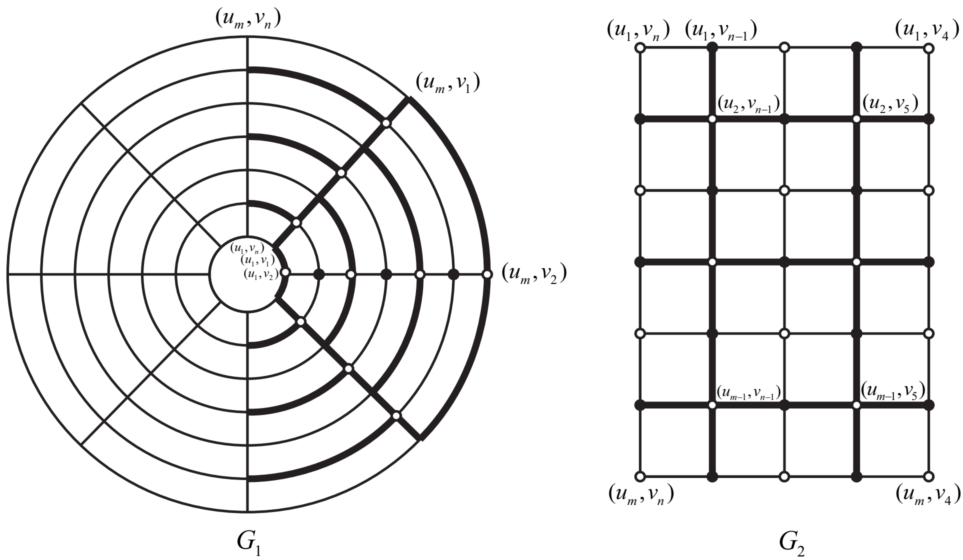


Figure 7: m is odd and n is even.

Let

$$\begin{aligned}D_1 =&\{(u_{2i+1},v_1)(u_{2i+1},v_2),(u_{2i+1},v_2)(u_{2i+1},v_3) \mid i=0,1,2,\ldots,\frac{m-1}{2}\}\cup\\ &\{(u_j,v_1)(u_{j+1},v_1),(u_j,v_3)(u_{j+1},v_3) \mid j=1,2,\ldots,m-1\}\cup\\ &\{(u_{2k},v_n)(u_{2k},v_1),(u_{2k},v_3)(u_{2k},v_4) \mid k=1,2,\ldots,\frac{m-1}{2}\}\end{aligned}$$

(see bold edges of G_1 of Figure 7). Then D_1 is an e-cut of $G_1 = P_m \times C_n$. We use D_1 to do an ED operation on G_1 and obtain $G_2 = P_m \times P_{n-3}$ (see Figure 7). Let $D_2, D_3, \ldots, D_{\frac{(m-1)(n-4)}{4}+1}$ be $\partial_{G_2}((u_{2i},v_5))$, $\partial_{G_2}((u_{2i},v_7))$,..., $\partial_{G_2}((u_{2i},v_{n-1}))$ ($i = 1,2,\ldots,\frac{m-1}{2}$), respectively. Then we continue to do ED operations from G_2 by $D_2, D_3, \ldots, D_{\frac{(m-1)(n-4)}{4}+1}$ in turn and obtain $G_{\frac{(m-1)(n-4)}{4}+2}$. Note that D_i is an e-cut of G_i for $i = 1,2,\ldots,\frac{(m-1)(n-4)}{4}+1$. We find that $G_{\frac{(m-1)(n-4)}{4}+2}$ can be obtained by subdividing every edge of $P_{\frac{m-1}{2}+1} \times P_{\frac{n-4}{2}+1}$ as shown in the thin edges of G_2 in Figure 7. Let C be a cycle of $G_{\frac{(m-1)(n-4)}{4}+2}$. Suppose that C encloses some region R in the plane, let A be the area of R, b be the number of vertices of G_2 on C, and i be the number of vertices of G_2 in the interior of C. Then A is divisible by 4. We can see that C is obtained by subdividing every edge of a cycle C' of $P_{\frac{m-1}{2}+1} \times P_{\frac{n-4}{2}+1}$. Since C' is a cycle of even length and $|V(C)| = 2|V(C')|$, b is divisible by 4. By Lemma 4.5, i is odd. Then $G_1 - V(C)$ has no perfect matchings. So each cycle of $G_{\frac{(m-1)(n-4)}{4}+2}$ is not a nice cycle of G_1. By Lemma 2.3, $D_1 \cup D_2 \cup \ldots D_{\frac{(m-1)(n-4)}{4}+1}$ is a complete forcing set of G_1. Since $|D_1| = (m+1)+2(m-1)+(m-1) = 4m-2$ and $|D_i| = 4$ for $i = 2,3,\ldots,\frac{(m-1)(n-4)}{4}+1$, $cf(G_1) \le |D_1 \cup D_2 \cup \ldots D_{\frac{(m-1)(n-4)}{4}+1}| = mn-n+2$. Consequently, $cf(P_m \times C_n) = mn-n+2$.

Case 2. Both m and n are even ($m \ge 2, n \ge 4$).

By Lemma 4.4, it suffices to construct a complete forcing set of $P_m \times C_n$ of size $mn - \frac{n}{2}$. Let

$$\begin{aligned}D_1 =&\{(u_{2i+1},v_1)(u_{2i+1},v_2),(u_{2i+1},v_2)(u_{2i+1},v_3) \mid i=0,1,2,\ldots,\frac{m-2}{2}\}\cup\\ &\{(u_j,v_1)(u_{j+1},v_1),(u_j,v_3)(u_{j+1},v_3) \mid j=1,2,\ldots,m-1)\}\cup\\ &\{(u_{2k},v_n)(u_{2k},v_1),(u_{2k},v_3)(u_{2k},v_4) \mid k=1,2,\ldots,\frac{m}{2}\}.\end{aligned}$$

Then D_1 is an e-cut of $G_1 = P_m \times C_n$. We use D_1 to do an ED operation on $G_1 = P_m \times C_n$ and obtain $G_2 = P_m \times P_{n-3}$ (see Figure 8). Let $D_2, D_3, \ldots, D_{\frac{m(n-4)}{4}+1}$ be $\partial_{G_2}((u_{2i},v_5))$, $\partial_{G_2}((u_{2i},v_7))$,..., $\partial_{G_2}((u_{2i},v_{n-1}))$ ($i = 1,2,\ldots,\frac{m}{2}$), respectively. Continuously doing ED operations from G_2 by $D_2, D_3, \ldots, D_{\frac{m(n-4)}{4}+1}$ in turn, we obtain $G_{\frac{m(n-4)}{4}+2}$. Note that $G_{\frac{m(n-4)}{4}+2}$ can be obtained by subdividing every edge of $P_{\frac{m-2}{2}+1} \times P_{\frac{n-4}{2}+1}$ as shown in Figure 8. Let C be a cycle of $G_{\frac{m(n-4)}{4}+2}$. Suppose that C encloses some region R in the plane, let A be the area of R, b the number of vertices of G_2 on C, and i be the number of vertices of G_2 in the interior of C. Then A is divisible by 4. We can see that C is obtained by subdividing every edge of a cycle C' of $P_{\frac{m-2}{2}+1} \times P_{\frac{n-4}{2}+1}$. Since C' is a cycle of even length and $|V(C)| = 2|V(C')|$, b is divisible by 4. By Lemma 4.5, i is

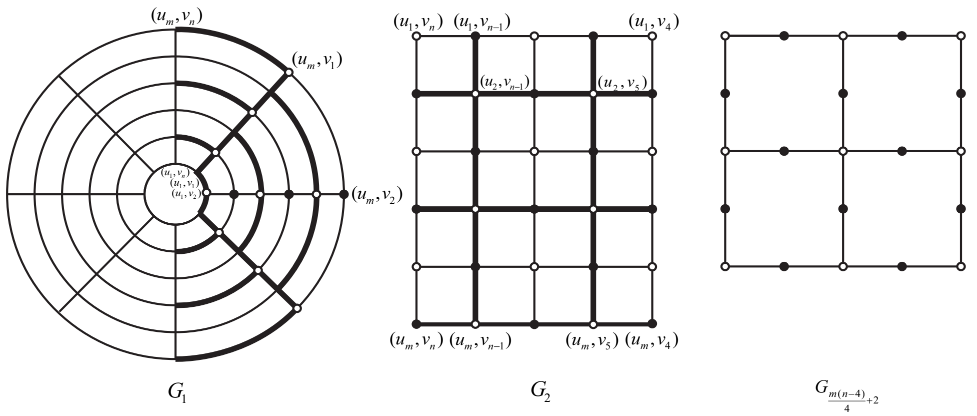


Figure 8: Both m and n are even.

odd. So $G_1 - V(C)$ has no perfect matchings. Thus each cycle of $G_{\frac{m(n-4)}{4}+2}$ is not a nice cycle of G_1. By Lemma 2.3, $D_1 \cup D_2 \cup \dots D_{\frac{m(n-4)}{4}+1}$ is a complete forcing set of G_1. Since $|D_1| = m + 2(m-1) + m = 4m - 2$, $|D_i| = 4$ for $i = 2, 3, \dots, \frac{(m-2)(n-4)}{4} + 1$ and $|D_j| = 3$ for $j = \frac{(m-2)(n-4)}{4} + 2, \frac{(m-2)(n-4)}{4} + 3, \dots, \frac{m(n-4)}{4} + 1$, $cf(G_1) \leq |D_1 \cup D_2 \cup \dots D_{\frac{(m-1)(n-4)}{4}+1}| = mn - \frac{n}{2}$. Consequently, $cf(P_m \times C_n) = mn - \frac{n}{2}$.

Case 3. m is even and n is odd $(m \geq 2, n \geq 3)$.

By Lemma 4.4, it suffices to prove $cf(G_1) \leq \frac{2mn+m-n-1}{2}$.

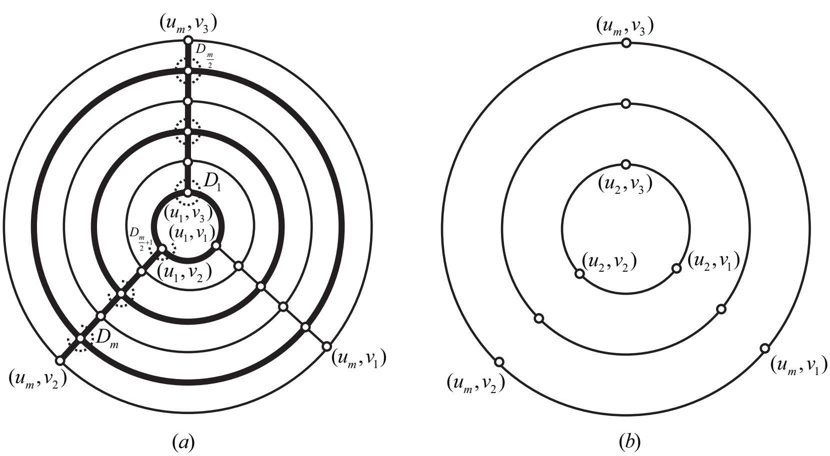


Figure 9: m is even and $n = 3$.

Subcase 3.1. $n = 3$.

Let $G_1 = P_m \times C_3$ and $D_1, D_2, \ldots, D_{\frac{m}{2}}$ be $\partial_{G_1}((u_{2i+1}, v_3))$ $(i = 0, 1, \ldots, \frac{m-2}{2})$, respectively (see Figure 9(a)). Then we use $D_1, D_2, \ldots, D_{\frac{m}{2}}$ to do ED operations on G_1 in turn and obtain $G_{\frac{m}{2}+1}$. Let $D_{\frac{m}{2}+1}, D_{\frac{m}{2}+2}, \ldots, D_m$ be $\partial_{G_{\frac{m}{2}+1}}((u_{2i+1}, v_2))$ $(i = 0, 1, \ldots, \frac{m-2}{2})$, respectively. Then we use $D_{\frac{m}{2}+1}, D_{\frac{m}{2}+2}, \ldots, D_m$ to do ED operations in turn and obtain G_{m+1}. We can see that G_{m+1} consists of $\frac{m}{2}$ disjoint cycles of length 3 and $c(G_{m+1}) = \frac{m}{2}$ as shown in Figure 9(b). Thus each cycle of G_{m+1} is not a nice cycle of G_1. By Lemma 2.3, $cf(G_1) \le c(G_1) + m - c(G_{m+1}) = 3(m-1) + 1 + m - \frac{m}{2} = \frac{7m-4}{2}$. Consequently, $cf(P_m \times C_{2n}) = \frac{7m-4}{2}$.

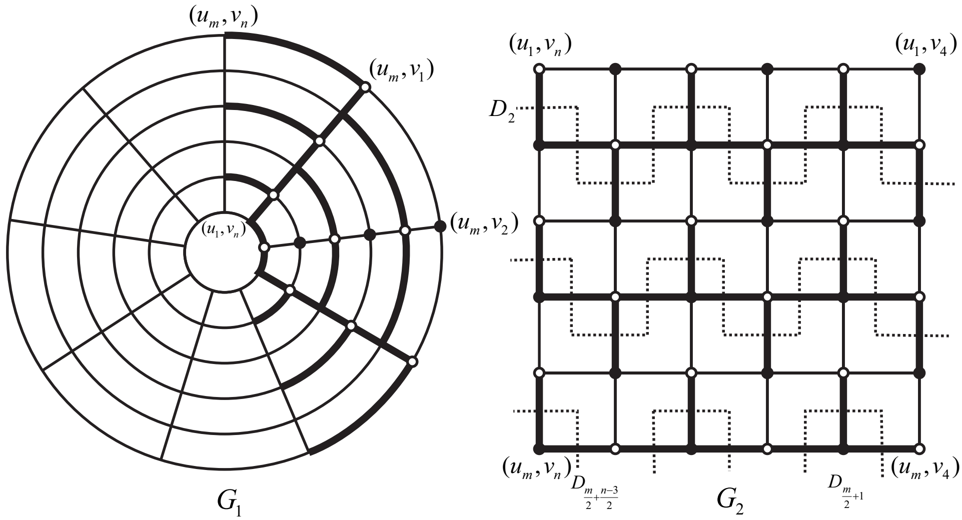


Figure 10: m is even and n is odd $(n \ge 5)$.

Subcase 3.2. $n \ge 5$.

Let

$$\begin{aligned} D_1 = &\{(u_{2i+1}, v_1)(u_{2i+1}, v_2), (u_{2i+1}, v_2)(u_{2i+1}, v_3) \mid i = 0, 1, 2, \ldots, \frac{m-2}{2})\} \cup \\ &\{(u_j, v_1)(u_{j+1}, v_1), (u_j, v_3)(u_{j+1}, v_3) | j = 1, 2, \ldots, m-1\} \cup \\ &\{(u_{2k}, v_n)(u_{2k}, v_1), (u_{2k}, v_3)(u_{2k}, v_4) | k = 1, 2, \ldots, \frac{m}{2}\}. \end{aligned}$$

Then we use D_1 to do an ED operation on $G_1 = P_m \times C_{2n}$ and obtain $G_2 = P_m \times P_{n-3}$ (see Figure 10). Let D_t $(t = 2, 3, \ldots, \frac{m}{2})$ be

$$\begin{aligned} &\{(u_{2t-2}, v_{2i+2})(u_{2t-1}, v_{2i+2}) \mid i = 1, 2, \ldots, \frac{n-3}{2}\} \cup \\ &\{(u_{2t-2}, v_{j+3})(u_{2t-2}, v_{j+4}) \mid j = 1, 2, \ldots, n-4\} \cup \\ &\{(u_{2t-3}, v_{2k+3})(u_{2t-2}, v_{2k+3}) \mid k = 1, 2, \ldots, \frac{n-3}{2}\}. \end{aligned}$$

Then we use $D_2, D_3, \ldots, D_{\frac{m}{2}}$ to do ED operations on G_1 in turn and obtain $G_{\frac{m}{2}+1}$ which is $P_2 \times P_{n-3}$. Let $D_{\frac{m}{2}+1}, D_{\frac{m}{2}+2}, \ldots, D_{\frac{m}{2}+\frac{n-3}{2}}$ be $\partial_{G_{\frac{m}{2}+1}}((u_m, v_{2s+3}))$ ($s = 1, 2, \ldots, \frac{n-3}{2}$), respectively. Then we use $D_{\frac{m}{2}+1}, D_{\frac{m}{2}+2}, \ldots, D_{\frac{m}{2}+\frac{n-3}{2}}$ to do ED operations on $G_{\frac{m}{2}+1}$ in turn and obtain $G_{\frac{m}{2}+\frac{n-3}{2}+1}$ which is the empty graph. By Lemma 2.3, $D_1 \cup D_2 \cup \cdots \cup D_{\frac{m}{2}+\frac{n-3}{2}}$ is a complete forcing set of G_1 and $cf(G_1) \leq c(G_1) + \frac{m}{2} + \frac{n-3}{2} - 0 = \frac{2mn+m-n-1}{2}$. Consequently, $cf(P_m \times C_n) = \frac{2mn+m-n-1}{2}$. □

At the end of this paper, by some simple calculations, we present the relationship between the cyclomatic number and complete forcing number for wheels and cylinders. For a wheel W_{2n}, $c(W_{2n}) = |E(W_{2n})| - |V(W_{2n})| + 1 = 2(2n-1) - 2n + 1 = 2n - 1$. By Theorem 4.3, $cf(W_{2n}) = c(W_{2n})$. For a cylinder $P_m \times C_n$, $c(P_m \times C_n) = |E(P_m \times C_n)| - |V(P_m \times C_n)| + 1 = n(m-1) + mn - mn + 1 = mn - n + 1$. By Theorem 4.6, we can see that $cf(P_m \times C_n) = c(P_m \times C_n) + 1$ if m is odd and n is even, $cf(P_m \times C_n) = c(P_m \times C_n) + \frac{n}{2} - 1$ if both m and n are even, and $cf(P_m \times C_n) = c(P_m \times C_n) + \frac{m+n-3}{2}$ if m is even and n is odd.

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Locally s-arc-transitive graphs arising from product action*

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Abstract

We study locally s-arc-transitive graphs arising from the quasiprimitive product action (PA). We prove that, for any locally $(G, 2)$-arc-transitive graph with G acting quasiprimitively with type PA on both G-orbits of vertices, the group G does not act primitively on either orbit. Moreover, we construct the first examples of locally s-arc-transitive graphs of PA type that are not standard double covers of s-arc-transitive graphs of PA type, answering the existence question for these graphs.

Keywords: Locally s-arc-transitive graph, quasiprimitive group, product action.

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1 Introduction

For an integer $s \geqslant 1$, an s-arc in a graph Γ is an $(s+1)$-tuple $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ of vertices such that $\alpha_i \sim \alpha_{i+1}$ and $\alpha_i \neq \alpha_{i+2}$ for each i. We say that Γ is *s-arc-transitive* if Γ contains an s-arc and the automorphism group of Γ acts transitively on the set of all s-arcs.

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If Γ is s-arc-transitive and each $(s-1)$-arc can be extended to an s-arc then any s-arc-transitive graph is also $(s-1)$-arc-transitive. The study of s-arc-transitive graphs goes back to the pioneering work of Tutte [32, 33], who showed that if Γ has valency three then $s \leqslant 5$. Weiss [35] later showed that if the valency restriction is relaxed to allow valency at least three then $s \leqslant 7$, with equality holding for the generalised hexagons arising from the groups $G_2(q)$ for $q = 3^f$.

Praeger [25] initiated a programme for the study of finite connected s-arc-transitive graphs by first showing that if $G \leqslant \mathrm{Aut}(\Gamma)$ acts transitively on the set of all s-arcs of Γ and $N \lhd G$ has at least three orbits on the set of vertices, then the quotient graph Γ_N whose vertices are the orbits of N is also s-arc-transitive. Moreover, Γ is a cover of Γ_N. This reduces the study of finite connected (G, s)-arc-transitive graphs to two basic types:

- those where G is *quasiprimitive* on the set of vertices, that is, where all nontrivial normal subgroups of G are transitive on vertices;
- those where G is *biquasiprimitive* on the set of vertices, that is, where all nontrivial normal subgroups of G have at most two orbits on vertices and there is a normal subgroup with two orbits.

Praeger showed that of the eight types of finite quasiprimitive groups, only four — HA (affine), TW (twisted wreath), AS (almost simple) and PA (product action) — can act 2-arc-transitively on a graph [25]. We use the types of quasiprimitive groups as given in [27] and define type PA, the main focus of this paper, in Section 2. These are slight variations on the types of primitive permutation groups given by the O'Nan–Scott Theorem. All graphs of type HA were classified by Praeger and Ivanov [18] while those of type TW were studied by Baddeley [1]. The 2-arc-transitive graphs for some families of almost simple groups have all been classified, for example the Suzuki groups [9], Ree groups [8] and $\mathrm{PSL}(2, q)$ [16]. The first examples of 2-arc-transitive graphs of PA type were given by Li and Seress [22] and studied further by Li, Seress, and Song [23]. Another family of quasiprimitive 2-arc-transitive graphs of PA type were constructed by Li, Ling, and Wu in [21].

In the biquasiprimitive case the graph is bipartite and such graphs were investigated in [26, 28]. An alternative way to study such graphs is via the notion of local s-arc-transitivity. We say that a graph Γ is *locally* (G, s)*-arc-transitive* for a group $G \leqslant \mathrm{Aut}(\Gamma)$ if for each vertex α, the vertex stabiliser G_α acts transitively on the set of all s-arcs starting at α. If G also acts transitively on the set of vertices then Γ is s-arc-transitive. If Γ is locally (G, s)-arc-transitive but G is intransitive on the set of vertices, then G has two orbits on vertices and Γ is bipartite. One way to construct locally s-arc-transitive graphs is to start with an s-arc-transitive graph Γ and take its standard double cover Σ, which has vertex set $V\Gamma \times \{1, 2\}$ and $(\alpha, i) \sim (\beta, j)$ precisely when $i \neq j$ and $\alpha \sim \beta$ in Γ. Then $\mathrm{Aut}(\Gamma)$ acts as automorphisms on Σ with two orbits on vertices and Σ is locally $(\mathrm{Aut}(\Gamma), s)$-arc-transitive [11].

If Γ is a bipartite graph and $G \leqslant \mathrm{Aut}(\Gamma)$ acts transitively on the set of vertices, then Γ is locally (G^+, s)-arc-transitive where G^+ is the index two subgroup that stabilises each part of the bipartition. Hence the study of locally s-arc-transitive graphs encompasses the study of all bipartite s-arc-transitive graphs and hence the biquasiprimitive case in Praeger's programme. It is also a wider class of graphs as the known generalised octagons are locally 9-arc-transitive but not vertex-transitive, and it has been shown by van Bon and Stellmacher [34] that this is best possible.

A programme for the study of finite connected locally s-arc-transitive graphs was mapped out by Giudici, Li and Praeger [11]. If Γ is locally (G, s)-arc-transitive with G having two orbits on vertices and $N \lhd G$ is intransitive on both G-orbits, then the quotient graph Γ_N is also locally s-arc-transitive. Moreover, Γ is a cover of Γ_N. This reduces the study of finite connected locally (G, s)-arc-transitive graphs for which G is vertex-intransitive into two basic types:

- those where G is quasiprimitive on each of its two orbits on vertices;
- those where G is quasiprimitive on only one of its two orbits on vertices.

In the second case, it was shown [11] that the quasiprimitive action must be of type HA, HS, AS, PA or TW. These were further studied in [12] where all examples where the quasiprimitive action has type PA preserving a product structure or type HS were classified. An infinite family of examples where the quasiprimitive action has type TW was given by Kaja and Morgan [19]. In the first case, either the two quasiprimitive actions have the same quasiprimitive type and are one of HA, AS, TW or PA, or they are different with one of type SD and one of type PA [11]. All 2-arc-transitive graphs of the latter type were classified in [13] and there are locally 5-arc-transitive examples in this case [14]. It was shown in [17, Lemma 3.2] that all locally 2-arc-transitive graphs where the quasiprimitive action is of type HA on both orbits are actually vertex-transitive but a complete classification has not been obtained – see [18, Section 2] for further discussion. All locally $(G, 2)$-arc-transitive graphs have been classified in the cases where G is an almost simple group whose socle is a Ree group [7], Suzuki group [31], or $\mathrm{PSL}(2, q)$ [3], while the sporadic group case was studied in [20]. Examples also exist in the PA and TW cases as we can take standard double covers of s-arc-transitive graphs of type PA and TW respectively.

The aim of this paper is to study locally s-arc-transitive graphs of PA type. We prove that, for any locally $(G, 2)$-arc-transitive graph with G acting quasiprimitively with type PA on both G-orbits of vertices, the group G does not act primitively on either orbit. Moreover, in the spirit of [22], we solve the existence problem for locally 2-arc-transitive graphs of PA type. In particular, we construct the first examples of locally s-arc-transitive graphs of PA type that are not standard double covers of s-arc-transitive graphs of PA type.

2 PA type

Let G act quasiprimitively on a set Ω. We say that G has *type PA* if there exists a G-invariant partition $\mathcal{B}$ of Ω such that G acts faithfully on $\mathcal{B}$ and we can identify $\mathcal{B}$ with Δ^k for some set Δ and $k \geqslant 2$ such that $G \leqslant H \operatorname{wr} S_k$ acts in the usual product action of a wreath product on Δ^k, where $H \leqslant \mathrm{Sym}(\Delta)$ is an almost simple group acting quasiprimitively on Δ. Moreover, if $T = \mathrm{soc}(H)$ then G has a unique minimal normal subgroup $N = T^k$. Note that since G is quasiprimitive, N acts transitively on Ω and hence on $\mathcal{B}$. Thus $G = NG_\alpha = NG_B$, where $B \in \mathcal{B}$ is a block containing $\alpha \in \Omega$. As N is minimal normal in G we have that G transitively permutes the simple direct factors of N and hence so do both G_α and G_B. Thus given $B = (\delta, \ldots, \delta) \in \mathcal{B}$ we may assume that $N_B = T_\delta^k$ and for $\alpha \in B$ we have that N_α is a subdirect subgroup of N_B, that is, the projection of N_α onto each direct factor is isomorphic to T_δ.

Let $R = T_\delta$. Following the terminology of [22], if $N_\alpha \cong R$ then we call N_α a *diagonal subgroup* of $N_B = R^k$. Then there exists automorphisms $\varphi_2, \varphi_3, \ldots, \varphi_k$ of R such that

$$N_\alpha = \{(t, t^{\varphi_2}, \ldots, t^{\varphi_k}) \mid t \in R\}.$$

If each of the φ_i is the trivial automorphism then we call N_α a *straight diagonal subgroup* while if some φ_i is nontrivial then we call N_α a *twisted diagonal subgroup*. Furthermore, if $N_\alpha \ncong R$ then we refer to N_α as being a *nondiagonal subgroup.* We refer to the quasiprimitive permutation group G of type PA as being of *straight diagonal*, *twisted diagonal*, or *nondiagonal* type according to the type of N_α.

Note that unlike for primitive groups of type PA, G does not necessarily preserve a product structure on Ω, only on some G-invariant partition $\mathcal{B}$. Indeed the following result shows that for locally 2-arc-transitive graphs this partition must be nontrivial on each of the bipartite halves.

Theorem 2.1. *Let Γ be a locally $(G, 2)$-arc-transitive connected graph with G quasiprimitive of type PA on both orbits Ω_1 and Ω_2. Let $N = T^k = \operatorname{soc}(G)$ and for $i = 1, 2$, let $\mathcal{B}_i$ be a G-invariant partition of Ω_i such that G preserves a product structure Δ_i^k on each $\mathcal{B}_i$. Then $\mathcal{B}_i \neq \Omega_i$ for each i.*

Proof. Suppose that $\mathcal{B}_i = \Omega_i$ for some i. Without loss of generality suppose that $i = 1$. Also note that there is an almost simple group H with socle T such that $G \leqslant H \operatorname{wr} S_k$.

Let $\alpha = (\omega, \dots, \omega) \in \Omega_1$. Then $N_\alpha = T_\omega^k$ with $T_\omega \neq 1$ and $G_\alpha = G \cap (H_\omega \operatorname{wr} S_k)$. By [11, Lemma 3.2], $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive so either all neighbours of α lie in the same block of $\mathcal{B}_2$ or in distinct blocks. If they all lie in the same block then for each $\beta \in \Omega_1$ we have that the neighbours of β lie in the same block. However, this contradicts Γ being connected. Hence for each $\alpha \in \Omega_1$, the neighbours of α lie in distinct blocks. Hence G_α acts 2-transitively on the set X of blocks of $\mathcal{B}_2$ that contain neighbours of α. By [11, Lemma 6.2], $N_\alpha^{\Gamma(\alpha)}$ is a transitive subgroup of the 2-transitive group $G_\alpha^{\Gamma(\alpha)}$ and so N_α also acts transitively on X. Let $B = (\delta_1, \delta_2, \dots, \delta_k) \in \mathcal{B}_2$ be a block containing a neighbour γ of α. Then $X = (\delta_1, \delta_2, \dots, \delta_k)^{N_\alpha} = \delta_1^{T_\omega} \times \delta_2^{T_\omega} \times \cdots \times \delta_k^{T_\omega}$. By [29, Theorem 1.1(b)], the stabiliser G_1 in G of the first simple direct factor of N projects onto H in the first coordinate and so $(G_1)_\alpha$ projects onto H_ω in the first coordinate. Hence $\delta_1^{T_\omega} = \delta_1^{H_\omega}$. Since $G_\alpha \leqslant H_\omega \operatorname{wr} S_k$ and transitively permutes the k simple direct factors of N, it follows that $\delta_i^{T_\omega} = \delta_1^{T_\omega}$ for each i. In particular, $X = A^k$ for some set A and we could have chosen $B = (\delta, \dots, \delta)$ for some $\delta \in \Delta_2$. Thus $G_{\alpha\gamma} \leqslant G_{\alpha,B} \leqslant H_{\omega\delta} \operatorname{wr} S_k$. However, for $\delta' \in A \backslash \{\delta\}$ there is no element of $H_{\omega\delta} \operatorname{wr} S_k$ mapping $(\delta', \delta, \dots, \delta)$ to $(\delta', \delta', \delta, \dots, \delta)$, contradicting G_α acting 2-transitively on X. Thus $\mathcal{B}_1 \neq \Omega_1$. $\square$

Corollary 2.2. *Let Γ be a locally $(G, 2)$-arc-transitive connected graph with G quasiprimitive of type PA on both orbits. Then G is not primitive on either orbit.*

3 Constructions

Let G be a finite group with subgroups L and R. Let Δ_1 be the set $[G : L]$ of right cosets of L in G and Δ_2 be the set $[G : R]$ of right cosets of R in G. We define the *coset graph* $\Gamma = \operatorname{Cos}(G, L, R)$ to be the bipartite graph with vertex set the disjoint union $\Delta_1 \cup \Delta_2$ such that $\{Lx, Ry\}$ is an edge if and only if $Lx \cap Ry \neq \varnothing$, or equivalently $xy^{-1} \in LR$. Then G acts by right multiplication on both Δ_1 and Δ_2, and induces automorphisms of Γ. Note that the vertices in Δ_1 have valency $|L : L \cap R|$ while the vertices in Δ_2 have valency $|R : L \cap R|$. We say that Γ has *valency* $\{|L : L \cap R|, |R : L \cap R|\}$. Conversely, if Γ is a graph and $G \leqslant \operatorname{Aut}(\Gamma)$ acts transitively on the set of edges of Γ but not on the set of vertices then Γ can be constructed in this way [11, Lemma 3.7]. We refer to the triple $(L, R, L \cap R)$ as the associated *amalgam*.

We collect the following properties of coset graphs. We say that a subgroup H of a group G is *core-free* if $\cap_{g\in G} H^g = 1$.

Lemma 3.1 ([11, Lemma 3.7])**.** *Let G be a group with proper subgroups L and R, and let $\Gamma = \mathrm{Cos}(G, L, R)$.*

(1) *Γ is connected if and only if $G = \langle L, R\rangle$.*

(2) *G acts faithfully on both $[G : L]$ and $[G : R]$ if and only if both L and R are core free in G.*

(3) *G acts transitively on the set of edges of Γ.*

(4) *Γ is locally $(G, 2)$-arc-transitive if and only if L acts 2-transitively on $[L : L \cap R]$ and R acts 2-transitively on $[R : L \cap R]$.*

We also need the following result, which essentially follows from the definition of a *completion* (and the *universal completion*) of an amalgam (see [15]) and results on covers of graphs (see, e.g., [2, Chapter 19]). The result is truly "folklore": while it seems to be taken for granted in the field, we also cannot find an explicit proof in the literature. We have included a proof here provided by Luke Morgan [24].

Lemma 3.2. *If Γ is a locally s-arc-transitive graph with amalgam $(L, R, L \cap R)$ and $s \geqslant 2$, then any other graph with amalgam $(L, R, L \cap R)$ is locally s-arc-transitive.*

Proof. Let $G := L *_{L\cap R} R$ be the universal completion of $(L, R, L \cap R)$ and let Γ^* denote the universal tree on which G acts edge-transitively. We identify L and R with their images in G, and label an edge $\{\alpha, \beta\}$ so that $G_\alpha = L$, $G_\beta = R$, and $G_{\alpha\beta} = L \cap R$. Since Γ is locally s-arc-transitive for $s \geqslant 2$, it is locally 2-arc-transitive and so the actions of L on the set of right cosets of $L \cap R$ in L, and of R on the set of right cosets of $L \cap R$ in R are 2-transitive [11, Lemma 3.2]. In particular, Γ^* is locally $(G, 2)$-arc-transitive.

Now let Σ be a graph with edge-transitive group of automorphisms H such that the amalgam $(H_\gamma, H_\delta, H_{\gamma\delta})$ is isomorphic to $(L, R, L \cap R)$, where $\{\gamma, \delta\}$ is an edge of Σ. By the universal property of G and of Γ^*, there is a map $\phi\colon G \to H$ such that the following diagrams commute:

$$\begin{tikzcd} L \arrow[r] \arrow[d] & G \arrow[d] \\ H_\gamma \arrow[r] & H \end{tikzcd} \quad \begin{tikzcd} R \arrow[r] \arrow[d] & G \arrow[d] \\ H_\delta \arrow[r] & H \end{tikzcd} \quad \begin{tikzcd} L\cap R \arrow[r] \arrow[d] & G \arrow[d] \\ H_{\gamma\delta} \arrow[r] & H \end{tikzcd} .$$

Let N be the kernel of ϕ. Then, $\Sigma = \Gamma^*_N$, the quotient graph, and the kernel of the action of G on Σ is exactly N.

In particular, $\phi(G_\alpha) = H_\gamma$ and $\phi(G_\beta) = H_\delta$. Further, since $\phi(G_{\alpha\beta}) = H_{\gamma\delta}$, we have commutative diagrams of the following groups:

$$\begin{tikzcd} G_\alpha \arrow[r] \arrow[d] & G_\alpha^{\Gamma^*(\alpha)} \arrow[d] \\ H_\gamma \arrow[r] & H_\gamma^{\Sigma(\gamma)} \end{tikzcd} \quad \begin{tikzcd} G_\beta \arrow[r] \arrow[d] & G_\beta^{\Gamma^*(\beta)} \arrow[d] \\ H_\delta \arrow[r] & H_\delta^{\Sigma(\delta)} \end{tikzcd} ,$$

where $G_\alpha^{\Gamma^*(\alpha)}$ denotes the induced action of G_α on $\Gamma^*(\alpha)$, etc.

We now claim that for $\varepsilon = \gamma, \delta$ and $\zeta \in \Gamma^*(\varepsilon)$, we have $\zeta^N \cap \Gamma^*(\varepsilon) = \{\zeta\}$. Indeed, this follows since $|G_\alpha : G_{\alpha\beta}| = |H_\gamma : H_{\gamma\delta}|$ and $|G_\beta : G_{\alpha\beta}| = |H_\delta : H_{\gamma\delta}|$.

Now suppose Γ^* is locally (G, r)-arc-transitive and Σ is locally (H, t)-arc-transitive. By [11, Lemma 5.1(3)], we have $t \geqslant r$.

Assume that $r < t$. We will show that Γ^* would be locally $(G, r + 1)$-arc-transitive in this case, contradicting the maximality of r.

Suppose P and P' are $(r + 1)$-paths in Γ^* with initial vertex α or β. Since $r \geqslant 1$, without loss of generality we may assume $P = (\alpha, \beta_1, \ldots, \beta_r, \beta_{r+1})$ and $P' = (\alpha, \beta_1, \ldots, \beta_r, \beta'_{r+1})$, where $\beta_1 = \beta$.

Consider the images of P^N and $(P')^N$ in Σ. Note that the images are two $(r+1)$-paths, since the equality $\beta_{i-1}^N = \beta_{i+i}^N$ would contradict our claim above. Hence, there is $h \in H_\gamma$ such that $(P^N)^h = (P')^N$. Since $\phi(G_\alpha) = H_\gamma$, we can take $h = \phi(g)$ for $g \in G_\alpha$, so g fixes α. Now, $(P^N)^h = (P')^N$ implies $(\beta^N)^g = \beta^N$. Thus, g fixes β^N, and, since g fixes α, g fixes the unique vertex in $\Gamma^*(\alpha) \cap \beta^N$, which is β; so, $g \in G_{\alpha\beta}$. Continuing in this way, we see that $g \in G_{\alpha\beta_1\ldots\beta_r}$. Now, $(\beta_{r+1}^N)^h = (\beta'_{r+1})^N$, and so β_{r+1}^g lies in the N-orbit of β'_{r+1}, and at the same time must be adjacent to β_r, since $g \in G_{\beta_r}$. Once more, the claim implies $\beta_{r+1}^g = \beta'_{r+1}$.

We have thus shown that G_α is transitive on $(r + 1)$-arcs with initial vertex u. A similar argument establishes that same result for G_β, and hence Γ^* is locally $(G, r + 1)$-arc-transitive. This contradicts the maximality of r, and, therefore, $r = t$, as desired. In particular, taking $\Sigma = \Gamma$ we see that $r = s$. Hence Γ^*, and so any graph with amalgam $(L, R, L \cap R)$, is locally s-arc-transitive. □

Lemma 3.1 enables us to construct locally $(G, 2)$-arc-transitive graphs where G has two orbits Δ_1 and Δ_2 on vertices and acts quasiprimitively of type PA on each. Recall the three types straight diagonal, twisted diagonal and nondiagonal of quasiprimitive groups of type PA. Analogously to [22], we refer to a locally $(G, 2)$-arc-transitive graph Γ where G is quasiprimitive of type PA on each orbit by the type of the two PA actions. For example, if G is of straight diagonal type on Δ_1 and twisted diagonal type on Δ_2 then we refer to Γ as being of *straight-twisted type*.

3.1 Straight-twisted type

Construction 3.3. We begin with the following: let $(L, R, L \cap R)$ be an amalgam for a locally s-arc-transitive graph, and suppose further that $L = L_1 \rtimes K$ and $R = R_1 \rtimes K$ such that K acts trivially on R_1. Note that this implies $L \cap R = (L_1 \cap R_1)K$.

Let H be an almost simple group with socle T, and subgroups H_1 and H_2 such that

- $H_1 \cong L_1$, $H_2 \cong R_1$, $H_1 \cap H_2 \cong L_1 \cap R_1$, i.e., $\phi\colon H_1 \to L_1$, $\tau\colon H_2 \to R_1$ are isomorphisms with restrictions each sending $H_1 \cap H_2 \to L_1 \cap R_1$,
- $H = \langle H_1, H_2 \rangle$, and
- not all automorphisms of L_1 in K extend to automorphisms of T.

We will abuse notation slightly and assume $L_1, R_1 \leqslant H$. Let $k = |K|$ and let

$$F = \{f : K \to H\} \cong H^k.$$

For each $\ell \in L_1$ and $r \in R_1$, define $f_\ell, f_r \in F$ such that

$$f_\ell(\kappa) = \ell^\kappa,$$
$$f_r(\kappa) = r$$

for all $\kappa \in K$. Furthermore, we let

$$N_\alpha := \{f_\ell \mid \ell \in L_1\} \cong L_1,$$
$$N_\beta := \{f_r \mid r \in R_1\} \cong R_1.$$

Since K acts trivially on R_1, we have that

$$N_\alpha \cap N_\beta = \{f_r \mid r \in R_1 \cap L_1\} \cong L_1 \cap R_1.$$

Let $N := \langle N_\alpha, N_\beta \rangle$.

Now K acts on F via $f^\sigma(\kappa) = f(\sigma\kappa)$ for each $\sigma, \kappa \in K$. Then for $\ell \in L_1$ we have that

$$(f_\ell)^\sigma(\kappa) = f_\ell(\sigma\kappa) = \ell^{\sigma\kappa} = f_{\ell^\sigma}(\kappa).$$

Hence $(f_\ell)^\sigma = f_{\ell^\sigma}$ and so K normalises N_α. Similarly, $(f_r)^\sigma = f_r$ for all $r \in R_1$ so K normalises N_β and hence also N. Define

$$G_\alpha := N_\alpha \rtimes K,$$
$$G_\beta := N_\beta \rtimes K,$$
$$G := \langle G_\alpha, G_\beta \rangle.$$

Finally, we define $\Gamma := \mathrm{Cos}(G, G_\alpha, G_\beta)$.

Lemma 3.4. *Let* Γ *be a graph yielded by* Construction 3.3. *Then* Γ *is a connected locally* (G, s)*-arc-transitive graph such that* G *acts quasiprimitively with type* PA *on each orbit of vertices. Moreover, the action of* G *on* $[G : G_\beta]$ *is straight diagonal, and the action of* G *on* $[G : G_\alpha]$ *is twisted diagonal, that is,* Γ *is of straight-twisted type.*

Proof. Let $F_T = \{f \in F \mid f(\kappa) \in T \text{ for all } \kappa \in K\} \cong T^k$. For each $\kappa \in K$, let

$$\begin{array}{cccc} \pi_\kappa\colon & F & \to & H \\ & f & \mapsto & f(\kappa). \end{array}$$

Since $\langle R_1, L_1 \rangle = H$, we have that $\pi_\kappa(N) = H$ for all $\kappa \in K$ and so by [30, page 328, Lemma], $N \cap F_T$ is a direct product of diagonal subgroups, each isomorphic to T. Since there are elements $\kappa \in K$ that do not extend to an automorphism of T, it follows that $N \cap F_T$ is not itself a diagonal subgroup and so $N \cap F_T \cong T^j$ for some integer $2 \leqslant j \leqslant k$.

Since the action of K on N_α is isomorphic to the action of K on L_1 we see that $G_\alpha \cong L$ and similarly, $G_\beta \cong R$. Moreover, $G_\alpha \cap G_\beta \cong \langle L_1 \cap R_1, K \rangle = L \cap R$. Therefore $\Gamma := \mathrm{Cos}(G, G_\alpha, G_\beta)$ is a connected graph with amalgam $(L, R, L \cap R)$ and is thus a locally s-arc-transitive graph.

Finally, since K transitively permutes the simple direct factors of F_T it also transitively permutes the simple direct factors of $N \cap F_T$. Thus $\mathrm{soc}(G) \cong T^j$ and $G \lesssim H \operatorname{wr} S_j$ for some integer $j \geqslant 2$. Since $\pi_\kappa(N_\alpha) = L_1$ for all $\kappa \in K$ it follows that N_α is a subdirect subgroup of L_1^j and similarly, N_β is a subdirect subgroup of R_1^j. Therefore, G acts quasiprimitively with type PA on both $[G : G_\alpha]$ and $[G : G_\beta]$, and, by construction, the action of G on $[G : G_\beta]$ is straight diagonal, and the action of G on $[G : G_\alpha]$ is twisted diagonal. □

Example 3.5. This example is based on [22, Example 4.1]. First, $(\mathrm{AGL}(1,5) \times C_2, S_3 \times C_4, C_4 \times C_2)$ is an amalgam admitting a locally 2-arc-transitive connected graph of valency $\{3,5\}$: indeed, a GAP computation shows that in the group S_7 we can take $L = \langle (4,5,6,7), (3,4,5,7,6), (1,2)\rangle \cong \mathrm{AGL}(1,5) \times C_2$ and $R = \langle (1,2), (1,2,3), (4,5,6,7)\rangle \cong S_3 \times C_4$ such that $\langle L, R\rangle = S_7$, and $L \cap R \cong C_4 \times C_2$ [10].

Let $T = \mathrm{PSL}(2,p)$, where p is a prime and $p \equiv \pm 1 \pmod{60}$. Thus we may select $D < T$ such that $D \cong D_{60}$, with $D = \langle h, d \mid h^{30} = d^2 = 1, h^d = h^{-1}\rangle$. First, define $L_1 := \langle h^3\rangle \cong C_{10} \cong C_5 \times C_2$. Noting that D has a subgroup $B := \langle h^{15}, d\rangle \cong C_2^2$, there exists an element x of T such that $B^x = B$ and $d^x = h^{15}$ [6]. Define $R_1 := \langle (h^{10})^x, d^x\rangle$ to be a subgroup of H^x isomorphic to S_3. Hence $\langle L_1, R_1\rangle = T$ and $L_1 \cap R_1 = C_2$. Finally, the order four elements of $\mathrm{AGL}(1,5)$ cannot be extended to automorphisms of T since $\mathrm{Aut}(T) = \mathrm{PGL}(2,p)$ has no elements of order four normalising but not centralising a subgroup of order five. Thus we let $K = \langle k\rangle \cong C_4$ and $L = L_1 \rtimes K$. Note, as in [22, Example 4.1], that the action of k^2 on elements of T is the same as conjugation by d. Therefore, by Lemma 3.4, there is a locally 2-arc-transitive graph with amalgam $(\mathrm{AGL}(1,5) \times C_2, S_3 \times C_4, C_4 \times C_2)$ of straight-twisted type.

Theorem 3.6. *There is an infinite family of locally* 5*-arc-transitive graphs with valencies* $\{4,5\}$ *of straight-twisted type.*

Proof. By [20], there is an amalgam admitting a locally 5-arc-transitive connected graph of valency $\{4,5\}$ from the Mathieu group M_{24}, with $L = C_2^4 \rtimes (A_4 \times C_3)$, $R = A_5 \times A_4$, and $L \cap R = A_4 \times A_4$. Note that $L = L_1 \rtimes K$ and $R = R_1 \times K$ where $L_1 = C_2^4 \rtimes C_3$, $R_1 = A_5$ and $K = A_4$.

Let $n \geqslant 2$ be an integer and $T = \mathrm{PSL}(2, 2^{2n})$. Then T contains a subgroup $R_1 \cong A_5 \cong \mathrm{PSL}(2,4)$ (see [6], for instance). Furthermore, T contains a subgroup Y isomorphic to $C_2^{2n} \rtimes C_{2^{2n}-1}$, and $2^{2n} - 1 \equiv 0 \pmod 3$. Let $Y = Y_2 \rtimes Y_1$, where $Y_2 \cong C_2^{2n}$ and $Y_1 = \langle y_1\rangle \cong C_{2^{2n}-1}$. Thus Y_1 has a cyclic group of order three, which we will denote by $Y_3 = \langle y_1^{(2^{2n}-1)/3}\rangle$, acting semiregularly on the nonidentity elements of Y_2. Moreover, we may choose R_1 such that $Y_0 := R_1 \cap Y \cong A_4$ and $Y_3 \leqslant Y_0$. By [6, Theorem 260], we see that $N_T(Y_0) \leqslant Y$, and, noting that Y_1 acts regularly on the nonidentity elements of Y_2, we see that $N_T(Y_0) = Y_0$. By [6, Theorem 255], for each divisor m of $2n$, all subfield subgroups of T isomorphic to $\mathrm{PSL}(2, 2^m)$ are conjugate. This implies that Y_0 is contained in a unique subfield subgroup T_m isomorphic to $\mathrm{PSL}(2, 2^m)$ for each divisor m of $2n$, m even (if m is odd, then $2^2 - 1 = 3$ does not divide $2^m - 1$). Note also that this implies that the maximal subgroup of T_m isomorphic to $C_2^m \rtimes C_{2^m-1}$ is actually $T_m \cap Y$. We claim that no subfield subgroup T_m containing Y_0, for m a proper even divisor of $2n$, also contains $Y_0^{y_1}$. If some T_m contains $Y_0^{y_1}$, then, since the elements of order two in Y_0 and $Y_0^{y_1}$ commute and $Y_0 \cap Y_0^{y_1} = Y_3$, we have that $\langle Y_0, Y_0^{y_1}\rangle \leqslant T_m \cap Y \cong C_2^m \rtimes C_{2^m-1}$, where $T_m \cap Y_1$ acts regularly on the nonidentity elements of $T_m \cap Y_2$. However, Y_1 acts regularly on the nonidentity elements of Y_2, so y_1 is the unique element of Y_1 mapping, say, $y_2 \in Y_0 \cap Y_2$ to $y_2^{y_1} \in Y_0^{y_1} \cap Y_2$. On the other hand, $y_1 \notin T_m \cap Y_1 = \langle y_1^{(2^{2n}-1)/(2^m-1)}\rangle$, so we have a contradiction.

Let $L_1 := \langle Y_0, Y_0'^{y_1} \rangle$. Then $L_1 \cong 2^4{:}3$ (SmallGroup(48,50) in the GAP [10] small groups library) which is isomorphic to the subgroup L_1 in L, hence the abuse of notation. Moreover, $L_1 \cap R_1 \cong A_4$ and, since L_1 is not contained in any subfield subgroup, we have that $T = \langle L_1, R_1 \rangle$. Since $\mathrm{P\Gamma L}(2, 2^{2n})$ does not contain a subgroup isomorphic to L ([6, Theorem 260] and noting that the outer automorphism group of $\mathrm{PSL}(2, 2^{2n})$ is cyclic), it follows that not all automorphisms of L_1 in L extend to automorphisms of T. Hence by Lemma 3.4, Construction 3.3 yields a locally 5-arc-transitive graph of straight-twisted type. □

3.2 Twisted-twisted type

If G acts quasiprimitively with straight PA type on a set Ω, then there exists $\alpha \in \Omega$ such that $N_\alpha = \{(r, r, \ldots, r) \mid r \in R\}$, where $N = T^k$ is the unique minimal normal subgroup of G. If $g = (t_1, t_2, \ldots, t_k) \in R^k \leqslant N$ then $N_{\alpha^g} = (N_\alpha)^g = \{(r^{t_1}, r^{t_2}, \ldots, r^{t_k}) \mid r \in R\}$, which is a twisted diagonal subgroup if $t_i \notin C_T(R)$ for some i. Thus the examples given in the previous section can also be viewed as being of twisted-twisted type. However, if G acts quasiprimitively of type twisted PA on a set Ω then N_α is a twisted diagonal subgroup of R^k for some R but there may not be a $\beta \in \Omega$ such that N_β is a straight diagonal subgroup. Thus not all twisted-twisted type examples arise in this way. In this section we give an alternative construction.

Construction 3.7. Let $(L, R, L \cap R)$ be an amalgam for a locally s-arc-transitive graph, and suppose further that $L = L_1 \rtimes K$ and $R = R_1 \rtimes K$ such that $K = K_L \times K_R$ where $K_L \leqslant \mathrm{Aut}(L_1)$ such that $K_L \cap \mathrm{Inn}(L_1) = \{1\}$, K_L acts trivially on R_1, $K_R \leqslant \mathrm{Out}(R_1)$ and K_R acts trivially on L_1. Let H be an almost simple group with socle T, and subgroups H_1 and H_2 such that

- $H_1 \cong L_1$, $H_2 \cong R_1$, $H_1 \cap H_2 \cong L_1 \cap R_1$,
- $H = \langle H_1, H_2 \rangle$, and
- not all elements of K extend to automorphisms of T.

We will abuse notation slightly and assume $L_1, R_1 \leqslant H$. Let $k = |K|$ and let $F = \{f\colon K \to H\} \cong H^k$. For each $\ell \in L_1 \cup R_1$, define $f_\ell \in F$ such that $f_\ell(\kappa) = \ell^\kappa$ for all $\kappa \in K$. Furthermore, we let $N_\alpha := \{f_\ell \mid \ell \in L_1\} \cong L_1$ and $N_\beta = \{f_r \mid r \in R_1\} \cong R_1$. Moreover, $N_\alpha \cap N_\beta = \{f_r \mid r \in R_1 \cap L_1\} \cong L_1 \cap R_1$. Let $N := \langle N_\alpha, N_\beta \rangle$.

Now K acts on F via $f^\sigma(\kappa) = f(\sigma\kappa)$ for each $\sigma, \kappa \in K$. As in Construction 3.3, K normalises both N_α and N_β, and hence also N. Define $G_\alpha := N_\alpha \rtimes K$, $G_\beta := N_\beta \rtimes K$ and $G := \langle G_\alpha, G_\beta \rangle$. Let $\Gamma = \mathrm{Cos}(G, G_\alpha, G_\beta)$.

Lemma 3.8. *Let* Γ *be a graph yielded by* Construction 3.7. *Then* Γ *is a connected locally* (G, s)*-arc-transitive graph such that* G *acts quasiprimitively with type* PA *on each orbit on vertices. Moreover, the action of* G *on both* $[G : G_\alpha]$ *and* $[G : G_\beta]$ *is twisted diagonal, that is,* Γ *is of twisted-twisted type.*

Proof. The proof is analogous to that of Lemma 3.4. □

Example 3.9. First, $(C_{71}{:}C_{70} \times C_9, C_{19}{:}C_{18} \times C_{35}, C_{630})$ is an amalgam that admits a locally 2-arc-transitive graph; indeed, if $G = A_{89}$,

$$L := \langle (1,2,8,28,14,30,34,3,20,54,36,33,40,41,9,56,26,51,60,18,42,29,39,17,46,58, 47,10,15,70,62,13,32,59,57,31,66,22,24,67,48,27,35,50,45,12,23,11,52,4,64, 7,53,25,16,61,21,44,6,5,68,71,19,55,38,69,65,49,63,43,37), (2,3,4,\dots,71)(72,73,\dots,89)\rangle,$$

and

$$R := \langle (1,72,73,85,74,88,86,78,75,80,89,84,87,77,79,83,76,82,81), (2,3,4,\dots,71)(72,73,\dots,89)\rangle,$$

then, using GAP, we see that $L \cong C_{71}{:}C_{70} \times C_9$, $R \cong C_{19}{:}C_{18} \times C_{35}$, $L \cap R \cong C_{630}$, $\langle L, R\rangle = G$, and by Lemma 3.1, the coset graph $\mathrm{Cos}(G, L, R)$ is a connected locally $(G, 2)$-arc-transitive graph.

Let $T = \mathbb{M}$, the Monster Group. By [4], T contains subgroups $L_1 \cong D_{142}$ and $R_1 \cong D_{38}$, and L_1 and R_1 may be selected such that $L_1 \cap R_1 \cong C_2$ (here, the element of order two is of type 2B). By [36] we see that $\mathbb{M}$ does not have a maximal subgroup of order divisible by 71 and 19. Thus $\langle L_1, R_1\rangle = T$. Let $K = C_{315} = C_{35} \times C_9$, and since T does not contain an element of order 315 [5], not all elements of K lift to an automorphism of T. Therefore, by Lemma 3.8, Construction 3.7 yields a locally 2-arc-transitive graph Γ with amalgam $(C_{71}{:}C_{70} \times C_9, C_{19}{:}C_{18} \times C_{35}, C_{630})$ of twisted-twisted type with valencies $\{71, 19\}$.

3.3 Straight-nondiagonal type

We first include an example of an *equidistant linear code* from [22], which proves useful in later constructions. A *linear (n,k)-code* C over $\mathrm{GF}(q)$ is a k-dimensional subspace of $\mathrm{GF}(q)^n$, a codeword has *weight* w if it has exactly w nonzero coordinates, and a code C is *equidistant* if all nonzero codewords have the same weight.

Example 3.10 ([22, Example 5.1]). Let $V = \mathrm{GF}(3)^4$, and let

$$C = \langle (1,1,1,0), (1,2,0,1)\rangle < V.$$

Then, C is a linear $(4, 2)$-code, and it contains eight nonzero code words:

$$(1,1,1,0), (1,2,0,1), (2,0,1,1), (0,2,1,2), (2,2,2,0), (2,1,0,2), (1,0,2,2), (0,1,2,1),$$

and hence C is equidistant of weight 3.

Let $\tau = (\sigma, 1, \sigma, \sigma)(1, 2, 3, 4) \in \mathrm{GL}(1, 3) \,\mathrm{wr}\, S_4 < \mathrm{GL}(V)$. Then, $\tau^4 = (\sigma, \sigma, \sigma, \sigma)$, $|\tau| = 8$, and τ permutes the eight nonzero words of C in the order given above.

Our next result constructs examples of straight-nondiagonal type.

Theorem 3.11. *For each integer $n \geqslant 3$, there exists a locally 2-arc-transitive graph of straight-nondiagonal type with valencies $\{n, 9\}$.*

Proof. We adapt the construction of [22, Lemma 5.2]. Let $H = S_{n+2}$. Then H contains subgroups $L \cong S_2 \times S_n$ and $R \cong S_3 \times S_{n-1}$ such that $\langle L, R\rangle = H$ and $L \cap R \cong S_2 \times S_{n-1}$

(this is realized by letting L be the stabilizer of $\{1,2\}$ and letting R be the stabilizer of $\{1,2,3\}$).

Based on the equidistant linear code defined in Example 3.10, we define $N_\alpha := \langle(\ell,\ell,\ell,\ell) \mid \ell \in L\rangle$. Moreover, if $R = R_1 \times R_2$, where $R_1 \cong S_3$, $R_2 \cong S_{n-1}$, and $R_1 = \langle h,\sigma | h^3 = \sigma^2 = hh^\sigma = 1\rangle$, we define $N_\beta := \langle(h,h,h,1),(h,h^{-1},1,h),(x,x,x,x) | x \in \langle\sigma\rangle \times R_2\rangle$. By choosing $\sigma \in L$ we have $N_\alpha \cap N_\beta \cong S_2 \times S_{n-1}$, and, as in [22, Lemma 5.2], $N_\beta \cong (C_3^2{:}C_2) \times S_{n-1} \not\cong R$. Let $N := \langle N_\alpha, N_\beta\rangle$. Since $\langle L,R\rangle \cong S_{n+2}$ it follows that N projects onto S_{n+2} in each of its four coordinates. Moreover, given any two of the four coordinates, N_β contains an element that is the identity in one coordinate and a nonidentity element of A_{n+2} in another. Thus $A_{n+2}^4 \lhd N$. Note that N is not necessarily all of S_{n+2}^4; indeed, the elements of N_β that do not have all entries equal have even permutations as their entries.

Define $\tau := (\sigma,1,\sigma,\sigma)(1,2,3,4)$. Then $\tau^4 = (\sigma,\sigma,\sigma,\sigma)$ and so $\tau^8 = 1$. Furthermore, τ centralizes N_α and normalises N_β. Let $G_\alpha := \langle N_\alpha,\tau\rangle$, $G_\beta := \langle N_\beta,\tau\rangle$, and $G := \langle G_\alpha, G_\beta\rangle$. By similar reasoning as in [22, Lemma 5.2], $A_{n+2}^4 \lesssim G$ and G induces C_4 on the 4 simple direct factors. Moreover, $G_\beta \cong \mathrm{AGL}(1,3^2) \times S_{n-1}$. We also see that $G_\alpha \cong C_8 \times S_n$, and $G_\alpha \cap G_\beta \cong C_8 \times S_{n-1}$.

Let $\Gamma := \mathrm{Cos}(G,G_\alpha,G_\beta)$. Since G_α acts on $[G_\alpha{:}G_\alpha \cap G_\beta]$ as S_n does on n points and G_β acts on $[G_\beta{:}G_\alpha \cap G_\beta]$ as $\mathrm{AGL}(1,3^2)$ does on $\mathrm{GF}(3^2)$, we see that Γ is a connected locally 2-arc-transitive graph with valencies $\{n,9\}$. Clearly, the action of G on $[G{:}G_\alpha]$ is straight diagonal, and the action of G on $[G{:}G_\beta]$ is nondiagonal (as in [22, Lemma 5.2]). Therefore, Γ is a locally 2-arc-transitive graph of straight-nondiagonal type with vertex valencies $\{n,9\}$. □

3.4 Twisted-nondiagonal type

As discussed at the start of Section 3.2, the straight-nondiagonal examples given by Theorem 3.11 can also be viewed as twisted-nondiagonal examples. We also have the following construction of a graph of twisted-nondiagonal type.

Example 3.12. Let $T = \mathrm{PSL}(2,61)$. By [6], T contains a maximal subgroup $M \cong D_{60}$. Now, M contains a subgroup X isomorphic to C_2^2, and $N_T(X) \cong A_4$. Now, $N_T(X)$ contains an element g of order three that is not in M. Thus we may select subgroups $L \leqslant M$ and $R \leqslant M^g$ such that $L \cong C_{10} \cong C_5 \times C_2$, $R \cong C_3{:}C_2$, $\langle L,R\rangle = T$ and $L \cap R = X \cong C_2$. Note that we may select presentations $L = \langle \ell, x | \ell^5 = x^2 = 1\rangle$ and $R = \langle r,x | r^3 = x^2 = rr^x = 1\rangle$.

Note that L has an isomorphism ϕ defined by $\phi\colon \ell \mapsto \ell^2, x \mapsto x$. We define $\overline{\ell} := (\ell,\ell^\phi,\ell^{\phi^2},\ell^{\phi^3}) = (\ell,\ell^2,\ell^4,\ell^3)$ and $\overline{x} := (x,x,x,x)$. Furthermore, we define $N_\alpha := \langle\overline{\ell},\overline{x}\rangle$, $N_\beta := \langle(r,r,r,1),(r,r^{-1},1,r),\overline{x}\rangle$, and $N := \langle N_\alpha,N_\beta\rangle$. As in [22, Lemma 5.2], none of the coordinates of N_β can be linked, so $N \cong T^4$. Moreover, $N_\alpha \cong L \cong C_5 \times C_2$, $N_\beta \cong C_3^2{:}C_2$ and $N_\alpha \cap N_\beta \cong C_2$.

Define $\tau := (x,1,x,x)(1,2,3,4)$. Then $\tau^4 = (x,x,x,x)$ and so $\tau^8 = 1$. Let $G_\alpha := \langle N_\alpha,\tau\rangle$, $G_\beta := \langle N_\beta,\tau\rangle$, and $G := \langle G_\alpha,G_\beta\rangle$. We note that τ centralizes $\overline{x}$, whereas $\overline{\ell}^\tau = (\ell^3,\ell,\ell^2,\ell^4) = \overline{\ell^3}$, and so $G_\alpha \cong C_2.\mathrm{AGL}(1,5)$. By similar reasoning as in [22, Lemma 5.2], we deduce that $G \cong \mathrm{PSL}(2,61) \operatorname{wr} C_4$ and $G_\beta \cong \mathrm{AGL}(1,3^2)$. We also see that $G_\alpha \cap G_\beta \cong C_8$.

Let $\Gamma := \mathrm{Cos}(G,G_\alpha,G_\beta)$. Since G_α acts on $[G_\alpha{:}G_\alpha \cap G_\beta]$ as $\mathrm{AGL}(1,5)$ does on $\mathrm{GF}(5)$ and G_β acts on $[G_\beta{:}G_\alpha \cap G_\beta]$ as $\mathrm{AGL}(1,3^2)$ does on $\mathrm{GF}(3^2)$, we see that Γ is a connected locally 2-arc-transitive graph with vertex valencies $\{5,9\}$. Clearly, the action of G on $[G{:}G_\alpha]$ is twisted diagonal, and the action of G on $[G{:}G_\beta]$ is nondiagonal (as in [22, Lemma 5.2]). Therefore, Γ is a locally 2-arc-transitive graph of twisted-nondiagonal type with valencies $\{5,9\}$.

3.5 Nondiagonal-nondiagonal type

Finally, in this subsection, we include a construction of a graph of nondiagonal-nondiagonal type.

Example 3.13. Let $T = J_2$, the second Janko group. By [5], T has two conjugacy classes of elements of order three, labelled 3A and 3B, and two conjugacy classes of involutions, labelled 2A and 2B. Moreover, the elements of type 3A are contained in a maximal subgroup isomorphic to $A_5 \times D_{10}$ which contains involutions from class 2B, and the elements of type 3B are contained in a maximal subgroup isomorphic to A_5 which also contains involutions of type 2B. Furthermore, within each of these maximal subgroups the elements of order three are normalized by an involution of type 2B. Using GAP, there are subgroups $L, R < T$, each isomorphic to S_3, such that $L \cap R \cong C_2$, L contains an element of order three of type 3A, R contains an element of order three of type 3B, and $\langle L, R\rangle = T$. Furthermore, by [5], the two conjugacy classes of order three are not fused by any outer automorphism of T. Let $L = \langle \ell, x \mid \ell^3 = x^2 = \ell\ell^x = 1\rangle$ and $R = \langle r, x \mid r^3 = x^2 = rr^x = 1\rangle$.

We again use the equidistant linear code as defined in Example 3.10. Define $N_\alpha := \langle(\ell,\ell,\ell,1),(\ell,\ell^{-1},1,\ell),(x,x,x,x)\rangle$ and $N_\beta := \langle(r,r,r,1),(r,r^{-1},1,r),(x,x,x,x)\rangle$. Note that $L \cap R \cong C_2$, and, reasoning as in [22, Lemma 5.2], we deduce that $N_\alpha \cong N_\beta \cong C_3^2{:}C_2 \ncong L, R$. Also, given any two of the four coordinates, both N_α and N_β contain an element that is the identity in one coordinate and a nonidentity element in another, so $N := \langle N_\alpha, N_\beta\rangle \cong J_2^4$.

Define $\tau := (x,1,x,x)(1,2,3,4)$. Then $\tau^4 = (x,x,x,x)$ and so $\tau^8 = 1$. Let $G_\alpha := \langle N_\alpha, \tau\rangle$, $G_\beta := \langle N_\beta, \tau\rangle$, and $G := \langle G_\alpha, G_\beta\rangle$. By similar reasoning as in [22, Lemma 5.2], $G \cong J_2 \operatorname{wr} C_4$ and $G_\alpha \cong G_\beta \cong \mathrm{AGL}(1,3^2)$. We also see that $G_\alpha \cap G_\beta \cong C_8$.

Let $\Gamma := \mathrm{Cos}(G, G_\alpha, G_\beta)$. Since G_α (respectively G_β) acts on $[G_\alpha{:}G_\alpha \cap G_\beta]$ (respectively $[G_\beta{:}G_\alpha \cap G_\beta]$) as $\mathrm{AGL}(1,3^2)$ does on $\mathrm{GF}(3^2)$, we see that Γ is a connected locally $(G,2)$-arc-transitive graph with valencies $\{9,9\}$. Moreover, Γ cannot be a standard double cover of a $(G,2)$-arc-transitive graph since L and R are not conjugate subgroups in $\mathrm{Aut}(J_2)$. Clearly, the action of G on both $[G{:}G_\alpha]$ and $[G{:}G_\beta]$ is nondiagonal (as in [22, Lemma 5.2]). Therefore, Γ is a locally $(G,2)$-arc-transitive graph of nondiagonal-nondiagonal type that is regular of valency 9.

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MARKO PETKOVŠEK (1955 – 2023)

Marko Petkovšek was born in 1955 and died in 2023. He completed his PhD in 1991 at the School of Computer Science, Carnegie Mellon University, Pittsburgh, after which he worked at the University of Ljubljana until retirement in 2021 as a professor, researcher, the head of the mathematics department and vice-dean.

Marko Petkovšek has an outstanding worldwide reputation in the fields of discrete mathematics and theoretical computer science, which he earned through his research and work in the field of symbolic computation. He is best known as a coauthor of the well-known book *A=B* and the author of the "Hyper" algorithm for solving linear recursive difference equations with polynomial coefficients in terms of hypergeometric forms, nowadays called the Petkovšek's algorithm.

In addition to fundamental results and publications in symbolic computation, Marko's work in graph theory, where he intermittently collaborated with one of us over a period of several decades, also contributes to his visibility. Let us say a little more about his work in this area. He explored various classes of perfect graphs, graphs with nonempty intersections of longest paths, hereditary graph classes, Fibonacci and Lucas cubes, and attacked several related problems. One of the first challenges suggested by Marko was the problem of the intersection of longest paths in graphs. We wrote a joint paper that went mostly unnoticed for a quarter of a century, only to receive wide attention in the past decade. Marko's mathematical breadth was extremely helpful in the treatment of various problems, as it often led to unexpected insights. Of this kind were his contributions to the enumeration of the vertex and edge orbits of Fibonacci cubes and Lucas cubes. Marko's work also established new directions of development. In his paper [Marko Petkovšek, Letter graphs and well-quasi-order by induced subgraphs, *Discrete Mathematics* **244** (2002) 375–388] he introduced the notion of letter graphs and proved that the class of k-letter graphs is well-quasi-ordered by the induced subgraph relation, and that it has only finitely many minimal forbidden induced subgraphs. This visionary paper preceded developments in the field by a decade, and is today recognized as a fundamental reference on the topic.

Let us finish with a few personal thoughts about Marko. Our deep and unbroken friendship began more than 30 years ago. One of us was lucky to share an office with Marko as a freshly minted assistant, and the other as his student. He introduced us both to the world of research and transferred his enthusiasm for it to us. He was the best possible friend. Despite his broad mathematical knowledge and depth of thought, he was extremely modest and downplayed his strengths and contributions. His was always pleasant and soothing company, be it on mountain trails, Saturday evening bridge sessions, or just conversations at and around work. In addition to mathematics, he had a broad general outlook. He held himself to the highest ethical and moral standards, inspiring others to do the same. And on mountain hikes he could always name every flower we saw in his mother tongue Slovene, German, and Latin.

Unfortunately, Marko left us too soon. We shall always remember the beautiful moments we spent with him and keep him in our memories as a truly exceptional man.

Andrej Bauer and Sandi Klavžar

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