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A parametrisation for symmetric designs admitting a flag-transitive, point-primitive automorphism group with a product action*

Eugenia O'Reilly-Regueiro †

Instituto de Matemáticas, Universidad Nacional Autónoma de México (UNAM), Área de la Investigación Científica, Circuito Exterior, Ciudad Universitaria, Coyoacán, 04510, Mexico City, Mexico

José Emanuel Rodríguez-Fitta

Facultad de Estudios Superiores Acatlán, Universidad Nacional Autónoma de México (UNAM), Av. Alcanfores y San Juan Totoltepec s/n, Santa Cruz Acatlán, Naucalpan, 53150, Edo. de México, Mexico

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Abstract

We study (v, k, λ)-symmetric designs having a flag-transitive, point-primitive automorphism group, with $v = m^2$ and $(k, \lambda) = t > 1$, and prove that if D is such a design with m even admitting a flag-transitive, point-primitive automorphism group G, then either:

(1) D is a design with parameters $\left((2t + s - 1)^2, \frac{2t^2-(2-s)t}{s}, \frac{t^2-t}{s^2}\right)$ with $s \geq 1$ odd, or

(2) G does not have a non-trivial product action.

We observe that the parameters in (1), when $s = 1$, correspond to Menon designs.

We also prove that if D is a (v, k, λ)-symmetric design with a flag-transitive, point-primitive automorphism group of product action type with $v = m^l$ and $l \geq 2$ then the complement of D does not admit a flag-transitive automorphism group.

Keywords: Symmetric-designs, flag-transitivity, primitive groups, automorphism groups of designs.

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†Corresponding author.

E-mail addresses: eugenia@im.unam.mx (Eugenia O'Reilly-Regueiro), 887191@pcpuma.acatlan.unam.mx (José Emanuel Rodríguez-Fitta)

1 Introduction

If $D = (P, \mathcal{B})$ is a (v, k, λ)-symmetric design, a *flag* of D is an ordered pair (p, B) such that $p \in P$ is a point of D, $B \in \mathcal{B}$ is a block of D, and $p \in B$. The *order* of D is $n = k - \lambda$.

There are some symmetric designs in which the parameters are related in some special way, such as Hadamard designs in which $v = 4n + 3$, $k = 2n + 1$, and $\lambda = n$ $(n \in \mathbb{Z}^+)$, and Menon designs, in which $v = 4t^2$, $k = 2t^2 - t$, and $\lambda = t^2 - t$ for some positive integer t. These last ones will be relevant in the present work.

If $G = \mathrm{Aut}(D)$, then G is *point-transitive* if it is transitive on P (the set of points of D), and it is *flag-transitive* if it is transitive on the set of flags of D. If G is point-transitive, it can either be *point-primitive*, that is, there is no G-invariant non-trivial partition of P, or *point-imprimitive*, which is when there is a non-trivial partition of the points of D invariant under the action of G.

Primitive groups are classified by the O'Nan-Scott Theorem, we will use the classification in [4] by Liebeck, Praeger, and Saxl, with five types, namely affine, almost simple, product, simple diagonal, and twisted wreath.

Buekenhout, Delandtsheer, and Doyen proved in [1] that if a 2-design with $\lambda = 1$ (linear space) admits a point-primitive, flag-transitive automorphism group G, then it must be of affine or almost simple type. O'Reilly-Regueiro proved the same result for symmetric 2-designs with $2 \leq \lambda \leq 4$ in [5, 6]. All designs in this paper will be 2-designs.

In [7], Tian and Zhou extended this result to $\lambda \leq 100$, and conjectured that it holds for all values of λ. Having an upper bound on λ, in [7] they ruled out the simple diagonal, product, and twisted wreath action by finding possible groups and/or sets of parameters of designs and then ruling them out by arithmetic constraints and the use of GAP [2]. Additionally, in [3, 8, 9, 10], Zhou et al. have tackled this issue from different perspectives, and have ruled out the product action for flag-transitive (v, k, λ) symmetric designs in which $\lambda \geq (k, \lambda)^2$, as well as, for those cases in which λ is prime.

We have tried to prove that if D is a (v, k, λ)-symmetric design with $v = m^2$ even and any λ admitting a point-primitve, flag-transitive automorphism group G, then G does not have a product action. In this paper we present our results, namely, a parametrisation for such designs which in some cases correspond to Menon designs.

In 1998, Zieschang proved in [11] that if a (not necessarily symmetric) 2-design in which $(r, \lambda) = 1$ (where r is the number of blocks incident with any given point) admits a flag-transitive group G, then G is of affine or almost simple type. Given this result, in our work we will assume $(k, \lambda) = t > 1$.

2 Product action

We start with a result from [5], which will be useful later.

Corollary 2.1. *If G is a flag-transitive automorphism group of a (v, k, λ)-symmetric design $D = (P, \mathcal{B})$, then k divides $\lambda(v - 1, |G_x|)$ for every point-stabiliser G_x.*

The next lemma gives us an arithmetic condition that will be used throughout this work.

Suppose that the group G has a product action on the set of points P. Then there is a finite set Γ with $|\Gamma| \geq 5$ and a group H acting primitively on Γ, with an almost simple or simple diagonal action, such that

$$P = \Gamma^l \text{ and } G \leq H^l \rtimes S_l = H \mathrm{wr} S_l, \text{ with } l \geq 2.$$

Lemma 2.2. *If G is a point-primitive group acting flag-transitively on a (v, k, λ)-symmetric design $D = (P, \mathcal{B})$, with a product action on P, then k divides $\lambda l(|\Gamma| - 1)$ and $v = |\Gamma|^l \leq \lambda l^2(|\Gamma| - 1)^2$.*

Proof. Take $x \in P = \Gamma^l$. If $x = (\gamma_1, ..., \gamma_l)$, define for $1 \geq j \geq l$ the Cartesian line of the j^{th} parallel class through x to be the set:

$$G_{x,j} = \{(\gamma_1, ..., \gamma_{j-1}, \gamma, \gamma_{j+1}, ..., \gamma_l) | \gamma \in \Gamma\},$$

(So there are l Cartesian lines through x).

Denote $|\Gamma| = m$.

Since G is primitive, G_x is transitive on the l Cartesian lines through x. Denote by Δ the union of those lines (excluding x). Then Δ is a union of orbits of G_x, and so every block through x intersects it in the same number of points. Hence k divides $\lambda l(|\Gamma| - 1)$. Also, $k^2 > \lambda(m^l - 1)$, so $(m^l - 1) < \lambda l^2(m - 1)^2$.

Hence

$$v = m^l \leq \lambda l^2(m - 1)^2. \tag{2.1}$$

□

3 Results

In this section we will only consider $l = 2$, further work may be done for greater values of l.

When $l = 2$, $m = \frac{r^2+2r+4\lambda}{4\lambda - r^2}$ so

$$(m + 1)r^2 + 2r - 4\lambda(m - 1) = 0 \tag{3.1}$$

solving for r we have

$$r = \frac{-2 \pm \sqrt{4 + 16\lambda(m - 1)(m + 1)}}{2(m + 1)} = \frac{-1 \pm \sqrt{1 + 4\lambda(m^2 - 1)}}{m + 1}$$

therefore

$$r = \frac{2(k - 1)}{m + 1}. \tag{3.2}$$

Suppose that $(k, \lambda) = t > 1$ (the case where $(k, \lambda) = 1$ was done by Paul-Hermann Zieschang [11]), so there exist positive integers a and b such that

$$k = at, \quad \lambda = bt. \tag{3.3}$$

Then, by Lemma 2.2 we have

$$k = \frac{2\lambda(m - 1)}{r}, \tag{3.4}$$

and substituting (3.3) in the last one and also in $k(k - 1) = \lambda(v - 1)$ we obtain

$$a = \frac{2b(m - 1)}{r}, \tag{3.5}$$

$$a(at - 1) = b(m^2 - 1). \tag{3.6}$$

From (3.5) we can see that a divides $b(m-1)$. But $(k,\lambda)=t$ so $t=(at,bt)$ implies $(a,b)=1$. Therefore a divides $m-1$, that is, there exists a positive integer s such that $m-1=as$ and substituting in (3.5), we obtain $r=2bs$. Then since $(a,b)=1$, this forces $s=(m-1,\frac{r}{2})$.
We have te following results with respect to the new parameters a and s.

Lemma 3.1. *Let D be a (v,k,λ)-symmetric design with $v=m^2$ admitting a flag-transitive, point-primitive automorphism group with product action. If $k=at$ and $\lambda=bt$ with $t=(k,\lambda)$, then $a\neq 1$.*

Proof. If $a=1$ then $k=t$ and $\lambda=kb$ with $b\geq 1$. This is a contradiction because $k>\lambda$, therefore $a\neq 1$. □

Lemma 3.2. *Let D be a (v,k,λ)-symmetric design with $v=m^2$ admitting a flag-transitive, point-primitive automorphism group with product action. If $k=at$ and $\lambda=bt$ with $t=(k,\lambda)$, then $(a,s)=1$ where s is a positive integer such that $m-1=as$ and $r=2bs$.*

Proof. Note (3.2) can be rewritten as:

$$r+1=k-(m-1)\frac{r}{2}.$$

Using the expressions $k=at$, $\lambda=bt$, $m-1=as$ and $r=2bs$ we obtain $1=a(t-bs^2)-2bs$ and here we can see that $(a,s)=1$. □

The fact that the parameter $s=1$ is a necessary and sufficient condition for Menon designs is seen in the following result:

Lemma 3.3. *Let D be a (v,k,λ)-symmetric design with $v=m^2$ admitting a flag-transitive, point-primitive automorphism group with product action. If $t=(k,\lambda)$ and $s\in\mathbb{Z}^+$ is such that $m-1=as$ and $r=2bs$, then $s=1$ if and only if $v=4t^2$, $k=2t^2-t$, and $\lambda=t^2-t$.*

Proof. Suppose first that $s=1$, so $m-1=a$ which implies $k=(m-1)t$. We also have $\frac{r}{2}=b$, so $\lambda=\frac{r}{2}t$. Now from $m=\frac{r^2+2r+4\lambda}{4\lambda-r^2}$ we obtain

$$m=\frac{b+t+1}{t-b} \tag{3.7}$$

then

$$a=m-1=\frac{2b+1}{t-b}. \tag{3.8}$$

Now if $t=b$ then $\lambda=tb=b^2=\dfrac{r^2}{4}$, and substituting in (3.1) we obtain

$$r^2(m+1)+2r-r^2(m-1)=0,\text{ so } r(r+1)=0.$$

This forces $r=0$ or $r=-1$, which is a contradiction and so $t\neq b$. From (3.8) we have $t-b\geq 1$.

Suppose that $t-b>1$, and let $x>1$ be an integer such that $t=b+x$. We will prove that x is an odd number. If $x=2y$ for some $y\in\mathbb{Z}$, then $t=b+2y$, and substituting in (3.8) we obtain

$$a=\frac{2b+1}{2y},$$

which is a contradiction since $a\in\mathbb{Z}$, so x is odd. Therefore there exists a positive integer y such that $x=2y+1>1$ and with this we obtain $t=b+2y+1$, substituting in (3.8) results in

$$a=\frac{2b+1}{2y+1}.$$

Using the last expression for a together with $a>b$ we obtain $2b+1>b(2y+1)$ which results in

$$1>b(2y-1). \tag{3.9}$$

But we assumed $t-b>1$ so $x=2y+1>1$, that is, $2y-1>-1$. This together with the expression (3.9) implies that the equation $2y-1=0$ should hold. But that implies $y=\frac{1}{2}$, which is a contradiction since we assumed $y\in\mathbb{Z}$.

From the above we can conclude that $b=t-1$ and this implies $\lambda=t(t-1)$. Then substituting this expression in (3.8) we have

$$a=2(t-1)+1=2t-1,$$

so $k=t(2t-1)$ and $m=a+1=2t$, therefore $v=m^2=4t^2$.

Now, suppose that we have a symmetric design with parameters $v=4t^2$, $k=2t^2-t$ and $\lambda=t^2-t$. Then $a=2t-1$ and $b=t-1$. In addition, we have $m=2t$ and all of these combined imply $m-1=2t-1=a$. But $m-1=as$, and so $s=1$. Hence the result. □

Remark 3.4. When we fix $m-1$ and we vary $\frac{r}{2}$ we get many possible values for λ that satisfy the equation $m=\frac{r^2+2r+4\lambda}{4\lambda-r^2}$, at this point we observe that if $m-1$ is a power of an odd prime, then the parameters satisfy the conditions of Menon designs.

Lemma 3.5. *Let D be a (v,k,λ)-symmetric design with $v=m^2$ admitting a flag-transitive, point-primitive automorphism group with product action such that $t=(k,\lambda)$ and $s\in\mathbb{Z}^+$ such that $m-1=as$ and $r=2bs$. If $m-1=p^d$ with p an odd prime and $d\in\mathbb{N}$, then $v=4t^2$, $k=2t^2-t$, and $\lambda=t^2-t$.*

Proof. From $m-1=as=p^d$ then we have the following possible cases:

1. $s=p^i$ and $a=p^{d-i}$ for some natural number $i<d$.
   This case is not possible because this would imply that $(a,s)=p^j$ for some natural number j, and this contradicts Lemma 3.2.

2. $a=p^i$ and $s=p^{d-i}$ for some natural number $i<d$.
   This case is not possible because this would imply that $(a,s)=p^j$ for some natural number j, contradicting Lemma 3.2.

3. $s=p^d$ and $a=1$.
   This is not possible because it contradicts Lemma 3.1.

4. $a = p^d$ and $s = 1$.
Recall that $s = 1$ (Lemma 3.3), so in this case $v = 4t^2$, $k = 2t^2 - t$ and $\lambda = t^2 - t$ (these are the conditions for Menon designs). With this we have proved the lemma. □

Remark 3.6. We cannot claim the previous result for any odd $m - 1$ because the parameters $(4900, 3267, 2178)$, $(16900, 2752, 448)$ and $(44100, 8019, 1458)$ are counterexamples to that possible generalisation. However we have neither confirmed nor discarded the existence of designs with these parameters. These (and Menon designs) are the only admissible parameters for $v \leq (210)^2$.

Recall the definition of the Cartesian lines from Lemma 2.2. In the case we are studying, when $l = 2$, there are two Cartesian lines through any point in the design, so we have two possibilities. Either:

(i) there exists a point x and a block that contains it such that it intersects only one Cartesian line through x, or

(ii) for any point x in the design, every block that contains it intersects each one of the Cartesian lines through x.

We now study these cases separately. Although there are similarities between both proofs, due to their length and enough differences we present two theorems for clarity.

Theorem 3.7 (**Case (i)**). *Let D be a (v, k, λ)-symmetric design with $v = m^2$ admitting a flag-transitive, point-primitive automorphism group with product action. If there exists a flag (x, A) in the design such that A intersects only one Cartesian line through x then $r+1$ divides k.*

Proof. Let (x, A) be the flag such that A intersects a Cartesian line through $x := (a_0, b_0)$. Suppose that A intersects the second Cartesian line through x.

First, let us prove that for any element of the block A, A intersects only the second Cartesian line through that point. We have, two subcases: either a point $y \in A$ is in the second Cartesian line through x, or a point $y \in A$ is not in the second Cartesian line through x.

First subcase: we can see that if we take a point $y \in A$ so that it is also in the second Cartesian line through x, then $y = (a_0, \nu)$ for some $\nu \in \Gamma$. In this way, the set of elements in the second Cartesian line through y which are also in A is the same as the intersection of A with the second Cartesian line through x. Also, since by Lemma 2.2 the size of the intersection of A with the second Cartesian line through x is $r + 1$, the size of the set of elements in the second Cartesian line through y which are also in A is $r + 1$, and since the size of the intersection of A with the Cartesian lines through any point is $r + 1$, there are no more elements of any of the two Cartesian lines through y in A. In particular, there are no elements of the first Cartesian line through y in A and so the statement is proved for this subcase.
Second subcase: Now we are going to take $y \in A$ such that it is not in the second Cartesian line through x, in particular $y \neq x$, so if $y := (a_1, b_1)$ then $a_1 \neq a_0$. Let us consider the flag (y, A). Since the group G is flag-transitive, there is a $g \in G$ such that $g(x, A) = (y, A)$, that is, $g(x) = y$, so

$$g(a_0, b_0) = (a_1, b_1), \tag{3.10}$$

this implies $g|_\Gamma(a_0) = a_1$. So, for any $\mu \in \Gamma$ such that $(a_0, \mu) \in A$ we have $g(a_0, \mu) = (a_1, \nu)$ for some $\nu \in \Gamma$. Thus the element $g \in G$ sends every element of the second Cartesian line through x which is also in A to an element of the second Cartesian line through y which is also in A. In this way A intersects only the second Cartesian line through y. This is true for any y which is not in the second Cartesian line through x and by Lemma 2.2 the size of this intersection is $r + 1$ and with this the statement the second subcase is proved.

Let A_0 be the set of points in the second Cartesian line through x which are also in A, including x, the size of this set is $r + 1$. Now let us take an element $x_1 \in A \setminus A_0$. By previous arguments A intersects only the second Cartesian line through x_1, therefore, if A_1 is the set of points in the second Cartesian line through x_1 that are in A including x_1 then the size of this set is also $r + 1$. In the same way as before, we take $x_2 \in A \setminus (A_0 \cup A_1)$ and define the set A_2 as the set of points in the second Cartesian line through x_2 that are in A including x_2 and again its size is $r + 1$.

The process is continued in this way until no more points can be taken in A, thus we get a set of points $x_0, x_1, ..., x_i \in A$ along with a collection of sets $A_0, A_1, ..., A_i$ for some natural number i, such that A_j is the intersection of the second Cartesian line through x_j with A. So, the size of A_j is $r + 1$ for all j. Also, $A = \bigcup_{j=0}^{j=i} A_j$ and by construction if $x_g \neq x_h$ then $A_g \neq A_h$ with $1 \leq g, h \leq i$.

It remains to prove that each pair of sets in this collection is disjoint, that is, if $A_e \neq A_f$ are two sets in the collection that was previously constructed, we must prove that $A_e \cap A_f = \emptyset$ with $1 \leq e, f \leq i$, $e \neq f$. Suppose that there exists an element $p \in A_e \cap A_f$, with $x_e := (a_e, b_e)$ and $x_f := (a_f, b_f)$. Then $p = (a_e, \mu) = (a_f, \nu)$ for some $\mu, \nu \in \Gamma$. We can see that $a_e = a_f$, which implies that x_e is in the second Cartesian line through x_f. This is a contradiction since $A_e \neq A_f$.

From all of the above we can conclude that we obtain a partition of the block A. We know that the size of A is k, and on the other hand $A = \bigcup_{j=0}^{j=i} A_j$.

They are all disjoint and the size of each A_j is $r+1$, so $k = i(r+1)$, and $r+1$ divides k. □

Let (x, A) be a flag such as in Theorem 3.7, that is, A intersects only the second Cartesian line through x. We count the number of flags (y, C) such that $x \in C$ and $y \neq x$ is in the second Cartesian line through x.

The number of these flags is the same as the number of blocks that contain x as well as elements of the second Cartesian line through x, (we denote this number by z), multiplied by the number of elements of the second Cartesian line through x (excluding x) which are in these blocks, that is, r, therefore the number of such flags (y, C) is zr.

On the other hand, x and y are together in λ blocks and there are $m - 1$ points of the second Cartesian line through x, so when we count these flags (y, C) we obtain $\lambda(m - 1)$.

The above implies the equation $zr = \lambda(m - 1)$, but the equation $kr = 2\lambda(m - 1)$ also holds, hence $z = \frac{k}{2}$ and since $z \in \mathbb{N}$, k is even. This means that half of the blocks that contain x intersect with the second Cartesian line through x and the other half intersect with the first Cartesian line through x. This is possible since the previous argument is also valid for the first Cartesian line through x.

In the following theorem we examine Case (ii).

Theorem 3.8 **(Case (ii)).** *Let D be a (v, k, λ)-symmetric design with $v = m^2$ admitting a flag-transitive, point-primitive automorphism group with product action. If for every point*

x in the design, every block that contains it intersects with the two Cartesian lines through x, then $\frac{r}{2}+1$ divides k.

Proof. Let $x=(a_0,b_0)$ be an arbitrary point in the design, and let A be a block containing x, then there are r_1 elements of the first Cartesian line through x (excluding x) in A and there are r_2 elements of the second Cartesian line through x (excluding x) in A. The numbers r_1 and r_2 satisfy the equation $r=r_1+r_2$, by Lemma 2.2.

If C is another block containing x, then it must intersect the two Cartesian lines through x. Since G acts transitively on the flags, there is an element $g\in G$ such that $g(x,A)=(x,C)$ and from this we can see that $g(x)=x$, that is, $g|_\Gamma$ fixes a_0 and b_0.

First, we will prove that g sends the elements of the first Cartesian line through x which are also in A to elements of the first Cartesian line through x which are also in C. Let (μ,b_0) be an element of the first Cartesian line through x which is also in A. Then $g(\mu,b_0)=(\nu,b_0)\in C$ for some $\nu\in\Gamma$ since $g|_\Gamma$ fixes b_0. Similarly g sends the elements of the second Cartesian line through x which are also in A to elements of the second Cartesian line through x which are also in C. Let(a_0,μ) be an element of the first Cartesian line through x which is also in A, then $g(a_0,\mu)=(a_0,\nu)\in C$ for some $\nu\in\Gamma$ since $g|_\Gamma$ fixes a_0. Therefore the block C has as many elements of the first Cartesian line through x as A, and as many elements of the second Cartesian line through x as A. The above is true for every block that contains x.

Now let us count the number of flags (y,C) of the design such that $y\neq x$ is an element of the first Cartesian line through x and C is a block containing x. Every block contains r_1 elements of the first Cartesian line through x, when we exclude x, and there are k blocks containing x. All of them intersect the first Cartesian line through x, therefore there are kr_1 flags of this type. On the other hand y and x are together in λ blocks and there are $m-1$ elements of the first Cartesian line through x (excluding x), hence there are $\lambda(m-1)$ flags of this type. This yields the equation $kr_1=\lambda(m-1)$, but from Lemma 2.2 the equation $kr=2\lambda(m-1)$ also holds and we conclude that $r_1=\frac{r}{2}$. However $r_1+r_2=r$, so the intersection of every block containing x with the second Cartesian line through x (excluding x) has $r_2=\frac{r}{2}$ elements.

The above is true for every x, that is, for every point in the design, every block that contains it intersects the first Cartesian line through that point in $\frac{r}{2}$ other points and the same holds for the second Cartesian line through that point (excluding the point itself).

In what follows we will consider A_0 to be the set of points of the second Cartesian line through x which are also in A including x itself. The number of elements in that set is $\frac{r}{2}+1$. Let us consider $x_1\in A\setminus A_0$ so from the previous paragraphs A intersects the second Cartesian line through x_1 in $\frac{r}{2}$ elements, thus if A_1 is the set of points of the second Cartesian line through x_1 which are also in A including x_1 itself, the number of elements in A_1 is $\frac{r}{2}+1$. Now we take an element $x_2\in A\setminus(A_0\cup A_1)$ in the same way as before, and let A_2 be the set of points of the second Cartesian line through x_2 which are also in A including x_2 itself. The number of elements in A_2 is $\frac{r}{2}+1$.

We can continue this process in this way until there are no more elements in A, (everything is finite), so we obtain a collection of points $x_0,x_1,...,x_i\in A$ and a collection of sets $A_0,A_1,...,A_i$ for some natural number i such that for all $j=0,\dots,i$ A_j is the intersection of the second Cartesian line through x_j with A, and A_j has $\frac{r}{2}+1$ elements. By construction, $A=\bigcup_{j=0}^{j=i}A_j$ and the construction implies that if $x_g\neq x_h$ then $A_g\neq A_h$ with $1\leq g,h\leq i$.

It remains to prove that every two sets in this collection are disjoint, that is, we must prove that if $A_e \neq A_f$ then $A_e \cap A_f = \emptyset$ (with $1 \leq e, f \leq i$ and $e \neq f$). Suppose there is an element $p \in A_e \cap A_f$, with $x_e := (a_e, b_e)$ and $x_f := (a_f, b_f)$. Then $p = (a_e, \mu) = (a_f, \nu)$ for some $\mu, \nu \in \Gamma$. We can see that $a_e = a_f$, which implies that x_e is in the second Cartesian line through x_f, a contradiction since $A_e \neq A_f$.

Therefore we have a partition of the block $A = \bigcup_{j=0}^{j=i} A_j$. The size of $\bigcup_{j=0}^{j=i} A_j$ is $i(\frac{r}{2}+1)$ since they are all disjoint, and the size of A is k, therefore $k = i(\frac{r}{2}+1)$ and $\frac{r}{2}+1$ divides k. □

Now we will present some consequences of Theorem 3.8.

Corollary 3.9. *With the same hypotheses of* Theorem 3.8, $\frac{r}{2}+1$ divides m.

Proof. From (3.2) we have

$$k = \frac{r}{2}m + \frac{r}{2} + 1,$$

and there is an integer p such that $k = p(\frac{r}{2}+1)$. Substituting in the previous equation we obtain $(p-1)\left(\frac{r}{2}+1\right) = \frac{r}{2}m$. Since $\left(\frac{r}{2}+1, \frac{r}{2}\right) = 1$, $\frac{r}{2}+1$ necessarily divides m. □

Corollary 3.10. *With the same hypotheses of* Theorem 3.8, $\frac{r}{2}+1$ divides λ.

Proof. There is an integer p such that $k = p(\frac{r}{2}+1)$, and substituting this and (3.2) in $k(k-1) = \lambda(m-1)(m+1)$, we obtain

$$p\frac{r}{2}\left(\frac{r}{2}+1\right) = \lambda(m-1).$$

By Corollary 3.9, $\frac{r}{2}+1$ divides m, so $\left(\frac{r}{2}+1, m-1\right) = 1$ and $\frac{r}{2}+1$ divides λ. □

Since t is the greatest common divisor of k and λ, the following holds:

Corollary 3.11. *With the same hypotheses of Theorem 3.8, $\frac{r}{2}+1$ divides t.*

Proof. Since $\frac{r}{2}+1$ divides k and λ, and also $(k, \lambda) = t$ we conclude $\frac{r}{2}+1$ divides t. □

In the next results, we will introduce a particular case in which we have obtained the parameters of a Menon design, as a consequence of Corollary 3.11. Since $\frac{r}{2}+1$ divides $t > 1$, we will first consider the case in which t is a prime number. The following result is a first approach to our main result.

Lemma 3.12. *Let D be a (v, k, λ)-symmetric design with $(k, \lambda) = t > 1$ a prime number and $v = m^2$, admitting a flag-transitive, point-primitive automorphism group G. If for every point x in the design, every block that contains it intersects the two Cartesian lines through x, then either G does not have a product action or D is a Menon design.*

Proof. From Corollary 3.11, we have $\frac{r}{2}+1$ divides t. Since t is a prime we obtain $\frac{r}{2} = 0$ or $\frac{r}{2}+1 = t$.

If $\frac{r}{2} = 0$ then from (3.2) we have $k - 1 = 0$ and this is impossible.

If on the other hand $\frac{r}{2}+1 = t$, then $m = \frac{r^2+2r+4\lambda}{4\lambda - r^2}$ so

$$m = \frac{bs^2 + s + t}{t - bs^2} \tag{3.11}$$

which implies $t \geq bs^2$. If $t = bs^2$ then $\lambda = b^2s^2 = \frac{r^2}{4}$. Substituting this in (3.1) we obtain

$$r^2(m+1) + 2r - r^2(m-1) = 0, \text{ therefore } r(r+1) = 0,$$

so $r = 0$ or $r = -1$ which is a contradiction, so $t > bs^2$.

This forces $t = \frac{r}{2} + 1 = bs + 1 > bs^2$, so $1 > bs(s-1)$ and $s = 1$. From Lemma 3.3, $v = 4t^2$, $k = 2t^2 - t$ and $\lambda = t^2 - t$, which are the parameters of a Menon design. □

Remark 3.13. The triples of parameters $(4900, 3267, 2178)$, $(16900, 2752, 448)$, and $(44100, 8019, 1458)$, do not correspond to Menon designs but they satisfy all known necessary arithmetic conditions on the existence of a symmetric design with v even, so we do not prove the conjeture for $v \leq (210)^2$ (we have not tried computational methods).

Lemma 3.14. *Let D be a (v, k, λ)-symmetric design with $(k, \lambda) = t > 1$ and $v = m^2 \leq (210)^2$ with m even admitting a flag-transitive point-primitive automorphism group G, then either G does not have a non-trivial product action or one of the following conditions holds:*

1. *D is a Menon design with parameters $(4t^2, 2t^2 - t, t^2 - t)$, where $t > 1$, or*
2. *D has parameters $(16900, 2752, 448)$.*

Proof. For $m \leq 210$ the admissible parameters that do not satisfy the conditions of Menon designs and in which $k - \lambda$ is a square are $(4900, 3267, 2178)$, $(16900, 2752, 448)$, and $(44100, 8019, 1458)$. For these, $\frac{r}{2} + 1$ is $47, 22$ and 39 respectively, so they do not satisfy Theorem 3.8. Now for those parameters $r + 1$ is $93, 43$ and 78 respectively. The first and third of them do not satisfy Theorem 3.7, but the parameters $(16900, 2752, 448)$ do. Thus, these are the only possible parameters for $m^2 \leq (210)^2$.

In this case, k is even, which is consistent with Theorem 3.7. We also have $s = 3$ so from Lemma 3.3 we know that these parameters cannot correspond to a Menon design. □

Remark 3.15. The triple $(16900, 2752, 448)$ does not correspond to a Menon design since $s = 3$, although it satisfies all the arithmetic conditions for a symmetric design. We make no claim as to whether such a design exists, but perhaps it is not the case that when $l = 2$ only Menon designs are possible (if at all).

The following is our main result, the proof follows Cases (i) and (ii) from Theorems 3.7 and 3.8, that is, either: there is a flag (x, A), such that the block A intersects only one Cartesian line through x (Case (i)), or for every point x, every block that contains it intersects both of the Cartesian lines through x (Case (ii)). The proof based on Case (i) is similar to the proof of the case in which $m - 1$ is the power of an odd prime. In this sense it is a generalisation of this proof, but because of the existence of the parameter s this generalisation was not obtained in an obvious way. For this reason, we need an additional arithmetic condition, which is found in Corollary 3.11.

In the proof based on Case (ii) we also obtain an arithmetic condition for a and so also for k. We believe we do not necessarily obtain parameters for Menon designs for an arbitrary λ when we study symmetric designs admitting a flag-transitive point-primitive automorphism group with product action when $l = 2$. This case also gives us a parametrisation of (v, k, λ) in terms of t and s, and if $s = 1$ then the parameters correspond to Menon designs, that is, our parameterisation is a generalisation of the parameterisation of Menon designs.

Theorem 3.16. *Let D a (v, k, λ)-symmetric design admitting a flag-transitive, point-primitive, automorphism group G with $(k, \lambda) = t > 1$ and $v = m^2$ with m even. Then either:*

(i) *G does not have a non-trivial product action, or*

(ii) *D is a design with parameters $\left((2t+s-1)^2, \frac{2t^2-(2-s)t}{s}, \frac{t^2-t}{s^2}\right)$ with $s \geq 1$ odd.*

When $s = 1$ D is a Menon design and if $s > 1$ then t is even.

Proof. From the hypotheses and from Lemma 2.2 there are two possible cases. For any given point, either each block that contains it intersects the two Cartesian lines through it, or there is a point such that a block containing it only intersects one Cartesian line through it. As we have seen, the latter implies that every block intersects only one Cartesian line through each point it contains.

First we will study this last case. Here, Theorem 3.7 is satisfied, so $r+1$ divides k with $r > 1$ an integer such that $kr = 2\lambda(m-1)$. This implies there is an integer p such that $k = p(r+1)$. Also $k - 1 = \frac{r}{2}(m+1)$ holds, so $k = \frac{r}{2}m + \frac{r}{2} + 1$, and therefore

$$m - 1 = (r+1)(m+1-2p).$$

Then $r+1$ divides $m-1$, but $m-1 = as = x(r+1)$ where $x := m+1-2p$ and since $r = 2bs$ we have $(r+1, s) = 1$ so $r+1$ divides a. Also a divides $r+1$ which forces $r+1 = a$, and this implies $k = (r+1)t$. This all implies $t - bs^2 = 1$, so $b = \frac{t-1}{s^2}$, and we obtain the parameters $\lambda = \frac{t-1}{s^2}t$, $k = \frac{2t+s-2}{s}t$, $v = (2t+s-1)^2$.

The proof of Theorem 3.7 states that k should be an even number and since $r+1$ is an odd number then t should be an even number. Since $m = 2t+s-1$ and m is an even number then s is an odd number.

The triple $(16900, 2752, 448)$ satisfies the conditions we obtained, with $t = 64$ and $s = 3$, so this is not a Menon design.

When $s = 1$ we obtain the parametrisation for Menon designs with t an even number.

Now suppose that for every point, every block that contains it intersects both Cartesian lines through it. Here the hypotheses of Theorem 3.8 hold and from Corollary 3.11, there exists $x \geq 1$ such that

$$t = \left(\frac{r}{2} + 1\right)x = (bs+1)x. \tag{3.12}$$

From (3.11) we obtain $m = \frac{s(bs+1)+t}{t-bs^2} = \frac{s(bs+1)+(bs+1)x}{t-bs^2}$, that is

$$m = \left(\frac{s+x}{t-bs^2}\right)\left(\frac{r}{2}+1\right). \tag{3.13}$$

Using (3.12) we obtain

$$t - bs^2 = x + bs(x-s), \tag{3.14}$$

which we divide into the following cases:

1. $x < s$
   From Lemma 3.12, we have $t - bs^2 > 0$, and from (3.14) $x > bs(s-x) > bx(s-x)$. Since $x < s$ then $1 > b(s-x) > 0$, but this cannot be the case since $b(s-x)$ should be an integer.

2. $s < x$
   From (3.13), $s + x \geq t - bs^2$. If $s + x = t - bs^2$ we have from (3.13) that $m = \frac{r}{2} + 1$ so $as = m - 1 = \frac{r}{2} = bs$ and therefore $a = b$, but $(a, b) = 1$ so $a = 1$ this is impossible by Lemma 3.1. Therefore $s + x > t - bs^2$, and from (3.14) we have $s + x > x + bs(x - s)$ so $1 > b(x - s) > 0$ and this is also imposisble since $b(x - s)$ is an integer.

3. $s = x$
   From (3.12), s divides t, and from (3.2) $at - bs(as + 2) = 1$, so $(t, s) = 1$ and therefore $s = 1$. Also from Lemma 3.3, $v = 4t^2$, $k = 2t^2 - t$, and $\lambda = t^2 - t$

This concludes the proof. □

The parameters of Menon designs are not the only ones we can obtain when we assume that the automorphism group of the design has a product action on the points of the design, and the parameters $(16900, 2752, 448)$ are an example of this. However we note that a design with the possible parameters which arise and do not correspond to Menon designs must satisfy that each block only intersects one Cartesian line through each point in that block.

It is not the case that the way in which we consider product action to obtain possible Menon designs does not work because here is a potential counterexample, but rather that with this theorem we give explicit expressions for the parameters v, k, λ, in terms of parameters s, t, and when $s = 1$ they do correspond to Menon designs.

4 One further result

Here we present an additional result for any $l \geq 2$.

Theorem 4.1. *Let D a (v, k, λ)-symmetric design with $v = m^l$ admitting a flag-transitive, point-primitive, automorphism group G with a non-trivial product action. Then the complement of the design is not flag-transitive.*

Proof. Suppose that D' is the complement of the design D, so its parameters are $(v', k', \lambda') = (v, v - k, v - 2k + \lambda)$. If we also assume D' is flag-transitive, then the following equation holds:

$$(v - k)(v - k - 1) = (v - 2k + \lambda)(m - 1)(m^{l-1} + m^{l-2} + ... + 1). \tag{4.1}$$

If D has a point-primitive automorphisms group G, then D' has the same point-primitive automorphisms group G and we can consider the Cartesian lines through a point, since G is transitive on the points of D'. Thus k' divides $\lambda' l(m - 1)$, so there is an integer p such that

$$(v - k)p = l(v - 2k + \lambda)(m - 1). \tag{4.2}$$

Substituting this in (4.1) we obtain

$$l(v-k)(v-1-k) = (v-k)p(m^{l-1}+m^{l-2}+...+1)$$
$$\text{so} \quad l((m-1)(m^{l-1}+m^{l-2}+...+1)-k) = p(m^{l-1}+m^{l-2}+...+1),$$

hence

$$lk = q(m^{l-1}+m^{l-2}+...+1) \tag{4.3}$$

with $q = l(m-1)-p > 0$.

But for D we know that $k = \dfrac{l\lambda(m-1)}{r}$ and $k(k-1) = \lambda(v-1)$ so

$$k(k-1) = \frac{kr}{l(m-1)}(m^l-1)$$

and we obtain a generalisation of (3.2):

$$l(k-1) = r(m^{l-1}+m^{l-2}+...+1). \tag{4.4}$$

If we substitute (4.3) in (4.4) then

$$q(m^{l-1}+m^{l-2}+...+1)-l = r(m^{l-1}+m^{l-2}+...+1)$$
$$\text{so} \quad (q-r)(m^{l-1}+m^{l-2}+...+1) = l,$$

and therefore $m^{l-1}+m^{l-2}+...+1 \leq l$ if $m > 1$ and $l \geq m^{l-1}+m^{l-2}+...+1 > l$, which is impossible.

We conclude $m \leq 1$, but this is a contradiction since $m \geq 5$. □

ORCID iDs

Eugenia O'Reilly-Regueiro https://orcid.org/0000-0001-5867-7258

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