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ON THE SPECIFIC AREA OF INHOMOGENEOUS BOOLEAN MODELS. EXISTENCE RESULTS AND APPLICATIONS

ELENA VILLA

Dept. of Mathematics, University of Milan, Via Saldini 50, 20133, Milano, Italy
e-mail: elena.villa@unimi.it
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ABSTRACT

The problem of the evaluation of the so-called *specific area* of a random closed set, in connection with its mean boundary measure, is mentioned in the classical book by Matheron on random closed sets (Matheron, 1975, p. 50); it is still an open problem, in general. We offer here an overview of some recent results concerning the existence of the specific area of inhomogeneous Boolean models, unifying results from geometric measure theory and from stochastic geometry. A discussion of possible applications to image analysis concerning the estimation of the mean surface density of random closed sets, and, in particular, to material science concerning birth-and-growth processes, is also provided.

Keywords: geometric measure theory, mean surface density, outer Minkowski content, specific area, stochastic geometry.

INTRODUCTION

In this paper we offer an overview of some recent results concerning the existence of the specific area of inhomogeneous Boolean models, and its relationship with the mean surface density of the involved random sets. Applications to birth-and-growth stochastic processes are also discussed.

If a n-dimensional random closed set Θ in $\mathbb{R}^d$ is such that $\mathbb{E}[\mathscr{H}^n_{|\Theta}]$ is absolutely continuous with respect to $\mathscr{H}^d$, where $\mathscr{H}^n$ is the n-dimensional Hausdorff measure, and $\mathscr{H}^n_{|\Theta}$ is its restriction to Θ, then the density (or Radon-Nikodym derivative) of $\mathbb{E}[\mathscr{H}^n_{|\Theta}]$ with respect to $\mathscr{H}^d$ is called *mean density of* Θ. Whenever the mean density of Θ exists, we denote it by λ_Θ.

The problem of the evaluation of the mean densities of lower dimensional random closed sets, and in particular of the mean surface density, is of interest in several real applications.

In Ambrosio *et al.* (2009) the notion of *local mean n-dimensional Minkowski content* of a random closed set has been introduced in order to provide approximations of the mean density of n-dimensional random closed sets in $\mathbb{R}^d$. With regard to the approximation of the mean surface density of a d-dimensional random closed set, the concept of *specific area* introduced in Matheron (1975, p. 50) turns out to be closely related to the notion of *local mean outer Minkowski content* of a random closed set. We remind that, given a random closed set Θ in $\mathbb{R}^d$, the specific area $\sigma_\Theta(x)$ of Θ at a point x in $\mathbb{R}^d$ is defined as the following limit

$$\sigma_\Theta(x) := \lim_{r\downarrow 0} \frac{\mathbb{P}(x \in \Theta_{\oplus r} \setminus \Theta)}{r}\,, \qquad (1)$$

whenever it exists. ($\Theta_{\oplus r}$ denotes here the parallel set of Θ at distance r, *i.e.*, $\Theta_{\oplus r} := \{x \in \mathbb{R}^d : \mathrm{dist}(x,\Theta) \le r\}$.) It is mentioned in Matheron (1975) that the definition of $\sigma_\Theta(x)$ is the "translation" into probabilistic terms of the following limit

$$\lim_{r\downarrow 0} \frac{\mathscr{H}^d(K_{\oplus r} \setminus K)}{r}, \quad K \subset \mathbb{R}^d \text{ compact.} \qquad (2)$$

The limit above, whenever it exists finite, is called *area of* K in Matheron (1975), or *outer Minkowski content of* K in Ambrosio *et al.* (2008). As right derivative at $r = 0$ of the volume function $V(r) := \mathscr{H}^d(K_{\oplus r})$, it is intuitively clear that there exist compact subsets of $\mathbb{R}^d$ such that the limit in Eq. 2 equals the surface measure $\mathscr{H}^{d-1}(\partial K)$ of K (*e.g.*, if K is convex with nonempty interior), that explains the name *area of* K; actually, examples of subsets of $\mathbb{R}^d$, even closure of their interior, such that the above limit differs from the $\mathscr{H}^{d-1}$-measure of their boundary can be provided. Notice that a computer graphics representation of lower dimensional sets in $\mathbb{R}^2$ is anyway provided in terms of pixels, which can offer only a 2-D box approximation of points in $\mathbb{R}^2$ (an interesting discussion on this is contained in Kärkkäinen *et al.*, 2002); therefore, the possibility of evaluating and estimating the surface measure of a set (the mean surface density for random sets) by the

volume measure of the Minkowski enlargement of the involved set by Eq. 2 (by Eq. 1 in the stochastic case) might be a solution to problems of this kind.

SPECIFIC AREA AND LOCAL MEAN OUTER MINKOWSKI CONTENT

Throughout the paper $\mathscr{H}^n$ is the n-dimensional Hausdorff measure, dx stands for $\mathscr{H}^d(\mathrm{d}x)$, $\mathscr{B}_{\mathbb{R}^d}$ is the Borel σ-algebra of $\mathbb{R}^d$ and $\mathscr{H}^n_{|A}$ denotes the restriction of $\mathscr{H}^n$ to a $\mathscr{H}^n$-measurable set $A \subset \mathbb{R}^d$ (*i.e.*, $\mathscr{H}^n_{|A}(B) = \mathscr{H}^n(A \cap B)$ for all $B \in \mathscr{B}_{\mathbb{R}^d}$). $B_r(x)$ will denote the closed ball with centre x and radius r, whereas for any integer n we denote by b_n the volume of the unit ball in $\mathbb{R}^n$.

We recall that, given a subset S of $\mathbb{R}^d$ and an integer n with $0 \le n \le d$, the *n-dimensional Minkowski content of S* is defined by

$$\mathscr{M}^n(S) := \lim_{r \downarrow 0} \frac{\mathscr{H}^d(S_{\oplus r})}{b_{d-n} r^{d-n}},$$

whenever the limit exists finite.

Let $A \in \mathscr{B}_{R^d}$; the quantity $\mathscr{SM}(A)$ defined as (Ambrosio *et al.*, 2008)

$$\mathscr{SM}(A) := \lim_{r \downarrow 0} \frac{\mathscr{H}^d(A_{\oplus r} \setminus A)}{r},$$

provided that the limit exists finite, is called *outer Minkowski content of A*. Note that if A is lower dimensional, then $\mathscr{SM}(A) = 2\mathscr{M}^{d-1}(A)$, whereas if A is a d-dimensional set, closure of its interior, then $A_{\oplus r} \setminus A$ coincides with the outer Minkowski enlargement at distance r of ∂A.

In this section we show why the specific area can be interpreted as the translation into probabilistic terms of the outer Minkowski content. For basic definitions and results of stochastic geometry we refer to Stoyan *et al.* (1995), Baddeley *et al.* (2007), and Schneider and Weil (2008).

If the limit

$$\lim_{r \downarrow 0} \frac{\mathscr{H}^d((A_{\oplus r} \setminus A) \cap B)}{r}$$

exists finite for any $B \in \mathscr{B}_{\mathbb{R}^d}$ such that $\mathscr{H}^{d-1}(\partial A \cap \partial B) = 0$, then we say that A admits *local* outer Minkowski content. Now, let us consider a random closed set Θ in $\mathbb{R}^d$, that is a measurable map $\Theta : (\Omega, \mathfrak{F}, \mathbb{P}) \to (\mathbb{F}, \sigma_{\mathbb{F}})$ ($\mathbb{F}$ and $\sigma_{\mathbb{F}}$ denote here the class of the closed subsets in $\mathbb{R}^d$ and the σ-algebra generated by the so-called hit-or-miss topology (Matheron, 1975), respectively). If the limit

$$\lim_{r \downarrow 0} \frac{\mathbb{E}[\mathscr{H}^d((\Theta_{\oplus r} \setminus \Theta) \cap B)]}{r} \tag{3}$$

exists finite for any $B \in \mathscr{B}_{\mathbb{R}^d}$ such that $\mathbb{E}[\mathscr{H}^{d-1}(\partial\Theta \cap \partial B)] = 0$, then we say that Θ admits *local mean outer Minkowski content*. Let us notice that, by a straightforward application of Fubini's theorem, Eq. 3 can be written equivalently

$$\lim_{r \downarrow 0} \int_B \frac{\mathbb{P}(x \in \Theta_{\oplus r} \setminus \Theta)}{r} \mathrm{d}x\,, \tag{4}$$

if furthermore Θ is *stationary* (and so $\mathbb{P}(x \in \Theta_{\oplus r} \setminus \Theta)$ is constant), then, by choosing $B = [0,1]^d$ in Eq. 4, we have that the specific area σ_Θ is constant, given by

$$\sigma_\Theta = \lim_{r \downarrow 0} \frac{\mathbb{E}[\mathscr{H}^d((\Theta_{\oplus r} \setminus \Theta) \cap [0,1]^d)]}{r}\,. \tag{5}$$

More in general, as we shall see in the next sections, if the boundary ∂A of a d-dimensional Borel set $A \in \mathbb{R}^d$ is "sufficiently regular", then $\mathscr{SM}(A) = \mathscr{H}^{d-1}(\partial A)$; thus, it is intuitive that for "sufficiently regular" random closed sets we may have

$$\mathbb{E}[\mathscr{H}^{d-1}(\partial\Theta \cap B)] = \lim_{r \downarrow 0} \frac{\mathbb{E}[\mathscr{H}^d((\Theta_{\oplus r} \setminus \Theta) \cap B)]}{r}\,,$$

and

$$\lim_{r \downarrow 0} \int_B \frac{\mathbb{P}(x \in \Theta_{\oplus r \setminus \Theta})\mathrm{d}x}{r} = \int_B \sigma_\Theta(x)\mathrm{d}x\,,$$

so that $\sigma_\Theta(x)$ turns out to be the mean surface density of Θ. This is true for instance when Θ satisfies a (local) Steiner formula; in this case the limit in Eq. 5 can be studied in terms of the quermass densities (or Minkowski functionals) associated to Θ, and so by means of tools from integral geometry mainly (Hug, 2000; Weil, 2001; Hug *et al.*, 2006; Baddeley *et al.*, 2007, and references therein). For other related works see also Hug and Last (2000), Hug *et al.* (2004), and Kiderlen and Rataj (2006).

The passage from stationary to nonstationary random closed sets, and from convex to more general grains gives rise to nontrivial problems. We will focus here on the specific area of inhomogeneous Boolean models; for this kind of random closed sets (widely used in real applications in Material Science, as we shall mention in the last section) the relationship between specific area and outer Minkowski content (of the typical grain) is more evident. Let us see why briefly.

Let Ξ be a Boolean model in $\mathbb{R}^d$ (Stoyan *et al.*, 1995; Baddeley *et al.*, 2007; Schneider and Weil, 2008) whose typical grain Z_0 has topological boundary ∂Z_0 with Hausdorff dimension $d-1$ $\mathbb{P}$-a.s. It is well known that Boolean models in $\mathbb{R}^d$ can be described by marked Poisson point processes on $\mathbb{R}^d$ with marks in the space of centred compact sets. In many examples and applications Z_0 is uniquely determined by a random quantity, say S, in a suitable *mark space* $\mathbf{K}$; for instance, in the very simple case of random balls, $\mathbb{K}=\mathbb{R}_+$ and S is the radius, whereas in applications to birth-and-growth processes, in some models $\mathbb{K}=\mathbb{R}^d$ and S is the spatial location of the nucleus, in other models $\mathbb{K}=\mathbb{R}_+$ and S is the birth time of the nucleus (see last section). Therefore, we shall consider (inhomogeneous) Boolean models of the type

$$\Xi(\omega)=\bigcup_{(x_i,s_i)\in\Psi(\omega)} x_i+Z_0(s_i)\,,$$

where $Z_0(s)$ is a compact subset of $\mathbb{R}^d$ containing the origin for any $s\in\mathbf{K}$, and Ψ is the marked Poisson point process in $\mathbb{R}^d$ with marks in $\mathbf{K}$ associated to Ξ, with intensity measure

$$\Lambda(\mathrm{d}(x,s))=f(x)\,\mathrm{d}x\,Q(\mathrm{d}s)\,.$$

The function f and the probability measure Q on $\mathbf{K}$ are called *intensity of* Ξ and *mark distribution*, respectively, and they are commonly assumed to be such that the mean number of grains hitting any compact subset of $\mathbb{R}^d$ is finite:

$$\int_{\mathbf{K}}\int_{(-Z_0(s))_{\oplus R}} f(x)\mathrm{d}xQ(\mathrm{d}s)<\infty \qquad \forall R>0\,. \quad (6)$$

(For basic definitions and results on the theory of point processes, we refer to Baddeley *et al.* (2007) and references therein.)

We shall denote by $\mathrm{diam}(Z_0)$ the (random) diameter of Z_0, and by $\mathrm{disc}\,f$ the set of all the points of discontinuity of f.

From now on let $Z^x:=x-Z_0$ $\forall x\in\mathbb{R}^d$. By the explicit expression of the capacity functional of a Boolean model (Matheron, 1975), it is not difficult to get (Villa, 2010, Eq. 3.1) that

$$\sigma_\Theta(x):=\lim_{r\downarrow 0}\frac{\mathbb{P}(x\in\Xi_{\oplus r}\setminus\Xi)}{r}=e^{-\mathbb{E}_Q[\int_{Z^x}f(y)\mathrm{d}y]}\cdot \lim_{r\downarrow 0}\frac{\left(1-\exp\left\{-\mathbb{E}_Q\left[\int_{Z^x_{\oplus r}\setminus Z^x}f(y)\mathrm{d}y\right]\right\}\right)}{r}, \quad (7)$$

where $\mathbb{E}_Q$ denotes the expectation with respect to Q; thus, in order to obtain sufficient conditions for the existence of $\sigma_\Theta(x)$, existence results for limits of the type

$$\lim_{r\downarrow 0}\frac{1}{r}\int_{A_{\oplus r}\setminus A} f(x)\,\mathrm{d}x \quad (8)$$

have to be studied.

Note that in the particular case in which f is constant the limit above exists if and only if A admits outer Minkowski content.

In the next section we recall some recent results on the existence of the outer Minkowski content of Borel subsets of $\mathbb{R}^d$, which will be the starting point to obtain existence results for the specific area of random closed sets.

EXISTENCE RESULTS FOR THE OUTER MINKOWSKI CONTENT

We remind that a compact set $A\subset\mathbb{R}^d$ is called *n-rectifiable* ($0\le n\le d-1$ integer) if it is representable as the image of a compact subset of $\mathbb{R}^n$ by a Lipschitz map from $\mathbb{R}^n$ to $\mathbb{R}^d$; more in general, a closed subset A of $\mathbb{R}^d$ is called to be *countably $\mathscr{H}^n$-rectifiable* if there exist countably many Lipschitz maps $g_i:\mathbb{R}^n\to\mathbb{R}^d$ such that $A\setminus\bigcup_i g_i(\mathbb{R}^n)$ is $\mathscr{H}^n$-negligible. (For definitions and basic properties of Hausdorff measure and rectifiable sets see, *e.g.*, Federer, 1969, Falconer, 1985, and Ambrosio *et al.*, 2000.) We call *Radon measure* in $\mathbb{R}^d$ any nonnegative and σ-additive set function μ defined on $\mathscr{B}_{\mathbb{R}^d}$ which is finite on bounded sets.

Federer's theorem (Federer, 1969, p. 275) on the existence of the n-dimensional Minkowski content of n-rectifiable compact sets is well known. We recall a generalization of such theorem, proved in Ambrosio *et al.* (2000, p. 110), to countably $\mathscr{H}^n$-rectifiable compact sets and then we shall see that if the boundary of a Borel subset of $\mathbb{R}^d$ satisfies similar conditions, then it admits outer Minkowski content.

Theorem 1 *Let $A\subset\mathbb{R}^d$ be a countably $\mathscr{H}^n$-rectifiable compact set and assume that*

$$\eta(B_r(x))\ge\gamma r^n \qquad \forall x\in A,\ \forall r\in(0,1) \quad (9)$$

holds for some $\gamma>0$ and some Radon measure η in $\mathbb{R}^d$ absolutely continuous with respect to $\mathscr{H}^n$. Then $\mathscr{M}^n(A)=\mathscr{H}^n(A)$.

Condition (Eq. 9) is a kind of quantitative non-degeneracy condition which prevents A from being too sparse; simple examples (Ambrosio *et al.*, 2000; 2008) show that $\mathscr{M}^n(A)$ can be infinite, and $\mathscr{H}^n(A)$ arbitrarily small, when this condition fails.

It is reasonable to conjecture that, if the boundary ∂A of a subset A of $\mathbb{R}^d$ satisfies the assumptions of

the theorem above, then A admits outer Minkowski content; in Villa (2009a) it has been proved that the value of $\mathscr{SM}(A)$ depends on the density of A at its boundary points. We remind that the *d-dimensional density* (briefly, *density*) of A is defined by Ambrosio *et al.* (2000)

$$\theta_d(A,x) := \lim_{r\downarrow 0} \frac{\mathscr{H}^d(A\cap B_r(x))}{\mathscr{H}^d(B_r(x))},$$

provided that the limit exists. It is clear that $\theta_d(A,x)$ equals 1 for all x in the interior of A, and 0 for all x into the interior of the complement set of A, whereas different values can be attained at its boundary points. It is well known (*cf.* Ambrosio *et al.*, 2000, Theorem 3.61) that if $\mathscr{H}^{d-1}(\partial A) < \infty$, then A has density either 0 or 1 or 1/2 at $\mathscr{H}^{d-1}$-almost every point of its boundary. For every $t \in [0,1]$ and every $\mathscr{H}^d$-measurable set $A \subset \mathbb{R}^d$ let

$$A^t := \{x \in \mathbb{R}^d : \theta_d(A,x) = t\}.$$

Intuitively, a small neighborhood of a point $x \in A^1 \cap \partial A$ is "almost all contained" in A, so that it gives no contribution to the volume of $A_{\oplus r} \setminus A$; thus, roughly speaking, we may say that x has negligible weight in the computing of the outer Minkowski content of A. Conversely, if A has null density in $x \in \partial A$, then, in a small neighborhood of x, $A_{\oplus r} \setminus A$ "almost all coincides" with the Minkowski enlargement of ∂A, so that, roughly speaking, we may say that the weight of x in the computing of the outer Minkowski content of A is twice the weight of a point $y \in A^{1/2}$.

In Villa (2009a) the following class of sets (whose boundary satisfies the assumptions of Theorem 1) has been introduced:

Definition 2 (The class $\mathscr{O}$) *Let $\mathscr{O}$ be the class of Borel sets A of $\mathbb{R}^d$ with countably $\mathscr{H}^{d-1}$-rectifiable and bounded topological boundary, such that*

$$\eta(B_r(x)) \geq \gamma r^{d-1} \qquad \forall x \in \partial A,\ \forall r \in (0,1) \qquad (10)$$

holds for some $\gamma > 0$ and some probability measure η in $\mathbb{R}^d$ absolutely continuous with respect to $\mathscr{H}^{d-1}$.

Theorem 3 *(Villa, 2009a) The class $\mathscr{O}$ is stable under finite unions and any $A \in \mathscr{O}$ admits outer Minkowski content, given by*

$$\mathscr{SM}(A) = \mathscr{H}^{d-1}(A^{1/2}) + 2\mathscr{H}^{d-1}(\partial A \cap A^0).$$

Remark 4 The set of points

$$\partial^* A := \mathbb{R}^d \setminus (A^0 \cup A^1)$$

where the density of A is neither 0 nor 1 is called *essential boundary* of A. It is proved (*cf.* Ambrosio *et al.*, 2000) that all the sets A^t are Borel sets, and that

$$\mathscr{H}^{d-1}(\partial^* A \cap B) = \mathscr{H}^{d-1}(A^{1/2} \cap B)$$

for all $B \in \mathscr{B}_{\mathbb{R}^d}$.

Remark 5 (The class $\mathscr{O}'$) In Villa (2009a) it is also proved that the same conclusions of the above theorem hold for a class of Borel subsets of $\mathbb{R}^d$, named $\mathscr{O}'$, defined similarly to $\mathscr{O}$ by replacing the condition of absolutely continuity of η with the assumption that $\mathscr{M}^{d-1}(\partial A) = \mathscr{H}^{d-1}(\partial A)$; then it follows that this class of sets contains all Borel sets with $(d-1)$-rectifiable boundary (and so finite unions of sets with positive reach or with Lipschitz boundary, in particular).

We conclude this section by the following theorem proved in Villa (2010), which generalizes Theorem 3 providing sufficient conditions for the existence of the limit in Eq. 8.

Theorem 6 *Let μ be a positive measure in $\mathbb{R}^d$ absolutely continuous with respect to $\mathscr{H}^d$ with locally bounded density f. Let A belong to $\mathscr{O}$ (or $\mathscr{O}'$). If $\mathscr{H}^{d-1}(\operatorname{disc} f) = 0$, then*

$$\lim_{r\downarrow 0} \frac{\mu(A_{\oplus r} \setminus A)}{r} = \int_{\partial^* A} f(x)\mathscr{H}^{d-1}(\mathrm{d}x) + 2\int_{\partial A \cap A^0} f(x)\mathscr{H}^{d-1}(\mathrm{d}x).$$

EXISTENCE RESULTS FOR THE SPECIFIC AREA

For details of the proofs of the results presented in this section, and for further remarks and comments, we refer to Villa (2010).

Let us consider a Boolean model Ξ in $\mathbb{R}^d$ with the notation introduced above. In order to provide sufficient conditions on Ξ such that its specific area exists finite, it is intuitive by Eq. 7 and Theorem 6 why the following Assumptions implies Eq. 12 for $\sigma_\Xi(x)$ in Proposition 7.

In the assumption (A1) below the notation $\Theta \supseteq \partial Z_0$ with $\mathbb{E}_Q[\mathscr{H}^{d-1}(\Theta)] < \infty$ means that for any $s \in \mathbf{K}$ there exists a closed set $\Theta(s) \supseteq Z_0(s)$ such that $\int_{\mathbf{K}} \mathscr{H}^{d-1}\Theta(s)Q(\mathrm{d}s) < \infty$.

Assumptions: (A1) ∂Z_0 is countably $\mathscr{H}^{d-1}$-rectifiable and compact, and such that there exist

$\gamma > 0$ and a random closed set $\Theta \supseteq \partial Z_0$ with $\mathbb{E}_Q[\mathscr{H}^{d-1}(\Theta)] < \infty$ such that, for Q-a.e. $s \in \mathbf{K}$,

$$\mathscr{H}^{d-1}(\Theta(s) \cap B_r(x)) \geq \gamma r^{d-1} \ \forall x \in \partial Z_0(s), \ \forall r \in (0,1).$$

(A2) $\mathscr{H}^{d-1}(\operatorname{disc} f) = 0$ and f is locally bounded such that for any compact set $K \subset \mathbb{R}^d$

$$\sup_{y \in K_{\oplus \operatorname{diam}(Z_0)}} f(y) \leq \xi_K \,, \tag{11}$$

for some random variable ξ_K with $\mathbb{E}_Q[\mathscr{H}^{d-1}(\Theta)\xi_K] < \infty$.

Proposition 7 *Let Ξ be a Boolean model as in the Assumptions. Then*

$$\begin{aligned}\sigma_\Xi(x) = \exp\Big\{ -\mathbb{E}_Q\Big[\int_{Z^x} f(y)\mathrm{d}y\Big]\Big\} \\ \cdot \mathbb{E}_Q\Big[\int_{\partial^* Z^x} f(y)\mathscr{H}^{d-1}(\mathrm{d}y) \\ + 2\int_{\partial Z^x \cap (Z^x)^0} f(y)\mathscr{H}^{d-1}(\mathrm{d}y)\Big] \,, \end{aligned} \tag{12}$$

for all $x \in \mathbb{R}^d$.

Remark 8 The assumption (A1) is often fulfilled with $\Theta = \partial Z_0$ or $\Theta = \partial Z_0 \cup \widetilde{A}$ for some sufficiently regular random closed set $\tilde{A}$. As a matter of fact, it can be seen as the stochastic version of Eq. 10, which, in many applications, is satisfied with $\eta(\cdot) = \mathscr{H}^n(\widetilde{A} \cap \cdot)$ for some closed set $\widetilde{A} \supseteq A$, as proved in (Ambrosio *et al.*, 2000, p. 111) (see also Ambrosio *et al.*, 2008).

The other integrability assumptions in (A1) and (A2) are just technical assumptions in order to exchange limit and integral in Eq. 7. We may notice that if f is bounded, then ξ_K is constant for any K, and so $\mathbb{E}_Q[\mathscr{H}^{d-1}(\Theta)\xi_K]$ is finite by (A1).

Having now an explicit formula for the specific area of inhomogeneous Boolean models as in the Assumptions, we may ask when σ_Ξ coincides with $\lambda_{\partial\Xi}$, the *mean surface density of* Ξ. The next theorem shows that, without any further regularity assumption on Z_0, σ_Ξ may differ form the mean surface density $\lambda_{\partial\Xi}$ of Ξ, in general.

Theorem 9 *If Ξ is a Boolean model as in the Assumptions satisfying* Eq. 6, *then*

$$\sigma_\Xi(x) = \lambda_{\partial^*\Xi}(x) + 2\lambda_{\partial\Xi\cap\Xi^0}(x)$$

for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$, where $\lambda_{\partial^*\Xi}$ and $\lambda_{\partial\Xi\cap\Xi^0}$ are the densities of $\mathbb{E}[\mathscr{H}^{d-1}_{|\partial^*\Xi}]$ and $E[\mathscr{H}^{d-1}_{|\partial\Xi\cap\Xi^0}]$, respectively.

Remark 10 By the proof of Theorem 9 it follows in particular that Ξ admits *local mean outer Minkowski content, i.e.*, for any Borel set $B \subset \mathbb{R}^d$ such that $\mathbb{E}[\mathscr{H}^{d-1}(\partial\Theta \cap \partial B)] = 0$ it holds

$$\begin{aligned}\lim_{r\downarrow 0} \frac{\mathbb{E}[\mathscr{H}^d((\Xi_{\oplus r} \setminus \Xi) \cap B)]}{r} \\ = \mathbb{E}[\mathscr{H}^{d-1}_{|\Xi^{1/2}}(B)] + 2\mathbb{E}[\mathscr{H}^{d-1}_{|\partial\Xi\cap\Xi^0}(B)]. \end{aligned} \tag{13}$$

We point out that the Assumptions, which guarantee the equation above, are quite general (see also Remark 8); the random sets which don't admit local mean outer Minkowski content could be considered as "pathological" sets.

Theorem 9 tells us that, in general, it may well happen that the specific area does not coincide with the mean surface density. We mentioned in the Introduction that a problem of interest in image analysis is the estimation of the mean surface density of random sets, and that the 2-D box approximations of points in $\mathbb{R}^2$ by pixels in computer graphics suggests the use of σ_Ξ for the estimation of the mean surface density. Therefore we provide now sufficient conditions on Ξ such that $\sigma_\Xi(x) = \lambda_{\partial\Xi}$ for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$. The Assumptions above on Ξ imply that $\mathbb{E}[\mathscr{H}^{d-1}(\partial\Xi)] < \infty$ $\mathbb{P}$-a.s., so that its mean surface density can be decomposed as follows:

$$\lambda_{\partial\Xi} = \lambda_{\partial\Xi\cap\Xi^0} + \lambda_{\partial^*\Xi} + \lambda_{\partial\Xi\cap\Xi^1} \,.$$

Then, it follows that $\sigma_\Xi(x) = \lambda_{\partial\Xi}(x) = \lambda_{\partial^*\Xi}(x)$ for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$ if $\mathbb{E}[\mathscr{H}^{d-1}_{|\partial\Xi\cap(\Xi^0\cup\Xi^1)}] = 0$.

The following proposition provides a sufficient regularity condition on the typical grain in order to have $\sigma_\Xi = \lambda_{\partial\Xi}$.

Proposition 11 *Let Ξ be a Boolean model as in the Assumptions satisfying* Eq. 6, *such that*

$$\mathbb{E}_Q[\mathscr{H}^{d-1}(\partial^* Z_0)] = \mathbb{E}_Q[\mathscr{H}^{d-1}(\partial Z_0)] \,. \tag{14}$$

Then

$$\begin{aligned}\sigma_\Xi(x) = \exp\Big\{ -\mathbb{E}_Q\Big[\int_{Z^x} f(y)\mathrm{d}y\Big]\Big\} \cdot \\ \cdot \mathbb{E}_Q\Big[\int_{\partial Z^x} f(y)\mathscr{H}^{d-1}(\mathrm{d}y)\Big] = \lambda_{\partial\Xi}(x) \end{aligned} \tag{15}$$

for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$.

We point out that there exist (although a bit "pathological") subsets of $\mathbb{R}^d$ such that the $\mathscr{H}^{d-1}$-measure of their boundary differs from that one of their essential boundary (*cf.* Ambrosio *et al.*, 2008, Sect. 5); so, by choosing a set of this kind as deterministic typical grain, the condition Eq. 14 is not satisfied, and $\lambda_{\partial\Xi} \neq \sigma_\Xi$ for the corresponding Boolean model Ξ. Furthermore, it can be shown (Ambrosio *et al.*, 2008) that the class of subsets A of $\mathbb{R}^d$ such that $\mathscr{H}^{d-1}(\partial A) = \mathscr{H}^{d-1}(\partial^* A)$ is not stable under finite unions, that is, even if two subsets A_1 and A_2 of $\mathbb{R}^d$ are such that $\mathscr{H}^{d-1}(\partial A_i) = \mathscr{H}^{d-1}(\partial^* A_i)$, $i = 1,2$, it may well happen that $\mathscr{H}^{d-1}(\partial(A_1 \cup A_2)) \neq \mathscr{H}^{d-1}(\partial^*(A_1 \cup A_2))$. Noticing that, by condition Eq. 6, Ξ is almost surely a locally finite union of grains, Proposition 11 tells us that a sort of stability under finite unions for the expected value holds for Boolean models, because Eq. 14 implies that $\mathbb{E}[\mathscr{H}^{d-1}(\partial\Xi)] = \mathbb{E}[\mathscr{H}^{d-1}(\partial^*\Xi)]$.

Since $\partial^* Z_0 \subset \partial Z_0$, condition Eq. 14 is satisfied if $\mathscr{H}^{d-1}(\partial^* Z_0) = \mathscr{H}^{d-1}(\partial Z_0)$ $\mathbb{P}$-a.s.; we remind that any compact subset A of $\mathbb{R}^d$ with Lipschitz boundary satisfies $\mathscr{H}^{d-1}(\partial^* A) = \mathscr{H}^{d-1}(\partial A)$, and that the same holds also for a certain class of compact sets with positive reach, containing, in particular, all the d-dimensional convex bodies (*cf.* Ambrosio *et al.*, 2008; Villa, 2010). Therefore we may claim that $\sigma_\Xi = \lambda_{\partial\Xi}$ $\mathscr{H}^d$-a.e. for all Boolean models with typical grain Z_0 satisfying regularity conditions of this kind, and so, starting by the definition of σ_Ξ, estimators for the mean surface density $\lambda_{\partial\Xi}$ of Ξ can be provided (see section below). We also mention that examples of applications of the above results to Boolean models of balls and segments are provided in Villa (2010, Sect. 5.3).

Remark 12 It is clear that Proposition 7 and Proposition 11 can be easily specified for the particular cases in which Ξ is stationary (in such case f is constant, say c, and so σ_Ξ is constant as well, equal to $e^{-c\mathbb{E}_Q[\mathscr{H}^d(Z_0)]} c\mathbb{E}_Q[\mathscr{M}(Z_0)]$), or Ξ has deterministic typical grain (in such case it is sufficient to assume that Z_0 is a compact set in $\mathscr{O}$, or $\mathscr{O}'$, and that the intensity f is locally bounded such that $\mathscr{H}^{d-1}(\mathrm{disc}\, f) = 0$).

Let us also notice that $\sigma_\Xi(x)$ might be not trivial for $(d-1)$-dimensional Boolean models, as well; it can be shown that under similar assumptions to (A1) and (A2),

$$\sigma_\Xi(x) = 2\lambda_{\partial\Xi}(x) = \mathbb{E}_Q\Big[\int_{Z^x} f(y)\mathscr{H}^{d-1}(\mathrm{d}y)\Big],$$

$\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$.

We refer to Villa (2010, Sect. 5) (see also Villa, 2009b) for a more detailed discussion of these particular cases.

SOME APPLICATIONS

In this section we shall discuss a couple of applications of the results above; in particular, applications to birth-and-growth processes which are of interest in several application areas as Material Science, Biology, etc.

An estimator of the mean surface density

It could be of interest in Image Analysis to estimate the mean surface density of random closed sets by means of their Minkowski enlargements, because of the computer graphics representation of lower dimensional sets in terms of pixels. By repeating the same argument in Villa (2010, Prop. 6.1), where estimators for the mean density of lower dimensional random closed sets are provided, in Villa (2009b) a natural estimator for $\lambda_{\partial\Xi}$ has been introduced, being Ξ a Boolean model as in the assumptions of Proposition 11, and so

$$\lambda_{\partial\Xi}(x) = \lim_{r\downarrow 0} \frac{\mathbb{P}(x \in \Xi_{\oplus r} \setminus \Xi)}{r},$$

for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$.

Given an i.i.d. random sample $\Xi_1, \ldots, \Xi_N$ of Ξ, the equation above suggests the following estimator of $\lambda_{\partial\Xi}(x)$:

$$\begin{aligned}\widehat{\lambda}^N_{\partial\Xi}(x) &:= \frac{\sum_{i=1}^N \mathbf{1}_{\Xi_{i\oplus R_N}\setminus\Xi_i}(x)}{NR_N} \\ &= \frac{\sum_{i=1}^N (\mathbf{1}_{\Xi_i\cap B_{R_N}(x)\neq\emptyset} - \mathbf{1}_{\Xi_i\cap\{x\}\neq\emptyset})}{NR_N},\end{aligned}$$

with R_N such that

$$\lim_{N\to\infty} R_N = 0 \quad \text{and} \quad \lim_{N\to\infty} NR_N = \infty.$$

It is not difficult to check that $\widehat{\lambda}^N_{\partial\Xi}(x)$ is an asymptotically unbiased and consistent estimator of $\lambda_{\partial\Xi}(x)$ for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$.

Remark 13 We also mention that the outer Minkowski content concept as measure of the boundary of deterministic sets, has been recently used to provide estimators for the $\mathscr{H}^{d-1}$-measure of the boundary of compact subsets of $\mathbb{R}^d$ in Armendáriz *et al.* (2009). Therefore, open problems related to the estimation of the mean boundary measure of random closed sets might be the following: the study of the statistical properties of the estimator above, and the generalization to random closed sets of the results in Armendáriz *et al.* (2009).

Applications to birth-and-growth processes

Many real phenomena as crystallization processes (Vandermeer *et al.*, 1991; Capasso, 2003, and references therein), tumor growth (Anderson, 2003), *etc.*, may be modelled as evolving random closed sets, that is as full dimensional time dependent random closed sets. In particular, any real situation in which *nuclei* are born in time and are located in space randomly, and each nucleus generates a *grain* evolving in time according to a given growth law, may be modelled as space-time structured stochastic *birth-and-growth processes*. Such a process is described by a marked point process, say $N := \{(T_j, X_j)\}_{j\in\mathbb{N}}$, modelling births at random times $T_j \in \mathbb{R}_+$ and related random spatial locations (*nuclei*) $X_j \in \mathbb{R}^d$ ($d \geq 2$), and by a growth model.

Denoting by $\Theta^t_{t_0}(x_0)$ the grain born at some time t_0 and location x_0, and grown up to time t, under regularity assumptions on the birth and growth model, the union set

$$\Theta^t = \bigcup_{n:T_n\leq t} \Theta^t_{T_n}(X_n)\,,$$

of such grains at time t is then a locally finite union of random closed sets. The family $\{\Theta^t\}_t$ is called *birth-and-growth process*, and a problem of interest in many applications is to find evolution equations for the mean volume density, in terms of the mean surface density of Θ^t.

A lot of papers on this subject can be found in literature; in particular, the case in which N is given by a marked Poisson process has been studied extensively (Kolmogorov, 1937; Burger *et al.*, 2006, and references therein).

In this section we want to point out the role of the existence of the (local) mean outer Minkowski content and the specific area of Θ^t in the study of the mean volume and surface densities of Θ^t. We recall that the *mean volume density of* Θ^t, usually denoted by $V_V(t,\cdot)$, is the Radon-Nikodym derivative of the measure $\mathbb{E}[\mathscr{H}^d_{|\Theta^t}]$, *i.e.*,

$$\mathbb{E}[\mathscr{H}^d(\Theta^t \cap B)] = \int_B V_V(t,x)\mathrm{d}x, \qquad \forall B \in \mathscr{B}_{\mathbb{R}^d};$$

similarly, the mean surface density of Θ^t, usually denoted by $S_V(t,\cdot)$, is the Radon-Nikodym derivative of the measure $\mathbb{E}[\mathscr{H}^{d-1}_{|\partial\Theta^t}]$, *i.e.*,

$$\mathbb{E}[\mathscr{H}^{d-1}(\partial\Theta^t \cap B)] = \int_B S_V(t,x)\mathrm{d}x, \qquad \forall B \in \mathscr{B}_{\mathbb{R}^d}.$$

Let us notice that

$$V_V(t,x) = \mathbb{P}(x \in \Theta^t),$$

whereas $S_V(t,x) = \lambda_{\partial\Theta^t}(x)$, according to our previous notation.

We assume here that the nucleation process N is a Poisson marked point process (equivalently, a Poisson point process in $\mathbb{R}_+ \times \mathbb{R}^d$), with intensity measure Λ of the type

$$\Lambda(\mathrm{d}(t,x)) = \alpha(t,x)\mathrm{d}t\mathrm{d}x$$

and satisfying the usual condition (Eq. 6).

Models of volume growth have been studied extensively, since the pioneering work by Kolmogorov (1937). Clearly, different kinds of growth model give rise to different kinds of processes $\{\Theta^t\}_t$.

Let us first consider the *normal growth model* (Capasso and Villa, 2007, and references therein), according to which, at $\mathscr{H}^{d-1}$-almost every point of the actual grain surface at time t (*i.e.*, at $\mathscr{H}^{d-1}$-a.e. $x \in \partial\Theta^t_{T_j}(X_j)$), growth occurs with a given strictly positive normal velocity

$$v(t,x) = G(t,x)n(t,x), \tag{16}$$

where $G(t,x)$ is a given deterministic growth field, and $n(t,x)$ is the unit outer normal at point $x \in \partial\Theta^t_{T_0}(X_0)$. We assume that

$$0 < g_0 \leq G(t,x) \leq G_0 < \infty \qquad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

for some $g_0, G_0 \in \mathbb{R}$, and that $G(t,x)$ is sufficiently regular such that the evolution problem given by (16) for the growth front $\partial\Theta^t_{t_0}(x)$ is well posed. It follows that for any fixed $t \in \mathbb{R}_+$, the topological boundary of each grain is a random closed set with locally finite $\mathscr{H}^{d-1}$-measure $\mathbb{P}$-almost surely (Burger, 2002). This and the assumption that Θ^t is a locally finite union of grains imply that $\mathbb{P}$-almost surely $\mathscr{H}^{d-1}(\partial\Theta^t) < \infty$, and so that Θ^t has density 1 or 1/2 at $\mathscr{H}^{d-1}$-a.e. point of its boundary $\mathbb{P}$-almost surely. We can claim the assumptions on the growth model imply that $\mathscr{H}^{d-1}((\Theta^t)^0) = 0$ $\mathbb{P}$-a.s.

For the normal growth model above with Poissonian nucleation process, recent results (Capasso and Villa, 2008, Prop. 25; Capasso and Villa, 2007, Prop. 19; Villa, 2008, Prop. 2.3) show that, if Θ^t admits local mean outer Minkowski content for all $t > 0$ and if the random variable $T(x)$ defined by

$$T(x) := \min\{t > 0 : x \in \Theta^t\}$$

is continuous with density, then the following evolution equation holds in weak form:

$$\frac{\partial}{\partial t} V_V(t,x) = G(t,x)S_V(t,x)\,. \tag{17}$$

The random variable $T(x)$ is called *time of capture of point x*, and its probability density function, say $p_{T(x)}$, is just the partial derivative of V_V with respect to t (Capasso and Villa, 2008, Eq. 14). By repeating the same arguments of Proposition 25 in Capasso and Villa (2008) and assuming that Θ^t admits local mean outer Minkowski content, we can state the following slightly more general result. Note that the assumption that the boundary of Θ^t is sufficiently regular so that Θ^t admits local mean outer Minkowski content, is not so restrictive for applications (see Remark 10).

Proposition 14 *Let $\{\Theta^t\}_t$ be a birth-and-growth process with normal growth model as above, and sufficiently regular such that Θ^t admits local mean outer Minkowski content* Eq. 13. *Then the following equation holds in weak form*

$$\frac{\partial}{\partial t}V_V(t,x) = G(t,x)\lambda_{\partial^*\Theta^t}(x). \tag{18}$$

Proof. (Sketch)
The assumptions on the nucleation process and on the growth model imply (Villa, 2008, Theorem 3.3) that $T(x)$ admits probability density function $p_{T(x)}$. By Eqs. 17 and 24 in Capasso and Villa (2008), the following chain of equalities hold

$$\begin{aligned}&\frac{\partial}{\partial t}V_V(t,x) = p_{T(x)} = h(t,x)\mathbb{P}(x \notin \Theta^t)\\ &= G(t,x)\frac{\partial}{\partial r}\mathbb{P}(x \in \Theta^t_{\oplus r} \setminus \Theta^t)_{|r=0} = G(t,x)\sigma_{\Theta^t}(x),\end{aligned}$$

where $h(t,x)$ is the so-called *hazard function* associated to the point x at time t (Capasso and Villa, 2008, Def. 21). Eq. 13 can be written equivalently (in weak form) as

$$\begin{aligned}&\frac{\partial}{\partial r}\mathbb{P}(x \in \Theta^t_{\oplus r} \setminus \Theta^t)_{|r=0}\\ &\qquad = \lambda_{\partial^*\Theta^t}(x) + 2\lambda_{\partial\Theta^t\cap(\Theta^t)^0}(x) = \lambda_{\partial^*\Theta^t}(x),\end{aligned}$$

where the last equation follows by having observed that $\mathscr{H}^{d-1}((\Theta^t)^0) = 0$ $\mathbb{P}$-a.s., and so we obtain Eq. 18. □

Remark 15 Let us notice that the following decomposition of the mean surface density holds

$$\lambda_{\partial\Theta^t} = \lambda_{\partial^*\Theta^t} + \lambda_{\partial\Theta^t\cap(\Theta^t)^0} + \lambda_{\partial\Theta^t\cap(\Theta^t)^1}.$$

As a consequence, since $\lambda_{\partial\Theta^t\cap(\Theta^t)^0} = 0$, Eq. 18 can be equivalently written

$$\frac{\partial}{\partial t}V_V(t,x) = G(t,x)(\lambda_{\partial\Theta^t}(x) - \lambda_{\partial\Theta^t\cap(\Theta^t)^1});$$

assuming now that for a.e. $t \in \mathbb{R}_+$ $\mathbb{E}[\mathscr{H}^{d-1}_{|\partial\Theta^t\cap(\partial\Theta^t)^1}] = 0$, then for a.e. $t \in \mathbb{R}_+$ $\lambda_{\partial^*\Theta^t} = \lambda_{\partial\Theta^t}$ and Eq. 18 coincides with Eq. 17.

Let us now consider a different growth model. We assume now that each grain grows with the same growth law of a "typical grain" with its nucleus at the origin; in other words, the grain $\Theta^t_s(x)$ born at point x at time s and grown up to time t can be seen as the translation in x of a grain born in 0 at the same birth-time. By using the notation introduced for Boolean models, given the nucleation process $\{(T_n, X_n)\}_n$, where now the marks are the birth times T_n associated to the spatial locations X_n of the nuclei, we can model the crystallized region Θ^t at time t as the Boolean model

$$\Theta^t = \bigcup_{(T_n,X_n)\in N:T_n\le t} X_n + Z_0(T_n) = X_n + \Theta^t_{T_n}(0)\,. \tag{19}$$

Notice that such a model may be used to describe real situations in which the growth model is assumed to be such that the shape of the grains is preserved during the process (*e.g.*, prolates and spheroids in Vandermeer *et al.*, 1991). Clearly, the particular case of spherical growth (which corresponds to the previous growth model with $G = G(t)$), can be described also by means of the present model with $Z_0(T_n) = B_{R(T_n,t)}(0)$, where $R(T_n,t) = \int_{T_n}^t G(s)\,\mathrm{d}s$.

Thus, if the Boolean model Θ^t satisfies the Assumptions, then by Proposition 7 it follows that the specific area σ_{Θ^t} exists given by Eq. 12; if furthermore condition Eq. 14 is satisfied (usually true in many real applications), then an explicit formula for the mean surface density (equivalently S_V) can be obtained by Eq. 15. In order to apply such formula to the birth-and-growth process in Eq. 19, note that the mark distribution Q represents now the probability law of the birth time of the typical grain; for instance, if the nucleation rate is constant in time, then, for any fixed t, Q is the uniform distribution in $[0,t]$. More in general, for any fixed t, since the nucleation takes place during the time interval $[0,t]$, Eq. 15 will be of the type

$$\begin{aligned}\sigma_{\Theta^t}(x) = S_V(t,x) = \exp\Big\{-\int_0^t\int_{x-\Theta^t_s(0)} f(y)\mathrm{d}y Q\mathrm{d}s\Big\}\cdot\\ \cdot\int_0^t\int_{\partial(x-\Theta^t_s(0))} f(y)\mathscr{H}^{d-1}(\dot{\mathrm{y}})Q(\mathrm{d}s)\,.\end{aligned}$$

Remark 16 Whenever the nucleation is homogeneous in space, then f is constant and the above formula

simplify as follows

$$S_V(t,x) = \exp\Big\{ - \int_0^t c\mathscr{H}^d(\Theta_s^t(0))Q\mathrm{d}s \Big\} \cdot \cdot \int_0^t \mathscr{H}^{d-1}(\partial(\Theta_s^t(0))Q(\mathrm{d}s),$$

in accordance with Vandermeer *et al.* (1991, Eqs. 9-12)

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