ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 23 (2023) #P1.07
https://doi.org/10.26493/1855-3974.2144.9e3
(Also available at http://amc-journal.eu)

A note on the neighbour-distinguishing index of digraphs

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Received 9 October 2019, accepted 30 January 2022, published online 20 October 2022

Abstract

In this note, we introduce and study a new version of neighbour-distinguishing arc-colourings of digraphs. An arc-colouring γ of a digraph D is proper if no two arcs with the same head or with the same tail are assigned the same colour. For each vertex u of D, we denote by $S_\gamma^-(u)$ and $S_\gamma^+(u)$ the sets of colours that appear on the incoming arcs and on the outgoing arcs of u, respectively. An arc colouring γ of D is *neighbour-distinguishing* if, for every two adjacent vertices u and v of D, the ordered pairs $(S_\gamma^-(u), S_\gamma^+(u))$ and $(S_\gamma^-(v), S_\gamma^+(v))$ are distinct. The neighbour-distinguishing index of D is then the smallest number of colours needed for a neighbour-distinguishing arc-colouring of D.

We prove upper bounds on the neighbour-distinguishing index of various classes of digraphs.

Keywords: Digraph, arc-colouring, neighbour-distinguishing arc-colouring.

Math. Subj. Class. (2020): 05C15, 05C20

1 Introduction

A proper edge-colouring of a graph G is *vertex-distinguishing* if, for every two vertices u and v of G, the sets of colours that appear on the edges incident with u and v are distinct. Vertex-distinguishing proper edge-colourings of graphs were independently introduced by Burris and Schelp [2], and by Černy, Horňák and Soták [5]. Requiring only adjacent vertices to be distinguished led to the notion of *neighbour-distinguishing* edge-colourings, considered in [1, 3, 7].

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Vertex-distinguishing arc-colourings of digraphs have been recently introduced and studied by Li, Bai, He and Sun [4]. An arc-colouring of a digraph is proper if no two arcs with the same head or with the same tail are assigned the same colour. Such an arc-colouring is *vertex-distinguishing* if, for every two vertices u and v of G,

(i) the sets $S^-(u)$ and $S^-(v)$ of colours that appear on the incoming arcs of u and v, respectively, are distinct, and

(ii) the sets $S^+(u)$ and $S^+(v)$ of colours that appear on the outgoing arcs of u and v, respectively, are distinct.

In this paper, we introduce and study a neighbour-distinguishing version of arc-colourings of digraphs, using a slightly different distinction criteria: two neighbours u and v are distinguished whenever $S^-(u) \neq S^-(v)$ or $S^+(u) \neq S^+(v)$.

Definitions and notation are introduced in the next section. We prove a general upper bound on the neighbour-distinguishing index of a digraph in Section 3, and study various classes of digraphs in Section 4. Concluding remarks are given in Section 5.

2 Definitions and notation

All digraphs we consider are without loops and multiple arcs. For a digraph D, we denote by $V(D)$ and $A(D)$ its sets of vertices and arcs, respectively. The *underlying graph* of D, denoted $\mathrm{und}(D)$, is the simple undirected graph obtained from D by replacing each arc uv (or each pair of arcs uv, vu) by the edge uv.

If uv is an arc of a digraph D, u is the *tail* and v is the *head* of uv. For every vertex u of D, we denote by $N_D^+(u)$ and $N_D^-(u)$ the sets of *out-neighbours* and *in-neighbours* of u, respectively. Moreover, we denote by $d_D^+(u) = |N_D^+(u)|$ and $d_D^-(u) = |N_D^-(u)|$ the *outdegree* and *indegree* of u, respectively, and by $d_D(u) = d_D^+(u) + d_D^-(u)$ the *degree* of u.

For a digraph D, we denote by $\delta^+(D)$, $\delta^-(D)$, $\Delta^+(D)$ and $\Delta^-(D)$ the minimum outdegree, minimum indegree, maximum outdegree and maximum indegree of D, respectively. Moreover, we let

$$\Delta^*(D) = \max\{\Delta^+(D),\ \Delta^-(D)\}.$$

A (proper) *k-arc-colouring* of a digraph D is a mapping γ from $V(D)$ to a set of k colours (usually $\{1, \dots, k\}$) such that, for every vertex u,

(i) any two arcs with head u are assigned distinct colours, and

(ii) any two arcs with tail u are assigned distinct colours.

Note here that two consecutive arcs vu and uw, v and w not necessarily distinct, may be assigned the same colour. The *chromatic index* $\chi'(D)$ of a digraph D is then the smallest number k for which D admits a k-arc-colouring.

The following fact is well-known (see e.g. [4, 6, 8]).

Proposition 2.1. *For every digraph D, $\chi'(D) = \Delta^*(D)$.*

For every vertex u of a digraph D, and every arc-colouring γ of D, we denote by $S_\gamma^+(u)$ and $S_\gamma^-(u)$ the sets of colours assigned by γ to the outgoing and incoming arcs of u, respectively. From the definition of an arc-colouring, we get $d_D^+(u) = |S_\gamma^+(u)|$ and $d_D^-(u) = |S_\gamma^-(u)|$ for every vertex u.

We say that two vertices u and v of a digraph D are *distinguished* by an arc-colouring γ of D, if $(S_\gamma^+(u), S_\gamma^-(u)) \neq (S_\gamma^+(v), S_\gamma^-(v))$. Note that we consider here ordered pairs, so that $(A, B) \neq (B, A)$ whenever $A \neq B$. Note also that if u and v are such that $d_D^+(u) \neq d_D^+(v)$ or $d_D^-(u) \neq d_D^-(v)$, which happens in particular if $d_D(u) \neq d_D(v)$, then they are distinguished by every arc-colouring of D. We will write $u \nsim_\gamma v$ if u and v are distinguished by γ and $u \sim_\gamma v$ otherwise.

A k-arc-colouring γ of a digraph D is *neighbour-distinguishing* if $u \nsim_\gamma v$ for every arc $uv \in A(D)$. Such an arc-colouring will be called an *nd-arc-colouring* for short. The *neighbour-distinguishing index* $\mathrm{ndi}(D)$ of a digraph D is then the smallest number of colours required for an nd-arc-colouring of D.

The following lower bound is easy to establish.

Proposition 2.2. *For every digraph D, $\mathrm{ndi}(D) \geq \chi'(D) = \Delta^*(D)$. Moreover, if there are two vertices u and v in D with $d_D^+(u) = d_D^+(v) = d_D^-(u) = d_D^-(v) = \Delta^*(D)$, then* $\mathrm{ndi}(D) \geq \Delta^*(D) + 1$.

Proof. The first statement follows from the definitions. For the second statement, observe that $S_\gamma^+(u) = S_\gamma^+(v) = S_\gamma^-(u) = S_\gamma^-(v) = \{1, \dots, \Delta^*(D)\}$ for any two such vertices u and v and any $\Delta^*(D)$-arc-colouring γ of D. □

3 A general upper bound

If D is an *oriented graph*, that is, a digraph with no opposite arcs, then every proper edge-colouring φ of $\mathrm{und}(D)$ is an nd-arc-colouring of D since, for every arc uv in D, $\varphi(uv) \in S_\varphi^+(u)$ and $\varphi(uv) \notin S_\varphi^+(v)$, which implies $u \nsim_\varphi v$. Hence, we get the following upper bound for oriented graphs, thanks to classical Vizing's bound.

Proposition 3.1. *If D is an oriented graph, then*

$$\mathrm{ndi}(D) \leq \chi'(\mathrm{und}(D)) \leq \Delta(\mathrm{und}(D)) + 1 \leq 2\Delta^*(D) + 2.$$

However, a proper edge-colouring of $\mathrm{und}(D)$ may produce an arc-colouring of D which is not neighbour-distinguishing when D contains opposite arcs. Consider for instance the digraph D given by $V(D) = \{a, b, c, d\}$ and $A(D) = \{ab, bc, cb, dc\}$. We then have $\mathrm{und}(D) = P_4$, the path of order 4, and thus $\chi'(\mathrm{und}(D)) = 2$. It is then not difficult to check that for any 2-edge-colouring φ of $\mathrm{und}(D)$, $S_\varphi^+(b) = S_\varphi^+(c)$ and $S_\varphi^-(b) = S_\varphi^-(c)$.

We will prove that the upper bound given in Proposition 3.1 can be decreased to $2\Delta^*(D)$, even when D contains opposite arcs. Recall that a digraph D is *k-regular* if $d_D^+(v) = d_D^-(v) = k$ for every vertex v of D. A *k-factor* in a digraph D is a spanning k-regular subdigraph of D. The following result is folklore.

Theorem 3.2. *Every k-regular digraph can be decomposed into k arc-disjoint 1-factors.*

We first determine the neighbour-distinguishing index of a 1-factor.

Proposition 3.3. *If D is a digraph with $d_D^+(u) = d_D^-(u) = 1$ for every vertex u of D, then* $\mathrm{ndi}(D) = 2$.

Proof. Such a digraph D is a disjoint union of directed cycles and any such cycle needs at least two colours to be neighbour-distinguished. An nd-arc-colouring of D using two colours can be obtained as follows. For a directed cycle of even length, use alternately colours 1 and 2. For a directed cycle of odd length, use the colour 2 on any two consecutive arcs, and then use alternately colours 1 and 2. The so-obtained 2-arc-colouring is clearly neighbour-distinguishing, so that $\mathrm{ndi}(D) = 2$. $\square$

We are now able to prove the following general upper bound on the neighbour-distinguishing index of a digraph.

Theorem 3.4. *For every digraph D, $\mathrm{ndi}(D) \leq 2\Delta^*(D)$.*

Proof. Let D' be any $\Delta^*(D)$-regular digraph containing D as a subdigraph. If D is not already regular, such a digraph can be obtained from D by adding new arcs, and maybe new vertices.

By Theorem 3.2, the digraph D' can be decomposed into $\Delta^*(D') = \Delta^*(D)$ arc-disjoint 1-factors, say $F_1, \dots, F_{\Delta^*(D)}$. By Proposition 3.3, we know that D' admits an nd-arc-colouring γ' using $2\Delta^*(D') = 2\Delta^*(D)$ colours. We claim that the restriction γ of γ' to $A(D)$ is also neighbour-distinguishing.

To see that, let uv be any arc of D, and let t and w be the two vertices such that the directed walk $tuvw$ belongs to a 1-factor F_i of D' for some i, $1 \leq i \leq \Delta^*(D)$. Note here that we may have $t = w$, or $w = u$ and $t = v$. If $\gamma(uv) \neq \gamma'(vw)$, then $\gamma(uv) \in S^+_\gamma(u)$ and $\gamma(uv) \notin S^+_\gamma(v)$. Similarly, if $\gamma'(tu) \neq \gamma(uv)$, then $\gamma(uv) \in S^-_\gamma(v)$ and $\gamma(uv) \notin S^-_\gamma(u)$. Since neither three consecutive arcs nor two opposite arcs in a walk of a 1-factor of D' are assigned the same colour by γ', we get that $u \not\sim_\gamma v$ for every arc uv of D, as required.

This completes the proof. $\square$

4 Neighbour-distinguishing index of some classes of digraphs

We study in this section the neighbour-distinguishing index of several classes of digraphs, namely complete symmetric digraphs, bipartite digraphs and digraphs whose underlying graph is k-chromatic, $k \geq 3$.

4.1 Complete symmetric digraphs

We denote by K^*_n the complete symmetric digraph of order n. Observe first that any proper edge-colouring ϵ of K_n induces an arc-colouring γ of K^*_n defined by $\gamma(uv) = \gamma(vu) = \epsilon(uv)$ for every edge uv of K_n. Moreover, since $S^+_\gamma(u) = S^-_\gamma(u) = S_\epsilon(u)$ for every vertex u, γ is neighbour-distinguishing whenever ϵ is neighbour-distinguishing. Using a result of Zhang, Liu and Wang (see Theorem 6 in [7]), we get that $\mathrm{ndi}(K^*_n) = \Delta^*(K^*_n) + 1 = n$ if n is odd, and $\mathrm{ndi}(K^*_n) \leq \Delta^*(K^*_n) + 2 = n + 1$ if n is even.

We prove that the bound in the even case can be decreased by one (we recall the proof of the odd case to be complete).

Theorem 4.1. *For every integer $n \geq 2$, $\mathrm{ndi}(K^*_n) = \Delta^*(K^*_n) + 1 = n$.*

Proof. Note first that we necessarily have $\mathrm{ndi}(K^*_n) \geq n$ for every $n \geq 2$ by Proposition 2.2. Let $V(K^*_n) = \{v_0, \dots, v_{n-1}\}$. If $n = 2$, we obviously have $\mathrm{ndi}(K^*_2) = |A(K^*_2)| = 2$ and the result follows. We can thus assume $n \geq 3$. We consider two cases, depending on the parity of n.

Suppose first that n is odd, and consider a partition of the set of edges of K_n into n disjoint maximal matchings, say $M_0, \dots, M_{n-1}$, such that for each i, $0 \le i \le n-1$, the matching M_i does not cover the vertex v_i. We define an n-arc-colouring γ of K_n^* (using the set of colours $\{0, \dots, n-1\}$) as follows. For every i and j, $0 \le i < j \le n-1$, we set $\gamma(v_iv_j) = \gamma(v_jv_i) = k$ if and only if the edge v_iv_j belongs to M_k. Observe now that for every vertex v_i, $0 \le i \le n-1$, the colour i is the unique colour that does not belong to $S_\gamma^+(v_i) \cup S_\gamma^-(v_i)$, since v_i is not covered by the matching M_i. This implies that γ is an nd-arc-colouring of K_n^*, and thus $\mathrm{ndi}(K_n^*) = n$, as required.

Suppose now that n is even. Let K' be the subgraph of K_n^* induced by the set of vertices $\{v_0, \dots, v_{n-2}\}$ and γ' be the $(n-1)$-arc-colouring of K' defined as above. We define an n-arc-colouring γ of K_n^* (using the set of colours $\{0, \dots, n-1\}$) as follows:

1. for every i and j, $0 \le i < j \le n-2$, $j \not\equiv i+1 \pmod{n-1}$, we set $\gamma(v_iv_j) = \gamma'(v_iv_j)$,
2. for every i, $0 \le i \le n-2$, we set $\gamma(v_iv_{i+1}) = n-1$ and $\gamma(v_{i+1}v_i) = \gamma'(v_{i+1}v_i)$ (subscripts are taken modulo $n-1$),
3. for every i, $0 \le i \le n-2$, we set $\gamma(v_{n-1}v_i) = \gamma'(v_{i-1}v_i)$ and $\gamma(v_iv_{n-1}) = \gamma'(v_{i+1}v_i)$.

Since the colour $n-1$ belongs to $S_\gamma^+(v_i) \cap S_\gamma^-(v_i)$ for every i, $0 \le i \le n-2$, and does not belong to $S_\gamma^+(v_{n-1}) \cup S_\gamma^-(v_{n-1})$, the vertex v_{n-1} is distinguished from every other vertex in K_n^*. Moreover, for every vertex v_i, $0 \le i \le n-2$,

$$S_\gamma^+(v_i) = S_{\gamma'}^+(v_i) \cup \{n-1\} \text{ and } S_\gamma^-(v_i) = S_{\gamma'}^-(v_i) \cup \{n-1\},$$

which implies that any two vertices v_i and v_j, $0 \le i < j \le n-2$, are distinguished since γ' is an nd-arc-colouring of K'. We thus get that γ is an nd-arc-colouring of K_n^*, and thus $\mathrm{ndi}(K_n^*) \le n$, as required.

This completes the proof. □

4.2 Bipartite digraphs

A digraph D is *bipartite* if its underlying graph is bipartite. In that case, $V(D) = X \cup Y$ with $X \cap Y = \emptyset$ and $A(D) \subseteq X \times Y \cup Y \times X$. We then have the following result.

Theorem 4.2. *If D is a bipartite digraph, then* $\mathrm{ndi}(D) \le \Delta^*(D) + 2$.

Proof. Let $V(D) = X \cup Y$ be the bipartition of $V(D)$ and γ be any (not necessarily neighbour-distinguishing) optimal arc-colouring of D using $\Delta^*(D)$ colours (such an arc-colouring exists by Proposition 2.1).

If γ is an nd-arc-colouring we are done. Otherwise, let $M_1 \subseteq A(D) \cap (X \times Y)$ be a maximal matching from X to Y. We define the arc-colouring γ_1 as follows:

$$\gamma_1(uv) = \Delta^*(D) + 1 \text{ if } uv \in M_1,\ \gamma_1(uv) = \gamma(uv) \text{ otherwise.}$$

Note that if uv is an arc such that u or v is (or both are) covered by M_1, then $u \nsim_{\gamma_1} v$ since the colour $\Delta^*(D) + 1$ appears in exactly one of the sets $S_{\gamma_1}^+(u)$ and $S_{\gamma_1}^+(v)$, or in exactly one of the sets $S_{\gamma_1}^-(u)$ and $S_{\gamma_1}^-(v)$.

If γ_1 is an nd-arc-colouring we are done. Otherwise, let $A^\sim$ be the set of arcs $uv \in A(D)$ with $u \sim_{\gamma_1} v$ and $M_2 \subseteq A^\sim \cap (Y \times X)$ be a maximal matching from Y to X of $A^\sim$. We define the arc-colouring γ_2 as follows:

$$\gamma_2(uv) = \Delta^*(D) + 2 \text{ if } uv \in M_2,\ \gamma_2(uv) = \gamma_1(uv) \text{ otherwise.}$$

Again, note that if uv is an arc such that u or v is (or both are) covered by M_2, then $u \nsim_{\gamma_2} v$. Moreover, since M_2 is a matching of $A^\sim$, pairs of vertices that were distinguished by γ_1 are still distinguished by γ_2.

Hence, every arc uv such that u and v were not distinguished by γ_1 are now distinguished by γ_2 which is thus an nd-arc-colouring of D using $\Delta^*(D) + 2$ colours. This concludes the proof. □

The upper bound given in Theorem 4.2 can be decreased when the underlying graph of D is a tree.

Theorem 4.3. *If D is a digraph whose underlying graph is a tree, then* $\mathrm{ndi}(D) \leq \Delta^*(D) + 1$.

Proof. The proof is by induction on the order n of D. The result clearly holds if $n \leq 2$. Let now D be a digraph of order $n \geq 3$, such that the underlying graph $\mathrm{und}(D)$ of D is a tree, and $P = v_1 \dots v_k$, $k \leq n$, be a path in $\mathrm{und}(D)$ with maximal length. By the induction hypothesis, there exists an nd-arc-colouring γ of $D - v_k$ using at most $\Delta^*(D - v_k) + 1$ colours. We will extend γ to an nd-arc-colouring of D using at most $\Delta^*(D) + 1$ colours.

If $\Delta^*(D) = \Delta^*(D - v_k) + 1$, we assign the new colour $\Delta^*(D) + 1$ to the at most two arcs incident with v_k so that the so-obtained arc-colouring is clearly neighbour-distinguishing.

Suppose now that $\Delta^*(D) = \Delta^*(D - v_k)$. If all neighbours of v_{k-1} are leaves, the underlying graph of D is a star. In that case, there is at most one arc linking v_{k-1} and v_k, and colouring this arc with any admissible colour produces an nd-arc-colouring of D. If the underlying graph of D is not a star, then, by the maximality of P, we get that v_{k-1} has exactly one neighbour which is not a leaf, namely v_{k-2}. This implies that the only conflict that might appear when colouring the arcs linking v_k and v_{k-1} is between v_{k-2} and v_{k-1} (recall that two neighbours with distinct indegree or outdegree are necessarily distinguished).

Since $d_D^+(v_{k-2}) \leq \Delta^*(D)$ and $d_D^-(v_{k-2}) \leq \Delta^*(D)$, there necessarily exist a colour a such that $S_\gamma^+(v_{k-2}) \neq S_\gamma^+(v_{k-1}) \cup \{a\}$, and a colour b such that $S_\gamma^-(v_{k-2}) \neq S_\gamma^-(v_{k-1}) \cup \{b\}$. Therefore, the at most two arcs incident with v_k can be coloured, using a and/or b, in such a way that the so-obtained arc-colouring is neighbour-distinguishing.

This completes the proof. □

4.3 Digraphs whose underlying graph is k-chromatic

Since the set of edges of every k-colourable graph can be partitionned in $\lceil \log k \rceil$ parts each inducing a bipartite graph (see e.g. Lemma 4.1 in [1]), Theorem 4.2 leads to the following general upper bound:

Corollary 4.4. *If D is a digraph whose underlying graph has chromatic number $k \geq 3$, then* $\mathrm{ndi}(D) \leq \Delta^*(D) + 2\lceil \log k \rceil$.

Proof. Starting from an optimal arc-colouring of D with $\Delta^*(D)$ colours, it suffices to use two new colours for each of the $\lceil \log k \rceil$ bipartite parts (obtained from any optimal vertex-colouring of the underlying graph of D), as shown in the proof of Theorem 4.2, in order to get an nd-arc-colouring of D. □

5 Discussion

In this note, we have introduced and studied a new version of neighbour-distinguishing arc-colourings of digraphs. Pursuing this line of research, we propose the following questions.

1. Is there any general upper bound on the neighbour-distinguishing index of symmetric digraphs?

2. Is there any general upper bound on the neighbour-distinguishing index of not necessarily symmetric complete digraphs?

3. Is there any general upper bound on the neighbour-distinguishing index of directed acyclic graphs?

4. The general bound given in Corollary 4.4 is certainly not optimal. In particular, is it possible to improve this bound for digraphs whose underlying graph is 3-colourable?

We finally propose the following conjecture.

Conjecture 5.1. *For every digraph D,* $\mathrm{ndi}(D) \leq \Delta^*(D) + 1$.

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References

[1] P. N. Balister, E. Győri, J. Lehel and R. H. Schelp, Adjacent vertex distinguishing edge-colorings, *SIAM J. Discrete Math.* **21** (2007), 237–250, doi:10.1137/S0895480102414107, https://doi.org/10.1137/S0895480102414107.

[2] A. C. Burris and R. H. Schelp, Vertex-distinguishing proper edge-colorings, *J. Graph Theory* **26** (1997), 73–82, doi:10.1002/(SICI)1097-0118(199710)26:2⟨73::AID-JGT2⟩3.0.CO;2-C, https://doi.org/10.1002/(SICI)1097-0118(199710)26:2<73::AID-JGT2>3.0.CO;2-C.

[3] K. Edwards, M. Horňák and M. Woźniak, On the neighbour-distinguishing index of a graph, *Graphs Comb.* **22** (2006), 341–350, doi:10.1007/s00373-006-0671-2, https://doi.org/10.1007/s00373-006-0671-2.

[4] H. Li, Y. Bai, W. He and Q. Sun, Vertex-distinguishing proper arc colorings of digraphs, *Discrete Appl. Math.* **209** (2016), 276–286, doi:10.1016/j.dam.2015.10.039, https://doi.org/10.1016/j.dam.2015.10.039.

[5] J. Černý, M. Horňák and R. Soták, Observability of a graph, *Math. Slovaca* **46** (1996), 21–31.

[6] D. B. West, *Introduction to Graph Theory*, Prentice Hall, Inc., Upper Saddle River, NJ, 1996.

[7] Z. Zhang, L. Liu and J. Wang, Adjacent strong edge coloring of graphs, *Appl. Math. Lett.* **15** (2002), 623–626, doi:10.1016/S0893-9659(02)80015-5, https://doi.org/10.1016/S0893-9659(02)80015-5.

[8] M. Zwonek, On arc-coloring of digraphs, *Opuscula Math.* **26** (2006), 185–195.