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Groups of Ree type in characteristic 3 acting on polytopes[*]

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Abstract

Every Ree group $\mathrm{R}(q)$, with $q \neq 3$ an odd power of 3, is the automorphism group of an abstract regular polytope, and any such polytope is necessarily a regular polyhedron (a map on a surface). However, an almost simple group G with $\mathrm{R}(q) < G \leq \mathsf{Aut}(\mathrm{R}(q))$ is not a C-group and therefore not the automorphism group of an abstract regular polytope of any rank.

Keywords: Abstract regular polytopes, string C-groups, small Ree groups, permutation groups.

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1   Introduction

Abstract polytopes are certain ranked partially ordered sets. A polytope is called "regular" if its automorphism group acts (simply) transitively on (maximal) flags. It is a natural question to try to classify all pairs $(\mathcal{P}, G)$, where $\mathcal{P}$ is a regular polytope and G is an automorphism group acting transitively on the flags of $\mathcal{P}$. An interesting subclass is constituted

by the pairs $(\mathcal{P}, G)$ with G almost simple, as then a lot of information is available about the maximal subgroups, centralizers of involutions, etc., of these groups, making classification possible for some of these families of groups. Potentially this could also lead to new presentations for these groups, as well as a better understanding of some families of such groups using geometry.

The study of polytopes arising from families of almost simple groups has received a lot of attention in recent years and has been very successful. Mazurov [23] and Nuzhin [25, 26, 27, 28] established that most finite simple groups are generated by three involutions, two of which commute. These are precisely the groups that are automorphism groups of rank three regular polytopes. The exceptions are $PSL_3(q)$, $PSU_3(q)$, $PSL_4(2^n)$, $PSU_4(2^n)$, A_6, A_7, M_{11}, M_{22}, M_{23}, McL, $PSU_4(3)$, $PSU_5(2)$. The two latter, although mentioned by Nuzhin as being generated by three involutions, two of which commute, have been found to be exceptions recently by Martin Mačaj and Gareth Jones (personal communication). We refer to [18] for almost simple groups of Suzuki type (see also [16]); [4, 20, 21] for groups $PSL_2(q) \leq G \leq P\Gamma L_2(q)$; [2] for groups $PSL_3(q)$ and $PGL_3(q)$; [1] for groups $PSL(4, q)$; [9] for symmetric groups; [3, 10, 11] for alternating groups; and [13, 19, 22] for the sporadic groups up to, and including, the third Conway group Co_3, but not the O'Nan group. Recently, Connor and Leemans have studied the rank 3 polytopes of the O'Nan group using character theory [5], and Connor, Leemans and Mixer have classified all polytopes of rank at least 4 of the O'Nan group [6].

Several attractive results were obtained in this vein, including, for instance, the proof that Coxeter's 57-cell and Grünbaum's 11-cell are the only regular rank 4 polytopes with a full automorphism group isomorphic to a group $PSL_2(q)$ (see [20]). Another striking result is the discovery of the universal locally projective 4-polytope of type $\{\{5, 3\}_5, \{3, 5\}_{10}\}$, whose full automorphism group is $J_1 \times PSL_2(19)$ (see [14]); this is based on the classification of all regular polytopes with an automorphism group given by the first Janko group J_1.

The existing results seem to suggest that polytopes of arbitrary high rank are difficult to obtain from a family of almost simple groups. Only the alternating and symmetric groups are currently known to act on abstract regular polytopes of arbitrary rank. For the sporadic groups the highest known rank is 5.

The Ree groups $R(q)$, with $q = 3^{2e+1}$ and $e > 0$, were discovered by Rimhak Ree [29] in 1960. In the literature they are also denoted by $^2G_2(q)$. These groups have a subgroup structure quite similar to that of the Suzuki simple groups $Sz(q)$, with $q = 2^{2e+1}$ and $e > 0$. Suzuki and Ree groups play a somewhat special role in the theory of finite simple groups, since they exist because of a Frobenius twist, and hence have no counterpart in characteristic zero. Also, as groups of Lie-type, they have rank 1, which means that they act doubly transitively on sets of points without further apparent structure. However, the rank 2 groups which are used to define them, do impose some structure on these sets. For instance, the Suzuki groups act on "inversive planes". For the Ree groups, one can define a geometry known as a "unital". However, these unitals, called *Ree unitals*, have a very complicated and little accessible geometric structure (for instance, there is no geometric proof of the fact that the automorphism group of a Ree unital is an almost simple group of Ree type; one needs the classification of doubly transitive groups to prove this). Also, Ree groups seem to be misfits in a lot of general theories about Chevalley groups and their twisted analogues. For instance, there are no applications yet of the Curtis-Tits-Phan theory for Ree groups; all finite quasisimple groups of Lie type are known to be presented by two

elements and 51 relations, except the Ree groups in characteristic 3 [12]. Hence it may be clear that the Ree groups $R(q)$, with q a power of 3, deserve a separate treatment when investigating group actions on polytopes.

Now, the regular polytopes associated with Suzuki groups are quite well understood (see [16, 18]). But the techniques used for the Suzuki groups are not sufficient for the Ree groups. In the present paper, we carry out the analysis for the groups $R(q)$. In particular, we ask for the possible ranks of regular polytopes whose automorphism group is such a group, and we prove the following theorem.

Theorem 1.1. *Among the almost simple groups G with $R(q) \leq G \leq \mathrm{Aut}(R(q))$ and $q = 3^{2e+1} \neq 3$, only the Ree group $R(q)$ itself is a C-group. In particular, $R(q)$ admits a representation as a string C-group of rank 3, but not of higher rank. Moreover, the non-simple Ree group $R(3)$ is not a C-group.*

In other words, the groups $R(q)$ behave just like the Suzuki groups: they allow representations as string C-groups, but only of rank 3. Although Nuzhin proved in [27] that these groups allow representations as string C-groups of rank 3 for every q, we will describe a string C-group representation for $R(q)$, $q \neq 3$, for each value of q to make the paper self-contained. Also, almost simple groups $R(q) < G \leq \mathrm{Aut}(R(q))$ can never be C-groups (in characteristic 3).

Rephrased in terms of polytopes, Theorem 1.1 says that among the almost simple groups $R(q) \leq G \leq \mathrm{Aut}(R(q))$, only the groups $G := R(q)$ are automorphism groups of regular polytopes, and that these polytopes must necessarily have rank 3.

Ree groups can also be the automorphism groups of abstract chiral polytopes. In fact, Sah [30] showed that every Ree group $R(3^{2e+1})$, with $2e + 1$ an odd prime, is a Hurwitz group; and Jones [15] later extended this result to arbitrary simple Ree groups $R(q)$, proving in particular that the corresponding presentations give chiral maps on surfaces. Hence the groups $R(q)$ are also automorphism groups of abstract chiral polyhedra.

It is an interesting open problem to explore whether or not almost simple groups of Ree type also occur as automorphism groups of chiral polytopes of higher rank.

Note that the Ree groups in characteristic 2 are also very special: they are the only (finite) groups of Lie type arising from a Frobenius twist and having rank at least 2. This makes them special, in a way rather different from the way the Ree groups in characteristic 3 are special. We think that in characteristic 2, quite different geometric methods will have to be used in the study of polytopes related to Ree groups.

2 Basic notions

2.1 Abstract polytopes and string C-groups

For general background on (abstract) regular polytopes and C-groups we refer to McMullen & Schulte [24, Chapter 2].

A polytope $\mathcal{P}$ is a ranked partially ordered set whose elements are called *faces*. A polytope $\mathcal{P}$ of rank n has faces of ranks $-1, 0, \ldots, n$; the faces of ranks 0, 1 or $n - 1$ are also called *vertices*, *edges* or *facets*, respectively. In particular, $\mathcal{P}$ has a smallest and a largest face, of ranks -1 and n, respectively. Each flag of $\mathcal{P}$ contains $n + 2$ faces, one for each rank. In addition to being locally and globally connected (in a well-defined sense), $\mathcal{P}$ is *thin*; that is, for every flag and every $j = 0, \ldots, n - 1$, there is precisely *one* other (j-*adjacent*) flag with the same faces except the j-face. A polytope of rank 3 is a *polyhedron*.

A polytope $\mathcal{P}$ is *regular* if its (automorphism) group $\Gamma(\mathcal{P})$ is transitive on the flags. If $\Gamma(\mathcal{P})$ has exactly two orbits on the flags such that adjacent flags are in distinct orbits, then $\mathcal{P}$ is said to be *chiral*.

The groups of regular polytopes are string C-groups, and vice versa. A *C-group* of *rank* n is a group G generated by pairwise distinct involutions $\rho_0, \ldots, \rho_{n-1}$ satisfying the following *intersection property*:

$$\langle \rho_j \mid j \in J \rangle \cap \langle \rho_j \mid j \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle \quad (J, K \subseteq \{0, \ldots, n-1\}).$$

Moreover, G, or rather $(G, \{\rho_0, \ldots, \rho_{n-1}\})$, is a *string* C-group (of rank n) if the underlying Coxeter diagram is a string diagram; that is, if the generators satisfy the relations

$$(\rho_j \rho_k)^2 = 1 \qquad (0 \le j < k - 1 \le n - 2).$$

Let $G_i := \langle \rho_j \mid j \ne i \rangle$ for each $i = 0, 1, \ldots, n-1$, and let $G_{ij} := \langle \rho_k \mid k \ne i, j \rangle$ for each $i, j = 0, 1, \ldots, n-1$ with $i \ne j$.

Each string C-group G (uniquely) determines a regular n-polytope $\mathcal{P}$ with automorphism group G. The *i-faces* of $\mathcal{P}$ are the right cosets of the distinguished subgroup G_i for each $i = 0, 1, \ldots, n-1$, and two faces are incident just when they intersect as cosets; formally we must adjoin two copies of G itself, as the (unique) (-1)- and n-faces of $\mathcal{P}$. Conversely, the group $\Gamma(\mathcal{P})$ of a regular n-polytope $\mathcal{P}$ is a string C-group, whose generators ρ_j map a fixed, or *base*, flag Φ of $\mathcal{P}$ to the j-adjacent flag Φ^j (differing from Φ in the j-face).

2.2   The Ree groups in characteristic 3

We let C_k denote a cyclic group of order k and D_{2k} a dihedral group of order $2k$.

The Ree group $G := \mathrm{R}(q)$, with $q = 3^{2e+1}$ and $e \ge 0$, is a group of order $q^3(q-1)(q^3+1)$. It has a faithful permutation representation on a Steiner system $\mathcal{S} := (\Omega, \mathcal{B}) = S(2, q+1, q^3+1)$ consisting of a set Ω of $q^3 + 1$ elements, the *points*, and a family of $(q+1)$-subsets $\mathcal{B}$ of Ω, the *blocks*, such that any two points of Ω lie in exactly one block. This Steiner system is also called a *Ree unital*. In particular, G acts 2-transitively on the points and transitively on the incident pairs of points and blocks of $\mathcal{S}$.

The group G has a unique conjugacy class of involutions (see [29]). Every involution ρ of G has a block B of $\mathcal{S}$ as its set of fixed points, and B is invariant under the centralizer $C_G(\rho)$ of ρ in G. Moreover, $C_G(\rho) \cong C_2 \times \mathrm{PSL}_2(q)$, where $C_2 = \langle \rho \rangle$ and the $\mathrm{PSL}_2(q)$-factor acts on the $q + 1$ points in B as it does on the points of the projective line $PG(1, q)$.

The Ree groups $\mathrm{R}(q)$ are simple except when $q = 3$. In particular, $\mathrm{R}(3) \cong P\Gamma L_2(8) \cong \mathrm{PSL}_2(8) : C_3$ and the commutator subgroup $\mathrm{R}(3)'$ of $\mathrm{R}(3)$ is isomorphic to $\mathrm{PSL}_2(8)$.

A list of the maximal subgroups of G is available, for instance, in [32, p. 349] and [17]. Here we briefly review the list for $\mathrm{R}(q)$, with $q \ne 3$, as the maximal subgroups are required in the proof of Theorem 1.1; in parentheses we also note their characteristic properties relative to the Steiner system $\mathcal{S}$.

- $N_G(A) \cong A : C_{q-1}$ (stabilizer of a point), where A is a 3-Sylow subgroup of G;
- $C_G(\rho) \cong C_2 \times \mathrm{PSL}_2(q)$ (stabilizer of a block), where $C_2 = \langle \rho \rangle$ and ρ is an involution of G;
- $\mathrm{R}(q_0)$ (stabilizer of a sub-unital of $\mathcal{S}$), where $(q_0)^p = q$ and p is a prime;

- $N_G(A_i)$, for $i = 1, 2, 3$, where A_i is a cyclic subgroup of G of one of the following kinds:

  - $A_1 = C_{\frac{q+1}{4}}$, with $N_G(A_1) \cong (C_2^2 \times D_{\frac{q+1}{2}}) : C_3$;
  - $A_2 = C_{q+1-3^{e+1}}$, with $N_G(A_2) \cong A_2 : C_6$;
  - $A_3 = C_{q+1+3^{e+1}}$, with $N_G(A_3) \cong A_3 : C_6$.

Note here that $q \equiv 3 \bmod 8$, so $(q-1)/2$ is odd and $(q+1)/2$ is even. Moreover, since p is odd, $q_0 - 1$ and $q_0 + 1$ divide $q - 1$ and $q + 1$, respectively. Finally, $q + 1$ is divisible by 4 but not by 8.

The automorphism group $\mathsf{Aut}(\mathrm{R}(q))$ of $\mathrm{R}(q)$ is given by

$$\mathsf{Aut}(\mathrm{R}(q)) \cong \mathrm{R}(q) {:} C_{2e+1},$$

so in particular $\mathsf{Aut}(\mathrm{R}(3)) \cong \mathrm{R}(3)$.

In the proof of our theorem we need the following lemma about normalizers of dihedral subgroups of dihedral groups. The proof is straightforward.

Lemma 2.1. *Let $m, n > 1$ be integers such that $m \mid n$. The normalizer $N_{D_{2n}}(D_{2m})$ of any subgroup D_{2m} of D_{2n} coincides with D_{2m} if n/m is odd, or is isomorphic to a subgroup D_{4m} of D_{2n} if n/m is even.*

3   Proof of Theorem 1.1

The proof of Theorem 1.1 is based on a sequence of lemmas. We begin in Lemma 3.1 by showing that if $\mathrm{R}(q) < G \leq \mathsf{Aut}(\mathrm{R}(q))$ then G can not be a C-group (with any underlying Coxeter diagram). Thus only the Ree groups $\mathrm{R}(q)$ themselves need further consideration. Then we prove in Lemma 3.3 that $\mathrm{R}(q)$ does not admit a representation as a string C-group of rank at least 5. In the subsequent Lemmas 3.9, 3.11 and 3.12 we then extend this to rank 4 and show that $\mathrm{R}(q)$ can also not be represented as a string C-group of rank 4. Finally, in Lemma 3.15 we construct each group $\mathrm{R}(q)$ as a rank 3 string C-group.

All information that we use about the groups $\mathrm{R}(q)$ can found in [17].

We repeatedly make use of the following simple observation. If $A : B$ is a semi-direct product of finite groups A, B such that B has odd order, then each involution in $A : B$ must lie in A. In fact, if $\rho = \alpha\beta$ is an involution, with $\alpha \in A$, $\beta \in B$, then $1 = \rho^2 = \alpha(\beta\alpha\beta^{-1})\beta^2$, where $\alpha(\beta\alpha\beta^{-1}) \in A$ and $\beta^2 \in B$; hence $\beta^2 = 1$, so $\beta = 1$ and $\rho = \alpha \in A$.

3.1   Reduction to simple groups $\mathrm{R}(q)$

We begin by eliminating the almost simple groups of Ree type that are not simple.

Lemma 3.1. *Let $\mathrm{R}(q) < G \leq \mathsf{Aut}(\mathrm{R}(q))$, where $q = 3^{2e+1}$. Then G is not a C-group.*

Proof. Since $\mathsf{Aut}(\mathrm{R}(q)) \cong \mathrm{R}(q) {:} C_{2e+1}$ and $2e + 1$ is odd, every involution in $\mathsf{Aut}(\mathrm{R}(q))$ lies in $\mathrm{R}(q)$ (by the previous observation), and hence any subgroup of $\mathsf{Aut}(\mathrm{R}(q))$ generated by involutions must be a subgroup of $\mathrm{R}(q)$. Thus no subgroup G of $\mathsf{Aut}(\mathrm{R}(q))$ strictly above $\mathrm{R}(q)$ can be a C-group. (When $e = 0$ we have $\mathsf{Aut}(\mathrm{R}(3)) \cong \mathrm{R}(3)$, so the statement holds trivially.) □

3.2 String C-groups of rank at least five

By Lemma 3.1 we may restrict ourselves to Ree groups $G = \mathrm{R}(q)$. We first rule out the possibility that the rank is 5 or larger.

Lemma 3.2. *Let G be a simple group. Suppose G has a generating set $S := \{\rho_0, \ldots, \rho_{n-1}\}$ of n involutions such that (G, S) is a string C-group. Then $|\rho_i \rho_{i+1}| \geq 3$ for all $i = 0, \ldots, n - 2$.*

Proof. This is due to the fact that, as G is simple, G is not directly decomposable, that is, G cannot be written as the direct product of two nontrivial normal subgroups of G. $\quad\square$

Lemma 3.3. *Let $G = \mathrm{R}(q)$, where $q = 3^{2e+1} \neq 3$. Suppose G has a generating set S of n involutions such that (G, S) is a string C-group. Then $n \leq 4$.*

Proof. Let $S = \{\rho_0, \ldots, \rho_{n-1}\}$, so in particular, $G = \langle \rho_0, \ldots, \rho_{n-1} \rangle$. Then ρ_0 commutes with $\rho_2, \ldots, \rho_{n-1}$, since the underlying Coxeter diagram is a string. However, by Lemma 3.2, ρ_0 does not commute with ρ_1 and ρ_{n-1} does not commute with ρ_{n-2}. Now suppose $n \geq 5$ and consider the subgroup $H := \langle \rho_0, \rho_1, \rho_{n-2}, \rho_{n-1} \rangle$ of G. Then H must be isomorphic to $D_{2c} \times D_{2d}$ for some integers $c, d \geq 3$. Inspection of the list of maximal subgroups of $\mathrm{R}(q)$ described above shows that direct products of (non-abelian) dihedral groups never occur as subgroups in G. So n is at most 4. $\quad\square$

3.3 String C-groups of rank four

Next we eliminate the possibility that the rank is 4. We begin with a general lemma about string C-groups that are simple.

Lemma 3.4. *Let (G, S) be a string C-group of rank n, and let G be simple. Then*

$$N_G(G_{01}) \backslash N_G(G_0)$$

must contain an involution (namely ρ_0).

Proof. The involution ρ_0 centralizes G_{01} and hence must lie in $N_G(G_{01})$. On the other hand, ρ_0 cannot also lie in $N_G(G_0)$ for otherwise G_0 would have to be a nontrivial normal subgroup in the simple group G. $\quad\square$

The next two lemmas will be applied to dihedral subgroups in subgroups of type $\mathrm{PSL}_2(q)$ or $C_2 \times \mathrm{PSL}_2(q)$ of $\mathrm{R}(q)$, respectively.

Lemma 3.5. *Let $q = 3^{2e+1}$ and $e \geq 0$. Then the order $2d$ of a non-abelian dihedral subgroup of $\mathrm{PSL}_2(q)$ must divide $q - 1$ or $q + 1$. Moreover, $d \not\equiv 0 \bmod 4$, and d is even only if $2d$ divides $q + 1$.*

Proof. Suppose D_{2d} is a non-abelian dihedral subgroup of $\mathrm{PSL}_2(q)$, so $d \geq 3$. We claim that $2d$ must divide $q + 1$ or $q - 1$. Recall that under the assumptions on q, the order $2d$ must either be 6 or must divide $q - 1$ or $q + 1$. It remains to eliminate 6 as a possible order. In fact, since q is an odd power of 3, the only maximal subgroups of $\mathrm{PSL}_2(q)$ with an order divisible by 6 are subgroups $\mathrm{PSL}_2(q_0)$ with q_0 a smaller odd power of 3 (see [8] for a list of the subgroups of $\mathrm{PSL}(2, q)$). If we apply this argument over and over again with smaller odd powers of 3, we eventually are left with a subgroup $\mathrm{PSL}_2(3)$. However,

$\mathrm{PSL}_2(3) \cong A_4$ and hence cannot have a subgroup of order 6. Thus $2d$ must divide $q + 1$ or $q - 1$. This proves the first statement of the lemma. The second statement follows from the fact that $q \equiv 3 \bmod 8$. □

Lemma 3.6. *Let $q = 3^{2e+1}$ and $e \geq 0$, let $2d$ divide $q - 1$ or $q + 1$, and let D_{2d} be a non-abelian dihedral subgroup of a group $C := C_2 \times \mathrm{PSL}_2(q)$.*

(a) Then there exists a dihedral subgroup D in $\mathrm{PSL}_2(q)$ such that D_{2d} is a subgroup of $C_2 \times D$ of index 1 or 2, and $N_C(D_{2d}) = N_C(C_2 \times D) = C_2 \times N_{\mathrm{PSL}_2(q)}(D)$. Here the normalizer $N_{\mathrm{PSL}_2(q)}(D)$ must lie in a maximal subgroup D_{q+1} or D_{q-1} of $\mathrm{PSL}_2(q)$, and coincide with $N_{D_{q+1}}(D)$ or $N_{D_{q-1}}(D)$, according as $2d$ divides $q - 1$ or $q + 1$.

(b) Let $D_{2d} \cong D$ (that is, the index is 2). If $2d \mid (q-1)$ or if $2d \mid (q+1)$ and $(q+1)/2d$ is odd, then $N_{\mathrm{PSL}_2(q)}(D) = D$ and $N_C(D_{2d}) \cong C_2 \times D_{2d}$. If $2d \mid (q+1)$ and $(q+1)/2d$ is even, then $N_{\mathrm{PSL}_2(q)}(D) \cong D_{4d}$ and $N_C(D_{2d}) \cong C_2 \times D_{4d}$.

(c) If $D_{2d} = C_2 \times D$ (that is, d is even, $d/2$ is odd, $D \cong D_d$, and the index is 1), then $N_{\mathrm{PSL}_2(q)}(D) \cong D_{2d}$ and $N_C(D_{2d}) \cong C_2 \times D_{2d}$ (regardless of whether $2d \mid (q - 1)$ or $2d \mid (q + 1)$).

(d) The structure of $N_C(D_{2d})$ only depends on d and q, not on the way in which D_{2d} is embedded in C.

Proof. For the first part, suppose $C_2 = \langle \rho \rangle$ and $D_{2d} = \langle \sigma_0, \sigma_1 \rangle$ where σ_0, σ_1 are standard involutory generators for D_{2d}. Write $\sigma_0 = (\rho^i, \sigma_0')$ and $\sigma_1 = (\rho^j, \sigma_1')$ for some $i, j = 0, 1$ and involutions σ_0', σ_1' in $\mathrm{PSL}_2(p)$. Then $D := \langle \sigma_0', \sigma_1' \rangle$ is a dihedral subgroup of $\mathrm{PSL}_2(p)$, and D_{2d} lies in $C_2 \times D$. Since the period of $\sigma_0' \sigma_1'$ divides that of $\sigma_0 \sigma_1$, the order of D is at most $2d$ and D_{2d} has index 1 or 2 in $C_2 \times D$. If this index is 1 then $D_{2d} = C_2 \times D$ (and d is even and $D \cong D_d$). If the index of D_{2d} in $C_2 \times D$ is 2, then $D \cong D_{2d}$ and $D_{2d} \cap \{1\} \times D$ must have index 1 or 2 in $\{1\} \times D$. If the index of $D_{2d} \cap \{1\} \times D$ in $\{1\} \times D$ is 1 then clearly $D_{2d} = \{1\} \times D$ and D_{2d} can be viewed as a subgroup of $\mathrm{PSL}_2(q)$. If the index of $D_{2d} \cap \{1\} \times D$ is 2, then $D_{2d} \cap \{1\} \times D$ is of the form $\{1\} \times E$ where E is either the cyclic subgroup C_d of D, or d is even and E is one of the two dihedral subgroups of D of order d. (Note here that D_{2d} cannot itself be a direct product in which one factor is generated by ρ, since ρ cannot lie in D_{2d}.)

Next we investigate normalizers. First note that the normalizer of a direct subproduct in a direct product of groups is the direct product of the normalizers of the component groups. Thus $N_C(C_2 \times D) = C_2 \times N_{\mathrm{PSL}_2(q)}(D)$.

We now show that the normalizers in C of the subgroups D_{2d} and $C_2 \times D$ coincide. There is nothing to prove if $D_{2d} = C_2 \times D$ or $D_{2d} = \{1\} \times D$. Now suppose that D_{2d} has index 2 in $C_2 \times D$ and E is as above. Then it is convenient to write D_{2d} in the form

$$D_{2d} = (\{1\} \times E) \cup (\{\rho\} \times (D \backslash E)). \tag{3.1}$$

If $(\alpha, \beta) \in C$ then

$$(\alpha, \beta) D_{2d} (\alpha, \beta)^{-1} = (\{1\} \times \beta E \beta^{-1}) \cup (\{\rho\} \times \beta (D \backslash E) \beta^{-1}). \tag{3.2}$$

Now if $(\alpha, \beta) \in N_C(D_{2d})$ then the group on the left in (3.2) is just D_{2d} itself and therefore $\beta E \beta^{-1} = E$ and $\beta (D \backslash E) \beta^{-1} = D \backslash E$. It follows that β normalizes both E and D, so in particular $(\alpha, \beta) \in N_C(C_2 \times D)$. Hence $N_C(D_{2d}) \leq N_C(C_2 \times D)$.

Now suppose that $(\alpha, \beta) \in N_C(C_2 \times D)$. Then β normalizes D. But $\beta E \beta^{-1}$ must be a subgroup of D of index 2 isomorphic to E, and hence $\beta E \beta^{-1}$ and E are either both cyclic

or both are dihedral. Clearly, if both subgroups are cyclic then $\beta E \beta^{-1} = E$. However, the case when both subgroups are dihedral is more complicated. First recall that then d must be even. Now the normalizer $N_{\mathrm{PSL}_2(q)}(D)$ of the dihedral subgroup D of $\mathrm{PSL}_2(q)$ in $\mathrm{PSL}_2(q)$ either coincides with D (that is, D is self-normalized), or is a dihedral subgroup containing D as a subgroup of index 2. We claim that under the assumptions on q, the second possibility cannot occur. In fact, in this case the normalizer would have to be a group of order $4d$, and since d is even, its order would have to be divisible by 8; however, the order of $\mathrm{PSL}_2(q)$ is not divisible by 8 when q is an odd power of 3, so $\mathrm{PSL}_2(q)$ certainly cannot contain a subgroup with an order divisible by 8. Thus $N_{\mathrm{PSL}_2(q)}(D) = D$. But β belongs to the normalizer of D in $\mathrm{PSL}_2(q)$, so then β must lie in D. In particular, $\beta E \beta^{-1} = E$ since E is normal in D. Thus, in either case we have $\beta E \beta^{-1} = E$, and since $\beta D \beta^{-1} = D$, also $\beta (D \backslash E) \beta^{-1} = D \backslash E$. Hence, (3.2) shows that $(\alpha, \beta) \in N_C(D_{2d})$. Hence also $N_C(C_2 \times D) \leq N_C(D_{2d})$.

To complete the proof of the first part, note that D must lie in a maximal subgroup $D_{q \pm 1}$ of $\mathrm{PSL}_2(q)$ and $N_{\mathrm{PSL}_2(q)}(D) = N_{D_{q \pm 1}}(D)$.

The second and third part of the lemma follow from Lemma 3.2 applied to the dihedral subgroup D of $D_{q \pm 1}$. In particular, D is self-normalized in $D_{q \pm 1}$ if $(q \pm 1)/|D|$ is odd, and $N_{\mathrm{PSL}_2(q)}(D)$ is a dihedral subgroup of $D_{q \pm 1}$ of order $2|D|$ if $(q \pm 1)/|D|$ is even. Bear in mind that $(q - 1)/2$ is odd, and $(q + 1)/2$ is even but not divisible by 4.

To establish the last part of the lemma, note that $N_C(D_{2d}) \cong C_2 \times D_{2d}$, except when $D \cong D_{2d}$, $2d \mid (q+1)$ and $(q+1)/2d$ is even. However, since $q \equiv 3 \bmod 8$, if $2d \mid (q+1)$ and $(q + 1)/2d$ is even then d must be odd. In other words, the situation described in the third part of the lemma cannot occur as this would require d to be even. Thus, if $2d \mid (q+1)$ and $(q + 1)/2d$ is even, then we are necessarily in the situation described in second part of the lemma, and so necessarily $N_C(D_{2d}) \cong C_2 \times D_{4d}$. $\qquad \square$

Our next lemma investigates possible C-subgroups of $G = \mathrm{R}(q)$ of rank 3. The vertex-figure of a putative regular 4-polytope with automorphism group G would have to be a regular polyhedron with a group of this kind.

Lemma 3.7. *The only proper subgroups of* $\mathrm{R}(q)$ *that could have the structure of a C-group of rank* 3 *are Ree subgroups* $\mathrm{R}(q_0)$ *with* $q_0 \neq 3$ *or subgroups of the form* $\mathrm{PSL}_2(q_0)$, $C_2 \times \mathrm{PSL}_2(q_0)$, *or* $\mathrm{R}(3)' \cong \mathrm{PSL}_2(8)$.

Proof. It is straightforward (sometimes by applying Lemma 3.2) to verify that only subgroups of maximal subgroups of $\mathrm{R}(q)$ of the second and third type can have the structure of a rank 3 C-group. Therefore we are left with Ree subgroups $\mathrm{R}(q_0)$ and subgroups of groups $C_2 \times \mathrm{PSL}_2(q_0)$, with q_0 an odd power of 3 dividing q, as well as subgroups of type $\mathrm{R}(3)' \cong \mathrm{PSL}_2(8)$ inside a subgroup $\mathrm{R}(3)$. A forward appeal to Lemma 3.15 shows that Ree groups $R(q_0)$ with $q_0 \neq 3$ do in fact act flag-transitively on polyhedra, and by [31], so does $\mathrm{R}(3)' \cong \mathrm{PSL}_2(8)$. The complete list of subgroups of $\mathrm{PSL}_2(q_0)$ is available, for instance, in [20]. As q_0 is an odd power of 3, the group $\mathrm{PSL}_2(q_0)$ does not have subgroups isomorphic to A_5, S_4, or $\mathrm{PGL}_2(q_1)$ for some q_1. Hence, none of the subgroups of $\mathrm{PSL}_2(q_0)$, except for those isomorphic to a group $\mathrm{PSL}_2(q_1)$, with q_1 an odd power of 3 dividing q_0 (and hence q), admits flag-transitive actions on polyhedra. Now the maximal subgroups of $C_2 \times \mathrm{PSL}_2(q_0)$ consist of the factor $\mathrm{PSL}_2(q_0)$, as well as all subgroups of the form $C_2 \times H$ where H is a maximal subgroup of $\mathrm{PSL}_2(q_0)$ from the following list:

$$E_{q_0} : C_{\frac{q_0 - 1}{2}}, \ D_{q_0 - 1}, \ D_{q_0 + 1}, \ \mathrm{PSL}_2(q_1).$$

A subgroup of $C_2 \times \mathrm{PSL}_2(q_0)$ of the form $C_2 \times D_{q_0-1}$ is isomorphic to $D_{2(q_0-1)}$ (since $q_0 \equiv 3 \bmod 8$), so none of its subgroups (including the full subgroup itself) can act regularly on a non-degenerate polyhedron (that is a polyhedron with no 2 in the Schläfli symbol). Similarly, a subgroup of $C_2 \times \mathrm{PSL}_2(q_0)$ of the form $C_2 \times D_{q_0+1}$ is isomorphic to $C_2 \times C_2 \times D_{(q_0+1)/2}$, so again none of its subgroups (including the full subgroup itself) can act regularly on a non-degenerate polyhedron. Finally, a subgroup of $C_2 \times \mathrm{PSL}_2(q_0)$ of the forms $C_2 \times (E_{q_0} : C_{(q_0-1)/2})$ has an order not divisible by 4. Hence, as in the two other cases, none of its subgroups (including the full subgroup itself) can act regularly on a non-degenerate polyhedron.

In summary, the only possible candidates for rank 3 subgroups of $\mathrm{R}(q)$ are of the form $\mathrm{R}(q_0)$, $\mathrm{PSL}_2(q_0)$, $C_2 \times \mathrm{PSL}_2(q_0)$, and $\mathrm{R}(3)' \cong \mathrm{PSL}_2(8)$. We can further rule out a subgroup of type $\mathrm{R}(3)$, since $\mathrm{R}(3) \cong P\Gamma L_2(8)$ is not generated by involutions. $\qquad\square$

For a subgroup B of A we define $N_A^0(B) := \langle a \mid a \in N_A(B), a^2 = 1 \rangle$. If B is generated by involutions then $B \le N_A^0(B) \le N_A(B)$. We first state a lemma that will be useful in several places.

Lemma 3.8. *Let $H := \mathrm{R}(3) = P\Gamma L_2(8)$, and let $D := D_{2d}$ be a dihedral subgroup of H of order at least 6. Then $d = 3$, 7 or 9, and in all cases $N_H^0(D) = D$.*

Proof. Straightforward. $\qquad\square$

The following lemma considerably limits the ways in which Ree groups $\mathrm{R}(q)$ might be representable as C-groups of rank 4.

Lemma 3.9. *If the group $G := \mathrm{R}(q)$ can be represented as a string C-group of rank 4, then*

$$N_G^0(G_{01}) = N_{C_G(\rho_0)}^0(G_{01}). \tag{3.3}$$

Proof. Suppose that G admits a representation as a string C-group of rank 4. Thus

$$G = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle.$$

Since $\mathrm{R}(3)$ is not generated by involutions, we must have $q \neq 3$.

The subgroup $G_{01} = \langle \rho_2, \rho_3 \rangle$ is a dihedral subgroup D_{2d} (say) of the centralizer $C_G(\rho_0)$ of ρ_0, and $C_G(\rho_0) \cong \langle \rho_0 \rangle \times \mathrm{PSL}_2(q)$. Here $d \ge 3$, by arguments similar to those used in the proof of Lemma 3.3. Thus

$$D_{2d} \cong \langle \rho_2, \rho_3 \rangle = G_{01} \le G_1 = \langle \rho_0, \rho_2, \rho_3 \rangle \le C_G(\rho_0) \cong C_2 \times \mathrm{PSL}_2(q). \tag{3.4}$$

By Lemma 3.6 applied to G_{01} and $C_G(\rho_0)$, there exists a dihedral subgroup D in the $\mathrm{PSL}_2(q)$-factor of $C_G(\rho_0)$ such that G_{01} is a subgroup of $\langle \rho_0 \rangle \times D = C_2 \times D$ of index at most 2 and

$$N_{C_G(\rho_0)}(G_{01}) = N_{C_G(\rho_0)}(C_2 \times D) = C_2 \times N_{PSL_2(q)}(D).$$

In fact, the proof of Lemma 3.6 shows that this subgroup $C_2 \times D$ is just given by G_1. But $\rho_0 \notin G_{01}$, so G_{01} has index 2 in $C_2 \times D = G_1$, and $D \cong G_{01} \cong D_{2d}$. Then Lemma 3.5, applied to D, shows that $2d$ must divide either $q + 1$ or $q - 1$.

The structure of the normalizer $N_{C_G(\rho_0)}(G_{01})$ can be obtained from Lemma 3.6. In fact, $N_{C_G(\rho_0)}(G_{01}) \cong C_2 \times D_{2d}$, unless $2d \mid (q + 1)$ and $(q + 1)/2d$ is even; in the latter

case $N_{C_G(\rho_0)}(G_{01}) \cong C_2 \times D_{4d}$. In particular, $N_{C_G(\rho_0)}(G_{01})$ is generated by involutions and its order is divisible by 4. We will show that the normalizer of G_{01} in $C_G(\rho_0)$ captures all the information about the full normalizer $N_G(G_{01})$ of G_{01} in G that is relevant for us. A key step in the proof is the invariance of the structure of the normalizer of G_{01} in arbitrary subgroups of G of type $C_2 \times \mathrm{PSL}_2(q)$; more precisely, the structure only depends on d and q, not on the way in which G_{01} is embedded in a subgroup $C_2 \times \mathrm{PSL}_2(q)$ (see Lemma 3.6).

The full normalizer $N_G(G_{01})$ of G_{01} in G must certainly contain $N_{C_G(\rho_0)}(G_{01})$ and also have an order divisible by 8. We claim that all involutions of the full normalizer $N_G(G_{01})$ must already lie in $C_G(\rho_0)$ and hence in $N_{C_G(\rho_0)}(G_{01})$.

First note that $N_G(G_{01})$ must certainly lie in a maximal subgroup M of G and then coincide with $N_M(G_{01})$. (Since G is simple, the normalizer of a proper subgroup of G cannot coincide with G.) Inspection of the list of maximal subgroups of G shows that only maximal subgroups M of type $R(q_0)$, $C_2 \times \mathrm{PSL}_2(q)$ or $N_G(A_1)$ have an order divisible by 4. Only those maximal subgroups could perhaps contain $N_{C_G(\rho_0)}(G_{01})$ and hence $N_G(G_{01})$. We investigate the three possibilities for M separately.

Suppose M is a group of type $C_2 \times \mathrm{PSL}_2(q)$. Then the invariance of the structure of the normalizer of G_{01} shows that $N_M(G_{01}) \cong N_{C_G(\rho_0)}(G_{01})$. However, $N_{C_G(\rho_0)}(G_{01}) \leq N_G(G_{01})$ and $N_G(G_{01}) = N_M(G_{01})$, so this gives $N_G(G_{01}) = N_{C_G(\rho_0)}(G_{01})$. But $N_{C_G(\rho_0)}(G_{01})$ is generated by involutions, so $N_{C_G(\rho_0)}(G_{01}) = N^0_{C_G(\rho_0)}(G_{01})$ and (3.3) must hold as well.

Let M be a group of type $N_G(A_1) \cong (C_2^2 \times D_{(q+1)/2}) : C_3$ where A_1 is a group $C_{(q+1)/4}$ (recall that $(q+1)/4$ is odd). Then all involutions of M must lie in its subgroup $K := C_2^2 \times D_{(q+1)/2} = C_2 \times D_{q+1}$. In particular, all involutions of $N_M(G_{01})$ must lie in K and hence in $N_K(G_{01})$; that is, $N^0_M(G_{01}) \leq N_K(G_{01})$. Also, G_{01} itself must lie in K and its order $2d$ must divide $q + 1$. The subgroup K lies in the centralizer C of the involution generating the C_2-factor in the direct product factorization $C_2 \times D_{q+1}$ for K, and $N_K(G_{01}) \leq N_C(G_{01})$. This subgroup C is of type $C_2 \times \mathrm{PSL}_2(q)$, and so again the invariance of the structure of the normalizers implies that $N_C(G_{01}) \cong N_{C_G(\rho_0)}(G_{01})$. But $N_G(G_{01}) = N_M(G_{01})$ and therefore

$$N_{C_G(\rho_0)}(G_{01}) = N^0_{C_G(\rho_0)}(G_{01}) \leq N^0_G(G_{01}) = N^0_M(G_{01}) \leq N_K(G_{01}) \leq N_C(G_{01}).$$

Thus $N^0_G(G_{01}) = N^0_{C_G(\rho_0)}(G_{01})$, as required.

Now let M be a Ree group $R(q_0)$ where $(q_0)^p = q$ and p is a prime. We first cover the case when M is a Ree group $R(3) = \mathrm{PSL}_2(8) : C_3$, that is, $q = 3^p$ where p is a prime. In that case, by Lemma 3.8, $N^0_G(G_{01}) = N^0_M(G_{01}) = G_{01}$. Hence, since also $G_{01} \leq N^0_{C_G(\rho_0)}(G_{01}) \leq N^0_G(G_{01})$, we must have $N^0_G(G_{01}) \leq N^0_{C_G(\rho_0)}(G_{01})$.

Now suppose $q_0 \neq 3$, so in particular M is simple. Then $2d$ must divide $q_0 \pm 1$, since $N_{C_G(\rho_0)}(G_{01})$ lies in M and therefore $\rho_0 \in M$, giving $G_{01} \leq N_{C_M(\rho_0)}(G_{01}) \cong C_2 \times \mathrm{PSL}_2(q_0)$. Since the subgroup $N_G(G_{01})$ of M must have an order divisible by 4, it must lie in a maximal subgroup M' of M of type $\mathrm{R}(q_1)$, $C_2 \times \mathrm{PSL}_2(q_1)$, or $N_{\mathrm{R}(q_0)}(A'_1)$ with $A'_1 \cong C_{(q_0+1)/4}$. The maximal subgroups M' of $M = \mathrm{R}(q_0)$ of types $C_2 \times \mathrm{PSL}_2(q_0)$ and $N_{\mathrm{R}(q_0)}(A'_1)$, respectively, lie in maximal subgroups of G of type $C_2 \times \mathrm{PSL}_2(q)$ or $N_G(A_1)$, so they are subsumed under the previous discussion. (Alternatively we could dispose of these cases for M' directly, using arguments very similar to those in the two previous cases for M.) Then this leaves the possibility that M' is of type $R(q_1)$, in which case we are back at a Ree group. Now continuing in this fashion to smaller and smaller Ree subgroups that could perhaps contain $N_G(G_{01})$, we eventually arrive at either a Ree subgroup $M^{(k)}$

(say) whose parameter $q^{(k)} \pm 1$ (say) is no longer divisible by $2d$, or a Ree group $\mathrm{R}(3)$. In the first case, $\mathrm{R}(q_0)$ does not contribute anything new to $N_G^0(G_{01})$, and the normalizer $N_G(G_{01})$ must already lie in one of the maximal subgroups of type $C_2 \times \mathrm{PSL}_2(q)$ or $N_G(A_1)$ discussed earlier; in particular, $N_G^0(G_{01}) = N_{C_G(\rho_0)}^0(G_{01})$, as required. In the second case, the normalizer $N_G(G_{01})$ lies in a Ree subgroup $\mathrm{R}(3) \cong \mathrm{PSL}_2(8) : C_3$, and its involutory part $N_G^0(G_{01})$ must lie in the $\mathrm{PSL}_2(8)$ subgroup. Again, by Lemma 3.8, we have $N_G^0(G_{01}) = N_M^0(G_{01}) = G_{01}$, hence (3.3) must also hold in this case. $\qquad \square$

Lemma 3.10. *If the group $G := \mathrm{R}(q)$ can be represented as a string C-group of rank 4, then $q \neq 3$ and both the facet stabilizer G_3 and vertex stabilizer G_0 have to be isomorphic to $\mathrm{PSL}_2(8) = \mathrm{R}(3)'$ (i.e. the commutator subgroup of $\mathrm{R}(3)$) or a simple Ree group $\mathrm{R}(q_0)$ with $q = q_0^m$ for some odd integer m.*

Proof. We consider the possible choices for G_0 in the given C-group representation of G of rank 4. Our goal is to use Lemma 3.4 to limit the choices for G_0 to just $\mathrm{R}(3)'$ or $R(q_0)$. First recall from Lemma 3.7 that the only possible candidates for G_0 are either Ree subgroups $\mathrm{R}(q_0)$ with $q_0 \neq 3$ or subgroups of the form $\mathrm{PSL}_2(q_0)$, $C_2 \times \mathrm{PSL}_2(q_0)$, or $\mathrm{R}(3)' \cong \mathrm{PSL}_2(8)$. To complete the proof we must eliminate the second and third types of candidates. This is accomplished by means of Lemmas 3.4 and 3.6, proving in each case that $N_G(G_{01}) \backslash N_G(G_0)$ cannot contain an involution, or equivalently $N_G^0(G_{01}) \leq N_G(G_0)$. Bear in mind that $G_{01} \leq G_0$.

First observe that all subgroups of G of the form $C_2 \times \mathrm{PSL}_2(q_0)$ are self-normalized in G; and the normalizer of a subgroup of G of the form $\mathrm{PSL}_2(q_0)$ is isomorphic to $C_2 \times \mathrm{PSL}_2(q_0)$. In other words, $N_G(G_0) = G_0$ if G_0 is of type $C_2 \times \mathrm{PSL}_2(q_0)$, and $N_G(G_0) = C_2 \times G_0$ if G_0 is of type $\mathrm{PSL}_2(q_0)$. We show that $N_G^0(G_{01}) \leq N_G(G_0)$ for each of these two choices of G_0.

Suppose that $G_0 \cong C_2 \times \mathrm{PSL}_2(q_0)$. We first claim that then $2d \mid q_0 \pm 1$ (where $2d = |G_{01}|$). To see this, note that the intersection of G_{01} with the $\mathrm{PSL}_2(q_0)$-factor of G_0 is a subgroup of index 1 or 2 in G_{01}. If the index is 1, the statement is clear by Lemma 3.5, since then G_{01} lies in the $\mathrm{PSL}_2(q_0)$-factor; and if the index is 2 and the intersection is a cyclic group C_d, the statement follows by inspection of the possible orders of cyclic subgroups of $\mathrm{PSL}_2(q_0)$. Now if the index is 2 and the intersection is a dihedral group D_d, then Lemma 3.6 shows that d must be even, $2d \mid q + 1$, and $d/2$ must be odd; moreover, $d \mid q_0 + 1$ since D_d lies in $\mathrm{PSL}_2(q_0)$, and hence $2d \mid q_0 + 1$ since $q_0 + 1$ is divisible by 4. Thus $2d \mid q_0 \pm 1$, as claimed.

Now, since $G_0 \cong C_2 \times \mathrm{PSL}_2(q_0)$, the normalizer $N_{C_G(\rho_0)}(G_{01})$ coincides with the normalizer $N_H(G_{01})$ of G_{01} taken in a suitable subgroup H of $C_G(\rho_0)$ of type $C_2 \times \mathrm{PSL}_2(q_0)$. In fact, from Lemma 3.6 we know that

$$N_{C_G(\rho_0)}(G_{01}) \leq C_2 \times D_{q \pm 1} \leq C_G(\rho_0) \cong C_2 \times \mathrm{PSL}_2(q).$$

But $2d \mid q_0 \pm 1$, so we must have $N_{C_G(\rho_0)}(G_{01}) \leq C_2 \times D_{q_0 \pm 1}$. However, $C_2 \times D_{q_0 \pm 1}$ lies in a subgroup H of $C_G(\rho_0)$ isomorphic to $C_2 \times \mathrm{PSL}_2(q_0)$.

To complete the argument (for any given type of group G_0) we show that $N_G^0(G_{01})$ must lie in $N_{G_0}(G_{01})$ and therefore also in G_0 and $N_G(G_0)$. When G_0 is a group of type $C_2 \times \mathrm{PSL}_2(q_0)$, the normalizer $N_{G_0}(G_{01})$ can be determined using Lemma 3.6 (with q replaced by q_0). In fact, by the invariance of the normalizers of G_{01} we know that $N_{G_0}(G_{01})$

and $N_H(G_{01})$ are isomorphic and that both subgroups are generated by involutions. However, then by Lemma 3.9,

$$N_{G_0}(G_{01}) = N^0_{G_0}(G_{01}) \le N^0_G(G_{01}) = N^0_{C_G(\rho_0)}(G_{01}) = N^0_H(G_{01}) = N_H(G_{01}),$$

so clearly $N_{G_0}(G_{01}) = N_H(G_{01})$. Thus $N^0_G(G_{01}) = N_{G_0}(G_{01}) \le G_0 \le N_G(G_0)$.

Now let G_0 be of type $\mathrm{PSL}_2(q_0)$. Then $C := N_G(G_0)$ is a group of type $C_2 \times \mathrm{PSL}_2(q_0)$ containing G_0, so we can replace G_0 by C and argue as before. In fact, using the same subgroup H, we see that the normalizers $N_C(G_{01})$ and $N_H(G_{01})$ are isomorphic subgroups generated by involutions. In particular,

$$N_C(G_{01}) = N^0_C(G_{01}) \le N^0_G(G_{01}) = N^0_{C_G(\rho_0)}(G_{01}) = N^0_H(G_{01}) = N_H(G_{01}),$$

and therefore $N_C(G_{01}) = N_H(G_{01})$. Hence $N^0_G(G_{01}) = N_C(G_{01}) \le C = N_G(G_0)$. $\square$

Let us now show that $G_0 \not\cong R(3)'$.

Lemma 3.11. *If* $\mathrm{R}(q)$ *has a representation as a string C-group of rank 4 with* $G_0 \cong R(3)'$, *then* $q = 27$.

Proof. Suppose $G := \mathrm{R}(q)$ is represented as a string C-group of rank 4 with generators $\rho_0, \ldots, \rho_3$. Then we know that $G_{01} \le G_1 \le C_G(\rho_0) \cong C_2 \times \mathrm{PSL}_2(q)$.

The abstract regular polyhedra with automorphism group $R(3)' = \mathrm{PSL}_2(8)$ are all known and are available, for instance, in [22]. There are seven examples, up to isomorphism, but not all can occur in the present context. In fact, the dihedral subgroup G_{01} of G_0 must also lie $C_G(\rho_0) \cong C_2 \times \mathrm{PSL}_2(q)$ and hence cannot be a subgroup D_{18}. It follows that the polyhedron associated with G_0 (that is, the vertex-figure of the polytope for G) must have Schläfli symbol $\{3, 7\}$, $\{7, 3\}$, or $\{7, 7\}$. We can further rule out the possibility that $G_{01} \cong D_6$ by Lemmas 3.5 and 3.6, giving that $C_2 \times \mathrm{PSL}_2(q)$ has no dihedral subgroup of order 6. Hence $G_{01} \cong D_{14}$.

The fixed point set of every involution in G is a block of the corresponding Steiner system $S(2, q + 1, q^3 + 1)$, and vice versa, every block is the fixed point set of a unique involution. Hence, two involutions with two common fixed points must coincide, since their blocks of fixed points must coincide. Suppose B_0 denotes the block of fixed points of ρ_0. As ρ_2 and ρ_3 centralize ρ_0, they stabilize B_0 globally but not pointwise. However, ρ_2 cannot have a fixed point among the $q + 1$ points in B_0, since otherwise two points of B_0 would have to be fixed by ρ_2 since $q + 1$ is even. Thus ρ_2, and similarly ρ_3, does not fix any point in B_0. Moreover, in order for $G_{01} \cong D_{14}$ to lie in a subgroup of G of type $C_2 \times \mathrm{PSL}_2(q)$, we must have $7 \mid q + 1$ or $7 \mid q - 1$. Using $q = 3^{2e+1}$ and working modulo 7 the latter possibility is easily seen to be impossible. On the other hand, the former possibility occurs precisely when $e \equiv 1 \bmod 3$, and then $3 \mid 2e + 1$. Hence G must have subgroups isomorphic to $\mathrm{R}(27) = \mathrm{R}(3^3)$.

We claim that G itself is isomorphic to $\mathrm{R}(27)$, that is, $q = 27$. Now the subgroup $G_0 \cong R(3)'$ lies in a unique subgroup $K \cong \mathrm{R}(3)$ of G, namely its normalizer $N_G(G_0)$. Indeed, Figure 1 tells us that $G_0 \cong R(3)'$ is in a unique subgroup isomorphic to $R(27)$ (because of the lower 1's on the edges joining the boxes). This subgroup K, in turn, lies in a unique subgroup $H \cong \mathrm{R}(27)$ of G. All Ree subgroups of G are self-normalized in G, so in particular K and H are self-normalized. Relative to the Ree subgroup H, the normalizer $N_H(C_7)$ in H of the cyclic subgroup C_7 of G_{01} is a maximal subgroup of type

$N_H(A_1) = (C_2^2 \times D_{14}) : C_3$ in H, which also contains G_{01} (see Section 2.2 or [7, p. 123]). Note here that this subgroup C_7 is a 7-Sylow subgroup of both K and H, and is normalized by G_{01}. Thus, $N_H(C_7) = (C_2^2 \times D_{14}) : C_3$. We claim that $N_H(G_{01}) = N_H(C_7)$. Clearly, $N_H(G_{01}) \leq N_H(C_7)$. For the opposite inclusion observe that $(C_2^2 \times D_{14}) : C_3$ has four subgroups isomorphic to D_{14}, including G_{01}. The subgroup G_{01} is normalized by the C_3-factor, and the three others are permuted under conjugation by C_3. This is due to the fact that if it were otherwise, the number of subgroups $R(3)'$ containing G_{01} would not be an integer but 4/3. Hence, among these four subgroups only G_{01} is normal and can be thought of as the subgroup D_{14} occurring in the factorization of the semi-direct product. It follows that the subgroups C_2^2 and C_3 normalize G_{01}. Thus

$$N_H(G_{01}) = N_H(C_7) = (C_2^2 \times D_{14}) : C_3.$$

Figure 1 shows the sublattice of the subgroup lattice of G that is relevant to the current situation. Each box contains two pieces of information: a group that describes the abstract structure of the groups in the conjugacy class of subgroups of G depicted by the box, and a number in the lower left corner that gives the number of subgroups in the conjugacy class. This number is the order of G divided by the order of the normalizer in G of a representative subgroup of the conjugacy class. Two boxes are joined by an edge provided that the subgroups represented by the lower box are subgroups of some subgroups represented by the upper box. There are also two numbers on each edge. The number at the top gives the number of subgroups in the conjugacy class for the lower box that are contained in a *given* subgroup in the conjugacy class for the upper box. The number at the bottom similarly is the number of subgroups in the conjugacy class for the upper box that contain a *given* subgroup in the conjugacy class for the lower box. If we know the lengths of the conjugacy classes for the upper box and lower box, then knowing one of these two numbers on the connecting edge gives us the other. For instance, in Figure 1, if we know that there are 36 (conjugate) subgroups D_{14} in a given subgroup $R(3)'$, then there are

$$\frac{|G|}{|R(3)|} \cdot 36 / \frac{|G|}{|2^2 \cdot 3 \cdot 14|} = 4$$

(conjugate) subgroups $R(3)'$ containing a given subgroup D_{14}.

Returning to our line of argument, as already pointed out above, Figure 1 tells us that $G_0 \cong R(3)'$ is in a unique subgroup isomorphic to $R(27)$, namely H (because of the lower 1's on the edges joining the boxes). It also shows that G_{01} is contained in a unique subgroup $(C_2^2 \times D_{14}) : C_3$, which, in turn, is contained in a unique $R(27)$, namely H. As we saw above, this subgroup $(C_2^2 \times D_{14}) : C_3$ is necessarily the normalizer $N_H(G_{01})$ of G_{01} in H. Moreover, ρ_0 has to lie in this unique subgroup $(C_2^2 \times D_{14}) : C_3$, which itself is a subgroup of H, and therefore $\langle \rho_0, G_0 \rangle \leq H$. This holds because $N_G(G_{01}) = N_H(G_{01})$. That these normalizers coincide can be seen as follows. Clearly, $N_H(G_{01}) \leq N_G(G_{01})$. Now for the opposite inclusion observe that for $g \in N_G(G_{01})$ we have $G_{01} = gG_{01}g^{-1} \leq gN_H(G_{01})g^{-1}$ and (trivially) $G_{01} \leq N_H(G_{01})$. But then Figure 1 shows that a subgroup D_{14} of H must lie in a unique conjugate of $(C_2^2 \times D_{14}) : C_3 = N_H(G_{01})$, so necessarily $gN_H(G_{01})g^{-1} = N_H(G_{01})$. Similarly, since $N_H(G_{01}) \leq H$ and hence $N_H(G_{01}) = gN_H(G_{01})g^{-1} \leq gHg^{-1}$, Figure 1 (at box $R(27)$) gives $gHg^{-1} = H$, so $g \in H$ since H is self-normalized. Thus $G = \langle \rho_0, G_0 \rangle = H \cong R(27)$. □
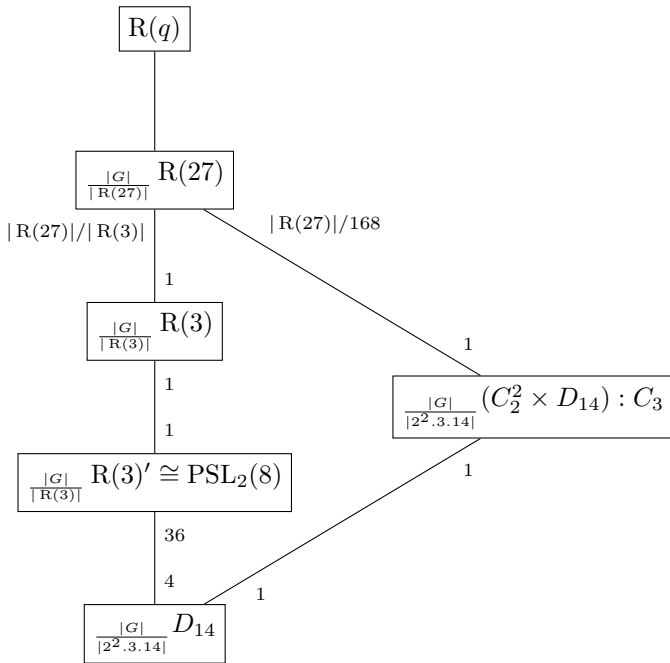
Figure 1: A sublattice of the subgroup lattice of $R(q)$.

Lemma 3.12. *The group* $R(27)$ *cannot be represented as a string C-group of rank 4.*

Proof. Let $G \cong R(27)$. By the previous lemmas we may assume that $G_0 \cong G_3 \cong$ $\mathrm{PSL}_2(8)$. In all other cases we know that G cannot be represented as a rank 4 string C-group. Moreover, from the proof of the previous lemma we already know that $G_{01} \cong D_{14}$ and $N_G(G_{01}) \cong (C_2^2 \times D_{14}) : C_3$. As there is a unique conjugacy class of subgroups $R(3)'$ in $R(27)$, and there is also a unique conjugacy class of subgroups D_{14} in $R(3)'$, the choice of ρ_2, ρ_3 is therefore unique up to conjugacy in $R(27)$. Once ρ_2, ρ_3 have been chosen, there are three candidates for ρ_0, namely the elements of the subgroup C_2^2 that centralizes D_{14}, and these are equivalent under conjugacy by C_3. Hence there is a unique choice for $\{\rho_0, \rho_2, \rho_3\}$ up to conjugacy. By similar arguments we also know that $G_3 \cong R(3)'$ and $G_{23} \cong D_{14}$, and that the pair (G_{23}, G_3) is related to (G_{01}, G_0) by conjugacy in $R(27)$. Hence there must exist an element $g \in R(27)$ such that

- $\rho_0^g = \rho_3, \rho_2^g = \rho_1, \rho_3^g = \rho_0$, or
- $\rho_0^g = \rho_3, \rho_2^g = \rho_0, \rho_3^g = \rho_1$.

The second case can be reduced to the first, as the centraliser of ρ_0 contains an element that swaps ρ_2 and ρ_3 (any two involutions in D_{14} are conjugate). Hence, we may assume without loss of generality that g swaps ρ_0 and ρ_3. In particular, $\langle \rho_0, \rho_3 \rangle$ is an elementary abelian group of order 4 normalized by g. All such subgroups are known to be conjugate and have as normalizer a group $(C_2^2 \times D_{14}) : C_3$. In this group, there is no element that will swap ρ_0 and ρ_3 under conjugation. All elements that will conjugate ρ_0 to ρ_3 will necessarily conjugate ρ_3 to $\rho_0\rho_3$. Hence we have a contradiction. □

We therefore know that if a string C-group representation of rank 4 exists for $\mathrm{R}(q)$, both G_0 and G_3 must be subgroups of Ree type. Thus from now on we can assume $G_0 \cong \mathrm{R}(q_0)$ with $q_0 > 3$.

In a Ree group, the dihedral subgroups D_{2n} are such that n must divide one of

$$9, \ q - 1, \ q + 1, \ \alpha_q := q + 1 - 3^{e+1}, \ \beta_q := q + 1 + 3^{e+1}.$$

Note that

$$\alpha_q \beta_q = q^2 - q + 1,$$

so in particular if H is a Ree subgroup $\mathrm{R}(q_0)$ of G then similarly $\alpha_{q_0} \beta_{q_0} = q_0^2 - q_0 + 1$.

Lemma 3.13. *Let $G \cong \mathrm{R}(q)$ with $q = 3^{2e+1}$ and $\langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$ be a string C-group representation of rank 4 of G. Then*

(1) *G_{01} is a dihedral subgroup D_{2d} with d a divisor of $q+1$ and of either α_{q_0} or β_{q_0} for some q_0 such that $q = q_0^m$ with m odd (where q_0 is determined by $G_0 = \mathrm{R}(q_0)$);*

(2) *$m = 3$, and G_0 and G_3 are conjugate Ree subgroups $\mathrm{R}(q_0)$ with $q = q_0^3$;*

(3) *G_{03} is a dihedral subgroup D_{2t} with t a divisor of α_{q_0} or β_{q_0}.*

Proof. (1) By Lemmas 3.10, 3.11 and 3.12, we may assume that G_0 is a simple Ree subgroup of G. Let G_0 be a Ree subgroup $\mathrm{R}(q_0)$, with $q_0 \neq 3$ such that $q_0^m = q$ with m a positive odd integer and let $G_{01} \cong D_{2d}$. As $G_{01} \leq C_G(\rho_0)$ we have that $2d \mid q \pm 1$ by Lemma 3.5. In order to have involutions in $N_G(G_{01}) \backslash N_G(G_0)$, the only possibility is that $N_G(G_{01})$ (of order divisible by 4) lies in a maximal subgroup of type $N_G(A_1)$ but not in a maximal subgroup $N_{G_0}(C_{\frac{q_0+1}{4}})$ of G_0; for otherwise, the same techniques as in Lemma 3.10 show that there is no involution in $N_G(G_{01}) \backslash N_G(G_0)$. Hence $2d$ divides $q + 1$. Observe that $N_G(A_1) \cong (C_2^2 \times D_{\frac{q+1}{2}}) : C_3$ has exactly four subgroups $D_{\frac{q+1}{2}}$ because of the subgroup C_2^2. These four subgroups are not all normalised by the C_3 because of the semi-direct product. Hence the C_3 must conjugate three of them and normalise the fourth one. Similarly, in $R(q_0)$ there are four subgroups D_{q_0+1} in each $N_{R(q_0)}(A_1')$ and it is obvious that $N_{R(q_0)}(D_{2d}) = N_{R(q)}(D_{2d})$ for every divisor d of $q_0 + 1$. Hence, in order to find some involutions in $N_G(G_{01}) \backslash N_G(G_0)$, we need to have that $2d$ does not divide $q_0 + 1$. Moreover, since $q_0 - 1$ divides $q - 1$, we have also that $(q_0 - 1, q + 1) = 2$. That forces d not to be a divisor of $q_0 - 1$ as $d > 2$. Hence, looking at the list of maximal subgroups of $R(q_0)$ we can conclude that d is a divisor of either α_{q_0} or β_{q_0} in order for D_{2d} to be a dihedral subgroup of $R(q_0)$.

(2) Observe that $q_0^3 + 1 = (q_0 + 1)\alpha_{q_0}\beta_{q_0}$ divides $q^3 + 1$. Let us first show that $2e + 1$ must be divisible by 3 in order for d to satisfy (1). Suppose $(3, 2e + 1) = 1$. Then $q_0 = 3^{2f+1}$ with $2e + 1 = m(2f + 1)$ and $(3, m) = 1$. Let p be an odd prime dividing $(\alpha_{q_0}\beta_{q_0}, q + 1)$ but not dividing $q_0 + 1$. Then p divides $(q_0^3 + 1, q + 1)$ and hence p divides

$$(q_0^6 - 1, q_0^{2m} - 1) = q_0^{2(3,m)} - 1 = q_0^2 - 1 = (q_0 + 1)(q_0 - 1)$$

and hence also $q_0 - 1$. As p divides $q + 1$, and $q_0 - 1$ divides $q - 1$, and since $(q - 1, q + 1) = 2$, we have that $p \mid 2$, a contradiction. Hence m must be divisible by 3 and so does $2e + 1$. Suppose $m \neq 3$. Then $m = 3m'$ and given a Ree subgroup $\mathrm{R}(q_0)$ of $\mathrm{R}(q)$ with $q_0^m = q$, there exists a Ree subgroup $\mathrm{R}(q_0^3)$ such that $\mathrm{R}(q_0) < \mathrm{R}(q_0^3) < \mathrm{R}(q)$. Using similar arguments as in the proof of Lemma 3.11, it is easy to show that, since $\alpha_{q_0}\beta_{q_0}$ divides

$q_0^3 + 1$, we must have $\langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle = \mathrm{R}(q_0^3)$ and therefore $m = 3$. Indeed, as we stated in (1), $N_{\mathrm{R}(q_0^3)}(D_{2d}) = N_{\mathrm{R}(q)}(D_{2d})$ for every divisor d of $q_0^3 + 1$. Hence $\rho_0 \in \mathrm{R}(q_0^3)$. This implies that $m = 3$ and $G_0 \cong \mathrm{R}(q_0)$ with $q_0^3 = q$. Dually, $G_3 \cong \mathrm{R}(q_0)$. As all subgroups $\mathrm{R}(q_0)$ are conjugate in $\mathrm{R}(q)$, we have that G_0 and G_3 are conjugate.

(3) is due to the fact that $G_0 \cap G_3 = G_{03}$ and that, by (2), G_0 and G_3 are conjugate in G. Hence, $N_G(G_{03}) \setminus G_0$ has to be nonempty and G_{03} must not be contained in a subgroup H of G_0 such that $N_G(H) \geq N_G(G_{03})$, for if such a subgroup H exists, then $G_0 \cap G_3 \geq H$. If t divides 9 or one of $q_0 \pm 1$, this does not happen. Hence t divides one of α_{q_0} or β_{q_0}. $\quad\square$

Lemma 3.14. *The small Ree groups have no string C-group representation of rank* 4.

Proof. Suppose G is a Ree group that has a string C-group representation of rank 4. By Lemma 3.10 and part (2) of Lemma 3.13 we may assume that $G := \mathrm{R}(q)$ where $q = q_0^3$ with $q_0 = 3^m$ for an odd integer m. Moreover, G_0 and G_3 are conjugate simple Ree subgroups isomorphic to $\mathrm{R}(q_0)$. By part (3) of Lemma 3.13, if $G_{03} = D_{2t}$ then t must be a divisor of either α_{q_0} or β_{q_0}, and since $q = q_0^3$, we also have

$$q + 1 = (q_0 + 1)(q_0^2 - q_0 + 1) = (q_0 + 1)\alpha_{q_0}\beta_{q_0}.$$

Thus t is also a divisor of $q + 1$. We claim that then $G_0 \cap G_3 > G_{03}$, which gives a contradiction to the intersection property. Indeed, since G_{03} lies in a subgroup $H := C_t : C_6$ of G_0, and the normaliser of G_{03} is not contained in G_0 (for otherwise, D_{2t} would have to lie in a unique subgroup $\mathrm{R}(q_0)$, whereas already G_0 and G_3 give two examples of such subgroups, by the previous lemma), we have $N_G(G_{03}) = (C_2 \times C_2 \times D_{2t}) : C_3$. This group contains $H = C_t : C_6 \cong D_{2t} : C_3$ as a normal subgroup, and G_{03} is normal in H. We also have that $N_G(H) = N_G(G_{03})$. But then, as G_{03} is normal in H, any subgroup $R(q_0)$ containing G_{03} must contain H. In particular this applies to G_3. Thus $G_0 \cap G_3 \geq H > G_{03}$, and the intersection property fails. $\quad\square$

3.4 String C-groups of rank 3

It remains to investigate the possibility of representing $\mathrm{R}(q)$ as a string C-group of rank 3. Nuzhin already showed in [27] that there exist triples of involutions, two of which commute, that generate $\mathrm{R}(q)$ for every q. This completes the proof of Theorem 1.1. However, we decided to give here another way to construct an example of a rank three regular polytope for $\mathrm{R}(q)$ for the paper to be self-contained.

Lemma 3.15. *Let* $G = \mathrm{R}(q)$, *with* $q \neq 3$ *an odd power of* 3. *Then there exists a triple of involutions* $S := \{\rho_0, \rho_1, \rho_2\}$ *in* G *such that* (G, S) *is a string C-group.*

Proof. Recall that the fixed point set of an involution in G is a block of the Steiner system $\mathcal{S} := S(2, q + 1, q^3 + 1)$. Pick two involutions ρ_0, ρ_1 from a maximal subgroup M of G of type $N_G(A_3)$ such that $\rho_0\rho_1$ has order $q + 1 + 3^{e+1}$, and let B_0, B_1, respectively, denote their blocks of fixed points. Obviously, $B_0 \cap B_1 = \emptyset$, for otherwise $\langle \rho_0, \rho_1 \rangle$ would lie in the stabilizer of a point in $B_0 \cap B_1$, which is not possible because of the order of $\rho_0\rho_1$. Recall here that the point stabilizers are maximal subgroups of the form $N_G(A) = A : C_{q-1}$, where A is a 3-Sylow subgroup of G. Now choose an involution ρ_2 in $C_G(\rho_0)$ distinct from ρ_0 such that its block of fixed points B_2 meets B_1 in a point. Such a ρ_2 exists as all involutions of $C_G(\rho_0)$ have pairwise disjoint blocks of size $q + 1$ and therefore they cover all of the $q^3 + 1$ points. Then $B_1 \cap B_2$ must consist of a single point p (say), and

$B_0 \cap B_2 = \emptyset$ since the stabilizer of a point does not contain Klein 4-groups. Then $\langle \rho_1, \rho_2 \rangle$ lies in the point stabilizer of p, and hence must a dihedral group D_{2n}, with n a power of 3. As $\langle \rho_0, \rho_1 \rangle$ is a subgroup of index 3 in M, and ρ_0 does not belong to M, we see that $\langle \rho_0, \rho_1, \rho_2 \rangle = G$. Moreover, since the orders of $\rho_0 \rho_1$ and $\rho_1 \rho_2$ are coprime, the intersection property must hold as well. Thus (G, S), with $S := \{\rho_0, \rho_1, \rho_2\}$, is a string C-group of rank 3. $\qquad \square$

We have not attempted to enumerate or classify all representations of $R(q)$ as a string C-group of rank 3.

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