

The dimension of the negative cycle vectors of signed graphs

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Abstract

A *signed graph* is a graph Γ with edges labeled “+” and “−”. The sign of a cycle is the product of its edge signs. Let $\text{SpecC}(\Gamma)$ denote the list of lengths of cycles in Γ . We equip each signed graph with a vector whose entries are the numbers of negative k -cycles for $k \in \text{SpecC}(\Gamma)$. These vectors generate a subspace of $\mathbb{R}^{|\text{SpecC}(\Gamma)|}$. Using matchings with a strong permutability property, we provide lower bounds on the dimension of this space; in particular, we show for complete graphs, complete bipartite graphs, and a few other graphs that this space is all of $\mathbb{R}^{|\text{SpecC}(\Gamma)|}$.

Keywords: Signed graph, negative cycle vector, permutable matching.

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1 Introduction

A *signed graph* Σ is a graph Γ whose edges have sign labels, either “+” or “−”. The sign of a cycle in the graph is the product of the signs of its edges. Write $c_l^-(\Sigma)$ for the number of negative cycles of length l in Σ and collect these numbers in the *negative cycle vector* $c^-(\Sigma) = (c_3^-, c_4^-, \dots, c_n^-) \in \mathbb{R}^{n-2}$, where n is the order of Σ . We are interested in the structure of the collection $\text{NCV}(\Gamma)$ of all negative cycle vectors of signings of a fixed underlying simple graph Γ .

The negative cycle numbers are of interest for several reasons. Ours is that, while the structure of a signed graph is more complex than that of an unsigned graph, much of that complexity is traceable to the distribution of negative cycles. We think negative cycle vectors are a step towards better understanding of those cycles. Beyond this, negative cycle numbers have been an object of interest since the first days of signed graph theory. When

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signed graphs were introduced by Harary [2] to be applied to a problem in social psychology by Cartwright and Harary [1], one of their concerns was to measure how unbalanced a signed graph is. One measure they proposed was the proportion of negative cycles, i.e., $[\sum_l c_l^-(\Sigma)]/[\sum_l c_l(\Gamma)]$, where c_l denotes the total number of l -cycles in the graph. This proportion is hard to calculate even for signed complete graphs, since the number of cycles can be exponential in the order n and the negative cycle numbers are also complicated.

There are at least three natural questions raised by the existence of the collections $\text{NCV}(\Gamma)$. Most simply, since any set of points in \mathbb{R}^{n-2} lies in a smallest subspace, what subspace do they span? That is the question we address here. The *cycle spectrum* $\text{SpecC}(\Gamma)$ is the list of lengths of cycles in Γ . The finite set $\text{NCV}(\Gamma)$ generates a subspace of \mathbb{R}^{n-2} that is contained in the subspace $\mathbb{R}_{\text{NCV}(\Gamma)}$ consisting of all vectors that are 0 in the coordinates that correspond to cycle lengths not in the cycle spectrum of Γ . We develop a general approach to the dimension question in terms of “permutable matchings” (see Section 2.3) that allows us to prove that, for $\Gamma = K_n, K_{m,n}$, and the Petersen graph, $\text{NCV}(\Gamma)$ spans $\mathbb{R}_{\text{SpecC}(\Gamma)}$; it also gives us a lower bound on dimension for the Heawood graph and one other graph family. We also solve a few examples with an *ad hoc* method.

Knowing the span of the negative cycle vectors, what is their convex hull? In [5] and [8] Popescu and Tomescu gave inequalities bounding the numbers of negative cycles in a signed complete graph, which may be a step towards the answer for K_n (see Section 5). A related question: Do the facets of the convex cone generated by $\text{NCV}(\Gamma)$ have combinatorial meaning?

The ultimate question: Which vectors in the convex hull are actually the vectors of signed graphs? Kittipassorn and Mészáros [3], inspired by the theory of two-graphs from finite group theory and geometry (see [7]) gave strong restrictions on the number of negative triangles in a signed K_n . This is a step towards the answer for K_n .

We discuss these questions further in Section 5.

Our work was originally focused on complete graphs and complete bipartite graphs. Those cases and others led the first author to the following conjecture, to which we do not know any counterexample.

Conjecture 1.1 (Schaefer, 2015). *For any graph Γ , $\dim \text{NCV}(\Gamma) = |\text{SpecC}(\Gamma)|$, the number of different lengths of cycles in Γ .*

2 Background

2.1 Graphs

A *graph* is a pair $\Gamma = (V, E)$, where $V = \{v_1, \dots, v_n\}$ is a finite set of *vertices* and E is a finite set of unordered pairs of vertices, called *edges*. Our graphs are all unlabeled, simple, and undirected. Thus, all cycle lengths are between 3 and n .

The cycle spectrum $\text{SpecC}(\Gamma)$ is the set of cycle lengths that appear in Γ . The number of cycles of length l in Γ is $c_l = c_l(\Gamma)$. The *cycle vector* of Γ is $c(\Gamma) = (c_3, c_4, \dots, c_n)$; sometimes we omit the components that correspond to lengths l not in the cycle spectrum. The number of cycle lengths in Γ , $|\text{SpecC}(\Gamma)|$, is clearly fundamental since $\dim \text{NCV}(\Gamma) \leq |\text{SpecC}(\Gamma)|$.

2.2 Signed graphs

A signed graph is a triple $\Sigma = (V, E, \sigma)$ where $\Gamma = (V, E)$ is a graph, called the *underlying graph* of Σ , and $\sigma: E \rightarrow \{+, -\}$ is the *sign function*. Two signed graphs are *isomorphic* if there is an isomorphism of underlying graphs that preserves edge signs. The *sign of a cycle* is the product of the signs of its edges; a signed graph in which every cycle is positive is called *balanced*. The *negative edge set* E^- is the set of negative edges of Σ and the *negative subgraph* is $\Sigma^- = (V, E^-)$, the spanning subgraph of negative edges. We sometimes write Γ_N for Γ signed so that N is its set of negative edges.

Switching Σ means choosing a vertex subset $X \subseteq V$ and negating all the edges between X and its complement. Switching yields an equivalence relation on the set of all signings of a fixed underlying graph. If Σ_2 is isomorphic to a switching of Σ_1 , we say that Σ_1 and Σ_2 are *switching isomorphic*. This relation is an equivalence relation on signed graphs; we denote the equivalence class of Σ by $[\Sigma]$. A signed graph is balanced if and only if it is switching isomorphic to the all-positive graph. Signed graphs that are switching isomorphic, like those in Figure 1, have the same negative cycle vector.

The negative cycle vector of Σ is $c^-(\Sigma) = (c_3^-(\Sigma), c_4^-(\Sigma), \dots, c_n^-(\Sigma))$, where $c_l^-(\Sigma) = c_l^-(\Sigma)$ is the number of negative cycles of length l . As with $c(\Gamma)$, we may omit the components of $c^-(\Sigma)$ that correspond to lengths l not in the cycle spectrum. Also, we may write either $c^-(\Sigma)$ or $c^-(\sigma)$, the latter when only the signature σ is varying.

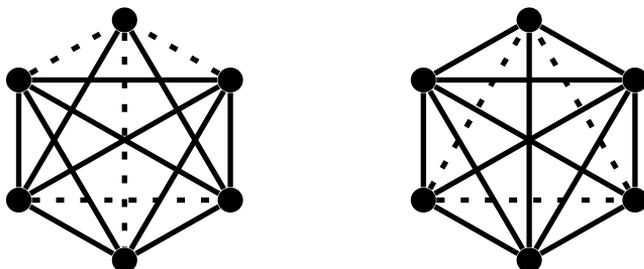


Figure 1: Two switching equivalent signings of K_6 , with the same negative cycle vector $(10, 18, 36, 36)$. Solid lines are positive, dashed lines are negative.

The *negation* of Σ is $-\Sigma = (V, E, -\sigma)$, in which the sign of every edge is negated. Sometimes Σ and $-\Sigma$ are switching isomorphic, e.g., when Σ is bipartite or when it is a signed complete graph whose negative subgraph is self-complementary.

2.3 Permutable matchings

A *matching* in Γ is a set M of pairwise nonadjacent edges; it is *perfect* if $V(M) = V$. A matching M —or any other edge set—is *permutable* if some subgroup of the automorphism group of Γ acts on the edges of M as the symmetric group $S_{|M|}$. We base our results largely on permutable matchings, having noticed their utility for complete and complete bipartite graphs. The advantage of permutability is that, in counting negative cycles using a permutable matching, any two equicardinal subsets belong to the same number of negative cycles of each length. That makes it feasible to calculate the numbers in the vectors we use to estimate the dimension of $\text{NCV}(\Gamma)$.

Our introduction of permutable matchings led to the question: Which graphs have per-

mutable matchings? That has been investigated by Schaefer and Swartz in [6]; they find large families of examples. On the other hand, there are only a few kinds of graph with permutable perfect matchings; Schaefer and Swartz determine them all.

3 Rank and dimension

The dimension of $\text{NCV}(\Gamma)$ is the rank of the matrix whose rows are the negative cycle vectors of all signatures of Γ . The columns of this matrix that correspond to lengths $k \in \{3, 4, \dots, n\} \setminus \text{SpecC}(\Gamma)$ are all zero; thus, we may ignore them. Since the rank cannot be greater than $|\text{SpecC}(\Gamma)|$, if we produce a submatrix of that rank we have proved that $\dim \text{NCV}(\Gamma) = |\text{SpecC}(\Gamma)|$. That is what we now endeavor to do with the aid of a permutable matching.

Even if permutable matchings fail to reach the spectral upper bound, they imply a lower bound. However, we are happy to say that in our three main examples, permutable matchings solve the dimension problem.

The rank of a matrix A is written $\text{rk}(A)$.

3.1 Any negative edge set

We begin with the most general calculation. Given a signed graph Γ_N with an arbitrary negative edge set $N \subseteq E$, how many negative cycles are there of each length? For $X \subseteq N$ let $f_l(X)$ = the number of l -cycles that intersect N precisely in X . We get a formula for f_l by Möbius inversion from $g_l(X)$ = the number of l -cycles that contain X , since

$$g_l(X) = \sum_{X \subseteq Y \subseteq N} f_l(Y),$$

which implies that

$$f_l(X) = \sum_{X \subseteq Y \subseteq N} (-1)^{|Y|-|X|} g_l(Y).$$

The number of negative l -cycles is the number of l -cycles that intersect N in an odd number of edges; therefore,

$$\begin{aligned} c_l^-(\Gamma_N) &= \sum_{X \subseteq N, |X| \text{ odd}} f_l(X) = \sum_{X \subseteq Y \subseteq N, |X| \text{ odd}} \sum (-1)^{|Y|-|X|} g_l(Y) \\ &= \sum_{Y \subseteq N} g_l(Y) \sum_{X \subseteq Y, |X| \text{ odd}} (-1)^{|Y|-|X|} \\ &= \sum_{\emptyset \neq Y \subseteq N} (-2)^{|Y|-1} g_l(Y). \end{aligned} \tag{3.1}$$

This applies to every underlying graph Γ .

3.2 A matrix calculation

Now assume we have a graph Γ of order n together with m unbalanced sign functions $\sigma_1, \dots, \sigma_m$ in addition to the all-positive function $\sigma_0 \equiv +$. To avoid redundancy we want the associated signed graphs to be switching nonisomorphic. For instance, choosing more than half the edges at a vertex to be negative is switching equivalent to choosing fewer than

half, so we would not want the negative edge set to contain more than $\frac{1}{2} \deg(v)$ of the edges incident with any vertex v .

We construct the matrix of the negative cycle vectors of all signings σ_s and their negatives, with columns segregated by parity. The rows are one for $+\Gamma$ (i.e., $\sigma_0 \equiv +$), then m rows for the unbalanced signatures $\sigma_s, 0 < s \leq m$, then $-\Gamma$ (the signature $-\sigma_0 \equiv -$), then the m negations $-\sigma_s$. The relationship between the upper and lower halves is that

$$c_l^-(-\sigma_s) = \begin{cases} c_l - c_l^-(\sigma_s) & \text{if } l \text{ is odd,} \\ c_l^-(\sigma_s) & \text{if } l \text{ is even.} \end{cases}$$

The resulting matrix is

$$\left(\begin{array}{ccc|ccc} 0 & 0 & \cdots & 0 & 0 & \cdots \\ c_3^-(\sigma_1) & c_5^-(\sigma_1) & \cdots & c_4^-(\sigma_1) & c_6^-(\sigma_1) & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ c_3^-(\sigma_m) & c_5^-(\sigma_m) & \cdots & c_4^-(\sigma_m) & c_6^-(\sigma_m) & \cdots \\ c_3 & c_5 & \cdots & 0 & 0 & \cdots \\ c_3 - c_3^-(\sigma_1) & c_5 - c_5^-(\sigma_1) & \cdots & c_4^-(\sigma_1) & c_6^-(\sigma_1) & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ c_3 - c_3^-(\sigma_m) & c_5 - c_5^-(\sigma_m) & \cdots & c_4^-(\sigma_m) & c_6^-(\sigma_m) & \cdots \end{array} \right). \quad (3.2)$$

The last column in the left half is that of $n - 1$ or n depending on whether n is even or odd; in the right half it is that of n or $n - 1$, respectively. Row operations reduce this matrix to

$$\left(\begin{array}{ccc|ccc} 0 & 0 & \cdots & 0 & 0 & \cdots \\ c_3^-(\sigma_1) & c_5^-(\sigma_1) & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ c_3^-(\sigma_m) & c_5^-(\sigma_m) & \cdots & 0 & 0 & \cdots \\ c_3 & c_5 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & \cdots & c_4^-(\sigma_1) & c_6^-(\sigma_1) & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & \cdots & c_4^-(\sigma_m) & c_6^-(\sigma_m) & \cdots \end{array} \right). \quad (3.3)$$

Ignoring the first row of zeros, this is a block matrix

$$A = \begin{pmatrix} U & O \\ c_{\text{odd}}(\Gamma) & \mathbf{0} \\ O & R \end{pmatrix}.$$

The middle row $c_{\text{odd}}(\Gamma)$, consisting of the odd-cycle numbers of Γ , corresponds to $-\Gamma$. The upper left block U is the matrix of negative odd-cycle vectors of the unbalanced signatures σ_s , and the lower right block R is the matrix of negative even-cycle vectors of the same signatures. We infer the fundamental fact that:

Lemma 3.1. *The rank of the negative cycle matrix (3.2) equals the sum of the ranks of $\begin{pmatrix} U \\ c_{\text{odd}}(\Gamma) \end{pmatrix}$ and R .*

For a bipartite graph $U = O$ and $c_{\text{odd}} = \mathbf{0}$, so only R needs to be considered.

3.3 Permutable negative matchings

Henceforth we assume we have chosen a fixed permutable matching M_m of m edges in Γ . For each $s = 1, 2, \dots, m$ we choose a submatching $M_s \subseteq M_m$ of s edges and we define the signature σ_s as that of the signed graph Γ_{M_s} . (It does not matter which M_s we use, because M_m is permutable.) This generates a matrix of negative cycle vectors as in (3.2).

Permutability implies that $g_l(Y)$ depends only on $|Y|$ so we may define $G_l(k) = g_l(Y)$ for any one k -edge subset $Y \subseteq M_m$. Then (3.1) becomes

$$c_l^-(\Gamma_{M_s}) = \sum_{k=1}^s (-2)^{k-1} \binom{s}{k} G_l(k) = \sum_{k=1}^n (-2)^{k-1} \frac{G_l(k)}{k!} (s)_k, \tag{3.4}$$

where $(x)_k$ denotes the falling factorial, $(x)_k = x(x-1) \cdots (x-[k-1])$. We may let k run up to n in the second summation because if $k > s$, the falling factorial equals 0. Formula (3.4) gives $c_l^-(\Gamma_{M_s})$ as a polynomial function $p_l(s)$ without constant term, of degree d_l where d_l is the largest integer k for which $G_l(k) > 0$; that is, d_l is the largest size of a submatching of M_m that is contained in some cycle of length l . We leave d_l undefined if no l -cycle intersects M_m . Clearly, $d_l \leq m$.

(This method works equally well for subsets of any permutable edge set N in any graph. It is easy to see that there are only three possible kinds of permutable set: a matching, a subset of the edges incident to a vertex, and the three edges of a triangle. In K_n a permutable set of edges at a vertex is useless since then the entire matrix (3.2) has rank 1. We have not seen a graph where a triangle’s edges might help find the dimension.)

A column of U or R is not all zero if and only if it corresponds to a cycle length l for which there exists an l -cycle in Γ that intersects M_m . Such a column contains m values of the polynomial $p_l(s)$. Since p_l has degree at most m and no constant term, these values determine p_l completely.

Now a nonzero column in U or R for cycle length l looks like this:

$$\begin{pmatrix} p_l(1) \\ p_l(2) \\ \vdots \\ p_l(m) \end{pmatrix} = \begin{pmatrix} \alpha_l 1^{d_l} + \cdots \\ \alpha_l 2^{d_l} + \cdots \\ \vdots \\ \alpha_l m^{d_l} + \cdots \end{pmatrix}, \tag{3.5}$$

since p_l is a polynomial of degree d_l ; here $\alpha_l = (-2)^{d_l-1} G_l(d_l)/d_l!$. Moreover, $d_l = \mu(l) > 0$ for a nonzero column, where we define

$$\mu(l) = \max_{C_l} |C_l \cap M_m|, \tag{3.6}$$

maximized over all l -cycles C_l .

Define δ_{odd} to be the number of distinct degrees d_l for odd lengths l whose column in U is not zero, and let δ_{even} be the number of distinct degrees d_l for even lengths whose column in R is not zero. If some values of d_l for, e.g., odd lengths l happen to be equal, they are counted only once. Thus, δ_{odd} may be less than the number of nonzero columns. The number of distinct polynomial degrees represented in the columns of U is δ_{odd} , and similarly for R the number is δ_{even} . Let Δ_{odd} be the set of distinct degrees d_l counted by δ_{odd} , and similarly for Δ_{even} .

Lemma 3.2. *The rank of U is at least δ_{odd} and that of R is at least δ_{even} .*

The rank of $\begin{pmatrix} U \\ c_{\text{odd}} \end{pmatrix}$ is $\text{rk}(U) + 1$ if there is an odd length l such that an l -cycle exists in Γ but no l -cycle intersects M_m .

Proof. In U choose one column of each different degree d_l . Divide by the leading coefficient α_l , which is necessarily nonzero; this does not affect the rank. Now add columns of the form $\begin{pmatrix} s^d \\ s=1 \end{pmatrix}^m$ for every $d = 1, 2, \dots, m$ that is not in Δ_{odd} . Column operations allow us to eliminate the lower-degree terms of the column (3.5), leaving a Vandermonde matrix with 1^d in the top row and m^d in the bottom row of column d for each $d = 1, 2, \dots, m$, the rank of which is m . Now reverse the column operations; the rank remains the same, so the columns of U must have full column rank.

The same reasoning applies to R .

The extra 1 in the rank of $\begin{pmatrix} U \\ c_{\text{odd}} \end{pmatrix}$ arises from the fact that, under the assumption, it has a column that is zero in U but is nonzero in c_{odd} . □

3.4 Theorems

Lemma 3.2 yields our principal general theorem. Given a matching M_m and a cycle length $l \in \text{SpecC}(\Gamma)$, define $\mu(l)$ by Equation (3.6).

Theorem 3.3. *Let M_m be a permutable m -matching in Γ . Then*

$$|\{\mu(l) : \text{odd } l \in \text{SpecC}(\Gamma)\}| + |\{\mu(l) > 0 : \text{even } l \in \text{SpecC}(\Gamma)\}| \leq \dim \text{NCV}(\Gamma) \leq |\text{SpecC}(\Gamma)|. \tag{3.7}$$

Suppose that all values $\mu(l)$ for even lengths $l \in \text{SpecC}(\Gamma)$ are distinct and positive, and all values $\mu(l)$ for odd lengths $l \in \text{SpecC}(\Gamma)$ are distinct. Then $\text{NCV}(\Gamma)$ spans $\mathbb{R}_{\text{SpecC}(\Gamma)}$.

Proof. The first part follows directly from Lemma 3.2 since

$$\dim \text{NCV}(\Gamma) \geq \text{rk}(A) = \text{rk} \begin{pmatrix} U \\ c_{\text{odd}} \end{pmatrix} + \text{rk}(R) \geq \delta_{\text{odd}} + \delta_{\text{even}}.$$

Moreover, if there is an odd length l such that $\mu(l) = 0$, then $\text{rk} \begin{pmatrix} U \\ c_{\text{odd}} \end{pmatrix} = \text{rk}(U) + 1 \geq \delta_{\text{even}} + 1$; that explains why we do not exclude $\mu(l) = 0$ from being counted in the odd-length part of (3.7).

In the second part, δ_{even} = the number of even cycle lengths in Γ and δ_{odd} or (if some odd $l \in \text{SpecC}(\Gamma)$ has $\mu(l) = 0$) $\delta_{\text{odd}} + 1$ = the number of odd cycle lengths in Γ . Then the left-hand side of Formula (3.7) equals $|\text{SpecC}(\Gamma)|$. □

There is a simpler statement that applies to graphs with a permutable matching that is sufficiently omnipresent, i.e., meeting the condition of Theorem 3.4. Given m , define $\nu_{\text{odd}}(m)$ = the number of odd lengths $l < 2m$ in $\text{SpecC}(\Gamma)$, +1 if there is an odd cycle length $l \geq 2m$, and define $\nu_{\text{even}}(m)$ = the number of even lengths $l < 2m$ in $\text{SpecC}(\Gamma)$, +1 if there is an even cycle length $l \geq 2m$.

Theorem 3.4. *Suppose M_m is a permutable m -matching in Γ and for every length $l \in \text{SpecC}(\Gamma)$ there exists a cycle C_l such that $|C_l \cap M_m| = \min(m, \lfloor l/2 \rfloor)$. Then*

$$\dim \text{NCV}(\Gamma) \geq \nu_{\text{odd}}(m) + \nu_{\text{even}}(m).$$

The hypothesis can be lessened since, if there is any cycle length $l \geq 2m$, it suffices to have one length $l \geq 2m$ for which there is a C_l with $|C_l \cap M_m| = m$.

Proof. The hypotheses imply that

$$d_l = \begin{cases} \lfloor l/2 \rfloor & \text{if } l \leq 2m, \\ m & \text{if } l \geq 2m. \end{cases}$$

We count the number of distinct values d_l for odd and even cycle lengths. For odd l we get $(l - 1)/2$ if $l \in \text{SpecC}(\Gamma)$ and $l < 2m$, and we get m if and only if there exists a cycle length $l \geq 2m$. The total is $\nu_{\text{odd}}(m)$. The computation of $\nu_{\text{even}}(m)$ is similar.

The values of $\mu(l)$ in Theorem 3.3 are the same as those of d_l unless there is a cycle length for which no l -cycle intersects M_m ; but that is ruled out by our hypotheses. Theorem 3.4 follows. □

A connected graph is *bipancyclic* if it is bipartite with vertex classes of size p and q and has a cycle of every even length from 4 to $2 \min(p, q)$. (This extends the usual definition, which assumes $p = q$.) This is the bipartite analog of pancyclicity, in which the graph has a cycle of every length from 3 to n , the order of the graph.

Corollary 3.5. *Assume Γ is pancyclic and has a permutable m -matching M_m , and for every l with $3 \leq l \leq n$ there is an l -cycle C_l with $|C_l \cap M_m| = \min(m, \lfloor l/2 \rfloor)$. Then*

$$\begin{aligned} \dim \text{NCV}(\Gamma) &= n - 2 && \text{if } 2m \geq n - 1, \\ n - 2 \geq \dim \text{NCV}(\Gamma) &\geq 2m - 1 && \text{if } 2m \leq n - 2. \end{aligned}$$

Assume Γ is bipancyclic and has vertex class sizes p, q with $p \leq q$, and it has a permutable m -matching M_m such that for every k with $2 \leq k \leq p$ there is a $2k$ -cycle C_{2k} with $|C_{2k} \cap M_m| = \min(m, k)$. Then

$$\begin{aligned} \dim \text{NCV}(\Gamma) &= p - 1 && \text{if } m = p, \\ p - 1 \geq \dim \text{NCV}(\Gamma) &\geq m - 1 && \text{if } m \leq p - 1. \end{aligned}$$

The hypotheses can be lessened in the same way as those of Theorem 3.4.

Proof. If Γ is pancyclic, ν_{odd} counts all the numbers $3, 5, \dots, 2m - 1$ plus 1 for $2m + 1$ if $n > 2m$, and ν_{even} counts the numbers $4, 6, \dots, 2m - 2$ plus 1 for $2m$ since $n \geq 2m$. Thus

$$\nu_{\text{odd}} + \nu_{\text{even}} = \begin{cases} (m) + (m - 1) = 2m - 1 & \text{if } n > 2m, \\ (m - 1) + (m - 1) = 2m - 2 & \text{if } n = 2m. \end{cases}$$

The conclusion follows easily.

If Γ is bipancyclic, then $\nu_{\text{even}} = m - 1$ and the conclusion follows easily. □

The two most complete graphs are easy consequences of any of the preceding results, but especially of Corollary 3.5.

Corollary 3.6. *For a complete graph K_n with $n \geq 3$,*

$$\dim \text{NCV}(K_n) = n - 2.$$

For a complete bipartite graph $K_{p,q}$ with $p, q \geq 2$,

$$\dim \text{NCV}(K_{p,q}) = \min(p, q) - 1.$$

4 Examples

4.1 The complete graph

Our original example was K_n . The biggest permutable edge set is a perfect or near-perfect matching. This turns out to be “perfect” for our purposes. But first, let us see the negative cycle vectors of all signings of small complete graphs.

The vectors for K_3 are

$$(0), (1)$$

(from the balanced and unbalanced triangle). The vectors for K_4 are

$$(0, 0), (2, 2), (4, 0)$$

(the all-positive graph, one negative edge, and two nonadjacent negative edges). Here are the vectors for K_5 :

$$(0, 0, 0), (3, 6, 6), (4, 8, 8), (5, 10, 6), (6, 8, 4), (7, 6, 6), (10, 0, 12);$$

and for K_6 :

$$\begin{aligned} &(0, 0, 0, 0), & (4, 12, 24, 24), & (6, 18, 36, 36), & (8, 20, 32, 24), \\ &(10, 18, 36, 36), & (8, 24, 40, 32), & (10, 22, 36, 28), & (12, 24, 24, 32), \\ &(10, 26, 36, 28), & (8, 24, 48, 32), & (14, 18, 36, 36), & (12, 24, 32, 32), \\ &(12, 20, 40, 24), & (10, 30, 36, 20), & (16, 12, 48, 24), & (20, 0, 72, 0). \end{aligned}$$

The number of switching isomorphism classes of complete graphs grows super-exponentially [4]. Since two signed graphs which yield different vectors must belong to different classes, one naturally wonders about the converse property, that the vector uniquely identifies a switching class. This is true up through K_7 but false for K_8 : see Figure ?? below (found by Gary Greaves, whose assistance we greatly appreciate). Thus when $n = 8$ there are fewer vectors than classes; for $n > 8$ see Question 5.5.

Now we compute the function G_l of Section 3.3. Consider the signed K_n 's whose negative edges are s nonadjacent edges, for $0 \leq s \leq \lfloor n/2 \rfloor$. It is straightforward to compute g_l . For a fixed $k \geq 1$ and set Y with $|Y| = k$, we need to form an l -cycle using Y and $l - k$ other edges. (Since Y is a matching, we know that $l \geq 2k$.) So we choose $l - 2k$ of the remaining $n - 2k$ vertices, and then create our cycle as follows: imagine contracting the edges in Y ; the resultant vertices, together with the other $l - 2k$ vertices, will form an $l - k$ -cycle in the contracted graph (which will eventually give an l -cycle in K_n). Cyclically order these $l - k$ “vertices”; this orders the vertices in our actual cycle while ensuring the

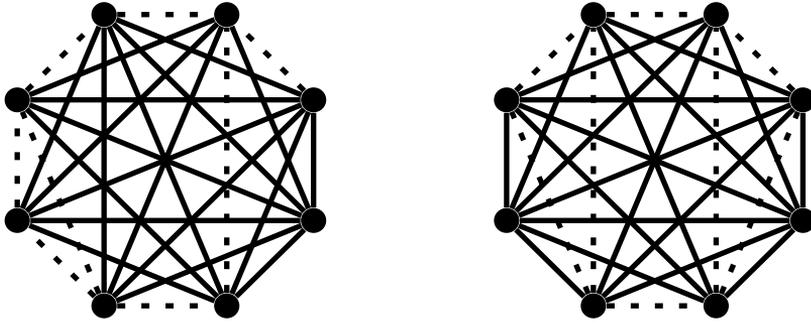


Figure 2: Two switching inequivalent signings of K_8 with the same negative cycle vector (28, 108, 336, 848, 1440, 1248).

edges from Y remain. There are $\frac{(l-k-1)!}{2}$ ways to do this. Then, we expand the contracted edges to regain them; there are 2 ways to do this for each edge. So we have

$$g_l(Y) = \binom{n-2k}{l-2k} (l-k-1)! \cdot 2^{k-1},$$

whence

$$G_l(k) = \binom{n-2k}{l-2k} (l-k-1)! \cdot 2^{k-1}.$$

By Equation (3.4), $c_l^-(s)$ is a polynomial in s of degree $d_l = \lfloor l/2 \rfloor$ and the general formula is

$$c_l^-(s) = \sum_{k=1}^n \binom{s}{k} (-4)^{k-1} \binom{n-2k}{l-2k} (l-k-1)!,$$

For example, $c_3^-(s) = s(n-2)$ and $c_4^-(s) = s(n^2 + 5n + 8) - 2s^2$. This formula for $c_l^-(s)$ demonstrates that the degrees d_l of the odd polynomials are all distinct, and the same for the even polynomials; consequently our main Theorem 3.3 itself implies that the matrix of negative cycle vectors $c^-(s)$ has full rank $n - 2$.

Alternatively, in K_n with a maximum matching, $\Delta_{\text{odd}} = \{3, 5, \dots\}$ (odd numbers up to n) and $\Delta_{\text{even}} = \{4, 6, \dots\}$ (even numbers up to n). So, by Lemma 3.2, for K_n the ranks of U and R are $\lceil n/2 \rceil - 1$ and $\lfloor n/2 \rfloor - 1$, respectively, which sum to $n - 2$.

4.2 Complete bipartite graphs

We now examine $K_{p,q}$, which always has $p \leq q$. We use a maximum matching M_p , i.e., we set $m = p$.

To get $c_{2l}^-(K_{p,q})$ we compute g_{2l} , where the subscript is now $2l$ because all cycles have even length. Call the two independent vertex sets $A = \{a_1, \dots, a_p\}$ and $B = \{b_1, \dots, b_q\}$. For a fixed k -edge set $Y = \{a_{i_1} b_{j_1}, \dots, a_{i_k} b_{j_k}\} \subseteq M_p$, where $k \leq l$, we need to form a $2l$ -cycle using Y and $2l - 2k$ other vertices. Fix one edge $y_1 \in Y$, say $y_1 = a_{i_1} b_{j_1}$. Choose $l - k$ of the remaining $p - k$ vertices from A , in order, in one of $(p - k)_{l-k}$ ways; $l - k$ of the remaining $q - k$ vertices from B , also in order, in one of $(q - k)_{l-k}$ ways; and shuffle the sequences together as $(a_{i_{k+1}}, b_{j_{k+1}}, \dots, a_{i_l}, b_{j_l})$. Insert Y into this $2(l - k)$ -sequence

by inserting y_1 before $a_{i_{k+1}}$, which we may do because each Y edge must be between an A vertex and a B vertex; treating the resulting sequence as cyclically ordered, which can be done in only one way since the A neighbor of y_1 appears after y_1 ; then ordering $Y \setminus \{y_1\}$ in one of $(k - 1)!$ ways as (y_2, \dots, y_k) ; and finally inserting y_2, \dots, y_k anywhere into the cycle in one of

$$\binom{[2(l - k) + 1] + [k - 1] - 1}{[2(l - k) + 1] - 1} = \binom{2l - k - 1}{k - 1}$$

ways. Note that when those edges are inserted into the cycle, there is only one way to orient each edge. The net result is that

$$G_{2l}(k) = g_{2l}(Y) = (p - k)_{l-k}(q - k)_{l-k} \cdot (k - 1)! \binom{2l - k - 1}{k - 1}.$$

Then by Equation (3.4), for $2 \leq l \leq p$,

$$c_{2l}^-(s) = \sum_{k=1}^p \binom{s}{k} \frac{(-2)^{k-1}}{k} (p - k)_{l-k}(q - k)_{l-k} \binom{2l - k - 1}{k - 1}.$$

This explicit formula for the negative cycle vectors $c^-(s)$, with Theorem 3.3, implies that $\dim \text{NCV}(K_{p,q}) = p = \min(p, q)$.

4.3 The Petersen graph

Next we consider the Petersen graph P , which has four cycle lengths, 5, 6, 8, and 9, so $\dim \text{NCV}(P) \leq 4$. It lacks a permutable 4-matching. In fact:

Theorem 4.1. *A 3-regular graph that is arc transitive cannot have a permutable 4-matching.*

Proof. By [6, Theorem 1.1] an arc-transitive graph with a permutable m -matching, where $m \geq 4$, must have degree at least m . □

The Petersen graph does have a permutable 3-matching, in fact, two kinds.

The first kind consists of alternate edges of a C_6 . In the language of Theorem 3.3, we must compute $\mu(l) = |\max\{C_l \cap M_3\}|$ for each cycle length. We find with little difficulty that $\mu(5) = 2$, $\mu(6) = 3$, $\mu(8) = 2$, and $\mu(9) = 3$. Therefore $|\Delta_{\text{odd}}| = 2$ and $|\Delta_{\text{even}}| = 2$, whence, despite only having a 3-matching, we can deduce that $\dim \text{NCV}(P) = 4$. We even know the negative cycle vectors corresponding to negative 0-, 1-, 2-, and 3-submatchings and the negated signatures; they are (in order of matching size)

$$\begin{matrix} (0, 0, 0, 0), & (4, 4, 8, 12), & (6, 6, 8, 10), & (6, 10, 0, 10) \\ (12, 0, 0, 20), & (8, 4, 8, 8), & (6, 8, 8, 10), & (6, 10, 0, 10). \end{matrix}$$

The bottom vector in each column corresponds to the negated signing.

The second kind of permutable 3-matching consists of three edges at distance 3. The first matching type also is three equally spaced edges in a C_9 , but not every such subset of a C_9 is also a set of alternating edges of a C_6 ; the other such subsets are 3-matchings of the second kind. This second kind generates negative cycle vectors from negated submatchings and the corresponding negated sign functions whose dimension is only 3, not 4, since with this matching the negated signatures are switching isomorphic to unnegated signatures. This shows that not all permutable m -matchings in a graph are equally useful.

4.4 The Heawood graph

The Heawood graph H is bipartite and has five cycle lengths, 6, 8, 10, 12, and 14, so $\dim \text{NCV}(H) \leq 5$. It has a permutable 3-matching, indeed three different kinds, for instance alternate edges of a 6-cycle. Using that 3-matching we find that $\mu(6) = 3$, $\mu(8) = 2$, $\mu(10) = 3$, $\mu(12) = 3$, and $\mu(14) = 3$. These are two different values, thus $\dim \text{NCV}(H) \geq 2$. The results for the other two kinds of permutable 3-matching are the same except that $\mu(6) = 2$. In every case μ has two values.

Our matching method, in principle, cannot prove more because H has no permutable 4-matching (see Theorem 4.1). Nonetheless we suspect the dimension equals $|\text{SpecC}(H)|$.

4.5 Other graphs with permutable perfect matchings, and the cube

Schaefer and Swartz found all graphs that have a permutable perfect matching. Besides K_n and $K_{p,p}$ they are the hexagon C_6 , the octahedron graph O_3 , and three general examples: the join $K_p \vee \overline{K_p}$ of a complete graph with its complement, the *matching join* $K_p \underline{\vee} K_p$ obtained from two copies of K_p by inserting a perfect matching between the two copies, and the matching join $K_p \underline{\vee} \overline{K_p}$, obtained by hanging a pendant edge from each vertex of K_p .

Our treatment of them leads us to one other family, the cyclic prisms $C_p \square K_2$.

4.5.1 The simple four

Trivially,

$$\dim \text{NCV}(C_6) = 1 = |\text{SpecC}(C_6)|.$$

It is easy to verify by hand that O_3 satisfies the conditions of Corollary 3.5, so

$$\dim \text{NCV}(O_3) = |\text{SpecC}(O_3)| = 4.$$

As for $K_p \underline{\vee} \overline{K_p}$, since the pendant edges contribute nothing to cycles, $\text{SpecC}(K_p \underline{\vee} \overline{K_p}) = \text{SpecC}(K_p)$ and $\text{NCV}(K_p \underline{\vee} \overline{K_p}) = \text{NCV}(K_p)$; thence

$$\dim \text{NCV}(K_p \underline{\vee} \overline{K_p}) = |\text{SpecC}(K_p \underline{\vee} \overline{K_p})| = p.$$

It is also easy to show that $K_p \vee \overline{K_p}$ satisfies the conditions of Corollary 3.5. Thus,

$$\dim \text{NCV}(K_p \vee \overline{K_p}) = |\text{SpecC}(K_p \vee \overline{K_p})| = 2p.$$

4.5.2 The matching join of two complete graphs

This graph, $K_p \underline{\vee} K_p$, is pancyclic, but its permutable matchings are peculiar. One kind is any matching in a K_p . A maximum matching $M_{\lfloor p/2 \rfloor}$ in K_p has $\mu(l) = \min(p, \lfloor l/2 \rfloor)$, hence $\dim \text{NCV}(K_p \underline{\vee} K_p) \geq p$ by reasoning similar to that for K_p . The matching M_p^\vee that joins the copies of K_p prevents a permutable matching from having edges in both copies. The only other permutable matchings are subsets of M_p^\vee . This matching only generates $\lfloor p/2 \rfloor$ switching nonisomorphic signatures since negating a subset of M_p^\vee switches to negating the complementary subset. By itself, therefore, choosing our grand matching M_m to be M_p^\vee does not give a better lower bound than p . Nonetheless we feel the dimension is likely to be $n - 2 = 2p - 2$.

The smallest case, $K_3 \vee K_3$, is the triangular prism. There are four cycle lengths; the cycle count vector is $(c_3, c_4, c_5, c_6) = (2, 3, 6, 3)$. The required dimension can be found directly. There are four unbalanced signatures; see Figure 3. The negative cycle vectors are linearly independent so $\dim \text{NCV}(K_3 \vee K_3) = |\text{SpecC}(v)|$, in agreement with Conjecture 1.1.

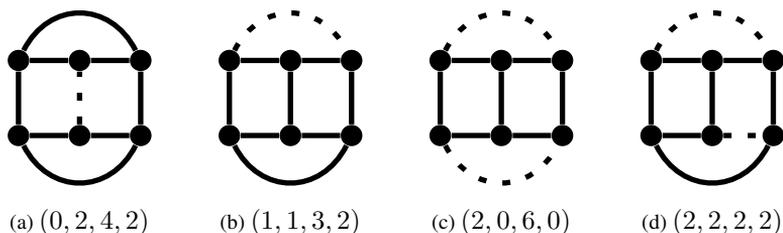


Figure 3: The four unbalanced switching classes of the prism $K_3 \vee K_3$ and their negative cycle vectors.

As for permutable matchings in the triangular prism, M_3^\vee gives $\mu(3) = 0, \mu(4) = \mu(5) = \mu(6) = 2$, thus $\dim \text{NCV}(K_3 \vee K_3) \geq 3$, less than the true value. A strange permutable matching gives the right dimension. Choose M_2 to consist of one edge from each triangle, not both in a C_4 . Then $\mu(3) = \mu(4) = 1$ and $\mu(5) = \mu(6) = 2$, so by Theorem 3.3, $\dim \text{NCV}(K_3 \vee K_3) = 4$, the exact value. This example and the Petersen graph demonstrate that useful permutable matchings need not be perfect matchings.

4.5.3 Prisms, with cube

The triangular prism lends support to our belief that $\dim \text{NCV}(K_p \vee K_p) = 2p - 2$. However, it is atypical since it is also a prism, the Cartesian product $C_p \square K_2$ with $p = 3$.

Prisms with $p > 3$ do not have permutable perfect matchings but they make good examples, especially the next case, the cube $Q_3 = C_4 \square K_2$. It is bipartite and has only three cycle lengths: 4, 6, and 8. Three unbalanced signatures whose negative cycle vectors are linearly independent are

- σ_1 , with one negative edge, e . It has $c^-(\sigma_1) = (2, 8, 4)$;
- σ_2 , with a second negative edge, parallel to e and sharing a quadrilateral with it. It has $c^-(\sigma_2) = (2, 12, 4)$;
- σ_3 , with a second negative edge, also parallel to e but not in a common quadrilateral. It has $c^-(\sigma_3) = (2, 4, 2)$.

Thus, $\dim \text{NCV}(Q_3) = |\text{SpecC}(Q_3)|$, again agreeing with Conjecture 1.1.

5 Questions

Here are what we consider the principal open questions concerning negative cycle numbers and vectors. The purpose is to find connections between the structure of Γ and the signed cycle structure of signatures of Γ . We list them in order of increasing refinement. Complete

graphs seems to be the simplest example with interesting properties so we recommend them as the first object of study, except of course in Question 5.1.

5.1 Dimension

Resolve Conjecture 1.1. If it is false, can $\dim \text{NCV}(\Gamma)$ be determined in terms of structural properties of Γ ?

5.2 Cone

The zero vector is the most obvious negative cycle vector of every graph. That suggests looking at the convex cone generated by $\text{NCV}(\Gamma)$. In particular, we wonder whether the facets or edges of that cone have combinatorial meaning.

5.3 Polytope

The convex hull $\text{conv NCV}(\Gamma)$ is a natural object of interest, and in particular its facets, which represent the complete set of inequalities satisfied by all negative cycle vectors. Almost nothing is known about these inequalities even for K_n . We looked at complete graphs of orders up to 6 but they were too small to suggest a conjecture.

If Σ is a signed K_n with frustration index $m = l(\Sigma) \leq n/2$, the negative cycle numbers for lengths $l < n/2$ (approximately) must satisfy bounds found by Popescu and Tomescu [5, Corollary 1]; the lower bounds occur when E^- is an m -edge star and the upper bounds when E^- is an m -edge matching. Since the bounds depend on the frustration index, they do not appear to constrain $\text{conv NCV}(\Gamma)$, but perhaps something relevant can be made of them.

5.4 Characterization

The negative cycle numbers of a signed K_n , Σ , must satisfy divisibility conditions found by Popescu and Tomescu [5, Section 4]. Aside from that and the work of Kittpassorn and Mészáros [3] on $c_3^-(\Sigma)$ —that is, sizes of n -vertex two-graphs—it is not known which integral vectors in $\text{conv NCV}(K_n)$ can be negative cycle vectors. Surely, a characterization will be difficult if not impossible.

We know of no partial results for other graphs.

5.5 Collapsing pairs

Concerning Gary Greaves' counterexample mentioned in Section 4.1, we propose:

Conjecture 5.1. *For every $n \geq 8$ there are pairs of switching nonisomorphic signed K_n 's that have the same negative cycle vector.*

In a related question, we ask whether the number I_n of switching isomorphism types of signed complete graphs [4] is asymptotic to the number $|\text{NCV}(K_n)|$ of negative cycle vectors of those graphs; that is, whether $|\text{NCV}(K_n)|/I_n \rightarrow 1$. If not, does it approach 0?

5.6 Conclusion

Evidently, there is much to discover before we can say the negative cycles in signed graphs are well understood.

References

- [1] D. Cartwright and F. Harary, Structural balance: a generalization of Heider's theory, *Psychol. Rev.* **63** (1956), 277–293, doi:10.1037/h0046049.
- [2] F. Harary, On the notion of balance of a signed graph, *Michigan Math. J.* **2** (1953–54), 143–146, doi:10.1307/mmj/1028989917.
- [3] T. Kittipassorn and G. Mészáros, Frustrated triangles, *Discrete Math.* **338** (2015), 2363–2373, doi:10.1016/j.disc.2015.06.006.
- [4] C. L. Mallows and N. J. A. Sloane, Two-graphs, switching classes and Euler graphs are equal in number, *SIAM J. Appl. Math.* **28** (1975), 876–880, doi:10.1137/0128070.
- [5] D. R. Popescu and I. Tomescu, Negative cycles in complete signed graphs, *Discrete Appl. Math.* **68** (1996), 145–152, doi:10.1016/0166-218x(95)00010-o.
- [6] A. Schaefer and E. Swartz, Graphs that contain multiply transitive matchings, submitted, arXiv:1706.08964 [math.CO].
- [7] J. J. Seidel, A survey of two-graphs, in: *Colloquio Internazionale sulle Teorie Combinatorie, Tomo I*, Accademia Nazionale dei Lincei, Rome, pp. 481–511, 1976, proceedings of the Convegno Lincei, No. 17, held in Rome, September 3 – 15, 1973.
- [8] I. Tomescu, Sur le nombre des cycles négatifs d'un graphe complet signé, *Math. Sci. Humaines* **53** (1976), 63–67, http://www.numdam.org/item/MSH_1976__53__63_0.