ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 23 (2023) #P2.02
https://doi.org/10.26493/1855-3974.2691.0b7
(Also available at http://amc-journal.eu)

Double generalized majorization and diagrammatics*

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Received 7 September 2021, accepted 16 May 2022, published online 11 November 2022

Abstract

In this paper we show that the generalized majorization of partitions of integers has a surprising completing-squares property. Together with the previously obtained transitivity-like property, this enables a compelling diagrammatical interpretation. Apart from purely combinatorial interest, the main result has applications in matrix completion problems, and representation theory of quivers.

Keywords: Partitions, majorization, diagrammatics, inequalities.

Math. Subj. Class. (2020): 05A17

1 Introduction

By a partition we mean a finite non-increasing sequence of integers. Let $a_1 \geq \ldots \geq a_s$ be integers, then we can define the corresponding partition $\mathbf{a} = (a_1, \ldots, a_s)$. For a partition

*We would like to thank the Referee for the valuable comments and suggestions. This work was done within the activities of CEAFEL and was partially supported by FCT, project UIDB/04721/2020, and Exploratory Grant EXPL/MAT-PUR/0584/2021. In the final stages, this work was also partially supported by the Science Fund of the Republic of Serbia, Projects no. 7744592, MEGIC – "Integrability and Extremal Problems in Mechanics, Geometry and Combinatorics" (M.D.) and no. 7749891, GWORDS – "Graphical Languages" (M.S.).

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$\mathbf{a} = (a_1, \dots, a_s)$ we shall assume that $a_i := +\infty$, for $i \leq 0$, and $a_i := -\infty$, for $i > s$. The following notation will be used throughout the paper

$$\mathbf{a} = (a_1, \dots, a_s), \tag{1.1}$$
$$\mathbf{b} = (b_1, \dots, b_k), \tag{1.2}$$
$$\mathbf{c} = (c_1, \dots, c_m), \tag{1.3}$$
$$\mathbf{d} = (d_1, \dots, d_{m+s}), \tag{1.4}$$
$$\mathbf{g} = (g_1, \dots, g_{m+k}), \tag{1.5}$$
$$\mathbf{f} = (f_1, \dots, f_{m+k+s}). \tag{1.6}$$

Arguably, the most famous comparison between two partitions of integers is a classical majorization in Hardy-Littlewood-Polya sense [16]. In this paper we deal with its generalisation given in [2, 9, 10]. More precisely, we compare three partitions of integers in the following way:

Definition 1.1. Let $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{g}$ be partitions (1.2), (1.3) and (1.5), respectively. If

$$c_i \geq g_{i+k}, \qquad i = 1, \dots, m, \tag{1.7}$$
$$\sum_{i=1}^{h_j} g_i - \sum_{i=1}^{h_j - j} c_i \leq \sum_{i=1}^{j} b_i, \qquad j = 1, \dots, k \tag{1.8}$$
$$\sum_{i=1}^{m+k} g_i = \sum_{i=1}^{m} c_i + \sum_{i=1}^{k} b_i, \tag{1.9}$$

where

$$h_j := \min\{i | c_{i-j+1} < g_i\}, \quad j = 1, \dots, k,$$

then we say that $\mathbf{g}$ is *majorized* by $\mathbf{c}$ and $\mathbf{b}$. This type of majorization we call *the generalized majorization*, and we write

$$\mathbf{g} \prec' (\mathbf{c}, \mathbf{b}).$$

The generalized majorization generalizes the classical majorization. Indeed, if $m = 0$, i.e. if the partition $\mathbf{c}$ is empty, the generalized majorization becomes the classical majorization between the partitions $\mathbf{g}$ and $\mathbf{b}$. Many intrinsic, purely combinatorial properties of generalized majorization, including generalizations of some of the well-known properties of the classical majorization, have been obtained in [10, 11, 14]. These results demonstrate rich structure of generalized majorization as an independent combinatorial object.

Apart from purely combinatorial interest, this relationship between three partitions of integers naturally appears in Matrix and Matrix Pencils completion problems [2, 7, 9, 12], as well as in Representation Theory of Quivers [22], and Perturbation Theory [1, 12].

In this paper we go further, and show that generalized majorization, apart from transitivity-like property that has been shown in [10, Theorem 8], also has certain completing-squares property. This novel property of generalized majorization is motivated by the study of two problems given below, that naturally appear both from matrix pencils completions, and representation theory of quivers point of view.

The first problem has appeared in [9, 11] and turned out to be very challenging and the key point in solving many perturbation and completion problems of Matrix Pencils, see e.g. [6, 7, 9, 12, 13].

Problem 1.2 (Double general majorization problem)**.** Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$, and $\mathbf{g}$ be partitions (1.1), (1.2), (1.4) and (1.5). Find necessary and sufficient conditions for the existence of a partition $\mathbf{f} = (f_1, \dots, f_{m+k+s})$, such that

$$\mathbf{f} \prec' (\mathbf{d}, \mathbf{b}) \quad \text{and} \quad \mathbf{f} \prec' (\mathbf{g}, \mathbf{a}). \tag{1.10}$$

We note that in the case of classical majorization there always exists a minimal partition of a given sum, i.e. for any two partitions of the same length and total sum, there exists a partition that is majorized by both of them. However, here the problem is much more complicated, and involved. A complete solution to Problem 1.2 was obtained in [11, 14]:

Theorem 1.3 ([14, Theorem 3])**.** *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{d}$*, and* $\mathbf{g}$ *be partitions* (1.1), (1.2), (1.4) *and* (1.5)*. There exists a partition* $\mathbf{f} = (f_1, \dots, f_{m+k+s})$*, such that*

$$\mathbf{f} \prec' (\mathbf{d}, \mathbf{b}) \quad \textit{and} \quad \mathbf{f} \prec' (\mathbf{g}, \mathbf{a})$$

if and only if

$$\sum_{i=1}^{m+s} d_i + \sum_{i=1}^{k} b_i = \sum_{i=1}^{m+k} g_i + \sum_{i=1}^{s} a_i$$

and the condition $\bar{\Omega}(\mathbf{g}, \mathbf{d}, \mathbf{b}, \mathbf{a})$ *holds.*

The explicit form of the condition $\bar{\Omega}(\mathbf{g}, \mathbf{d}, \mathbf{b}, \mathbf{a})$ is given in [11, 14], and consists of inequalities between the elements of the partitions $\mathbf{g}, \mathbf{d}, \mathbf{b}, \mathbf{a}$. These involve very technical explicit definition of certain sets S and Δ, and we dismiss it here. We refer the interested reader to [11, 14] for all details and properties on these sets, and for the explicit form of $\bar{\Omega}$.

The second problem has showed its importance when studying bounded rank one perturbations of matrix pencil [12]. Also, it naturally appears in the study of the possible Kronecker invariants of a partially prescribed Matrix Pencil, see e.g. [13, 17]. Apart of the case $k = s = 1$ which has been solved in [12], the following problem is still open:

Problem 1.4 (Pseudo double majorization problem)**.** Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$, and $\mathbf{g}$ be partitions (1.1), (1.2), (1.4) and (1.5). Find necessary and sufficient conditions for the existence of a partition $\mathbf{c} = (c_1, \dots, c_m)$, such that

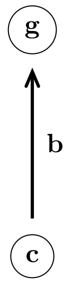
$$\mathbf{g} \prec' (\mathbf{c}, \mathbf{b}) \quad \text{and} \quad \mathbf{d} \prec' (\mathbf{c}, \mathbf{a}). \tag{1.11}$$

The goal of the paper is to prove the relationship between the double majorization Problem 1.2 and pseudo double majorization Problem 1.4. In Theorem 3.2, as the main result of the paper, we prove that Problem 1.4 implies Problem 1.2. That is, we prove that for four partitions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$, and $\mathbf{g}$ as in (1.1), (1.2), (1.4) and (1.5), the existence of a partition $\mathbf{c}$ satisfying (1.11) implies the existence of a partition $\mathbf{f}$ satisfying (1.10). In addition, we explicitly construct such partition $\mathbf{f}$. This is a surprising, and nontrivial property of the generalized majorization. Also, in Section 4 we give a counterexample that the converse does not hold.

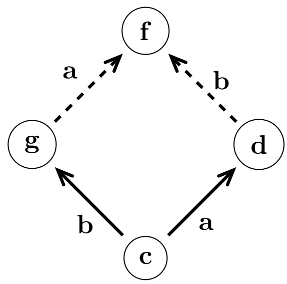
This purely combinatorial result has several interpretations. First, let us introduce some diagrammatics into the story, and denote general majorization by an arrow, i.e. let us denote

$$\mathbf{g} \prec' (\mathbf{c}, \mathbf{b})$$

by



Now, as a direct corollary to our main result we obtain the following commutative diamond-like diagram:



In other words, the lower half of the square (represented by full lines) can always be completed to a full square. More details on diagrammatics are given in Section 4.2.

In addition, the above completion up to a commutative diagram is related to various classical Linear Algebra problems. First of all, both Problems 1.2 and 1.4 naturally appear as cornerstones in solving the classical General Matrix Pencils Completion Problem [17]. In particular, a solution to Problem 1.2 is a key result in obtaining a full description of the possible Kronecker invariants of a quasi-regular matrix pencil with a prescribed subpencil in [13]. For similar contributions and importance of Problems 1.2 and 1.4 in matrix pencils completion problems see [6, 7, 9]. The close relationship between Problems 1.2 and 1.4 obtained in this paper, should have a significant impact in obtaining a complete solution of the General Matrix Pencils Completion Problem. Similar applications are expected in the study of representation of Kronecker quivers, [22].

Another area of applications of results on generalized majorizations is in Bounded Rank Perturbation problems [3–5, 18–21]. In the case when partitions $\mathbf{a}$ and $\mathbf{b}$ are both of length one (i.e. when $s = k = 1$), Problems 1.2 and 1.4 have been addressed and solved separately in [12], and were crucial in solving the rank one perturbation problem for matrix pencils. It is expected that the main result of this paper should lead to a solution of the arbitrary rank perturbation problem in the future. Some steps in this direction have already been done in [8]. Indeed, in [8] we have studied and resolved the classical bounded rank perturbation problem for quasi-regular matrix pencils (pencils with full normal rank). For all details on matrix pencils see [15]. This is a very general result in low rank perturbation theory, and has been open for a long time. The milestone in its solution is the main result of the paper – Theorem 3.2. It allows to choose a special, preferred form of the low rank matrix pencil that performs the perturbation. We expect more impact of Theorem 3.2 in the study of bounded rank perturbations of different classes of matrix pencils in the future.

2 Partitions and generalized majorization

For any two partitions $\mathbf{a} = (a_1, \dots, a_s)$ and $\mathbf{b} = (b_1, \dots, b_k)$ by $\mathbf{a} \cup \mathbf{b}$ we mean a partition obtained as a non-increasing ordering of $\{a_1, \dots, a_s, b_1, \dots, b_k\}$. If $a > b$ are nonnegative integers, then we assume $\sum_{i=a}^{b} a_i := 0$.

Now we shall list some of the basic properties of the auxiliary numbers, h_j, that appear in the definition of the generalized majorization. Below we use the notation from Definition 1.1.

Since $h_j = \min\{i | c_{i-j+1} < g_i\}$, for $j = 1, \dots, k$, we have

$$m + k + 1 > h_k > \cdots > h_2 > h_1 > 0, \tag{2.1}$$

and so in particular

$$h_j \geq j, \quad j = 1, \dots, m + k. \tag{2.2}$$

Also from the definition of h_j we have

$$c_{i-j+1} \geq g_i, \quad \text{for} \quad i < h_j, \quad \text{for any} \quad j = 1, \dots, k. \tag{2.3}$$

We notice that in Definition 1.1, if (1.9) is satisfied, then (1.8) is equivalent to the following:

$$\sum_{i=h_j+1}^{m+k} g_i \geq \sum_{i=h_j-j+1}^{m} c_i + \sum_{i=j+1}^{k} b_i, \quad j = 1, \dots, k. \tag{2.4}$$

The generalized majorization implies *weak majorization* given by the following definition:

Definition 2.1. If partitions $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{g}$ from (1.2), (1.3), and (1.5), respectively, satisfy conditions (1.7), (2.4) and

$$\sum_{i=1}^{m+k} g_i \geq \sum_{i=1}^{m} c_i + \sum_{i=1}^{k} b_i,$$

then we say that $\mathbf{g}$ is *weakly majorized* by $\mathbf{c}$ and $\mathbf{b}$, and we write

$$\mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}).$$

Lemma 2.2 ([7, Theorem 2.5])**.** *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{d}$*, and* $\mathbf{g}$ *from* (1.1), (1.2), (1.4)*, and* (1.5)*, respectively, satisfy*

$$\sum_{i=1}^{m+s} d_i + \sum_{i=1}^{k} b_i = \sum_{i=1}^{m+k} g_i + \sum_{i=1}^{s} a_i.$$

If there exists a partition $\bar{\mathbf{f}} = (\bar{f}_1, \dots, \bar{f}_{m+k+s})$ *such that*

$$\bar{\mathbf{f}} \prec'' (\mathbf{d}, \mathbf{b}) \quad \textit{and} \quad \bar{\mathbf{f}} \prec'' (\mathbf{g}, \mathbf{a}), \tag{2.5}$$

then there exists a partition $\mathbf{f} = (f_1, \dots, f_{m+k+s})$ *such that*

$$\mathbf{f} \prec' (\mathbf{d}, \mathbf{b}) \quad \textit{and} \quad \mathbf{f} \prec' (\mathbf{g}, \mathbf{a}). \tag{2.6}$$

Moreover, if the partition $\bar{\mathbf{f}}$ *satisfying* (2.5) *consists of nonnegative integers, and*

$$\sum_{i=1}^{m+s} d_i + \sum_{i=1}^{k} b_i \geq 0,$$

then there exists a partition $\mathbf{f}$ *consisting of nonnegative integers satisfying* (2.6)*.*

We also cite the result from [10] which shows the transitivity property of generalized majorization. More on this topic is given in Section 4.

Theorem 2.3 ([10])**.** *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{d}$ *and* $\mathbf{f}$ *be partitions* (1.1)*,* (1.2)*,* (1.4) *and* (1.6)*, respectively. If*

$$\mathbf{f} \prec' (\mathbf{d}, \mathbf{b}) \quad \textit{and} \quad \mathbf{d} \prec' (\mathbf{c}, \mathbf{a}),$$

then

$$\mathbf{f} \prec' (\mathbf{c}, \mathbf{a} \cup \mathbf{b}).$$

3 Main result

We start this section by giving one auxiliary result:

Lemma 3.1. *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{d}$ *and* $\mathbf{g}$ *be the partitions* (1.1)*,* (1.2)*,* (1.4) *and* (1.5)*, respectively. Let* $\mathbf{c} = (c_1, \dots, c_m)$ *be a partition such that*

$$\mathbf{d} \prec' (\mathbf{c}, \mathbf{a}) \quad \textit{and} \quad \mathbf{g} \prec' (\mathbf{c}, \mathbf{b}). \tag{3.1}$$

Let $h_j = \min\{i | c_{i-j+1} < g_i\}$*,* $j = 1, \dots, k$*, and* $\bar{h}_j = \min\{i | c_{i-j+1} < d_i\}$*,* $j = 1, \dots, s$*. Let* $\mathbf{g}' = (g'_1, \dots, g'_m)$ *be a partition obtained from* $\mathbf{g}$ *after removing* $g_{h_1}, \dots, g_{h_k}$*, i.e.*

$$\{g'_1, \dots, g'_m\} = \{g_1, \dots, g_{m+k}\} \setminus \{g_{h_1}, \dots, g_{h_k}\},$$

and let $\mathbf{d}' = (d'_1, \dots, d'_m)$ *be a partition obtained from* $\mathbf{d}$ *after removing* $d_{\bar{h}_1}, \dots, d_{\bar{h}_s}$*, i.e.*

$$\{d'_1, \dots, d'_m\} = \{d_1, \dots, d_{m+s}\} \setminus \{d_{\bar{h}_1}, \dots, d_{\bar{h}_s}\}.$$

Then

$$c_i \geq \max(g'_i, d'_i), \quad i = 1, \dots, m. \tag{3.2}$$

Proof. Fix $i \in \{1, \dots, m\}$. Let $h_0 := 0$, $h_{k+1} := m + k + 1$. Then there exists $j \in \{0, \dots, k\}$ such that

$$h_{j+1} - (j+1) \geq i > h_j - j.$$

This is true since $h_{u+1} > h_u$, and so $h_{u+1} - (u+1) \geq h_u - u$, for all $u = 0, \dots, k$, as well as $h_0 - 0 = 0$ and $h_{k+1} - (k+1) = m$.

Then

$$h_{j+1} > i + j > h_j,$$

and so by the definition of $\mathbf{g}'$ we have

$$g_{i+j} = g'_i.$$

If $j < k$, by (2.3) we have that $c_{l-j} \geq g_l$ for all $l < h_{j+1}$, and so

$$c_i \geq g_{i+j} = g'_i.$$

If $j = k$, by (3.1) and definition of the generalized majorization, we again obtain

$$c_i \geq g_{i+k} = g_{i+j} = g'_i.$$

By replacing the partitions $\mathbf{g}'$ by $\mathbf{d}'$ we shall also obtain

$$c_i \geq d'_i.$$

Altogether we have obtained (3.2), as desired. □

Now we can give the main result of the paper:

Theorem 3.2. *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{d}$ *and* $\mathbf{g}$ *be partitions* (1.1), (1.2), (1.4) *and* (1.5)*, respectively. If there exists a partition* $\mathbf{c} = (c_1, \dots, c_m)$ *such that*

$$\mathbf{d} \prec' (\mathbf{c}, \mathbf{a}) \quad \textit{and} \quad \mathbf{g} \prec' (\mathbf{c}, \mathbf{b}), \tag{3.3}$$

then there exists a partition $\mathbf{f} = (f_1, \dots, f_{m+k+s})$ *such that*

$$\mathbf{f} \prec' (\mathbf{d}, \mathbf{b}) \quad \textit{and} \quad \mathbf{f} \prec' (\mathbf{g}, \mathbf{a}). \tag{3.4}$$

Proof. By the definition of the generalized majorization (Definition 1.1) and by (2.4), we have that (3.3) is equivalent to:

$$c_i \geq g_{i+k}, \qquad i = 1, \dots, m, \tag{3.5}$$

$$\sum_{i=h_j+1}^{m+k} g_i - \sum_{i=h_j-j+1}^{m} c_i \geq \sum_{i=j+1}^{k} b_i, \qquad j = 1, \dots, k, \tag{3.6}$$

$$\sum_{i=1}^{m+k} g_i = \sum_{i=1}^{m} c_i + \sum_{i=1}^{k} b_i, \tag{3.7}$$

and

$$c_i \geq d_{i+s}, \qquad i = 1, \dots, m, \tag{3.8}$$

$$\sum_{i=\bar{h}_j+1}^{m+s} d_i - \sum_{i=\bar{h}_j-j+1}^{m} c_i \geq \sum_{i=j+1}^{s} a_i, \qquad j = 1, \dots, s, \tag{3.9}$$

$$\sum_{i=1}^{m+s} d_i = \sum_{i=1}^{m} c_i + \sum_{i=1}^{s} a_i, \tag{3.10}$$

where

$$h_j := \min\{i | c_{i-j+1} < g_i\}, \quad j = 1, \dots, k,$$

and

$$\bar{h}_j := \min\{i | c_{i-j+1} < d_i\}, \quad j = 1, \dots, s.$$

Equalities (3.7) and (3.10) together give

$$\sum_{i=1}^{m+s} d_i + \sum_{i=1}^{k} b_i = \sum_{i=1}^{m+k} g_i + \sum_{i=1}^{s} a_i. \tag{3.11}$$

Let us denote by $\mathbf{g}' = (g'_1, \dots, g'_m)$ a partition obtained from $\mathbf{g}$ after removing $\{g_{h_1}, \dots, g_{h_k}\}$. Also, let us denote by $\mathbf{d}' = (d'_1, \dots, d'_m)$, a partition obtained from $\mathbf{d}$ after removing $\{d_{\bar{h}_1}, \dots, d_{\bar{h}_s}\}$. By Lemma 3.1 we have that

$$c_i \geq \max(g'_i, d'_i), \quad i = 1, \dots, m. \tag{3.12}$$

In order to prove the existence of a partition $\mathbf{f} = (f_1, \dots, f_{m+k+s})$ satisfying (3.4), by (3.11) and by Lemma 2.2 it is enough to prove the existence of a partition $\bar{\mathbf{f}} = (\bar{f}_1, \dots, \bar{f}_{m+k+s})$ satisfying

$$\bar{\mathbf{f}} \prec'' (\mathbf{d}, \mathbf{b}) \quad \text{and} \quad \bar{\mathbf{f}} \prec'' (\mathbf{g}, \mathbf{a}). \tag{3.13}$$

We shall define the partition $\bar{\mathbf{f}} = (\bar{f}_1, \dots, \bar{f}_{m+k+s})$ as a non-increasing ordering of integers $\min(g'_1, d'_1), \dots, \min(g'_m, d'_m), g_{h_1}, \dots, g_{h_k}, d_{\bar{h}_1}, \dots, d_{\bar{h}_s}$, i.e.

$$\bar{\mathbf{f}} := \{\min(g'_1, d'_1), \dots, \min(g'_m, d'_m)\} \cup \{g_{h_1}, \dots, g_{h_k}\} \cup \{d_{\bar{h}_1}, \dots, d_{\bar{h}_s}\}.$$

By Definition 2.1, we are left with proving the following:

$$g_i \geq \bar{f}_{i+s}, \qquad i = 1, \dots, m+k, \tag{3.14}$$

$$\sum_{i=l_j+1}^{m+k+s} \bar{f}_i \geq \sum_{i=l_j-j+1}^{m+k} g_i + \sum_{i=j+1}^{s} a_i, \qquad j = 1, \dots, s, \tag{3.15}$$

$$\sum_{i=1}^{m+k+s} \bar{f}_i \geq \sum_{i=1}^{m+k} g_i + \sum_{i=1}^{s} a_i, \tag{3.16}$$

$$d_i \geq \bar{f}_{i+k}, \qquad i = 1, \dots, m+s, \tag{3.17}$$

$$\sum_{i=\bar{l}_j+1}^{m+k+s} \bar{f}_i \geq \sum_{i=\bar{l}_j-j+1}^{m+s} d_i + \sum_{i=j+1}^{k} b_i, \qquad j = 1, \dots, k, \tag{3.18}$$

$$\sum_{i=1}^{m+k+s} \bar{f}_i \geq \sum_{i=1}^{m+s} d_i + \sum_{i=1}^{k} b_i, \tag{3.19}$$

where

$$l_j := \min\{i | g_{i-j+1} < \bar{f}_i\}, \quad j = 1, \dots, s,$$

and

$$\bar{l}_j := \min\{i | d_{i-j+1} < \bar{f}_i\}, \quad j = 1, \dots, k.$$

In fact, we shall prove only (3.14) – (3.16). By replacing the partition $\mathbf{g}$ by $\mathbf{d}$, and the partition $\mathbf{a}$ by $\mathbf{b}$, the formulas (3.17) – (3.19) will follow.

To that end, let us denote by $\bar{\mathbf{f}}' = (\bar{f}'_1, \dots, \bar{f}'_{m+k})$ the following partition:

$$\bar{\mathbf{f}}' := \{\min(g'_1, d'_1), \dots, \min(g'_m, d'_m)\} \cup \{g_{h_1}, \dots, g_{h_k}\}.$$

Then

$$\bar{\mathbf{f}} = \bar{\mathbf{f}}' \cup \{d_{\bar{h}_1}, \dots, d_{\bar{h}_s}\},$$

and so

$$\bar{f}'_i \geq \bar{f}_{i+s}, \quad i = 1, \dots, m+k. \tag{3.20}$$

Since

$$\mathbf{g} = \mathbf{g}' \cup \{g_{h_1}, \dots, g_{h_k}\},$$

we also have

$$g_i \geq \bar{f}'_i, \quad i = 1, \dots, m+k. \tag{3.21}$$

Altogether, (3.20) and (3.21) give (3.14). By the definition of $\bar{\mathbf{f}}$ we have

$$\begin{aligned}
\sum_{i=1}^{m+k+s} \bar{f}_i &= \sum_{i=1}^{m} \min(g'_i, d'_i) + \sum_{i=1}^{k} g_{h_i} + \sum_{i=1}^{s} d_{\bar{h}_i} \\
&= \sum_{i=1}^{m} g'_i + \sum_{i=1}^{m} d'_i - \sum_{i=1}^{m} \max(g'_i, d'_i) + \sum_{i=1}^{k} g_{h_i} + \sum_{i=1}^{s} d_{\bar{h}_i} \\
&= \sum_{i=1}^{m+k} g_i + \sum_{i=1}^{m+s} d_i - \sum_{i=1}^{m} \max(g'_i, d'_i).
\end{aligned}$$

By applying (3.12), we get

$$\sum_{i=1}^{m+k+s} \bar{f}_i \geq \sum_{i=1}^{m+k} g_i + \sum_{i=1}^{m+s} d_i - \sum_{i=1}^{m} c_i,$$

which by (3.10) gives (3.16), as desired.

Hence, we are left with proving (3.15). First, we introduce by convention $h_0 := 0$, and $h_{k+1} := m+k+1$. Now, fix $j \in \{1, \dots, s\}$. Let $u_j \in \{0, \dots, k\}$ and $\alpha_j \in \{0, \dots, m+k\}$ be such that

$$g_{h_{u_j}} \geq d_{\bar{h}_j} > g_{h_{u_j+1}}, \tag{3.22}$$

$$g_{\alpha_j} \geq d_{\bar{h}_j} > g_{\alpha_j+1}. \tag{3.23}$$

Then

$$h_{u_j+1} > \alpha_j \geq h_{u_j}. \tag{3.24}$$

From the definition of h_i we have that $h_i \geq i$, for all $i = 1, \dots, k$, (see (2.2)). This, together with (3.24) gives

$$\alpha_j \geq u_j.$$

Also, by the definition of $\mathbf{g}'$, from (3.22) and (3.23) we obtain that

$$g'_{\alpha_j-u_j} \geq d_{\bar{h}_j} > g'_{\alpha_j-u_j+1}. \tag{3.25}$$

Moreover, from the definition of $\bar{h}_j$, and from (3.12), we have that

$$d_{\bar{h}_j} > c_{\bar{h}_j-j+1} \geq g'_{\bar{h}_j-j+1}.$$

Thus,

$$g'_{\alpha_j-u_j} > g'_{\bar{h}_j-j+1},$$

and so

$$\alpha_j - u_j \leq \bar{h}_j - j.$$

Hence,

$$\min(\alpha_j - u_j, \bar{h}_j - j) = \alpha_j - u_j. \tag{3.26}$$

Next, we shall prove that

$$l_j = \alpha_j + j, \tag{3.27}$$

and

$$\bar{f}_{l_j} = d_{\bar{h}_j}. \tag{3.28}$$

(Recall that $l_j = \min\{i | g_{i-j+1} < \bar{f}_i\}$). Indeed, we have:

$$g_{h_1} \geq \cdots \geq g_{h_{u_j}} \geq d_{\bar{h}_j} > g_{\alpha_j+1}, \tag{3.29}$$

$$d_{\bar{h}_1} \geq \cdots \geq d_{\bar{h}_{j-1}} \geq d_{\bar{h}_j} > g_{\alpha_j+1}, \tag{3.30}$$

$$g'_1 \geq \cdots \geq g'_{\alpha_j-u_j} \geq d_{\bar{h}_j} > g_{\alpha_j+1}, \tag{3.31}$$

$$d'_1 \geq \cdots \geq d'_{\bar{h}_j-j} \geq d_{\bar{h}_j} > g_{\alpha_j+1}. \tag{3.32}$$

From the definition of $\bar{\mathbf{f}}$, and by (3.26), we have that there are at least $u_j + j + \min(\alpha_j - u_j, \bar{h}_j - j) = \alpha_j + j$ elements of $\bar{\mathbf{f}}$ that are bigger or equal than $d_{\bar{h}_j}$. Therefore $\bar{f}_{\alpha_j+j} \geq d_{\bar{h}_j} > g_{\alpha_j+1}$, and so $l_j \leq \alpha_j + j$.

For the other inequality, first suppose that $\bar{f}_{l_j} > d_{\bar{h}_j}$. Then among $\{\bar{f}_1, \ldots, \bar{f}_{l_j}\}$, there would be at most $j-1$ $d_{\bar{h}_i}$'s, while all other elements would be less than or equal to some of the elements of the partition $\mathbf{g}$. Therefore, we would have that for all $i = 1, \ldots, l_j$, $\bar{f}_i \leq g_{i-(j-1)}$, and so $\bar{f}_{l_j} \leq g_{l_j-j+1}$, which contradicts the definition of l_j.

Hence $\bar{f}_{l_j} \leq d_{\bar{h}_j}$, and so by (3.23) and the definition of l_j

$$g_{\alpha_j} \geq d_{\bar{h}_j} \geq \bar{f}_{l_j} > g_{l_j-j+1},$$

and so $l_j \geq \alpha_j + j$. Altogether, this proves (3.27) and (3.28).

In addition, by (3.29) – (3.32), we have also shown that

$$\sum_{i=1}^{l_j} \bar{f}_i = \sum_{i=1}^{\alpha_j+j} \bar{f}_i = \sum_{i=1}^{j} d_{\bar{h}_i} + \sum_{i=1}^{u_j} g_{h_i} + \sum_{i=1}^{\alpha_j-u_j} \min(g'_i, d'_i). \tag{3.33}$$

Now, we have

$$\sum_{i=l_j+1}^{m+k+s} \bar{f}_i = \sum_{i=\alpha_j+j+1}^{m+k+s} \bar{f}_i = \sum_{i=j+1}^{s} d_{\bar{h}_i} + \sum_{i=u_j+1}^{k} g_{h_i} + \sum_{i=\alpha_j-u_j+1}^{m} \min(g_i', d_i')$$
$$= \sum_{i=j+1}^{s} d_{\bar{h}_i} + \sum_{i=u_j+1}^{k} g_{h_i} + \sum_{i=\alpha_j-u_j+1}^{m} g_i' + \sum_{i=\alpha_j-u_j+1}^{m} d_i' - \sum_{i=\alpha_j-u_j+1}^{m} \max(g_i', d_i').$$

We note that by (3.22), (3.23) and (3.25) we have

$$\sum_{i=u_j+1}^{k} g_{h_i} + \sum_{i=\alpha_j-u_j+1}^{m} g_i' = \sum_{i=\alpha_j+1}^{m+k} g_i = \sum_{i=l_j-j+1}^{m+k} g_i.$$

Also,

$$\sum_{i=j+1}^{s} d_{\bar{h}_i} + \sum_{i=\alpha_j-u_j+1}^{m} d_i' - \sum_{i=\alpha_j-u_j+1}^{m} \max(g_i', d_i') =$$
$$\sum_{i=j+1}^{s} d_{\bar{h}_i} + \sum_{i=\alpha_j-u_j+1}^{\bar{h}_j-j} d_i' + \sum_{i=\bar{h}_j-j+1}^{m} d_i' - \sum_{i=\alpha_j-u_j+1}^{\bar{h}_j-j} \max(g_i', d_i') - \sum_{i=\bar{h}_j-j+1}^{m} \max(g_i', d_i').$$

For all $i \in \{\alpha_j - u_j + 1, \ldots, \bar{h}_j - j\}$, by (3.32) and (3.25) we have

$$d_i' \geq d_{\bar{h}_j} > g_i',$$

and so

$$\max(g_i', d_i') = d_i'.$$

We also have

$$\sum_{i=j+1}^{s} d_{\bar{h}_i} + \sum_{i=\bar{h}_j-j+1}^{m} d_i' = \sum_{i=\bar{h}_j+1}^{m+s} d_i.$$

Altogether we have

$$\sum_{i=l_j+1}^{m+k+s} \bar{f}_i = \sum_{i=l_j-j+1}^{m+k} g_i + \sum_{i=\bar{h}_j+1}^{m+s} d_i - \sum_{i=\bar{h}_j-j+1}^{m} \max(g_i', d_i')$$
$$\geq \sum_{i=l_j-j+1}^{m+k} g_i + \sum_{i=\bar{h}_j+1}^{m+s} d_i - \sum_{i=\bar{h}_j-j+1}^{m} c_i,$$

where the last inequality follows from (3.12). Finally by (3.9) we obtain (3.15), as desired. This finishes our proof. □

Remark 3.3. We note that if both $\mathbf{d}$ and $\mathbf{g}$ are partitions consisting of nonnegative integers, such that

$$\sum_{i=1}^{m+s} d_i + \sum_{i=1}^{k} b_i \geq 0,$$

then by Lemma 2.2 the partition $\mathbf{f}$ also consists of nonnegative integers.

In the course of the proof of Theorem 3.2, we have also proved the following result

Corollary 3.4. *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{d}$ *and* $\mathbf{g}$ *be partitions* (1.1), (1.2), (1.4) *and* (1.5)*, respectively. Let* $\mathbf{c} = (c_1, \dots, c_m)$ *be a partition such that*

$$\mathbf{d} \prec'' (\mathbf{c}, \mathbf{a}) \quad \textit{and} \quad \mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}), \tag{3.34}$$

then there exists a partition $\mathbf{f} = (f_1, \dots, f_{m+k+s})$ *such that*

$$\mathbf{f} \prec'' (\mathbf{d}, \mathbf{b}) \quad \textit{and} \quad \mathbf{f} \prec'' (\mathbf{g}, \mathbf{a}). \tag{3.35}$$

Also, by Theorem 2.3 we have

Corollary 3.5. *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{d}$ *and* $\mathbf{g}$ *be partitions* (1.1), (1.2), (1.4) *and* (1.5)*, respectively. If there exists a partition* $\mathbf{c} = (c_1, \dots, c_m)$*, such that*

$$\mathbf{d} \prec' (\mathbf{c}, \mathbf{a}) \quad \textit{and} \quad \mathbf{g} \prec' (\mathbf{c}, \mathbf{b})$$

then there exists a partition $\mathbf{f} = (f_1, \dots, f_{m+k+s})$ *such that*

$$\mathbf{f} \prec' (\mathbf{c}, \mathbf{a} \cup \mathbf{b}).$$

Finally, by combining Theorem 1.3 with the result of Corollary 3.4, we obtain necessary conditions for the pseudo double majorization problem.

Corollary 3.6. *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{d}$ *and* $\mathbf{g}$ *be partitions* (1.1), (1.2), (1.4) *and* (1.5)*, respectively. If there exists a partition* $\mathbf{c} = (c_1, \dots, c_m)$*, such that*

$$\mathbf{d} \prec'' (\mathbf{c}, \mathbf{a}) \quad \textit{and} \quad \mathbf{g} \prec'' (\mathbf{c}, \mathbf{b})$$

then the condition $\bar{\Omega}(\mathbf{g}, \mathbf{d}, \mathbf{b}, \mathbf{a})$ *holds.*

4 Some comments and more on diagrammatics of generalized majorization

4.1 A counter example for the converse of Theorem 3.2

In the following example we show that the converse of Theorem 3.2 does not hold:

Example 4.1. Let us consider the following partitions of integers:

$$\mathbf{d} = (7, 2, 1) \tag{4.1}$$

$$\mathbf{g} = (7, 2, 1) \tag{4.2}$$

$$\mathbf{a} = (3, 1) \tag{4.3}$$

$$\mathbf{b} = (2, 2) \tag{4.4}$$

The partition

$$\mathbf{f} = (4, 4, 3, 2, 1) \tag{4.5}$$

satisfies

$$\mathbf{f} \prec' (\mathbf{g}, \mathbf{a}) \quad \text{and} \quad \mathbf{f} \prec' (\mathbf{d}, \mathbf{b}). \tag{4.6}$$

Indeed, (4.6) is equivalent to

$$\min(g_i, d_i) \geq f_{i+2}, \qquad i = 1, \dots, 3, \tag{4.7}$$

$$\sum_{i=l_j+1}^{5} f_i \geq \sum_{i=l_j-j+1}^{3} g_i + \sum_{i=j+1}^{2} a_i, \qquad j = 1, 2, \tag{4.8}$$

$$\sum_{i=1}^{5} f_i = \sum_{i=1}^{3} g_i + \sum_{i=1}^{2} a_i = \sum_{i=1}^{3} d_i + \sum_{i=1}^{2} b_i, \tag{4.9}$$

$$\sum_{i=\bar{l}_j+1}^{5} f_i \geq \sum_{i=\bar{l}_j-j+1}^{3} d_i + \sum_{i=j+1}^{2} b_i, \qquad j = 1, 2, \tag{4.10}$$

where

$$l_1 = \bar{l}_1 = 2, \quad l_2 = \bar{l}_2 = 3.$$

By (4.1) – (4.5) we directly get that all of (4.7) – (4.10) hold. Hence we have (4.6), as announced.

However, there is no partition $\mathbf{c}$ satisfying

$$\mathbf{g} \prec' (\mathbf{c}, \mathbf{b}) \quad \text{and} \quad \mathbf{d} \prec' (\mathbf{c}, \mathbf{a}). \tag{4.11}$$

Indeed, by the definition of generalized majorization, we would have that such a partition $\mathbf{c}$ would be of length one, i.e. $\mathbf{c} = (c_1)$ for certain integer c_1, and that

$$c_1 = \sum_{i=1}^{3} g_i - \sum_{i=1}^{2} b_i = \sum_{i=1}^{3} d_i - \sum_{i=1}^{2} a_i = 6$$

Then

$$h_1 = \min\{i | c_i < g_i\} = 2,$$

and hence we would need that

$$\sum_{i=h_1+1}^{3} g_i \geq \sum_{i=h_1-1+1}^{1} c_i + \sum_{i=1+1}^{2} b_i$$

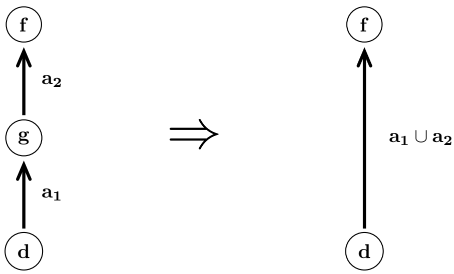
which gives

$$g_3 = 1 \geq b_2 = 2,$$

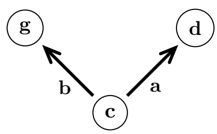
which is a contradiction. Hence there is no partition $\mathbf{c}$ satisfying (4.11), as announced.

4.2 Diagrammatics

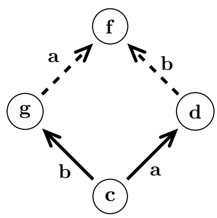
By using diagrammatics introduced in Section 1, Theorem 2.3 implies the following transitivity-like property of the generalized majorization:



The main result of the paper, Theorem 3.2, can be described diagrammatically, by stating that every diagram of the form



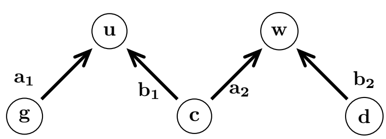
can be completed to a square



The two properties allow various combinations. For example, by combining the result from Theorem 3.2 with the result from Theorem 2.3 we can get the following. Let $\mathbf{c}$, $\mathbf{u}$, $\mathbf{w}$, $\mathbf{g}$, $\mathbf{d}$, $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{b_1}$ and $\mathbf{b_2}$ be partitions such that

$$\mathbf{u} \prec' (\mathbf{g}, \mathbf{a_1}), \quad \mathbf{u} \prec' (\mathbf{c}, \mathbf{b_1}), \quad \mathbf{w} \prec' (\mathbf{c}, \mathbf{a_2}), \quad \mathbf{w} \prec' (\mathbf{d}, \mathbf{b_2}),$$

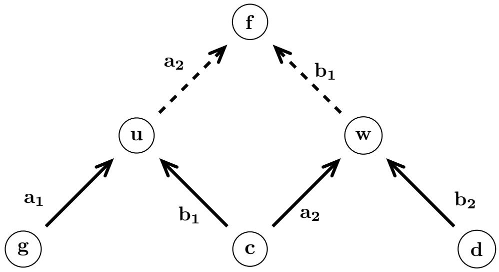
i.e.



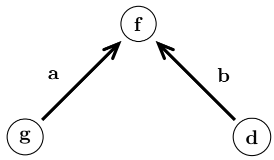
Then by Theorem 3.2 there exists a partition $\mathbf{f}$ such that

$$\mathbf{f} \prec' (\mathbf{u}, \mathbf{a_2}) \quad \text{and} \quad \mathbf{f} \prec' (\mathbf{w}, \mathbf{b_1}).$$

Diagrammatically this gives



Finally, by Theorem 2.3, such $\mathbf{f}$ satisfies



where

$$\mathbf{a} = \mathbf{a_1} \cup \mathbf{a_2}, \quad \mathbf{b} = \mathbf{b_1} \cup \mathbf{b_2}.$$

5 Conclusions

In this paper we study new properties of generalized majorization. The main result of the paper is the proof that the generalized majorization has a completing-squares property. More precisely, we have introduced pseudo double majorization problem for two pairs of partitions (Problem 1.4), and we relate it with double majorization problem (Problem 1.2). In particular, we prove that the existence of a partition $\mathbf{c}$ satisfying (1.11) implies the existence of a partition $\mathbf{f}$ satisfying (1.10). By introducing diagrammatical interpretation of generalized majorization, our main result has an elegant geometric interpretation, which also complements the previous results on transitivity-like property of generalized majorization [10].

Finally, the obtained results are expected to have strong impact in solving the General Matrix Pencil Completion Problem, as well as in solving Bounded Rank Perturbation Problems for matrix pencils.

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