Image Anal Stereol 2012;31:109-120
Original Research Paper
doi:10.5566/ias.v31.p109-120

ON RANDOM ITERATED FUNCTION SYSTEMS WITH GREYSCALE MAPS

MATTHEW DEMERS[1], HERB KUNZE[✉,2] AND DAVIDE LA TORRE[3]

[1]University of Guelph, Department of Mathematics & Statistics, Guelph, Ontario, Canada; [2]University of Guelph, Department of Mathematics & Statistics, Guelph, Ontario, Canada; [3]University of Milan, Department of Economics, Business and Statistics, Milan, Italy
e-mail: mdemers@uoguelph.ca; hkunze@uoguelph.ca; davide.latorre@unimi.it
(Received January 18, 2012; revised April 27, 2012; accepted May 8, 2012)

ABSTRACT

In the theory of Iterated Function Systems (IFSs) it is known that one can find an IFS with greyscale maps (IFSM) to approximate any target signal or image with arbitrary precision, and a systematic approach for doing so was described. In this paper, we extend these ideas to the framework of random IFSM operators. We consider the situation where one has many noisy observations of a particular target signal and show that the greyscale map parameters for each individual observation inherit the noise distribution of the observation. We provide illustrative examples.

Keywords: random fixed point equations, random iterated function systems, collage theorem.

INTRODUCTION

In fractal imaging compression and coding based on Generalized Fractal Transforms (GFT), one seeks to approximate a target image or signal by the fixed points of a contractive fractal transform operator. The usual formulation involves a fixed set of geometric contraction maps along with a corresponding set of greyscale adjustment maps. The inverse problem requires one to find the best greyscale map parameters for a given target image. The solution process is referred to as "collage coding" because of the visual effect of the contraction maps; it applies the collage theorem, a simple consequence of Banach's fixed point theorem. The approximation of the target image or signal can be generated through iteration of the fractal transform (see Hutchinson, 1981; Barnsley *et al.*, 1985; Barnsley and Demko, 1985; Barnsley, 1989; Barnsley and Hurd, 1993; Forte and Vrscay, 1995; 1999; Iacus and La Torre, 2005b;a; Kunze *et al.*, 2008; La Torre *et al.*, 2009; La Torre and Vrscay, 2011 for more details and applications).

In Forte and Vrscay (1995), the authors showed that one can find an iterated function system with greyscale maps (IFSM) to approximate any target signal or image with arbitrary precision, and they provided a suboptimal but systematic approach for doing so.

The goal of this paper is to extend the work in Forte and Vrscay (1995) to the framework of random IFSM operators. We consider the situation where one has many noisy observations of a particular target signal and show that the greyscale map parameters for each individual observation inherit the noise distribution of the observation.

Prior to these results, we present several short background sections. First, we discuss the deterministic collage coding framework and quickly summarize the related IFSM setting. Then we present the random collage coding framework. Finally, we formulate the random IFSM operator framework and present some illustrative numerical examples.

FIXED POINT EQUATIONS AND COLLAGE THEOREM

In this section, we present some basic contraction map results that will be used in later sections. Let (X,d_X) denote a complete metric space. Then $T: X \to X$ is contractive if there exists a $c \in [0,1)$ such that

$$d_X(Tx,Ty) \leq c d_X(x,y) \quad \text{for all} \quad x,y \in X\,. \tag{1}$$

We normally refer to the infimum of all c values satisfying Eq. 1 as the *contraction factor* of T.

Theorem 1 *(Banach): Let (X,d_X) be a complete metric space. Also let $T: X \to X$ be a contraction mapping with contraction factor $c \in [0,1)$. Then there exists a unique $\bar{x} \in X$ such that $\bar{x} = T\bar{x}$. Moreover, for any $x \in X$, $d_X(T^n x, \bar{x}) \to 0$ as $n \to \infty$.*

Theorem 2 *(Continuity theorem for fixed points, Centore and Vrscay, 1994): Let (X,d_X) be a complete metric space and T_1, T_2 be two contractive mappings* *with contraction factors c_1 and c_2 and fixed points $\bar{y}_1$ and $\bar{y}_2$, respectively. Then*

$$d_X(\bar{y}_1,\bar{y}_2) \leq \frac{1}{1-\min\{c_1,c_2\}} d_{X,\sup}(T_1,T_2) \quad (2)$$

where

$$d_{X,\sup}(T_1,T_2) = \sup_{x \in X} d_X(T_1(x),T_2(x)) \; . \quad (3)$$

Banach's fixed point theorem provides a mechanism to solve the *forward* problem of the fixed point equation $x = Tx$: Given a contraction T, construct, or at least approximate, its fixed point $\bar{x}$. And the continuity theorem establishes that small changes in a contraction mapping produce small changes in the associated fixed points.

A formal mathematical *inverse problem* associated with the fixed point equation $x = Tx$ is: Given a target element x and an $\varepsilon > 0$, find a contraction map $T(\varepsilon)$ (perhaps from a suitable family of operators) with fixed point $\bar{x}(\varepsilon)$ such that $d(x,\bar{x}(\varepsilon)) < \varepsilon$. In the case that one is able to solve such an inverse problem to arbitrary precision, i.e., $\varepsilon \to 0$, then one may identify the target x as the fixed point of a contractive operator T on X.

In practical applications, however, it is not generally possible to find such solutions to arbitrary accuracy nor is it even feasible to search for such contraction maps. Instead, one makes use of the following result, which is a simple consequence of Banach's fixed point theorem.

Theorem 3 *("Collage theorem", Barnsley et al., 1985): Let (X,d) be a complete metric space and $T : X \to X$ a contraction mapping with contraction factor $c \in [0,1)$. Then for any $x \in X$,*

$$d(x,\bar{x}) \leq \frac{1}{1-c} d(x,Tx) \; , \quad (4)$$

where $\bar{x}$ is the fixed point of T.

The *approximation error* $d(x,\bar{x})$ is bounded above by the so-called *collage distance* $d(x,Tx)$. Most practical methods of solving such inverse problems, for example, fractal image coding (Fisher, 1995; Lu, 2003), search for an operator T for which the collage distance is as small as possible, while ensuring that c is sufficiently far away from one. In other words, these methods seek an operator T that maps the target x as close as possible to itself. This inverse problem solution procedure, often referred to as *collage coding*, is most often performed by considering a parametrized family of contraction maps T_λ, $\lambda \in \Lambda \subset \mathbb{R}^n$, and then minimizing the collage distance $d(x,T_\lambda x)$.

Finally, we mention another interesting result which is a simple consequence of Banach's fixed point theorem.

Theorem 4 *("Anti-collage theorem", Vrscay and Saupe, 1999): Let (X,d) be a complete metric space and $T : X \to X$ a contraction mapping with contraction factor $c \in [0,1)$. Then for any $x \in X$,*

$$d(x,\bar{x}) \geq \frac{1}{1+c} d(x,Tx) \; , \quad (5)$$

where $\bar{x}$ is the fixed point of T.

FORMAL SOLUTION TO THE INVERSE PROBLEM FOR ITERATED FUNCTION SYSTEMS WITH GREYSCALE MAPS

The method of iterated function systems with greyscale maps (IFSM), as formulated by Forte and Vrscay (1995), can be used to approximate a given element u of $L^2([0,1])$. We consider the case in which $u : [0,1] \to [0,1]$ and the space

$$X = \left\{ u : [0,1] \to [0,1], u \in L^2[0,1] \right\} \; . \quad (6)$$

The ingredients of an N-map IFSM on X are (Forte and Vrscay, 1995; 1999)

1. a set of N contractive mappings $w = \{w_1,w_2,\ldots,w_N\}$, $w_i : [0,1] \to [0,1]$, satisfying the covering condition $\bigcup_{i=1}^N w_i([0,1]) = [0,1]$, and most often affine in form:

$$w_i(x) = s_i x + a_i, \quad 0 \leq s_i < 1, \quad i = 1,2,\ldots,N; \quad (7)$$

2. a set of associated functions – the greyscale maps – $\phi = \{\phi_1,\phi_2,\ldots,\phi_N\}$, $\phi_i : \mathbb{R} \to \mathbb{R}$. Affine maps are usually employed:

$$\phi_i(t) = \alpha_i t + \beta_i \, , \quad (8)$$

with the conditions

$$\alpha_i, \beta_i \in [0,1] \quad (9)$$

and

$$0 \leq \sum_{i=1}^{N} \alpha_i + \beta_i < 1 \; . \quad (10)$$

Associated with the N-map IFSM (w,ϕ) is the *fractal transform* operator T, the action of which on a function $u \in X$ is given by

$$(Tu)(x) = \sum_{i=1}^{N}{}' \phi_i(u(w_i^{-1}(x))) , \qquad (11)$$

where the prime means that the sum operates on all those terms for which w_i^{-1} is defined.

Theorem 5 *(Forte and Vrscay, 1995): $T : X \to X$ and for any $u, v \in X$ we have*

$$d_2(Tu,Tv) \le Cd_2(u,v) \qquad (12)$$

where

$$C = \sum_{i=1}^{N} s_i^{\frac{1}{2}} \alpha_i . \qquad (13)$$

When $C < 1$, then T is contractive on X, implying the existence of a unique fixed point $\bar{u} \in X$ such that $\bar{u} = T\bar{u}$.

The squared collage distance function associated with an N-map IFSM may be written as a quadratic form,

$$\Delta^2 = z^T Az + b^T z + c , \qquad (14)$$

where $z = (\alpha_1, \dots \alpha_k, \beta_1, \dots, \beta_k)$. We observe that in the special that the sets $w_i([0,1])$, $i = 1,\dots,N$, are non-overlapping, for each i the values of α_i and β_i that minimize Δ^2 can be determined by solving a separate minimization problem using least squares.

The inverse problem associated with IFSM can, in principle, be solved to arbitrary accuracy, using a procedure defined in Forte and Vrscay (1995). The maps w_k are chosen from an infinite set W of fixed affine contraction maps on $[0,1]$ which satisfy the following properties.

Definition 1 *We say that W generates an m-dense and nonoverlapping family A of subsets of I if for every $\varepsilon > 0$ and every $B \subset I$ there exists a finite set of integers i_k, $i_k \ge 1$, $1 \le k \le N$, such that*

(i) $A = \cup_{k=1}^{N} w_{i_k}(I) \subset B$,

(ii) $m(B \backslash A) < \varepsilon$, and

(iii) $m(w_{i_k}(I) \cap w_{i_l}(I)) = 0$ if $k \ne l$,

where m denotes Lebesgue measure.

Let

$$W^N = \{w_1, \dots w_N\} \qquad (15)$$

be the N truncations of w. Let $\Phi^N = \{\phi_1,\dots,\phi_N\}$ be the N-vector of affine grey level maps. Let z_N be the solution of the previous quadratic optimization problem and $\Delta^2_{N,min} = \Delta^2_N(z_N)$. In Forte and Vrscay (1995), the following result was proved.

Theorem 6

$$\Delta^2_{N,min} \to 0 \text{ as } N \to \infty.$$

Using the Collage Theorem, the inverse problem may be solved to arbitrary accuracy. A practical choice for the contraction maps w on $X = [0,1]$ is

$$w_{ij}(x) = 2^{-i}(x + j - 1),\ i = 1,2,\dots,\ j = 1,2,\dots,2^i.$$

RANDOM FIXED POINT EQUATIONS

Let $(\Omega, \mathscr{F}, \mathbb{P})$ denote a probability space and (X,d) a metric space. A mapping $T : \Omega \times X \to X$ is called a *random operator* if for any $x \in X$ the function $T(.,x)$ is measurable. A measurable mapping $x : \Omega \to X$ is called a *random fixed point* of a random operator T if x is a solution of the equation

$$T(w,x(w)) = x(w) . \qquad (16)$$

for a.e. $\omega \in \Omega$. A random operator T is called continuous (Lipschitz, contraction) if for a.e. $w \in \Omega$ we have that $T(w,.)$ is continuous (Lipschitz, contraction). There are many papers in the literature that deal with such equations for single-valued and set-valued random operators (see Bharucha-Reid, 1972; Itoh, 1979; Papageorgiou, 1988; Lin, 1995; Liu, 1997; Shahzad, 2001; 2004a;b). Solutions of random fixed point equations and inclusions are usually produced by means of stochastic generalizations of classical fixed point theory.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let (X, d_X) be a complete metric space. Let $T : \Omega \times X \to X$ be a given operator. We look for the solution of the equation

$$T(\omega,x(\omega)) = x(\omega) \qquad (17)$$

for a.e. $\omega \in \Omega \backslash A$ and $\mathbb{P}(A) = 0$. Suppose that the operator T satisfies the inequality

$$d(T(\omega,x),T(\omega,y)) \le c(\omega)d(x,y) , \qquad (18)$$

where $c(\omega) : \Omega \to X$. When T satisfies this property, we say that T is a $c(\omega)$-Lipschitz operator. If $c(\omega) \le c < 1$ a.e., with $\omega \in \Omega \backslash A$ and $\mathbb{P}(A) = 0$ then we say that T is a $c(\omega)$-contraction. In this case for $\omega \in \Omega \backslash A$ there exists a unique point $x(\omega) \in X$. It is clear that the uniqueness makes sense except for sets of measure zero. Once again, the inverse problem can be formulated as: Given a function $x : \Omega \to X$ and a family of operators $T_\lambda : \Omega \times X \to X$, find λ such that x is the solution of random fixed point equation

$$T_\lambda(\omega,x(\omega)) = x(\omega) . \qquad (19)$$

We may now state the following random analogs to Theorems 1-4.

Corollary 1 *("Regularity conditions"): Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space and (X,d_X) be a complete metric space. Let $T:\Omega\times X\to X$ be a given $c(\omega)$-contraction. Then $d_X(x(\omega_1),x(\omega_2))\leq \frac{1}{1-c}\sup_{x\in X} d(T(\omega_1,x),T(\omega_2,x))$*

Corollary 2 *("Continuity Theorem"): Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space and let (X,d_X) be a complete metric space. Let $T_1:\Omega\times X\to X$ and $T_2:\Omega\times X\to X$ be $c_1(\omega)$- and $c_2(\omega)$-contractions respectively. Then*

$$d_X(x_1(\omega),x_2(\omega))\leq \frac{1}{1-\min\{c_1,c_2\}}\sup_{x\in X} d_X(T_1(\omega,x),T_2(\omega,x))\ .$$

Theorem 7 *("Collage Theorem"): Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space and let (X,d_X) be a complete metric space. Let $T:\Omega\times X\to X$ be a given $c(\omega)$-contraction. Then*

$$\frac{1}{1+c}d_X(x(\omega),T(\omega,x(\omega)))\leq d_X(x(\omega),\bar{x}(\omega))\leq \frac{1}{1-c}d_X(x(\omega),T(\omega,x(\omega)))\ ,$$

a.e. $\omega\in\Omega$, where $\bar{x}(\omega)$ is the solution of $T(\omega,\bar{x}(\omega))=\bar{x}(\omega)$.

The above described approach to deal with random fixed point equations does not guarantee, in general, the measurability of the function $x:\Omega\to X$ with respect to the sigma algebra $\mathscr{F}$. In order to overcome this difficulty, we provide an abstract formulation of the same problem which guarantees the measurability of x.

Recall that (X,d) is said to be a Polish space if it is a separable complete metric space. In random fixed point theory, separability plays an important role. Two important examples of Polish metric spaces which will be useful in the following are $C([a,b])$ and $L^2([a,b])$. Consider now the space Y of all measurable functions $x:\Omega\to X$. If we define the operator $\tilde{T}:Y\to Y$ as $(\tilde{T}y)(\omega)=T(\omega,x(\omega))$ the solutions of this fixed point equation on Y are the solutions of the random fixed point equation $T(\omega,x(\omega))=x(\omega)$. Suppose that the metric d_X is bounded, that is $d_X(x_1,x_2)\leq K$ for all $x_1,x_2\in X$. So the function $\psi(\omega)=d_X(x_1(\omega),x_2(\omega)):\Omega\to\mathbb{R}$ is an element of $L^1(\Omega)$ for all $x_1,x_2\in\Omega$. We can then define on the space Y the following function

$$d_Y(x_1,x_2)=\int_\Omega d_X(x_1(\omega),x_2(\omega))\,\mathrm{d}\omega\ . \qquad (20)$$

It can be shown that the space (Y,d_Y) is a complete metric space. The following result holds.

Theorem 8 *(Itoh, 1977): Let X be a Polish space, and $T:\Omega\times X\to X$ be a mapping such that for each $\omega\in\Omega$ the function $T(\omega,.)$ is $c(\omega)$-Lipschitz and for each $x\in X$ the function $T(.,x)$ is measurable. Let $x:\Omega\to X$ be a measurable mapping; then the mapping $\xi:\Omega\to X$ defined by $\xi(\omega)=T(\omega,x(\omega))$ is measurable.*

The previous theorem (Eq. 8) holds if $T(\omega,\cdot)$ is only continuous (see, for instance, Himmelberg, 1975).

Corollary 3 *Let X be a Polish space and $T:\Omega\times X\to X$ be a mapping such that for each $\omega\in\Omega$ the function $T(\omega,\cdot)$ is a $c(\omega)$-contraction. Suppose that for each $x\in X$ the function $T(\cdot,x)$ is measurable. Then there exists a unique solution of the equation $\tilde{T}\bar{x}=\bar{x}$ that is $T(\omega,\bar{x}(\omega))=\bar{x}(\omega)$ for a.e. $\omega\in\Omega$.*

In these results separability plays a crucial role. In fact using this hypothesis and Theorem 8 one can prove that the function $\xi(\omega)=T(\omega,x(\omega))$ is measurable, that is $\tilde{T}:Y\to Y$. Other random fixed point theorems for contraction mappings in Polish spaces can be found in Spaček (1955); Hanš (1957); Bharucha-Reid (1972). The inverse problem can be formulated as: Given a function $\bar{x}:\Omega\to X$ and a family of operators $\tilde{T}_\lambda:Y\to Y$ find λ such that $\bar{x}$ is the solution of random fixed point equation

$$\tilde{T}_\lambda\bar{x}=\bar{x}\ , \qquad (21)$$

that is,

$$T_\lambda(\omega,\bar{x}(\omega))=\bar{x}(\omega)\ . \qquad (22)$$

Now, with the collage and continuity theorems, we may state the following.

Corollary 4 *Let X be a Polish space and $T:\Omega\times X\to X$ be a mapping such that for each $\omega\in\Omega$ the function $T(\omega,\cdot)$ is a $c(\omega)$-contraction. Suppose that for each $x\in X$ the function $T(\cdot,x)$ is measurable. Then for any $x\in Y$,*

$$\frac{1}{1+c}d_Y(x,\tilde{T}x)\leq d_Y(x,\bar{x})\leq\frac{1}{1-c}d_Y(x,\tilde{T}x)\ , \qquad (23)$$

where $\bar{x}$ is the fixed point of $\tilde{T}$, that is, $\bar{x}(\omega):=T(\omega,\bar{x}(\omega))$.

FORMAL SOLUTION TO THE INVERSE PROBLEM FOR RANDOM ITERATED FUNCTION SYSTEMS WITH GREYSCALE MAPS

We now formulate a *Random IFSM* fractal transform that is analogous to the deterministic fractal transform in Eq. 11.

Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space and consider now the following operator $T : X\times\Omega\to X$ defined by

$$\begin{aligned}T(\omega,u) &= (Tu)(x)\\ &= \sum_{i=1}^{N}{}'\alpha_i(u(w_i^{-1}(x)))+\beta_i(\omega)\\ &= (T^*u)+\beta(\omega)\ ,\end{aligned}$$

where $\beta(\omega)=\sum_{i=1}^{N}{}'\beta_i(\omega)$. That is, we suppose that the greyscale maps take the form

$$\phi_i(\omega,t)=\alpha_i t+\beta_i(\omega)\ , \tag{24}$$

where β_i are random variables with mean μ_i and $|\beta_i(\omega)|<\delta$ a.e. $\omega\in\Omega$, and that

$$\alpha_i,\delta\in[0,1] \tag{25}$$

with

$$0\le\sum_{i=1}^{N}(\alpha_i+\delta)<1\ . \tag{26}$$

We randomize only the β_i parameters since the randomness in the other term of $T(\omega,u)$ is transmitted by the function u. Let

$$Y=\{u:\Omega\to X, u \text{ is measurable}\} \tag{27}$$

and consider the function

$$\psi(\omega):=d_X(u_1(\omega),u_2(\omega))=\|u_1(\omega)-u_2(\omega)\|_2.$$

From the hypotheses it is clear that $\psi(\omega)\in L^1(\Omega)$. We know that (Y,d_Y) is a complete metric space where

$$d_Y(u_1,u_2)=\int_\Omega\|u_1(\omega)-u_2(\omega)\|_2 d\omega\ . \tag{28}$$

Obviously, the function $\xi(\omega):=(T^*u)(\omega)+\beta(\omega)$ belongs to Y because T^* is Lipschitz on X and the β is measurable. If we define $\tilde{T}:Y\to Y$ where $\tilde{T}u=\xi$ we have

$$\begin{aligned}&d_Y(\tilde{T}u_1,\tilde{T}u_2)=\\ &\int_\Omega\|(T^*u_1)(\omega)+\beta(\omega)-(T^*u_2)(\omega)-\beta(\omega)\|_2\,\mathrm{d}\omega=\\ &\int_\Omega\|u_1(\omega)-u_2(\omega)\|_2\,\mathrm{d}\omega=Cd_Y(u_1,u_2)\ .\end{aligned}$$

So there exists $\bar{u}\in Y$ such that $\tilde{T}\bar{u}=\bar{u}$, that is, $\bar{u}(\omega)=T(\omega,\bar{u}(\omega))$ for a.e. $\omega\in\Omega$. That is,

$$\begin{aligned}\bar{u}(\omega,x) &= (T^*\bar{u}(\omega))(x)+\beta(\omega)\\ &= \sum_{i=1}^{N}{}'\alpha_i(\bar{u}(\omega,w_i^{-1}(x)))+\beta_i(\omega)\end{aligned} \tag{29}$$

for a.e. $x\in[0,1]$. The inverse problem of finding the random variables $\beta_i(\omega)$, perhaps to arbitrary accuracy, is note possible given the random nature of the problem. Instead, we suppose that the function $\bar{u}(\omega,x):\Omega\times[0,1]\to\mathbb{R}$ is integrable and let

$$\bar{u}(x)=\int_\Omega[\bar{u}(\omega)](x)\,\mathrm{d}\omega=\int_\Omega\bar{u}(\omega,x)\,\mathrm{d}\omega\ . \tag{30}$$

Thus Eq. 29 becomes

$$\bar{u}(x)=\sum_{i=1}^{N}{}'\alpha_i(\bar{u}(w_i^{-1}(x)))+\mu_i\ . \tag{31}$$

That is, the expectation value of $\bar{u}(\omega,x)$ is the solution of a deterministic N-map IFSM on X (cf. Eq. 11). We observe in Eq. 31 another motivation for not randomizing the parameters α_i: the expectation of the product $\alpha_i u(\cdot)$ becomes complicated.

Suppose that we have N observations, $u(\omega_1)$, $u(\omega_2)$, ..., $u(\omega_n)$, of the variable $u(\omega)$. Choose a set of greyscale maps, as in Eq. 15. We collage code each observation, $u(\omega_k)$, $k=1,\ldots,n$, to find the set of minimal collage parameters α_{ik} and β_{ik}, $k=1,\ldots,n$, $i=1,\ldots,N$. As a result of our earlier discussion, if the observations differ only by a uniform additive noise factor, we expect that the distribution of the β_{ik} parameters, $k=1\ldots,n$ will inherit that distribution, with mean μ_i.

NUMERICAL EXAMPLES

This section provides several numerical examples which illustrate the above formulation and "mean property."

Example: We define the IFS maps

$$w_{ij}(x)=2^{-i}(x+j-1),\quad i=1,\ldots,4,\ j=1,2,\ldots,2^i,$$

with associated greyscale maps $\phi_{ij}(t)=\alpha_{ij}t+\beta_{ij}$. In this example, we consider the target function

$$u(x)=0.8x^2+0.1\ . \tag{32}$$

We construct N observations of the function by adding constant noise to it:

$$u_k(x)=u(x)+\nu_k,\quad k=1,\ldots,N\ ,$$

where $|\nu_k|<0.1$ with mean zero. The picture in Fig. 1 presents ten realizations with constant and place-dependent noise.

Table 1. *Greyscale map mean parameters $\alpha_{i,j}$ in the case of constant additive noise.*

coefficient	50	100	200
$\alpha_{1,1}$	0.25000	0.25000	0.25000
$\alpha_{1,2}$	0.25732	0.25730	0.25730
$\alpha_{2,1}$	0.06250	0.06250	0.06250
$\alpha_{2,2}$	0.06983	0.06982	0.06982
$\alpha_{2,3}$	0.07715	0.07715	0.07715
$\alpha_{2,4}$	0.08447	0.08447	0.08447
$\alpha_{3,1}$	0.01563	0.01563	0.01563
$\alpha_{3,2}$	0.02295	0.02295	0.02295
$\alpha_{3,3}$	0.03028	0.03028	0.03028
$\alpha_{3,4}$	0.03760	0.03760	0.03760
$\alpha_{3,5}$	0.04493	0.04493	0.04493
$\alpha_{3,6}$	0.05226	0.05226	0.05226
$\alpha_{3,7}$	0.05958	0.05958	0.05958
$\alpha_{3,8}$	0.06691	0.06691	0.06691
$\alpha_{4,1}$	0.00391	0.00391	0.00391
$\alpha_{4,2}$	0.01124	0.01124	0.01124
$\alpha_{4,3}$	0.01857	0.01857	0.01857
$\alpha_{4,4}$	0.02590	0.02590	0.02590
$\alpha_{4,5}$	0.03323	0.03323	0.03323
$\alpha_{4,6}$	0.04056	0.04056	0.04056
$\alpha_{4,7}$	0.04790	0.04789	0.04789
$\alpha_{4,8}$	0.05523	0.05523	0.05523
$\alpha_{4,9}$	0.06256	0.06256	0.06256
$\alpha_{4,10}$	0.06989	0.06989	0.06989
$\alpha_{4,11}$	0.07722	0.07722	0.07722
$\alpha_{4,12}$	0.08455	0.08455	0.08455
$\alpha_{4,13}$	0.09188	0.09188	0.09188
$\alpha_{4,14}$	0.09921	0.09921	0.09921
$\alpha_{4,15}$	0.10655	0.10650	0.10650
$\alpha_{4,16}$	0.11388	0.11390	0.11390

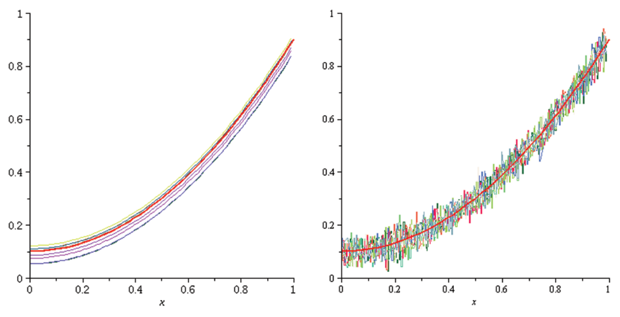

Fig. 1. *Ten realizations for constant additive noise and place-dependent additive noise.*

Tables 1 and 2 presents the mean value of each greyscale map parameter obtained by collage coding the separate realizations. The calculations were performed for 50, 100, and 200 realizations, using Maple.

Table 2. *Greyscale map mean parameters $\beta_{i,j}$ in the case of constant additive noise.*

coefficient	50	100	200
$\beta_{1,1}$	0.07105	0.07261	0.07199
$\beta_{1,2}$	0.07157	0.07312	0.07250
$\beta_{2,1}$	0.08881	0.09077	0.08998
$\beta_{2,2}$	0.08948	0.09142	0.09065
$\beta_{2,3}$	0.09054	0.09247	0.09170
$\beta_{2,4}$	0.09200	0.09391	0.09315
$\beta_{3,1}$	0.09325	0.09530	0.09448
$\beta_{3,2}$	0.09451	0.09655	0.09573
$\beta_{3,3}$	0.09733	0.09935	0.09854
$\beta_{3,4}$	0.10171	0.10372	0.10292
$\beta_{3,5}$	0.10766	0.10965	0.10885
$\beta_{3,6}$	0.11517	0.11715	0.11635
$\beta_{3,7}$	0.12424	0.12620	0.12542
$\beta_{3,8}$	0.13487	0.13682	0.13604
$\beta_{4,1}$	0.09436	0.09644	0.09561
$\beta_{4,2}$	0.09796	0.10002	0.09920
$\beta_{4,3}$	0.10781	0.10986	0.10904
$\beta_{4,4}$	0.12391	0.12594	0.12513
$\beta_{4,5}$	0.14626	0.14828	0.14747
$\beta_{4,6}$	0.17486	0.17686	0.17606
$\beta_{4,7}$	0.20971	0.21170	0.21090
$\beta_{4,8}$	0.25081	0.25278	0.25200
$\beta_{4,9}$	0.29816	0.30012	0.29934
$\beta_{4,10}$	0.35176	0.35371	0.35293
$\beta_{4,11}$	0.41161	0.41354	0.41277
$\beta_{4,12}$	0.47771	0.47963	0.47886
$\beta_{4,13}$	0.55006	0.55196	0.55120
$\beta_{4,14}$	0.62866	0.63055	0.62979
$\beta_{4,15}$	0.71351	0.71538	0.71463
$\beta_{4,16}$	0.80461	0.80647	0.80573

Tables 3 and 4 present the greyscale parameters obtained upon collage coding the mean observation

$$u^*(x) = \frac{1}{N}\sum_{k=1}^{N} u_k(x) \,.$$

We see that the corresponding values in the tables agree increasingly well as the number of realizations increase. In all cases, for any fixed choices of i and j, the value of $\alpha_{i,j}$ is the same across all simulations and across the two tables. In fractal imaging, the α parameters represent a contrast adjustment and the β parameters represent a brightness adjustment. In this example, the different realizations differ only by a vertical shift, so the corresponding fractal codes differ only in the brightness parameters. We also observe that values in the tables agree to at least two decimal places with the IFSM parameters for the *unnoised* signal, which makes sense since the additive noise has mean zero.

Table 3. *Greyscale map parameters $\alpha_{i,j}$ for the mean observation in the case of constant additive noise.*

coefficient	50	100	200
$\alpha_{1,1}$	0.25000	0.25000	0.25000
$\alpha_{1,2}$	0.25730	0.25730	0.25730
$\alpha_{2,1}$	0.06250	0.06250	0.06250
$\alpha_{2,2}$	0.06982	0.06982	0.06982
$\alpha_{2,3}$	0.07715	0.07715	0.07715
$\alpha_{2,4}$	0.08447	0.08447	0.08447
$\alpha_{3,1}$	0.01562	0.01562	0.01562
$\alpha_{3,2}$	0.02295	0.02295	0.02295
$\alpha_{3,3}$	0.03028	0.03028	0.03028
$\alpha_{3,4}$	0.03760	0.03760	0.03760
$\alpha_{3,5}$	0.04493	0.04493	0.04493
$\alpha_{3,6}$	0.05226	0.05226	0.05226
$\alpha_{3,7}$	0.05958	0.05958	0.05958
$\alpha_{3,8}$	0.06691	0.06691	0.06691
$\alpha_{4,1}$	0.00391	0.00391	0.00391
$\alpha_{4,2}$	0.01124	0.01124	0.01124
$\alpha_{4,3}$	0.01857	0.01857	0.01857
$\alpha_{4,4}$	0.02590	0.02590	0.02590
$\alpha_{4,5}$	0.03323	0.03323	0.03323
$\alpha_{4,6}$	0.04056	0.04056	0.04056
$\alpha_{4,7}$	0.04789	0.04789	0.04789
$\alpha_{4,8}$	0.05523	0.05523	0.05523
$\alpha_{4,9}$	0.06256	0.06256	0.06256
$\alpha_{4,10}$	0.06989	0.06989	0.06989
$\alpha_{4,11}$	0.07722	0.07722	0.07722
$\alpha_{4,12}$	0.08455	0.08455	0.08455
$\alpha_{4,13}$	0.09188	0.09188	0.09188
$\alpha_{4,14}$	0.09921	0.09921	0.09921
$\alpha_{4,15}$	0.10650	0.10650	0.10650
$\alpha_{4,16}$	0.11390	0.11390	0.11390

Table 4. *Greyscale map parameters $\beta_{i,j}$ for the mean observation in the case of constant additive noise.*

coefficient	50	100	200
$\beta_{1,1}$	0.07105	0.07261	0.07199
$\beta_{1,2}$	0.07157	0.07312	0.07250
$\beta_{2,1}$	0.08881	0.09076	0.08998
$\beta_{2,2}$	0.08948	0.09142	0.09065
$\beta_{2,3}$	0.09054	0.09247	0.09170
$\beta_{2,4}$	0.09200	0.09391	0.09315
$\beta_{3,1}$	0.09325	0.09530	0.09448
$\beta_{3,2}$	0.09451	0.09654	0.09573
$\beta_{3,3}$	0.09733	0.09935	0.09854
$\beta_{3,4}$	0.10171	0.10372	0.10292
$\beta_{3,5}$	0.10766	0.10965	0.10885
$\beta_{3,6}$	0.11517	0.11715	0.11635
$\beta_{3,7}$	0.12424	0.12620	0.12542
$\beta_{3,8}$	0.13487	0.13682	0.13604
$\beta_{4,1}$	0.09436	0.09644	0.09561
$\beta_{4,2}$	0.09796	0.10002	0.09920
$\beta_{4,3}$	0.10781	0.10986	0.10904
$\beta_{4,4}$	0.12391	0.12594	0.12513
$\beta_{4,5}$	0.14626	0.14828	0.14747
$\beta_{4,6}$	0.17486	0.17686	0.17606
$\beta_{4,7}$	0.20971	0.21170	0.21090
$\beta_{4,8}$	0.25081	0.25278	0.25200
$\beta_{4,9}$	0.29816	0.30012	0.29934
$\beta_{4,10}$	0.35176	0.35371	0.35293
$\beta_{4,11}$	0.41161	0.41354	0.41277
$\beta_{4,12}$	0.47771	0.47963	0.47886
$\beta_{4,13}$	0.55006	0.55196	0.55120
$\beta_{4,14}$	0.62866	0.63055	0.62979
$\beta_{4,15}$	0.71351	0.71538	0.71463
$\beta_{4,16}$	0.80461	0.80647	0.80573

Example: We repeat the preceding example, this time adding place-dependent noise to produce the N realizations:

$$u_k(x) = u(x) + \nu_k(x), \quad k = 1, \ldots, N ,$$

where $u(x) = 0.8x^2 + 0.1$, as before, $|\nu_k(x)| < 0.1$ for all x, and, for each fixed x, $\nu_k(x)$ are samples from the re-scaled distribution $\mathcal{N}(0, 0.01)$. The right picture in Fig. 1 presents ten realizations. The results are presented in Tables 5–8, with the first table presenting the means of the greyscale map parameters obtained by collage coding the realizations separately and the second table presenting the greyscale map parameters obtained by collage coding the mean observation, $u^*(x)$.

Table 5. *Greyscale map mean parameters $\alpha_{i,j}$ in the case of place-dependent additive noise with a normal distribution.*

coefficient	50	100	200
$\alpha_{1,1}$	0.24783	0.24817	0.24656
$\alpha_{1,2}$	0.25505	0.25544	0.25412
$\alpha_{2,1}$	0.06121	0.06223	0.06291
$\alpha_{2,2}$	0.06938	0.07046	0.07104
$\alpha_{2,3}$	0.07708	0.07851	0.07958
$\alpha_{2,4}$	0.08467	0.08460	0.08452
$\alpha_{3,1}$	0.01528	0.01578	0.01762
$\alpha_{3,2}$	0.02500	0.02240	0.02327
$\alpha_{3,3}$	0.03001	0.03025	0.0279
$\alpha_{3,4}$	0.03565	0.03681	0.03749
$\alpha_{3,5}$	0.04367	0.04528	0.04503
$\alpha_{3,6}$	0.05363	0.05390	0.05448
$\alpha_{3,7}$	0.06107	0.05783	0.05697
$\alpha_{3,8}$	0.06887	0.06726	0.06449
$\alpha_{4,1}$	0.00145	0.00175	0.00103
$\alpha_{4,2}$	0.01221	0.00770	0.00599
$\alpha_{4,3}$	0.01686	0.02246	0.02139
$\alpha_{4,4}$	0.02663	0.02713	0.02261
$\alpha_{4,5}$	0.03320	0.02775	0.02702
$\alpha_{4,6}$	0.03973	0.03700	0.04394
$\alpha_{4,7}$	0.05133	0.05263	0.05698
$\alpha_{4,8}$	0.05110	0.05331	0.05169
$\alpha_{4,9}$	0.06367	0.06415	0.05847
$\alpha_{4,10}$	0.07064	0.07165	0.07154
$\alpha_{4,11}$	0.08009	0.08117	0.08127
$\alpha_{4,12}$	0.08196	0.08116	0.08289
$\alpha_{4,13}$	0.08840	0.08420	0.08868
$\alpha_{4,14}$	0.09788	0.09744	0.09734
$\alpha_{4,15}$	0.10892	0.11101	0.11136
$\alpha_{4,16}$	0.11107	0.11086	0.10584

Table 6. *Greyscale map mean parameters $\beta_{i,j}$ in the case of place-dependent additive noise with a normal distribution.*

coefficient	50	100	200
$\beta_{1,1}$	0.07567	0.07569	0.07636
$\beta_{1,2}$	0.07616	0.07611	0.07662
$\beta_{2,1}$	0.09393	0.09383	0.09400
$\beta_{2,2}$	0.09424	0.09417	0.09423
$\beta_{2,3}$	0.09511	0.09500	0.09509
$\beta_{2,4}$	0.09649	0.09693	0.09764
$\beta_{3,1}$	0.09838	0.09843	0.09808
$\beta_{3,2}$	0.09875	0.09962	0.09961
$\beta_{3,3}$	0.10234	0.10268	0.10412
$\beta_{3,4}$	0.10740	0.10717	0.10756
$\beta_{3,5}$	0.11273	0.11245	0.11306
$\beta_{3,6}$	0.11915	0.11994	0.12040
$\beta_{3,7}$	0.12841	0.12973	0.13083
$\beta_{3,8}$	0.13923	0.14004	0.14117
$\beta_{4,1}$	0.10043	0.09999	0.09986
$\beta_{4,2}$	0.10254	0.10497	0.10664
$\beta_{4,3}$	0.11348	0.11149	0.11200
$\beta_{4,4}$	0.12806	0.12842	0.13100
$\beta_{4,5}$	0.15160	0.15332	0.15423
$\beta_{4,6}$	0.18088	0.18174	0.17955
$\beta_{4,7}$	0.21230	0.21107	0.20814
$\beta_{4,8}$	0.25745	0.25764	0.25772
$\beta_{4,9}$	0.30314	0.30231	0.30508
$\beta_{4,10}$	0.35598	0.35589	0.35696
$\beta_{4,11}$	0.41463	0.41407	0.41363
$\beta_{4,12}$	0.48285	0.48257	0.48226
$\beta_{4,13}$	0.55575	0.55723	0.55612
$\beta_{4,14}$	0.63338	0.63320	0.63350
$\beta_{4,15}$	0.71795	0.71599	0.71530
$\beta_{4,16}$	0.81047	0.81016	0.81133

Table 7. *Greyscale map parameters $\alpha_{i,j}$ for the mean observation in the case of place-dependent additive noise with a normal distribution.*

coefficient	50	100	200
$\alpha_{1,1}$	0.24995	0.25030	0.24878
$\alpha_{1,2}$	0.25722	0.25759	0.25626
$\alpha_{2,1}$	0.06152	0.06250	0.06312
$\alpha_{2,2}$	0.06984	0.07086	0.07153
$\alpha_{2,3}$	0.07725	0.07870	0.07967
$\alpha_{2,4}$	0.08506	0.08504	0.08488
$\alpha_{3,1}$	0.01534	0.01568	0.01744
$\alpha_{3,2}$	0.02503	0.02235	0.02321
$\alpha_{3,3}$	0.03006	0.03031	0.02814
$\alpha_{3,4}$	0.03564	0.03679	0.03740
$\alpha_{3,5}$	0.04381	0.04543	0.04513
$\alpha_{3,6}$	0.05365	0.05386	0.05433
$\alpha_{3,7}$	0.06127	0.05801	0.05702
$\alpha_{3,8}$	0.06899	0.06750	0.06477
$\alpha_{4,1}$	0.00143	0.00178	0.00084
$\alpha_{4,2}$	0.01229	0.00781	0.00629
$\alpha_{4,3}$	0.01675	0.02249	0.02141
$\alpha_{4,4}$	0.02668	0.02705	0.02266
$\alpha_{4,5}$	0.03321	0.02780	0.02719
$\alpha_{4,6}$	0.03973	0.03692	0.04373
$\alpha_{4,7}$	0.05143	0.05280	0.05710
$\alpha_{4,8}$	0.05118	0.05356	0.05175
$\alpha_{4,9}$	0.06371	0.06419	0.05859
$\alpha_{4,10}$	0.07061	0.07159	0.07149
$\alpha_{4,11}$	0.08008	0.08115	0.08136
$\alpha_{4,12}$	0.08206	0.08141	0.08320
$\alpha_{4,13}$	0.08855	0.08437	0.08873
$\alpha_{4,14}$	0.09792	0.09758	0.09738
$\alpha_{4,15}$	0.10897	0.11110	0.11133
$\alpha_{4,16}$	0.11104	0.11083	0.10592

Table 8. *Greyscale map parameters $\beta_{i,j}$ for the mean observation in the case of place-dependent additive noise with a normal distribution.*

coefficient	50	100	200
$\beta_{1,1}$	0.07489	0.07491	0.07554
$\beta_{1,2}$	0.07536	0.07532	0.07583
$\beta_{2,1}$	0.09382	0.09373	0.09392
$\beta_{2,2}$	0.09407	0.09402	0.09405
$\beta_{2,3}$	0.09504	0.09493	0.09505
$\beta_{2,4}$	0.09634	0.09677	0.09751
$\beta_{3,1}$	0.09837	0.09847	0.09814
$\beta_{3,2}$	0.09874	0.09964	0.09963
$\beta_{3,3}$	0.10232	0.10265	0.10403
$\beta_{3,4}$	0.10740	0.10717	0.10759
$\beta_{3,5}$	0.11268	0.11239	0.11303
$\beta_{3,6}$	0.11914	0.11996	0.12046
$\beta_{3,7}$	0.12833	0.12967	0.13081
$\beta_{3,8}$	0.13918	0.13996	0.14106
$\beta_{4,1}$	0.10045	0.10002	0.09993
$\beta_{4,2}$	0.10251	0.10494	0.10653
$\beta_{4,3}$	0.11352	0.11147	0.11200
$\beta_{4,4}$	0.12803	0.12844	0.13098
$\beta_{4,5}$	0.15160	0.15330	0.15416
$\beta_{4,6}$	0.18087	0.18177	0.17964
$\beta_{4,7}$	0.21227	0.21101	0.20810
$\beta_{4,8}$	0.25742	0.25755	0.25768
$\beta_{4,9}$	0.30313	0.30230	0.30504
$\beta_{4,10}$	0.35599	0.35590	0.35697
$\beta_{4,11}$	0.41464	0.41408	0.41360
$\beta_{4,12}$	0.48282	0.48248	0.48216
$\beta_{4,13}$	0.55569	0.55716	0.55611
$\beta_{4,14}$	0.63338	0.63317	0.63351
$\beta_{4,15}$	0.71793	0.71595	0.71530
$\beta_{4,16}$	0.81049	0.81017	0.81130

Example: We repeat the preceding example a last time, with the place-dependent noise $\nu_k(x)$ being samples from a beta distribution $\beta(1,2)$. The results are presented in Tables 9–12, with the first table presenting the means of the greyscale map parameters obtained by collage coding the realizations separately and the second table presenting the greyscale map parameters obtained by collage coding the mean observation, $u^*(x)$.

Table 9. *Greyscale map mean parameters* $\alpha_{i,j}$ *in the case of place-dependent additive noise with a beta distribution.*

coefficient	50	100	200
$\alpha_{1,1}$	0.25000	0.25000	0.25000
$\alpha_{1,2}$	0.25732	0.25732	0.25732
$\alpha_{2,1}$	0.06250	0.06250	0.06250
$\alpha_{2,2}$	0.06983	0.06983	0.06983
$\alpha_{2,3}$	0.07715	0.07715	0.07715
$\alpha_{2,4}$	0.08447	0.08447	0.08447
$\alpha_{3,1}$	0.01563	0.01563	0.01563
$\alpha_{3,2}$	0.02295	0.02295	0.02295
$\alpha_{3,3}$	0.03028	0.03028	0.03028
$\alpha_{3,4}$	0.03760	0.03760	0.03760
$\alpha_{3,5}$	0.04493	0.04493	0.04493
$\alpha_{3,6}$	0.05226	0.05226	0.05226
$\alpha_{3,7}$	0.05958	0.05958	0.05958
$\alpha_{3,8}$	0.06691	0.06691	0.06691
$\alpha_{4,1}$	0.00391	0.00391	0.00391
$\alpha_{4,2}$	0.01124	0.01124	0.01124
$\alpha_{4,3}$	0.01857	0.01857	0.01857
$\alpha_{4,4}$	0.02590	0.02590	0.02590
$\alpha_{4,5}$	0.03323	0.03323	0.03323
$\alpha_{4,6}$	0.04056	0.04056	0.04056
$\alpha_{4,7}$	0.04790	0.04790	0.04790
$\alpha_{4,8}$	0.05523	0.05523	0.05523
$\alpha_{4,9}$	0.06256	0.06256	0.06256
$\alpha_{4,10}$	0.06989	0.06989	0.06989
$\alpha_{4,11}$	0.07722	0.07722	0.07722
$\alpha_{4,12}$	0.08455	0.08455	0.08455
$\alpha_{4,13}$	0.09188	0.09188	0.09188
$\alpha_{4,14}$	0.09921	0.09921	0.09921
$\alpha_{4,15}$	0.10655	0.10655	0.10655
$\alpha_{4,16}$	0.11388	0.11388	0.11388

Table 10. *Greyscale map mean parameters* $\beta_{i,j}$ *in the case of place-dependent additive noise with a beta distribution.*

coefficient	50	100	200
$\beta_{1,1}$	0.27162	0.28736	0.31950
$\beta_{1,2}$	0.27019	0.28577	0.31760
$\beta_{2,1}$	0.33953	0.35920	0.39937
$\beta_{2,2}$	0.33824	0.35776	0.39762
$\beta_{2,3}$	0.33735	0.35671	0.39625
$\beta_{2,4}$	0.33684	0.35605	0.39528
$\beta_{3,1}$	0.35650	0.37716	0.41934
$\beta_{3,2}$	0.35580	0.37630	0.41817
$\beta_{3,3}$	0.35666	0.37701	0.41856
$\beta_{3,4}$	0.35909	0.37928	0.42052
$\beta_{3,5}$	0.36308	0.38311	0.42404
$\beta_{3,6}$	0.36863	0.38851	0.42912
$\beta_{3,7}$	0.37574	0.39547	0.43577
$\beta_{3,8}$	0.38441	0.40399	0.44397
$\beta_{4,1}$	0.36075	0.38165	0.42433
$\beta_{4,2}$	0.36239	0.38313	0.42550
$\beta_{4,3}$	0.37028	0.39087	0.43292
$\beta_{4,4}$	0.38442	0.40485	0.44660
$\beta_{4,5}$	0.40481	0.42509	0.46652
$\beta_{4,6}$	0.43145	0.45158	0.49269
$\beta_{4,7}$	0.46434	0.48431	0.52511
$\beta_{4,8}$	0.50348	0.52330	0.56378
$\beta_{4,9}$	0.54887	0.56853	0.60870
$\beta_{4,10}$	0.60051	0.62002	0.65988
$\beta_{4,11}$	0.65840	0.67776	0.71730
$\beta_{4,12}$	0.72254	0.74174	0.78097
$\beta_{4,13}$	0.79293	0.81198	0.85089
$\beta_{4,14}$	0.86957	0.88846	0.92706
$\beta_{4,15}$	0.95245	0.97120	1.00950
$\beta_{4,16}$	1.04160	1.06020	1.09820

Table 11. *Greyscale map parameters $\alpha_{i,j}$ for the mean observation in the case of place-dependent additive noise with a beta distribution.*

coefficient	50	100	200
$\alpha_{1,1}$	0.25000	0.28736	0.25000
$\alpha_{1,2}$	0.25732	0.28577	0.25732
$\alpha_{2,1}$	0.06250	0.35920	0.06250
$\alpha_{2,2}$	0.06983	0.35776	0.06983
$\alpha_{2,3}$	0.07715	0.35671	0.07715
$\alpha_{2,4}$	0.08447	0.35605	0.08447
$\alpha_{3,1}$	0.01563	0.37716	0.01563
$\alpha_{3,2}$	0.02295	0.37630	0.02295
$\alpha_{3,3}$	0.03028	0.37701	0.03028
$\alpha_{3,4}$	0.03760	0.37928	0.03760
$\alpha_{3,5}$	0.04493	0.38311	0.04493
$\alpha_{3,6}$	0.05226	0.38851	0.05226
$\alpha_{3,7}$	0.05958	0.39547	0.05958
$\alpha_{3,8}$	0.06691	0.40399	0.06691
$\alpha_{4,1}$	0.00391	0.38165	0.00391
$\alpha_{4,2}$	0.01124	0.38313	0.01124
$\alpha_{4,3}$	0.01857	0.39087	0.01857
$\alpha_{4,4}$	0.02590	0.40485	0.02590
$\alpha_{4,5}$	0.03323	0.42509	0.03323
$\alpha_{4,6}$	0.04056	0.45158	0.04056
$\alpha_{4,7}$	0.04790	0.48431	0.04790
$\alpha_{4,8}$	0.05523	0.52330	0.05523
$\alpha_{4,9}$	0.06256	0.56853	0.06256
$\alpha_{4,10}$	0.06989	0.62002	0.06989
$\alpha_{4,11}$	0.07722	0.67776	0.07722
$\alpha_{4,12}$	0.08455	0.74174	0.08455
$\alpha_{4,13}$	0.09188	0.81198	0.09188
$\alpha_{4,14}$	0.09921	0.88846	0.09921
$\alpha_{4,15}$	0.10655	0.97120	0.10655
$\alpha_{4,16}$	0.11388	1.06020	0.11388

Table 12. *Greyscale map parameters $\beta_{i,j}$ for the mean observation in the case of place-dependent additive noise with a beta distribution.*

coefficient	50	100	200
$\beta_{1,1}$	0.27162	0.28736	0.31950
$\beta_{1,2}$	0.27019	0.28577	0.31760
$\beta_{2,1}$	0.33953	0.35920	0.39937
$\beta_{2,2}$	0.33824	0.35776	0.39762
$\beta_{2,3}$	0.33735	0.35671	0.39625
$\beta_{2,4}$	0.33684	0.35605	0.39528
$\beta_{3,1}$	0.35650	0.37716	0.41934
$\beta_{3,2}$	0.35580	0.37630	0.41817
$\beta_{3,3}$	0.35666	0.37701	0.41856
$\beta_{3,4}$	0.35909	0.37928	0.42052
$\beta_{3,5}$	0.36308	0.38311	0.42404
$\beta_{3,6}$	0.36863	0.38851	0.42912
$\beta_{3,7}$	0.37574	0.39547	0.43577
$\beta_{3,8}$	0.38441	0.40399	0.44397
$\beta_{4,1}$	0.36075	0.38165	0.42433
$\beta_{4,2}$	0.36239	0.38313	0.42550
$\beta_{4,3}$	0.37028	0.39087	0.43292
$\beta_{4,4}$	0.38442	0.40485	0.44660
$\beta_{4,5}$	0.40481	0.42509	0.46652
$\beta_{4,6}$	0.43145	0.45158	0.49269
$\beta_{4,7}$	0.46434	0.48431	0.52511
$\beta_{4,8}$	0.50348	0.52330	0.56378
$\beta_{4,9}$	0.54887	0.56853	0.60870
$\beta_{4,10}$	0.60051	0.62002	0.65988
$\beta_{4,11}$	0.65840	0.67776	0.71730
$\beta_{4,12}$	0.72254	0.74174	0.78097
$\beta_{4,13}$	0.79293	0.81198	0.85089
$\beta_{4,14}$	0.86957	0.88846	0.92706
$\beta_{4,15}$	0.95245	0.97120	1.00950
$\beta_{4,16}$	1.04160	1.06020	1.09820

We remark that, in all examples, the mean function $u^*(x)$ constructed from the N observations may be viewed as a kind of denoising of the noisy signals.

In closing, we mention that the extension of the ideas in this paper to images (as opposed to signals) is the focus of some current work. We mention here that the ideas link naturally to the idea of image denoising, which has been the focus of much work in image science, including work from a fractal-based direction Ghazel *et al.* (2003).

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