Immersing small complete graphs

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Abstract

Following in the spirit of the Hadwiger and Hajós conjectures, Abu-Khzam and Langston have conjectured that every k-chromatic graph contains an immersion of K_k. They proved this for $k \leq 4$. Much before that, Lescure and Meyniel [10] obtained a stronger result that included also the values $k = 5$ and 6, by proving that every simple graph of minimum degree $k - 1$ contains an immersion of K_k. They noted that they also have a proof of the same result for $k = 7$ but have not published it due to the length of the proof. We give a simple proof of this result. This, in particular, proves the conjecture of Abu-Khzam and Langston for every $k \leq 7$.

Keywords: Immersion, Hadwiger Conjecture.

Math. Subj. Class.: 05C35, 05C83

1   Introduction

In this paper, all graphs are finite and may have loops and multiple edges. A graph H is a *minor* of a graph G if (a graph isomorphic to) H can be obtained from a subgraph of G by contracting edges. A graph H is a *topological minor* of a graph G if G contains a subgraph which is isomorphic to a graph that can be obtained from H by subdividing some edges. In such a case, we also say that G contains a *subdivision* of H. The chromatic number of G, denoted $\chi(G)$, is the minimum number of colors required by G in any proper coloring of its vertices. The graph G is *k-chromatic* if $\chi(G) = k$.

Our paper is motivated by two famous conjectures concerning the chromatic number and the minor and topological order, namely, Hadwiger's conjecture and Hajós' conjecture.

Hadwiger's Conjecture from 1943 suggests a far-reaching generalization of the Four Color Theorem [2, 3, 11] and is considered to be one of the deepest open problems in graph theory. It states that every loopless graph without a K_k-minor is $(k-1)$-colorable. The special case of the Hadwiger Conjecture when $k = 5$ was shown earlier (Wagner [16]) to be equivalent to the Four Color Theorem. In 1993, Robertson, Seymour and Thomas [14] proved that the case $k = 6$ also follows from the Four Color Theorem. The cases $k \geq 7$ are open; for the case $k = 7$, a partial result in [9] is all that is known.

Hajós proposed a stronger conjecture that for all $k \geq 1$, every k-chromatic graph contains a subdivision of the complete graph on k vertices. He already considered the conjecture in the 1940's in connection with attacks on the Four Colour Conjecture (now theorem). For $k \leq 4$, the conjecture is true, for $k = 5, 6$, it still remains open. But for every $k \geq 7$, it was disproved by Catlin [6]. In fact, Erdős and Fajtlowicz [7] proved that the conjecture is false for almost all graphs, see also Bollobás and Catlin [4]. Recently, Thomassen provided a variety of interesting natural examples where Hajós' Conjecture fails [15].

In this paper, we consider a different containment relation – graph immersions. A pair of adjacent edges uv and vw with $u \neq w$ is *lifted* by deleting the edges uv and vw, and adding the edge uw (possibly in parallel to an existing edge). A graph H is said to be *immersed* in a graph G if a graph isomorphic to H can be obtained from a subgraph of G by lifting pairs of edges. If H is immersed in a graph G, then we also say that G *contains an H-immersion.*

Previous investigation on immersions has been mainly conducted from an algorithmic standpoint. We refer the reader to [5, 8]. On the other hand, it would be interesting to consider structural issues, since the notions of an immersion and a minor seem to be quite similar, and structural approach concerning graph minors has been extremely successful. In fact, Robertson and Seymour extended their proof of the famous Wagner's conjecture [12] to prove that graphs are well-quasi-ordered by the immersion relation [13]. This proves a conjecture of Nash-Williams. The proof is based on a significant part of the graph minors project. Hence, we may expect that structural approach concerning immersions is difficult, maybe as difficult as structure results concerning graph minors.

Immersion containment is quite different from that of topological or minor containment. It is immediate that a topological minor of H implies both a minor and an immersion of H, but the converse is not generally true. The existence of an H minor and an H immersion are easily seen to be incomparable. Therefore, it would be interesting to investigate relations between the chromatic number of a graph G and the largest size of a complete graph immersed in G. In fact, Abu-Khzam and Langston conjectured the following in [1].

Conjecture 1. The complete graph K_k can be immersed in any k-chromatic graph.

This conjecture, like Hadwiger's conjecture and Hajós' conjecture, is trivially true for $k \leq 4$. In fact, since Hajós' conjecture is true if $k \leq 4$, this immediately implies Conjecture 1 for the cases $k \leq 4$.

On the other hand, the case $k = 5$ does not seem to be trivial. Abu-Khzam and Langston [1] proved a weaker statement that K_5^- is immersed in every 5-chromatic graph. They also pointed out that structural investigations are extensively studied for graphs without a K_4-immersion (see [5]), but almost nothing is done for graphs without a K_5-immersion.

Let us observe that every k-chromatic graph contains a k-critical subgraph and that the minimum degree of every k-critical graph is at least $k - 1$. This shows that the following question for complete immersions is of importance in relation to Conjecture 1.

Problem 2. Let k be a positive integer. Find the smallest value $f(k)$ such that every simple graph of minimum degree at least $f(k)$ contains an immersion of K_k.

Clearly, $f(k) \geq k - 1$, and if $f(k) = k - 1$, then Conjecture 1 holds since every k-chromatic graph has a simple subgraph of minimum degree $k - 1$. Again, this problem is trivial for $k \leq 4$.

The following example, due to Paul Seymour (private communication), shows that $f(k) \geq k$ for every $k \geq 10$. Let G be the graph obtained from the complete graph K_{12} by deleting the edges of four disjoint triangles. Then G has minimum degree 9, but it is easy to argue that it does not contain an immersion of K_{10}.

The purpose of this paper is to prove the following theorem which establishes Conjecture 1 and resolves Problem 2 for the intermediate values of k.

Theorem 3. Let $f(k)$ be the smallest integer such that every simple graph of minimum degree at least $f(k)$ contains an immersion of K_k. Then $f(k) = k - 1$ for $k \in \{5, 6, 7\}$.

As mentioned before, every k-chromatic graph contains a subgraph of minimum degree $k - 1$. Therefore, Theorem 3 also shows that Conjecture 1 holds for $k \leq 7$. Although the methods of our proof may extend to the next case ($k = 8$), it appears that there may be too many cases to make this approach feasible.

After completion of the original version of this paper, it came to our attention that Lescure and Meyniel [10] solved Problem 2 for $k = 5, 6$. They also report in their paper that they also have a proof of the same result for $k = 7$ but have not published it due to the length of the proof. Our proof turns out to have some similarities with the proof of Lescure and Meyniel, but the hardest case when $k = 7$ is simple enough to be presented on a few pages only.

2 Immersing complete graphs

In this section we shall prove the main theorem. Actually, for inductive purposes, we shall prove a slightly stronger statement. We begin with a little notation.

Our graphs may in general have parallel edges; If two vertices are joined by more than one edge, the set of edges joining them is called a *proper parallel class*. The *degree* of a vertex v, denoted $deg(v)$, is the number of edges incident with v (counting loops twice). We let $N(v)$ denote the set of vertices adjacent to v and we let $\overline{N}(v) = N(v) \cup \{v\}$.

Theorem 4. Let $d \in \{4, 5, 6\}$, let $G = (V, E)$ be a loopless graph, and let $u \in V$. Assume further that G satisfies the following properties:

- $|V| \geq d$.
- $deg(v) \geq d$ for every $v \in V \setminus \{u\}$.
- There are at most $d-2$ proper parallel classes, and every edge in such a parallel class is incident with u.

Then there is an immersion of K_{d+1} in G.

Proof. Suppose (for a contradiction) that G is a counterexample to the theorem with $|V| + |E|$ minimum. We shall prove properties of G in several steps.

(1) $|V| \geq d+1$.

It follows from the assumptions that there exists a vertex in $V \setminus \{u\}$ not incident with any proper parallel class. Consequently, $|V| \geq d+1$.

(2) Every edge has an end of degree at most d.

If e has no end of degree $\leq d$, then $G - e$ is a smaller counterexample.

(3) $deg(v) = d$ for every $v \in N(u)$.

If $v \in N(u)$ has degree $> d$, then $G - uv$ is a smaller counterexample.

(4) $deg(v) \leq d+1$ for every $v \in V \setminus \{u\}$.

If there exists $v \in V \setminus \{u\}$ with $deg(v) > d+1$, then either the neighbors of v form a complete graph (giving us an immersion of K_{d+1} in G) or there exist $w_1, w_2 \in N(v)$ which are nonadjacent, and the graph obtained from G by lifting vw_1 and vw_2 to form the edge $w_1 w_2$ is a smaller counterexample.

(5) $N(u)$ induces a complete graph.

If $v_1, v_2 \in N(u)$ are nonadjacent, then the graph obtained by lifting uv_1 and uv_2 to form the edge $v_1 v_2$ is a smaller counterexample.

(6) $|N(u)| \geq 3$.

If $|N(u)| \leq 1$, then $G - u$ is a smaller counterexample. If $|N(u)| = 2$, then we may let $\{v_1, v_2\} = N(u)$ and assume that there are s edges between u, v_1 and t between u, v_2 with $s \leq t$. Now G immerses the graph G' obtained from G by deleting u and adding s new parallel edges between v_1, v_2, so G' (with the special vertex v_2) is a smaller counterexample.

(7) $|N(u)| \leq d-2$.

If $|N(u)| \geq d$, then (5) implies that $\overline{N}(u)$ contains K_{d+1} as a subgraph, so G immerses K_{d+1}. If $|N(u)| = d-1$, let G' be the graph obtained from G by identifying $\overline{N}(u)$ to a single new vertex and deleting any resulting loops. More precisely, we replace $V(\overline{N}(u))$ by a single vertex u' and replace each edge vw ($v \in N(u)$, $w \notin \overline{N}(u)$) by the edge $u'w$, possibly obtaining multiple edges incident with u'. By using (3) and (5), it is easy to see that each vertex in $N(u)$ has precisely one neighbor outside $\overline{N}(u)$ and that the degree of u' is $d-1$. This implies that G immerses the graph G'. Since immersion is a transitive

relation, we conclude that G' (with the special vertex u') is a smaller counterexample. This contradiction proves (7).

(8) $d = 6$ and $|N(u)| = 3$.

Suppose (for a contradiction) that (8) fails. By (6) and (7) we must have $d \in \{5, 6\}$ and $|N(u)| = d - 2$. Again in this case, we form a new graph G' from G by identifying $\overline{N}(u)$ to a single new vertex u' and deleting any resulting loops. By the minimality of G we find that G' immerses K_{d+1}. This immersed K_{d+1} in G' can be used to find an immersion of K_{d+1} in G. The "hardest" case to verify is when $d = 6$, u' has degree 8, and the immersed K_7 uses all of these edges, and has u' as a vertex. Up to symmetries, there are two cases to be considered, and they are conveniently displayed in Figure 1, where the black vertex and the bold paths show the neighborhood of the immersed vertex of K_7 and the dotted edges show the segment of the additional subdivided edge that is passing through u' in G'. This yields a contradiction and completes the proof of (8).
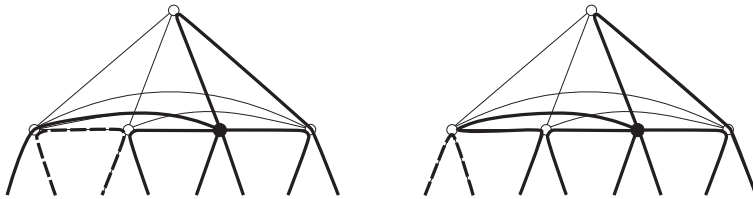

Figure 1: Extending K_7-immersion from G' to G when $|N(u)| = 4$

(9) G does not have exactly one proper parallel class.

Suppose (for a contradiction) that $N(u) = \{v_1, v_2, v_3\}$ and that the edges between u and v_1 form the only proper parallel class in G. Now, form a new graph G' by deleting u and adding edges $v_1 v_2$ and $v_1 v_3$. The graph G' is now immersed in G, and gives us a smaller counterexample (using the special vertex v_1).

(10) There do not exist $w_1, w_2 \in \bigcap_{v \in N(u)} N(v) \setminus \{u\}$ so that either $w_1 w_2 \in E$ or $deg(w_1) = 6 = deg(w_2)$.

Suppose that such a pair w_1, w_2 exists and let us form the graph G' from G by identifying $\overline{N}(u) \cup \{w_1, w_2\}$ to a single new vertex u'. It follows from (2)–(5) and (8) that u' will have degree ≤ 9 (so G' will have at most four proper parallel classes). Since $N(u)$ has at least one and at most three neighbors outside of $N(u) \cup \{u, w_1, w_2\}$, it also follows that G has a vertex that is not adjacent to $N(u)$. This implies that G' has at least 6 vertices. Consequently, G' contains an immersion of K_7. This immersion of K_7 in G' can be used to find an immersion of K_7 in G. The "hardest" case is when eight of the edges incident with u' are used in the K_7 and u' is a vertex of the immersed K_7. As for the most general subcase, we may also assume that $deg(u') = 9$. This case can be divided into subcases depending on which pair of edges incident with u' are lifted to form a new edge (drawn as a broken bold curve in Figure 2). Up to symmetries, there are four subcases as shown in Figure 2: both edges are incident with w_1, one is incident with w_1 and the other one with w_2, one with w_1 and the other with $N(u)$, or both with $N(u)$. In Figure 2, bold paths represent partial routings for each of these four cases, up to symmetries. In each subcase, one or
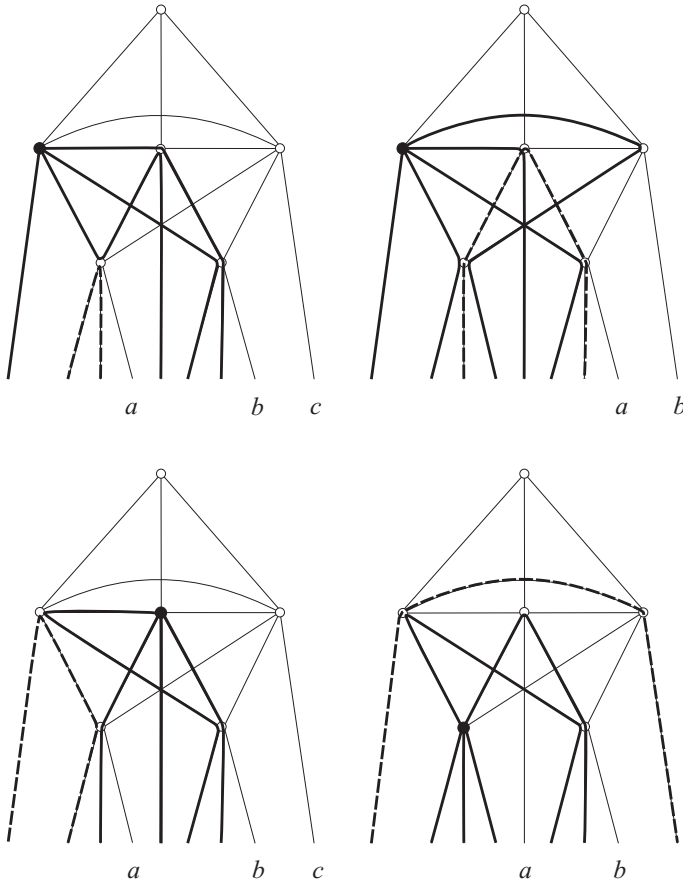
Figure 2: Extending K_7-immersion from G' to G for claim (10)

two of the edges incident with the vertex of degree 6 in K_7 (drawn as the black vertex) are still missing. In this way, each of these figures describes two or three additional subcases, corresponding to which of the remaining edges a, b (or c) is not used in the K_7-immersion. It is easy to verify that all of these subcases can be completed to form an immersion of K_7 in G.

(11) $|\bigcap_{v \in N(u)} N(v) \setminus \{u\}| \leq 2$.

    Suppose (for a contradiction) that $w_1, w_2, w_3 \in \bigcap_{v \in N(u)} N(v) \setminus \{u\}$ are distinct. It follows from (10) that $\{w_1, w_2, w_3\}$ is an independent set, and that at most one of these vertices has degree 6. Now, form a new graph G' from G by deleting $\overline{N}(u)$ and then adding the new edges $w_1 w_2$, $w_2 w_3$, and $w_3 w_1$. The graph G' is simple with at most one vertex of degree < 6 and has at least 6 vertices, so it immerses K_7. But then G immerses K_7 as well, since G' is immersed in G. This yields a contradiction.

(12) Every $v \in N(u)$ satisfies $|N(v)| \leq 5$.

Let $N(u) = \{v_1, v_2, v_3\}$ and suppose (for a contradiction) that $N(v_3) = \{u, v_1, v_2, w_1, w_2, w_3\}$. First suppose that $v_1 w_1, v_2 w_2 \notin E$. Then the graph G' obtained from G by deleting v_3 and adding the edges $v_1 w_1$, $v_2 w_2$, and $u w_3$ is immersed in G, and yields a smaller counterexample, a contradiction. Next suppose that $v_1 w_1, v_1 w_2 \notin E$. By the previous case, we may assume that $v_2 w_1, v_2 w_2, v_2 w_3 \in E$. It follows from this that v_2 and v_3 have six distinct neighbors, so by (9) the graph G is simple. Now, form the graph G'' from G by deleting u and v_3, then adding the edges $v_1 w_1$ and $v_1 w_2$, and then adding a new edge $v_2 w_3$ in parallel to the existing edge. The graph G'' is immersed in G, but then G'' with the special vertex v_2 is a smaller counterexample, giving us a contradiction.

Using the two properties just demonstrated, we may now assume that $v_1 w_2, v_1 w_3, v_2 w_2, v_2 w_3 \in E$. Further, by (11), we may assume that $v_1 w_1 \notin E$, and by (10) we may assume that $w_2 w_3 \notin E$. Now we form the graph G''' from G by deleting v_3 and then adding the edges $v_1 w_1$, $w_2 w_3$, and $u v_2$. Now G''' is immersed in G, but then G''' is a smaller counterexample, giving us a contradiction. This proves (12).

Now, (12) implies that every neighbor of u is incident with a proper parallel class. It follows from this and (3), (5), (8) that the graph obtained from G by identifying $\overline{N}(u)$ to a single new vertex and deleting loops is immersed in G. This new graph is a smaller counterexample, thus completing the proof. $\square$

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