Tight orientably-regular polytopes

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Abstract

It is known that every equivelar abstract polytope of type $\{p_1, \ldots, p_{n-1}\}$ has at least $2p_1 \cdots p_{n-1}$ flags. Polytopes that attain this lower bound are called *tight*. Here we investigate the conditions under which there is a tight orientably-regular polytope of type $\{p_1, \ldots, p_{n-1}\}$. We show that it is necessary and sufficient that whenever p_i is odd, both p_{i-1} and p_{i+1} (when defined) are even divisors of $2p_i$.

Keywords: Abstract regular polytope, equivelar polytope, flat polytope, tight polytope.

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1 Introduction

Abstract polytopes are ranked partially-ordered sets that resemble the face-lattice of a convex polytope in several key ways. Many discrete geometric objects can be viewed as an abstract polytope by considering their face-lattices, but there are also many new kinds of structures that have no immediate geometric analogue.

A *flag* of an abstract polytope is a chain in the poset that contains one element of each rank. In many ways, it is more natural to work with the flags of a polytope rather than the faces themselves. For example, every automorphism (order-preserving bijection) of a polytope is completely determined by its effect on any single flag.

Regular polytopes are those for which the automorphism group acts transitively on the set of flags. The automorphism group of a regular polytope is a quotient of some string Coxeter group $[p_1, \ldots, p_{n-1}]$, and conversely, every sufficiently nice quotient of a string

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Coxeter group appears as the automorphism group of a regular polytope. Indeed, it is actually possible to reconstruct a regular polytope from its automorphism group, so that much of the study of regular polytopes is purely group-theoretic.

We say a regular polytope $\mathcal{P}$ is of *type* (or has *Schläfli symbol*) $\{p_1, \ldots, p_{n-1}\}$ if $[p_1, \ldots, p_{n-1}]$ is the minimal string Coxeter group that covers the automorphism group of $\mathcal{P}$, in a way that $p_1, \ldots, p_{n-1}$ are the orders of the relevant generators. There is an equivalent formulation of this property that is entirely combinatorial, and hence it is possible to define a Schläfli symbol for many non-regular polytopes, including chiral polytopes (see [10]) and other two-orbit polytopes (see [7]). Any polytope with a well-defined Schläfli symbol is said to be *equivelar*.

In [3], the first author determined the smallest regular polytope (by number of flags) in each rank. To begin with, he showed that every regular polytope of type $\{p_1, \ldots, p_{n-1}\}$ has at least $2p_1 \cdots p_{n-1}$ flags. Polytopes that meet this lower bound are called *tight*. He then exhibited a family of tight polytopes, one in each rank, of type $\{4, \ldots, 4\}$. Using properties of the automorphism groups of regular polytopes, he showed that each polytope was the smallest regular polytope in rank $n \geq 9$, and that in smaller ranks, the minimum was also attained by a tight polytope (with type or dual type $\{3\}$, $\{3, 4\}$, $\{4, 3, 4\}$, $\{3, 6, 3, 4\}$, $\{4, 3, 6, 3, 4\}$, $\{3, 6, 3, 6, 3, 4\}$ or $\{4, 3, 6, 3, 6, 3, 4\}$, respectively).

The second author showed in [5] that the bound on the number of flags extended to any equivelar polytope, regardless of regularity. Accordingly, it makes sense to extend the definition of tight polytopes to include any polytope of type $\{p_1, \ldots, p_{n-1}\}$ with $2p_1 \cdots p_{n-1}$ flags. An alternate formulation was proved as well, showing that an equivelar polytope is tight if and only if every face is incident with all faces (if any) that are 2 ranks higher.

Tightness is a restrictive property, and not every Schläfli symbol is possible for a tight polytope. In order for there to be a tight polytope of type $\{p_1, \ldots, p_{n-1}\}$, it is necessary that no two adjacent values p_i and p_{i+1} are odd. Theorem 5.1 in [5] shows that this condition is sufficient in rank 3. In higher ranks, the question of sufficiency is still open.

Constructing non-regular polytopes in high ranks is difficult. In order to determine which Schläfli symbols are possible for a tight polytope, it is helpful to begin by considering only regular polytopes. If every p_i is even, then Theorem 5.3 in [3] and Theorem 6.3 in [5] show that there is a tight regular polytope of type $\{p_1, \ldots, p_{n-1}\}$. Also the computational data from [2, 6] led the second author to conjecture that if p is odd and $q > 2p$, there is no tight regular polyhedron of type $\{p, q\}$. Although we are currently unable to prove this conjecture, we can show that for tight *orientably*-regular polyhedra, if p is odd then q must divide $2p$. Moreover, we are able to prove the following generalisation in higher ranks:

Theorem 1.1. *There is a tight orientably-regular polytope of type $\{p_1, \ldots, p_{n-1}\}$ if and only if each of the integers p_{i-1} and p_{i+1} (when defined) is an even divisor of $2p_i$ whenever p_i is odd, for $1 \leq i < n$.*

2 Background

Our background information is mostly taken from [8, Chs. 2, 3, 4], with a few small additions.

2.1 Definition of a polytope

Let $\mathcal{P}$ be a ranked partially-ordered set, the elements of which are called *faces*, and suppose that the faces of $\mathcal{P}$ range in rank from -1 to n. We call each face of rank j a *j-face*, and we

say that two faces are *incident* if they are comparable. We also call the 0-faces, 1-faces and $(n-1)$-faces the *vertices*, *edges* and *facets* of $\mathcal{P}$, respectively. A *flag* is a maximal chain in $\mathcal{P}$. We say that two flags are *adjacent* if they differ in exactly one face, and that they are *j-adjacent* if they differ only in their j-faces.

If F and G are faces of $\mathcal{P}$ such that $F \leq G$, then the *section* G/F consists of those faces H such that $F \leq H \leq G$. If F is a j-face and G is a k-face, then we say that the *rank* of the section G/F is $k-j-1$. If removing G and F from the Hasse diagram of G/F leaves us with a connected graph, then we say that G/F is *connected*. That is, for any two faces H and H' in G/F (other than F and G themselves), there is a sequence of faces

$$H = H_0, H_1, \ldots, H_k = H'$$

such that $F < H_i < G$ for $0 \leq i \leq k$ and the faces H_{i-1} and H_i are incident for $1 \leq i \leq k$. By convention, we also define all sections of rank at most 1 to be connected.

We say that $\mathcal{P}$ is an (*abstract*) *polytope of rank n*, or briefly, an *n-polytope*, if it satisfies the following four properties:

(a) There is a unique greatest face F_n of rank n, and a unique least face F_{-1} of rank -1.

(b) Each flag has $n+2$ faces.

(c) Every section is connected.

(d) Every section of rank 1 is a diamond — that is, whenever F is a $(j-1)$-face and G is a $(j+1)$-face for some j, with $F < G$, there are exactly two j-faces H with $F < H < G$.

Condition (d) is known as the *diamond condition.* Note that this condition ensures that for $0 \leq j < n$, every flag Φ has a unique j-adjacent flag, which we denote by Φ^j.

In ranks $-1, 0$, and 1, there is a unique polytope up to isomorphism. Abstract polytopes of rank 2 are also called *abstract polygons*, and for each $2 \leq p \leq \infty$, there is a unique abstract polygon with p vertices and p edges, denoted by $\{p\}$.

If F is a j-face and G is a k-face of a polytope with $F \leq G$, then the section G/F itself is a $(k-j-1)$-polytope. We may identify a face F with the section F/F_{-1}, and call the section F_n/F the *co-face at F*. The co-face at a vertex F_0 is also called a *vertex-figure* at F_0.

If $\mathcal{P}$ is an n-polytope, F is an $(i-2)$-face of $\mathcal{P}$, and G is an $(i+1)$-face of $\mathcal{P}$ with $F < G$, then the section G/F is an abstract polygon. If it happens that for $1 \leq i < n$, each such section is (isomorphic to) the same polygon $\{p_i\}$, no matter which $(i-2)$-face F and incident $(i+1)$-face G we choose, then we say that $\mathcal{P}$ has *Schläfli symbol* $\{p_1, \ldots, p_{n-1}\}$, or that $\mathcal{P}$ is *of type* $\{p_1, \ldots, p_{n-1}\}$. Also when this happens, we say that $\mathcal{P}$ is *equivelar*.

All sections of an equivelar polytope are themselves equivelar polytopes. In particular, if $\mathcal{P}$ is an equivelar polytope of type $\{p_1, \ldots, p_{n-1}\}$, then all its facets are equivelar polytopes of type $\{p_1, \ldots, p_{n-2}\}$, and all its vertex-figures are equivelar polytopes of type $\{p_2, \ldots, p_{n-1}\}$.

Next, let $\mathcal{P}$ and $\mathcal{Q}$ be two polytopes of the same rank. A surjective function $\gamma : \mathcal{P} \to \mathcal{Q}$ is called a *covering* if it preserves incidence of faces, ranks of faces, and adjacency of flags. If there exists such a covering $\gamma : \mathcal{P} \to \mathcal{Q}$, then we say that $\mathcal{P}$ *covers* $\mathcal{Q}$.

The *dual* of a polytope $\mathcal{P}$ is the polytope obtained by reversing the partial order. If $\mathcal{P}$ is an equivelar polytope of type $\{p_1, \ldots, p_{n-1}\}$, then the dual of $\mathcal{P}$ is an equivelar polytope of type $\{p_{n-1}, \ldots, p_1\}$.

2.2   Regularity

For polytopes $\mathcal{P}$ and $\mathcal{Q}$, an *isomorphism* from $\mathcal{P}$ to $\mathcal{Q}$ is an incidence- and rank-preserving bijection. By connectedness and the diamond condition, every polytope isomorphism is uniquely determined by its effect on a given flag. An isomorphism from $\mathcal{P}$ to itself is an *automorphism* of $\mathcal{P}$, and the group of all automorphisms of $\mathcal{P}$ is denoted by $\Gamma(\mathcal{P})$. We will denote the identity automorphism by ε.

We say that $\mathcal{P}$ is *regular* if the natural action of $\Gamma(\mathcal{P})$ on the flags of $\mathcal{P}$ is transitive (and hence regular, in the sense of being sharply-transitive). For convex polytopes, this definition is equivalent to any of the usual definitions of regularity.

Now let $\mathcal{P}$ be any regular polytope, and choose a flag Φ, which we call a *base flag*. Then the automorphism group $\Gamma(\mathcal{P})$ is generated by the *abstract reflections* $\rho_0, \ldots, \rho_{n-1}$, where ρ_i maps Φ to the unique flag Φ^i that is i-adjacent to Φ. These generators satisfy $\rho_i^2 = \varepsilon$ for all i, and $(\rho_i \rho_j)^2 = \varepsilon$ for all i and j such that $|i - j| \geq 2$. Every regular polytope is equivelar, and if its Schläfli symbol is $\{p_1, \ldots, p_{n-1}\}$, then the order of each $\rho_{i-1}\rho_i$ is p_i. Note that if $\mathcal{P}$ is a regular polytope of type $\{p_1, \ldots, p_{n-1}\}$, then $\Gamma(\mathcal{P})$ is a quotient of the string Coxeter group $[p_1, \ldots, p_{n-1}]$, which is the abstract group generated by n elements $x_0, \ldots, x_{n-1}$ subject to the defining relations $x_i^2 = 1$, $(x_{i-1}x_i)^{p_i} = 1$, and $(x_i x_j)^2 = 1$ whenever $|i - j| \geq 2$.

Next, if Γ is any group generated by elements $\rho_0, \ldots, \rho_{n-1}$, we define $\Gamma_I = \langle \rho_i \mid i \in I \rangle$ for each subset I of the index set $\{0, 1, \ldots, n - 1\}$. If Γ is the automorphism group $\Gamma(\mathcal{P})$ of a regular polytope $\mathcal{P}$, then these subgroups satisfy the following condition, known as the *intersection condition*:

$$\Gamma_I \cap \Gamma_J = \Gamma_{I \cap J} \quad \text{for all } I, J \subseteq \{0, 1, \ldots, n - 1\}. \tag{2.1}$$

More generally, if Γ is any group generated by elements $\rho_0, \ldots, \rho_{n-1}$ of order 2 such that $(\rho_i \rho_j)^2 = 1$ whenever $|i - j| \geq 2$, then we say that Γ is a *string group generated by involutions*, and abbreviate this to say that Γ is an *sggi*. If the sggi Γ also satisfies the intersection condition (2.1) given above, then we call Γ a *string C-group*.

There is a natural way of building a regular polytope $\mathcal{P}(\Gamma)$ from a string C-group Γ such that $\Gamma(\mathcal{P}(\Gamma)) \cong \Gamma$ and $\mathcal{P}(\Gamma(\mathcal{P})) \cong \mathcal{P}$. In particular, the i-faces of $\mathcal{P}(\Gamma)$ are taken to be the cosets of the subgroup $\Gamma_i = \langle \rho_j \mid j \neq i \rangle$, with incidence of faces $\Gamma_i \varphi$ and $\Gamma_j \psi$ given by

$$\Gamma_i \varphi \leq \Gamma_j \psi \quad \text{if and only if} \quad i \leq j \text{ and } \Gamma_i \varphi \cap \Gamma_j \psi \neq \emptyset.$$

This construction is also easily applied when Γ is any sggi (not necessarily a string C-group), but in that case, the resulting poset is not always a polytope.

The following theory from [8] helps us determine when a given sggi is a string C-group:

Proposition 2.1. *Let $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ be an sggi, and $\Lambda = \langle \lambda_0, \ldots, \lambda_{n-1} \rangle$ a string C-group. If there is a homomorphism $\pi : \Gamma \to \Lambda$ sending each σ_i to λ_i, and if π is one-to-one on the subgroup $\langle \rho_0, \ldots, \rho_{n-2} \rangle$ or the subgroup $\langle \rho_1, \ldots, \rho_{n-1} \rangle$, then Γ is a string C-group.*

Proposition 2.2. *Let $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ be an sggi. If both $\langle \rho_0, \ldots, \rho_{n-2} \rangle$ and $\langle \rho_1, \ldots, \rho_{n-1} \rangle$ are string C-groups, and $\langle \rho_0, \ldots, \rho_{n-2} \rangle \cap \langle \rho_1, \ldots, \rho_{n-1} \rangle \subseteq \langle \rho_1, \ldots, \rho_{n-2} \rangle$, then Γ is a string C-group.*

Given a regular n-polytope $\mathcal{P}$ with automorphism group $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, we define the *abstract rotations* $\sigma_1, \ldots, \sigma_{n-1}$ by setting $\sigma_i = \rho_{i-1}\rho_i$ for $1 \leq i < n$. Then the subgroup $\langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ of $\Gamma(\mathcal{P})$ is denoted by $\Gamma^+(\mathcal{P})$, and called the *rotation subgroup of* $\mathcal{P}$. The index of $\Gamma^+(\mathcal{P})$ in $\Gamma(\mathcal{P})$ is at most 2, and when the index is exactly 2, then we say that $\mathcal{P}$ is *orientably-regular*. Otherwise, if $\Gamma^+(\mathcal{P}) = \Gamma(\mathcal{P})$, then we say that $\mathcal{P}$ is *non-orientably-regular*. (This notation comes from the study of regular maps.) A regular polytope $\mathcal{P}$ is orientably-regular if and only if $\Gamma(\mathcal{P})$ has a presentation in terms of the generators $\rho_0, \ldots, \rho_{n-1}$ such that all of the relators have even length. Note that every section of an orientably-regular polytope is itself orientably-regular.

2.3 Flat and tight polytopes

The theory of abstract polytopes accommodates certain degeneracies not present in the study of convex polytopes. For example, the face-poset of a convex polytope is a lattice (which means that any two elements have a unique supremum and infimum), but this need not be the case with abstract polytopes. The simplest abstract polytope that is not a lattice is the digon $\{2\}$, in which both edges are incident with both vertices. This type of degeneracy can be generalised as follows. If $\mathcal{P}$ is an n-polytope, and $0 \leq k < m < n$, then we say that $\mathcal{P}$ is (k, m)-*flat* if every one of its k-faces is incident with every one of its m-faces. If $\mathcal{P}$ has rank n and is $(0, n-1)$-flat, then we also say simply that $\mathcal{P}$ is a *flat polytope*. Note that if $\mathcal{P}$ is (k, m)-flat, then $\mathcal{P}$ must also be (i, j)-flat whenever $0 \leq i \leq k < m \leq j < n$. In particular, if $\mathcal{P}$ is (k, m)-flat, then it is also flat.

We will also need the following, taken from [8, Lemma 4E3]:

Proposition 2.3. *Let $\mathcal{P}$ be an n-polytope, and let $0 \leq k < m < i < n$. If each i-face of $\mathcal{P}$ is (k, m)-flat, then $\mathcal{P}$ is also (k, m)-flat. Similarly, if $0 \leq i < k < m < n$ and each co-i-face of $\mathcal{P}$ is $(k-i-1, m-i-1)$-flat, then $\mathcal{P}$ is (k, m)-flat.*

It is easy to see that the converse is also true. In other words, if $\mathcal{P}$ is (k, m)-flat, then for $i > m$ each i-face of $\mathcal{P}$ is (k, m)-flat, and for $j < k$ each co-j-face of $\mathcal{P}$ is $(k-j-1, m-j-1)$-flat.

Next, we consider tightness. An equivelar polytope $\mathcal{P}$ of type $\{p_1, \ldots, p_{n-1}\}$ has at least $2p_1 \cdots p_{n-1}$ flags, by [5, Proposition 3.3]. Whenever $\mathcal{P}$ has exactly this number of flags, we say that $\mathcal{P}$ is *tight*. It is clear that $\mathcal{P}$ is tight if and only if its dual is tight, and that in a tight polytope, every section of rank 3 or more is tight. Also we will need the following, taken from [5, Theorem 4.4]:

Theorem 2.4. *Let $n \geq 3$ and let $\mathcal{P}$ be an equivelar n-polytope. Then $\mathcal{P}$ is tight if and only if it is $(i, i+2)$-flat for $0 \leq i \leq n-3$.*

Later in this paper we will build polytopes inductively, and for that, the following approach is useful. We say that the regular n-polytope $\mathcal{P}$ has the *flat amalgamation property* (or FAP) with respect to its k-faces, if adding the relations $\rho_i = \varepsilon$ for $i \geq k$ to $\Gamma(\mathcal{P})$ yields a presentation for $\langle \rho_0, \ldots, \rho_{k-1} \rangle$. Similarly, we say that $\mathcal{P}$ has the FAP with respect to its co-k-faces if adding the relations $\rho_i = \varepsilon$ for $i \leq k$ yields a presentation for $\langle \rho_{k+1}, \ldots, \rho_{n-1} \rangle$.

We will also use the following, taken from [8, Theorem 4F9]:

Theorem 2.5. *Suppose $m, n \geq 2$, and $0 \leq k \leq m-2$ where $k \geq m-n$. Let $\mathcal{P}_1$ be a regular m-polytope, and let $\mathcal{P}_2$ be a regular n-polytope such that the co-k-faces of $\mathcal{P}_1$ are*

isomorphic to the $(m-k-1)$-faces of $\mathcal{P}_2$. Also suppose that $\mathcal{P}_1$ has the FAP with respect to its co-k-faces, and that $\mathcal{P}_2$ has the FAP with respect to its $(m-k-1)$-faces. Then there exists a regular $(k+n+1)$-polytope $\mathcal{P}$ such that $\mathcal{P}$ is (k,m)-flat, and the m-faces of $\mathcal{P}$ are isomorphic to $\mathcal{P}_1$, while the co-k-faces of $\mathcal{P}$ are isomorphic to $\mathcal{P}_2$. Furthermore, $\mathcal{P}$ has the FAP with respect to its m-faces and its co-k-faces.

3 Tight orientably-regular polyhedra

We now consider the values of p and q for which there is a tight orientably-regular polyhedron of type $\{p, q\}$. By [5, Proposition 3.5], there are no tight polyhedra of type $\{p, q\}$ when p and q are both odd. Also by [3, Theorem 5.3] and [5, Theorem 6.3], if p and q are both even then there exists a tight orientably-regular polyhedron whose automorphism group is the quotient of the Coxeter group $[p, q]$ obtained by adding the extra relation $(x_0 x_1 x_2 x_1)^2 = 1$. (Note that this is the group of the polyhedron $\{p, q \,|\, 2\}$ described in [8, p. 196]; see [9] for more information on this and related polyhedra.) Indeed, for even p and q, there can often be multiple (non-isomorphic) tight polyhedra; for example, there are two of type $\{4, 8\}$ that are non-isomorphic (see [2]).

When p is odd and q is even (or vice-versa), the situation is more complicated. Evidence from [2] and [6] led the second author to conjecture that there are no tight regular polyhedra of type $\{p, q\}$ if p is odd and $q > 2p$ (see [5]). We will show that this is true in the orientably-regular case. In fact, we will prove something stronger, namely that q must divide $2p$.

We start by showing that if p is odd and q is an even divisor of $2p$, then there is a tight orientably-regular polyhedron of type $\{p, q\}$. To do this, we define $\Gamma(p, q)$ as the group $\langle\, \rho_0, \rho_1, \rho_2 \,|\, \rho_0^2, \rho_1^2, \rho_2^2, (\rho_0\rho_1)^p, (\rho_1\rho_2)^q, (\rho_0\rho_2)^2, (\rho_0\rho_1\rho_2\rho_1)^2 \,\rangle$, which is obtainable by adding one extra relator to the Coxeter group $[p, q]$. (Note that this is the group of the polyhedron $\{p, q\}_{\cdot,2}$ described in [8, p. 196].)

Theorem 3.1. *Let $p \geq 3$ be odd, and let q be an even divisor of $2p$. Then there is a tight orientably-regular polyhedron $\mathcal{P}$ of type $\{p, q\}$ such that $\Gamma(\mathcal{P}) \cong \Gamma(p, q)$.*

Proof. Let $\Gamma(p, q) = \langle \rho_0, \rho_1, \rho_2 \rangle$. In light of the construction in Section 2.2, all we need to do is show that $\Gamma(p, q)$ is a string C-group of order $2pq$, in which the order of $\rho_0\rho_1$ is p and the order of $\rho_1\rho_2$ is q.

First, note that the element $\omega = (\rho_1\rho_2)^2$ generates a cyclic normal subgroup N of $\Gamma(p, q)$, since each of ρ_1 and ρ_2 conjugates ω to its inverse, and the extra relation $(\rho_0\rho_1\rho_2\rho_1\rho_2)^2 = 1$ implies that ρ_0 does the same. Factoring out N gives quotient $\Gamma(p, 2)$, in which the extra relation $(\rho_0\rho_1\rho_2\rho_1\rho_2)^2 = 1$ is redundant. In fact $\Gamma(p, 2)$ is isomorphic to the string Coxeter group $[p, 2]$, which is an extension of the dihedral group of order $2p$, and has order $4p$.

In particular, $\Gamma(p, q)$ covers $\Gamma(p, 2) \cong [p, 2]$, and it follows that $\rho_0\rho_1$ has order p (rather than some proper divisor of p). Also the cover from $\Gamma(p, q)$ to $\Gamma(p, 2)$ is one-to-one on $\langle \rho_0, \rho_1 \rangle$, and so by Proposition 2.1, we find that $\Gamma(p, q)$ is a string C-group.

Next, we observe that the dihedral group $D_q = \langle\, y_1, y_2 \,|\, y_1^2, y_2^2, (y_1 y_2)^q \,\rangle$ is a quotient of $\Gamma(p, q)$, via an epimorphism taking $\rho_1 \mapsto y_1$, $\rho_2 \mapsto y_2$ and $\rho_0 \mapsto (y_1 y_2)^{p-1} y_1$. (Note that the defining relations for $\Gamma(p, q)$ are satisfied by their images in D_q, since $(y_1 y_2)^{p-1} y_1$ has order 2, the order of $(y_1 y_2)^{p-1}$ divides p (as $p-1$ is even and q divides $2p$), the order of $(y_1 y_2)^{p-1} y_1 y_2 = (y_1 y_2)^p$ divides 2 (as q divides $2p$), and $(y_1 y_2)^{p-1} y_2 y_1 y_2$ has order 2.) In particular, the image of $\rho_1\rho_2$ is $y_1 y_2$, which has order q, and hence $\rho_1\rho_2$ has order q.

Finally, $|\Gamma(p,q)| = |\Gamma(p,q)/N||N| = |\Gamma(p,2)||N| = 4p(q/2) = 2pq$, since $N = \langle(\rho_1\rho_2)^2\rangle$ has order $q/2$. $\qquad\square$

We will show that in fact, the only tight orientably-regular polyhedra of type $\{p,q\}$ with p odd are those given in Theorem 3.1. We proceed with the help of a simple lemma.

Lemma 3.2. *Let $\mathcal{P}$ be an orientably-regular polyhedron of type $\{p,q\}$, with p odd, and with automorphism group $\Gamma(\mathcal{P})$ generated by the reflections ρ_0, ρ_1, ρ_2. If $\omega = (\rho_1\rho_2)^2 = \sigma_2^2$ generates a normal subgroup of $\Gamma^+(\mathcal{P})$, then ω is central, and q divides $2p$.*

Proof. For simplicity, let $x = \sigma_1 = \rho_0\rho_1$ and $y = \sigma_2 = \rho_1\rho_2$, so that $xy = \rho_0\rho_2$ and hence $x^p = y^q = (xy)^2 = 1$, and also $\omega = y^2$. By hypothesis, $\langle y^2\rangle$ is normal, and so $xy^2x^{-1} = y^{2k}$ for some k. It follows that $y^2 = x^p y^2 x^{-p} = y^{2k^p}$ and that $y^2 = (xy)^2 y^2 (xy)^{-2} = y^{2k^2}$, and therefore $2 \equiv 2k^2 \equiv 2k^p \bmod q$. Then also $2 \equiv 2k^2 k^{p-2} \equiv 2k^{p-2} \bmod q$, and by induction $2 \equiv 2k^p \equiv 2k^{p-2} \equiv \cdots \equiv 2k \bmod q$, since p is odd. Thus $xy^2x^{-1} = y^{2k} = y^2$, and so $\omega = y^2$ is central. Moreover, since y^2 commutes with x (which has order p) and $xy = y^{-1}x^{-1}$, we find that $y^{2p} = x^p y^{2p} = (xy^2)^p = (y^{-1}x^{-1}y)^p = y^{-1}x^{-p}y = 1$, and so $2p$ is a multiple of q. $\qquad\square$

We also utilise a connection between tight polyhedra and regular Cayley maps, as is explained in [4]. Specifically, suppose that the finite group G is generated by two non-involutory elements x and y such that xy has order 2, and that G can be written as AY where $Y = \langle y\rangle$ is core-free in G (that is, Y contains no non-trivial normal subgroup of G), and A is a subgroup of G such that $A \cap Y = \{1\}$. Then G is the group $\Gamma^+(M)$ of orientation-preserving automorphisms group of a regular Cayley map M for the group A. Furthermore, this map M is reflexible if and only if G admits an automorphism taking $x \mapsto xy^2 (= y^{-1}x^{-1}y)$ and $y \mapsto y^{-1}$.

Theorem 3.3. *Let $p \geq 3$ be odd. If $\mathcal{P}$ is a tight orientably-regular polyhedron of type $\{p,q\}$, then q is an even divisor of $2p$, and $\Gamma(\mathcal{P})$ is isomorphic to $\Gamma(p,q)$.*

Proof. Let $G = \Gamma^+(\mathcal{P})$, and let $\sigma_1 = \rho_0\rho_1$ and $\sigma_2 = \rho_1\rho_2$ be its standard generators. Also take $F = \langle\sigma_1\rangle$ and $V = \langle\sigma_2\rangle$, which are the stabilisers in $\Gamma^+(\mathcal{P})$ of a 2-face and incident vertex of $\mathcal{P}$. Then $F \cap V = \langle\varepsilon\rangle$ since $\mathcal{P}$ is a polytope, and $G = FV$ since $\mathcal{P}$ is tight.

Now, let N be the core of V in G (which is the largest normal subgroup of G contained in V), and let $\overline{G} = G/N$, $\overline{V} = V/N$ and $\overline{F} = FN/N$. Then $\overline{G} = \overline{V}\,\overline{F}$, and $\overline{V} \cap \overline{F}$ is trivial, and also $\overline{V}$ is core-free. Thus $\overline{G}$ is the orientation-preserving automorphism group of a regular Cayley map M for the cyclic group $\overline{F}$. Furthermore, since $\mathcal{P}$ is an orientably-regular polyhedron, the group $G = \Gamma^+(\mathcal{P})$ has an automorphism taking $\sigma_1 \mapsto \sigma_1\sigma_2^2$ and $\sigma_2 \mapsto \sigma_2^{-1}$, and $\overline{G}$ has the analogous property. Hence M is reflexible.

On the other hand, by [4, Theorem 3.7] we know that the only reflexible regular Cayley map for a cyclic group of odd order p is the equatorial map on the sphere, with p vertices of valence 2. Thus $|\overline{V}| = 2$, and so $q = |V|$ is even, and $N = \langle\sigma_2^2\rangle$.

In particular, $\langle\sigma_2^2\rangle$ is a normal subgroup of $\Gamma^+(\mathcal{P})$, and hence by Lemma 3.2 we also find that q divides $2p$, and that σ_2^2 is central. But $\sigma_2^2 = (\rho_1\rho_2)^2$ is inverted under conjugation by ρ_1, and now centralised by $\sigma_1 = \rho_0\rho_1$, and therefore also inverted under conjugation by ρ_0. Hence the relation $(\rho_0\rho_1\rho_2\rho_1\rho_2)^2 = \varepsilon$ holds in $\Gamma(\mathcal{P})$, so $\Gamma(\mathcal{P})$ is a quotient of $\Gamma(p,q)$. Then finally, since $\mathcal{P}$ is tight we have $|\Gamma(\mathcal{P})| = 2pq = |\Gamma(p,q)|$, and it follows that $\Gamma(\mathcal{P}) \cong \Gamma(p,q)$. $\qquad\square$

Combining Theorem 3.1 with Theorem 3.3 and [5, Theorem 6.3], we can now draw the following conclusion:

Theorem 3.4. *There is a tight orientably-regular polyhedron of type $\{p, q\}$ if and only if one of the following is true*:

(a) *p and q are both even, or*

(b) *p is odd and q is an even divisor of $2p$, or*

(c) *q is odd and p is an even divisor of $2q$.*

4 Tight orientably-regular polytopes in higher ranks

Now that we know which Schläfli symbols appear among tight orientably-regular polyhedra, we can proceed to classify the tight orientably-regular polytopes of arbitrary rank. We will say that the $(n-1)$-tuple $(p_1, \ldots, p_{n-1})$ is *admissible* if each of p_{i-1} and p_{i+1} (when defined) is an even divisor of $2p_i$ whenever p_i is odd.

Theorem 4.1. *If $\mathcal{P}$ is a tight orientably-regular polytope of type $\{p_1, \ldots, p_{n-1}\}$, then the $(n-1)$-tuple $(p_1, \ldots, p_{n-1})$ is admissible.*

Proof. If $\mathcal{P}$ is tight and orientably-regular, then by [5, Proposition 3.8] we know that all of its sections of rank 3 are tight and orientably-regular. Hence in particular, if p_i is odd then the sections of $\mathcal{P}$ of type $\{p_{i-1}, p_i\}$ and $\{p_i, p_{i+1}\}$ are tight and orientably-regular. The rest now follows from Theorem 3.3. □

We will prove that this necessary condition is also sufficient, which will then complete the proof of Theorem 1.1. We do this by constructing the automorphism group of a tight orientably-regular polytope of the given type.

Let $(p_1, \ldots, p_{n-1})$ be an admissible $(n-1)$-tuple. Then we define the group $\Gamma(p_1, \ldots, p_{n-1})$ to be the quotient of the string Coxeter group $[p_1, \ldots, p_{n-1}]$ obtained by adding $n-2$ extra relations $r_1 = \cdots = r_{n-2} = 1$, where

$$r_i = \begin{cases} (x_{i-1}x_ix_{i+1}x_i)^2 & \text{if } p_i \text{ and } p_{i+1} \text{ are both even, or} \\ (x_{i-1}x_ix_{i+1}x_ix_{i+1})^2 & \text{if } p_i \text{ is odd and } p_{i+1} \text{ is even, or} \\ (x_{i+1}x_ix_{i-1}x_ix_{i-1})^2 & \text{if } p_i \text{ is even and } p_{i+1} \text{ is odd.} \end{cases}$$

Note that if $n = 3$, this definition of $\Gamma(p_1, p_2)$ coincides with the one in the previous section. For $n \geq 4$, the group $\Gamma(p_1, \ldots, p_{n-1})$ is the amalgamation of $\Gamma(p_1, \ldots, p_{n-2})$ with $\Gamma(p_2, \ldots, p_{n-1})$ in the obvious way, subject to the extra relation $(x_0x_{n-1})^2 = 1$.

Also let $\mathcal{P}(p_1, \ldots, p_{n-1})$ be the poset obtained from $\Gamma(p_1, \ldots, p_{n-1})$, using the construction in Section 2.2. We will show that $\Gamma(p_1, \ldots, p_{n-1})$ is a string C-group of order $2p_1p_2 \ldots p_{n-1}$, and then since every relator of $\Gamma(p_1, \ldots, p_{n-1})$ has even length, it follows that $\mathcal{P}(p_1, \ldots, p_{n-1})$ is a tight orientably-regular polytope of type $\{p_1, \ldots, p_{n-1}\}$.

We start by considering the order of $\Gamma(p_1, \ldots, p_{n-1})$.

Proposition 4.2. *Let p_i be even. Then every element of $\Gamma(p_1, \ldots, p_{n-1})$ either commutes with $(x_{i-1}x_i)^2$ or inverts it by conjugation. In particular, the square of every element of $\Gamma(p_1, \ldots, p_{n-1})$ commutes with $(x_{i-1}x_i)^2$.*

Proof. Let $\omega = (x_{i-1}x_i)^2$. If $j \leq i - 3$ or $j \geq i + 2$, then x_j commutes with both x_{i-1} and x_i, and so commutes with ω. Also it is clear that x_{i-1} and x_i both conjugate ω to ω^{-1}, so it remains to consider only x_{i-2} and x_{i+1}. Now since p_i is even, the relator r_{i-1} is either $(x_{i-2}x_{i-1}x_ix_{i-1})^2$ or $(x_{i-2}x_{i-1}x_ix_{i-1}x_i)^2$. In the first case, x_{i-2} commutes with $x_{i-1}x_ix_{i-1}$ and hence with $(x_{i-1}x_ix_{i-1})x_i = \omega$, while in the second case, we have $(x_{i-2}\omega)^2 = 1$ and so x_{i-2} conjugates ω to ω^{-1}. Similarly, the relator r_i is $(x_{i-1}x_ix_{i+1}x_i)^2$ or $(x_{i+1}x_ix_{i-1}x_ix_{i-1})^2$, and in these two cases we find that $x_{i+1}\omega x_{i+1} = \omega$ or ω^{-1}, respectively. Thus every generator x_j of $\Gamma(p_1, \ldots, p_{n-1})$ either commutes with $(x_{i-1}x_i)^2$ or inverts it by conjugation, and it follows that the same is true for every element of $\Gamma(p_1, \ldots, p_{n-1})$. The rest follows easily. □

Proposition 4.3. *Let $(p_1, \ldots, p_{n-1})$ be an admissible $(n-1)$-tuple with the property that for every i strictly between 1 and $n-1$, either $p_i = 2$ or $p_{i-1} = p_{i+1} = 2$. Then $y_i = x_{i-1}x_i$ has order p_i for all i, and $|\Gamma(p_1, \ldots, p_{n-1})| = 2p_1 \cdots p_{n-1}$.*

Proof. We use induction on n, together with the observation that if $p_j = 2$, then $1 = (x_{j-1}x_j)^2$, so that x_{j-1} commutes with x_j, and therefore $\langle x_0, \ldots, x_{j-1} \rangle$ centralises $\langle x_j, \ldots, x_{n-1} \rangle$. First, if $p_1 = 2$, then $\Gamma(p_1, p_2, \ldots, p_{n-1}) = \Gamma(2, p_2, \ldots, p_{n-1}) \cong \langle x_0 \rangle \times \Gamma(p_2, \ldots, p_{n-1})$, and so $|\Gamma(p_1, \ldots, p_{n-1})| = 2|\Gamma(p_2, \ldots, p_{n-1})| = 4p_2 \cdots p_{n-1} = 2p_1p_2 \cdots p_{n-1}$. Otherwise $p_2 = 2$ and $\Gamma(p_1, p_2, \ldots, p_{n-1}) = \Gamma(p_1, 2, p_3, \ldots, p_{n-1}) \cong \Gamma(p_1) \times \Gamma(p_3, \ldots, p_{n-1})$, and therefore $|\Gamma(p_1, \ldots, p_{n-1})| = |\Gamma(p_1)|\, |\Gamma(p_3, \ldots, p_{n-1})| = 2p_1\, 2p_3 \cdots p_{n-1} = 2p_1p_2 \cdots p_{n-1}$. The claim about the orders of the elements $y_i = x_{i-1}x_i$ follows easily by induction as well. □

Lemma 4.4. *Let q_i be the order of $x_{i-1}x_i$ in $\Gamma(p_1, \ldots, p_{n-1})$, for $1 \leq i < n$. Then $q_i = p_i$ whenever p_i is odd, and also $|\Gamma(p_1, \ldots, p_{n-1})| = 2q_1 \cdots q_{n-1}$, which divides $2p_1 \cdots p_{n-1}$.*

Proof. Let $k_i = p_i$ when p_i is odd, or 2 when p_i is even. Then since k_i divides p_i for all i, there exists an epimorphism $\pi : \Gamma(p_1, \ldots, p_{n-1}) \to \Gamma(k_1, \ldots, k_{n-1})$. Also the $(n-1)$-tuple $(k_1, \ldots, k_{n-1})$ is admissible, and indeed k_{i-1} and k_{i+1} are both 2 whenever k_i is odd (since p_{i-1} and p_{i+1} are both even whenever p_i is odd). Thus $(k_1, \ldots, k_{n-1})$ satisfies the hypotheses of Proposition 4.3, and so $|\Gamma(k_1, \ldots, k_{n-1})| = 2k_1 \cdots k_{n-1}$.

Moreover, Proposition 4.3 tells us that when p_i is odd, the order of the image of x_i in $\Gamma(k_1, \ldots, k_{n-1})$ is $k_i = p_i$, and so $q_i = p_i$; on the other hand, if p_i is even, then the order of the image of x_i in $\Gamma(k_1, \ldots, k_{n-1})$ is $k_i = 2$, and so q_i is even in that case.

Now the kernel of the epimorphism π is the smallest normal subgroup of $\Gamma(p_1, \ldots, p_{n-1})$ containing the elements $(x_{i-1}x_i)^2$ for those i such that p_i is even. By Proposition 4.2, however, the subgroup N generated by these elements is normal in $\Gamma(p_1, \ldots, p_{n-1})$, and abelian. Hence in particular, $N = \ker \pi$, and also by the intersection condition, $|N|$ is the product of the numbers $q_i/2$ over all i for which p_i is even. Thus $|\Gamma(p_1, \ldots, p_{n-1})| = 2q_1 \cdots q_{n-1}$. □

In order to use Theorem 2.5 to build our tight regular polytopes recursively, we need two more observations. The first concerns the flat amalgamation property (FAP):

Proposition 4.5. *If p_2 is even, then $\mathcal{P}(p_1, \ldots, p_{n-1})$ has the FAP with respect to its 2-faces, and if p_{n-2} is even, then $\mathcal{P}(p_1, \ldots, p_{n-1})$ has the FAP with respect to its co-$(n-3)$-faces.*

Proof. Let p_2 be even, and consider the effect of killing the generators x_i of $\Gamma(p_1, \ldots, p_{n-1})$, for $i \geq 2$ (that is, by adding the relations $x_i = 1$ to the presentation for $\Gamma(p_1, \ldots, p_{n-1})$). Each of the relators $r_3, \ldots, r_{n-2}$ contains only generators x_i with $i \geq 2$, so becomes redundant, and may be removed. The relator r_2 reduces to x_1^2 or x_1^4, while r_1 reduces to $(x_0 x_1^2)^2$, which is equivalent to x_0^2, and hence all of these become redundant too. Thus adding the relations $x_i = 1$ to $\Gamma(p_1, \ldots, p_{n-1})$ has the same effect as adding the relations $x_i = 1$ to the string Coxeter group $[p_1, \ldots, p_{n-1}]$. It is easy to see that this gives the quotient group with presentation $\langle\, x_0, x_1 \,|\, x_0^2, x_1^2, (x_0 x_1)^{p_1} \,\rangle$, which is the automorphism group of the 2-faces of $\mathcal{P}(p_1, \ldots, p_{n-1})$. Thus $\mathcal{P}(p_1, \ldots, p_{n-1})$ has the FAP with respect to its 2-faces. The second claim can be proved by a dual argument. $\qquad\square$

Proposition 4.6. *Let $\mathcal{P}$ be an equivelar n-polytope with tight m-faces and tight co-k-faces, where $m \geq k + 3$. Then $\mathcal{P}$ is tight.*

Proof. Since the m-faces are tight, they are $(i, i+2)$-flat for $0 \leq i \leq m-3$, by Theorem 2.4, and then by Proposition 2.3, the polytope $\mathcal{P}$ is $(i, i+2)$-flat for $0 \leq i \leq m-3$. Similarly, the co-k-faces are $(i, i+2)$-flat for $0 \leq i \leq n-k-4$, and $\mathcal{P}$ is $(i, i+2)$-flat for $k+1 \leq i \leq n-3$. Finally, since $m \geq k + 3$, we see that $\mathcal{P}$ is $(i, i+2)$-flat for $0 \leq i \leq n-3$, and again Theorem 2.4 applies, to show that $\mathcal{P}$ is tight. $\qquad\square$

We can now prove the following.

Theorem 4.7. *Let $(p_1, \ldots, p_{n-1})$ be an admissible $(n-1)$-tuple, with $n \geq 4$. Also suppose that p_{i-1} and p_{i+1} are both even, for some i (with $2 \leq i \leq n-2$). If $\mathcal{P}(p_1, \ldots, p_i)$ is a tight orientably-regular polytope of type $\{p_1, \ldots, p_i\}$, and $\mathcal{P}(p_i, \ldots, p_{n-1})$ is a tight orientably-regular polytope of type $\{p_i, \ldots, p_{n-1}\}$, then $\mathcal{P}(p_1, \ldots, p_{n-1})$ is a tight orientably-regular polytope of type $\{p_1, \ldots, p_{n-1}\}$.*

Proof. Let $\mathcal{P}_1 = \mathcal{P}(p_1, \ldots, p_i)$ and $\mathcal{P}_2 = \mathcal{P}(p_i, \ldots, p_{n-1})$, which by hypothesis are tight orientably-regular polytopes of the appropriate types. Since p_{i-1} and p_{i+1} are even, Proposition 4.5 tells us that $\mathcal{P}_1$ has the FAP with respect to its co-$(i-2)$-faces, and $\mathcal{P}_2$ has the FAP with respect to its 2-faces. Then by Theorem 2.5, there exists a regular polytope $\mathcal{P}$ with $(i+1)$-faces isomorphic to $\mathcal{P}_1$ and co-$(i-2)$-faces isomorphic to $\mathcal{P}_2$. Moreover, since $\mathcal{P}_1$ and $\mathcal{P}_2$ are both tight, Proposition 4.6 implies that $\mathcal{P}$ is also tight, and then since $\mathcal{P}$ is of type $\{p_1, \ldots, p_{n-1}\}$, we find that $|\Gamma(\mathcal{P})| = 2p_1 \cdots p_{n-1}$. But also the $(i+1)$-faces of $\mathcal{P}$ are isomorphic to $\mathcal{P}_1$, and the co-$(i-2)$-faces are isomorphic to $\mathcal{P}_2$, and so the standard generators of $\Gamma(\mathcal{P})$ must satisfy all the relations of $\Gamma(p_1, \ldots, p_{n-1})$. In particular, $\Gamma(\mathcal{P})$ is a quotient of $\Gamma(p_1, \ldots, p_{n-1})$, and so $|\Gamma(p_1, \ldots, p_{n-1})| \geq 2p_1 \cdots p_{n-1}$. On the other hand, $|\Gamma(p_1, \ldots, p_{n-1})| \leq 2p_1 \cdots p_{n-1}$ by Lemma 4.4. Thus $|\Gamma(p_1, \ldots, p_{n-1})| = 2p_1 \cdots p_{n-1}$, and hence also $\Gamma(\mathcal{P}) \cong \Gamma(p_1, \ldots, p_{n-1})$, and $\mathcal{P} \cong \mathcal{P}(p_1, \ldots, p_{n-1})$. Thus $\mathcal{P}(p_1, \ldots, p_{n-1})$ is a tight polytope of type $\{p_1, \ldots, p_{n-1}\}$, and finally, since all the defining relations of $\Gamma(p_1, \ldots, p_{n-1})$ have even length, we find that $\mathcal{P}(p_1, \ldots, p_{n-1})$ is orientably-regular. $\qquad\square$

Note that the above theorem helps us deal with a large number of possibilities, once we have enough 'building blocks' in place. We are assuming that the $(n-1)$-tuple $(p_1, \ldots, p_{n-1})$ is admissible, so that p_{i-1} and p_{i+1} are even divisors of $2p_i$ whenever p_i is odd. Now suppose that $n \geq 6$. If p_2 and p_4 are both even, then Theorem 4.7 will apply, and if not, then one of them is odd, say p_2, in which case p_1 and p_3 must both be even, and

again Theorem 4.7 will apply. Hence this leaves us with just a few cases to verify, namely admissible $(n-1)$-tuples $(p_1, \ldots, p_{n-1})$ with $n = 4$ or 5 for which there is no i such that p_{i-1} and p_{i+1} are both even.

The only such cases are as follows:

- $n = 4$, with p_1 odd, p_2 even and p_3 even, or dually, p_1 even, p_2 even and p_3 odd,
- $n = 4$, with p_1 odd, p_2 even and p_3 odd,
- $n = 5$, with p_1 odd, p_2 even, p_3 even and p_4 odd.

We start with the cases where $n = 4$:

Proposition 4.8. *If p_1 is odd, p_2 is an even divisor of $2p_1$, and $p_3 \geq 2$, then $\Gamma(p_1, p_2, p_3)$ is a string C-group, and $\mathcal{P}(p_1, p_2, p_3)$ is a tight orientably-regular polytope of type $\{p_1, q, p_3\}$, for some even q dividing p_2.*

Proof. Let $\Gamma = \Gamma(p_1, p_2, p_3) = \langle x_0, x_1, x_2, x_3 \rangle$, and let $\overline{\Gamma} = \Gamma(p_1, 2, p_3) = \langle y_0, y_1, y_2, y_3 \rangle$, where $y_i = \overline{x_i}$ is the image of x_i under the natural epimorphism $\pi : \Gamma(p_1, p_2, p_3) \to \Gamma(p_1, 2, p_3)$, for $0 \leq i \leq 3$. By Proposition 4.3, we know that $\Gamma(p_1, 2, p_3) \cong [p_1, 2, p_3]$. Also the subgroup $\langle x_0, x_1, x_2 \rangle$ of Γ covers $[p_1, 2]$, and this cover is one-to-one on $\langle x_0, x_1 \rangle$, so $\langle x_0, x_1, x_2 \rangle$ is a string C-group, by Proposition 2.1. A similar argument shows that $\langle x_1, x_2, x_3 \rangle$ is a string C-group. Now the intersection of these two string C-groups is $\langle x_1, x_2 \rangle$, since the intersection of their images in $\overline{\Gamma}$ is $\langle y_0, y_1, y_2 \rangle \cap \langle y_1, y_2, y_3 \rangle = \langle y_1, y_2 \rangle$, and the kernel of π is $\langle (x_1 x_2)^2 \rangle$. Hence by Proposition 2.2, $\Gamma(p_1, p_2, p_3)$ is a string C-group, and the rest follows easily from Lemma 4.4. $\square$

Lemma 4.9. *If $p_i = 2$ for some i, then in $\Gamma(p_1, \ldots, p_{n-1}) = \langle x_0, \ldots, x_{n-1} \rangle$, we have $\langle x_0, \ldots, x_i \rangle \cong \Gamma(p_1, \ldots, p_i)$ and $\langle x_{i-1}, \ldots, x_{n-1} \rangle \cong \Gamma(p_i, \ldots, p_{n-1})$.*

Proof. First consider $\Lambda = \langle x_0, \ldots, x_i \rangle$. This is obtainable as a quotient of $\Gamma(p_1, \ldots, p_i)$ by adding extra relations. But also it can be obtained from the group $\Gamma(p_1, \ldots, p_{n-1})$ by killing the unwanted generators $x_{i+1}, \ldots, x_{n-1}$. (Note that for $i+1 \leq j \leq n-1$, the relation $r_j = 1$ and all relations of the form $(\rho_j \rho_k)^m = 1$ become redundant and may be removed, and the same holds for the relations $r_i = 1$ and $r_{i-1} = 1$ since the assumption that $p_i = 2$ implies that $[x_{i-1}, x_i] = (x_{i-1}, x_i)^2 = 1$ and hence that $x_0, \ldots, x_{i-1}$ commute with $x_i, \ldots, x_{n-1}$.) It follows that $\Lambda \cong \Gamma(p_1, \ldots, p_i)$. Also $\langle x_{i-1}, \ldots, x_{n-1} \rangle \cong \Gamma(p_i, \ldots, p_{n-1})$, by the dual argument. $\square$

Theorem 4.10. *If p_1 is odd, p_2 is an even divisor of $2p_1$, and p_3 is even, then $\mathcal{P}(p_1, p_2, p_3)$ is a tight orientably-regular polytope of type $\{p_1, p_2, p_3\}$.*

Proof. First, the group $\Gamma(p_1, p_2, p_3) = \langle x_0, x_1, x_2, x_3 \rangle$ covers $\Gamma(p_1, p_2, 2) = \langle y_0, y_1, y_2, y_3 \rangle$, say, since p_3 is even. Also by Lemma 4.9 we know that $\langle y_0, y_1, y_2 \rangle$ is isomorphic to $\Gamma(p_1, p_2)$, and hence in particular, $y_1 y_2$ has order p_2. It follows that the order of $x_1 x_2$ is also p_2, and then the conclusion follows from Proposition 4.8. $\square$

Theorem 4.11. *If p_1 and p_3 are odd, and p_2 is an even divisor of both $2p_1$ and $2p_3$, then $\mathcal{P}(p_1, p_2, p_3)$ is a tight orientably-regular polytope of type $\{p_1, p_2, p_3\}$.*

Proof. Under the given assumptions, the group $\Gamma(p_1, p_2, p_3)$ is obtained from the string Coxeter group $[p_1, p_2, p_3] = \langle x_0, x_1, x_2, x_3 \rangle$ by adding the two extra relations $(x_0 x_1 x_2 x_1 x_2)^2 = 1$ and $(x_3 x_2 x_1 x_2 x_1)^2 = 1$. These imply that the element $\omega = (x_1 x_2)^2$

is inverted under conjugation by each of x_0 and x_3, and hence by all the x_i. Now let $y_i = x_{i-1}x_i$ for $1 \leq i \leq 3$. These elements generate the orientation-preserving subgroup $\Gamma^+(p_1, p_2, p_3)$, and they all centralise $\omega = y_2^2$. It follows that $\Gamma^+(p_1, p_2, p_3)$ has presentation

$$\langle\, y_1, y_2, y_3 \mid y_1^{p_1}, y_2^{p_2}, y_3^{p_3}, (y_1y_2)^2, (y_2y_3)^2, (y_1y_2y_3)^2, [y_1, y_2^2], [y_3, y_2^2]\,\rangle.$$

We now exhibit a permutation representation of this group on the Cartesian product $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$, by letting each y_i induce the permutation π_i, where

$$(j, k)^{\pi_2} = (j, k+1) \text{ for all } (j, k),$$

$$(j, k)^{\pi_1} = \begin{cases} (j+1, k) & \text{for } k \text{ even} \\ (j-1, k-2) & \text{for } k \text{ odd,} \end{cases} \qquad (j, k)^{\pi_3} = \begin{cases} (j, k-2j) & \text{for } k \text{ even} \\ (j, k+2(j-1)) & \text{for } k \text{ odd.} \end{cases}$$

It is easy to see that π_1 and π_2 have orders p_1 and p_2 respectively (since p_2 divides $2p_1$), and that the order of π_3 divides $p_2/2$ and hence divides p_3. It is also easy to verify that they satisfy the other defining relations for $\Gamma^+(p_1, p_2, p_3)$, and thus we do have a permutation representation.

In particular, since π_2 has order p_2, so does $y_2 = x_1x_2$, and again the conclusion follows from Proposition 4.8. $\qquad\square$

We now handle the remaining case.

Theorem 4.12. *If p_1 and p_4 are odd, p_2 is an even divisor of $2p_1$, and p_3 is an even divisor of $2p_4$, then $\mathcal{P}(p_1, p_2, p_3, p_4)$ is a tight orientably-regular polytope of type $\{p_1, \ldots, p_4\}$.*

Proof. Take $\Gamma = \Gamma(p_1, \ldots, p_4) = \langle x_0, \ldots, x_4 \rangle$, and $\Lambda = \Gamma(p_1, p_2, 2, p_4) = \langle y_0, \ldots, y_4 \rangle$. Then Γ covers Λ, and this induces a cover from $\langle x_0, \ldots, x_3 \rangle$ to $\langle y_0, \ldots, y_3 \rangle$, which is isomorphic to $\Gamma(p_1, p_2, 2)$ by Lemma 4.9. Similarly we have a cover from $\langle x_0, x_1, x_2 \rangle$ to $\langle y_0, y_1, y_2 \rangle$, which is isomorphic to $\Gamma(p_1, p_2)$. But on the other hand, $\langle x_0, x_1, x_2 \rangle$ is a quotient of $\Gamma(p_1, p_2)$, and hence these two groups are isomorphic. In particular, the cover from $\langle x_0, \ldots, x_3 \rangle$ to $\Gamma(p_1, p_2, 2)$ is one-to-one on the facets, so $\langle x_0, \ldots, x_3 \rangle$ is a string C-group. By a dual argument, $\langle x_1, \ldots, x_4 \rangle$ is also a string C-group.

Next, let $\Delta = \Gamma(p_1, 2, 2, p_4) = \langle z_0, \ldots, z_4 \rangle$, and let π be the covering homomorphism from Γ to Δ. The kernel of π is the subgroup generated by $(x_1x_2)^2$ and $(x_2x_3)^2$, since the defining relations for $\Gamma = \Gamma(p_1, \ldots, p_4)$ imply that these two elements are centralised or inverted under conjugation by each generator x_i. In particular, $\ker \pi \subseteq \langle x_1, x_2, x_3 \rangle$. As also the intersection of $\langle z_0, \ldots, z_3 \rangle$ and $\langle z_1, \ldots, z_4 \rangle$ in Δ is $\langle z_1, z_2, z_3 \rangle$, it follows that intersection of $\langle x_0, \ldots, x_3 \rangle$ and $\langle x_1, \ldots, x_4 \rangle$ is $\langle x_1, x_2, x_3 \rangle$. Hence Γ is a string C-group.

Now $\langle x_0, x_1, x_2 \rangle \cong \Gamma(p_1, p_2)$, and by Theorem 3.1 we know the polytope $\mathcal{P}(p_1, p_2)$ has type $\{p_1, p_2\}$. Similarly $\langle x_2, x_3, x_4 \rangle \cong \Gamma(p_3, p_4)$, and $\mathcal{P}(p_3, p_4)$ has type $\{p_3, p_4\}$. It follows that $\mathcal{P}(p_1, p_2, p_3, p_4)$ is an orientably-regular polytope of type $\{p_1, \ldots, p_4\}$. In particular, the order of $x_{i-1}x_i$ is p_i (for $1 \leq i \leq 4$), and so by Lemma 4.4, $\mathcal{P}(p_1, p_2, p_3, p_4)$ is tight. $\qquad\square$

This gives us all the building blocks we need. With the help of Theorem 4.7, we now know that $\mathcal{P}(p_1, \ldots, p_{n-1})$ is a tight orientably-regular polytope of type $\{p_1, \ldots, p_{n-1}\}$ whenever $(p_1, \ldots, p_{n-1})$ is admissible, and the proof of Theorem 1.1 is complete.

5   Tight non-orientably-regular polytopes

We have not yet been able to completely characterise the Schläfli symbols of tight, non-orientably-regular polytopes, but we have made some partial progress. For example, we can easily find an infinite family of tight, non-orientably-regular polyhedra.

Theorem 5.1. *For every odd positive integer k, there exists a non-orientably-regular tight polyhedron of type $\{3k, 4\}$, with automorphism group $\Lambda(k)$ having presentation*

$$\langle\, \rho_0, \rho_1, \rho_2 \mid \rho_0^2, \rho_1^2, \rho_2^2, (\rho_0\rho_1)^{3k}, (\rho_0\rho_2)^2, (\rho_1\rho_2)^4, \rho_0\rho_1\rho_2\rho_1\rho_0\rho_1\rho_2\rho_1\rho_2 \,\rangle.$$

Proof. We note that $\Lambda(1)$ is the automorphism group of the hemi-octahedron (of type $\{3, 4\}$), and that $\Lambda(k)$ covers $\Lambda(1)$, for every k. Hence in each $\Lambda(k)$, the order of $\rho_1\rho_2$ is 4 (and not 1 or 2). Next, because the covering is one-to-one on the vertex-figures, it follows from Proposition 2.1 that $\Lambda(k)$ is a string C-group. Also $\Lambda(k)$ has a relation of odd length, and so it must be the automorphism group of a non-orientably-regular polyhedron.

Now let N be the subgroup generated by the involutions ρ_2 and $\rho_1\rho_2\rho_1$. Since their product has order 2, this is a Klein 4-group. Moreover, N is normalised by ρ_1 and ρ_2, and also by ρ_0 since ρ_0 centralises ρ_2 and the final relation in the definition of $\Lambda(k)$ gives $(\rho_1\rho_2\rho_1)^{\rho_0} = \rho_1\rho_2\rho_1\rho_2$. It is now easy to see that N is the normal closure of $\langle\rho_2\rangle$. The quotient $\Lambda(k)/N$ is isomorphic to $\langle\, \rho_0, \rho_1 \mid \rho_0^2, \rho_1^2, (\rho_0\rho_1)^{3k} \,\rangle$, with the final relator for $\Lambda(k)$ becoming trivial, and so $\Lambda(k)/N$ is dihedral of order $6k$. In particular, this shows that $\rho_0\rho_1$ has order $3k$ (and that $|\Lambda(k)| = |\Lambda(k)/N||N| = 24k$). $\qquad\square$

The computational data that we have on polytopes with up to 2000 flags (obtained with the help of MAGMA [1]) suggests that these polyhedra are the only tight non-orientably-regular polyhedra of type $\{p, q\}$ with p odd.

Using Theorem 5.1, it is possible to build tight, non-orientably-regular polytopes in much the same way as we did in Theorem 4.7. In particular, the regular polytope with automorphism group $\Lambda(k)$ has the FAP with respect to its 2-faces, and its dual has the FAP with respect to its vertex-figures (co-0-faces). Then by Theorem 2.5, we know there are tight, non-orientably-regular polytopes of type $\{4, 3k, 4\}$ for each odd k, and of type $\{4, 3k, r\}$ for each odd k and each even r dividing $3k$. It is possible to continue in this fashion, building tight non-orientably-regular polytopes of every rank.

Finally, just as with orientably-regular polytopes, there are some kinds of Schläfli symbol for which no examples can be constructed using Theorem 2.5. In fact (and in contrast with the situation for orientably-regular polytopes), there seem to be no tight non-orientably-regular polytopes of some of these types at all. For example, there are no tight regular polytopes of type $\{3, 4, r\}$ with $r \geq 3$ and with 2000 flags or fewer, but the reason for this is not clear.

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