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Tomáš Kaiser Ondřej Rucký
Matěj Stehlík Riste Škrekovski

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Strong parity vertex coloring of plane graphs*

Tomáš Kaiser† Ondřej Rucký‡ Matěj Stehlík§ Riste Škrekovski¶

Abstract

Czap and Jendroľ introduced the notions of strong parity vertex coloring and the corresponding strong parity chromatic number χ_s. They conjectured that there is a constant bound K on χ_s for the class of 2-connected plane graphs. We prove that the conjecture is true with $K = 97$, even with an added restriction to proper colorings. Next, we provide simple examples showing that the sharp bound is at least 8, or at least 10 for proper strong parity vertex colorings.

Keywords: graph, strong parity vertex coloring, strong parity chromatic number, proper coloring, face, discharging

1 Introduction

Czap and Jendroľ [1] defined the notions of strong parity vertex coloring and the corresponding strong parity chromatic number χ_s equivalently to what follows. Let G be a connected plane graph, and let f be its face. A (clockwise) *facial walk* of f is a shortest closed walk in G with the embeddings of its edges defining a closed oriented curve $\mathcal{C}(t)$ such that for every t_0, a sufficiently small left-side neighborhood of $\mathcal{C}(t)$ at t_0 belongs entirely to the interior of f. Note that $\mathcal{C}(t)$ is precisely the topological boundary of f since G is connected, and that distinct facial walks of f differ only in the choice of their starting vertex. Consider a fixed coloring of G; the face f satisfies the *strong parity vertex coloring condition* (the *spv-condition* briefly) with respect to the coloring if for each color c of the coloring, there is zero or an odd number of occurrences of vertices colored with c on a facial walk of f. The coloring is a *strong parity vertex coloring* (or, shortly, an *spv-coloring*) if the spv-condition holds for every face of G. Assume now that G is 2-connected. The minimum number of colors that admits an spv-coloring of G is called the *strong parity chromatic number* of G and is denoted by $\chi_s(G)$.

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†Department of Mathematics and Institute for Theoretical Computer Science, University of West Bohemia, Univerzitní 8, 306 14 Plzeň, Czech Republic. E-mail: kaisert@kma.zcu.cz. Institute for Theoretical Computer Science is supported by the Czech Ministry of Education as project 1M0545.

‡Department of Mathematics, University of West Bohemia, Univerzitní 8, 306 14 Plzeň, Czech Republic. E-mail: rucky@kma.zcu.cz.

§Laboratoire G-SCOP, Grenoble INP, UJF, CNRS, 46 avenue Félix Viallet, 38031 Grenoble Cedex, France. E-mail: matej.stehlik@g-scop.inpg.fr.

¶Department of Mathematics, University of Ljubljana, Jadranska 21, 1000 Ljubljana, Slovenia. Supported in part by ARRS Research Program P1-0297.

We make two remarks regarding these definitions. First, one can obviously extend the notion of spv-coloring and χ_s to the class of connected and 2-connected plane multigraphs respectively. Second, the assumption of 2-connectedness of G is crucial in the definition of $\chi_s(G)$; Czap and Jendrol' [1] mention a simple example of a non-2-connected plane graph, namely two triangles sharing precisely one vertex, for which no spv-coloring obviously exists. Clearly, a sufficient condition for G to have an spv-coloring is that facial walks of all faces in G be cycles, which indeed holds if G is 2-connected. Nevertheless, the spv-condition of the face f with respect a particular coloring could be modified to the requirement that for each color c of the coloring, there is zero or an odd number of vertices colored with c in the boundary of f. Then the spv-coloring as well as $\chi_s(G)$ are well-defined for every plane graph G, the latter following from the fact that a coloring assigning a different color to each vertex of G is clearly an spv-coloring, and both the notions coincide with their original definitions in the class of connected and 2-connected plane graphs respectively.

Czap and Jendrol' [1] conjectured that there is a constant bound K on χ_s for the class of 2-connected plane graphs. Furthermore, they suggested that the best possible bound equals 6, providing an infinite family of graphs with $\chi_s = 6$. The main result of our paper confirms the conjecture for the class of 2-connected plane multigraphs with an added restriction to proper colorings:

Theorem 1.1. *Every 2-connected plane multigraph has a proper spv-coloring with at most 97 colors.*

The proof is given in Section 2. During the preparation of this paper, another proof (for a somewhat worse constant) was independently found by Czap, Jendrol and Voigt [2]. In Section 3, we present examples demonstrating that the sharp bound of the conjecture is greater than or equal to 8 or, with a restriction to proper colorings, at least 10. We remark above that the definitions of the spv-coloring and χ_s can be extended to the class of all plane multigraphs, and thus also the conjecture generalized this way might be considered. However, we do not pursue this problem in our paper; it remains open.

It should be noted that even before the introduction of parity vertex colorings, certain edge-coloring versions of these notions were introduced in [3, 4]. Recently, facial parity edge-colorings has been studied in [5].

In the remainder of the section, we establish basic notation used throughout the paper; the notions not mentioned here are standard in graph theory [6]. For brevity, we will always refer to multigraphs as graphs, unless a confusion could arise. Let G be a connected plane graph, v its vertex, e_1 and e_2 its edges, and f its face. Then $F(v)$ or $F(e_1)$ denotes the set of faces incident with v or e_1 respectively. The *boundary vertices* or *boundary edges* of f are all the vertices or edges of G respectively which are incident with f; the sets of these are written as $V(f)$ and $E(f)$ respectively. We refer to $|V(f)|$ as the *length* of f. The degree of v, i.e., the number of edges incident with v, is denoted by $d(v)$. If $d(v) = 2$, the vertex v is a 2-*vertex*; otherwise, we call v a *high-degree* vertex or a vertex of *high* degree. The edges e_1 and e_2 of G are *parallel* when they are not loops and share their endvertices. If e_1 and e_2 are parallel and constitute the boundary of f, the face f is a *digon*. For a path P, every its vertex not being its endvertex is called an *internal vertex* of P. Finally, we remark that for all the notation defined, the relevant graph may be referred to by a subscript whenever confusion could arise. For example, we write $F_G(v)$ or $d_G(v)$ if that graph is G.

2 Upper bound

This section is devoted to the proof of the main result, Theorem 1.1.

Proof of Theorem 1.1. We proceed by contradiction, taking a minimal counterexample G with respect to the number of vertices first and to the number of edges next, and applying the well-known discharging method to it. This is done in Section 2.2.

The discharging argument relies on the fact that G, being a minimal counterexample, must satisfy certain structural conditions. These are listed, and the fact itself is proved, in Section 2.1.

Before proceeding to Sections 2.1 and 2.2, we define some ad hoc notation and include several basic claims used afterwards.

Let G be a 2-connected plane graph; let f be a face of G. The number of high-degree vertices on the boundary of f is called the *weight* of f, written as $w(f)$. The face f is a *pseudodigon* if $w(f) = 2$ and f is not a digon; it is a *small* or *large* if $w(f) < 20$ or $w(f) \geq 20$ respectively. The *modified weight* $w'(f)$ of f is defined as 3 if $w(f) = 2$, and $w(f)$ otherwise. Suppose that v is a vertex of G. Then the *configuration* of v, or *f-reduced configuration* of v, is the tuple obtained by ordering the elements from the multiset $\{w(g) \colon g \in F(v)\}$ or $\{w(g) \colon g \in F(v),\, g \neq f\}$, respectively, in a nondecreasing manner. The *modified configuration* of v and *modified f-reduced configuration* of v are defined analogously, with $w'(f_i)$ replaced by $w(f_i)$. The *(open) face-vertex neighborhood* of v, denoted by $N^F(v)$, is defined as $\left(\bigcup_{g \in F(v)} V(g)\right) - \{v\}$, and the *$f$-reduced (open) face-vertex neighborhood* of v, referred to as $N^F(v, f)$, is the set $\left(\bigcup_{g \in F(v),\, g \neq f} V(g)\right) - \{v\}$; we call the sizes of these sets the *face degree* and *f-reduced face degree* respectively, writing $d^F(v)$ and $d^F(v, f)$ respectively. As before, we add the name of the graph in question as the subscript of the symbolic notation defined here to avoid confusion if necessary. For instance, we write $N^F_G(v)$ or $d^F_G(v, f)$ for the graph G.

We now introduce a graph operation to be used in the proof of Lemma 2.4. Let G be a plane graph and v its vertex of degree $d > 0$, there is no loop at v, and the edges incident with v are enumerated in a clockwise order as $e_i := vv_i$, $i \in \mathbb{Z}_d$; note that, in general, some of the vertices v_i may coincide. By the *annihilation* of v we mean the construction of a plane graph G' from G as follows:

(1) add edges $e'_i := v_i v_{i+1}$, $i \in \mathbb{Z}_d$, embedded in the plane so that for each i, the edges e_i, e_{i+1}, and e'_i, in this order, constitute a facial walk;

(2) delete v together with all the edges e_i.

Intuitively, one may achieve the desired embeddings of the edges e'_i by drawing each e'_i 'close enough' to the curve consisting of the embeddings of e_i and e_{i+1}. See Figure 2.1 for an example of a properly conducted annihilation. Regarding the faces of G and G', it is obvious that the following holds:

Observation 2.1.

(1) *Every face of G not in $F_G(v)$ is also a face of G';*

(2) *each face $g \in F_G(v)$ has its counterpart g' in G' such that a facial walk of g' may be obtained from a facial walk W of g by replacing each subsequence of W of the form $e_i v e_{i+1}$ with e'_i, and hence $V(g') = V(g) - \{v\}$;*

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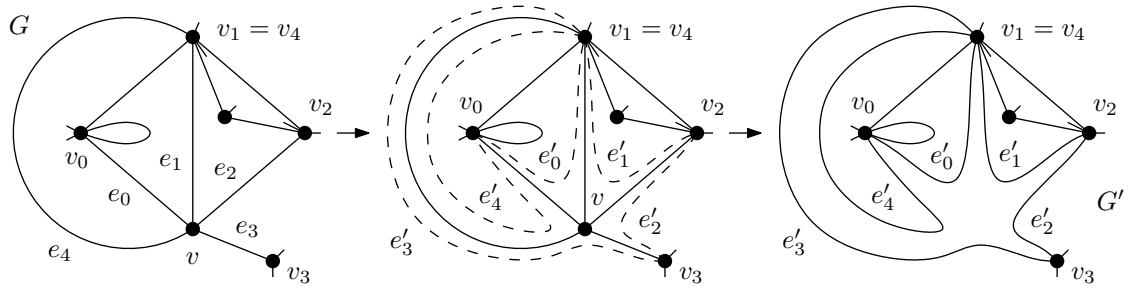


Figure 2.1. The annihilation of a vertex v. The original graph G is on the left, the resulting graph G' on the right.

(3) *there is precisely one more face in G', having the sequence $v_0e'_{d-1}v_{d-1}e'_{d-2}\dots v_1e'_0v_0$ as its facial walk.*

If G is 2-connected, any of its vertices may be annihilated. Therefore, it makes sense to ask whether the class of 2-connected plane graphs is closed under the operation of annihilation of a vertex. One can easily see that the answer is negative: if $v_i = v_j$ for some $i \neq j$ in the definition above, i.e., there is a pair of parallel edges incident with v in G, then the vertex v_i becomes a cut-vertex in G'. However, it turns out that unless this situation occurs or G is too small, 2-connectedness is preserved:

Lemma 2.2. *If v is a vertex of a 2-connected plane graph G, $|V(G)| \geq 4$, such that it is incident with no pair of parallel edges, then a graph G' obtained after the annihilation of v is 2-connected.*

Proof. Every plane graph on at least three vertices is 2-connected precisely when a facial walk of each of its faces is a cycle; we use this criterion for both G and G' in the following. Take an arbitrary face f' of G', and let W' be a facial walk of f'. We may assume that f' is not a face of G, otherwise there is nothing to show. By Observation 2.1, W' is either the walk $v_0e'_{d-1}v_{d-1}e'_{d-2}\dots v_1e'_0v_0$, or it arises from a facial walk W in G by replacing each subsequence e_ive_{i+1} of W with e'_i. In both cases, it follows from the assumptions that W' is a cycle. □

Finally, we include a technical lemma that significantly shortens the case analysis in the proof of Claim 1.

Lemma 2.3. *Let (l_i), (l'_i), $i = 0, \dots, k$, be tuples of positive integers such that $l_j \leq l'_j$ for every $j \neq k$, and $l'_k \geq l'_j$ for every j for which $l_j < l'_j$. Then $\sum_{i=0}^k l'_i \leq \sum_{i=0}^k l_i$ or $\sum_{i=0}^k 1/l'_i \leq \sum_{i=0}^k 1/l_i$.*

Proof. We prove first that if $\sum_{i=0}^k l'_i = \sum_{i=0}^k l_i$ and the tuples are distinct, then $\sum_{i=0}^k 1/l'_i < \sum_{i=0}^k 1/l_i$. We proceed by induction on the size of the set $J := \{j : j \neq k, l_j < l'_j\}$. From the assumptions, it follows that J is nonempty; fix j_0 as any index in J, and let $d := l'_{j_0} - l_{j_0}$. Consider a tuple (l''_i) such that $l''_{j_0} = l'_{j_0} = l_{j_0} + d$, $l''_k = l_k - d$, and $l''_j = l_j$ for the remaining indices. Clearly, $l''_k \geq l'_k$, and the number of pairs of different elements l''_j, l'_j for $j \neq k$ equals $|J| - 1$. It holds

$$\sum_{i=0}^k \frac{1}{l''_i} = \sum_{i=0}^k \frac{1}{l_i} + \left(\frac{1}{l''_{j_0}} - \frac{1}{l_{j_0}}\right) + \left(\frac{1}{l''_k} - \frac{1}{l_k}\right) = \sum_{i=0}^k \frac{1}{l_i} - d\left(\frac{1}{l_{j_0}(l_{j_0}+d)} - \frac{1}{l''_k(l''_k+d)}\right),$$

and since

$$l''_k \geq l'_k \geq l'_{j_0} > l_{j_0}$$

by the assumptions and the choice of j_0, it follows immediately that

$$\sum_{i=0}^{k} \frac{1}{l''_i} < \sum_{i=0}^{k} \frac{1}{l_i}. \tag{2.1}$$

In the trivial case, i.e., when $|J| = 1$, the tuple (l''_i) equals (l'_i), and there is nothing more to prove. Otherwise, we apply the induction for (l''_i) and (l'_i), in this order, obtaining that $\sum_{i=0}^{k} 1/l'_i < \sum_{i=0}^{k} 1/l''_i$; this together with (2.1) gives the desired conclusion.

Now we prove the claim. We may suppose that $\sum_{i=0}^{k} l'_i > \sum_{i=0}^{k} l_i$. If $l'_k \geq l_k$, then trivially $\sum_{i=0}^{k} 1/l'_i < \sum_{i=0}^{k} 1/l_i$. Otherwise, there clearly exists a tuple (l''_i) such that $l_j \leq l''_j \leq l'_j$ for every $j \neq k$, $l''_k = l'_k$, and $\sum_{i=0}^{k} l''_i = \sum_{i=0}^{k} l_i$. Note that (l''_i) is distinct from both (l_i) and (l'_i). Then it follows from above that $\sum_{i=0}^{k} 1/l''_i < \sum_{i=0}^{k} 1/l_i$, and by the conditions on (l''_i) with respect to (l'_i), it holds $\sum_{i=0}^{k} 1/l'_i < \sum_{i=0}^{k} 1/l''_i$; the proof is complete. □

2.1 Reducibility

We infer several constraints applying to the graph G as Lemma 2.4, and based on these, we derive bounds for the (reduced) face degree of a vertex in G in terms of the sum of its (reduced) configuration in Lemma 2.5. We remark that, according to established terminology, a graph contradicting Lemma 2.4 is called *reducible*.

Lemma 2.4. *The graph G has the following properties:*

(1) $|G| > 97$*;*

(2) *G does not contain parallel edges; in particular, G is without digons;*

(3) *no facial walk of a face of G contains four consecutive 2-vertices;*

(4) *for every vertex v of G, $d^F(v) > 96$;*

(5) *for every two vertices u and v of G such that $F(u) \cap F(v) = \{f\}$, $d^F(u, f) + d^F(v, f) > 95$.*

Proof. By assumption, G is a 2-connected graph. We prove each of the assertions by contradiction.

The statement (1) is straightforward; consider an assignment of a different color to each vertex of G.

We proceed to show assertion (2); let e_1 and e_2 denote two parallel edges in G. We distinguish two cases. If e_1 together with e_2 delimit a digon f, we simply delete one of the two edges, say, e_1, obtaining a (2-connected) graph G'. By the minimality of G, the graph G' has a proper spv-coloring c with at most 97 colors. As all faces of G except f have their counterparts in G' with the same sets of boundary vertices, and since c is proper, c is also a proper spv-coloring of G.

When, on the other hand, the curve $\mathcal{C}$ comprising the embeddings of e_1 and e_2 is not a boundary of a digon, we produce two graphs G_1 and G_2 by deleting the interior and exterior of $\mathcal{C}$, respectively, from G. Both these graphs are clearly 2-connected (note that

each has at least three vertices), and smaller than G with respect to our order; thus they have proper spv-colorings with at most 97 colors. The colorings can be chosen such that they coincide at the common vertices, i.e., the endvertices of e_1, and that the number of colors in their union c is minimal. Then c is a proper spv-coloring of G using at most 97 colors.

Next, suppose $x_1x_2x_3x_4$ is a path contradicting statement (3). Let v_1 (respectively v_4) be the neighbour of x_1 (respectively x_4) other than x_2 (respectively x_3). The vertices v_1 and v_4 are distinct and different from all x_i, $i = 1, \ldots, 4$, otherwise the facial walk in question would contain just four or five vertices, and by the 2-connectedness of G these would be the only vertices of G; a contradiction to assertion (1). We construct a graph G' by contracting the path $x_1 \ldots x_4v_4$ into a single vertex v_4'. It remains 2-connected due to statement (1), and hence by assumption, G' has a proper spv-coloring c of at most 97 colors. If we use the coloring for the corresponding vertices of G and assign the (distinct) colors $c(v_1)$ to x_2, x_4, and $c(v_4')$ to x_1, x_3, v_4 respectively, we obviously have a proper spv-coloring of G with not more than 97 colors.

We now focus on assertion (4). We perform the annihilation of v, obtaining a graph G'. By parts (1), (2) and Lemma 2.2, G' is 2-connected. This means, by the minimality of G, that G' has a proper spv-coloring c with at most 97 colors. Using c for G and assigning a color not used by c on any vertex in $N_G^F(v)$, but, if possible, present in c, to v, we clearly obtain a proper coloring of G of cardinality less than or equal to 97.

Now, by Observation 2.1, the only faces of G whose sets of boundary vertices differ from those of their counterparts in G' are the elements of $F_G(v)$, but, by the choice of the color of v, the spv-condition is maintained for them. Hence, the coloring of G is also an spv-coloring, a contradiction.

Finally we deal with statement (5). We construct a graph G'' by the annihilation of u. As above, G'' is 2-connected, and hence we may annihilate v in G'' to obtain a graph G'. Since u and v have precisely one common incident face in G, they are not adjacent; therefore, the annihilation of u does not create any new edges at v. This means, by part (2), that there is no pair of parallel edges incident with v in G'', and thus, considering statement (1) again, Lemma 2.2 can be applied for the annihilation of v. We conclude that G' is 2-connected. As it is also smaller than G with respect to our order, there is a proper spv-coloring c' of G' using at most 97 colors.

We extend the coloring to G as follows. If there exists a color used by c' on a vertex in $V_G(f) - N_G^F(u, f) - N_G^F(v, f)$ but on no vertex in $N_G^F(u, f) \cup N_G^F(v, f)$, we assign this color to both u and v. If the opposite is true, we color u and v each with a different color not used by c' on any vertex in $N_G^F(u, f) \cup N_G^F(v, f)$, but, if possible, appearing in c'. Obviously, either case yields a coloring c of G with not more than 97 colors.

For a desired contradiction, it remains to show that c is a proper spv-coloring of G. As all neighbors of both u and v belong to $N_G^F(u, f) \cup N_G^F(v, f)$, the coloring is indeed proper. Next, by Observation 2.1 and the assumption about $F(u) \cap F(v)$, we see that each face g of G has its counterpart g' in G' with $V_G(g)$ being equal to $V_{G'}(g') \cup \{u, v\}$ if $f = g$, $V_{G'}(g') \cup \{u\}$ or $V_{G'}(g') \cup \{v\}$ if $g \in F_G(u) - \{f\}$ or $g \in F_G(v) - \{f\}$ respectively, and $V_{G'}(g')$ otherwise. Then, considering the particular choice of the colors of u and v in either case, it is straightforward that the spv-condition holds for every face of G. □

Lemma 2.5. *Let v be a vertex of G.*

(1) *Assume that (l_i), $i \in \mathbb{Z}_k$, is the configuration of v. Then:*

(a) $d^F(v) \leq 4(l_0 + l_1)$ *if* $k = 2$;

(b) $d^F(v) \leq 4\sum_{i=0}^{2} l_i - 15 - 3|\{i : l_i + l_{i+1} \leq 25\}|$ *if* $k = 3$;

(c) $d^F(v) \leq 4\sum_{i=0}^{k-1} l_i - 5k$ *if* $k \geq 4$;

(d) (l_i) *does not equal* $(2, 8, 19)$ *nor* $(3, 8, 19)$.

(2) *Let* f *be a face of* G *incident with* v. *Suppose that* (l_i), $i = 1, \ldots, k$, *is the* f*-reduced configuration of* v. *Then:*

(a) $d^F(v, f) \leq 4\sum_{i=1}^{k} l_i - 5k + 4$;

(b) $d^F(v, f) \leq 4l_1 - 1$ *if* $k = 1$;

(c) $d^F(v, f) \leq 4(l_1 + l_2) - 9$ *if* $k = 2$ *and* $l_1 + l_2 \leq 25$.

Proof. For convenience, we construct a graph G' from G by replacing each nontrivial path P in G having both endvertices of high degree and all internal vertices of degree 2 with an edge. (Note that P may be a single edge.) The correspondence between G and G' is then as follows. Since G is 2-connected and it is not a cycle by Lemma 2.4 (1) and Lemma 2.4 (3), the vertices of G' are precisely the high-degree vertices of G. Furthermore, for every edge e of G' there is a corresponding path P_e in G with the same endvertices and all its internal vertices being of degree 2; and every face f' of G' has its counterpart f in G (and vice versa) with a facial walk of f arising from a facial walk of f' by replacing each edge e with P_e, and consequently, it holds that $w(f) = w(f')$. Note that, by Lemma 2.4 (3), P_e has at most three internal vertices for every $e \in E(G')$.

We start with the following observation, which directly implies statement (1a).

Let $u \in V(G)$ *have degree* 2. *Denote by* f_1, f_2 *the faces incident with* u, *and by* f_1', f_2' *the corresponding faces in* G'. *Call* e_u *the edge of* G' *such that* $u \in V(P_{e_u})$, *and let* u' *be one of the endvertices of* P_{e_u}. *Finally, let* e_1, e_2 *denote the edge of* $E(f_1')$, $E(f_2')$ *respectively such that it is incident with* u' *and is different from* e_u. *Then* $d^F(u) \leq 4(w(f_1) + w(f_2)) - 6$, *and if both* P_{e_1} *and* P_{e_2} *are single edges, then* $|d^F(u)| \leq 4(w(f_1) + w(f_2)) - 12$. (2.2)

We prove the observation. There are $|V(f_1') \cup V(f_2')| \leq w(f_1') + w(f_2') - 2$ vertices and $|E(f_1') \cup E(f_2')| - 1 \leq w(f_1') + w(f_2') - 2$ edges distinct from e_u incident with f_1' or f_2' in G'. Therefore, $N^F(u)$ consists of at most $w(f_1') + w(f_2') - 2$ high-degree vertices, at most $3(w(f_1') + w(f_2') - 2)$ vertices of degree 2 contained in some P_e, $e \in E(f_1') \cup E(f_2') - \{e_u\}$, and at most two extra 2-vertices adjacent to u. In total, we obtain the desired bound for $d^F(u)$. Clearly, if both P_{e_1} and P_{e_2} have no internal vertices, the maximum number of 2-vertices in $N^F(u)$, and hence the bound, drops by 6.

Now, the statement (2b) is straightforward by focusing only on the vertices and edges incident with f_1' in the proof of (2.2).

The rest of the proof is based on similar but generalized reasoning. Now v is of high degree (denoted by d), hence it is also in $V(G')$. Let the edges incident with v in G' be enumerated in a clockwise order as $e_i := vv_i$, $i \in \mathbb{Z}_d$; let f_i' denote the face of G' whose facial walk contains $e_i v e_{i+1}$ as a subsequence and let f_i be the corresponding face of G; let E be the union of $E_{G'}(f_i')$ and let E_j be the set $\bigcup_{i, i \neq j} E_{G'}(f_i')$; and finally, let E_v and V_v denote the set of all the edges e_i and vertices v_i respectively.

We claim that for every vertex v' in $N^F_{G'}(v)$, there exists an integer k such that $v' \in V(f_k') - \{v, v_{k+1}\}$, and similarly, if $v' \in N^F_{G'}(v, f_j') - \{v_j\}$ for a fixed j, then there exists an integer k, $k \neq j$, such that $v' \in V(f_k') - \{v, v_{k+1}\}$. These assertions are trivially true when $v' \notin V_v$ or there is an integer k such that $e_k = vv'$ and f_k' is not a digon. Assume

therefore the contrary. But then it is easy to see that all the faces f'_i are digons. By observation (2.2) and Lemma 2.4 (4), every P_{e_i} is a single edge; hence a pair of parallel edges occurs in G, a contradiction to Lemma 2.4 (2).

Thus $N^F_{G'}(v) = \bigcup_i (V(f'_i) - \{v, v_{i+1}\})$ and $N^F_{G'}(v, f'_j) = \bigcup_{i,\, i\neq j} (V(f'_i) - \{v, v_{i+1}\}) \cup \{v_j\}$. We conclude that $d^F_{G'}(v)$ (respectively $d^F_{G'}(v, f'_j)$), i.e., the number of high-degree vertices in $N^F_G(v)$ (respectively $N^F_G(v, f_j)$), is bounded by $\sum_i (w(f'_i) - 2)$ (respectively $\sum_{i,\, i\neq j} (w(f'_i) - 2) + 1$) from above. Obviously, $|E - E_v| \leq \sum_i (w(f'_i) - 2)$ and $|E_j - E_v| \leq \sum_{i,\, i\neq j} (w(f'_i) - 2)$; therefore the number of 2-vertices in $N^F_G(v)$, $N^F_G(v, f_j)$ not contained in any P_{e_i} is at most $3\sum_i (w(f'_i) - 2)$, $3\sum_{i,\, i\neq j} (w(f'_i) - 2)$ respectively. Finally, there are up to three vertices of degree 2 in every P_{e_i}, hence at most $3d$ of them in total. Summing the respective bounds, we obtain that

$$d^F_G(v) \leq 4 \sum_i w(f'_i) - 5d, \tag{2.3}$$

$$d^F_G(v, f_j) \leq 4 \sum_{i,\, i\neq j} w(f'_i) - 5d + 9. \tag{2.4}$$

Statements (1c) and (2a) then immediately follow.

We now focus on assertions (1b) and (1d). By their assumptions, $d = 3$; we may, up to the orientation of the plane, choose the labeling above such that $w(f_i) = l_i$. Then for each j such that $l_j + l_{j+1} \leq 25$, the path P_{e_j} must be a single edge by (2.2) and Lemma 2.4 (4), and the right side of (2.3) decreases by 3. This proves statement (1b). Let now $(l_i) = (3, 8, 19)$. Then both P_{e_0} and P_{e_1} are single edges by the preceding analysis. Considering this, we infer that P_{e_2} also has no internal vertex by (2.2) and Lemma 2.4 (4) again. Hence bound (2.3) drops from 105 by 9, which proves assertion (1d).

Finally, suppose the assumptions of statement (2c) hold. Again, up to the orientation of the plane, we may assume the labeling above to be chosen such that $w(f_j) = l_j$, $j = 1, 2$, and $f_0 = f$. As before, $l_0 + l_1 \leq 25$ implies that the path P_{e_1} is without internal vertices. Hence the bound for $j = 2$ from (2.4) decreases by 3, which completes the proof. □

2.2 Discharging

Having explored the properties of the graph G, we are ready to derive a contradiction using the discharging method.

In the first phase, we charge the vertices and faces of G as follows:

- each vertex v receives $d(v) - 6$ units of charge;
- each face f receives $2|V(f)| - 6$ units of charge.

The following observation is a standard corollary of the Euler formula.

Observation 2.6. *The sum of the charges defined above is* -12.

In the second phase, we redistribute the charges according to Rules 1–3 applied in sequence.

Rule 1. *Every face not being a pseudodigon sends two units of charge to each incident* 2*-vertex. Each pseudodigon does the same, except that one of the respective* 2*-vertices receives no charge.*

Rule 2. *Every small face distributes its whole charge evenly to all incident high-degree vertices. Each large face behaves the same way, except that it retains the charge of* 4.

Rule 3. *Every large face sends each incident vertex with a negative charge c after the application of Rule 2 the charge of $-c$.*

The aim is to show that the resulting charge of every vertex and every face in G is nonnegative, a contradiction to Observation 2.6.

First, we analyze how much charge a vertex v of high degree d receives by Rule 2. Denote the faces incident with v by f_i, $i = 0, \ldots, d-1$, and let n_i be the number of 2-vertices incident with f_i. After the application of Rule 1, each f_i has charge $2|V(f_i)| - 6 - 2n_i$ if f_i is not a pseudodigon, and $2|V(f_i)| - 6 - 2(n_i - 1)$ otherwise. Since, by Lemma 2.4 (2), f_i is not a digon, the charge can be clearly written as $2w'(f_i) - 6$ in either case. Hence, when f_i is a small face, it sends v the charge of

$$\frac{2w'(f_i) - 6}{w(f_i)} = 2 - \frac{6}{w'(f_i)}.$$

The equality is true as $w(f_i) = w'(f_i)$ if f_i is not a pseudodigon, and $2w'(f_i) - 6 = 0$ otherwise. On the other hand, when f_i is a large face, it sends v

$$\frac{2w'(f_i) - 6 - 4}{w(f_i)} = 2 - \frac{10}{w'(f_i)}$$

units of charge. Note that, in both cases, the charge received by v from f_i is nonnegative. In total, the vertex v obtains the nonnegative charge of

$$\sum_{\substack{i \\ w'(f_i)<20}} \left(2 - \frac{6}{w'(f_i)}\right) + \sum_{\substack{i \\ w'(f_i)\geq 20}} \left(2 - \frac{10}{w'(f_i)}\right) =$$

$$= 2d - 6 \sum_{\substack{i \\ w'(f_i)<20}} \frac{1}{w'(f_i)} - 10 \sum_{\substack{i \\ w'(f_i)\geq 20}} \frac{1}{w'(f_i)}. \tag{2.5}$$

Now we establish the following two essential claims. For convenience, we refer to the vertices with a negative charge prior to the application of Rule 3 as *special* vertices. Note that, considering the initial charge of vertices and the fact that each vertex receives a nonnegative charge during the application of Rule 2, every special vertex has degree at most 5.

Claim 1. *Every special vertex is incident with a large face.*

Proof. We proceed by contradiction, assuming that v is a special vertex not incident with any large face. Suppose first that $d(v) = 2$; let the two faces incident with v be denoted by f_1 and f_2. As the initial charge of v is -4, it is clear by Rule 1 that at least one of these faces, f_1 say, is a pseudodigon. By assumption, $w(f_2) \leq 19$. Hence by Lemma 2.5 (1a) we conclude that $d^F(v) \leq 78$, a contradiction to Lemma 2.4 (4).

Therefore, v is a special vertex of high degree d. Summing its initial charge and charge (2.5) sent to v during the application of Rule 2, we obtain that

$$3d - 6 - 6\sum_i 1/w'(f_i) < 0,$$

or equivalently,

$$\sum_i 1/w'(f_i) > d/2 - 1. \tag{2.6}$$

We proceed by case analysis; let (l_i) denote the modified configuration of v. Assume first that v is of degree 3. Then $l_0 \leq 5$, otherwise (2.6) fails. As l_1, $l_2 \leq 19$, it follows by Lemma 2.5 (1b) that $d^F(v) \leq 4\sum_i l_i - 21$, and consequently, by Lemma 2.4 (4),

$$\sum_i l_i \geq 30. \tag{2.7}$$

If $l_0 = 3$ and $l_1 \leq 9$, then, by (2.7), (l_i) is one of the three tuples $(3, 8, 19)$, $(3, 9, 18)$, and $(3, 9, 19)$. But the first of these is excluded by Lemma 2.5 (1d), and the remaining two contradict (2.6). Next, if $l_0 = 3$ and $l_1 > 9$, then, by Lemma 2.3 applied to the tuples $(3, 10, 15)$ and (l_i), in this order, it must hold $\sum_i l_i \leq 28$ or $\sum_i 1/l_i \leq 1/2$. However, that is a contradiction to (2.7) or (2.6) respectively.

Hence $l_0 \geq 4$. If l_0, $l_1 = 4$, then $\sum_i l_i \leq 27$, which is impossible by (2.7). Otherwise we may use Lemma 2.3 for the tuples $(4, 5, 20)$ and (l_i), and obtain contradiction to (2.7) or (2.6) again.

Thus $d \geq 4$; as we have remarked above, $d \leq 5$. Then Lemma 2.5 (1c) together with Lemma 2.4 (4) imply (2.7) again. In particular, (l_i) cannot be of the form $(3, 3, 3, x)$. Thus by Lemma 2.3 applied to the tuple $(3, 3, 4, 12)$ (respectively $(3, 3, 3, 3, 6)$, depending on d) and (l_i), $\sum_i l_i \leq 22$ (respectively $\sum_i l_i \leq 18$) or $\sum_i 1/l_i \leq 1$ (respectively $\sum_i 1/l_i \leq 3/2$). By (2.7) and (2.6), this is a contradiction in either case. □

Claim 2. *Every large face has a nonnegative charge after the application of Rule 3.*

Proof. Let f be an arbitrary large face of G. We start by listing the possible f-reduced or modified f-reduced configurations of special vertices incident with f, and for each case we note the upper bound on the charge of these vertices. Take such a vertex v. If v is a 2-vertex, then, by Rule 1, its f-reduced configuration is (2) and the charge equals -2.

Suppose now that v is of high degree d. Let (l_i), $i = 1, \ldots, d-1$ be its modified f-reduced configuration, and let d' denote the number of large faces incident with v. As noted above, $d \leq 5$. Considering the initial charge of v and (2.5), we see that after the application of Rule 2, v has charge

$$3d - 6 - 6 \sum_{\substack{i \\ w'(f_i)<20}} \frac{1}{w'(f_i)} - 10 \sum_{\substack{i \\ w'(f_i)\geq 20}} \frac{1}{w'(f_i)}.$$

As the charge is negative by assumption, we obtain that

$$3d - 6 - 6 \sum_{\substack{i \\ w'(f_i)<20}} \frac{1}{w'(f_i)} - d'/2 < 0 \tag{2.8}$$

by the definition of large face. Furthermore, $w'(f_i) \geq 3$ always, and hence

$$d - 6 + 3/2d' = 3d - 6 - 2(d - d') - d'/2 < 0.$$

From this, we immediately deduce that $d' = 1$, i.e., f is the only large face incident with v, and $d \leq 4$.

(modified) f-reduced configuration	*charge*
(2)	-2
$(3, x)$, $x \leq 11$	$\geq 1/2 - 6/x \geq -3/2$
$(4, x)$, $x \leq 5$	$\geq 1 - 6/x \geq -1/2$
$(3, 3, 3)$	$\geq -1/2$

Table 1. The proof of Claim 2: the list of possible f-reduced (the first line) or modified f-reduced (the other lines) configurations of special vertices incident with the face f, together with the charge of these vertices.

Assume first that $d = 3$. Then (2.8) reduces to

$$5/2 - 6(1/l_1 + 1/l_2) < 0.$$

We easily infer that there are only the following two possibilities: either $l_1 = 3$ and $l_2 \leq 11$, or $l_1 = 4$ and $l_2 \leq 5$. The charge of v is at least $1/2 - 6/l_2$ in the former, and at least $1 - 6/l_2$ in the latter case.

Now, let $d = 4$. Then, by (2.8),

$$11/2 - 6\sum_i l_i < 0.$$

If some l_i were greater than or equal to 4, then this inequality would not hold. Hence $(l_i) = (3, 3, 3)$, and the charge of v is at least $-1/2$.

We summarize the results in Table 1.

Now, let S denote the set of all special vertices incident with f, and R denote the total charge of these vertices. We observe the following:

Every two vertices $u, v \in S$ are incident with at least two common faces. (2.9)

Suppose the contrary. We consider the possible f-reduced or modified f-reduced configurations of u and v listed above, and use Lemma 2.5 (2a), (2b), or (2c) to show that both $d^F(u, f)$ and $d^F(v, f)$ are at most 47. This is a contradiction to Lemma 2.4 (5).

We proceed by contradiction from now on, i.e., we assume that

$$R < -4. \tag{2.10}$$

Obviously, $|S| \geq 3$, as can be seen in Table 1.

Let $v \in S$ be a 2-vertex; then the other face f' incident with v is a pseudodigon. By Rule 1 and (2.9), every other vertex v' in S is one of the two high-degree vertices v_1, v_2 incident with f'. Therefore $S = \{v, v_1, v_2\}$; it follows that v_1 and v_2 are each of degree 3 by assumption (2.10). This means that $F(v_1) = F(v_2)$, and hence the configurations of both the vertices are the same. Considering (2.10) again, it is clear that the modified f-reduced configuration of v_1 as well as v_2 is $(3, 3)$.

We digress by making an auxiliary observation:

Let g be a face of G with $w(g) \leq 3$. Then the intersection of the boundaries of f and g consists of pairwise disjoint paths, of which at most one is nontrivial; the internal vertices of all these paths are of degree 2 in G. Furthermore, every two vertices of degree 3 incident with both f and g are precisely the endvertices of such a nontrivial path, and hence, there are at most two such vertices. (2.11)

To prove this, let us denote the intersection of the boundaries of f and g by H. The assertion about the structure of H is obvious by considering the 2-connectedness of G and the weights of both f and g. Let $\mathcal{P}$ denote the set of the respective paths. If v is a vertex of degree 3 incident with f as well as g, then H must contain an edge incident with v, i.e., v lies on, and hence is an endvertex of, a nontrivial path in $\mathcal{P}$. The second assertion easily follows.

Applying (2.11) to the face incident with v_1 different from both f and f', and recalling that f' is a pseudodigon, we infer that $w(f) = 2$; a contradiction.

Thus all vertices in S are of high degree. Suppose that $S = \{v_1, v_2, v_3\}$. Then, by assumption (2.10), each of v_1, v_2, and v_3 has $(3, 3)$ as its modified f-reduced configuration. Hence there is no face except f incident with all the three vertices by (2.11). Consequently, by (2.9), G contains three different faces f_{ij}, $1 \leq i < j \leq 3$, each incident with both v_i and v_j and different from f. However, by (2.11), the boundary of f is precisely $\bigcup \mathcal{P}$ then; thus $w(f) = 3$, a contradiction to the assumptions.

Therefore, $|S| \geq 4$. We claim that G contains a face $f' \neq f$ incident with all the vertices in S. To prove this, take two vertices u, v from S nonadjacent in the boundary of f. By (2.9), there is a curve $\mathcal{C}_{uv}$ connecting the embeddings of u and v through a face $f_{uv} \neq f$ of G. Consider any two other vertices x, y from S such that u, x, v, and y appear in a facial walk of f in this order. Again, there is a curve $\mathcal{C}_{xy}$ joining the embeddings of x and y and contained, with the exception of its ends, in a face $f_{xy} \neq f$. Clearly, $\mathcal{C}_{uv} \cap \mathcal{C}_{xy} \neq \emptyset$, otherwise we would obtain an outerplanar embedding of K_4 in the plane, and hence $f_{xy} = f_{uv}$; the assertion easily follows. Now, let $k := |S|$. Then $w(f') \geq k$, and consequently, the f-reduced configuration of each v in S contains a number greater than or equal to k. From Table 1, we see that

$$R \geq k(1/2 - 6/k) = k/2 - 6,$$

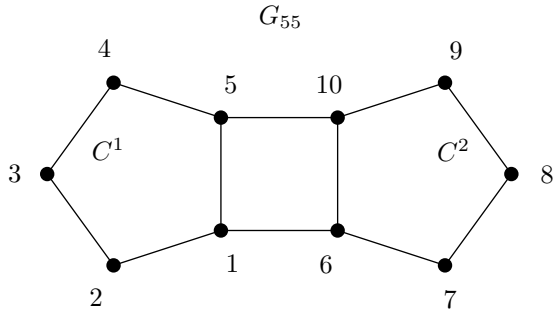
the right side of which is at least -4 by the condition on k. This is, however, a contradiction to assumption (2.10). □

With the help of the two preceding claims, we can easily finish the proof. By Claim 1 and Rule 3, each special vertex, and hence every vertex, ends up with a nonnegative charge. The final charge of every face is nonnegative as well; Rule 2 and Claim 2 guarantee this for small and large faces respectively. However, as already mentioned, this is a contradiction to Observation 2.6.

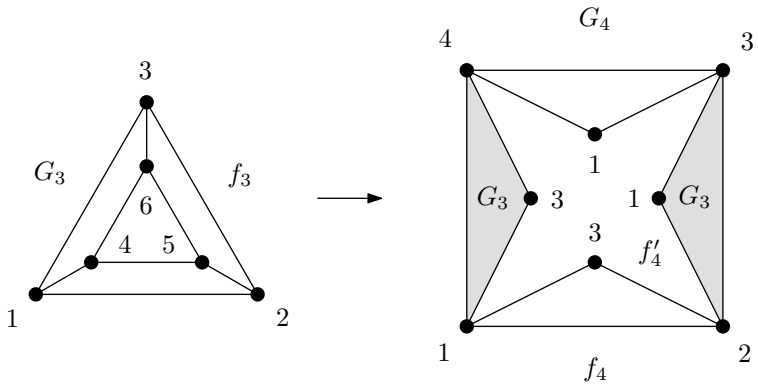
3 Lower bound

In this section, we provide examples showing that the best possible constant bound on χ_s for the class of 2-connected plane simple graphs is at least 8, and the corresponding bound for proper spv-colorings is at least 10. Note that for the class of 2-connected plane multigraphs instead, both versions of the bound coincide and are limited by the latter estimate: this follows from the fact that every spv-coloring of a plane multigraph G' arising from a plane multigraph G by replacing each edge of G by a (suitably embedded) digon is a proper spv-coloring of G.

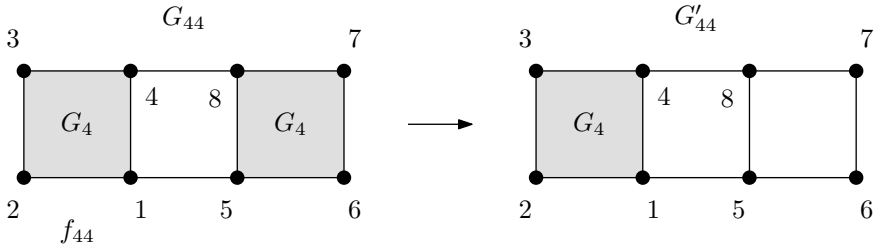
First, we focus on the bound for proper spv-colorings. We construct a graph G_{55} on ten vertices by linking two disjoint cycles C^1, C^2 on five vertices with two edges in such a way that their endvertices on C^1 and also on C^2 are adjacent. See Figure 3.1 (a).



(a) The graph G_{55}.



(b) The graphs G_3 (left) and G_4 (right).



(c) The graphs G_{44} (left) and G_{44}' (right).

Figure 3.1. Illustrations for Section 3. The labeled gray areas represent the respective subgraphs not depicted in detail. For each graph, the relevant coloring is unique up to symmetry; it is indicated by numeric labels.

By the spv-conditions for the two faces of G_{55} of length 5, every spv-coloring c must assign each vertex of C^1 a different color; the same holds for C^2. The spv-condition for the remaining face of G_{55} then implies that c uses each color precisely once.

Second, we consider the bound on χ_s. Take a three-sided prism G_3 embedded in the plane so that one of its triangular faces is the outer face f_3. Then, as observed by Czap and Jendrol' [1, proof of Lemma 5.1], every coloring c of G_3, such that each face of G_3 distinct from f_3 satisfies the spv-condition with respect to c, colors the boundary vertices of f_3 each with a different color.

Now construct a graph G_4 from a cycle on four vertices by replacing every second edge with a copy of G_3 in such a way that the outer face f_4 of G_4 is of length 4; see Figure 3.1 (b). Every coloring c' of G_4, such that the spv-condition is satisfied for each face of G_4 different from f_4, has, when restricted to the vertices of any of the copies of G_3, the property of c discussed above. This and the spv-conditions for f_4 and the remaining

face of G_4 of length 8 then imply that c' assigns a different color to each boundary vertex of f_4.

Finally we reproduce the construction of G_{55} with copies of G_4 in place of the cycles on five vertices. Thereby we obtain a graph G_{44} with the outer face f_{44} of length 8, such that, clearly, every spv-coloring of G_{44} is injective on $V(f_{44})$. We remark that this property is preserved even if we reduce G_{44} to a graph G'_{44} by replacing one of the copies of G_4 with a cycle on four vertices, as can be easily checked. The graph G_{44} and G'_{44} are shown in Figure 3.1 (c). □

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