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Blacklists vs. Whitelists

In the age of open access (OA) it is sometimes hard to distinguish between honest, genuine research journals and fake, predatory journals run by predatory publishers whose only interest is to make money and who are ready to publish anything under the APC (article processing charges) business model.

There exist several blacklists of publishers and journals with such unethical practices. There are well-known cases where the whole editorial board of a prominent journal resigns over an unethical and greedy policy of a publisher and creates another, unblemished journal. It is not uncommon for individual mathematicians and other scientists to boycott certain publishers for the same reason.

Not publishing a paper in a predatory journal is certainly a legitimate choice for any researcher. However, maintaining a public blacklist is more dangerous. It may be challenged in court and may result in heavy penalties for the author of such a list.

We think that the solution lies in whitelists in which learned societies and trustworthy individuals can endorse high-quality OA journals that are free both for readers and authors. In a sense both MathSciNet and zbMATH form such whitelists. To a certain extent even the Web of Knowledge represents a whitelist. A journal that does not appear on these lists is either not a mathematical journal, is too young or has some ethical issues.

Several journals, including ours, declare that they follow the EMS Code of Practice. Unfortunately, no one really checks whether this is indeed the case. The Ethics Committee should look at such journals and confirm their claims when appropriate. This explicit addition to the whitelist would be of great importance for any emerging good journal. It would also serve authors when faced with the problem of choosing a venue for their publication.

Klavdija Kutnar, Dragan Marušič and Tomaž Pisanski
Editors in Chief

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Calculating genus polynomials via string operations and matrices

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Abstract

To calculate the genus polynomials for a recursively specifiable sequence of graphs, the set of cellular imbeddings in oriented surfaces for each of the graphs is usually partitioned into *imbedding-types*. The effects of a recursively applied graph operation τ on each imbedding-type are represented by a *production matrix*. When the operation τ amounts to constructing the next member of the sequence by attaching a copy of a fixed graph H to the previous member, Stahl called the resulting sequence of graphs an H*-linear family*. We demonstrate herein how representing the imbedding types by strings and the operation τ by *string operations* enables us to automate the calculation of the production matrices, a task requiring time proportional to the square of the number of imbedding-types.

Keywords: Graph imbedding, genus polynomial, production matrix, transfer matrix method.

Math. Subj. Class.: 05A15, 05A20, 05C10

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1 Introduction

The ***genus polynomial*** of a graph G is defined to be the generating function

$$\Gamma_G(z) = \sum_{i \geq 0} g_i(G) z^i,$$

where $g_i(G)$ counts the cellular imbeddings of G in the closed oriented surface S_i of genus i. Following their introduction by [12] in 1987, and starting with the work of [6], the genus polynomials for a recursively constructed sequence of graphs have most frequently been calculated, as in [8, 9, 13], by partitioning the imbeddings according to the cyclic orderings of occurrences of root-vertices on the face-boundary walks (abbr. ***fb-walks***) of the imbeddings. In this paper, we describe how to expedite such calculations.

1.1 Rotation systems

All our graphs come with a labeling of the edges. All graph imbeddings in this paper are assumed to be cellular, that is, each component of the complement of the imbedded graph is homeomorphic to the interior of the unit disk. All surfaces are assumed to be closed and oriented.

To describe the imbeddings of a graph G, we assign $+$ and $-$ orientations to the edges, including self-loops. Then any imbedding defines, for each vertex, a cyclic order of the signed edge-ends initiating at that vertex, which is called the ***rotation*** at that vertex. The rotations collectively form a ***rotation system*** (e.g., see [19]), which acts as a permutation ρ on the oriented edge set. If λ is the involution that reverses the orientation of each edge, then the face boundary walks of the imbedding are the orbits of the permutation $\rho\lambda$.

A rotation system for a graph has also been called a "ribbon graph" or a "fat graph", especially in the context of algebraic geometry, Riemann surfaces, and the theory of dessins ([3, 20, 24]). We use the Euler polyhedral formula

$$|V| - |E| + |F| \; = \; 2 - 2\gamma(S)$$

to compute the genus $\gamma(S)$ of the imbedding surface S.

Two imbeddings $\iota_1, \iota_2 \colon G \to S$ determine the same rotation system if and only if there is a homeomorphism of the surface S taking $\iota_1(G)$ to $\iota_2(G)$ that acts as the identity isomorphism on the graph G (i.e., respects the labeling of edges). Accordingly, there is a bijection from the set of imbeddings of G to the set of rotation systems.

A problem in calculating genus polynomials is that the number of possible cyclic orderings of the edge-ends incident at a d-valent vertex is $(d-1)!$. Thus, the number of imbeddings of a graph G is the product $\Pi(d_v - 1)!$, taken over all vertices v of G, where d_v is the valence of v. It is well-known [30] that the problem of calculating the minimum genus of a graph is NP-hard, even when the graph is 3-regular. It follows that calculating the genus polynomial is at least that hard. For example, the number of rotation systems for the complete graph K_7 is

$$(5!)^7 \approx 3.6 \times 10^{14},$$

and the genus polynomial for K_7 has only recently been computed [2]. Table 1 gives the list of coefficients.

Table 1: Genus distribution of the complete graph K_7.

i	g_i
0	0
1	240
2	3,396,960
3	3,746,107,320
4	594,836,922,960
5	20,761,712,301,960
6	158,500,382,165,280
7	178,457,399,105,280

1.2 Context

Genus polynomials for recursively specified families of graphs have been computed mostly within a general paradigm in which the recursive operation occurs in the vicinity on a small number of vertices or edges designated as *roots*. The set of all imbeddings of each graph in the family is *partitioned* into what we now call *imbedding-types*, according to incidence of the fb-walks on the roots, a technique for calculating genus polynomials that was introduced by [6]. This basic paradigm is exemplified by [8, 13] for root-vertices, and by [25, 26] for root-edges.

This paper integrates several embellishments of the basic paradigm:

- The genus polynomial for a graph is partitioned into a ***pgd-vector***, with one coordinate for each imbedding type, such that each coordinate is a polynomial that gives the number of oriented imbeddings of that imbedding type in every orientable surface.
- The recursively applied topological operation is represented by a *production system*, as developed by Gross, Khan, and Poshni, in a series of papers [8, 13, 25, 26], that transforms the pgd-vector for a given graph into the pgd-vector for the graph resulting from an application of the recursive operation used to specify the graph family. In those papers, the productions were calculated with the aid of a multiplicity of drawings of rotation projections.
- The representation of production systems by matrices was introduced by Stahl [27], for application to pgd-vectors of some graphs in what he called H*-linear families*. Such matrices are now called ***production matrices***, and the graph sequences are now called H*-linear sequences*, or simply *linear sequences*. Stahl used what he called *permutation-partition pairs* to derive production matrices.
- The representation of *imbedding-types* by strings of root-vertices, as developed by Gross [11].
- Using string operations directly to calculate the production matrices, as suggested subsequently by Mohar [23].

The general idea of a ***linear sequence*** is that a copy of a graph H is attached to each graph in the sequence to form the next graph in the sequence. It is necessary to attach each copy of H in the same way, as described precisely by [4].

1.3 Outline of this paper

Our main focus in this paper is the calculation of production matrices. Since the size of the matrix increases with the number of imbedding types, and since the number of imbedding-types grows exponentially with the number of roots and with the valences of the roots, most of the calculations of genus polynomials have been for sequences of graphs with at most two roots and valences no larger than 4.

The string notation by which we concisely represent imbedding types allows us to automate the bookkeeping used in tracking the way imbedding types are changed by the addition of paths between root vertices. The advantages of this system are many. It allows us to derive in a few lines (see Subsection 4.3) the computation of production matrices that formerly involved many figures [10] or detailed paper-and-pencil applications of what Stahl [28] called the "Walkup reduction" for permutation-partition pairs. String notation facilitates the computer calculation of production matrices whose derivation would be unfeasibly tedious by hand (see the 12×12 matrix in Section 5). Finally, it reveals ways of combining different imbedding types to get smaller matrices (see Subsection 5.1).

Following a review in Section 2 of the representation of imbedding-types by strings, Section 3 introduces the representation of topological and vertex-labeling operations on imbeddings by string operations. Section 3 also introduces the concept of grouping two or more i-types into a "super-type". As an illustration of how the string operations are used in calculations of genus distributions, Section 4 applies these representations to two previously published examples, one of which (the iterated claw) we have adapted here to give a detailed example of grouping. Also, we explain in Section 4 how our use of productions to calculate pgd-vectors is interpretable as an embellishment of the *transfer matrix method*, along the lines described by [29].

Section 5 explores issues related to computation. It uses the theory developed in the previous sections to calculate genus polynomials for a vertex-amalgamation path of copies of K_4 and for an edge-amalgamated path of copies of K_4. Without string operations, both derivations would be long and tedious. We used two computational aids while preparing this paper.

- The computational system Maple®.
- A computer program, based on string operations, that calculates production matrices.

Such kinds of aids are what we have in mind in various comments here, rather than a state-of-the-art computer. Section 5 includes an additional example of the grouping of i-types into a super-type.

In Section 6, we use Burnside's Lemma to derive a formula for the maximum number of imbedding types for a graph with two roots of any possible combination of valences. We generalize the formula to more that two roots. From the rapid growth rate of the number of imbedding-types, as valences and the number of roots of the graphs at issue increases, it becomes clear that programmable computation tools are a virtual necessity when seeking to calculate genus polynomials.

2 Representing imbedding-types by strings

In this section, we develop a system of notation that uses strings of root-labels, so that representing the addition of an edge to a graph becomes a simple matter of applying a few string-processing rules.

2.1 Face-boundary-walks

We assign labels $0, 1, 2, \ldots$ to the roots of a graph G. Given an imbedding of G, we represent a face as a string of roots, in the order they are encountered in a traversal of its fb-walk following the orientation of the surface. If an fb-walk does not contain any roots, we call its string *empty*. Two strings are ***equivalent representations of an fb-walk*** if one is a cyclic shift of the other. We denote an entire equivalence class of strings by putting a representative string of labels inside parentheses. The ***canonical*** representative for the equivalence class of fb-walks is the one with minimum lexicographic order with respect to the labels $0, 1, \ldots$.

Remark 2.1. Vertices that are not roots do not appear in the string representing a face. Accordingly, the appearance of consecutive labels $\ldots 12 \ldots$ within a string would not imply that there is an edge between vertices 1 and 2. Also, since any labeled vertex may appear more than once around an fb-walk, the corresponding cyclic list of root-labels is not a permutation.

2.2 Imbedding types

The collection of non-empty strings for all the fb-walks of an oriented imbedding of a rooted graph G is called an ***imbedding-type*** of G (abbr. ***i-type***). The collection of all imbedding types over all imbeddings of G is called the ***full*** collection of imbedding types for G.

In order to compare imbedding types for the same rooted graph, we usually use the shortlex order [31] on canonical representatives to make a list of fb-walks (rather than a set): shorter faces are listed before longer ones, and if two faces have the same length, the one with shortlexically least canonical representative is listed first. We call such a list the ***canonical form*** for the i-type.

Example 2.1. Figure 1 shows an imbedding of K_4 in the sphere with roots $0, 1, 2$, and 3. If the "interior" fb-walks are oriented counterclockwise (which forces the "exterior" fb-walk to appear as clockwise, from the perspective of vertex 0), then the i-type (in canonical form) is

$$(012)(023)(031)(132).$$

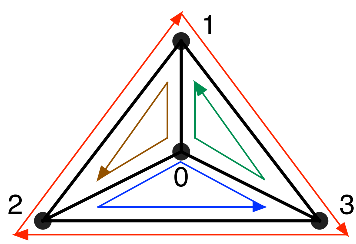

Figure 1: An imbedding of K_4 in S_0.

Notice that each face is represented by its canonical form (cyclic shift with least lexicographic order) and that the faces are listed in shortlex order. Since for this example every vertex is a root, it follows that two consecutive vertices (with respect to cyclic order) in the representation of a face actually does represent a directed edge. For any two roots i and j, the directed edge ij appears exactly once. If $i = 0$ and $j = 1$, we could suppress the labels $2, 3$ to obtain the i-type

$$(01)(0)(01)(1) = (0)(1)(01)(01)$$

for the imbedding of Figure 1. If the only root is 0, then the imbedding type would be $(0)(0)(0)$. Notice in the last imbedding type, the number of strings is less than the number of faces, because the fb-walk (132) contains no instances of vertex 0, and we do not list empty faces. If we reverse the orientation of the sphere and have all four vertices $0, 1, 2, 3$ as roots, then the i-type in canonical form would be

$$(021)(032)(013)(123) = (013)(021)(032)(123).$$

Observe that the shortlex order for the faces differs from the previous orientation. However, the i-type for roots $0, 1$ is the same as before, as is the i-type for root 0, when labels $1, 2,$ and 3 are suppressed.

Example 2.2. Considering all 2^4 rotation systems for K_4, we get the following census of i-types for roots $0, 1$, given in shortlex order:

- 2 of i-type $(0)(1)(01)(01)$
- 2 of i-type $(0)(01011)$
- 2 of i-type $(1)(00101)$
- 2 of i-type $(01)(0011)$
- 8 of i-type $(01)(0101)$

Notice that since there is only one edge 01, only one of the substrings 01 in an i-type, for example $(01)(0101)$, comes from an edge. The other juxtapositions of 0 and 1 come from suppressing incidences of the roots 2 and 3. We conclude that

$$\{(0)(1)(01)(01), (0)(01011), (1)(00101), (01)(0011), (01)(0101)\}$$

is a full set of i-types for K_4 with roots 0 and 1. In Section 6 of this paper, we shall see that the maximum number of i-types for a pair of 3-valent roots is 38.

Remark 2.2. We observe that within the string representation of any i-type, each root-vertex appears as many times as its valence. If there is an edge between roots i and j, then both ij and ji must appear at least once in every i-type. On the other hand, as we have noted, the appearance of ij in a string does not imply that there is an edge between i and j.

Remark 2.3. Suppose that G has no multi-edges or self-loops, and suppose that every vertex is a root. Then each rotation system for the graph G determines a unique i-type, since each i-type determines a rotation system for the dual graph. In this circumstance, the number of i-types would be the same as the number of rotation systems. At the opposite extreme, the set of imbeddings for a tree with one root-vertex has only one i-type.

Remark 2.4. When there are multi-edges or loops and every vertex is a root, it happens that different rotation systems can determine the same i-type. For example, the bouquet B_n has only one vertex 0 and has n loops at that vertex. Then an i-type is simply a partition of $2n$ into k parts, where k is the opposite parity of n (k is the number of faces, so the Euler characteristic $1 - n + k$ must be even). Thus, the number of i-types for imbeddings of B_n with k faces is at most the Stirling subset number $\left\{ {2n \atop k} \right\}$ (i.e., the Stirling number of the second kind), where k and n have opposite parities.

2.3 String notational conventions

We adopt two notational conventions for strings:

- The concatenation of a string S with a string T is denoted by ST.
- The reverse string for a string S is denoted by S^{-1}.

We emphasize that SS^{-1} is not the empty string, but rather the concatenation of S with its reverse (which forms a palindrome). This notation does satisfy the relations

$$\begin{aligned} (ST)^{-1} &= T^{-1}S^{-1} \quad \text{and} \\ (S^{-1})^{-1} &= S \end{aligned}$$

as if in a group, even though our strings are not permutations (since roots can repeat), and even though they do not form a group.

2.4 Pgd-vectors

Given an i-type t, we write its ***partial genus polynomial*** in the form

$$\sum a_i z^i$$

where a_i is the number of type-t imbeddings of G of genus i.

If the i-types are listed in shortlex order, then we can associate the set of partitioned genus polynomials for G with a column vector whose r^{th} coordinate is the partial genus polynomial for the r^{th} i-type. This is called a ***pgd-vector*** for the graph G. For instance, the partitioned genus distribution for the complete graph K_4 given by Example 2.2 corresponds to the vector

$$\begin{bmatrix} 2 & 2z & 2z & 2z & 8z \end{bmatrix}^T$$

where the superscript T denotes the transpose.

3 Operations on imbedding-types

In this section, we describe how a path-adding operation affects the i-types. We also describe the relabeling of root-vertices, and the suppression of some root-labels, which are used, for instance, when there are no more paths to be added at a root-vertex.

3.1 Adding a path within a face and between faces

Let G be a rooted graph and let iUj be a path whose endpoints i, j are roots of G but all other vertices of U are not in G. If U is empty, we have simply the edge ij. The effect of adding iUj into a face with fb-walk $(iSjT)$ is given by the following operation:

$$(iSjT) + iUj \;\rightarrow\; (iSjU^{-1})(iUjT). \tag{3.1}$$

In calculations, we may denote the right-hand side by $\mathrm{Add}_{iUj}[iSjT]$. If the i-type in which the fb-walk $(iSjT)$ occurs is of the form

$$(iSjT)W_1W_2 \ldots W_k,$$

which includes other fb-walks, then applying Operation (3.1) to that i-type yields the i-type

$$(iSjU^{-1})(iUjT)W_1W_2\dots W_k.$$

That is, the other fb-walks of the i-type are simply recopied.

The effect of adding the path iUj between two faces (iS) and (jT) is given by this operation:

$$[(iS),(jT)] + iUj \;\rightarrow\; z(iSiUjTjU^{-1}). \tag{3.2}$$

The right-hand side may be expressed as $\mathrm{Add}_{iUj}[(iS),(jT)]$. When applying Operation (3.2) to an i-type with fb-walks (iS) and (jT), any other fb-walks of the i-type are simply recopied, the same as for Operation (3.1). The multiplier z indicates that the genus of the imbedding rises by 1 when a handle is added to the surface.

For the circumstance in which the faces (iS) and (jT) lie within (disjoint) imbeddings ι and ι' of separate graphs G and G', the effect of joining the imbeddings by adding the path iUj between the two faces (iS) and (jT) is given by this operation:

$$[(iS),(jT)] + iUj \;\rightarrow\; (iSiUjTjU^{-1}). \tag{3.3}$$

The non-presence of the multiplier z signifies the fact that the genus of the surface in which the resulting graph is imbedded is simply the sum of the genera of the imbeddings ι and ι'.

Example 3.1. Consider an imbedding of the 4-cycle 0213 in the sphere. There are two faces, one with fb-walk (0213) and the other with fb-walk (0312). Thus, the initial i-type is (0213)(0312). There are four ways to add a path 0451 to such an imbedding, one within the face (0213), one within the face (0312) and two between the faces (0213) and (0312). Figure 2 shows the four possible ways to add the path $0U1$ and the resulting i-type for each.

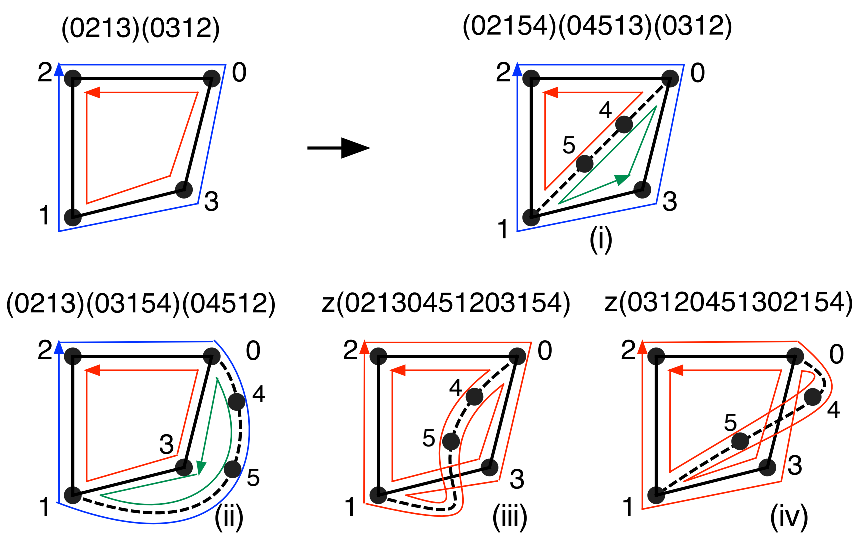

Figure 2: Adding the path 0451 to an imbedding of a 4-cycle in the sphere.

(i) Inserting path 0451 into the face (0213) yields the imbedding type

$$(02154)(04513)(0312),$$

as per Operation (3.1). We now have three faces. Root-vertices 0 and 1 now have valence 3, so they now appear three times in this representation of the i-type.

(ii) Inserting the path 0451 instead into the face (0312) yields i-type

$$(0213)(03154)(04512).$$

(iii) If we join the two faces, from endpoint 0 inside the face (0213), to endpoint 2 inside the face (0312), then the resulting string expression is

$$z(02130451203154).$$

(iv) If we add the path 0451 with edge-end 0 now inside the face (0312) and edge-end 1 inside the face (0213), we get the string expression

$$z(02154031204513).$$

It follows that the net result of adding the path 0451 to the i-type $(0213)(0312)$ is the following linear combination of i-types taken over the ring $\mathbb{Z}[z]$ of polynomials with integer coefficients:

$$(02154)(04513)(0312) \;+\; (0213)(03154)(04512) \\ \;+\; z(0213045120354) \;+\; z(03120451302154).$$

Remark 3.1. The path ii for adding a self-loop is simply a special case. As a variation on Operation (3.1), we have

$$(iS) + ii \;\rightarrow\; (i)(iSi)$$

As a variation on Operation (3.2), we have

$$[(iS),(iT)] \;\rightarrow\; z(iSiiTi)$$

Remark 3.2. If a graph already has an edge ij, then adding the path $P = ij$ creates a multiple adjacency.

3.2 Suppressing roots and relabeling roots

Given a subset of roots $\{i, j, \dots\}$, the ***root-suppression operator*** $\mathrm{Sup}_{i,j,\dots}$ acts to suppress every occurrence of the root-labels $i, j, \dots$ within an i-type t. For example,

$$\mathrm{Sup}_{1,2}[(1)(12)(0212)(0231303)] = (0)(03303).$$

Observe that we delete empty pairs of parentheses as a final step in suppressing roots.

Example 3.1, continued. Suppressing roots 2 and 3 as well as any roots along U transforms the i-type $(021U^{-1})(0U13)(0312)$ into the i-type $(01)(01)(01)$. Similarly,

$$\mathrm{Sup}_{1,2,U}[z(021U^{-1})(0U13)(0312)] = z(010101).$$

Moreover, when root-suppression is applied to a linear combination of i-types, it can reduce the number of terms. For instance,

$$\begin{aligned}\mathrm{Sup}_{2,3,U}[&(021U^{-1})(0U13)(0312) + (0213)(031U^{-1})(0U12)\\ &+ z(02130U1203U^{-1}) + z(03120U13021U^{-1})]\\ = {}& 2(01)(01)(01) + 2z(010101).\end{aligned}$$

We can also relabel roots, by using the ***root-relabeling operator***. Suppose that the label i appears in i-type t and label j does not. Then $\mathrm{Lab}_{ij}[t]$ is the i-type obtained by replacing in t all occurrences of i by j. Thus,

$$\mathrm{Lab}_{24}[(1)(2)(22)(1323)] = (1)(4)(44)(1343).$$

We denote by $\mathrm{Lab}_{ii',jj',\ldots}[t]$ the result of relabeling i by i', j by j' etc.

3.3 Reversing orientation

If the orientation of a graph imbedding is reversed, the effect on i-types is as follows:

- the cyclic order of each fb-walk is reversed;
- the genus of the imbedding stays the same.

We call this the ***i-type reversal operator***. Given an i-type t, we denote by t^{-1} the i-type in which each fb-walk string is reversed. Note that if (ST) is an fb-walk within i-type t, then the corresponding fb-walk in t^{-1} is $(T^{-1}S^{-1})$, for which a cyclic shift gives $(S^{-1}T^{-1})$. On the other hand, the i-type $(R^{-1}S^{-1}T^{-1})$ is not a cyclic shift of the i-type $(RST)^{-1} = (T^{-1}S^{-1}R^{-1})$.

Proposition 3.3. *The i-type reversal operator commutes with the operators* Add*,* Sup*, and* Lab*.*

Proof. Clearly, we can reverse lists either before of after suppressing or relabeling vertices, and the result is the same. Using Rule (3.1) for adding a path within a face, we have

$$\mathrm{Add}_P[(iSjT)]^{-1} = [(SP^{-1})(PT)] = (T^{-1}P^{-1})(S^{-1}P) \quad \text{and} \tag{3.4}$$

$$\mathrm{Add}_P[(iSjT)^{-1}] = \mathrm{Add}_P[iT^{-1}jS^{-1}] = (T^{-1}P^{-1})(S^{-1}P) \tag{3.5}$$

Using Rule (3.2) for adding an edge between two faces, we have

$$\mathrm{Add}_P[(iS),(jT)]^{-1} = z(PTP^{-1}S)^{-1} = z(S^{-1}PT^{-1}P^{-1}) \quad \text{and} \tag{3.6}$$

$$\mathrm{Add}_P[(iS)^{-1},(jT)^{-1}] = \mathrm{Add}_P[(iS^{-1}),(jT^{-1})] = z(PT^{-1}P^{-1}S^{-1}) \tag{3.7}$$

□

3.4 Combining i-types into super-types

As we have observed, the number of i-types grows exponentially with the valence and the number of roots, so any way of reducing the number of i-types is welcome. For example, in building a graph by path-addition, we can always group an i-type with its reverse, since i-type reversal commutes with edge path-adding. Indeed, root-suppression is also a way of grouping many i-types together.

Suppose that the rooted graph H is obtained from the rooted graph G by a sequence Op of the following kinds of operations:

path-additions, root-suppression, and root-relabeling.

Let $\mathcal{T}$ be the full collection of i-types for G, and let $\mathcal{S}$ be the full collection of i-types for H, both in shortlex order. Then for any i-type $t \in \mathcal{T}$, we see that

> The expression $Op(t)$ is a linear combination of elements of $\mathcal{S}$, with coefficients taken from the ring $\mathbb{Z}[z]$ of polynomials in z.

We represent Op, therefore, as a matrix M whose columns are labeled by i-types in $\mathcal{S}$, and whose rows are labeled by i-types in $\mathcal{T}$, where $M_{s,t}$ is the coefficient of i-type s in the expression $Op(t)$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of $\mathcal{S}$ and $\mathcal{T}$, respectively. Suppose that we order the i-types within $\mathcal{S}$ and the i-types within $\mathcal{T}$ so that the i-types within each cell of $\mathcal{P}$ and within each cell of $\mathcal{Q}$ are contiguous in the respective orderings, inducing a partitioning of the production matrix M into blocks that satisfy this criterion:

> Within each block, the column sums are the same. (This requirement applies also to the blocks that span only a single row of the matrix M, which implies that the entries in such a row are identical.)

Then we call the partitions $\mathcal{P}$ and $\mathcal{Q}$ ***compatible*** with M. Moreover, we call each part of $\mathcal{P}$ and $\mathcal{Q}$ a ***super-type*** for the operation Op. We can then condense the matrix M to a smaller one whose columns are indexed by $\mathcal{P}$ and rows by $\mathcal{Q}$, and whose entries are the constant column sum of the block of M determined by the respective parts.

We have already encountered super-types in two contexts: type-reversal and root-suppression. For type-reversal, we partition a full collection of i-types into parts by grouping together an i-type and its reverse. Since type-reversal commutes with path-adding, root-suppression, and root-relabeling, it is compatible with any sequence of those operations. We can also view root-suppression $\mathrm{Sup}_{i,j,\dots}$ itself as creating super-types. In this case, we have $\mathcal{S} = \mathcal{T}$. The parts of $\mathcal{P}$ are just singletons; i-types s, t are in the same part of $\mathcal{Q}$ if and only if $\mathrm{Sup}_{i,j,\dots}(s) = \mathrm{Sup}_{i,j,\dots}(t)$. Notice in this case, the matrix M is just the identity matrix and each block is a part of a single column of M. The condensed matrix has a single 1 in each column.

Another way to create super-types is to exploit any symmetry between roots. With $H, G, \mathcal{S}, \mathcal{T}$ as before, suppose there is a graph automorphism f of H that permutes the roots of H and G. Then f also induces a permutation of the $\mathcal{S}$ and $\mathcal{T}$. We can then use orbits of that permutation as super-types.

Grouping types into super-types by graph automorphisms and reversal is illustrated particularly well in the family of iterated claws in Subsection 4.3, where 12 i-types are reduced to three super-types. For now we consider an example that provides a clear illustration of the theory underlying the reduction.

Example 3.2. Suppose that $G = K_4$, as in Example 2.2, with roots 0 and 1, and that the graph H is obtained from G by the operation of adding a second edge between 0 and 1. Since there is an automorphism of the graph G interchanging 0 and 1, we have the partition given in Table 2 for the full set $\mathcal{T}$ of i-types of the graph G, under the partition $\mathcal{Q}$ (induced by this automorphism), with the parts of $\mathcal{Q}$ indicated by square brackets.

Table 2: Partitioning the i-types for $(K_4, \{0, 1\})$.

$\mathcal{T}$		$\mathcal{T}/\mathcal{Q}$
$(0)(1)(01)(01)$		$(0)(1)(01)(01)$
$(0)(01011)$		$[(0)(01011),\ (1)(00101)]$
$(1)(00101)$	$\longrightarrow$	$(01)(0011)$
$(01)(0011)$		$(01)(0101)$
$(01)(0101)$		

We can construct the full set $\mathcal{S}$ of 13 i-types for the graph H, by adding the path 01 to the i-types in $\mathcal{T}$ for the graph G. In Table 3, we again use square brackets to enclose the parts of the partition $\mathcal{P}$.

Table 3: Partitioning the i-types for $(K_4 + 01, \{0, 1\})$.

$\mathcal{S}$		$\mathcal{S}/\mathcal{P}$
$(0)(1)(01)(01)(01)(0)(1)(010101)$		$(0)(1)(01)(01)(01)$
$(0)(01)(01011)$		$(0)(1)(010101)$
$(1)(01)(00101)$		$[(0)(01)(01011),\ (1)(01)(00101)]$
$(0)(011)(0101)$		$[(0)(011)(0101),\ (1)(001)(0101)]$
$(1)(001)(0101)$		$(01)(01)(0011)$
$(01)(01)(0011)$		$(01)(01)(0101)$
$(01)(01)(0101)$	$\longrightarrow$	$(01)(001)(011)$
$(01)(001)(011)$		$[(00101011),\ (00110101)]$
(00101011)		(00101101)
(00110101)		(01010101)
(00101101)		
(01010101)		

Applying the string operation for adding the edge 01 to K_4, we obtain the matrix M, which maps the pgd-vector for K_4 to the pgd-vector for the graph $K_4 + 01$, as follows:

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 2z & 0 & 0 & 0 & 0 \\ 2z & 4 & 0 & 0 & 0 \\ 2z & 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ z & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & z & z & 2z & 0 \\ 0 & z & z & 2z & 0 \\ 0 & z & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 4z \end{bmatrix} \begin{bmatrix} 2 \\ 2z \\ 2z \\ 2z \\ 8z \end{bmatrix} = \begin{bmatrix} 4 \\ 4z \\ 12z \\ 12z \\ 4z \\ 4z \\ 8z \\ 40z \\ 4z \\ 8z^2 \\ 8z^2 \\ 4z^2 \\ 32z^2 \end{bmatrix} \tag{3.8}$$

Our partition $\mathcal{Q}$ of $\mathcal{T}$ groups columns 2 and 3 and represents combining the i-types $(0)(01011)$ and $(1)(00101)$, which corresponds to the automorphism on K_4 that swaps roots 0 and 1. Our partition $\mathcal{P}$ of $\mathcal{S}$ involves three pairings: rows 3 and 4, rows 5 and 6, and rows 10 and 11, which correspond to the automorphism on $K_4 + 01$ that swaps roots 0 and 1. This compresses the 13×5 matrix of (3.8) down to the 10×4 matrix on the left side of Equation (3.9).

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 2z & 0 & 0 & 0 \\ 4z & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ z & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 2 & 0 \\ 0 & 2z & 4z & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & 0 & 4z \end{bmatrix} \begin{bmatrix} 2 \\ 4z \\ 2z \\ 8z \end{bmatrix} = \begin{bmatrix} 4 \\ 4z \\ 24z \\ 8z \\ 8z \\ 40z \\ 4z \\ 16z^2 \\ 4z^2 \\ 32z^2 \end{bmatrix} \tag{3.9}$$

4 Two examples of linear families

In this section, we examine the application of the string operations Sup, Add, and Lab to two linear sequences previously studied elsewhere.

4.1 Production matrices

Given a linear family $\{G_n : n = 0, 1, \ldots\}$ of graphs, constructed by recursive application of the topological operator $\tau\colon G_n \to G_{n+1}$, and with the pgd-vector $V_n(z)$ for G_n, for $n = 0, 1, \ldots$, the associated ***production matrix*** $M_\tau(z)$ is a matrix such that we have the recursion

$$V_n(z) = M_\tau(z)V_{n-1}(z), \text{ for } n = 1, 2, \ldots \tag{4.1}$$

and, consequently, the equation

$$V_n(z) = M_\tau(z)^n V_0(z), \text{ for } n = 1, 2, \ldots \tag{4.2}$$

Here, as in some previous papers (e.g., [14, 17]), our production matrices record a system of rules that computer scientists might call *productions*.

4.2 X-ladders

An ***X-ladder*** is envisioned as a ladder with evenly many rungs, such that the rungs are paired, and such that within a pair, they cross each other in a planar drawing, as illustrated in Figure 3. This example was first given by [28].

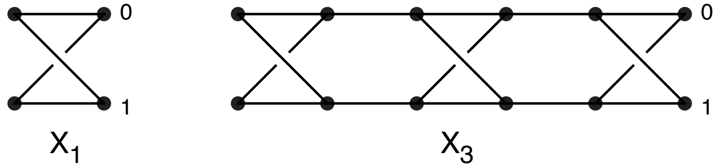

Figure 3: The X-ladders X_1 and X_3.

To represent the construction of X_n from X_{n-1}, we use the following procedure (a sequence of i-type operations) to add the next X:

Procedure 4.1. Add the next X to an X-ladder.

$$\mathrm{Sup}_{0,1} \circ \mathrm{Add}_{02431} \tag{4.3}$$
$$\mathrm{Sup}_{2,3} \circ \mathrm{Add}_{253} \tag{4.4}$$
$$\mathrm{Lab}_{40,51} \tag{4.5}$$

We denote the composition of the steps of Procedure 4.1 by Rec_X.

Since the X-ladder X_1 is simply a 4-cycle with labeled vertices 0 and 1, its one and only i-type is $(01)(01)$. To obtain the pgd-vector for X_2 from the pgd-vector for X_1, we apply Procedure 4.1. In this non-machine calculation, we separate Step (4.4) into two parts.

$$\begin{aligned} \mathrm{Sup}_{0,1}[\mathrm{Add}_{02431}[(01)(01)]] &= 2(234)(243) + 2z(224334) \\ \mathrm{Sup}_{2,3}[\mathrm{Add}_{253}[2(234)(243)]] &= 4(4)(5)(45) + 4z(4545) \\ \mathrm{Sup}_{2,3}[\mathrm{Add}_{253}[2z(224334)]] &= 8z(45)(45) \end{aligned}$$

By then applying $\mathrm{Lab}_{40,51}$, we obtain the production

$$\mathrm{Rec}_X[(01)(01)] = 4(0)(1)(01) + 8z(01)(01) + 4z(0101). \tag{4.6}$$

In general, a ***production*** for an i-type associates to it a linear combination of all the i-types, taken over the ring of polynomials in the indeterminate z.

Thus, there are three possible i-types for imbeddings of the X-ladder X_2. Since two of them are not i-types for X_1, we need to continue with the i-types of X_3, to see whether there are any additional i-types, before we write the production matrix.

To compute the effect of Rec_X on X_2, we need to compute its effect on the three imbedding-types $(0)(1)(01)$, $(01)(01)$, and (0101). We already know the production (4.6) for the imbedding-type $(01)(10)$.

We begin with i-type $(0)(1)(01)$.

$$\begin{aligned}\mathrm{Sup}_{0,1}[\mathrm{Add}_{02431}[(0)(1)(01)] &= (234)(243) + 3z(224334)\\ \mathrm{Sup}_{23}[\mathrm{Add}_{253}[(234)(243)] &= 2(4)(5)(45) + 2z(4545)\\ \mathrm{Sup}_{2,3}[\mathrm{Add}_{253}[3z(224334)]] &= \{12z(45)(45)\}\end{aligned}$$

By relabeling with $\mathrm{Lab}_{40,51}$, we obtain the production

$$\mathrm{Rec}_X[(0)(1)(01)] = 2(0)(1)(01) + 12z(01)(01) + 2z(0101). \tag{4.7}$$

We continue with the effect of Rec_X on i-type (0101).

$$\begin{aligned}\mathrm{Sup}_{0,1}[\mathrm{Add}_{02431}[(0101)]] &= 4(243)(342)\\ \mathrm{Sup}_{2,3}[\mathrm{Add}_{253}[4(234)(243)] &= 8(45)(4)(5) + 8z(4545)\end{aligned}$$

Applying $\mathrm{Lab}_{40,51}$, we obtain the production

$$\mathrm{Rec}_X[(0101)] = 8(0)(1)(01) + 8z(0101). \tag{4.8}$$

We see that no new types arise when applying Rec_X to X_2. Thus, the only possible i-types for any X-ladder X_n arising from application of Rec_X are

$$(0)(1)(01),\ (01)(01),\ \text{and}\ (0101).$$

Accordingly, we may write the pgd-vectors of X_1, X_2, and X_3 as

$$V_{X_1} = \begin{bmatrix}0\\1\\0\end{bmatrix} \quad V_{X_2} = \begin{bmatrix}4\\8z\\4z\end{bmatrix} \quad V_{X_3} = \begin{bmatrix}8+64z\\48z+64z^2\\8z+64z^2\end{bmatrix}$$

By recording the coefficients of the i-types in productions (4.6), (4.7), and (4.8) as columns of the production matrix $M_X(z)$ for Rec_X we have

$$M_X(z) = \begin{bmatrix}2 & 4 & 8\\ 12z & 8z & 0\\ 2z & 4z & 8z\end{bmatrix}$$

We see that $M_X(z)V_{X_1}(z) = V_{X_2}(z)$ and that $M_X(z)V_{X_2}(z) = V_{X_3}(z)$.

Proposition 4.1 enables us to check for possible errors.

Proposition 4.1. *Suppose that* $\{G_n : n = 0, 1, \ldots\}$ *is a linear family with production matrix* $M(z)$*. Then substituting* $z = 1$ *gives a matrix whose column sums are the same constant* s*, where the number of imbeddings of* G_{n+1} *equals* s *times the number of imbeddings of* G_n*.*

Proof. Substituting $z = 1$ in any column of $M(z)$ counts the number s of ways that the extra paths can be added between the roots of G_n and the roots of G_{n+1}. This number is the same for each imbedding-type and hence for each column of $M(z)$. Clearly, s also tells us the growth factor in the number of imbeddings from G_n to G_{n+1}. □

As Proposition 4.1 indicates, the substitution $z = 1$ in $M_X(z)$ gives column sums of $s = 16$, implying that any imbedding of X_n of a given type generates 16 imbeddings of X_{n+1}. This makes sense since X_{n+1} has four more 3-valent vertices than L_n, so it should have $(2!)^4 = 16$ times as many imbeddings.

4.3 Iterated claws

This example is adapted from [14] and [17].

The iterated claw Y_1 is obtained from the complete bipartite graph $K_{3,3}$ as follows:

1. Choose one vertex of $K_{3,3}$ to be the root-vertex 0.
2. Subdivide each of the edges incident with 0.
3. Assign labels 1, 2, and 3 to the resulting three 2-valent vertices.

To obtain the graph $(Y_n, 0)$ from the graph $(Y_{n-1}, 0)$, we join a new 3-valent vertex 7 to the vertices 1, 2, and 3 by paths 741, 752 and 763. We then suppress labels 1, 2, 3, and 0 and relabel vertex 4 as 1, vertex 5 as 2, vertex 6 as 3, and vertex 7 as 0.

Figure 4 illustrates the graph Y_3. We observe that the graph Y_1 is homeomorphic to $K_{3,3}$.

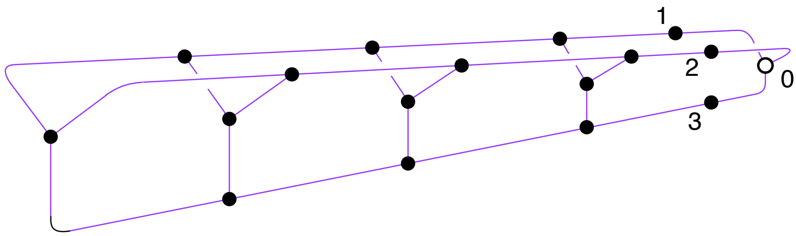

Figure 4: The iterated claw Y_3.

To obtain the pgd-vector of Y_n from the pgd-vector of Y_{n-1}, we now describe how to construct Y_n from Y_{n-1} with this procedure.

Procedure 4.2. Add the next claw to an iterated claw.

$$\mathrm{Sup}_{0,1,2} \circ \mathrm{Add}_{14752} \tag{4.9}$$

$$\mathrm{Sup}_{3} \circ \mathrm{Add}_{367} \tag{4.10}$$

$$\mathrm{Lab}_{41,52,63,70} \tag{4.11}$$

We denote the composition of the steps of Procedure 4.2 by Rec_Y.

We note that at the root vertex 0, there must be face corners 102, 203, and 301. We partition the set of i-types according to the number of faces incident with the root-vertex 0:

(a) three faces: the i-type must be $(013)(021)(032)$ or its reverse;

(b) two faces: the imbedding must be of one of the types $(013)(022031)$, $(021)(012033)$, $(032)(011023)$, or of their reverses;

(c) one face: the imbedding must be of types (011022033), (012031023), or of their reverses.

Thus, we have 12 i-types in all.

Grouping each i-type with its inverse yields six "super-types". To reduce from six to three, we notice that the dihedral $\mathbb{D}_3$ symmetry of the claw is visible within the notation for the types. For instance, from the one group (b) i-type $(013)(022031)$, we could obtain any of the other the other i-types by a permutation of $1, 2, 3$ and a possible reversal. Thus, we need to consider only how path-adding affects the i-type $(013)(022031)$. On the other hand, the two one-face i-types are not related by a permutation of $1, 2, 3$. Nevertheless, we will see that grouping the two together does provide a compatible partition for the production matrix. We denote the three super-types simply by listing the face structure at 0:

(a) three faces: $(0)(0)(0)$;

(b) two faces: $(0)(00)$;

(c) one face: (000).

We now calculate $\mathrm{Rec}_Y[t]$ for one representative t from each of the three super-types. For i-type $t = (013)(021)(032)$ from super-type $(0)(0)(0)$, we obtain

$$\begin{aligned}\mathrm{Sup}_{0,1,2}[\mathrm{Add}_{14752}[t]] &= (475)(574)(3)(3) + z(4753574)(3)\\ &\quad + z(3475574)(3) + z(34753574)\end{aligned}$$

By applying $\mathrm{Sup}_{3456} \circ \mathrm{Add}_{367}$ to the right side, we obtain

$$4z(77)(7) + z[2(77)(7) + 2z(777)] + z[2(77)(7) + 2z(777)] + z[4(77)(7)].$$

Collecting terms, we obtain

$$12z(7)(77) + 4z^2(777).$$

Relabeling 7 by 0 then yields the production

$$\mathrm{Rec}_Y[(102)(203)(301)] = 0(0)(0)(0) + 12z(0)(00) + 4z^2(000) \quad (4.12)$$

For i-type $t = (013)(022031)$ from super-type $(0)(00)$, we have:

$$\mathrm{Sup}_{012}[\mathrm{Add}_{14752}[t]] = 2(475)(3574)(3) + 2z(34753574).$$

Applying $\mathrm{Sup}_{3456} \circ \mathrm{Add}_{367}$ to the right side, we obtain:

$$2[(7)(7)(7) + 2z(7)(77) + z(777)] + 2z[4(7)(77)].$$

Then relabeling 7 by 0 yields the production

$$\mathrm{Rec}_Y[(013)(022031)] = 2(0)(0)(0) + 12z(00)(0) + 2z(000) \quad (4.13)$$

It is easily verified we would get the same result beginning instead with alternative representatives $t = (021)(012033)$ or $t = (032)(011023)$.

For i-type $t = (011022033)$ from super-type (000), we have

$$\mathrm{Sup}_{012}[\mathrm{Add}_{14752}[t]] = 4(475)(33547)$$

Applying $\mathrm{Sup}_{3456} \circ \mathrm{Add}_{367}$ to the right side yields

$$4[2(7)(7)(7) + 2z(777)].$$

Then relabeling 7 by 0, we obtain the production

$$\mathrm{Rec}_Y[(102203301] \quad = \quad 8(0)(0)(0) + 0(0)(00) + 8z(000) \tag{4.14}$$

It is easily verified that we get the same result when we begin with type $t = (012031023)$. That is what enables us to group them together in a super-type, even though they are not related by a permutation of $1, 2, 3$ or by reversal.

We copy the coefficients from (4.12), (4.13), and (4.14) into the columns of the production matrix $M_Y(z)$ for Rec_Y, with input and output basis $\{(0)(0)(0), (00)(0), (000)\}$.

$$M_Y(z) = \begin{bmatrix} 0 & 2 & 8 \\ 12z & 12z & 0 \\ 4z^2 & 2z & 8z \end{bmatrix}$$

We note that the column sums with $z = 1$ are $16 = 2^4$ and that Y_{n+1} has four extra vertices of valence 3. We observe that the power of string notation for i-types has allowed us to compute the recursion matrix for this family in only a page, while the original calculation [14] requires many pages and many figures. As in [14], we obtain the pgd-vectors

$$V_{Y_1} = \begin{bmatrix} 16z \\ 24z \\ 24z^2 \end{bmatrix} \quad V_{Y_2} = \begin{bmatrix} 48z + 192z^2 \\ 480z^2 \\ 48z^2 + 256z^3 \end{bmatrix} \quad V_{Y_3} = \begin{bmatrix} 1344z^2 + 2048z^3 \\ 576z^2 + 8064z^3 \\ 1536z^3 + 2816z^4 \end{bmatrix}$$

Of course, since $\mathbb{Z}[z]$ is a ring, rather than a field, a "pgd-vector" is more accurately described as an r-tuple than as a vector, where r is the number of i-types.

The functor relating a string operation $\tau \colon G \to H$ to the corresponding production matrix $M_\tau(z) \colon V_G(z) \to V_H(z)$, is represented by the commutative diagram in Figure 5.

$$\begin{array}{ccc} G & \xrightarrow{\quad \tau \quad} & H \\ \downarrow & & \downarrow \\ V_G(z) & \xrightarrow{M_\tau(z)} & V_H(z) \end{array}$$

Figure 5: Functor from the category of graphs and string operations to the category of ring modules and matrices with integer polynomial coefficients.

4.4 Polynomial matrix and transfer matrix methods

There are models in the physical sciences where the computational process uses polynomial matrix entries, like our production matrices. Some such models in chemistry were explored in [21, 22], which uses the terminology *polynomial matrix method*. This method was adapted by [1] for application to matching polynomials of *polygraphs*.

As described by [7], the *transfer matrix method* for various mathematical contexts concerns the transformation of a given problem into a matter of counting walks in a digraph. We observe that if A is the adjacency matrix of a digraph, then the ij entry of the matrix A^k counts the numbers of paths from vertex v_i to vertex v_j.

A generalization of this problem (see [29]) is concerned with a digraph in which the arc from vertex i to vertex j, for all i and j, is labeled with the element $m_{i,j}$ of a commutative ring, with $M = (m_{i,j})$. Instead of counting the paths of length k, we are calculating the sum of the products of all length-k paths from v_i to v_j. Of course, the ij entry of the matrix M^k gives this sum for v_i and v_j. In [5] and [23], the matrix M is called a "transfer matrix".

When calculating pgd-vectors for a graph sequence $\{G_n : n = 0, 1, \ldots\}$ that is specified by recursive application of a topological operation τ, we take the imbedding types as vertices of the digraph. We label the arc from type-i to type-j by the coefficient of type-j in the production for type-i.

5 Machine computation of production matrices

In this section, we give two examples of linear sequences whose production matrices have been calculated with the aid of a computer program. It should be clear that calculating these production matrices by hand would be daunting. Heretofore, such calculations have been done mostly by hand, which has limited us to calculating the genus polynomials only for relatively few graph families. As a consequence, we have very little data to study deep issues, such as the log-concavity conjecture, that the genus distribution of every graph is a log-concave polynomial (see [18, 16]).

5.1 Vertex-amalgamation path of copies of K_4

We define the graph T_1 to be the complete graph on four vertices, with a single root, labeled 0. The graph T_n is obtained from T_{n-1} by vertex-amalgamating a new copy of K_4 to T_{n-1}. The graphs T_2 and T_3 are illustrated in Figure 6.

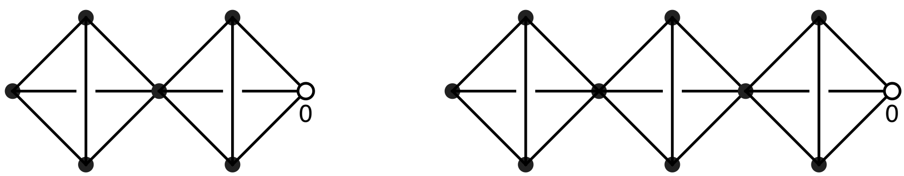

Figure 6: The graphs T_2 and T_3.

Following the paradigm of [13], we could obtain T_n from T_{n-1} by vertex-amalgamating a doubly rooted copy of K_4 to a singly rooted copy of T_{n-1}. However, whereas a pair of 2-valent root-vertices involves at most 10 i-types, it can be seen in Table 5 that for two 3-valent root-vertices, the number of i-types could be as large as 38. Moreover, the potential number of productions for amalgamating two graphs with 38 i-types could be as large as $38^2 = 1444$. In what follows, we see that using the string-operation paradigm enables us to reduce the number of i-types from 38 to 3.

The topological operation of vertex-amalgamating an additional copy of K_4 to the rooted graph $(T_{n-1}, 0)$ can be represented by the following sequence of string operations.

Procedure 5.1. Add the next copy of K_4 by vertex-amalgamation.

$$\mathrm{Add}_{01230} \tag{5.1}$$

$$\mathrm{Add}_{02} \tag{5.2}$$

$$\mathrm{Add}_{13} \tag{5.3}$$

$$\mathrm{Sup}_{0,1,3} \tag{5.4}$$

$$\mathrm{Lab}_{20} \tag{5.5}$$

We see that the i-types for a graph with a single 3-valent root-vertex named 0 are

$$(0)(0)(0) \quad (0)(00) \quad (000)$$

More generally, the number of i-types for a graph with a single k-valent root-vertex equals at most the number of partitions of the integer k. Nonetheless, even though only three productions would be needed, deriving them with pencil-and-paper calculations would be tedious work. Just for a start, there are 12 ways to insert the path 01230 into an imbedding of T_{n-1}, two ways between each of the three pairs of distinct corners at root-vertex 0 and two ways at each corner. The total number of imbeddings of T_n that are consistent with each imbedding of T_{n-1} is 480.

Theorem 5.1. *The pdg-vector of the graph T_n is $M^{n-1}\mathbf{V_1}$, where the initial pgd-vector $\mathbf{V_1}$ is $\begin{bmatrix}2 & 12z & 2z\end{bmatrix}^T$ and the production matrix is*

$$M_T(z) \;=\; \begin{bmatrix} 96z+18 & 80z+30 & 60 \\ 48z^2+156z & 220z & 360z \\ 144z^2+18z & 120z^2+30z & 60z \end{bmatrix} \tag{5.6}$$

Proof. The initial pgd-vector $\mathbf{V_1}$ for $(K_4, 0)$ and the production matrix are best calculated by a computer program. □

5.2 Edge-amalgamation path of copies of K_4

Here we define $\overline{T}_1$ to be the complete graph K_4 with a single root-edge 01. The graph $\overline{T}_n$ is obtained from $\overline{T}_n$ by edge-amalgamating a copy of K_4. The new root-edge is the edge in the new copy that is independent of the edge amalgamated to the previous root-edge. The graphs $\overline{T}_2$ and $\overline{T}_3$ are illustrated in Figure 7.

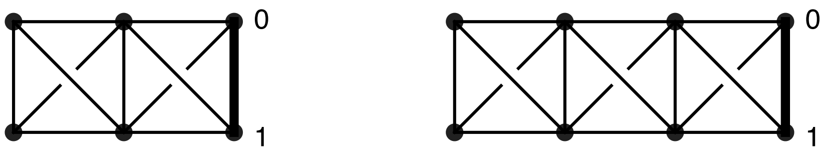

Figure 7: The graphs $\overline{T}_2$ and $\overline{T}_3$.

The topological operation of extending the graph $\overline{T}_{n-1}$ by edge-amalgamating an additional copy of K_4 can be represented by the following sequence of string operations.

Procedure 5.2. Add the next copy of K_4 by edge-amalgamation.

$$\mathrm{Add}_{0231} \tag{5.7}$$
$$\mathrm{Add}_{03} \tag{5.8}$$
$$\mathrm{Add}_{12} \tag{5.9}$$
$$\mathrm{Sup}_{0,1} \tag{5.10}$$
$$\mathrm{Lab}_{20,31} \tag{5.11}$$

We determine that the i-types for the graphs $\overline{T}_n$ are as follows, grouped by classes under the automorphism interchanging 0 and 1 and listed in shortlex order:

1. $(0)(1)(01)(01)$
2. $(0)(1)(0011)$
3. $(0)(01)(011), (1)(01)(001)$
4. $(0)(00111), (1)(00011)$
5. $(0)(01011), (1)(00101)$
6. $(01)(01)(01)$
7. $(01)(0011)$
8. $(01)(0101)$
9. $(001)(011)$
10. (000111)
11. $(001011), (001101)$
12. (010101)

Each imbedding of $\overline{T}_{n-1}$ in each of these 12 super-types has 576 possible extensions to an imbedding of $\overline{T}_n$.

Theorem 5.2. *The pdg-vector of the graph $\overline{T}_n$ is $\overline{M}^{n-1}(z)\mathbf{V}(z)$, where the production matrix is*

$$\begin{bmatrix} 4 & 18 & 8 & 36 & 40 & 6 & 20 & 22 & 12 & 72 & 80 & 84 \\ 8z & 0 & 16z & 0 & 0 & 24z & 32z & 32z & 32z & 0 & 0 & 0 \\ 64z & 96z & 96z & 96z & 96z & 96z & 128z & 128z & 128z & 0 & 0 & 0 \\ 48z^2 & 32z^2 & 32z^2 & 0 & 0 & 48z^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8z & 36z & 16z & 72z & 80z & 12z & 40z & 44z & 24z & 144z & 160z & 168z \\ 60z & 56z & 72z & 48z & 48z & 60z & 64z & 64z & 96z & 0 & 0 & 0 \\ 104z^2+4z & 48z^2+18z & 64z^2+8z & 36z & 40z & 72z^2+6z & 20z & 22z & 12z & 72z & 80z & 84z \\ 16z & 72z & 32z & 144z & 128z & 24z & 80z & 72z & 48z & 288z & 256z & 240z \\ 104z^2 & 48z^2 & 64z^2 & 0 & 0 & 72z^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 32z^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 64z^2 & 96z^2 & 96z^2 & 96z^2 & 96z^2 & 96z^2 & 128z^2 & 128z^2 & 128z^2 & 0 & 0 & 0 \\ 60z^2 & 56z^2 & 72z^2 & 48z^2 & 48z^2 & 60z^2 & 64z^2 & 64z^2 & 96z^2 & 0 & 0 & 0 \end{bmatrix}$$

The initial graph $(\overline{T}_1, 0)$ has the pgd-vector

$$\overline{\mathbf{V}}(z) = \begin{bmatrix} 2 & 0 & 0 & 0 & 4z & 0 & 2z & 8z & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

Proof. The initial pgd-vector and the production matrix were calculated by our computer program. □

If follows that

$$\overline{\mathbf{T}}_{\mathbf{2}} = \begin{bmatrix} 8+376z \\ 16z+320z^2 \\ 128z+1664z^2 \\ 96z^2 \\ 16z+752z^2 \\ 120z+832z^2 \\ 584z^2+8z \\ 32z+1248z^2 \\ 208z^2 \\ 64z^3 \\ 128z^2+1664z^3 \\ 120z^2+832z^3 \end{bmatrix} \quad \text{and} \quad \overline{\mathbf{T}}_{\mathbf{3}} = \begin{bmatrix} 32+5040z+119552z^2+207616z^3 \\ 64z+9216z^2+111872z^3 \\ 512z+56064z^2+612864z^3 \\ 384z^2+28416z^3+103424z^4 \\ 64z+10080z^2+239104z^3+415232z^4 \\ 480z+43200z^2+365568z^3 \\ 5872z^2+32z+176256z^3+389376z^4 \\ 128z+19136z^2+414464z^3+644096z^4 \\ 832z^2+56704z^3+181760z^4 \\ 256z^3+12032z^4 \\ 512z^2+56064z^3+612864z^4 \\ 480z^2+43200z^3+365568z^4 \end{bmatrix}.$$

6 Enumerating possible imbedding types

Various previously published genus polynomial calculations have involved recursive constructions of families of graphs with two 2-valent root-vertices, for which ten i-types are sufficient. As we progress toward more general results, most especially in regard to the LCGD conjecture, we are encountering recursive graph constructions for which we use arbitrarily many vertex roots, of arbitrary degrees.

In this section, we first use Burnside's Lemma to calculate the number of i-types that can occur for two 2-valent roots. Then we generalize to obtain lower and upper bounds on the number of i-types for arbitrarily many root-vertices or arbitrary valences. Interestingly, our method provides a formula for calculating the number of possible cyclic partitions of a multi-set. Thus, it is a generalization of Stirling numbers of the first kind.

6.1 Two 2-valent roots

Early papers on genus polynomial calculations via pgd-vectors used ten mnemonics for the i-types for graphs with two 2-valent roots. The following table lists the ten mnemonics and their corresponding type-names:

dd^0	dd'	dd''	ds^0	ds'
$(0)(0)(1)(1)$	$(0)(01)(1)$	$(01)(01)$	$(0)(0)(11)$	$(0)(011)$
sd^0	sd'	ss^0	ss^1	ss^2
$(00)(1)(1)$	$(001)(1)$	$(00)(11)$	(0101)	(0011)

An ad hoc examination confirms that the ten type-names contain all the possible partitions of the multi-set $\{0,0,1,1\}$ into cyclic cells. We now undertake a reconfirmation of this calculation of ten possible i-types, using Burnside's Lemma.

Our set of objects is the set of disjoint cycle decompositions of the 24 permutations in the symmetric group Σ_4, with domain $\{0,1,2,3\}$. Our permutation group on them has the permutations

$$\epsilon \text{ (identity)} \quad (0\ 2) \quad (1\ 3) \quad (0\ 2)(1\ 3) \tag{6.1}$$

where we regard the numbers 2 and 3 as second copies of the numbers 0 and 1, respectively. Under the action of this permutation group, the orbit of the permutation $(0\ 1)(2)(3)$ is

$$(0)(1)(2\ 3) \quad (0)(3)(1\ 2) \quad (1)(2)(0\ 3) \quad (2)(3)(0\ 1)$$

This orbit corresponds to the imbedding-type $(0)(1)(01)$.

The identity permutation ϵ fixes all 24 disjoint cycle representations of Σ_4. The permutation $(0\ 2)$ fixes the subgroup of disjoint cycle representations in which both 0 and 2 are fixed or transposed, whose cardinality is 4. The permutation $(1\ 3)$ fixes the same subgroup of cardinality 4. The permutation $(0\ 2)(1\ 3)$ fixes that same subgroup, plus the set

$$(0\ 1)(2\ 3) \quad (0\ 3)(1\ 2) \quad (0\ 1\ 2\ 3) \quad (0\ 3\ 2\ 1)$$

for a total of 8 fixed points. Applying Burnside's Lemma, we divide the sum of the sizes of the fixed-point sets by the cardinality of the permutation group (6.1) to obtain

$$\frac{24+4+4+8}{4} = \frac{40}{4} = 10$$

as the maximum number of i-types for two 2-valent roots.

6.2 Two roots, 2-valent and 3-valent

Suppose that root 0 is 2-valent and root 1 is 3-valent. Then there are 18 imbedding-types, as in Table 4.

Table 4: Table of the i-types for two roots, one 2-valent and one 3-valent.

structure	imbedding types		
1^5	$(0)(0)(1)(1)(1)$		
$1^3\,2$	$(0)(0)(1)(11)$	$(0)(1)(1)(01)$	$(1)(1)(1)(00)$
$1\,2^2$	$(0)(01)(11)$	$(1)(00)(11)$	$(1)(01)(01)$
$1^2\,3$	$(0)(0)(111)$	$(0)(1)(011)$	$(1)(1)(001)$
$2\,3$	$(00)(111)$	$(01)(011)$	$(11)(001)$
$1\,4$	$(0)(0111)$	$(1)(0011)$	$(1)(0101)$
5	(00111)	(01011)	

The action of the permutation group $\Sigma_{\{0,2\}} \times \Sigma_{\{1,3,4\}}$ on the elements of $\Sigma_{\{0,1,2,3,4\}}$ has the cycle index

$$\frac{1}{12}\left[t_1^5 + 4t_1^3t_2 + 3t_1t_2^2 + 2t_2t_3\right].$$

We now consider the number of fixed points for each of the four permutation types.

Type t_1^5. The identity permutation fixes all 120 elements of $\Sigma_{\{0,1,2,3,4\}}$.

Type $t_1^3t_2$. Each permutation of structure $t_1^3t_2$ fixes 12 elements of $\Sigma_{\{0,1,2,3,4\}}$. For instance, $(0\ 2)$ fixes each of the six elements with the 1-cycles (0) and (2) and each of the six with the 2-cycle (02), for a total of 12. The sum of the sized of the fixed-point sets of the four permutations of structure $t_1^3t_2$ is 48.

Type $t_1t_2^2$. Each permutation of structure $t_1t_2^2$ fixes 8 elements of $\Sigma_{\{0,1,2,3,4\}}$. For instance, $(0\ 2)(1\ 3)$ fixes both of the elements with the 1-cycles (0), (2), and (4), both with the 2-cycle (02) and the 1-cycle (4), and also the four elements

$$(0\ 1)(2\ 3),\ (0\ 3)(1\ 2),\ (0\ 1\ 2\ 3),\ \text{and}\ (0\ 3\ 2\ 1)$$

for a total of 8. The sum of the sized of the fixed-point sets of the four permutations of structure $t_1t_2^2$ is 24.

Type $t_1^2t_3$. Each permutation of structure $t_1^2t_3$ fixes 6 elements of $\Sigma_{\{0,1,2,3,4\}}$. In particular, $(0)(2)(134)$ fixes $\mathbb{Z}_{\{0,2\}} \times \mathbb{Z}_{\{1,3,4\}}$, as does $(0)(2)(1\ 4\ 3)$. Together, they make a contribution of 12 to the sum of the sizes of the fixed point sets.

Type t_2t_3. These two permutations each fix the same 6 elements of $\Sigma_{\{0,1,2,3,4\}}$ as in the preceding case, for a net contribution of 12.

Applying Burnside's Lemma, we infer that the number of orbits is

$$\frac{120 + 48 + 24 + 12 + 12}{12} = \frac{216}{12} = 18.$$

6.3 Several roots of arbitrary degrees

We now calculate lower and upper bounds on the number of i-types.

Theorem 6.1. *For a class of graphs with roots* $0, 1, \ldots, k-1$ *of respective degrees* $d_0, d_1, \ldots, d_{k-1}$, *the number of i-types is at least*

$$\frac{(d_0 + d_1 + \cdots + d_{k-1})!}{d_0! d_1! \cdots d_{k-1}!} \tag{6.2}$$

Proof. In addition to their respective primary names $0, 1, \ldots, k-1$, each root j has $d_j - 1$ aliases chosen from among the numbers

$$k,\ k+1,\ \ldots,\ d_0{+}d_1{+}\cdots{+}d_{k-1}$$

with no two different primary names having any aliases in common. Accordingly, our set of objects is the set of disjoint cycle representations of the symmetric group Σ_K, where $K = d_0 + d_1 + \cdots + d_{k-1}$. The permutation group that acts on them is isomorphic to

$$\Sigma_{d_0} \times \Sigma_{d_1} \times \cdots \times \Sigma_{d_{k-1}}$$

Since the identity permutation fixes all the cycle forms of Σ_K, the sum of the sizes of the sets of fixed points is at least $K!$. The cardinality of the permutation group is $d_1! d_2! \cdots d_k!$. Thus, by Burnside's Lemma, a lower bound on the number of i-types is given by (6.2). □

Theorem 6.2. *For a class of graphs with roots* 0 *and* 1, *of respective degrees* a *and* b, *the number of i-types is at most*

$$\sum_c \prod_{k=1}^{n} k^{c_k} c_k! \sum_{\forall i, p_i + q_i = c_i} \sum_{(1^{p_1} 2^{p_2} \cdots a^{p_a}) \in P_a} \sum_{(1^{q_1} 2^{q_2} \cdots b^{q_b}) \in P_b} \frac{1}{\prod_{i=1}^{a} i^{p_i} p_i! \prod_{j=1}^{b} j^{q_j} q_j!},$$

where the sum $\sum_c$ *is over all partitions* $1^{c_1} 2^{c_2} \cdots n^{c_n} \in P_n$ *and* P_n *is the set of all partitions of the number* n.

Proof. The action of the permutation group

$$\Sigma_{\{1,3,4,\ldots,a+1\}} \times \Sigma_{\{2,a+2,a+3,\ldots,a+b\}}$$

on the elements of $\Sigma_{\{1,2,\ldots,n\}}$, where $n = a + b$, has the cycle index

$$C_{a,b} = \sum_{(1^{p_1} 2^{p_2} \cdots a^{p_a}) \in P_a} \sum_{(1^{q_1} 2^{q_2} \cdots b^{q_b}) \in P_b} \frac{\prod_{i=1}^{a} t_i^{p_i} \prod_{j=1}^{b} t_j^{q_j}}{\prod_{i=1}^{a} i^{p_i} p_i! \prod_{j=1}^{b} j^{q_j} q_j!},$$

where P_m is the set of all partitions of m. The number of fixed points for a permutation of cycle type $1^{c_1} 2^{c_2} \cdots n^{c_n}$ is given by

$$a! b! C_{a,b}(1^{c_1} 2^{c_2} \cdots n^{c_n}) \prod_{k=1}^{n} k^{c_k} c_k!,$$

where $C_{a,b}(1^{c_1}2^{c_2}\cdots n^{c_n})$ is the coefficient of $t_1^{c_1}t_2^{c_2}\cdots t_n^{c_n}$ in the polynomial $C_{a,b}$. Thus, each permutation of structure $t_1^{c_1}t_2^{c_2}\cdots t_n^{c_n}$ fixes

$$\prod_{k=1}^{n} k^{c_k}c_k! \sum_{\forall i, p_i+q_i=c_i} \sum_{(1^{p_1}2^{p_2}\cdots a^{p_a})\in P_a} \sum_{(1^{q_1}2^{q_2}\cdots b^{q_b})\in P_b} \frac{a!b!}{\prod_{i=1}^{a} i^{p_i}p_i! \prod_{j=1}^{b} j^{q_j}q_j!}.$$

elements of $\Sigma_{\{1,2,\ldots,n\}}$.

Applying Burnside's Lemma, we conclude that the number of orbits is given by

$$\sum_{c} \frac{1}{a!b!} \prod_{k=1}^{n} k^{c_k}c_k! \sum_{\forall i, p_i+q_i=c_i} \sum_{(1^{p_1}2^{p_2}\cdots a^{p_a})\in P_a} \sum_{(1^{q_1}2^{q_2}\cdots b^{q_b})\in P_b} \frac{a!b!}{\prod_{i=1}^{a} i^{p_i}p_i! \prod_{j=1}^{b} j^{q_j}q_j!}$$

which equals

$$\sum_{c} \prod_{k=1}^{n} k^{c_k}c_k! \sum_{\forall i, p_i+q_i=c_i} \sum_{(1^{p_1}2^{p_2}\cdots a^{p_a})\in P_a} \sum_{(1^{q_1}2^{q_2}\cdots b^{q_b})\in P_b} \frac{1}{\prod_{i=1}^{a} i^{p_i}p_i! \prod_{j=1}^{b} j^{q_j}q_j!},$$

where the sum $\sum_c$ is over all partitions $1^{c_1}2^{c_2}\cdots n^{c_n} \in P_n$. □

Applying our formula for $a, b \le 10$, we obtain Table 5.

Table 5: The maximum number of i-types for two root-vertices, of valences a and b.

$a\backslash b$	1	2	3	4	5	6	7	8	9	10
1	2	4	7	12	19	30	45	67	97	139
2	4	10	18	34	56	94	146	228	340	506
3	7	18	38	74	133	233	385	623	977	1501
4	12	34	74	158	297	550	951	1614	2627	4202
5	19	56	133	297	602	1166	2133	3775	6437	10692
6	30	94	233	550	1166	2382	4551	8424	14953	25835
7	45	146	385	951	2133	4551	9142	17639	32680	58659
8	67	228	623	1614	3775	8424	17639	35492	68356	127443
9	97	340	977	2627	6437	14953	32680	68356	136936	264747
10	139	506	1501	4202	10692	25835	58659	127443	264747	530404

Theorem 6.3. *The formula corresponding to that of Theorem 6.2 for m roots of degrees $(a_1, a_2, \ldots, a_m)$ is given by*

$$\sum_{c} \prod_{k=1}^{n} k^{c_k}c_k! \sum_{\forall i, p_{1i}+p_{2i}+\cdots+p_{di}=c_i} \sum_{\substack{\forall d=1,2,\ldots,m,\\ \left(1^{p_{d1}}2^{p_{d2}}\cdots a_d^{p_{da_d}}\right)\in P_{a_d}}} \frac{1}{\prod_{d=1}^{m}\prod_{i=1}^{a_d} i^{p_{di}}p_{di}!},$$

where the sum $\sum_c$ is over all partitions $1^{c_1}2^{c_2}\cdots n^{c_n} \in P_n$.

Proof. This proof uses the same arguments as for Theorem 6.2. □

Using the formula from Theorem 6.3 for the calculations, we present in Table 6 the maximum number of imbedding-types for triply rooted graphs with root-vertices of valences $1 \le i, j, k \le 5$.

Table 6: The maximum number of imbedding-types for three roots, of valences i, j, k for $i = 1, 2, 3, 4, 5$.

$i = 1$

$j\backslash k$	1	2	3	4	5
1	6	14	28	52	90
2	14	38	84	170	316
3	28	84	206	450	899
4	52	170	450	1058	2254
5	90	316	899	2254	5110

$i = 2$

$j\backslash k$	1	2	3	4	5
1	14	38	84	170	316
2	38	120	290	644	1284
3	84	290	788	1886	4074
4	170	644	1886	4868	11214
5	316	1284	4074	11214	27556

$i = 3$

$j\backslash k$	1	2	3	4	5
1	28	84	206	450	899
2	84	290	788	1886	4074
3	206	788	2370	6146	14302
4	450	1886	6146	17170	42696
5	899	4074	14302	42696	112966

$i = 4$

$j\backslash k$	1	2	3	4	5
1	52	170	450	1058	2254
2	170	644	1886	4868	11214
3	450	1886	6146	17170	42696
4	1058	4868	17170	51630	137070
5	2254	11214	42696	137070	387146

$i = 5$

$j\backslash k$	1	2	3	4	5
1	90	316	899	2254	5110
2	316	1284	4074	11214	27556
3	899	4074	14302	42696	112966
4	2254	11214	42696	137070	387146
5	5110	27556	112966	387146	1161498

7 Conclusions

We have focused here primarily on the computational aspects involved in applying string operations toward the determination of genus polynomials of graphs. We recognize the following two immediate benefits of the string-operations paradigm:

1. It enables us to reduce the number of partial genus polynomials (one for each imbedding-type) into which a genus polynomial must be partitioned.
2. The imbedding-types, the production matrix, and the partial genus polynomials (which are the coordinates of a pgd-vector) can be calculated by a computer program, which enables us to generate a much larger set of experimental data.

Beyond using string operations in new calculations of enumerative results on graph imbeddings, some new theoretical insights may arise from them. One may reasonably consider how the paradigm of string operations relates to the log-concavity conjecture, that every genus polynomial is log-concave (see [16, 18]). We observe that using Theorem 4.7.2 of [29] could give generating functions for the individual entries of a power of a production matrix.

In a sequel [15], we regard a linear family of graphs as a Markov process is which the states are i-types and a slightly modified form of the production matrix is the transition matrix. We explore the properties of such Markov processes.

The methods described here seem amenable to extension. Suppose that instead of a fixed production matrix $M(z)$ for a graph sequence $\{G_n : n = 0, 1, \ldots\}$, with pgd-vectors $V_n(z)$ we had a sequence of production matrices $M_n(z)$, such that Recursion (4.1) was generalized to

$$M_n(z)v_n(z) = V_{n+1}(z),$$

and Equation (4.2) to

$$V_n(z) = M_{n-1}(z)M_{n-2}(z) \cdots M_0(z)V_0(z).$$

A tractable recursion or a closed formula for $M_n(z)$ would enable us to calculate the pgd-vector $V_n(z)$ reasonably rapidly. Of course, such a sequence of production matrices corresponds to a non-stationary Markov process.

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Finite actions on the 2-sphere, the projective plane and I-bundles over the projective plane

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Abstract

In this paper, we consider the finite groups which act on the 2-sphere $\mathbb{S}^2$ and the projective plane $\mathbb{P}^2$, and show how to visualize these actions which are explicitly defined. We obtain their quotient types by distinguishing a fundamental domain for each action and identifying its boundary. If G is an action on $\mathbb{P}^2$, then G is isomorphic to one of the following groups: $\mathbb{S}_4$, $\mathbb{A}_5$, $\mathbb{A}_4$, $\mathbb{Z}_m$ or $\mathrm{Dih}(\mathbb{Z}_m)$. For each group, there is only one equivalence class (conjugation), and G leaves an orientation reversing loop invariant if and only if G is isomorphic to either $\mathbb{Z}_m$ or $\mathrm{Dih}(\mathbb{Z}_m)$. Using these preliminary results, we classify and enumerate the finite groups, up to equivalence, which act on $\mathbb{P}^2 \times I$ and the twisted I-bundle over $\mathbb{P}^2$. As an example, if $m > 2$ is an even integer and $m/2$ is odd, there are three equivalence classes of orientation reversing $\mathrm{Dih}(\mathbb{Z}_m)$-actions on the twisted I-bundle over $\mathbb{P}^2$. However if $m/2$ is even, then there are two equivalence classes.

Keywords: Achiral symmetry, chiral symmetry, equivalence of actions, finite group action, isometry, orbifold, symmetry.

Math. Subj. Class.: 57S25, 05E18, 57M60, 57R18, 58D19, 57M20

1 Introduction

The finite orientation preserving groups which act effectively on $\mathbb{S}^2$ are known. (See for example Gross and Tucker [5] and Zimmermann [9].) They are the octahedral symmetric group $\mathbb{S}_4$, the dodecahedral/icosahedral alternating group $\mathbb{A}_5$, the tetrahedral alternating

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group $\mathbb{A}_4$, the cyclic group $\mathbb{Z}_m$ or the dihedral group $\mathrm{Dih}(\mathbb{Z}_m)$. Using this classification, the actions on the projective plane $\mathbb{P}^2$ are also known as folklore, and one can easily compute them by this theorem of Singerman [7] and Tucker [8].

Theorem. *Let F be a closed non-orientable surface and let $p\colon \tilde{F} \to F$ be the orientable double cover with covering translation $t\colon \tilde{F} \to \tilde{F}$. Then any finite group G acting on F, lifts to an orientation preserving action of G on $\tilde{F}$ that commutes with t. Moreover, the action of G on F is determined by the action of $G \times \langle t \rangle$ on $\tilde{F}$.*

If $t\colon \mathbb{S}^2 \to \mathbb{S}^2$ is the covering translation such that $\mathbb{S}^2/\langle t \rangle = \mathbb{P}^2$, one checks that any rotation of $\mathbb{S}^2$ commutes with t. Therefore since these groups consists of rotations, it follows that the orientation preserving actions on $\mathbb{S}^2$ project to $\mathbb{P}^2$, giving the following corollary.

Corollary. *Any finite group acting on $\mathbb{P}^2$ is isomorphic to one of the following groups: $\mathbb{S}_4$, $\mathbb{A}_5$, $\mathbb{A}_4$, $\mathbb{Z}_m$ or $\mathrm{Dih}(\mathbb{Z}_m)$.*

A finite G-action on a manifold M is a monomorphism $\varphi\colon G \to \mathrm{Homeo}(M)$, where G is a finite group, and $\mathrm{Homeo}(M)$ is the group of homeomorphisms of M. Two actions φ_1 and φ_2 are equivalent if there exists a homeomorphism h of M such that $h\varphi_1(G)h^{-1} = \varphi_2(G)$. For an action φ, the quotient space M/φ is an orbifold which is referred to as the *quotient type* of the action.

In this paper, we describe how to visualize the finite groups which act on the 2-sphere $\mathbb{S}^2$ and the projective plane $\mathbb{P}^2$, and show how to obtain their quotient types. Our approach, for the groups which are not cyclic or dihedral, is to view these groups as subgroups of the symmetric group S_n for an appropriate n, tiling the 2-sphere with appropriate polygons with n vertices for each group, and explicitly defining each action. As for the cyclic and dihedral groups, we use spherical coordinates to precisely describe their actions on $\mathbb{S}^2$. For all these groups, we can easily identify an explicit fundamental region for each action and see its quotient type, which is obtained by identifying the boundary of the fundamental region. In this way, it is easy to see the actions on $\mathbb{S}^2$, $\mathbb{P}^2$ and their quotient types. This part of the paper may be considered expository, and we obtain the following theorem where the description of these quotient types may be found in Figure 1.

Theorem 7.1. *Let $\varphi\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ be a finite group action on $\mathbb{P}^2$. Then G is isomorphic to one of the following groups: $\mathbb{S}_4$, $\mathbb{A}_5$, $\mathbb{A}_4$, $\mathbb{Z}_m$ or $\mathrm{Dih}(\mathbb{Z}_m)$. The orbifold quotient $\mathbb{P}^2/\varphi$ is an orbifold homeomorphic to one of the following orbifolds: O^h, I^h, T^v, Z^h_m, S^{2m}, D^v_m or D^h_m. There is only one equivalence class for each group.*

(1) $G \simeq \mathbb{S}_4$ *if and only if* $\mathbb{P}^2/\varphi = O^h$.

(2) $G \simeq \mathbb{A}_5$ *if and only if* $\mathbb{P}^2/\varphi = I^h$.

(3) $G \simeq \mathbb{A}_4$ *if and only if* $\mathbb{P}^2/\varphi = T^v$.

(4) $G \simeq \mathbb{Z}_m$ *and m is even if and only if* $\mathbb{P}^2/\varphi = Z^h_m$.

(5) $G \simeq \mathbb{Z}_m$ *and m is odd if and only if* $\mathbb{P}^2/\varphi = S^{2m}$.

(6) $G \simeq \mathrm{Dih}(\mathbb{Z}_m)$ *and m odd if and only if* $\mathbb{P}^2/\varphi = D^v_m$.

(7) $G \simeq \mathrm{Dih}(\mathbb{Z}_m)$ *and m even if and only if* $\mathbb{P}^2/\varphi = D^h_m$.

This approach relates to topics in topological graph theory found in Gross and Tucker [5]. There, graphs are embedded on surfaces and finite groups act on these spaces with quotient spaces, branch covering maps and branch points, relating to orbifold covering maps and cone points.

Using the above result, we classify in Theorem 7.4 the finite group actions, up to equivalence, on $\mathbb{P}^2 \times I$ for $I = [0, 1]$. If G is an action on $\mathbb{P}^2 \times I$, then G is isomorphic to one of the following groups: $\mathbb{S}_4$, $\mathbb{S}_4 \times \mathbb{Z}_2$, $\mathbb{A}_5$, $\mathbb{A}_5 \times \mathbb{Z}_2$, $\mathbb{A}_4$, $\mathbb{A}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_m$, $\mathbb{Z}_m \times \mathbb{Z}_2$, $\mathrm{Dih}(\mathbb{Z}_m)$ or $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2$. We indicate the number of equivalence classes for each group in Theorem 7.4. If W is the twisted I-bundle over the projective plane $\mathbb{P}^2$, then we obtain the following results:

Corollary 8.12. *Let $\varphi\colon G \to \mathrm{Homeo}(W)$ be a finite orientation preserving G-action on W. Then G is isomorphic to one of the following groups: $\mathbb{S}_4$, $\mathbb{A}_5$, $\mathbb{A}_4$, $\mathbb{Z}_m$ or $\mathrm{Dih}(\mathbb{Z}_m)$. The orbifold quotient for each action is a twisted I-bundle orbifold over the following 2-orbifolds: O^h (for $\mathbb{S}_4$), I^h (for $\mathbb{A}_5$), T^v (for $\mathbb{A}_4$), Z^h_m (for $\mathbb{Z}_m$ and m even), S^{2m} (for $\mathbb{Z}_m$ and m odd), D^v_m (for $\mathrm{Dih}(\mathbb{Z}_m)$ and m odd) and D^h_m (for $\mathrm{Dih}(\mathbb{Z}_m)$ and m even). There is one equivalence class for each quotient type.*

Theorem 9.4. *Let $\varphi\colon G \to \mathrm{Homeo}(W)$ be an orientation reversing G-action. Then G is isomorphic to one of the following groups: $\mathbb{S}_4$, $\mathbb{Z}_m$ with m even, $\mathrm{Dih}(\mathbb{Z}_m)$, $\mathbb{S}_4 \times \mathbb{Z}_2$, $\mathbb{A}_5 \times \mathbb{Z}_2$, $\mathbb{A}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_m \times \mathbb{Z}_2$ or $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2$.*

(1) If G is either $\mathbb{S}_4$, $\mathbb{S}_4 \times \mathbb{Z}_2$, $\mathbb{A}_5 \times \mathbb{Z}_2$, $\mathbb{A}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_m \times \mathbb{Z}_2$ with m even or $\mathrm{Dih}(\mathbb{Z}_m)$ with m odd, there is only one equivalence class.

(2) If G is $\mathbb{Z}_m$ with $m > 2$ even and $m/2$ odd, then there are two equivalence classes of $\mathbb{Z}_m = \mathbb{Z}_{m/2} \times \mathbb{Z}_2$-actions on W.

(3) If G is $\mathbb{Z}_m$ with either $m/2$ even or $m = 2$, then there is only one equivalence class.

(4) If G is $\mathrm{Dih}(\mathbb{Z}_m)$ with $m > 2$ and $m/2$ even, there are two equivalence classes of $\mathrm{Dih}(\mathbb{Z}_m)$-actions on W.

(5) If G is $\mathrm{Dih}(\mathbb{Z}_m)$ with $m > 2$ and $m/2$ odd, there are three equivalence classes of $\mathrm{Dih}(\mathbb{Z}_m)$-actions on W.

(6) If G is $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2$ with m even, there is only one equivalence class.

(7) If G is $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2$ with m odd, then $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2 \simeq \mathrm{Dih}(\mathbb{Z}_{2m})$ and there are three equivalence classes of $\mathrm{Dih}(\mathbb{Z}_{2m})$-actions on W.

We list all the closed 2-orbifolds with positive Euler number, of which there are 14. (See Figure 1.) In referring to these orbifolds, we use Schönflies notation found in Coxeter and Moser [1], and Dunbar [3].

There are five orientable 2-orbifolds with positive Euler number which have as their underlying space a 2-sphere with the cone points indicated in the notation. They are $\Sigma(2, 2, n) = D_n$, $\Sigma(2, 3, 3) = T$, $\Sigma(2, 3, 4) = O$, $\Sigma(2, 3, 5) = I$, and $\Sigma(n, l) = C_{n,l}$. These double cover the following nine non-orientable 2-orbifolds where the double lines are reflector lines. The superscripts h and v stand for horizontal and vertical reflections in their orientable double covers. Except for $\Sigma(n, l) = C_{n,l}$ where the cone points are at the north and south poles, all the cone points are located on the equator.

In this article, where appropriate and depending on the context, we use the same symbol to denote the quotient space and the group acting on $\mathbb{S}^2$. For example, $O = \Sigma(2, 3, 4)$ and O also denote the octahedral group.

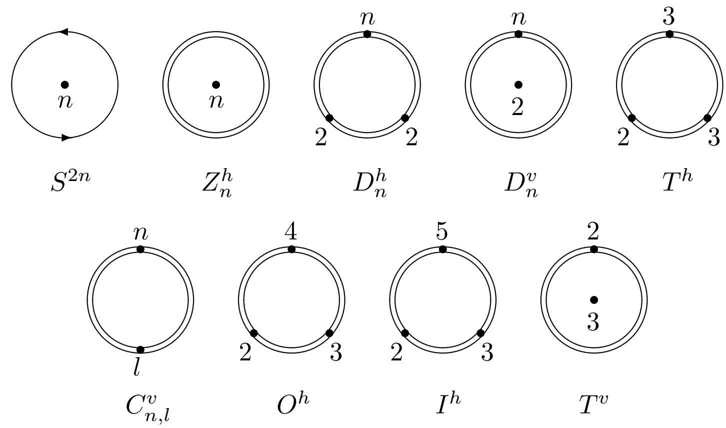

Figure 1: The nine non-orientable 2-orbifolds of positive Euler number.

Here is a brief outline of the paper. We consider each of these orbifolds in Sections 2 through 6, and give model maps which we consider as standard actions, to obtain each quotient type. Summarizing we give the main results for finite actions on $\mathbb{P}^2$ and $\mathbb{P}^2 \times I$ in Section 7. Sections 8 and 9 are devoted to classifying the finite actions on the twisted I-bundle W over $\mathbb{P}^2$.

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2 Chiral octahedral symmetry O and achiral octahedral symmetry O^h

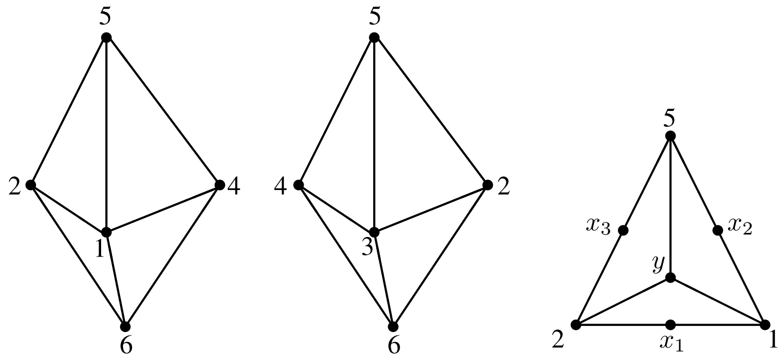
We describe the groups $O = \mathbb{S}_4$ and $O^h = \mathbb{S}_4 \times \mathbb{Z}_2$ acting on the 2-sphere $\mathbb{S}^2$, and show how O acts on the projective plane $\mathbb{P}^2$. We view $\mathbb{S}^2$ as an octahedron which has eight triangles (faces): $\triangle 125$, $\triangle 145$, $\triangle 126$, $\triangle 146$, $\triangle 235$, $\triangle 236$, $\triangle 345$ and $\triangle 346$. (See Figure 2.)

Consider elements of $\mathbb{S}_6$ where $a = (1,2)(3,4)(5,6)$ and $b = (1,2,5)(3,4,6)$. The two elements act on the octahedron. We can see that a is a $180°$ rotation about the axis passing through the midpoint of edges $\overline{1,2}$ and $\overline{3,4}$. On the other hand, b is a $120°$ rotation about the axis passing through the barycenter of $\triangle 125$ and $\triangle 346$ respectively. Further, $ab = (2,6,4,5)$ where ab is a $90°$ rotation about the axis passing through vertices 1 and 3. As a result, the two elements a and b generate a group isomorphic to $\mathbb{S}_4$, and we denote this group by $O = \langle a, b \mid a^2 = b^3 = (ab)^4 = 1 \rangle$, the octahedral group.

Next, we use Σ to denote the quotient space of $\mathbb{S}^2$ by O, and we will find a fundamental region for Σ on $\mathbb{S}^2$. We first claim that $\triangle 125$ will tile the whole octahedron $\mathbb{S}^2$ by the action of O. Observe that the action by a sends $\triangle 125$ to $\triangle 216$. Further, $b^2(ab)b^{-2} = (1,4,3,2)$ is a $90°$ rotation about the axis passing through vertices 5 and 6, which shows our claim.

Note that the number of fundamental regions for Σ on $\mathbb{S}^2$ must be 24 as the number is the order of the octahedral group $O = \mathbb{S}_4$. Since the $\mathbb{S}^2$ currently has eight faces, we will have to triangulate them further. Our approach is that we will add one more vertex on the barycenter on each triangle. For instance, one of the triangulations on $\triangle 125$ is shown in the Figure 2.

We now show that $\triangle 12y$ becomes a fundamental region for Σ. Since a rotational axis of b passes the vertex y, the barycenter of $\triangle 125$, one can see that b permutes those three triangles $\triangle 12y$, $\triangle 51y$ and $\triangle 25y$. In the meantime, edges $\overline{1, x_1}$ and $\overline{2, x_1}$ are identified



(a) Front of $\mathbb{S}^2$. (b) Back of $\mathbb{S}^2$. (c) Triangle 125.

Figure 2: $\mathbb{S}^2$ as an octahedron.

by a. Likewise, edges $\overline{1,y}$ and $\overline{2,y}$ are identified by b. Points 1, x_1 and y will be cone points under the action. Each has an order 4, 2 and 3 respectively. Consequently, we obtain $\Sigma = \mathbb{S}^2/O = \Sigma(2,3,4)$.

In order to obtain O^h, we consider an action $i = (1,3)(2,4)(5,6)$ which is the antipodal map on $\mathbb{S}^2$. Notice that the antipodal map commutes with the elements in O, hence it induces the reflection map on $\mathbb{S}^2/O = \Sigma(2,3,4)$. Now, we choose a triangle whose vertices are 1, x_1 and y. Apply $(ab)^2b(ab)i$ on the triangle gives us the triangle with vertices 2, x_1 and y. Notice that segments $1x_1$ and $2x_1$; $1y$ and $2y$ have been identified under O-action and the segment x_1y has been fixed under the map $(ab)^2b(ab)i$. This argument shows that $\triangle 1x_1y$ is a fundamental region for $O \times \langle i \rangle$-action on $\mathbb{S}^2$. The vertices of $\triangle 1x_1y$ become corner reflectors, and the edges minus the vertices become the reflector lines. As a result, $\mathbb{S}^2/[O \times \langle i \rangle] = O^h$, where $O \times \langle i \rangle = \mathbb{S}_4 \times \mathbb{Z}_2 = \pi_1(O^h)$.

We remark that $\mathbb{S}^2/\langle i \rangle = \mathbb{P}^2$ is the projective plane. Since the antipodal map i commutes with O, the octahedral action on $\mathbb{S}^2$ induces the action generated by $\bar{a}$ and $\bar{b}$ on $\mathbb{P}^2$, which is isomorphic to the octahedral group O. As a result, we also obtain $\mathbb{P}^2/\langle \bar{a}, \bar{b} \rangle = O^h$. We will now describe the octahedral action O on $\mathbb{P}^2$.

The left diagram in Figure 3 illustrates a fundamental region on $\mathbb{S}^2$ used to obtain $\mathbb{P}^2$ under the antipodal map $i = (1,3)(2,4)(5,6)$. For any arc $\overline{x,y,z}$, we let $[\overline{x,y,z}]$ be its projection in $\mathbb{P}^2$. The arc $\overline{1,2,3}$ (or $\overline{3,4,1}$ etc) on $\mathbb{S}^2$ projects to an orientation reversing loop $[\overline{1,2,3}]$ on $\mathbb{P}^2$. The generator $\bar{a}$ maps the loop $[\overline{1,2,3}]$ onto $[\overline{2,1,4}] = [\overline{2,1}][\overline{1,4}] = [\overline{2,1}][\overline{3,2}]$, which traces the same loop as $[\overline{1,2,3}]$ starting at a different point. Thus $\bar{a}$ leaves the loop $[\overline{1,2,3}]$ invariant and restricted to this loop is a rotation. On the other hand, the map $\bar{b}$ maps the loop $[\overline{1,2,3}]$ onto $[\overline{2,5,4}]$ whose image is shown as a bold line in the middle diagram in Figure 3 above. Moreover, $\bar{b}^2$ maps the loop $[\overline{1,2,3}]$ onto the loop $[\overline{1,3,5}]$. Thus the $\mathbb{Z}_3$-action generated by $\bar{b}$ does not leave the orientation reversing loop $[\overline{1,2,3}]$ on $\mathbb{P}^2$ invariant.

However, it is important to emphasize that this does not imply the $\mathbb{Z}_3$-action leaves no orientation reversing loops invariant. In fact, we can find another orientation reversing loop on $\mathbb{P}^2$ which is left invariant under the map $\bar{b}$. It can be found by looking at the octahedron $\mathbb{S}^2$ which double covers $\mathbb{P}^2$. Consider the circle on $\mathbb{S}^2$ which contains the vertices consisting of the midpoints of $\overline{4,5}$, $\overline{5,3}$, $\overline{3,2}$, $\overline{2,6}$, $\overline{6,1}$ and $\overline{1,4}$. One can check that this circle is left invariant under b and the covering translation i, hence it projects to an orientation reversing loop on $\mathbb{P}^2$ left invariant under $\bar{b}$. It follows that the entire $\mathbb{S}_4$-action on $\mathbb{P}^2$ does not leave any orientation reversing loop invariant.

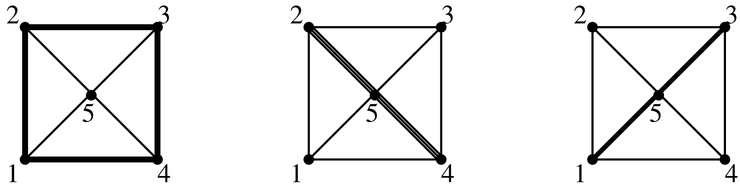


Figure 3: Fundamental region on octahedron.

Lemma 2.1. *Let $\mathbb{Z}_2$ be a subgroup of $\pi_1(O^h)$ such that $\mathbb{P}^2 \to O^h$ is the covering corresponding to $\mathbb{Z}_2$. Then $\mathbb{Z}_2 = \langle i \rangle$.*

Proof. We will show that there is only one element of order two in $O^h = \mathbb{S}_4 \times \mathbb{Z}_2$ acting on $\mathbb{S}^2$ which is fixed point free and orientation reversing, and that element is i. Since the elements in $\mathbb{S}_4$ and $\mathbb{Z}_2$ commute, we will first look at all elements of order two in $\mathbb{S}_4$. In this group, there are nine such elements. Six of them are a rotation of 180° where their rotational axes are on midpoints of edges. For example, one rotational axis passes the midpoint of $\overline{1,4}$ and $\overline{2,3}$. Another one passes the midpoint of $\overline{2,5}$ and $\overline{4,6}$. Notice that all six types of these rotations are conjugate in $\mathbb{S}_4$. Moreover, there are three types of 90° rotations, call them r_1, r_2 and r_3, where $r_1 = (1,2,3,4)$, $r_2 = (1,6,3,5)$ and $r_3 = (2,5,4,6)$ respectively. Clearly, they generate three kinds of 180° rotations which are conjugate in $\mathbb{S}_4$. As a result, $\mathbb{S}_4$ has two conjugacy classes of order two elements, and we will choose a and $(ab)^2$ from the group to represent each class. There is an easy way to verify if two elements in $\mathbb{S}_n$ are conjugate for $n \in \mathbb{N}$ by checking their cycle types. (See [2, Chapter 4].) Now, we compose them with the antipodal map i to obtain $ai = (1,4)(2,3)$ and $(ab)^2 i = (1,3)$. Since both maps have a fixed point, if $\mathbb{P}^2 \to O^h$ is the covering corresponding to any $\mathbb{Z}_2$, then $\mathbb{Z}_2 = \langle i \rangle$. □

Proposition 2.2. *Let $\varphi\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ be a finite action such that $\mathbb{P}^2/\varphi$ is homeomorphic to O^h. Then $G \simeq \mathbb{S}_4$ and φ is conjugate to the standard action $\mathbb{S}_4 = \langle \bar{a}, \bar{b} \rangle$. Moreover, no orientation reversing loop is left invariant by the G-action.*

Proof. Let $\nu\colon \mathbb{P}^2 \to \mathbb{P}^2/\langle \bar{a}, \bar{b} \rangle$ and $\nu_\varphi\colon \mathbb{P}^2 \to \mathbb{P}^2/\varphi$ be the orbifold covering maps. By assumption there exists a homeomorphism $h\colon \mathbb{P}^2/\langle \bar{a}, \bar{b} \rangle \to \mathbb{P}^2/\varphi$. By Lemma 2.1, the $\mathbb{Z}_2$ subgroup of $\pi_1(\mathbb{P}^2/\varphi)$ giving rise to a covering $\mathbb{P}^2 \to \mathbb{P}^2/\varphi$ is unique. Hence h lifts to a homeomorphism $\tilde{h}\colon \mathbb{P}^2 \to \mathbb{P}^2$ and we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\tilde{h}} & \mathbb{P}^2 \\ \downarrow_{\nu} & & \downarrow_{\nu_\varphi} \\ \mathbb{P}^2/\langle \bar{a}, \bar{b} \rangle & \xrightarrow{h} & \mathbb{P}^2/\varphi \end{array}$$

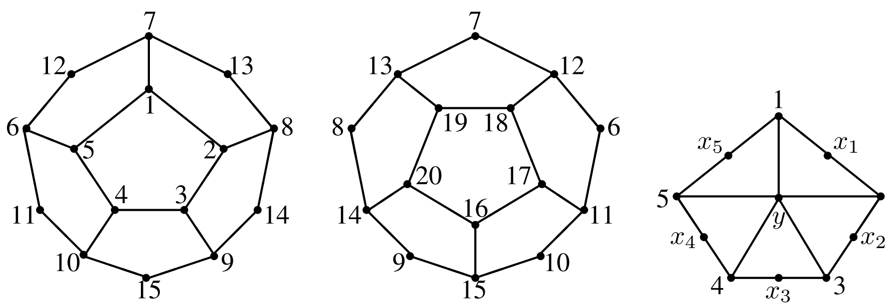
This implies that $G \simeq \mathbb{S}_4$ and $\tilde{h}$ conjugates φ to the standard action $\langle \bar{a}, \bar{b} \rangle$. □

3 Chiral dodecahedral/icosahedral symmetry I and achiral dodecahedral/icosahedral symmetry I^h

We describe the groups $I = \mathbb{A}_5$ and $I^h = I \times \mathbb{Z}_2 = \mathbb{A}_5 \times \mathbb{Z}_2$ acting on the 2-sphere $\mathbb{S}^2$, and show how $\mathbb{A}_5$ acts on $\mathbb{P}^2$.

We view $\mathbb{S}^2$ as a dodecahedron consisting of 12 pentagons as shown in the first two figures from the left in Figure 4. We also consider two elements a and b in $\mathbb{S}_{20}$ where $a = (1,2)(3,7)(4,13)(5,8)(6,14)(9,12)(10,19)(11,20)(15,18)(16,17)$ and the element $b = (2,5,7)(3,6,13)(4,12,8)(9,11,19)(10,18,14)(15,17,20)$. The two elements act on the dodecahedron $\mathbb{S}^2$, and we can see that a is a $180°$ rotation about the axis passing through the midpoint of edges $\overline{1,2}$ and $\overline{16,17}$. On the other hand, b is a $120°$ rotation about the axis passing through the vertices 1 and 16. Moreover, ab^{-1} is a $72°$ rotation about the axis passing through the barycenter of the pentagon whose vertices are 1, 2, 3, 4, 5 and 16, 17, 18, 19, 20 respectively since $ab^{-1} = (1,2,3,4,5)(6,7,8,9,10)(11,12,13,14,15)$ $(16,17,18,19,20)$. Consequently, a and b generate a group isomorphic to $I = \mathbb{A}_5$ written by $I = \langle a, b \mid a^2 = b^3 = (ab^{-1})^5 = 1\rangle$.

We use Σ to denote the quotient space of $\mathbb{S}^2$ by I, and we will look for a fundamental region for Σ on $\mathbb{S}^2$. We will first observe that one of the pentagons consists of vertices 1, 2, 3, 4 and 5 tiles the remaining pentagons on $\mathbb{S}^2$. This pentagon is sent to the pentagon with the vertices 1, 5, 6, 12, 7 by b. Then, ab^{-1} permutes the remaining pentagons in the front of $\mathbb{S}^2$. On the other hand, $(ab^{-1})^2$ sends the vertices 1, 5, 6, 12, 7 to the vertices 4, 3, 9, 15, 10. Then, the map a sends them to the vertices 13, 7, 12, 18, 19 on the back of $\mathbb{S}^2$. At this stage, one can see that ab^{-1} permutes all pentagons on the back of $\mathbb{S}^2$ except the one on the center whose vertices are 16, 17, 18, 19, 20. However, it can be obtained by applying the map $b^{-1}(ab^{-1})^2$ on the vertices 4, 3, 9, 15, 10.



(a) Front of $\mathbb{S}^2$. (b) Back of $\mathbb{S}^2$. (c) Triangulated face.

Figure 4: $\mathbb{S}^2$ as a dodecahedron.

Next, we will add a vertex on the barycenter of the pentagon 1, 2, 3, 4, 5 (see Figure 4), which we denote by y. We also add vertices x_i $(1 \leq i \leq 5)$ on this pentagon. We can see that $\triangle 12y$ tiles the remaining triangles on the pentagon 1, 2, 3, 4 and 5 (see Figure 4) by the map ab^{-1}. By the argument above, this proves that $\triangle 12y$ is a fundamental region for Σ. Now the edges $\overline{1, x_1}$ and $\overline{2, x_1}$ are identified by $a \in I$. Likewise, $ab^{-1} \in I$ identifies edges $\overline{1, y}$ and $\overline{2, y}$. The vertices 1, x_1 and y are fixed by the elements b, a and ab^{-1} respectively. Thus, the vertices project to the cone points on Σ of orders 3, 2 and 5 respectively. Consequently, $\Sigma = \mathbb{S}^2/I = \Sigma(2,3,5)$.

In order to obtain I^h, we consider an antipodal map on $\mathbb{S}^2$ defined by $i = (1,16)(2,17)$ $(3,18)(4,19)(5,20)(6,14)(7,15)(8,11)(9,12)(10,13)$. It is easy to check that the antipodal map commutes with the elements of I, hence $I \times \mathbb{Z}_2 = \langle a, b, i \mid a^2 = b^3 = (ab^{-1})^5 = i^2 = 1, [a,i] = [b,i] = 1\rangle$. As a result, i induces the reflection map on $\mathbb{S}^2/I = \Sigma(2,3,5)$.

We choose a triangle whose vertices are 1, x_1 and y on the fundamental region $\triangle 12y$. When the map $a(ab^{-1})^3 b^{-1}(ab)i$ is applied on $\triangle 1x_1y$, its image is $\triangle 4x_3y$. Since $\triangle 4x_3y$ is identified to $\triangle 2x_1y$ by $(ab^{-1})^{-2} \in I$, this shows the antipodal map i on $\mathbb{S}^2$ induces the reflection map on $\Sigma(2,3,5)$ at its equator line. Consequently, $\mathbb{S}^2/[I \times \langle i\rangle] = I^h$, where $I \times \langle i\rangle = \mathbb{A}_5 \times \mathbb{Z}_2$. The quotient space $\mathbb{S}^2/[I \times \langle i\rangle]$ is a mirrored disk where cone points of order 2, 3 and 5 are on the mirror.

Note that $\mathbb{S}^2/\langle i\rangle = \mathbb{P}^2$ is the projective plane and the map i commutes with I. Thus, the icosahedral action on $\mathbb{S}^2$ induces the action generated by $\bar{a}$ and $\bar{b}$ on $\mathbb{P}^2$, which is isomorphic to the icosahedral group I. Hence, we obtain $\mathbb{P}^2/\langle \bar{a}, \bar{b}\rangle = I^h$.

We will now describe the $I = \mathbb{A}_5$-action on $\mathbb{P}^2$, and show that it is unique up to conjugation. The front and back of $\mathbb{S}^2$ in Figure 4 describe a fundamental region used to obtain $\mathbb{P}^2 = \mathbb{S}^2/\langle i\rangle$ where i is the antipodal map on $\mathbb{S}^2$. Note that the boundary of each region in the diagram is left invariant and the interior of each region is exchanged under i. The arc $\overline{7,13,8,14,9,15}$ (or $\overline{15,10,11,6,12,7}$ etc) projects to an orientation reversing loop $[\overline{7,13,8,14,9,15}]$ on $\mathbb{P}^2$. The map ab^{-1} leaves the outer most loop containing the arc invariant up to the covering translation i. Thus, the induced map $\overline{ab^{-1}}$ in $\mathbb{P}^2$ leaves this orientation reversing loop invariant. On one hand, a leaves the circle containing vertices 1, 2, 3, 9, 15, 16, 17, 18, 12, 7, 1 in $\mathbb{S}^2$ invariant which double covers an orientation reversing loop on $\mathbb{P}^2$. Note that $\overline{a}$ leaves this orientation reversing loop invariant. However, the orientation reversing loops $[\overline{7,13,8,14,9,15}]$ and $[\overline{3,4,5,6,12,18}] = [\overline{3,4,5,6,9,3}]$ in $\mathbb{P}^2$ are exchanged by $\overline{a}$. Finally, b will induce a map $\overline{b}$ on $\mathbb{P}^2$. One can see this since three orientation reversing loops in $\mathbb{P}^2$, namely $[\overline{7,13,8,14,9,15}]$, $[\overline{2,3,4,10,11,17}] = [\overline{2,3,4,10,8,2}]$ and $[\overline{5,6,12,18,4,5}] = [\overline{5,6,9,3,4,5}]$, are permuted under $\overline{b}$.

Note that although we can find an orientation reversing loop left invariant under $\overline{b}$, no common orientation reversing loop exists which is left invariant by both $\overline{a}$ and $\overline{b}$ since the two maps generate an $\mathbb{A}_5$-action on $\mathbb{P}^2$.

Lemma 3.1. *Let $\mathbb{Z}_2$ be a subgroup of $\pi_1(I^h)$ such that $\mathbb{P}^2 \to I^h$ is the covering corresponding to $\mathbb{Z}_2$. Then $\mathbb{Z}_2 = \langle i\rangle$.*

Proof. We claim that there is only one element of order two in $I^h = \mathbb{A}_5 \times \mathbb{Z}_2$ acting on $\mathbb{S}^2$ which is fixed point free and orientation reversing up to a conjugacy. Notice that all elements in I^h have the form of $a^l b^m i^n$ for some $l, m, n \in \mathbb{Z}$ where $a, b \in \mathbb{A}_5$ and $i \in \mathbb{Z}_2$. Since a corresponding covering space must be regular, the group generated by $a^l b^m i^n$ must be a normal subgroup in I^h. In particular, $a^l b^m$ generates a normal subgroup of $\mathbb{A}_5$ which is impossible unless $l = m = 0$. Therefore, a covering space of the orbifold I^h corresponding to a $\mathbb{Z}_2$ subgroup in $\pi_1(I^h) = \mathbb{A}_5 \times \mathbb{Z}_2$ is $\mathbb{S}^2/\langle i\rangle = \mathbb{P}^2$. Therefore, an $\mathbb{A}_5$-action on $\mathbb{P}^2$ with quotient type I^h is unique up to conjugacy. □

Proposition 3.2. *Let $\varphi\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ be a finite action such that $\mathbb{P}^2/\varphi$ is homeomorphic to I^h. Then $G \simeq \mathbb{A}_5$ and φ is conjugate to the standard action $I = \langle \bar{a}, \bar{b}\rangle$. Moreover, no orientation reversing loop is left invariant by the G-action.*

Proof. The proof is similar to that of Proposition 2.2 and uses the above Lemma 3.1. □

4 Chiral tetrahedral symmetry T and pyritohedral symmetry T^v

We consider the groups $T = \mathbb{A}_4$ and $T^v = T \times \mathbb{Z}_2 = \mathbb{A}_4 \times \mathbb{Z}_2$ acting on the 2-sphere $\mathbb{S}^2$ and describe how T acts on the projective plane.

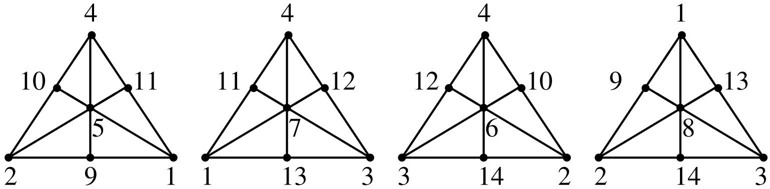


Figure 5: $\mathbb{S}^2$ as a tetrahedron.

We view $\mathbb{S}^2$ as a tetrahedron which has four faces: $\triangle 124$, $\triangle 134$, $\triangle 234$ and $\triangle 123$ (see Figure 5). We add a total of 14 vertices on the faces to triangulate the tetrahedron, and $\triangle 123$ in the Figure 5 illustrates a "bottom" of the tetrahedron.

Consider elements of $\mathbb{S}_{14}$ where $a = (1,2,3)(5,6,7)(9,14,13)(10,12,11)$ and $b = (1,2)(3,4)(5,8)(6,7)(10,13)(11,14)$. These two generators act on the tetrahedron $\mathbb{S}^2$. For instance, a is a $120°$ rotation about the axis passing throught the vertices 4 and 8; and b is a $180°$ rotation about the axis passing throught the vertices 9 and 12. It is easy to see that $ab = (1,3,4)(5,8,6)(9,14,10)(11,13,12)$ is a $120°$ rotation about the axis passing through the vertices 2 and 7. Hence, the two elements a and b generate a group isomorphic to $\mathbb{A}_4$, and we call this group by $T = \langle a, b \mid a^3 = b^2 = (ab)^3 = 1\rangle$, which is the tetrahedral group.

Let Σ be the quotient space of $\mathbb{S}^2$ by the group T. We will observe that the face $\triangle 123$ on the "bottom" of this tetrahedron tiles the rest of its faces. To understand this, we look at the map b which sends from $\triangle 123$ to $\triangle 124$. Then, the map a permutes $\triangle 124$ by $120°$ each time to tile the whole tetrahedron. However, this argument shows that we may choose $\triangle 128$ for a fundamental region for Σ since the map a permutes within the three triangles $\triangle 128$, $\triangle 238$ and $\triangle 318$ on the "bottom" face $\triangle 123$ of $\mathbb{S}^2$.

Notice that b, which has the order two, fixes the vertex 9. Hence, it becomes an exceptional point of order two. Further, b identifies edges $\overline{1,9}$ and $\overline{2,9}$. On the other hand, a fixes the vertex 8, hence this vertex becomes a cone point of order three. Also, a identifies edges $\overline{1,8}$ and $\overline{2,8}$. Moreover, the map ab, which has an order three, fixes the vertex 2 to obtain an additional cone point of order three. Consequently, $\Sigma = \mathbb{S}^2/T = \Sigma(2,3,3)$.

Next, we will discuss how to obtain T^v. An antipodal map defined by $i = (1,6)(2,7)(3,5)(4,8)(9,12)(10,13)(11,14)$ on $\mathbb{S}^2$ commutes with the elements in T, hence we have $T \times \mathbb{Z}_2 = \langle a, b, i \mid a^3 = b^2 = (ab)^3 = i^2 = 1, [a,i] = [b,i] = 1\rangle$ and i induces a map on $\mathbb{S}^2/T = \Sigma(2,3,3)$. However, it requires some work to analyze what map i induces on the orbifold $\Sigma(2,3,3)$.

First, let x and y be the mid-point of the edge $\overline{1,8}$ and $\overline{2,8}$ respectively. Since the points x and y are identified in $\Sigma(2,3,3)$ by $a \in T$, we may view the union of $\overline{x,9}$ and $\overline{9,y}$ as the *vertical equator line* on $\Sigma(2,3,3)$. Notice that the induced map i on $\Sigma(2,3,3)$ fixes all points on the vertical equator line. It can be checked by observing that $a^{-1}(ab)^{-1}(a^2b)i$ fixes the points on the line $\overline{x,9}$; and $(ab)(a^2b)i$ fixes the points on the line $\overline{9,y}$.

Secondly, we will show that the induced map i on $\Sigma(2,3,3)$ is a reflection on the vertical equator line $\overline{x,9} \cup \overline{9,y}$. To see this, consider $\triangle 189$ lying on our fundamental region $\triangle 128$. Apply the map $a(ab)^2 i$ on $\triangle 189$ gives us $\triangle 819$ which is a reflection on $\overline{x, 9}$. Likewise, $\triangle 289$ is reflected on $\overline{9, y}$ by the map $a(ab)^2 bi$ to get $\triangle 829$. As a result, the induced map i on $\Sigma(2,3,3)$ is a reflection at the vertical equator line on the orbifold.

By the argument above, $\mathbb{S}^2/[T \times \langle i \rangle] = T^v$, where $T \times \langle i \rangle = \mathbb{A}_4 \times \mathbb{Z}_2$ and the quotient space $\mathbb{S}^2/[T \times \langle i \rangle]$ is a mirrored disk containing a corner reflector containing one exceptional points of order 2 and 3 on it and one exceptional point of order 3 in its interior.

Recall that the antipodal map i commutes with T on $\mathbb{S}^2$, hence $a, b \in T$ induce maps $\bar{a}$ and $\bar{b}$ on $\mathbb{S}^2/\langle i \rangle = \mathbb{P}^2$. Moreover, $\mathbb{P}^2/\langle \bar{a}, \bar{b} \rangle = \mathbb{S}^2/[T \times \langle i \rangle] = T^v$, where $\langle \bar{a}, \bar{b} \rangle$ is isomorphic to T.

We will now describe the $T = \mathbb{A}_4$-action on $\mathbb{P}^2$ and show that it is unique up to conjugacy.

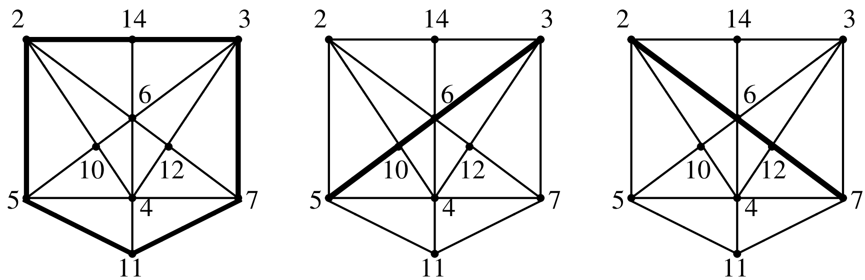


Figure 6: Fundamental region on tetrahedron.

The left diagram in Figure 6 above illustrates a fundamental region on $\mathbb{S}^2$ used to obtain $\mathbb{P}^2$ under the antipodal map i, where $\mathbb{S}^2$ is viewed as in Figure 5 and $\mathbb{S}^2/\langle i \rangle = \mathbb{P}^2$. This can be seen by observing the circle containing vertices, 5, 2, 14, 3, 7, 11 is left invariant by i, and the vertices 4, 10, 6, 12 are sent to 8, 13, 1, 9 respectively. The projective plane is obtained by identifying the opposite side in this polygon.

Recall a and b are generators of the tetrahedral group $T = \mathbb{A}_4$ operating on $\mathbb{S}^2$. Furthermore $i \notin T = \mathbb{A}_4$. Thus T induces an action on $\mathbb{P}^2$ and the elements $a, b \in T$ induce maps $\overline{a}, \overline{b}$ on $\mathbb{P}^2$. Notice that the generator $\overline{a}$ maps the loop $[\overline{5,2,14,3}] = [\overline{5,2}][\overline{2,14}][\overline{14,3}]$ in $\mathbb{P}^2$ onto $[\overline{6,3}][\overline{3,13}][\overline{13,1}] = [\overline{6,3}][\overline{5,10}][\overline{10,6}] = [\overline{6,3,5,10,6}] = [\overline{5,10,6,3}]$, and $\overline{a}^2$ maps this loop onto $[\overline{2,6,12,7}]$. Each image is expressed as a bold line in the Figure 6 above. Thus, the map $\overline{a}$ does not leave this orientation reversing loop invariant in $\mathbb{P}^2$. Likewise, $\overline{ab}$ and $\overline{(ab)}^2$ map the loop $[\overline{5,2,14,3}] = [\overline{2,14,3,7}]$ onto $[\overline{2,10,4,7}]$ and $[\overline{2,6,12,7}]$ respectively. Furthermore, $\overline{b}$ maps the loop $[\overline{5,2,14,3}]$ onto $[\overline{11,4,6,14}]$. The loop consists of vertices 2, 6, 12, 7, 1 and 9 on the tetrahedron $\mathbb{S}^2$ is left invariant by the map $b \in T$ and the covering translation i, hence the arc having vertices 2, 6, 12 and 7 projects to an orientation reversing loop on $\mathbb{P}^2$. There is no orientation reversing loop in $\mathbb{P}^2$ which is left invariant by both $\overline{a}$ and $\overline{b}$.

Lemma 4.1. *Let $\mathbb{Z}_2$ be a subgroup of $\pi_1(T^v)$ such that $\mathbb{P}^2 \to T^v$ is the covering corresponding to $\mathbb{Z}_2$. Then $\mathbb{Z}_2 = \langle i \rangle$.*

Proof. We will show that the orbifold T^v has only one $\mathbb{P}^2$ covering space up to a conjugacy. Notice that $\mathbb{A}_4$ has three elements of order two. These elements are b, aba^{-1} and a^2ba^{-2}, which are all equivalent. Thus, $\mathbb{A}_4 \times \mathbb{Z}_2$ has two conjugacy classes of order two elements which reverse orientation, namely bi and i. Since $bi = (1,7)(2,6)(3,8)(4,5)(9,12)$ fixing vertices 10, 11, 13 and 14, we have a desired conclusion. □

Proposition 4.2. *Let $\varphi\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ be a finite action such that $\mathbb{P}^2/\varphi = T^v$. Then $G \simeq \mathbb{A}_4$ and φ is conjugate to the standard action generated by $\langle \bar{a}, \bar{b}\rangle$. Moreover, no orientation reversing loop is left invariant by the G-action.*

Proof. The proof follows as in Proposition 2.2 and uses Lemma 4.1. □

We remark that [4] contains excellent figures to show us how each element in $\mathbb{A}_4$ acts on a tetrahedron.

5 Achiral tetrahedral symmetry T^h

In Section 1, we have seen the $O = \mathbb{S}_4$-action on $\mathbb{S}^2$ where $\mathbb{S}^2/O$ is $\Sigma(2,3,4)$, which is an orientable orbifold. In this section, we will investigate another $O = \mathbb{S}_4$-action on $\mathbb{S}^2$. However, the resulting quotient space $\mathbb{S}^2/O = T^h$ will be non-orientable this time. More specifically, it will be a mirrored disk which contains two cone points of order three and one cone point of order two on the mirror. Note that we will triangulate $\mathbb{S}^2$ as shown in Section 4 which is a tetrahedron.

First, we will begin by providing generators to define a group isomorphic to $\mathbb{S}_4$. Consider two elements $a = (1,2)(6,7)(10,11)(13,14)$ and $b = (2,4,3)(5,7,8)(9,11,13)$ $(10,12,14)$ in $\mathbb{S}_{14}$. We can see that a is a reflection on the circle containing vertices 4, 5, 9, 8, 3 and 12 in $\mathbb{S}^2$. On the other hand, b is a $120°$ rotation about the axis passing through vertices 1 and 6. It is easy to check $ab = (1,2,4,3)(5,6,7,8)(9,10,12,13)(11,14)$. Although ab reverses an orientation, it is called *improper rotation*. As a result, $\mathbb{S}_4 = \langle a, b \mid a^2 = b^3 = (ab)^4 = 1\rangle$.

Secondly, $\mathbb{A}_4$ is an index two subgroup of $\mathbb{S}_4$ and the subgroup can be expressed by using the two generators for $\mathbb{S}_4$. In order to get a presentation for $\mathbb{A}_4$, consider $(ab)^2 = (1,4)(2,3)(5,7)(6,8)(9,12)(10,13)$ which is a $180°$ rotation about the axis passing through vertices 11 and 14. Then, $b(ab)^2 = (1,3,4)(5,8,6)(9,14,10)(11,13,12)$ is a $120°$ rotation about the axis passing through vertices 2 and 7. Consequently, we obtain a desired subgroup $\mathbb{A}_4 = \langle b, (ab)^2 \mid [(ab)^2]^2 = b^3 = [b(ab)^2]^3 = 1\rangle$.

Thirdly, we will look for a fundamental region for $\mathbb{S}^2/\mathbb{A}_4$. It is easy to compute that the map b permutes $\triangle 134$, $\triangle 123$ and $\triangle 142$. Further, $b(ab)^2$ maps from $\triangle 123$ to $\triangle 432$. Thus, we will look at $\triangle 134$. However, $\triangle 137$ tiles $\triangle 134$ using the element $b(ab)^2$. Then, the vertices 1 and 7 become order 3 cone points since they are fixed by b and $b(ab)^2$ respectively. Thus, we may choose $\triangle 137$ for our fundamental region. Notice that the vertex 11 is fixed under $(ab)^2$, which becomes the order 2 cone point, and it is identified to the vertex 13 by $b(ab)^2 \in \mathbb{A}_4$. Now, $b(ab)^2$ identifies $\overline{1,7}$ and $\overline{3,7}$; $b(ab)^2b^{-1}$ identifies $\overline{1,13}$ and $\overline{3,13}$. As a result, the quotient space $\mathbb{S}^2/\mathbb{A}_4$ is indeed $\Sigma(2,3,3)$.

Finally, we will discuss how to obtain the orbifold T^h. Recall the map $a \in \mathbb{S}_4$ reflects on the circle containing vertices 4, 5, 9, 8, 3 and 12 in $\mathbb{S}^2$. We compose this map by a covering translation $(ab)^{-1}[(ab)^2b]^2(ab) = (1,2,3)(5,6,7)(9,14,13)(10,12,11) \in \mathbb{A}_4$, which is a $120°$ rotation about the axis passing through vertices 4 and 8. Then, $(ab)^{-1}[(ab)^2b]^2(ab)a$ sends the triangle containing vertices 1, 13 and 7 to the triangle containing vertices 3, 7 and 13. Notice that $\overline{1,7}$ and $\overline{3,7}$ are identified in $\Sigma(2,3,3)$. Likewise, $\overline{1,13}$, and $\overline{3,13}$ are identified. Thus, the circle containing vertices 1, 7, 13 becomes the line of reflection under the map induced by a on $\Sigma(2,3,3)$. Consequently, we obtain $\mathbb{S}^2/\mathbb{S}_4 = T^h$.

Unlike the previous orbifolds, T^h is not covered by a projective plane. Notice that $\pi_1(T^h) = \mathbb{S}_4$ contains six elements of order two which are orientation reversing. All of them are a reflection at a plane whose intersection with the tetrahedron is a triangle containing either vertices 2, 11, 3; vertices 1, 12, 2; vertices 3, 10, 1; vertices 4, 9, 3; vertices 4, 13, 2; or vertices 4, 14, 1. Clearly, none of them give a fixed point free action on the tetrahedron $\mathbb{S}^2$, and hence this yields the following lemma.

Lemma 5.1. *The orbifold T^h is not covered by a projective plane.*

6 Cyclic and dihedral actions

We describe the cyclic and dihedral actions on $\mathbb{S}^2$ and the projective plane $\mathbb{P}^2$. In describing these actions, it is convenient to use spherical coordinates. Therefore for any point $(x,y,z) \in \mathbb{S}^2$, we let $x = \sin\phi\cdot\cos\theta$, $y = \sin\phi\cdot\sin\theta$ and $z = \cos\phi$.

We begin by defining a rotation of order m on $\mathbb{S}^2$ as follows:

$$r(x,y,z) = \big(\sin\phi\cdot\cos(\theta+\tfrac{2\pi}{m}),\ \sin\phi\cdot\sin(\theta+\tfrac{2\pi}{m}),\ \cos\phi\big).$$

Note that r fixes only the points $(0,0,1)$ and $(0,0,-1)$.

A spinning map s which rotates through an angle of π about the y-axis is defined by $s(x,y,z) = (-x,y,-z)$. In terms of the spherical coordinate system, the map is defined by

$$s(x,y,z) = \big(\sin(\phi+\pi)\cdot\cos(-\theta),\ \sin(\phi+\pi)\cdot\sin(-\theta),\ \cos(\phi+\pi)\big).$$

One can check that $s\circ r\circ s^{-1} = r^{-1}$, and therefore $\langle r,s\rangle$ generates a dihedral group $\mathrm{Dih}(\mathbb{Z}_m)$ acting on $\mathbb{S}^2$.

Finally we define the antipodal map i on $\mathbb{S}^2$ by $i(x,y,z) = (-x,-y,-z)$. In terms of the spherical coordinate system,

$$i(x,y,z) = \big(\sin(\phi+\pi)\cdot\cos\theta,\ \sin(\phi+\pi)\cdot\sin\theta,\ \cos(\phi+\pi)\big).$$

We have $\mathbb{S}^2/\langle i\rangle = \mathbb{P}^2$. Observe that $i\circ s\circ i^{-1} = s$ and $i\circ r\circ i^{-1} = r$. Hence i commutes with r and s which implies the following lemma:

Lemma 6.1. *The maps r and s induce homeomorphisms $\bar{r}$ and $\bar{s}$ on $\mathbb{P}^2$ respectively.*

Let $k(x,y,z) = \big(\sin\phi\cdot\cos(\theta+\frac{\pi}{m}),\ \sin\phi\cdot\sin(\theta+\frac{\pi}{m}),\ \cos\phi\big)$. A computation shows that $k\circ s\circ k^{-1} = r\circ s$, $k\circ r\circ k^{-1} = r$ and $k\circ i = i\circ k$. This implies that the induced map $\bar{k}$ on $\mathbb{P}^2$ conjugates $\bar{s}$ to $\bar{r}\circ\bar{s}$ and commutes with $\bar{r}$.

Notice that we can express the three maps above in terms of a PL-category. Let $m\in\mathbb{N}$. We assume that vertices from 1 to $2m$ are located on the equator line of $\mathbb{S}^2$. The vertices $2m+1$ and $2m+2$ are on the poles. As a result, we obtain $4m$ many faces (triangles) from these vertices on $\mathbb{S}^2$.

If $m>1$ is odd, then the rotation r is expressed by $r = (1,3,\dots,2m-1)(2,4,\dots,2m)$ whose order is m. On one hand, if m is even, then $r = (1,2,3,\dots,2m)$ whose order is $2m$. In each case, the vertices $2m+1$ and $2m+2$ are fixed under r since they are the north and the south poles. The spinning map for $m>1$ passing through the y-axis is defined by $s = (2,2m)(3,2m-1)\cdots(m,m+2)(2m+1,2m+2)$. The vertices 1 and $m+1$ are fixed under s. The antipodal map for $m > 1$ is defined by $i = (1, m+1)(2, m+2) \cdots (m, 2m)(2m+1, 2m+2)$.

For the case when r has order two, we place vertices 1 to 4 on the equator of $\mathbb{S}^2$ and vertices 5 and 6 are on the poles (see the Figure 2 in Section 2). Then $r = (1,3)(2,4)$, $s = (2,4)(5,6)$, and $i = (1,3)(2,4)(5,6)$. Define a map $j = (1,6,3,5)$. A computation shows that $j \circ s \circ j^{-1} = r$, $j \circ r \circ j^{-1} = s$ and $j \circ i \circ j^{-1} = i$. Therefore $\bar{s}$ is conjugate to $\bar{r}$ on $\mathbb{P}^2$. Summarizing we have the following lemma:

Lemma 6.2. *There exists a homeomorphism $\bar{k}$ on $\mathbb{P}^2$ which conjugates $\bar{s}$ to $\bar{r} \circ \bar{s}$ and commutes with $\bar{r}$. When $\bar{r}$ has order two, there exists a homeomorphism $\bar{j}$ on $\mathbb{P}^2$ which conjugates $\bar{r}$ to $\bar{s}$ and $\bar{s}$ to $\bar{r}$.*

6.1 Quotient types $\Sigma(0, m, m)$, $\Sigma(2, 2, m)$, D^{ν}_m and D^h_m

The space $\Sigma(0, m, m)$ is an orbifold whose underlying space is a 2-sphere with two cone points each of order m. Similarly $\Sigma(2, 2, m)$ is an orbifold whose underlying space is a 2-sphere with three cone points, two of order 2 and one of order m. The orbifold D^{ν}_m is a mirrored disk containing a cone point of order m and 2 on the mirror and its interior respectively. The orbifold D^h_m is a mirrored disk with three cone points on the mirror, one of order m and two of order 2.

Observe that we obtain $\mathbb{S}^2/\langle r\rangle = \Sigma(0, m, m)$, which double covers $\mathbb{S}^2/\langle r, s\rangle = \Sigma(2, 2, m)$. Since i commutes with r and s, we have $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2 = [\langle r\rangle \circ_{-1} \langle s\rangle] \times \langle i\rangle$ acting on $\mathbb{S}^2$. Now r and s acting on $\mathbb{S}^2$ induce a $\mathrm{Dih}(\mathbb{Z}_m)$-action on $\mathbb{S}^2/\langle i\rangle = \mathbb{P}^2$. Furthermore, i operating on $\mathbb{S}^2$ induces an orientation reversing involution $\bar{i}$ on $\Sigma(2, 2, m)$, and we have $\Sigma(2, 2, m)/\langle \bar{i}\rangle = \mathbb{P}^2/\langle \bar{r}, \bar{s}\rangle = \mathbb{S}^2/(\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2)$. Thus the fundamental group of the quotient space $\mathbb{P}^2/\langle \bar{r}, \bar{s}\rangle$ is $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2 = [\langle r\rangle \circ_{-1} \langle s\rangle] \times \langle i\rangle$.

Let $p\colon \mathbb{S}^2 \to \mathbb{S}^2/\langle r, s\rangle = \Sigma(2, 2, m)$ be the orbifold covering map and note that $p(0, 0, 1) = p(0, 0, -1)$ is the cone point of order m. Since $i(0, 0, 1) = (0, 0, -1)$ and $s(0, 0, -1) = (0, 0, 1)$, it follows that $\bar{i}(p(0, 0, 1)) = p(0, 0, 1)$, and thus $\bar{i}$ fixes the cone point of order m in $\Sigma(2, 2, m)$. Hence $\bar{i}$ is a reflection. If m is odd, $r^k(0, 1, 0) \neq (0, -1, 0)$ for any k. Thus $p(0, 1, 0)$ and $p(0, -1, 0)$ are the two distinct cone points of order 2 in $\Sigma(2, 2, m)$. If m is even, then $p(0, 1, 0) = p(0, -1, 0)$ is a cone point of order 2 since $r^{\frac{m}{2}}(0, 1, 0) = (0, -1, 0)$. We will consider the cases m odd and m even separately.

Suppose m is odd. Then since $i(0, 1, 0) = (0, -1, 0)$, it follows that $\bar{i}(p(0, 1, 0)) = p(0, -1, 0)$ and thus $\bar{i}$ exchanges the two cone points of order 2. Since $\bar{i}$ fixes the cone point of order m, it follows that $\mathbb{P}^2/\langle \bar{r}, \bar{s}\rangle = D^{\nu}_m$. The order two elements in $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2$ are: i, s, $r^j si$. One can check that

$$r^j si(x, y, z) = \big(\sin\phi \cdot \cos(-\theta + \tfrac{2\pi j}{m}),\ \sin\phi \cdot \sin(-\theta + \tfrac{2\pi j}{m}),\ \cos\phi\big).$$

By choosing $\phi = 0$ or π, the map fixes the points $(0, 0, \pm 1)$ on $\mathbb{S}^2$. Note that $\Sigma(0, 2, 2) = \mathbb{S}^2/\langle s\rangle$ is not a regular covering space of D^{ν}_m since $\langle s\rangle$ is not a normal subgroup of $\pi_1(D^{\nu}_m) = \mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2$. Thus i is the only orientation reversing element which is fixed-point free. This implies that when m is odd, $\pi_1(D^{\nu}_m)$ has a unique normal $\mathbb{Z}_2$ subgroup generated by a fixed-point free orientation reversing element, and the covering of D^{ν}_m corresponding to this subgroup is $\mathbb{P}^2$.

Next we suppose m is even and show how to obtain D^{ν}_m. Write $m = 2n$ and observe that the rotation r of order $2n$ on $\mathbb{S}^2$ is defined as follows:

$$r(x, y, z) = \big(\sin\phi \cdot \cos(\theta + \tfrac{\pi}{n}),\ \sin\phi \cdot \sin(\theta + \tfrac{\pi}{n}),\ \cos\phi\big).$$

Note that r fixes only the points $(0,0,\pm 1)$, and since $r^n s(1,0,0) = (1,0,0)$ it follows that $p(1,0,0)$ is one of the cone points of order 2 in $\Sigma(2,2,2n)$.

Consider the point $\big(\sin(\frac{\pi}{2})\cdot\cos(\frac{\pi}{2n}),\ \sin(\frac{\pi}{2})\cdot\sin(\frac{\pi}{2n}),\ \cos(\frac{\pi}{2})\big)$. We see that $r^{n+1}s$ fixes $\big(\sin(\frac{\pi}{2})\cdot\cos(\frac{\pi}{2n}),\ \sin(\frac{\pi}{2})\cdot\sin(\frac{\pi}{2n}),\ \cos(\frac{\pi}{2})\big)$, and so it follows that the point $p\big(\big(\sin(\frac{\pi}{2})\cdot\cos(\frac{\pi}{2n}),\ \sin(\frac{\pi}{2})\cdot\sin(\frac{\pi}{2n}),\ \cos(\frac{\pi}{2})\big)\big)$ is the other cone point of order 2 in $\Sigma(2,2,m)$.

Define a reflection $l\colon \mathbb{S}^2 \to \mathbb{S}^2$ by

$$l(x,y,z) = \big(\sin(-\phi)\cdot\cos(-\theta+\tfrac{\pi}{2n}),\ \sin(-\phi)\cdot\sin(-\theta+\tfrac{\pi}{2n}), \cos(-\phi)\big).$$

A calculation shows that $lsl^{-1} = rs$ and $lrl^{-1} = r^{-1}$. Thus we have $\mathrm{Dih}(\mathbb{Z}_{2n})\circ\mathbb{Z}_2 = [\langle r\rangle \circ_{-1} \langle s\rangle]\circ\langle l\rangle$ acting on $\mathbb{S}^2$ and an induced map $\bar{l}$ acting on $\Sigma(2,2,m) = \mathbb{S}^2/\langle r,s\rangle$. A further computation shows that $l(1,0,0) = l(\sin(\frac{\pi}{2})\cdot\cos(0), \sin(\frac{\pi}{2})\cdot\sin(0), \cos(\frac{\pi}{2})) = (\sin(\frac{-\pi}{2})\cdot\cos(\frac{\pi}{2n}), \sin(\frac{-\pi}{2})\cdot\sin(\frac{\pi}{2n}), \cos(\frac{-\pi}{2}))$. Applying r^n to this element, we see that

$$r^n\big(\sin(\tfrac{-\pi}{2})\cdot\cos(\tfrac{\pi}{2n}),\ \sin(\tfrac{-\pi}{2})\cdot\sin(\tfrac{\pi}{2n}),\ \cos(\tfrac{-\pi}{2})\big) = \big(\sin(\tfrac{\pi}{2})\cdot\cos(\tfrac{\pi}{2n}),\ \sin(\tfrac{\pi}{2})\cdot\sin(\tfrac{\pi}{2n}),\ \cos(\tfrac{\pi}{2})\big).$$

Hence the induced map $\bar{l}$ exchanges the two cone points of order two. In addition, consider a set $F \subseteq \mathbb{S}^2$ defined by

$$F = \big\{(\sin\varphi\cdot\cos(\tfrac{\pi}{4n}+\pi),\ \sin\varphi\cdot\sin(\tfrac{\pi}{4n}+\pi),\ \cos\varphi) \mid \varphi\in\mathbb{R}\big\}.$$

Notice that

$$\begin{aligned} l\big(&\sin\varphi\cdot\cos(\tfrac{\pi}{4n}+\pi), \sin\varphi\cdot\sin(\tfrac{\pi}{4n}+\pi), \cos\varphi\big) \\ &= \big(\sin(-\varphi)\cdot\cos(-\tfrac{\pi}{4n}-\pi+\tfrac{\pi}{2n}), \sin(-\varphi)\cdot\sin(-\tfrac{\pi}{4n}-\pi+\tfrac{\pi}{2n}), \cos(-\varphi)\big) \\ &= \big(-\sin\varphi\cdot\cos(\tfrac{\pi}{4n}+\pi), -\sin\varphi\cdot\sin(\tfrac{\pi}{4n}+\pi), \cos(\varphi)\big) \\ &= r^{\frac{m}{2}}\big(\sin\varphi\cdot\cos(\tfrac{\pi}{4n}+\pi), \sin\varphi\cdot\sin(\tfrac{\pi}{4n}+\pi), \cos\varphi\big). \end{aligned}$$

Therefore, $p(F) = \mathrm{fix}\{\bar{l}\}$ in $\Sigma(2,2,m)$ where p denotes the covering map. Consequently, $\bar{l}$ is a reflection exchanging the cone points of order 2. Thus $\Sigma(2,2,m)/\langle\bar{l}\rangle = D^{v}_{2n}$, and $\pi_1(D^{v}_{2n}) = \mathrm{Dih}(\mathbb{Z}_{2n})\circ\mathbb{Z}_2 = [\langle r\rangle\circ_{-1}\langle s\rangle]\circ\langle l\rangle$ where $lsl^{-1} = rs$ and $lrl^{-1} = r^{-1}$. The elements of order two are: r^n, $r^k s$, $r^k l$ (for any integer $k = 0,1,\dots,2n-1$). The only orientation reversing elements of order two are $r^k l$, and they all fix the points $(0,0,1)$ and $(0,0,-1)$. Thus there is no orbifold covering $\mathbb{P}^2 \to D^{v}_{2n}$. We summarize the above in the following theorem.

Theorem 6.3. *Let $\varphi\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ be a finite action such that $\mathbb{P}^2/\varphi = D^{\nu}_m$. Then m is odd, $G \simeq \mathrm{Dih}(\mathbb{Z}_m)$ and φ is conjugate to the standard action generated by $\langle\bar{r},\bar{s}\rangle$.*

Proof. By the above m is odd. Let $\nu\colon \mathbb{P}^2 \to D^{\nu}_m = \mathbb{P}^2/\langle\bar{r},\bar{s}\rangle$ be the covering map corresponding to the standard action. For the action $\varphi\colon G\to\mathrm{Homeo}(\mathbb{P}^2)$ with $\mathbb{P}^2/\varphi = D^{\nu}_m$, let $\mu\colon \mathbb{P}^2 \to \mathbb{P}^2/\varphi$ be the covering map and $h\colon D^{\nu}_m \to \mathbb{P}^2/\varphi$ be a homeomorphism. By the above the subgroup $\mu_*(\pi_1(\mathbb{P}^2))$ in $\pi_1(\mathbb{P}^2/\varphi)$ is unique. Thus h lifts to a homeomorphism $\tilde{h}$ of $\mathbb{P}^2$ such that $h\nu = \mu\tilde{h}$. This implies that the two actions are conjugate by $\tilde{h}$. □

Suppose m is even. Since $r^{\frac{m}{2}}(0,1,0) = (0,-1,0)$ and $i(0,1,0) = (0,-1,0)$, it follows that $\bar{i}$ fixes the cone point $p(0,1,0)$. Since $\bar{i}$ also fixes the cone point of order m, we have that $\bar{i}$ is a reflection leaving each cone point fixed and $\mathbb{P}^2/\langle \bar{r}, \bar{s}\rangle = D^h_m$. The order two elements in $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2$ are: i, s, $r^j si$ or $r^{\frac{m}{2}} i$. Since $r^j si(0,0,1) = (0,0,1)$, we only need to consider

$$r^{\frac{m}{2}} i(x,y,z) = \big(\sin(\phi+\pi)\cdot\cos(\theta+\pi),\ \sin(\phi+\pi)\cdot\sin(\theta+\pi),\ \cos(\phi+\pi)\big).$$

Letting $\phi = \frac{\pi}{2}$ and $\theta = 0$, we see that the point $(1,0,0)$ is fixed by $r^{\frac{m}{2}} i$. Thus D^h_m has a unique $\mathbb{P}^2$ covering up to conjugation. This implies that when m is even, $\pi_1(D^h_m)$ has a unique normal $\mathbb{Z}_2$ subgroup generated by a fixed-point free orientation reversing element, and the covering of D^h_m corresponding to this subgroup is $\mathbb{P}^2$.

We now suppose m is odd and show how to obtain D^h_m. Define a reflection $l_0 \colon \mathbb{S}^2 \to \mathbb{S}^2$ by

$$l_0(x,y,z) = \big(\sin(-\phi)\cdot\cos(-\theta),\ \sin(-\phi)\cdot\sin(-\theta),\ \cos(-\phi)\big) = (-x,y,z).$$

One can check that $l_0 s l_0^{-1} = s$ and $l_0 r l_0^{-1} = r^{-1}$. Hence $\mathrm{Dih}(\mathbb{Z}_m)\circ\mathbb{Z}_2 = [\langle r\rangle \circ_{-1} \langle s\rangle]\circ\langle l_0\rangle$ acting on $\mathbb{S}^2$ and an induced map $\bar{l}_0$ acting on $\Sigma(2,2,m) = \mathbb{S}^2/\langle r, s\rangle$. Clearly l_0 fixes the points $(0,1,0)$ and $(0,-1,0)$. Recall $p(0,1,0) \neq (0,-1,0)$. Hence the induced map $\bar{l}_0$ on $\Sigma(2,2,m)$ is a reflection which fixes each cone point. Thus $\Sigma(2,2,m)/\langle \bar{l}_0\rangle = D^h_m$ and $\pi_1(D^h_m) = \mathrm{Dih}(\mathbb{Z}_m)\circ\mathbb{Z}_2 = [\langle r\rangle \circ_{-1} \langle s\rangle]\circ\langle l_0\rangle$.

The elements of order two are: s, $r^k s$, $r^k l_0$ (any integer $k = 0, 1, \ldots, m-1$), and sl_0. The only orientation reversing elements of order two are $r^k l_0$ and sl_0, but they all have fix-points. Thus there is no orbifold covering $\mathbb{P}^2 \to D^h_m$. We summarize the above in the following theorem whose proof is similar to Theorem 6.3.

Theorem 6.4. *Let $\varphi \colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ be a finite action such that $\mathbb{P}^2/\varphi = D^h_m$. Then m is even, $G \simeq \mathrm{Dih}(\mathbb{Z}_m)$ and φ is conjugate to the standard action generated by $\langle \bar{r}, \bar{s}\rangle$.*

6.2 Quotient types S^{2m} and Z^h_m

We use S^{2m} and Z^h_m to denote a projective plane that has one cone point of order m and a mirrored disk containing an order m cone point in its interior respectively. The orbifold Z^h_0 denotes a mirrored disk without an exceptional point within its interior, and if $m = 1$, then $S^{2(1)} = \mathbb{P}^2$. Recall $\langle r\rangle \times \langle i\rangle = \mathbb{Z}_m \times \mathbb{Z}_2$ acts on $\mathbb{S}^2$. Hence, the involutions in this group are either i, $r^{\frac{m}{2}}$ or $ir^{\frac{m}{2}}$ for an even number m.

If m is even, then $ir^{\frac{m}{2}}(x,y,z) = \big(\sin\phi\cdot\cos\theta,\ \sin\phi\cdot\sin\theta,\ -\cos\phi\big)$. The fixed point set of this map is the circle at the equator on $\mathbb{S}^2$ and occurs when $\phi = \frac{\pi}{2}$. Thus, $\mathbb{S}^2/\langle ir^{\frac{m}{2}}\rangle = \mathbb{S}^2/\mathbb{Z}_2 = Z^h_0$. Furthermore, r on $\mathbb{S}^2$ induces a rotation $\bar{r}$ on Z^h_0 fixing a point not on the mirror, and inducing an action $\widehat{r}$ acts on $\mathbb{P}^2 = \mathbb{S}^2/\langle i\rangle$. In the meantime, i on $\mathbb{S}^2$ induces a reflection $\bar{i}$ on $\Sigma(0,m,m) = \mathbb{S}^2/\langle r\rangle$ since $r^{\frac{m}{2}}(-1,0,0) = (1,0,0)$ and $i(1,0,0) = (-1,0,0)$. As a result, we obtain $Z^h_m = Z^h_0/\langle\bar{r}\rangle = \mathbb{P}^2/\langle\widehat{r}\rangle = \Sigma(0,m,m)/\langle\bar{i}\rangle$ for m is even. Note that $\pi_1(Z^h_m) \simeq \mathbb{Z}_m \times \mathbb{Z}_2$ is generated by r and i, where i is the only fixed-point free orientation reversing element. This implies that when m is even, $\pi_1(Z^h_m)$ has a unique normal $\mathbb{Z}_2$ subgroup generated by a fixed-point free orientation reversing element, and the covering of Z^h_m corresponding to this subgroup is $\mathbb{P}^2$.

We now show how to obtain Z^h_m when m is odd. Let ρ be a homeomorphism of $\mathbb{S}^2$ defined by $\rho(x,y,z) = \big(\sin\phi\cdot\cos\theta,\ \sin\phi\cdot\sin\theta, -\cos\phi\big)$. A computation shows that

ρ and r commute. We obtain an orbifold covering map $\mathbb{S}^2 \to \Sigma(0,m,m) = \mathbb{S}^2/\langle r\rangle$ with ρ inducing a reflection $\overline{\rho}$ on $\Sigma(0,m,m)$. The quotient space $\Sigma(0,m,m)/\langle\overline{\rho}\rangle = Z^h_m$ and $\pi_1(Z^h_m) \simeq \mathbb{Z}_m \times \mathbb{Z}_2$ is generated by r and ρ. Since m is odd, the only element of order 2 in $\pi_1(Z^h_m)$ is ρ which has fixed points. Thus there is no orbifold covering $\mathbb{P}^2 \to Z^h_m$ when m is odd. Consequently, the following theorem is obtained:

Theorem 6.5. *Let $\varphi\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ be a finite action. If $\mathbb{P}^2/\varphi = Z^h_m$, then m is even, $G \simeq \mathbb{Z}_m$ and φ is conjugate to the standard action generated by $\langle\widehat{r}\rangle$.*

If m is odd, we again have r inducing a map $\widehat{r}$ on $\mathbb{P}^2 = \mathbb{S}^2/\langle i\rangle$, and one can check that the induced map $\overline{i}$ on $\Sigma(0,m,m) = \mathbb{S}^2/\langle r\rangle$ is the antipodal map. Consequently, we obtain $S^{2m} = \mathbb{P}^2/\langle\widehat{r}\rangle = \Sigma(0,m,m)/\langle\overline{i}\rangle$. Furthermore $\pi_1(S^{2m}) \simeq \mathbb{Z}_m \times \mathbb{Z}_2$ is generated by r and i, where the only order two fixed-point free orientation reversing element is i. Hence when m is odd, $\pi_1(S^{2m})$ has a unique normal $\mathbb{Z}_2$ subgroup generated by a fixed-point free orientation reversing element, and the covering of S^{2m} corresponding to this subgroup is $\mathbb{P}^2$.

To obtain S^{2m} when m is even, we write $m = 2n$ and define a homeomorphism h of $\mathbb{S}^2$ by $h(x,y,z) = \big(\sin(\phi+\pi)\cdot\cos(\theta+\frac{\pi}{2n}),\ \sin(\phi+\pi)\cdot\sin(\theta+\frac{\pi}{2n}),\ \cos(\phi+\pi)\big)$. Observe that h is a composition of the antipodal map and a rotation through $\pi/2n$, and h generates a $\mathbb{Z}_{2(2n)}$-action on $\mathbb{S}^2$ and $\mathbb{S}^2/\langle h^2\rangle = \Sigma(0,2n,2n)$. It follows that the induced map $\overline{h}$ on $\Sigma(0,2n,2n)$ is the antipodal map and $\Sigma(0,2n,2n)/\langle\overline{h}\rangle = S^{2(2n)}$. Furthermore $\pi_1(S^{2(2n)}) \simeq \mathbb{Z}_{2(2n)}$ is generated by h. The only element of order 2 is h^{2n}, and $h^{2n}(x,y,z) = \big(\sin\phi\cdot\cos(\theta+\pi),\ \sin\phi\cdot\sin(\theta+\pi),\ \cos\phi\big)$ has fixed-points. Thus there is no orbifold covering $\mathbb{P}^2 \to S^{2m}$ when m is even. Summarizing these results we obtain the following theorem:

Theorem 6.6. *Let $\varphi\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ be a finite action. If $\mathbb{P}^2/\varphi = S^{2m}$, then m is odd, $G \simeq \mathbb{Z}_m$ and φ is conjugate to the standard action generated by $\langle\widehat{r}\rangle$.*

6.3 Nonexistence of quotient type $C^{\nu}_{m,m}$

The orbifold $C^{\nu}_{m,m}$ is a mirrored disk with two cone points on the mirror of order m. We will show that the orbifold $C^{\nu}_{m,m}$ is obtained by some covering translations on $\mathbb{S}^2$. Recall the reflection map on the yz-plane defined on $\mathbb{R}^3$ by $l_0(x,y,z) = (-x,y,z)$ and the rotation $r(x,y,z) = \big(\sin\phi\cdot\cos(\theta+\frac{2\pi}{m}),\ \sin\phi\cdot\sin(\theta+\frac{2\pi}{m}),\ \cos\phi\big)$. It is easy to check that $\mathrm{Dih}(\mathbb{Z}_m) = \langle r\rangle \circ_{-1} \langle l_0\rangle$. Then, we obtain $\Sigma(0,m,m) = \mathbb{S}^2/\langle r\rangle$ and the reflection on $\mathbb{S}^2$ induces a reflection $\overline{l_0}$ on $\Sigma(0,m,m)$. As a result, $C^{\nu}_{m,m} = \Sigma(0,m,m)/\langle\overline{l_0}\rangle$ where $\pi_1(C^{\nu}_{m,m}) = \mathrm{Dih}(\mathbb{Z}_m)$. The order two elements in $\pi_1(C^{\nu}_{m,m})$ are $r^j l_0$ for $0 \le j \le m$, or $r^{\frac{m}{2}}$ for m even. A calculation shows that

$$r^j l_0(x,y,z) = \big(\sin(-\phi)\cdot\cos(-\theta+\tfrac{2\pi}{m}),\ \sin(-\phi)\cdot\sin(-\theta+\tfrac{2\pi}{m}),\ \cos(-\phi)\big),$$

which has fixed points at $(0,0,\pm1) \in \mathbb{S}^2$ when $\phi = 0$ or π. Since l_0 and $r^{\frac{m}{2}}$ when m is even, have fixed points, $C^{\nu}_{m,m}$ is not covered by $\mathbb{P}^2$. We therefore have shown the following proposition:

Proposition 6.7. *The projective plane does not cover $C^{\nu}_{m,m}$.*

7 Finite group actions on $\mathbb{P}^2$ and $\mathbb{P}^2 \times I$

In this section, we summarize the above results and classify the finite group actions on $\mathbb{P}^2$ and $\mathbb{P}^2 \times I$.

Theorem 7.1. *Let $\varphi\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ be a finite group action on $\mathbb{P}^2$. Then G is isomorphic to one of the following groups: $\mathbb{S}_4$, $\mathbb{A}_5$, $\mathbb{A}_4$, $\mathbb{Z}_m$ or $\mathrm{Dih}(\mathbb{Z}_m)$. The orbifold quotient $\mathbb{P}^2/\varphi$ is an orbifold homeomorphic to one of the following orbifolds: O^h, I^h, T^v, Z^h_m, S^{2m}, D^v_m or D^h_m. There is only one equivalence class for each group.*

(1) $G \simeq \mathbb{S}_4$ if and only if $\mathbb{P}^2/\varphi = O^h$.

(2) $G \simeq \mathbb{A}_5$ if and only if $\mathbb{P}^2/\varphi = I^h$.

(3) $G \simeq \mathbb{A}_4$ if and only if $\mathbb{P}^2/\varphi = T^v$.

(4) $G \simeq \mathbb{Z}_m$ and m is even if and only if $\mathbb{P}^2/\varphi = Z^h_m$.

(5) $G \simeq \mathbb{Z}_m$ and m is odd if and only if $\mathbb{P}^2/\varphi = S^{2m}$.

(6) $G \simeq \mathrm{Dih}(\mathbb{Z}_m)$ and m odd if and only if $\mathbb{P}^2/\varphi = D^v_m$.

(7) $G \simeq \mathrm{Dih}(\mathbb{Z}_m)$ and m even if and only if $\mathbb{P}^2/\varphi = D^h_m$.

Proof. Let $\varphi\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ be a finite group action. Then $\mathbb{P}^2/\varphi$ is a non-orientable 2-orbifold with positive euler number $\chi(\mathbb{P}^2/\varphi)$. The non-orientable good orbifolds (orbifolds which have manifolds for their universal covering spaces) with positive euler numbers are the following: $C^v_{m,m}$, S^{2m}, Z^h_m, D^h_m, D^v_m, T^h, O^h, I^h and T^v. The result then follows by the above. □

Theorem 7.2. *Let $\varphi\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ be a finite group action. The action $\varphi(G)$ does not leave any orientation reversing loop in $\mathbb{P}^2$ invariant if and only if G is isomorphic to $\mathbb{S}_4$, $\mathbb{A}_5$ or $\mathbb{A}_4$. Furthermore, φ is equivalent to one of these standard actions.*

Proof. This follows from Sections 2 through 6. □

Theorem 7.3. *Let $\varphi\colon G \to \mathrm{Homeo}(\mathbb{P}^2 \times I)$ be a finite group action. If for every $g \in G$ $\varphi(g)(\mathbb{P}^2 \times \{0\}) = \mathbb{P}^2 \times \{0\}$, then G is isomorphic to one of the following groups: $\mathbb{S}_4$, $\mathbb{A}_5$, $\mathbb{A}_4$, $\mathbb{Z}_m$ or $\mathrm{Dih}(\mathbb{Z}_m)$. Furthermore, there is only one equivalence class for each group which is represented by one of the standard actions.*

Proof. By the comment following Theorem 8.1 in [6], we may conjugate $\varphi(G)$ so that it is a product action. This implies that there is a G-action $\varphi_1\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ such that for any $g \in G$, we have $\varphi(g)(z,t) = (\varphi_1(g)(z), t)$. By Theorem 7.1, there exists a homeomorphism k of $\mathbb{P}^2$ such that $k\varphi_1(G)k^{-1}$ is one of the standard actions (1) through (7) listed there. Conjugating this action further by $k \times \mathrm{id}$ proves the result. □

Theorem 7.4. *Let $\varphi\colon G \to \mathrm{Homeo}(\mathbb{P}^2 \times I)$ be a finite group action. Then G is isomorphic to one of the following groups: $\mathbb{S}_4$, $\mathbb{S}_4 \times \mathbb{Z}_2$, $\mathbb{A}_5$, $\mathbb{A}_5 \times \mathbb{Z}_2$, $\mathbb{A}_4$, $\mathbb{A}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_m$, $\mathbb{Z}_m \times \mathbb{Z}_2$, $\mathrm{Dih}(\mathbb{Z}_m)$ or $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2$.*

(1) If G is isomorphic to $\mathbb{S}_4$, then there are two equivalence classes.

(2) If G is isomorphic to either $\mathbb{S}_4 \times \mathbb{Z}_2$, $\mathbb{A}_5$, $\mathbb{A}_5 \times \mathbb{Z}_2$, $\mathbb{A}_4$, $\mathbb{A}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_m \times \mathbb{Z}_2$ or $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2$, then there is one equivalence class.

(3) Suppose G is isomorphic to $\mathbb{Z}_m$. If m is odd, then there is one equivalence class. If m is even, then there are two equivalence classes.

(4) Suppose $G \simeq \mathrm{Dih}(\mathbb{Z}_m)$. If $m = 2$ or if m is odd, then there are two equivalence classes. If $m > 2$ and m is even, there are 3 equivalence classes.

Proof. Again by [6], we may assume $\varphi\colon G \to \mathrm{Homeo}(\mathbb{P}^2 \times I)$ is a product action. Thus there exists a homomorphism $\varphi_1\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ such that for any $g \in G$, $\varphi(g)(z,t) = (\varphi_1(g)(z), t)$ or $\varphi(g)(z,t) = (\varphi_1(g)(z), 1-t)$. There exists a $\mathbb{Z}_2$-action on $\mathbb{P}^2 \times I$ generated by a map R defined by $R(z,t) = (z, 1-t)$.

Suppose first that $\varphi_1\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ is not one-to-one. This implies there exists an element $g_0 \neq 1 \in G$ such that $\varphi_1(g_0)(z) = z$ for all $z \in \mathbb{P}^2$, and so $\varphi(g_0)(z,t) = (z, 1-t)$ or $\varphi(g_0) = R$. Since R commutes with $(\varphi_1(g)(z), t)$ and $(\varphi_1(g)(z), 1-t)$, it follows that g_0 commutes with every element of G. Let $H = \{g \in G \mid \varphi(g)(z,t) = (\varphi_1(g)(z), t)\}$. If $\varphi(g)(z,t) = (\varphi_1(g)(z), 1-t)$, then $\varphi(g^2)(z,t) = (\varphi_1(g^2)(z), t)$ showing $g^2 \in H$. It follows that H is an index two normal subgroup of G, and $G = H \times \mathbb{Z}_2$ where $\mathbb{Z}_2 = \langle g_0 \rangle$. Furthermore $\varphi_1|_H\colon H \to \mathrm{Homeo}(\mathbb{P}^2)$ is one-to-one. By Theorem 7.1, H is isomorphic to $\mathbb{S}_4$, $\mathbb{A}_5$, $\mathbb{A}_4$, $\mathbb{Z}_m$ or $\mathrm{Dih}(\mathbb{Z}_m)$ and conjugate to one of the standard actions. As in Theorem 7.3, we may conjugate $\varphi|_H\colon H \to \mathrm{Homeo}(\mathbb{P}^2 \times I)$ by a homeomorphism $k \times \mathrm{id}$ to a standard action, proving the result in this case.

Suppose $\varphi_1\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ is one-to-one, and hence is a G-action. Note that in this case $R \notin \varphi(G)$. By Theorem 7.1, G is isomorphic to $\mathbb{S}_4$, $\mathbb{A}_5$, $\mathbb{A}_4$, $\mathbb{Z}_m$ or $\mathrm{Dih}(\mathbb{Z}_m)$ and conjugate to one of the standard actions. Thus as above by conjugating $\varphi(G)$, we may assume that φ_1 is one of the standard actions. Suppose $G = \mathbb{S}_4$ and $\varphi_1(G) = \langle \bar{a}, \bar{b} \mid \bar{a}^2 = \bar{b}^3 = (\bar{a}\bar{b})^4 = 1\rangle$. Let A and B be actions on $\mathbb{P}^2 \times I$ defined by $A(z,t) = (\bar{a}(z), t)$ and $B(z,t) = (\bar{b}(z), t)$. If $B \circ R \in \varphi(G)$, then $(B \circ R)^3 = R \in \varphi(G)$, and this would imply that $\varphi_1\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ is not one-to-one. Thus $B \in \varphi(G)$ and we either have $A \in \varphi(G)$ or $A \circ R \in \varphi(G)$. Consequently there are two possibilities $\varphi(G) = \langle A, B\rangle$ or $\varphi(G) = \langle A \circ R, B\rangle$, both isomorphic to $\mathbb{S}_4$. They are not conjugate since the quotient space $(\mathbb{P}^2 \times I)/\langle A, B\rangle$ has two boundary components while the quotient space $(\mathbb{P}^2 \times I)/\langle A \circ R, B\rangle$ has only one boundary component. Suppose $G = \mathbb{A}_5$ and let $\varphi_1(G) = \langle \bar{a}, \bar{b} \mid \bar{a}^2 = \bar{b}^3 = (\bar{a}\bar{b}^{-1})^5 = 1\rangle$. As above we obtain actions A and B on $\mathbb{P}^2 \times I$ defined by $A(z,t) = (\bar{a}(z), t)$ and $B(z,t) = (\bar{b}(z), t)$. We see as in the previous case that $B \circ R \notin \varphi(G)$. Furthermore since $(A \circ R \circ B^{-1})^5 = R$, it follows that $A \circ R \notin \varphi(G)$. Thus $\varphi(G) = \langle A, B\rangle$ with only one equivalence class. The proof is similar for $\mathbb{A}_4$. If $G \simeq \mathbb{Z}_m$, then when m is odd the action is conjugate to $(r \times \mathrm{id})$; and when m is even the action is conjugate to either $(r \times \mathrm{id})$ or $(r \times \mathrm{id}) \circ R$. We now suppose $\varphi_1(G) = \mathrm{Dih}(\mathbb{Z}_m) = \langle \bar{r}, \bar{s}\rangle$. We first suppose m is even. The possible groups for $\varphi(G)$ are: $H_1 = \langle (\bar{r} \times \mathrm{id}), (\bar{s} \times \mathrm{id})\rangle$, $H_2 = \langle (\bar{r} \times \mathrm{id}) \circ R, (\bar{s} \times \mathrm{id})\rangle$, $H_3 = \langle (\bar{r} \times \mathrm{id}), (\bar{s} \times \mathrm{id}) \circ R\rangle$, $H_4 = \langle (\bar{r} \times \mathrm{id}) \circ R, (\bar{s} \times \mathrm{id}) \circ R\rangle$. Clearly, H_1 is not conjugate to any of the other groups since no element of H_1 exchanges the boundary components of $\mathbb{P}^2 \times I$. The element of order two $(\bar{s} \times \mathrm{id})$ in H_2 does not exchange boundary components, however every element of order two in H_3 exchanges boundary components showing H_2 is not conjugate to H_3. Similarly, the element $(\bar{r} \times \mathrm{id})$ of order m in H_3 cannot be conjugate to $(\bar{r} \times \mathrm{id}) \circ R$ in H_4, showing H_3 and H_4 are not conjugate. Notice $H_4 = \langle (\bar{r} \times \mathrm{id}) \circ R, (\bar{s} \times \mathrm{id}) \circ R\rangle = \langle (\bar{r} \times \mathrm{id}) \circ R, (\bar{r}\bar{s} \times \mathrm{id})\rangle$. Using Lemma 6.2, it follows that H_2 is conjugate to H_4, showing there are three equivalence classes when $m > 2$. When $m = 2$, Lemma 6.2 also shows that H_2 and H_3 are conjugate, and so we have only two equivalence classes in this case. When m is odd, the only two possibilities are $\langle (\bar{r} \times \mathrm{id}), (\bar{s} \times \mathrm{id})\rangle$ and $\langle (\bar{r} \times \mathrm{id}), (\bar{s} \times \mathrm{id}) \circ R\rangle$. □

8 Finite actions on twisted I-bundle over $\mathbb{P}^2$

For $\mathbb{S}^2 \times I$, define a fixed-point free orientation preserving involution $\alpha\colon \mathbb{S}^2 \times I \to \mathbb{S}^2 \times I$ by $\alpha(z,t) = (i(z), 1-t)$. The manifold $S^2 \times I/\langle\alpha\rangle = W$ is a *twisted I-bundle over the one-sided projective plane* $\mathbb{P}^2$. Let $\nu\colon \mathbb{S}^2 \times I \to W$ be the covering map and note that $\nu(\mathbb{S}^2 \times \{1/2\}) = \mathbb{P}^2$ is a one-sided projective plane. The levels of W are $\nu(\mathbb{S}^2 \times \{t\})$, and a homeomorphism h of W is *level preserving* if $h(\nu(\mathbb{S}^2 \times \{t\})) = \nu(\mathbb{S}^2 \times \{t\})$. We may view W as the set of equivalence classes $\{[z,t] \mid (z,t)$ is equivalent to $(i(z), 1-t)\}$.

Let $\mathrm{Homeo}(W, \mathbb{P}^2)$ be the group of homeomorphisms which leave the projective plane $\mathbb{P}^2$ invariant. Denote by $\mathrm{Cent}^p(\alpha)$ the subgroup of the centralizer of α which leaves $\mathbb{S}^2 \times \{1/2\}$ invariant and preserves the sides of $\mathbb{S}^2 \times \{1/2\}$. Every homeomorphism which leaves $\mathbb{P}^2$ invariant lifts to two homeomorphisms of $\mathbb{S}^2 \times I$, one of which preserves the sides of $\mathbb{S}^2 \times \{1/2\}$ while the other doesn't. Thus for any homeomorphism $f \in \mathrm{Homeo}(W, \mathbb{P}^2)$ there is a unique lift $\tilde{f} \in \mathrm{Cent}^p(\alpha)$, and we obtain an isomorphism $\mathcal{L}\colon \mathrm{Homeo}(W, \mathbb{P}^2) \to \mathrm{Cent}^p(\alpha)$. Note that since $\nu|_{\mathbb{S}^2 \times \{0\}}\colon \mathbb{S}^2 \times \{0\} \to \partial W$ is a homeomorphism, it follows that f is orientation preserving if and only if $\tilde{f}$ is orientation preserving. We obtain the following proposition.

Proposition 8.1. $\mathcal{L}\colon \mathrm{Homeo}(W, \mathbb{P}^2) \to \mathrm{Cent}^p(\alpha)$ *is an isomorphism.*

There exists a map $R\colon \mathrm{Homeo}(W, \mathbb{P}^2) \to \mathrm{Homeo}(\mathbb{P}^2)$ defined by restricting any homeomorphism to $\mathbb{P}^2$.

Proposition 8.2. *Let* $\varphi\colon G \to \mathrm{Homeo}(W, \mathbb{P}^2)$ *be an effective orientation preserving G-action. Then the restriction* $R\varphi\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ *is an effective G-action.*

Proof. Let $\tilde{\varphi} = \mathcal{L} \circ \varphi\colon G \to \mathrm{Cent}^p(\alpha)$ be an orientation preserving G-action on $\mathbb{S}^2 \times I$. Suppose there exists an element $g \in G$ such that $R\varphi(g) = \mathrm{id}$, and thus $\tilde{\varphi}(g)|_{\mathbb{S}^2 \times \{1/2\}} = \mathrm{id}$ or i. Since $\tilde{\varphi}(g)$ does not reverse the sides of $\mathbb{S}^2 \times \{1/2\}$ and $\tilde{\varphi}(g)$ is orientation preserving, we have that $\tilde{\varphi}(g)|_{\mathbb{S}^2 \times \{1/2\}} = \mathrm{id}$ implying $\tilde{\varphi}(g) = \mathrm{id}$. This implies that $\varphi(g) = \mathrm{id}$ proving the result. □

Remark 8.3. The assumption that the G-action in Proposition 8.2 be orientation preserving is necessary. For if we define an involution ρ of W by $\rho[z,t] = [z, 1-t] = [i(z), t]$, then $R\rho = \mathrm{id}|_{\mathbb{P}^2}$ but $\rho \neq \mathrm{id}$ on W.

Proposition 8.4. *Let* $\varphi_1, \varphi_2\colon G \to \mathrm{Homeo}(W, \mathbb{P}^2)$ *be two orientation preserving G-actions such that* $R\varphi_1$ *and* $R\varphi_1$ *are effective G-actions on* $\mathbb{P}^2$ *with* $R\varphi_1(G) = R\varphi_2(G)$*. Then there exists a homeomorphism* k *of* W *isotopic to the identity such that* $k\varphi_1(G)k^{-1} = \varphi_2(G)$.

Proof. Let $R\varphi_1 = \bar{\varphi}_1$ and $R\varphi_2 = \bar{\varphi}_2$ be the effective G-actions on $\mathbb{P}^2$. Replacing φ_2 by $\varphi_2\bar{\varphi}_2^{-1}\bar{\varphi}_1$, we may assume $\bar{\varphi}_1 = \bar{\varphi}_2$. Let $\mathcal{L}\varphi_1 = \tilde{\varphi}_1$ and $\mathcal{L}\varphi_2 = \tilde{\varphi}_2$. Since φ_1 and φ_2 are both orientation preserving, it follows that $\tilde{\varphi}_1(g)|_{\mathbb{S}^2 \times \{1/2\}} = \tilde{\varphi}_2(g)|_{\mathbb{S}^2 \times \{1/2\}}$ for any $g \in G$.

Consider $\tilde{\varphi}_i(G)|_{\mathbb{S}^2 \times [0,1/2]}\colon \mathbb{S}^2 \times [0,1/2] \to \mathbb{S}^2 \times [0,1/2]$. Again using [1], both these actions are conjugate to product actions, and hence there exist homeomorphisms k_i such that $k_i\tilde{\varphi}_i(G)k_i^{-1}$ is a product action. The conjugating maps k_i may be chosen to be the identity on $\mathbb{S}^2 \times \{1/2\}$. Since both actions agree on $\mathbb{S}^2 \times \{1/2\}$, we have $k_1\tilde{\varphi}_1(G)k_1^{-1} = k_2\tilde{\varphi}_2(G)k_2^{-1}$. Letting $h = k_2^{-1}k_1$, we obtain a homeomorphism

$h\colon \mathbb{S}^2 \times [0,1/2] \to \mathbb{S}^2 \times [0,1/2]$, isotopic to the identity relative to $\mathbb{S}^2 \times \{1/2\}$, such that $h(\tilde{\varphi}_1(G)|_{\mathbb{S}^2\times[0,1/2]})h^{-1} = \tilde{\varphi}_1(G)|_{\mathbb{S}^2\times[0,1/2]}$. Extend h to $\mathbb{S}^2 \times [1/2,1]$ by letting $h|_{\mathbb{S}^2\times[1/2,1]} = (\alpha|_{\mathbb{S}^2\times[0,1/2]}) \circ (h|_{\mathbb{S}^2\times[0,1/2]}) \circ (\alpha|_{\mathbb{S}^2\times[1/2,1]})$ and note that h and α commute. Let $g \in G$ and $z \in \mathbb{S}^2 \times [1/2,1]$. Now we have

$$h\tilde{\varphi}_1(g)h^{-1}(z) = h\tilde{\varphi}_1(g)(\alpha \circ h^{-1}|_{\mathbb{S}^2\times[0,1/2]} \circ \alpha)(z) = \alpha(h\tilde{\varphi}_1(g)h^{-1})(\alpha(z)) = \alpha\tilde{\varphi}_2(g')\alpha(z)$$

for some $g' \in G$. Since $\alpha\tilde{\varphi}_2(g')\alpha(z) = \tilde{\varphi}_2(g')(z)$, we have shown $h\tilde{\varphi}_1(G)h^{-1} = \tilde{\varphi}_2(G)$. Letting $\mathcal{L}^{-1}(h) = k$, we have $h\varphi_1(G)h^{-1} = \varphi_2(G)$ proving the result. □

Remark 8.5. The assumption in Proposition 8.4 that the actions are both orientation preserving is necessary. For example, let $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ be an orientation preserving homeomorphism commuting with i such that $f^{2n} = \mathrm{id}$. Define two $\mathbb{Z}_{2n}$-actions on W by $F[z,t] = [f(z),t]$ and $G[z,t] = [if(z),t]$. Clearly $G|_{\mathbb{P}^2} = F|_{\mathbb{P}^2}$ since i projects to the identity on $\mathbb{P}^2$, however they are not conjugate as F is orientation preserving, and G is orientation reversing.

Corollary 8.6. *Let $\varphi_1, \varphi_2\colon G \to \mathrm{Homeo}(W,\mathbb{P}^2)$ be effective orientation preserving G-actions such that $R\varphi_1(G) = R\varphi_2(G)$. Then there exists a homeomorphism k of W isotopic to the identity such that $k\varphi_1(G)k^{-1} = \varphi_2(G)$.*

Proof. By Proposition 8.2 $R\varphi_1$ and $R\varphi_1$ are effective G-actions on $\mathbb{P}^2$, and so the result follows by Proposition 8.4. □

Proposition 8.7. *Let $\varphi\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ be an effective G-action. Then φ extends to an effective level preserving orientation preserving G-action $\widehat{\varphi}$ on W.*

Proof. By [7] and [8], there exists an action $\tilde{\varphi}\colon G \to \mathrm{Cent}_+(i)$ where $\mathrm{Cent}_+(i)$ consists of orientation preserving elements in the centralizer $\mathrm{Cent}(i)$ of i in $\mathrm{Homeo}(\mathbb{S}^2)$. Define an action $\theta\colon G \to \mathrm{Cent}^p(\alpha)$ by $\theta(g)(x,t) = (\tilde{\varphi}(g)(z),t)$. Then $\mathcal{L}^{-1}\theta\colon G \to \mathrm{Homeo}(W,\mathbb{P}^2)$ is the extension. □

Let $\mathcal{E}(\mathbb{P}^2,G)$ be the set of equivalence classes of effective G-actions on $\mathbb{P}^2$, and let $\mathcal{E}_+(W,G)$ be the set of equivalence classes of effective orientation preserving G-actions on W. Denote by $\mathcal{E}_+((W,\mathbb{P}^2),G)$ the subset of $\mathcal{E}_+(W,G)$ which have a representative that leaves a one-sided projective plane invariant.

Proposition 8.8. *Let $\varphi\colon G \to \mathrm{Homeo}(W)$ be a finite action on W. Then there exists a one-sided projective plane P such that $\varphi(g)(P) = P$ for all $g \in G$.*

Proof. Let $\mathrm{Homeo}(\mathbb{S}^2 \times I, \mathbb{S}^2 \times \{0\})$ be the group of homeomorphisms which leave $\mathbb{S}^2 \times \{0\}$ invariant. There exists an injection $\mathcal{L}_0\colon \mathrm{Homeo}(W) \to \mathrm{Homeo}(\mathbb{S}^2 \times I, \mathbb{S}^2 \times \{0\}) \cap \mathrm{Cent}(\alpha)$ defined by lifting any homeomorphism to a homeomorphism of $\mathbb{S}^2 \times I$ leaving $\mathbb{S}^2 \times \{0\}$ invariant. Letting $\mathcal{L}_0\varphi = \tilde{\varphi}\colon G \to \mathrm{Homeo}(\mathbb{S}^2 \times I, \mathbb{S}^2 \times \{0\}) \cap \mathrm{Cent}(\alpha)$, we obtain a $G \times \mathbb{Z}_2$ action on $\mathbb{S}^2 \times I$ where the $\mathbb{Z}_2$-action is generated by α, which projects to the φ-action on W. This action is equivalent to a product action by [1], and thus there is an $G \times \mathbb{Z}_2$-invariant 2-sphere S in $\mathrm{int}(\mathbb{S}^2 \times I)$. Furthermore, $\alpha(S) = S$ and $\nu(S)$ is a $\varphi(G)$-invariant projective plane in W. □

Corollary 8.9. $\mathcal{E}_+((W,\mathbb{P}^2),G) = \mathcal{E}_+(W,G)$.

Proposition 8.10. *Let P be a one-sided projective plane in W. Then there exists a homeomorphism k of W, isotopic to the identity, such that $k(P) = \mathbb{P}^2$.*

Proof. Isotope P to intersect $\mathbb{P}^2$ in simple closed curves. We may assume the number of curves in $P \cap \mathbb{P}^2$ is minimal. We will show that the number of simple closed curves in $P \cap \mathbb{P}^2$ is one. Note first that $P \cap \mathbb{P}^2 \neq \emptyset$, for otherwise $P \subset W - \mathbb{P}^2$ which is isomorphic to $\mathbb{S}^2 \times [0,1)$, and this is impossible. If the number of intersections of $P \cap \mathbb{P}^2$ exceeds one, and hence the number of simple closed curves in P exceeds one, then there is a simple closed curve $\delta \in P \cap \mathbb{P}^2$ which bounds a disk Δ in P. We may assume Δ is innermost in P, in the sense that $\operatorname{int}(\Delta) \cap \mathbb{P}^2 = \emptyset$. Since δ bounds a disk in W, it follows that δ is an orientation preserving loop in $\mathbb{P}^2$, and thus bounds a disk D in $\mathbb{P}^2$. Now $D \cup \Delta$ is a separating 2-sphere. If $D \cup \Delta$ bounds a ball, then we may isotope P to eliminate δ.

We therefore assume $D \cup \Delta$ does not bound a ball, and is therefore parallel to the sphere boundary $\partial W = \nu(\mathbb{S}^2 \times \{0\})$. Lift P to an α-invariant 2-sphere S in $\mathbb{S}^2 \times I$, let Δ_1 and Δ_2 be the two lifts of Δ in S, and let D_1 and D_2 be the two lifts of D in $\mathbb{S}^2 \times \{1/2\}$. Denote $\partial\Delta_i$ by δ_i. We may assume $D_1 \cup \Delta_1 \subset \mathbb{S}^2 \times [0, 1/2]$ and $D_2 \cup \Delta_2 \subset \mathbb{S}^2 \times [1/2, 1]$. Furthermore, there is an α-invariant simple closed curve $\gamma \in S \cap \mathbb{S}^2 \times \{1/2\}$, separating δ_1 and δ_2. Note that $(\mathbb{S}^2 \times \{1/2\} - \operatorname{int}(D_1))$ is a disk in $\mathbb{S}^2 \times \{1/2\}$ whose boundary is the boundary of the disk Δ_1 in S. Now $D_1 \cup \Delta_1$ is parallel to $\mathbb{S}^2 \times \{0\}$, which implies that $\Delta_1 \cup (\mathbb{S}^2 \times \{1/2\} - \operatorname{int}(D_1))$ bounds a ball in $\mathbb{S}^2 \times [0, 1/2]$. Thus we may construct an α-equivariant isotopy, relative to γ, which eliminates the intersections δ_1 and δ_2. Projecting this isotopy to W eliminates the δ-intersection of $P \cap \mathbb{P}^2$.

Thus we have shown that $S \cap \mathbb{S}^2 \times \{1/2\}$ is a single simple closed curve γ which projects to a non-contractable simple closed curve $\overline{\gamma}$ in $P \cap \mathbb{P}^2$. By an argument similar to the one above, there is an α-equivariant isotopy, relative to γ, which isotopes S to $\mathbb{S}^2 \times \{1/2\}$. Projecting this isotopy to W, we obtain an isotopy taking P to $\mathbb{P}^2$. □

Theorem 8.11. *The map $\Gamma\colon \mathcal{E}(\mathbb{P}^2, G) \to \mathcal{E}_+((W, \mathbb{P}^2), G)$ defined by extending G-actions from $\mathbb{P}^2$ to W is a bijection.*

Proof. Let $[\varphi] \in \mathcal{E}(\mathbb{P}^2, G)$. By Proposition 8.7, φ can be extended to a G-action $\widehat{\varphi}$ on W. Define $\Gamma([\varphi]) = [\widehat{\varphi}]$. Suppose ψ is a G-action on $\mathbb{P}^2$ such that $[\psi] = [\varphi] \in \mathcal{E}(\mathbb{P}^2, G)$. Then there exists a homeomorphism h of $\mathbb{P}^2$ such that $h\varphi(G)h^{-1} = \psi(G)$. Lift h to an orientation preserving homeomorphism k of $\mathbb{S}^2$ and note that $ik = ki$. Extend k to a homeomorphism $\widehat{k}$ by $\widehat{k}(x,t) = (k(x), t)$. Letting $\widehat{h} = \mathcal{L}^{-1}\widehat{k}$. we see that $\widehat{h}$ is an extension of h. Since $R(\widehat{h}\widehat{\varphi}(G)\widehat{h}^{-1}) = h\varphi(G)h^{-1} = \psi(G) = R(\widehat{\psi}(G))$, it follows by Proposition 8.4 that $[\widehat{\varphi}] = [\widehat{\psi}]$, and thus Γ is well defined.

Let $[\delta] \in \mathcal{E}((W, \mathbb{P}^2), G)$. Thus there is a one-sided projective plane P such that $\delta(g)(P) = P$ for all $g \in G$. By Proposition 8.10, there exists a homeomorphism of W taking P to $\mathbb{P}^2$. This implies that we may choose a representative δ' in $[\delta]$ such that $\delta'(g)(\mathbb{P}^2) = \mathbb{P}^2$ for all $g \in G$. By Proposition 8.2, the restriction $R\delta'$ is an effective G-action on $\mathbb{P}^2$ and therefore represents an element in $\mathcal{E}(\mathbb{P}^2, G)$. Let $\Gamma([R\delta']) = [\widehat{R\delta'}]$. Since $R(\widehat{R\delta'}) = R\delta'$, it follows by Proposition 8.4 that $[\widehat{R\delta'}] = [\delta']$, and thus $\Gamma([R\delta']) = [\delta']$ showing Γ is a surjection.

To show Γ is one-to-one, suppose that $[\varphi], [\theta] \in \mathcal{E}(\mathbb{P}^2, G)$ are such that their level preserving extensions $[\widehat{\varphi}] = [\widehat{\theta}]$ in $\mathcal{E}_+((W, \mathbb{P}^2), G)$. Now W/φ and W/θ are homeomorphic twisted I-bundle orbifolds over one of the following 2-orbifolds: O^h, I^h, T^v, Z_m^h, S^{2m}, D_m^v or D_m^h. Since by Theorem 7.1, there is only one equivalence class for each action on $\mathbb{P}^2$ which determines a unique quotient type, it follows that $[\varphi] = [\theta]$ showing Γ is one-to-one. □

Corollary 8.12. *Let $\varphi\colon G \to \mathrm{Homeo}(W)$ be a finite orientation preserving G-action on W. Then G is isomorphic to one of the following groups: $\mathbb{S}_4$, $\mathbb{A}_5$, $\mathbb{A}_4$, $\mathbb{Z}_m$ or $\mathrm{Dih}(\mathbb{Z}_m)$. The orbifold quotient for each action is a twisted I-bundle orbifold over the following 2-orbifolds: O^h (for $\mathbb{S}_4$) , I^h (for $\mathbb{A}_5$), T^v (for $\mathbb{A}_4$), Z^h_m (for $\mathbb{Z}_m$ and m even), S^{2m} (for $\mathbb{Z}_m$ and m odd), D^v_m (for $\mathrm{Dih}(\mathbb{Z}_m)$ and m odd) and D^h_m (for $\mathrm{Dih}(\mathbb{Z}_m)$ and m even). There is one equivalence class for each quotient type.*

9 Orientation reversing finite actions on twisted I-bundle over $\mathbb{P}^2$

Recall that $W = \{[z,t] \mid (z,t) \text{ is equivalent to } (i(z), 1-t)\}$ with $\mathbb{P}^2 = \{[z, 1/2] \in W\}$, and $\mathcal{L}\colon \mathrm{Homeo}(W, \mathbb{P}^2) \to \mathrm{Cent}^p(\alpha)$ is an isomorphism. Let f_1 be a homeomorphism of $\mathbb{P}^2$, and let $\tilde{f}_1$ be a lift of f_1 to $\mathbb{S}^2$. We remark that $\tilde{f}_1$ commutes with i. A homeomorphism $f\colon W \to W$ is a *product homeomorphism* if $f[z,t] = [\tilde{f}_1(z), t]$. Note that $f|_{\mathbb{P}^2} = f_1$. Let $\mathrm{Homeo}(\mathbb{S}^2 \times I, \mathbb{S}^2 \times \{1/2\})$ be the group of homeomorphisms of $\mathbb{S}^2 \times I$ which leave $\mathbb{S}^2 \times \{1/2\}$ invariant. Define the map $\tilde{R}\colon \mathrm{Homeo}(\mathbb{S}^2 \times I, \mathbb{S}^2 \times \{1/2\}) \to \mathrm{Homeo}(\mathbb{S}^2)$ by restricting any homeomorphism to $\mathbb{S}^2 \times \{1/2\}$.

Lemma 9.1. *Let $\varphi\colon G \to \mathrm{Homeo}(W, \mathbb{P}^2)$ be an effective G-action and let $\tilde{\varphi} = \mathcal{L}\varphi\colon G \to \mathrm{Cent}^p(\alpha) \subset \mathrm{Homeo}(\mathbb{S}^2 \times I, \mathbb{S}^2 \times \{1/2\})$. Then $\tilde{R}\tilde{\varphi}\colon G \to \mathrm{Homeo}(\mathbb{S}^2)$ is an effective G-action.*

Proof. Suppose there exists an element $g \in G$ such that $\tilde{R}\tilde{\varphi}(g) = id|_{\mathbb{S}^2\times\{1/2\}}$. Since $\tilde{R}\tilde{\varphi}(g)$ does not reverse the sides of $\mathbb{S}^2 \times \{1/2\}$, it follows that $\tilde{\varphi}(g) = \mathrm{id}$. □

Remark 9.2. Note that the involution ρ of W defined by $\rho[z,t] = [i(z), t] = [z, 1-t]$, has the property that $R\rho = id|_{\mathbb{P}^2}$, but $\tilde{R}\mathcal{L}\rho(z,t) = (i(z), t)$ and thus does not restrict to the identity on $\mathbb{S}^2 \times \{1/2\}$.

Theorem 9.3. *Let $\varphi\colon G \to \mathrm{Homeo}(W)$ be an effective G-action. Then φ is conjugate to a product action on W.*

Proof. By Propositions 8.8 and 8.10, we may assume $\varphi(g)(\mathbb{P}^2) = \mathbb{P}^2$ for every $g \in G$. Let $\tilde{\varphi} = \mathcal{L}\varphi\colon G \to \mathrm{Cent}^p(\alpha)$. By Lemma 9.1, $\tilde{R}\tilde{\varphi}\colon G \to \mathrm{Homeo}(\mathbb{S}^2)$ is an effective G-action which commutes with i. Define an action $\tilde{\theta}\colon G \to \mathrm{Cent}^p(\alpha) \subset \mathrm{Homeo}(\mathbb{S}^2 \times I, \mathbb{S}^2 \times \{1/2\})$ by $\tilde{\theta}(g) = \tilde{R}\tilde{\varphi}(g) \times id$. Thus $\tilde{\theta}(g)|_{\mathbb{S}^2\times\{1/2\}} = \tilde{\varphi}(g)|_{\mathbb{S}^2\times\{1/2\}}$ for any $g \in G$. Projecting this action to W, we obtain an effective product action $\theta\colon G \to \mathrm{Homeo}(W, \mathbb{P}^2)$. We now use the proof in Proposition 8.4 to construct a homeomorphism h which commutes with α and conjugates $\tilde{\theta}(G)$ to $\tilde{\varphi}(G)$. The homeomorphism h projects to a homeomorphism of W which conjugates $\theta(G)$ to $\varphi(G)$, thus completing the proof. □

We will now define the *standard actions* $\mathbb{S}_4 \times \mathbb{Z}_2$, $\mathbb{A}_4 \times \mathbb{Z}_2$, $\mathbb{A}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_m \times \mathbb{Z}_2$ or $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2$ on W. Consider first the group $\mathbb{S}_4 = \langle a, b \mid a^2 = b^3 = (ab)^4 = 1\rangle$ acting on $\mathbb{S}^2$ commuting with i, and its projection $\mathbb{S}_4 = \langle \bar{a}, \bar{b} \mid \bar{a}^2 = \bar{b}^3 = (\bar{a}\bar{b})^4 = 1\rangle$ to $\mathbb{P}^2$. Define the product maps $A, B\colon W \to W$ by $A[z,t] = [a(z), t]$ and $B[z,t] = [b(z), t]$. Note that $\langle A, B, \rho\rangle = \mathbb{S}_4 \times \mathbb{Z}_2$. The other standard group actions on W are defined in a similar fashion.

Theorem 9.4. *Let $\varphi\colon G \to \mathrm{Homeo}(W)$ be an orientation reversing G-action. Then G is isomorphic to one of the following groups: $\mathbb{S}_4$, $\mathbb{Z}_m$ with m even, $\mathrm{Dih}(\mathbb{Z}_m)$, $\mathbb{S}_4 \times \mathbb{Z}_2$, $\mathbb{A}_5 \times \mathbb{Z}_2$, $\mathbb{A}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_m \times \mathbb{Z}_2$ or $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2$.*

(1) If G is either $\mathbb{S}_4$, $\mathbb{S}_4 \times \mathbb{Z}_2$, $\mathbb{A}_5 \times \mathbb{Z}_2$, $\mathbb{A}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_m \times \mathbb{Z}_2$ with m even or $\mathrm{Dih}(\mathbb{Z}_m)$ with m odd, there is only one equivalence class.

(2) If G is $\mathbb{Z}_m$ with $m > 2$ even and $m/2$ odd, then there are two equivalence classes of $\mathbb{Z}_m = \mathbb{Z}_{m/2} \times \mathbb{Z}_2$-actions on W.

(3) If G is $\mathbb{Z}_m$ with either $m/2$ even or $m = 2$, then there is only one equivalence class.

(4) If G is $\mathrm{Dih}(\mathbb{Z}_m)$ with $m > 2$ and $m/2$ even, there are two equivalence classes of $\mathrm{Dih}(\mathbb{Z}_m)$-actions on W.

(5) If G is $\mathrm{Dih}(\mathbb{Z}_m)$ with $m > 2$ and $m/2$ odd, there are three equivalence classes of $\mathrm{Dih}(\mathbb{Z}_m)$-actions on W.

(6) If G is $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2$ with m even, there is only one equivalence class.

(7) If G is $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2$ with m odd, then $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2 \simeq \mathrm{Dih}(\mathbb{Z}_{2m})$ and there are three equivalence classes of $\mathrm{Dih}(\mathbb{Z}_{2m})$-actions on W.

Proof. Let $\varphi\colon G \to \mathrm{Homeo}(W)$ be an effective orientation reversing G-action. We may assume by Theorem 9.3, that there exists G-actions $\tilde{\varphi}_1\colon G \to \mathrm{Homeo}(\mathbb{S}^2)$ and $\varphi_1\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$, such that $\varphi(g)[z,t] = [\tilde{\varphi}_1(g)(z), t]$ and $\tilde{\varphi}_1(g)$ is a lift of $\varphi_1(g)$. Note that $\tilde{\varphi}_1\colon G \to \mathrm{Homeo}(\mathbb{S}^2)$ is an effective orientation reversing G-action on $\mathbb{S}^2$ by Lemma 9.1.

Suppose that $\varphi_1\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ is not an effective G-action, and so there exists an element $g_0 \neq 1 \in G$ such that $\varphi_1(g_0) = \mathrm{id}\colon \mathbb{P}^2 \to \mathbb{P}^2$. Since $\tilde{\varphi}_1(g_0)$ is a lift of $\varphi_1(g_0)$ and $\tilde{\varphi}$ is an effective action, we see that $\tilde{\varphi}_1(g_0)(z) = i(z)$ and $\varphi(g_0)[z,t] = [i(z),t] = [z, 1-t]$. Thus $\varphi(g_0) = \rho \in \varphi(G)$, and note that ρ commutes with every element in $\varphi(G)$. Let $H = \{g \in G \mid \varphi(g) \text{ is orientation preserving}\}$, and observe that $G = H \times \langle g_0 \rangle$. Since $(\varphi|_H)\colon H \to \mathrm{Homeo}(W, \mathbb{P}^2)$ is an effective orientation preserving action, it follows by Proposition 8.2 that $(\varphi_1|_H)\colon H \to \mathrm{Homeo}(\mathbb{P}^2)$ is an effective action. By Theorem 7.1, there exists a homeomorphism $k_1\colon \mathbb{P}^2 \to \mathbb{P}^2$ such that $k_1\varphi_1(H)k_1^{-1}$ is one of the standard actions $\mathbb{S}_4$, $\mathbb{A}_5$, $\mathbb{A}_4$, $\mathbb{Z}_m$ or $\mathrm{Dih}(\mathbb{Z}_m)$ on $\mathbb{P}^2$. Lifting k_1 to a homeomorphism $\tilde{k}_1\colon \mathbb{S}^2 \to \mathbb{S}^2$, we see that $\tilde{k}_1\tilde{\varphi}_1(H)\tilde{k}_1$ is the same standard action $\mathbb{S}_4$, $\mathbb{A}_5$, $\mathbb{A}_4$, $\mathbb{Z}_m$ or $\mathrm{Dih}(\mathbb{Z}_m)$ on $\mathbb{S}^2$. Define a homeomorphism $k\colon W \to W$ by $k[z,t] = [\tilde{k}_1(z), t]$. Since $\rho[z,t] = [z, 1-t]$, it follows that $k\rho k^{-1} = \rho$. Therefore $k\varphi(G)k^{-1}$ is one of the standard actions: $\mathbb{S}_4 \times \mathbb{Z}_2$, $\mathbb{A}_5 \times \mathbb{Z}_2$, $\mathbb{A}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_m \times \mathbb{Z}_2$ or $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2$, where the $\mathbb{Z}_2$ group is generated by ρ.

Assume first that $\varphi_1\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ is an effective G-action on $\mathbb{P}^2$. Hence by Theorem 7.1, $\varphi_1(G)$ is conjugate to one of the following standard actions on $\mathbb{P}^2$: $\mathbb{S}_4$, $\mathbb{A}_5$, $\mathbb{A}_4$, $\mathbb{Z}_m$ or $\mathrm{Dih}(\mathbb{Z}_m)$.

Suppose there exists a homeomorphism k_1 of $\mathbb{P}^2$ such that $k_1\varphi_1(G)k^{-1} = \langle \bar{a}, \bar{b} \mid \bar{a}^2 = \bar{b}^3 = (\bar{a}\bar{b})^4 = 1\rangle = \mathbb{S}_4$. Let $\tilde{k}_1$ be a lift of k_1 to $\mathbb{S}^2$, and note that $\tilde{k}_1\tilde{\varphi}_1(G)\tilde{k}_1^{-1}$ projects to $\langle \bar{a}, \bar{b}\rangle$, and is therefore one of the following groups: $\langle ai, b\rangle$, $\langle a, bi\rangle$ or $\langle ai, bi\rangle$. The group $\langle a, b\rangle$ is not in the list, since it is orientation preserving on $\mathbb{S}^2$. Since $(bi)^3 = i$, it follows that $\langle a, bi\rangle = \langle ai, bi\rangle = \langle a, b\rangle \times \langle i\rangle = \mathbb{S}_4 \times \mathbb{Z}_2$, and hence must be excluded. Thus, $\tilde{k}_1\tilde{\varphi}_1(G)\tilde{k}_1^{-1} = \langle ai, b\rangle$. Define a homeomorphism k of W by $k[z,t] = [\tilde{k}_1(z), t]$, and note that $k\varphi(G)k^{-1} = \langle A\rho, B\rangle \simeq \mathbb{S}_4$.

We now suppose there exists a homeomorphism k_1 of $\mathbb{P}^2$ such that $k_1\varphi_1(G)k^{-1} = \langle \bar{a}, \bar{b} \mid \bar{a}^2 = \bar{b}^3 = (\bar{a}\bar{b})^5 = 1\rangle = \mathbb{A}_5$. As in the previous paragraph, there exists a lift $\tilde{k}_1$

such that $\tilde{k}_1\tilde{\varphi}_1(G)\tilde{k}_1^{-1}$ is one of the following groups: $\langle ai, b\rangle$, $\langle a, bi\rangle$ or $\langle ai, bi\rangle$. We see that $(bi)^3 = i$ and $(aib)^5 = i$. This implies $\langle ai, b\rangle = \langle a, bi\rangle = \langle ai, bi\rangle = \mathbb{A}_5 \times \mathbb{Z}_2$, and therefore must be excluded. A similar argument shows that $\varphi_1(G)$ cannot be conjugate to $\mathbb{A}_4$.

Next, assume that G is isomorphic to $\mathbb{Z}_m$, and so we have a homeomorphism k_1 of $\mathbb{P}^2$ such that $k_1\varphi_1(G)k_1^{-1} = \langle\bar{r}\rangle$. For the lifted homeomorphism $\tilde{k}_1$ on $\mathbb{S}^2$, we have $\tilde{k}_1\tilde{\varphi}_1(G)\tilde{k}_1 = \langle ri\rangle$. However if m is odd, then $(ri)^m = i$, implying G isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_2$, which is a contradiction. Thus m is even. Let k be a homeomorphism of W defined by $k[z,t] = [\tilde{k}_1(z), t]$. Observe that $k\varphi(G)k^{-1} = \langle R\rho\rangle$ where $R[z,t] = [r(z), t]$ and $R^m = \mathrm{id}$. If $m/2$ is odd and greater than one, then $\mathbb{Z}_m = \mathbb{Z}_{m/2} \times \mathbb{Z}_2 = \langle R^2\rangle \times \langle R^{m/2}\rho\rangle$, and note that ρ is not an element of this group. Thus when $m/2$ is odd, there are two non-equivalent $\mathbb{Z}_m$-actions on W. They are $\langle R^2\rangle \times \langle R^{m/2}\rho\rangle$ and $\langle R^2\rangle \times \langle\rho\rangle$, the first in which no element restricts to the identity on $\mathbb{P}^2$, and the second that has an element which restricts to the identity on $\mathbb{P}^2$. If either $m = 2$ or $m/2$ is even, there is only one equivalence class.

Finally, we assume G is isomorphic to $\mathrm{Dih}(\mathbb{Z}_m)$. Again we have a homeomorphism k_1 of $\mathbb{P}^2$ such that $k_1\varphi_1(G)k_1^{-1} = \langle\bar{r}\rangle \circ_{-1} \langle\bar{s}\rangle$, and its lift $\tilde{k}_1$ such that $\tilde{k}_1\tilde{\varphi}_1(G)\tilde{k}_1^{-1}$ is one of the following groups: $\langle ri\rangle \circ_{-1} \langle s\rangle = H_1$, $\langle r\rangle \circ_{-1} \langle si\rangle = H_2$, $\langle ri\rangle \circ_{-1} \langle si\rangle = H_3$. Consider first the case when m is odd. Since $(ri)^m = i$, we obtain $H_1 = H_3 = \mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2$, and so these cases are excluded. Therefore we only consider H_2. Likewise $\varphi(G)$ is conjugate to the group $\langle R, S\rho\rangle$ where $S[z,t] = [s(z), t]$, and there is one equivalence class. Suppose that m is even. There exists a homeomorphism k of $\mathbb{S}^2$ commuting with r and i such that $ksk^{-1} = rs$ (see Section 6 after Lemma 6.1). Therefore for ri and si in H_3, $krik^{-1} = ri \in H_1$ and $ksik^{-1} = rsi \in H_1$, showing H_3 is conjugate to H_1. If $m = 2$, there exists a homeomorphism j of $\mathbb{S}^2$ commuting with i such that $jrj^{-1} = s$ and $jsj^{-1} = r$ (see Section 6 before Lemma 6.2). This implies that we may conjugate $\varphi(G)$ to either $\langle R\rho, S\rangle$ or $\langle R, S\rho\rangle$ when $m > 2$, or to $\langle R\rho, S\rangle$ when $m = 2$. If $m > 2$, then any generator of $\mathbb{Z}_m$ in $\langle R\rho, S\rangle$ is an odd power of $R\rho$ relatively prime to m, and thus orientation reversing. On the other hand, any generator of $\mathbb{Z}_m$ in $\langle R, S\rho\rangle$ is orientation preserving. Hence these groups cannot be conjugate. This implies that if $m/2$ is even, there are two equivalence classes of $\mathrm{Dih}(\mathbb{Z}_m)$-actions.

We note that when $m/2$ is odd and not equal to one, there are three equivalence classes of $\mathrm{Dih}(\mathbb{Z}_m)$-actions. They are $\langle R\rho\rangle \circ_{-1} \langle S\rangle$, $\langle R\rangle \circ_{-1} \langle S\rho\rangle$ and $\langle R^2\rho\rangle \circ_{-1} \langle S\rangle$. The last group has an element $(R^2\rho)^{m/2} = \rho$ restricting to the identity on $\mathbb{P}^2$, and the group may be viewed as $(\langle R^2\rangle \circ_{-1} \langle S\rangle) \times \langle\rho\rangle = \mathrm{Dih}(\mathbb{Z}_{m/2}) \times \mathbb{Z}_2$.This group was identified in the second paragraph of this proof when we assumed $\varphi_1\colon G \to \mathrm{Homeo}(\mathbb{P}^2)$ was not an effective G-action.

The proof is completed by noting that if G is isomorphic to $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2$ and m is odd, then $\mathrm{Dih}(\mathbb{Z}_m) \times \mathbb{Z}_2$ is isomorphic to $\mathrm{Dih}(\mathbb{Z}_{2m})$, and this case has already been dealt with. □

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A combinatorial problem and numerical semigroups*

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Abstract

Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be two n-tuples of positive integers, let X be a set of positive integers, and let g be a positive integer. In this work we show an algorithmic process in order to compute all the sets C of positive integers that fulfill the following conditions:

1. The cardinality of C is equal to g;
2. If $x, y \in \mathbb{N} \setminus \{0\}$ and $x + y \in C$, then $C \cap \{x, y\} \neq \emptyset$;
3. If $x \in C$ and $\frac{x-b_i}{a_i} \in \mathbb{N} \setminus \{0\}$ for some $i \in \{1, \dots, n\}$, then $\frac{x-b_i}{a_i} \in C$;
4. $X \cap C = \emptyset$.

Keywords: Combinatorial problems, numerical semigroups, Frobenius varieties, Frobenius pseudo-varieties.

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1 Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of integers and non-negative integers, respectively. Let us suppose that we want to compute a set $C \subset \mathbb{N}$ of six elements such that the following conditions are fulfilled.

(C1) If x, y are positive integers such that $x + y \in C$, then $C \cap \{x, y\} \neq \emptyset$.

(C2) If $x \in C$ and $x - 4$ is a positive integer, then $x - 4 \in C$.

(C3) If $x \in C$ and $\frac{x-1}{2}$ is a positive integer, then $\frac{x-1}{2} \in C$.

(C4) $5 \notin C$.

The purpose of this work will be to give an answer to this type of combinatorial problems.

Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two n-tuples (with $n \geq 1$) of positive integers, let X be a non-empty subset of $\mathbb{N} \setminus \{0\}$, and let g be a positive integer. Let us denote by $\mathrm{P}(a, b, X, g)$ the problem of computing all the subsets C of $\mathbb{N} \setminus \{0\}$ that fulfill the following conditions.

(P1) The cardinality of C is equal to g.

(P2) If $x, y \in \mathbb{N} \setminus \{0\}$ and $x + y \in C$, then $C \cap \{x, y\} \neq \emptyset$.

(P3) If $x \in C$ and $\frac{x-b_i}{a_i} \in \mathbb{N} \setminus \{0\}$ for some $i \in \{1, \ldots, n\}$, then $\frac{x-b_i}{a_i} \in C$.

(P4) $X \cap C = \emptyset$.

With the previous notation, we observe that the problem proposed at the beginning is just $\mathrm{P}\big((1,2),(4,1),\{5\},6\big)$.

A *numerical semigroup* (see [6]) is a submonoid S of $(\mathbb{N}, +)$ such that $\mathbb{N} \setminus S$ is finite. The cardinality of $\mathbb{N} \setminus S$ is the so-called *genus* of S and is denoted by $\mathrm{g}(S)$.

It is easy to see that C is a solution of $\mathrm{P}(a, b, X, g)$ if and only if $S = \mathbb{N} \setminus C$ is a numerical semigroup that fulfills the following conditions.

(S1) $\mathrm{g}(S) = g$.

(S2) If $s \in S \setminus \{0\}$, then $as + b \in S^n$ (where $as + b = (a_1 s + b_1, \ldots, a_n s + b_n)$).

(S3) $X \subseteq S$.

Let us denote by $\mathcal{N}(a, b, X)$ the set

$$\big\{S \mid S \text{ is a numerical semigroup, } X \subseteq S, \text{ and } as + b \in S^n \text{ for all } s \in S \setminus \{0\}\big\}.$$

With this notation, the solutions of $\mathrm{P}(a, b, X, g)$ are the elements of the set

$$\big\{\mathbb{N} \setminus S \mid S \in \mathcal{N}(a, b, X) \text{ and } \mathrm{g}(S) = g\big\}.$$

Let S be a numerical semigroup. The *Frobenius number* of S (see [2]), denoted by $\mathrm{F}(S)$, is the maximum integer that does not belong to S.

A *Frobenius variety* (see [5]) is a non-empty family of numerical semigroups $\mathcal{V}$ that fulfills the following conditions.

(V1) If $S, T \in \mathcal{V}$, then $S \cap T \in \mathcal{V}$.

(V2) If $S \in \mathcal{V}$ and $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{V}$.

Before to show the structure of this paper, let us see some remarks in order to delimit the problem $\mathrm{P}(a, b, X, g)$ inside the theory of numerical semigroups.

Remark 1.1. If we only impose condition (C1) (equivalently, condition (P2)), then we are considering the family $\mathcal{S}$ of all numerical semigroups that, obviously, is a Frobenius variety. Moreover, in [6] it is shown how to arrange the elements of $\mathcal{S}$ in a tree with root $\mathbb{N}$.

Remark 1.2. Keeping in mind only conditions (C1) and (C4) (equivalently, conditions (P2) and (P4) or condition (S3)), we have the family $\mathcal{S}(X)$ of all numerical semigroups containing X. Once again, it is not difficult to check that $\mathcal{S}(X)$ is a Frobenius variety. In addition, following the ideas of this paper, it will be clear how we can arrange the elements of $\mathcal{S}(X)$ in a tree with root $\mathbb{N}$.

Remark 1.3. Now let us consider only conditions (C1), (C2) and (C3) (equivalently, conditions (P2) and (P3) or condition (S2)). In this case we get families of numerical semigroups satisfying a set non-homogeneous patterns (see [1]). This case is related to the results of [1], where the concept of m-variety makes possible to arrange the elements of certain families in trees with root $S_m = \{0, m, \rightarrow\} = \{0\} \cup \{n \in \mathbb{N} \mid n \geq m\}$. Precisely, m-varieties are examples of Frobenius pseudo-varieties (see [4]).

In Section 2 we will see that $\mathcal{N}(a, b, X)$ is a Frobenius variety. In addition, we will show that such a variety is finite if and only if $\gcd(X \cup \{b_1, \ldots, b_n\}) = 1$ (where, as usual, $\gcd(A)$ is the *greatest common divisor* of the elements in A).

Let us denote by $M(a, b, X)$ the intersection of all the elements in $\mathcal{N}(a, b, X)$. Observe that $M(a, b, X)$ is always a submonoid of $(\mathbb{N}, +)$. In addition, we will prove that $M(a, b, X)$ is a numerical semigroup if and only if $\mathcal{N}(a, b, X)$ has finitely many elements.

In Section 2 we will show that $\mathrm{P}(a, b, X, g)$ has a solution if and only if the cardinality of $\mathbb{N} \setminus M(a, b, X)$ is greater than or equal to g. Moreover, we will give an algorithm in order to compute $M(a, b, X)$. Therefore, we will have an algorithmic process to decide whether $\mathrm{P}(a, b, X, g)$ has a solution.

In Section 3, with the help of some results from [5], we will arrange the elements of $\mathcal{N}(a, b, X)$ in a tree with root $\mathbb{N}$. Moreover, we will characterize the children of a vertex in such a tree and, consequently, will have a recursive procedure in order to build $\mathcal{N}(a, b, X)$. Accordingly, we will have an algorithmic process to compute all the elements of $\mathcal{N}(a, b, X)$ with a fixed genus and, in particular, an algorithm to compute all the solutions of $\mathrm{P}(a, b, X, g)$.

Finally, using the concept of Frobenius pseudo-variety, we will state and solve a generalization of $\mathrm{P}(a, b, X, g)$ in Section 4.

2 (a, b)-monoids

In this work, unless stated otherwise, $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ denote two n-tuples of positive integers. If $X \subseteq \mathbb{N}$, then $\mathcal{N}(a, b, X)$ is the set

$$\big\{S \mid S \text{ is a numerical semigroup, } X \subseteq S, \text{ and } as + b \in S^n \text{ for all } s \in S \setminus \{0\}\big\},$$

with $as + b = (a_1 s + b_1, \ldots, a_n s + b_n)$.

Proposition 2.1. *$\mathcal{N}(a,b,X)$ is a Frobenius variety.*

Proof. It is clear that $\mathbb{N} \in \mathcal{N}(a,b,X)$ and, therefore, $\mathcal{N}(a,b,X) \neq \emptyset$. It is also clear that, if $S,T \in \mathcal{N}(a,b,X)$, then $S \cap T \in \mathcal{N}(a,b,X)$. Now, let $S \in \mathcal{N}(a,b,X)$ such that $S \neq \mathbb{N}$. In order to show that $S \cup \{\mathrm{F}(S)\} \in \mathcal{N}(a,b,X)$, it is enough to see that $a\mathrm{F}(S)+b \in (S \cup \{\mathrm{F}(S)\})^n$. Observe that, if $i \in \{1,\dots,n\}$, then $a_i\mathrm{F}(S)+b_i > \mathrm{F}(S)$ and, therefore, $a_i\mathrm{F}(S)+b_i \in S \cup \{\mathrm{F}(S)\}$. Consequently, $a\mathrm{F}(S)+b \in (S \cup \{\mathrm{F}(S)\})^n$. □

We will say that M is an (a,b)*-monoid* if M is a submonoid of $(\mathbb{N},+)$ fulfilling that $am+b \in M^n$ for all $m \in M \setminus \{0\}$.

Proposition 2.2. *Let X be a non-empty subset of $\mathbb{N}$. Then M is an (a,b)-monoid that contains X if and only if there exists $J \subseteq \mathcal{N}(a,b,X)$ such that $M = \bigcap_{S \in J} S$.*

Proof. The sufficient condition is trivial. For the necessary one, let $M_k = M \cup \{k,\rightarrow\}$, for all $k \in \mathbb{N}$ (where $\{k,\rightarrow\} = \{n \in \mathbb{N} \mid n \geq k\}$). Then it is clear that $M_k \in \mathcal{N}(a,b,X)$ and that $M = \bigcap_{k \in \mathbb{N}} M_k$. □

Let us observe that, if we denote by $M(a,b,X) = \bigcap_{S \in \mathcal{N}(a,b,X)} S$, then $M(a,b,X)$ is the smallest (a,b)-monoid containing X.

Theorem 2.3. *Let X be a non-empty subset of $\mathbb{N} \setminus \{0\}$ and let g be a positive integer. Then the problem $\mathrm{P}(a,b,X,g)$ has a solution if and only if the cardinality of $\mathbb{N} \setminus M(a,b,X)$ is greater than or equal to g.*

Proof. (Necessity.) If C is a solution of $\mathrm{P}(a,b,X,g)$ and we take $S = \mathbb{N} \setminus C$, then $S \in \mathcal{N}(a,b,X)$ and $\mathrm{g}(S) = g$. Since $M(a,b,X) \subseteq S$, we have that $\mathbb{N} \setminus S \subseteq \mathbb{N} \setminus M(a,b,X)$ and, thereby, the cardinality of $\mathbb{N} \setminus M(a,b,X)$ is greater than or equal to g.

(Sufficiency.) Let us suppose that $\mathbb{N} \setminus M(a,b,X) = \{c_1 < \cdots < c_g < \cdots\}$. If we take $S = M(a,b,X) \cup \{c_g+1,\rightarrow\}$, then it is clear that $S \in \mathcal{N}(a,b,X)$ and $\mathrm{g}(S) = g$. Therefore, $C = \mathbb{N} \setminus S$ is a solution of $\mathrm{P}(a,b,X,g)$. □

Let us observe that, if $\mathrm{P}(a,b,X,g)$ has a solution and, in addition, we have computed $M(a,b,X)$, then the proof of the sufficient condition in the previous theorem gives us a method to compute a solution of $\mathrm{P}(a,b,X,g)$.

If X is a non-empty subset of $\mathbb{N}$, then we denote by $\langle X \rangle$ the submomoid of $(\mathbb{N},+)$ generated by X, that is,

$$\langle X \rangle = \{\lambda_1 x_1 + \cdots + \lambda_k x_k \mid k \in \mathbb{N} \setminus \{0\},\ x_1,\dots,x_k \in X, \text{ and } \lambda_1,\dots,\lambda_k \in \mathbb{N}\}.$$

If $M = \langle X \rangle$, then we will say that M is generated by X or, equivalently, that X is a *system of generators* of M. The next result is well known (see, for instance, [6]).

Lemma 2.4. *If $X \subseteq \mathbb{N}$, then $\langle X \rangle$ is a numerical semigroup if and only if $\gcd(X) = 1$.*

We know that $M(a,b,X)$ is a submonoid of $(\mathbb{N},+)$. From the following proposition we will get that, if $X \subseteq \mathbb{N} \setminus \{0\}$, then $M(a,b,X)$ is a numerical semigroup if and only if $\gcd(X \cup \{b_1,\dots,b_n\}) = 1$.

Proposition 2.5. *If $X \subseteq \mathbb{N} \setminus \{0\}$, then $\gcd(M(a,b,X)) = \gcd(X \cup \{b_1,\dots,b_n\})$.*

Proof. Let $d = \gcd(M(a,b,X))$ and $d' = \gcd(X \cup \{b_1,\dots,b_n\})$. In order to prove the proposition, we will show that $d' \mid d$ and $d \mid d'$. (As usual, if p, q are positive integers, then $p \mid q$ means that p divides q.)

First of all, it is clear that $\langle\{d'\}\rangle$ is an (a,b)-monoid containing X. Thus, $M(a,b,X) \subseteq \langle\{d'\}\rangle$ and, consequently, $d' \mid d$.

Now, let us take $x \in X$. Then $\{x, a_1x + b_1, \dots, a_nx + b_n\} \subseteq M(a,b,X)$ and $\gcd\{x, a_1x + b_1, \dots, a_nx + b_n\} = \gcd\{x, b_1, \dots, b_n\}$. Therefore, we have that $X \cup \left(\bigcup_{i=1}^{n}\{a_ix + b_i \mid x \in X\}\right) \subseteq M(a,b,X)$ and $\gcd\left(X \cup \left(\bigcup_{i=1}^{n}\{a_ix + b_i \mid x \in X\}\right)\right) = \gcd(X \cup \{b_1,\dots,b_n\}) = d'$. Accordingly, $d \mid d'$. □

In the next result we show when the Frobenius variety $\mathcal{N}(a,b,X)$ is finite.

Theorem 2.6. *Let X be a subset of $\mathbb{N} \setminus \{0\}$. Then the following conditions are equivalent.*

1. *$\mathcal{N}(a,b,X)$ is finite.*
2. *$M(a,b,X)$ is a numerical semigroup.*
3. *$\gcd(X \cup \{b_1,\dots,b_n\}) = 1$.*

Proof. The equivalence between conditions *2* and *3* is a consequence of Lemma 2.4 and Proposition 2.5. Now, let us see the equivalence between conditions *1* and *2*.

(1. ⇒ 2.) It is enough to observe that the finite intersection of numerical semigroups is another numerical semigroup.

(2. ⇒ 1.) If $S \in \mathcal{N}(a,b,X)$, then $M(a,b,X) \subseteq S$. Thus, $S = M(a,b,X) \cup Y$ with $Y \subseteq \mathbb{N} \setminus M(a,b,X)$. Since $\mathbb{N} \setminus M(a,b,X)$ is finite, we conclude that $\mathcal{N}(a,b,X)$ is finite. □

Our next aim in this section will be to give an algorithm in order to compute $M(a,b,X)$. For this is fundamental the following result.

Proposition 2.7. *Let M be a submonoid of $(\mathbb{N},+)$ generated by $X \subseteq \mathbb{N} \setminus \{0\}$. Then M is an (a,b)-monoid if and only if $ax + b \in M^n$ for all $x \in X$.*

Proof. The necessary condition is trivial. For the sufficiency, let $m \in M \setminus \{0\}$. Then there exist $x_1,\dots,x_t \in X$ such that $m = x_1 + \cdots + x_t$. If $t = 1$, then $m = x_1$ and $am + b = ax_1 + b \in M^n$. If $t \geq 2$, then

$$am + b = a(x_1 + \cdots + x_t) + b = a(x_1 + \cdots + x_{t-1}) + ax_t + b.$$

Since $a(x_1 + \cdots + x_{t-1})$, $ax_t + b \in M^n$, we finish the proof. □

The above proposition will be useful in order to determine whether a submonoid M of $(\mathbb{N},+)$ is or is not an (a,b)-monoid. Let us see an example.

Example 2.8. $S = \langle\{4,5,11\}\rangle$ is an $\big((1,2),(4,1)\big)$-monoid because $(1,2)4 + (4,1) = (8,9) \in S^2$, $(1,2)5 + (4,1) = (9,11) \in S^2$, and $(1,2)11 + (4,1) = (15,23) \in S^2$. Nevertheless, $T = \langle\{5,7,9\}\rangle$ is not an $\big((1,2),(4,1)\big)$-monoid because $(1,2)5 + (4,1) = (9,11) \notin T^2$ (observe that $11 \notin T$).

With the help of Proposition 2.7, it would be possible to give an algorithm in order to compute $M(a,b,X)$. However, we are going to postpone such an algorithm because, as we will see now, we can focus on case in which $\gcd(X \cup \{b_1,\dots,b_n\}) = 1$, and thus simplify the computations.

We say that an integer d divides an n-tupla of integers $c = (c_1,\dots,c_n)$ if $d \mid c_i$ for all $i \in \{1,\dots,n\}$. In such a case, we denote by $\frac{c}{d} = \left(\frac{c_1}{d},\dots,\frac{c_n}{d}\right)$. If $A \subseteq \mathbb{Z}$ and $k \in \mathbb{Z}$, then $kA = \{ka \mid a \in A\}$. Finally, if $A \subseteq \mathbb{Z}$, $d \in \mathbb{Z}$, and $d \mid a$ for all $a \in A$, then $\frac{A}{d} = \left\{\frac{a}{d} \mid a \in A\right\}$.

Lemma 2.9. *Let M be an (a,b)-monoid such that $M \neq \{0\}$. If $\gcd(M) = d$, then*

1. *d divides b;*
2. *if $d' \in \mathbb{N} \setminus \{0\}$ and $d' \mid d$, then $\frac{M}{d'}$ is an $\left(a, \frac{b}{d'}\right)$-monoid;*
3. *if $k \in \mathbb{N} \setminus \{0\}$, then kM is an (a,kb)-monoid.*

Proof. 1. If we take $X = M \setminus \{0\}$, and having in mind that $M(a,b,X)$ is the smallest (a,b)-monoid containing X, then this item is a consequence of Proposition 2.5.

2. It is clear that $\frac{M}{d'}$ is a submonoid of $(\mathbb{N},+)$. In addition, if $x \in \frac{M}{d'} \setminus \{0\}$, then $d'x \in M \setminus \{0\}$ and, therefore, $ad'x + b \in M^n$. Consequently, $ax + \frac{b}{d'} \in \frac{M^n}{d'} = \left(\frac{M}{d'}\right)^n$.

3. It is clear that kM is a submonoid of $(\mathbb{N},+)$. Now, arguing as in the previous item, if $x \in kM \setminus \{0\}$, then $\frac{x}{k} \in M \setminus \{0\}$ and, therefore, $a\frac{x}{k} + b \in M^n$. Consequently, $ax + kb \in k \cdot M^n = (kM)^n$. □

The next result says us that, in order to compute $M(a,b,X)$, it is sufficient to calculate $d = \gcd\left(X \cup \{b_1,\dots,b_n\}\right)$ and $M\left(a, \frac{b}{d}, \frac{X}{d}\right)$. Observe that $\gcd\left(\frac{X}{d} \cup \left\{\frac{b_1}{d},\dots,\frac{b_n}{d}\right\}\right) = 1$ and, therefore, $M\left(a, \frac{b}{d}, \frac{X}{d}\right)$ is a numerical semigroup.

Proposition 2.10. *Let X be a subset of $\mathbb{N} \setminus \{0\}$. If $\gcd(X \cup \{b_1,\dots,b_n\}) = d$, then $M(a,b,X) = d \cdot M\left(a, \frac{b}{d}, \frac{X}{d}\right)$.*

Proof. From item 3 of Lemma 2.9, we have that $d \cdot M\left(a, \frac{b}{d}, \frac{X}{d}\right)$ is an (a,b)-monoid containing X. Therefore, $M(a,b,X) \subseteq d \cdot M\left(a, \frac{b}{d}, \frac{X}{d}\right)$.

On the other hand, from Proposition 2.5 and item 2 of Lemma 2.9, we deduce that $\frac{M(a,b,X)}{d}$ is an $\left(a, \frac{b}{d}\right)$-monoid containing $\frac{X}{d}$. Consequently, $M\left(a, \frac{b}{d}, \frac{X}{d}\right) \subseteq \frac{M(a,b,X)}{d}$, that is, $d \cdot M\left(a, \frac{b}{d}, \frac{X}{d}\right) \subseteq M(a,b,X)$. □

We are now ready to show the announced algorithm.

Algorithm 2.11.

INPUT: A non-empty finite set of positive integers X such that $\gcd\left(X \cup \{b_1,\dots,b_n\}\right) = 1$.

OUTPUT: $M(a,b,X)$.

(1) $A = \emptyset$ and $G = X$.
(2) If $G \setminus A = \emptyset$, then return $\langle G \rangle$ and stop the algorithm.
(3) $m = \min(G \setminus A)$.
(4) $H = \left\{a_i m + b_i \mid i \in \{1,\dots,n\} \text{ and } a_i m + b_i \notin \langle G \rangle\right\}$.
(5) If $H = \emptyset$, then go to (7).
(6) $G = G \cup H$.
(7) $A = A \cup \{m\}$ and go to (2).

In order to justify the performance of this algorithm, let us observe that, if the algorithm stops, then it returns $\langle G\rangle$ such that $ag+b \in \langle G\rangle^n$ for all $g \in G$. Therefore, by applying Proposition 2.7, we have that $\langle G\rangle$ is an (a,b)-monoid. In addition, by construction, it is clear that G must be a subset of every (a,b)-monoid which contains X. Thus, $\langle G\rangle$ is the smallest (a,b)-monoid containing X. Consequently, in order to justify the algorithm, it will be enough to see that the algorithm stops. In fact, when we arrive to step (7) at the first time, we have that $\gcd(G)=1$ and, thereby, $\langle G\rangle$ is a numerical semigroup. Therefore, $\mathbb{N}\setminus\langle G\rangle$ is finite and we can go to the step (6) only in a finite number of times.

Let us illustrate the performance of Algorithm 2.11 with two examples. In the first one $M(a,b,X)$ is a numerical semigroup.

Example 2.12. We are going to compute $M = M\big((1,2),(4,1),\{5\}\big)$.

- $A=\emptyset$ and $G=\{5\}$.
- $m=5$, $H=\{9,11\}$, $G=\{5,9,11\}$, and $A=\{5\}$.
- $m=9$, $H=\{13\}$, $G=\{5,9,11,13\}$, and $A=\{5,9\}$.
- $m=11$, $H=\emptyset$, $G=\{5,9,11,13\}$, and $A=\{5,9,11\}$.
- $m=13$, $H=\{17\}$, $G=\{5,9,11,13,17\}$, and $A=\{5,9,11,13\}$.
- $m=17$, $H=\emptyset$, $G=\{5,9,11,13,17\}$, and $A=\{5,9,11,13,17\}$.

Therefore, $M=\langle\{5,9,11,13,17\}\rangle$.

Going back to the problem $\mathrm{P}\big((1,2),(4,1),\{5\},6\big)$ of the introduction, we have that, since $\mathbb{N}\setminus M=\{1,2,3,4,6,7,8,12\}$ has cardinality equal to 8, then Theorem 2.3 asserts that the proposed problem has a solution. Moreover, the solutions will be some subsets, with cardinality equal to 6, of $\{1,2,3,4,6,7,8,12\}$. In addition, by the proof of the sufficiency of Theorem 2.3, we know that $\{1,2,3,4,6,7\}$ is a solution of such a problem.

Let us see now an example in which $M(a,b,X)$ is not a numerical semigroup.

Example 2.13. Let us see that $\mathrm{P}\big((2,3),(4,2),\{6,8\},9\big)$ has a solution. For that, we begin with the computation of $M\big((2,3),(4,2),\{6,8\}\big)$. By applying Proposition 2.10, since $\gcd(\{6,8,4,2\})=2$, we get that $M\big((2,3),(4,2),\{6,8\}\big)=2\cdot M\big((2,3),(2,1),\{3,4\}\big)$. Now, from Algorithm 2.11, $M\big((2,3),(2,1),\{3,4\}\big)=\langle\{3,4\}\rangle$. Therefore,

$$M\big((2,3),(4,2),\{6,8\}\big)=\langle\{6,8\}\rangle=\{0,6,8,12,14,16,\dots\}.$$

Since $\mathbb{N}\setminus M\big((2,3),(4,2),\{6,8\}\big)$ has infinitely many elements, its cardinality is greater than or equal to 9 and, consequently, Theorem 2.3 assures that $\mathrm{P}\big((2,3),(4,2),\{6,8\},9\big)$ has a solution. Moreover, by the proof of the sufficiency of Theorem 2.3, we have that $\{1,2,3,4,5,7,9,10,11\}$ is a solution.

Remark 2.14. If we suppose, for a moment, that $X=\emptyset$ (in a sense, we are removing condition (P4) in $\mathrm{P}(a,b,X,g)$ such as is observed in Remark 1.2), then it is obvious that $S_k=\{0,k,\rightarrow\}$, for all $k\in\mathbb{N}$, are numerical semigroups that belong to $\mathcal{N}(a,b,X)$ independently of the chosen n-tuples a,b. Thus $M(a,b,X)=\{0\}$, that is, the submonoid of $(\mathbb{N},+)$ generated by $X=\emptyset$.

Remark 2.15. Now, let us suppose that a,b are 0-tuples, that is, we remove condition (P3) in $\mathrm{P}(a,b,X,g)$ (see Remark 1.3). In this case, it is straightforward to show that $M(a,b,X)$ is just the monoid generated by X.

3 The tree associated to $\mathcal{N}(a, b, X)$

A *graph* G is a pair (V, E), where

- V is a non-empty set whose elements are called *vertices* of G,
- E is a subset of $\{(v, w) \in V \times V \mid v \neq w\}$ whose elements are called *edges* of G.

A *path (of length n)* connecting the vertices x and y of G is a sequence of different edges of the form $(v_0, v_1), (v_1, v_2), \dots, (v_{n-1}, v_n)$ such that $v_0 = x$ and $v_n = y$.

We say that a graph G is a *tree* if there exists a vertex r (known as the *root* of G) such that, for every other vertex x of G, there exists a unique path connecting x and r. If (x, y) is an edge of the tree, then we say that x is a *child* of y.

We define the graph $\mathrm{G}\big(\mathcal{N}(a, b, X)\big)$ in the following way.

- $\mathcal{N}(a, b, X)$ is the set of vertices of $\mathrm{G}\big(\mathcal{N}(a, b, X)\big)$;
- $(S, S') \in \mathcal{N}(a, b, X) \times \mathcal{N}(a, b, X)$ is an edge of $\mathrm{G}\big(\mathcal{N}(a, b, X)\big)$ if $S' = S \cup \{\mathrm{F}(S)\}$.

By Proposition 2.1 and [5, Theorem 27], we have that $\mathrm{G}\big(\mathcal{N}(a, b, X)\big)$ is a tree with root $\mathbb{N}$. Our first purpose in this section will be to establish what are the children of a vertex in such a tree. For this we need to introduce some concepts.

Let S be a numerical semigroup and let G be a system of generators of S. We say that G is a *minimal system of generators* of S if $S \neq \langle Y \rangle$ for all $Y \subset G$. It is well known (see [6]) that every numerical semigroup admits a unique minimal system of generators and that, in addition, such a system is finite. Observe that, if we denote by $\mathrm{msg}(S)$ the minimal system of generators of S, then $\mathrm{msg}(S) = \big(S \setminus \{0\}\big) \setminus \big((S \setminus \{0\}) + (S \setminus \{0\})\big)$. On the other hand, we have (see [6]) that, if S is a numerical semigroup and $s \in S$, then $S \setminus \{s\}$ is another numerical semigroup if and only if $s \in \mathrm{msg}(S)$.

An immediate consequence of [5, Proposition 24, Theorem 27] is the next result.

Theorem 3.1. *The graph $\mathrm{G}\big(\mathcal{N}(a, b, X)\big)$ is a tree with root $\mathbb{N}$. Moreover, the set of children of a vertex $S \in \mathcal{N}(a, b, X)$ is*

$$\big\{S \setminus \{m\} \mid m \in \mathrm{msg}(S),\ m > \mathrm{F}(S),\ \textit{and } S \setminus \{m\} \in \mathcal{N}(a, b, X)\big\}.$$

In the next result we will show the conditions that must satisfy $m \in \mathrm{msg}(S)$ in order to have $S \setminus \{m\} \in \mathcal{N}(a, b, X)$.

Proposition 3.2. *Let $S \in \mathcal{N}(a, b, X)$ and let $m \in \mathrm{msg}(S)$. Then $S \setminus \{m\} \in \mathcal{N}(a, b, X)$ if and only if $\frac{m-b_i}{a_i} \notin S \setminus \{0\}$ for all $i \in \{1, \dots, n\}$ and $m \notin X$.*

Proof. (Necessity.) Since $X \subseteq S \setminus \{m\}$, we have that $m \notin X$. Let us suppose that there exists $i \in \{1, \dots, n\}$ such that $\frac{m-b_i}{a_i} \in S \setminus \{0\}$. Since $\frac{m-b_i}{a_i} \neq m$, we have that $\frac{m-b_i}{a_i} \in S \setminus \{0, m\}$ and that $a_i\big(\frac{m-b_i}{a_i}\big) + b_i = m \notin S \setminus \{m\}$. Therefore, $S \setminus \{m\} \notin \mathcal{N}(a, b, X)$.

(Sufficiency.) If $S \setminus \{m\} \notin \mathcal{N}(a, b, X)$, then there exists $s \in S \setminus \{0, m\}$ and there exists $i \in \{1, \dots, n\}$ such that $a_i s + b_i \notin S \setminus \{m\}$. Since $S \in \mathcal{N}(a, b, X)$, we know that $a_i s + b_i \in S$. Therefore, $a_i s + b_i = m$ and, consequently, $\frac{m-b_i}{a_i} = s \in S \setminus \{0\}$. □

Our next purpose will be to build recurrently $\mathrm{G}\big(\mathcal{N}(a, b, X)\big)$ from its root and joining each vertex with its children by means of edges. In order to make easy that construction, we will study the relation between the minimal system of generators of a numerical semigroup S and the minimal system of generators of $S \setminus \{m\}$, where m is a minimal generator of S greater than $\mathrm{F}(S)$. First of all, it is clear to observe that, if S is minimally generated by $\{m, m+1, \ldots, 2m-1\}$ (that is, $S = \{0, m, \rightarrow\}$), then $S \setminus \{m\} = \{0, m+1, \rightarrow\}$ is minimally generated by $\{m+1, m+2, \ldots, 2m+1\}$. In other case we can apply the following result, which is [3, Corollary 18].

Proposition 3.3. *Let S be a numerical semigroup with minimal system of generators $\mathrm{msg}(S) = \{n_1 < n_2 < \cdots < n_p\}$. If $i \in \{2, \ldots, p\}$ and $n_i > \mathrm{F}(S)$, then*

$$\mathrm{msg}(S \setminus \{n_i\}) = \begin{cases} \{n_1, \ldots, n_p\} \setminus \{n_i\}, & \text{if there exists } j \in \{2, \ldots, i-1\} \\ & \text{such that } n_i + n_1 - n_j \in S; \\ \big(\{n_1, \ldots, n_p\} \setminus \{n_i\}\big) \cup \{n_i + n_1\}, & \text{in other case.} \end{cases}$$

Let us illustrate the previous results with an example.

Example 3.4. By Proposition 2.7, it is easy to see that $S = \langle\{5, 7, 8, 9, 11\}\rangle$ belongs to $\mathcal{N}\big((1,2), (4,1), \{5\}\big)$. On the other hand, from Theorem 3.1, we know that the set of children of S in the tree $\mathrm{G}\big(\mathcal{N}\big((1,2), (4,1), \{5\}\big)\big)$ is

$$\big\{S \setminus \{m\} \mid m \in \mathrm{msg}(S),\ m > \mathrm{F}(S),\ \text{and } S \setminus \{m\} \in \mathcal{N}\big((1,2), (4,1), \{5\}\big)\big\}.$$

Since $\mathrm{F}(S) = 6$, we have that $\{m \in \mathrm{msg}(S) \mid m > \mathrm{F}(S)\} = \{7, 8, 9, 11\}$. Furthermore, by Proposition 3.2, we know that $S \setminus \{m\} \in \mathcal{N}\big((1,2), (4,1), \{5\}\big)$ if and only if $m \notin \{5\}$ and $\big\{m - 4, \frac{m-1}{2}\big\} \cap \big(S \setminus \{0\}\big) = \emptyset$. Thus, since that $\big\{7 - 4, \frac{7-1}{2}\big\} \cap \big(S \setminus \{0\}\big) = \big\{8 - 4, \frac{8-1}{2}\big\} \cap \big(S \setminus \{0\}\big) = \emptyset$, $\big\{9 - 4, \frac{9-1}{2}\big\} \cap \big(S \setminus \{0\}\big) \neq \emptyset$, and $\big\{11 - 4, \frac{11-1}{2}\big\} \cap \big(S \setminus \{0\}\big) \neq \emptyset$, we conclude that $S = \langle\{5, 7, 8, 9, 11\}\rangle$ has two children. Namely, they are $\langle\{5, 7, 8, 9, 11\}\rangle \setminus \{7\} = \langle\{5, 8, 9, 11, 12\}\rangle$ and $\langle\{5, 7, 8, 9, 11\}\rangle \setminus \{8\} = \langle\{5, 7, 9, 11, 13\}\rangle$, where we have applied Proposition 3.3.

Following the idea of the previous example, we can build $\mathrm{G}\big(\mathcal{N}\big((1,2), (4,1), \{5\}\big)\big)$ in a recurrent way starting from its root, that is, from $\mathbb{N} = \langle\{1\}\rangle$ (see Figure 1).

Let us observe that, since $\gcd(\{5\} \cup \{4, 1\}) = 1$ and by Theorem 2.6, then we know that $\mathcal{N}((1,2), (4,1), \{5\})$ is a finite Frobenius variety and, thereby, we have been able of building it completely in a finite number of steps.

Let us also observe that, if $S \in \mathcal{N}(a, b, X)$, then $\mathrm{g}(S)$ is equal to the length of the path connecting S with $\mathbb{N}$ in the tree $\mathrm{G}\big((\mathcal{N}(a, b, X)\big)$. Therefore, in order to build the elements of $\mathcal{N}(a, b, X)$ with a fixed genus g, we only need to build the elements of $\mathcal{N}(a, b, X)$ which are connected to $\mathbb{N}$ through a path of length less than or equal to g. Consequently, we have an algorithmic process to compute all the solutions of the problem $\mathrm{P}(a, b, X, g)$.

For instance, in the tree $\mathrm{G}\big(\mathcal{N}\big((1,2), (4,1), \{5\}\big)\big)$, the numerical semigroups which are connected to $\mathbb{N}$ through a path of length 6 are $\langle\{5, 8, 9, 11, 12\}\rangle$, $\langle\{5, 7, 9, 11, 13\}\rangle$, and $\langle\{5, 6, 9, 13\}\rangle$. Therefore, the problem proposed in the introduction has three solutions. Namely, $\mathbb{N} \setminus \langle\{5, 8, 9, 11, 12\}\rangle = \{1, 2, 3, 4, 6, 7\}$, $\mathbb{N} \setminus \langle\{5, 7, 9, 11, 13\}\rangle = \{1, 2, 3, 4, 6, 8\}$, and $\mathbb{N} \setminus \langle\{5, 6, 9, 11, 13\}\rangle = \{1, 2, 3, 4, 7, 8\}$.

We finish with an example in which $\mathcal{N}(a, b, X)$ is an infinite Frobenius variety.

Example 3.5. Let us compute all the solutions of $\mathrm{P}\big((2,3), (4,2), \{6, 8\}, 4\big)$. First of all, from Example 2.13, we know that $M\big((2,3), (4,2), \{6, 8\}\big) = \langle\{6, 8\}\rangle$ and, by Theorem 2.3, that the problem has a solution. Moreover, since $\gcd(\{6, 8\} \cup \{4, 2\}) = 2 \neq 1$,

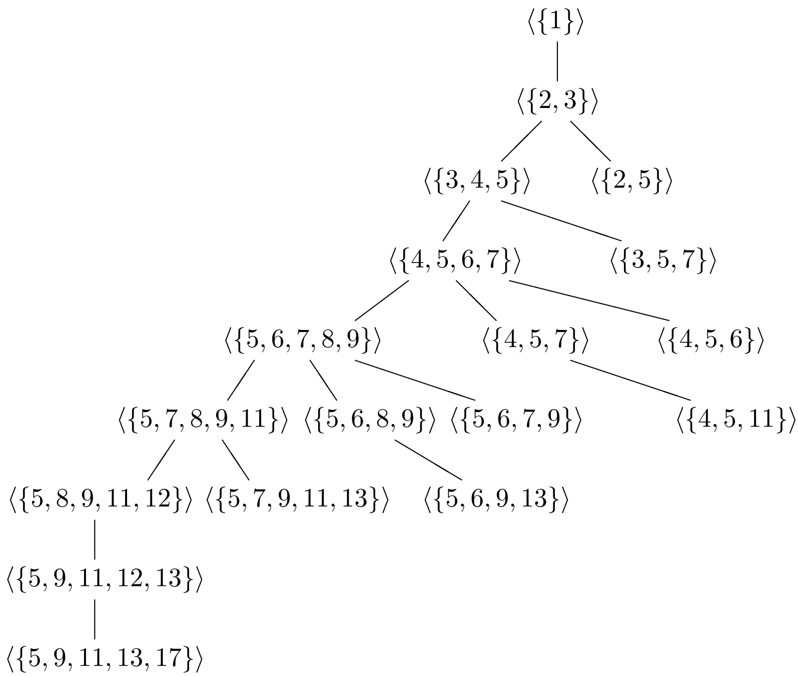

Figure 1: Tree associated to the finite Frobenius variety $\mathcal{N}\big((1,2),(4,1),\{5\}\big)$.

we have that $\mathcal{N}\big((2,3),(4,2),\{6,8\}\big)$ is a infinite Frobenius variety. However, in a finite number of steps, we can compute the elements of $\mathrm{G}\big(\mathcal{N}\big((2,3),(4,2),\{6,8\}\big)\big)$ which are connected to $\mathbb{N}$ through a path of length 4, such as we show in Figure 2.

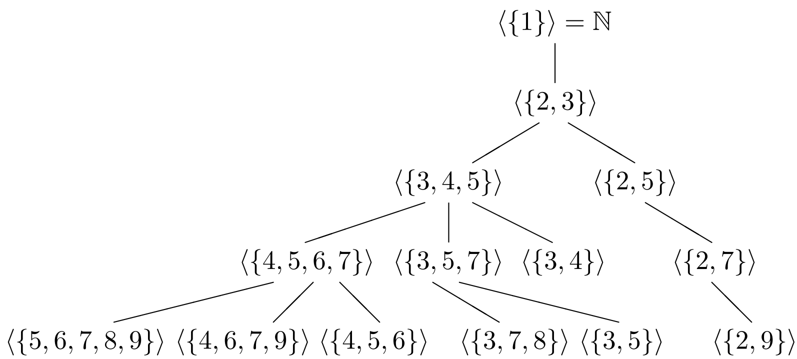

Figure 2: Five first levels of the tree associated to the infinite Frobenius variety $\mathcal{N}\big((2,3),(4,2),\{6,8\}\big)$.

Therefore, the sets $\mathbb{N}\setminus\langle\{5,6,7,8,9\}\rangle = \{1,2,3,4\}$, $\mathbb{N}\setminus\langle\{4,6,7,9\}\rangle = \{1,2,3,5\}$, $\mathbb{N}\setminus\langle\{4,5,6\}\rangle = \{1,2,3,7\}$, $\mathbb{N}\setminus\langle\{3,7,8\}\rangle = \{1,2,4,5\}$, $\mathbb{N}\setminus\langle\{3,5\}\rangle = \{1,2,4,7\}$, and $\mathbb{N}\setminus\langle\{2,9\}\rangle = \{1,3,5,7\}$ are the (six) solutions of $\mathrm{P}\big((2,3),(4,2),\{6,8\},4\big)$.

Remark 3.6. Let us observe that, in the construction of the trees, we can assume that $X = \emptyset$ or that a, b are 0-tuples (see Remarks 2.14 and 2.15). Then, we obtain all the possible solutions in each case. In particular, if we consider jointly such assumptions, then we get the tree associated to the full family of numerical semigroups (see Remark 1.1).

4 A generalization of the problem

Along this section r and g are non-negative integers, $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are n-tuples of positive integers, and X is a non-empty subset of $\{r+1, \rightarrow\}$. We will denote by $\mathrm{P}_r(a, b, X, g)$ the (generalised) problem of computing all the subsets C of $\{r+1, \rightarrow\}$ that fulfill the following conditions.

(GP1) The cardinality of C is equal to g.

(GP2) If $x, y \in \{r+1, \rightarrow\}$ and $x + y \in C$, then $C \cap \{x, y\} \neq \emptyset$.

(GP3) If $x \in C$ and $\frac{x-b_i}{a_i} \in \{r+1, \rightarrow\}$ for some $i \in \{1, \dots, n\}$, then $\frac{x-b_i}{a_i} \in C$.

(GP4) $X \cap C = \emptyset$.

Let us observe that $\mathrm{P}_0(a, b, X, g) = \mathrm{P}(a, b, X, g)$.

It is clear that a set C is a solution of $\mathrm{P}_r(a, b, X, g)$ if and only if $S = \{0, r+1, \rightarrow\} \setminus C$ is a numerical semigroup that fulfills the following conditions.

(GS1) $\mathrm{g}(S) = r + g$.

(GS2) If $s \in S \setminus \{0\}$, then $as + b \in S^n$.

(GS3) $X \subseteq S$.

Let us denote by $\mathcal{N}_r(a, b, X)$ the set of all numerical semigroups which are subsets of $\{0, r+1, \rightarrow\}$ and satisfy the conditions (GS2) and (GS3). Let us observe that, with this notation, the solutions of $\mathrm{P}_r(a, b, X, g)$ are the elements of the set

$$\big\{\{0, r+, 1 \rightarrow\} \setminus S \mid S \in \mathcal{N}_r(a, b, X) \text{ and } \mathrm{g}(S) = r + g\big\}.$$

Moreover, $\mathcal{N}_r(a, b, X) = \big\{S \in \mathcal{N}(a, b, X) \mid S \subseteq \{0, r+1, \rightarrow\}\big\}$.

The following proposition is analogous to Theorem 2.3.

Proposition 4.1. *Let us take $M_r(a, b, X) = \bigcap_{S \in \mathcal{N}_r(a,b,X)} S$. Then $\mathrm{P}_r(a, b, X, g)$ has a solution if and only if the cardinality of $\mathbb{N} \setminus M_r(a, b, X)$ is greater than or equal to $g + r$.*

Proof. *(Necessity.)* If C is a solution of $\mathrm{P}_r(a, b, X, g)$, then $S = \{0, r+1, \rightarrow\} \setminus C$ belongs to $\mathcal{N}_r(a, b, X)$ and $\mathrm{g}(S) = g + r$. Since $M_r(a, b, X) \subseteq S$, then we conclude that the cardinality of $\mathbb{N} \setminus M_r(a, b, X)$ is greater than or equal to $g + r$.

(Sufficiency.) If $\{0, r+1, \rightarrow\} \setminus M_r(a, b, X) = \{c_1 < \cdots < c_g < \cdots\}$ and $S = M_r(a, b, X) \cup \{c_g + 1, \rightarrow\}$, then it is easy to see that $S \in \mathcal{N}_r(a, b, X)$ and $\mathrm{g}(S) = g + r$. Therefore, $C = \{0, r+1, \rightarrow\} \setminus S$ is a solution of $\mathrm{P}_r(a, b, X, g)$. □

Observe that the cardinality of $\mathbb{N} \setminus M_r(a, b, X)$ is greater than or equal to $g + r$ if and only if the cardinality of $\{0, r+1, \rightarrow\} \setminus M_r(a, b, X)$ is greater than or equal to g.

Proposition 4.2. $M_r(a, b, X) = M(a, b, X)$.

Proof. Since $\mathcal{N}_r(a,b,X) \subseteq \mathcal{N}(a,b,X)$, then we have that

$$M(a,b,X) = \bigcap_{S\in\mathcal{N}(a,b,X)} S \subseteq \bigcap_{S\in\mathcal{N}_r(a,b,X)} S = M_r(a,b,X).$$

Let us now see the other inclusion. Since $\{0, r+1, \rightarrow\} \in \mathcal{N}(a,b,X)$ and $\mathcal{N}(a,b,X)$ is a Frobenius variety, we have that, if $S \in \mathcal{N}(a,b,X)$, then $S \cap \{0, r+1, \rightarrow\} \in \mathcal{N}(a,b,X)$. In addition, $S \cap \{0, r+1, \rightarrow\} \subseteq \{0, r+1, \rightarrow\}$ and, thus, $S \cap \{0, r+1, \rightarrow\} \in \mathcal{N}_r(a,b,X)$. In this way, $\mathcal{R} = \big\{S \cap \{0, r+1, \rightarrow\} \mid S \in \mathcal{N}(a,b,X)\big\} \subseteq \mathcal{N}_r(a,b,X)$. Consequently, $M_r(a,b,X) = \bigcap_{S\in\mathcal{N}_r(a,b,X)} S \subseteq \bigcap_{S\in\mathcal{R}} S = \bigcap_{S\in\mathcal{N}(a,b,X)} S = M(a,b,X)$. □

As an immediate consequence of Proposition 4.2 and Proposition 2.10, we have the next result.

Corollary 4.3. *If* $\gcd\big(X \cup \{b_1,\dots,b_n\}\big) = d$, *then* $M_r(a,b,X) = d \cdot M\big(a, \frac{b}{d}, \frac{X}{d}\big)$.

Let us observe that, as a consequence fo the previous corollary, we can use Algorithm 2.11 in order to compute $M_r(a,b,X)$.

The following result is the analogous to Theorem 2.6 for the current problem.

Corollary 4.4. *The following conditions are equivalent.*

1. $\mathcal{N}_r(a,b,X)$ *is finite.*
2. $M_r(a,b,X)$ *is a numerical semigroup.*
3. $\gcd(X \cup \{b_1,\dots,b_n\}) = 1$.

Proof. The equivalence between conditions *2* and *3* is a consequence of Proposition 4.2 and Theorem 2.6. Now, let us see the equivalence between conditions *1* and *2*.

(1. ⇒ 2.) It is enough to observe that the finite intersection of numerical semigroups is another numerical semigroup.

(2. ⇒ 1.) If $S \in \mathcal{N}_r(a,b,X)$, then $M_r(a,b,X) \subseteq S$. Thus, $S = M_r(a,b,X) \cup Y$ for some $Y \subseteq \mathbb{N} \setminus M_r(a,b,X)$. Since $\mathbb{N} \setminus M_r(a,b,X)$ is finite, then we can conclude that $\mathcal{N}_r(a,b,X)$ is finite. □

Let us illustrate the previous results with several examples.

Example 4.5. Let us see that $\mathrm{P}_r\big((1,2),(4,1),\{5\},6\big)$ has a solution if and only if $r \in \{0,1,2\}$. Since $\{5\} \subseteq \{r+1,\rightarrow\}$, then $r \in \{0,1,2,3,4\}$. By Proposition 4.2 and Example 2.12, we have that $M = M_r\big((1,2),(4,1),\{5\},6\big) = \langle\{5,9,11,13,17\}\rangle$. Since $\mathbb{N} \setminus M = \{1,2,3,4,6,7,8,12\}$ has cardinality equal to 8, by applying Proposition 4.1, we easily deduce that $\mathrm{P}_r\big((1,2),(4,1),\{5\},6\big)$ has a solution if and only if $r \in \{0,1,2\}$.

Example 4.6. If $r \in \{0,1,2,3,4,5\}$, then $\mathcal{N}_r\big((2,3),(4,2),\{6,8\}\big)$ is an infinite set. In fact, this is an immediate consequence of Corollary 4.4 and that $\gcd(\{4,2,6,8\}) = 2 \neq 1$.

Example 4.7. Let us compute $\mathrm{P}_3\big((2,3),(4,2),\{6,8\},9\big)$. By Proposition 4.2 and Example 2.13, we have that $M_3\big((2,3),(4,2),\{6,8\}\big) = \langle\{6,8\}\rangle$. Now, if we apply the construction given in the sufficiency of Proposition 4.1, we have that $\{4,5,7,9,10,11,13,15,17\}$ is a solution.

If G is a tree and u, v are two vertices of G such that there exists a path between them, then we will say that u is a *descendant* of v. The next result has an easy proof.

Proposition 4.8. *$\mathcal{N}_r(a,b,X)$ is the set of all descendants of $\{0, r+1, \rightarrow\}$ in the tree $\mathrm{G}\big(\mathcal{N}(a,b,X)\big)$.*

A Frobenius pseudo-variety (see [4]) is a non-empty family $\mathcal{P}$ of numerical semigroups that fulfills the following conditions.

(PV1) $\mathcal{P}$ has a maximum element $\max(\mathcal{P})$ (with respect to the inclusion order).

(PV2) If $S, T \in \mathcal{P}$, then $S \cap T \in \mathcal{P}$.

(PV3) If $S \in \mathcal{P}$ and $S \neq \max(\mathcal{P})$, then $S \cup \mathrm{F}(S) \in \mathcal{P}$.

As an immediate consequence of Proposition 4.8 and the comment above to Example 7 in [4], we have the following result.

Proposition 4.9. *$\mathcal{N}_r(a,b,X)$ is a Frobenius pseudo-variety.*

Let us observe that, if $r \geq 1$, then $\max(\mathcal{N}_r(a,b,X)) = \{0, r+1, \rightarrow\} \neq \mathbb{N}$. Therefore, by applying [4, Proposition 1], we have that $\mathcal{N}_r(a,b,X)$ is not a Frobenius variety.

Now, let us notice that the subgraph, of a tree, which is formed by a vertex and all its descendants is also a tree. We will denote by $\mathrm{G}\big(\mathcal{N}_r(a,b,X)\big)$ the subtree of $\mathrm{G}\big(\mathcal{N}(a,b,X)\big)$ formed by $\{0, r+1, \rightarrow\}$ and all its descendants.

Example 4.10. The root of $\mathrm{G}\big(\mathcal{N}_3\big((1,2),(4,1),\{5\}\big)\big)$ is $\{0,4,\rightarrow\} = \langle\{4,5,6,7\}\rangle$. Thus, from Example 3.4, we have that such a tree is given by Figure 3.

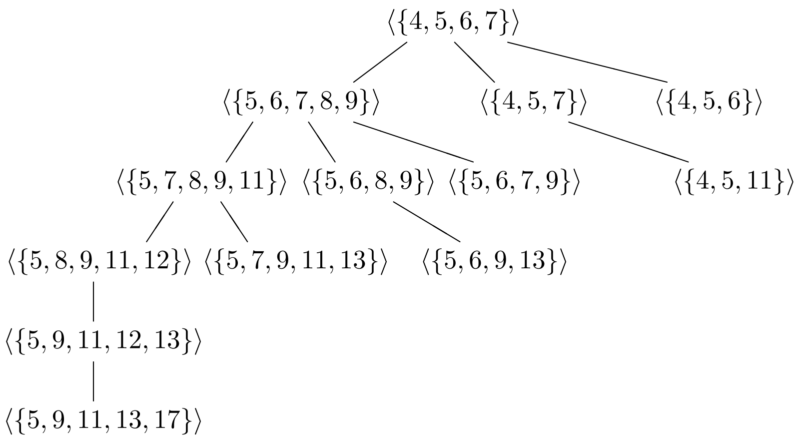

Figure 3: Tree associated to the Frobenius pseudo-variety $\mathcal{N}_3\big((1,2),(4,1),\{5\}\big)$.

Let us observe that $\mathcal{N}_r(a,b,X)$ is the set of vertices in $\mathrm{G}\big(\mathcal{N}_r(a,b,X)\big)$, and that $(S,S') \in \mathcal{N}_r(a,b,X) \times \mathcal{N}_r(a,b,X)$ is an edge of $\mathrm{G}\big(\mathcal{N}_r(a,b,X)\big)$ if and only if $S' = S \cup \{\mathrm{F}(S)\}$. It is also clear that, if $S \in \mathcal{N}_r(a,b,X)$, then the set formed by the children of S in $\mathcal{N}_r(a,b,X)$ is the same that the set formed by the children of S in $\mathcal{N}(a,b,X)$. In this way, by applying Theorem 3.1, we have the next result.

Proposition 4.11. *The graph $\mathrm{G}\big(\mathcal{N}_r(a,b,X)\big)$ is a tree with root $\{0, r+1, \rightarrow\}$. Moreover, the set of children of a vertex S in $\mathrm{G}\big(\mathcal{N}_r(a,b,X)\big)$ is*

$$\big\{S \setminus \{m\} \mid m \in \mathrm{msg}(S),\ m > \mathrm{F}(S),\ \textit{and}\ S \setminus \{m\} \in \mathcal{N}(a,b,X)\big\}.$$

Now, let us notice that, by using Propositions 3.2 and 3.3, we can compute the children of any vertex S in $\mathrm{G}\big(\mathcal{N}_r(a,b,X)\big)$ and, consequently, we have an algorithmic process to recurrently build the elements of $\mathcal{N}_r(a,b,X)$.

We finish this section with an illustrative example about the above comment.

Example 4.12. Let us compute all the solutions of $\mathrm{P}_3\big((2,3),(4,2),\{6,8\},4\big)$. In order to do this, we have to determine the vertices of $\mathrm{G}\big(\mathcal{N}_3\big((2,3),(4,2),\{6,8\}\big)\big)$ which are connected to $\{0,4,\rightarrow\} = \langle\{4,5,6,7\}\rangle$ through a path of length 4.

Let us observe that, if A is the set of vertices (of a tree) which are connected to the root through a path of length k, then the set formed by all vertices that are children of some vertex of A is just the set of vertices which are connected to the root through a path of length $k+1$. Thus, if we denote by A_i the set formed by the vertices of $\mathrm{G}\big(\mathcal{N}_3\big((2,3),(4,2),\{6,8\}\big)\big)$ which are connected to $\langle\{4,5,6,7\}\rangle$ through a path of length i, then (by applying Propositions 4.11, 3.2, and 3.3) we obtain recurrently the following sets.

- $A_0 = \big\{\langle\{4,5,6,7\}\rangle\big\}$
- $A_1 = \big\{\langle\{5,6,7,8,9\}\rangle, \langle\{4,6,7,9\}\rangle, \langle\{4,5,6\}\rangle\big\}$
- $A_2 = \big\{\langle\{6,7,8,9,10,11\}\rangle, \langle\{5,6,8,9\}\rangle, \langle\{5,6,7,8\}\rangle, \langle\{4,6,9,11\}\rangle, \langle\{4,6,7\}\rangle\big\}$
- $A_3 = \big\{\langle\{6,8,9,10,11,13\}\rangle, \langle\{6,7,8,10,11\}\rangle, \langle\{6,7,8,9,11\}\rangle, \langle\{6,7,8,9,10\}\rangle, \langle\{5,6,8\}\rangle, \langle\{4,6,11,13\}\rangle, \langle\{4,6,9\}\rangle\big\}$
- $A_4 = \big\{\langle\{6,8,10,11,13,15\}\rangle, \langle\{6,8,9,11,13\}\rangle, \langle\{6,8,9,10,13\}\rangle, \langle\{6,8,9,10,11\}\rangle, \langle\{6,7,8,11\}\rangle, \langle\{6,7,8,10\}\rangle, \langle\{6,7,8,9\}\rangle, \langle\{4,6,13,15\}\rangle, \langle\{4,6,11\}\rangle\big\}$

Therefore, the set of solutions of $\mathrm{P}_3\big((2,3),(4,2),\{6,8\},4\big)$ is

$$\big\{\langle\{4,5,6,7\}\rangle \setminus S \mid S \in A_4\big\} = \big\{\{4,5,7,9\},\{4,5,7,10\},\{4,5,7,11\},\{4,5,7,13\},\{4,5,9,10\},\{4,5,9,11\},\{4,5,10,11\},\{5,7,9,11\},\{5,7,9,13\}\big\}.$$

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Tilings of hyperbolic $(2 \times n)$-board with colored squares and dominoes

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Abstract

Several articles deal with tilings with squares and dominoes of the well-known regular square mosaic in Euclidean plane, but not any with the hyperbolic regular square mosaics. In this article, we examine the tiling problem with colored squares and dominoes of one type of the possible hyperbolic generalization of $(2 \times n)$-board.

Keywords: Tiling, domino, hyperbolic mosaic, Fibonacci numbers, combinatorial identity.

Math. Subj. Class.: 05A19, 05B45, 11B37, 11B39, 52C20

1 Introduction

In the hyperbolic plane there exist infinite types of regular mosaics, they are denoted by Schläfli's symbol $\{p, q\}$, where the positive integers p and q have the property $(p-2)(q-2) > 4$, see [5]. If $p = 4$ they are the regular square mosaics and each vertex of the mosaic is surrounded by q squares. Note that if $p = q = 4$ we obtain the Euclidean square mosaic.

Now we define the $(2 \times n)$-board on mosaic $\{4, q\}$, where $q \geq 4$. First we take a square S_1 with vertices A_0, A_1, B_1, B_0 according to Figure 1, and later to Figures 2 and 3. As the second step we consider the square S_2, which has a common edge A_1B_1 with S_1. The two new vertices are A_2, B_2. Similarly, we define the squares $S_3, \ldots, S_n$, their newly constructed vertices are A_i and B_i $(3 \leq i \leq n)$, respectively. The union of S_i $(1 \leq i \leq n)$

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forms the first level of the board. It is depicted with yellow colors in Figures 1-3. (On the left-hand side of Figure 2 the mosaic $\{4, 5\}$ and the (2×4)-board are illustrated in Poincaré disk model and on the right-hand side there is a “schematic” (2×4)-board from the mosaic.) The second level of the board is formed by the squares of the mosaic having at least one vertex from the set $\{A_1, A_2, \ldots, A_n\}$ and not from $\{B_1, B_2, \ldots, B_n, A_{n+1}\}$, where the last point is the appropriate point of the virtually joined square S_{n+1} (A_0 is not in the first set, see Figure 3). These are the light blue squares in the figures. In the first level, independently from q there are n squares, while the second level contains $n(q-3)$ squares (see Figure 3).

Let r_n be the number of the different tilings with (1×1)-squares and (1×2)-dominoes (two squares with a common edge) of a $(2 \times n)$-board of mosaic $\{4, q\}$. It is known that the tilings of a $(1 \times n)$-board on the Euclidean square mosaic can be counted by the Fibonacci numbers [2, 4]. In fact, $r_n = f_n$, where $\{f_n\}_{n=0}^{\infty}$ is the shifted Fibonacci sequence ($F_n = f_{n-1}$, where F_n is the n-th Fibonacci number, A000045 in OEIS [12]), so that

$$f_n = f_{n-1} + f_{n-2} \quad (n \geq 2)$$

holds with initial values $f_0 = f_1 = 1$ (and $f_{-1} = 0$).

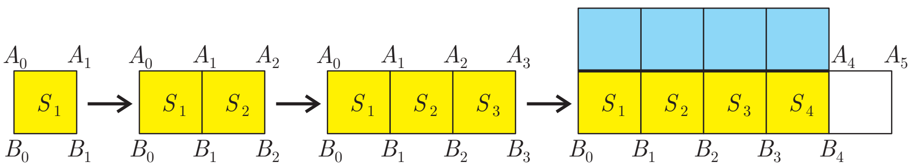

Figure 1: (2×4)-board on Euclidean mosaic $\{4, 4\}$.

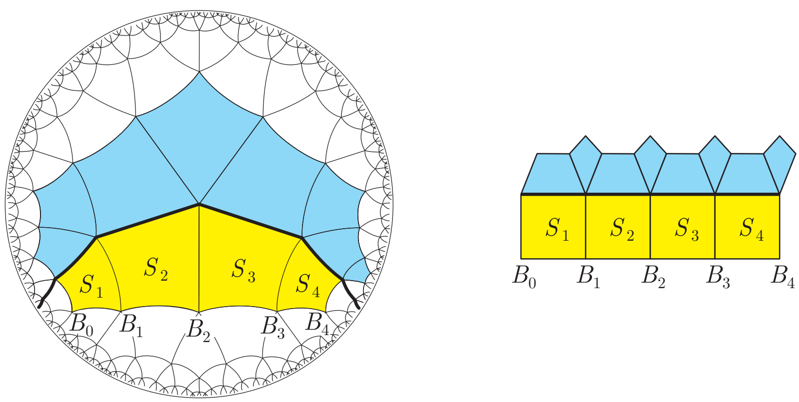

Figure 2: (2×4)-board on hyperbolic mosaic $\{4, 5\}$.

McQuistan and Lichtman [9] (generalizations in [6]) studied the tilings in case of the Euclidean square mosaic $\{4, 4\}$ and they proved that r_n satisfies the identity

$$r_n = 3r_{n-1} + r_{n-2} - r_{n-3} \tag{1.1}$$

for $n \geq 3$ with initial values $r_0 = 1$, $r_1 = 2$ and $r_2 = 7$ (A030186 in [12]).

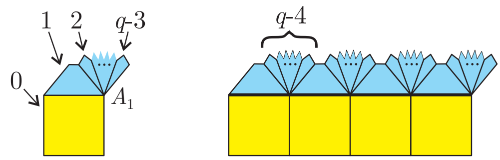

Figure 3: (2×1)-board and (2×4)-board on hyperbolic mosaic $\{4, q\}$ $(q \geq 5)$.

In the work [3], the generalized Fibonacci number u_n, where

$$u_n = au_{n-1} + bu_{n-2}, \quad (n \geq 2) \tag{1.2}$$

with initial values $u_0 = 1$, $u_1 = a$ (and $u_{-1} = 0$), is interpreted as the number of ways to tile a $(1 \times n)$-board using a colors of squares and b colors of dominoes. Obviously, if $a = b = 1$ then $u_n = f_n$. Belbachir and Belkhir proved a couple of general combinatorial identities related to u_n in [1].

Let R_n be the number of tilings of $(2 \times n)$-board of mosaic $\{4, q\}$ using a colors of squares and b colors of dominoes. When $q = 4$ Katz and Stenson [7] showed the recurrence rule

$$R_n = (a^2 + 2b)R_{n-1} + a^2 b\, R_{n-2} - b^3 R_{n-3}, \quad (n \geq 3) \tag{1.3}$$

with initial values $R_0 = 1$, $R_1 = a^2 + b$ and $R_2 = a^4 + 4a^2 b + 2b^2$.

In this article, we examine the tilings of $(2 \times n)$-board on mosaic $\{4, q\}$ $(q \geq 4)$ with colored squares and dominoes in a general way and we obtain the following main theorem.

Theorem 1.1. *Assume $q \geq 4$. The sequence $\{R_n\}_{n=0}^{\infty}$ can be described by the fourth order linear homogeneous recurrence relation*

$$R_n = \alpha_q\, R_{n-1} + \beta_q\, R_{n-2} + \gamma_q\, R_{n-3} - b^{2(q-2)} R_{n-4}, \quad (n \geq 4) \tag{1.4}$$

where (explicit formulas later)

$$\alpha_{q+2} = a\alpha_{q+1} + b\alpha_q, \tag{1.5}$$

$$\beta_{q+3} = (a^2 + b)\beta_{q+2} + b(a^2 + b)\beta_{q+1} - b^3 \beta_q, \tag{1.6}$$

$$\gamma_{q+2} = -ab\gamma_{q+1} + b^3 \gamma_q \tag{1.7}$$

with initial values

$$\alpha_4 = a^2 + b, \qquad \alpha_5 = a(a^2 + 3b),$$

$$\beta_4 = 2b(a^2 + b), \quad \beta_5 = b(a^2 + b)(a^2 + 2b), \quad \beta_6 = b(a^6 + 6a^4 b + 10a^2 b^2 + 2b^3),$$

$$\gamma_4 = b^2(a^2 - b), \quad \gamma_5 = -ab^3(a^2 + b),$$

moreover $R_0 = 1$, $R_1 = u_{q-2}$, $R_2 = u_{q-2}^2 + abu_{q-4}u_{q-3} + bu_{q-3}^2 + b^2 u_{q-4}^2$, $R_3 = (u_{q-2}^2 + 2abu_{q-4}u_{q-3} + 2bu_{q-3}^2 + 2b^2 u_{q-4}^2)u_{q-2} + b^2(u_{q-3}u_{q-4} + (a^2 + b)u_{q-4}u_{q-5} + au_{q-4}^2)u_{q-3} + ab^3 u_{q-4}^2 u_{q-5}$.

If $a = b = 1$, then Theorem 1.1 leads to the following corollary. Recall that $f_n = F_{n+1}$ (shifted Fibonacci numbers).

Corollary 1.2. *The sequence* $\{r_n\}_{n=0}^{\infty}$ *can be given by the fourth order linear homogeneous recurrence relation*

$$r_n = 2f_{q-3}\, r_{n-1} + \left(5f_{q-4}^2 + (-1)^{q-1}\right) r_{n-2} + 2(-1)^q f_{q-5}\, r_{n-3} - r_{n-4}, \quad (n \geq 4) \tag{1.8}$$

with initial values $r_0 = 1$, $r_1 = f_{q-2}$, $r_2 = 7f_{q-4}^2 + 7f_{q-4}f_{q-5} + 2f_{q-5}^2$ *and* $r_3 = 22f_{q-4}^3 + 36f_{q-4}^2 f_{q-5} + 19f_{q-4}f_{q-5}^2 + 3f_{q-5}^3$.

Observe, that if $q = 4$, then (1.4) returns with (1.3) (compute the sum of R_n and bR_{n-1}). Similarly, the extension of (1.1) is (1.8).

2 Tilings on mosaic $\{4, q\}$

We can see that our tiling exercise of the hyperbolic (2×1)-board on the mosaic $\{4, q\}$ $(q \geq 5)$ is the same as the tiling exercise of the Euclidean $\left(1 \times (q-2)\right)$-board. So $R_1 = u_{q-2}$ and $r_1 = f_{q-2}$ (Figure 3).

Before the discussion of the main result, we define the break-ability of a tiling. A tiling of a $(2 \times n)$-board is breakable in position i for $1 \leq i \leq n-1$, if this tiling is a concatenation of the tilings of a $(2 \times i)$-subboard and a $\left(2 \times (n-i)\right)$-subboard. Clearly, the number of colored tilings of such a board is $R_i R_{n-i}$. A tiling is unbreakable in position i in three different ways: if a domino covers the last square of the first subboard and the first square of the second subboard either in the first or the second level, or on both levels (see Figure 4).

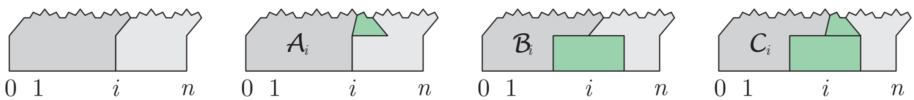

Figure 4: Breakable and unbreakable tilings in position i when $q = 7$.

Now, we define three subboards. Let $\mathcal{A}_i$, $\mathcal{B}_i$ and $\mathcal{C}_i$ be the subboards of $(2 \times i)$-board $(1 \leq i \leq n)$, respectively, where the last square from second level, the last square from first level and the last squares from both levels are deleted from $(2 \times i)$-board. In Figure 4 these subboards are illustrated. Let A_i, B_i and C_i denote the number of different colored tilings of $\mathcal{A}_i$, $\mathcal{B}_i$ and $\mathcal{C}_i$, respectively.

2.1 Proof of Theorem 1.1 and Corollary 1.2

Our proof is based on the connections among $(2 \times n)$-board, $\mathcal{A}_n$, $\mathcal{B}_n$ and $\mathcal{C}_n$ subboards. We can easily give the number of tilings if $n = 1$. They are $R_1 = u_{q-2}$, $A_1 = u_{q-4}$, $B_1 = u_{q-3}$ and $C_1 = u_{q-4}$. Moreover let $R_0 = 1$, $A_0 = B_0 = C_0 = 0$.

Generally, if $n \geq 2$, then Figure 5 shows the recurrence connections of the subboards. For example, let us see the first row. We can build a full $(2 \times n)$-board by four different ways from the full $\left(2 \times (n-1)\right)$-board or from the subboards $\mathcal{A}_{n-1}$, $\mathcal{B}_{n-1}$ and $\mathcal{C}_{n-1}$. If we join a suitable (2×1)-board to a $\left(2 \times (n-1)\right)$-board, then the coefficient u_{q-2} is obvious in case of the breakable tilings in position $n-1$. When we complete $\mathcal{A}_{n-1}$ to a full $(2 \times n)$-board, we have a domino in the second level with b different colors, and we put a square onto the first level with a colors. (If we replace the laid down domino in the second level with two squares, then these tilings would be a part of the first case when we completed the $\big(2 \times (n-1)\big)$-board.) The rest part can be tiled freely. Consequently, the coefficient of A_{n-1} is abu_{q-4} and these are unbreakable tilings in position $n-1$. Now, let us complete $\mathcal{B}_{n-1}$ and $\mathcal{C}_{n-1}$ to be full $(2 \times n)$-board with a domino in the first level or with two dominoes, one is in the first level and the other in the second level, respectively. The rest parts can be tiled freely. We obtain bu_{q-3} and b^2u_{q-4} new (unbreakable in position $n-1$) tilings. Summarising the result of the first row of Figure 5 we have the first equation of the system of recurrence equations (2.1). The determinations of the other rows can be explained similarly. We mention, that, for example, in the fourth row $\mathcal{B}_{n-1}$ does not appear, because when we complete it to $\mathcal{C}_n$ we do not have new tiling type, the tilings are in the first tiling types in the same row. (The yellow square would be in the grey $\big(2 \times (n-1)\big)$-board – see the last row in Figure 5.) Hence the recurrence equations for $n \geq 1$ satisfy the system

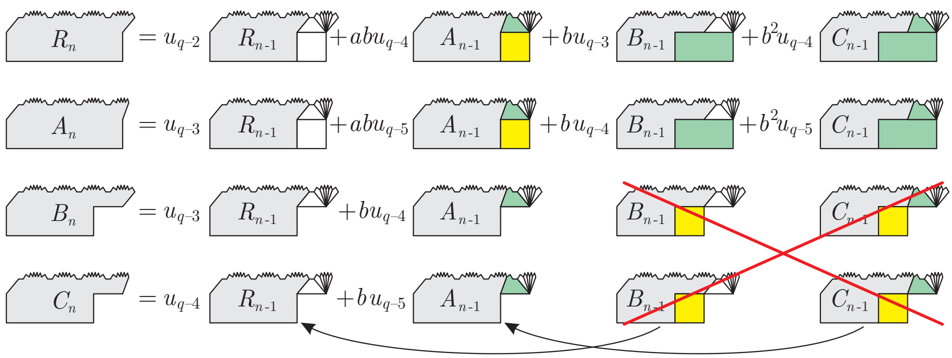

Figure 5: Base of recurrence connections of the subboards.

$$
\begin{aligned}
R_n &= u_{q-2}R_{n-1} + ab\,u_{q-4}A_{n-1} + b\,u_{q-3}B_{n-1} + b^2\,u_{q-4}C_{n-1} \\
A_n &= u_{q-3}R_{n-1} + ab\,u_{q-5}A_{n-1} + b\,u_{q-4}B_{n-1} + b^2\,u_{q-5}C_{n-1} \\
B_n &= u_{q-3}R_{n-1} + b\,u_{q-4}A_{n-1} \\
C_n &= u_{q-4}R_{n-1} + b\,u_{q-5}A_{n-1}.
\end{aligned}
\tag{2.1}
$$

Recall that the initial values are $R_0 = 1$, $A_0 = B_0 = C_0 = 0$. The matrix of the coefficients of (2.1) is

$$
\mathbf{M} = \begin{pmatrix} u_{q-2} & ab\,u_{q-4} & b\,u_{q-3} & b^2\,u_{q-4} \\ u_{q-3} & ab\,u_{q-5} & b\,u_{q-4} & b^2\,u_{q-5} \\ u_{q-3} & b\,u_{q-4} & 0 & 0 \\ u_{q-4} & b\,u_{q-5} & 0 & 0 \end{pmatrix}.
$$

As usual, the characteristic equation of $\mathbf{M}$ provides the recurrence relation for $\{R_n\}$ (and $\{A_n\}$, $\{B_n\}$, $\{C_n\}$; see the proof in [10]. The computation was made by the help of software MAPLE.) Thus we have

$$
R_n = \alpha_q\,R_{n-1} + \beta_q\,R_{n-2} + \gamma_q\,R_{n-3} + \delta_q R_{n-4} \quad (n \geq 4), \tag{2.2}
$$

where (with some calculation using (1.2))

$$\begin{aligned}
\alpha_q &= abu_{q-5} + u_{q-2},\\
\beta_q &= b(b^2u_{q-5}^2 - au_{q-5}u_{q-2} + 2bu_{q-4}^2 + au_{q-4}u_{q-3} + u_{q-3}^2),\\
\gamma_q &= -b^2(bu_{q-5}^2u_{q-2} - 2u_{q-4}u_{q-3}^2 + au_{q-5}u_{q-3}^2 + u_{q-4}^2u_{q-2}),\\
\delta_q &= -b^4(u_{q-5}^2u_{q-3}^2 - 2u_{q-5}u_{q-4}^2u_{q-3} + u_{q-4}^4).
\end{aligned}$$

Moreover, we obtain the initial values of the recurrence for $n = 1, 2, 3$ from system (2.1). They are $R_1 = u_{q-2}$, $R_2 = u_{q-2}^2 + abu_{q-4}u_{q-3} + bu_{q-3}^2 + b^2u_{q-4}^2$ and

$$\begin{aligned}
R_3 = {}&(u_{q-2}^2 + abu_{q-4}u_{q-3} + bu_{q-3}^2 + b^2u_{q-4}^2)u_{q-2}\\
&+ (abu_{q-2}u_{q-4} + a^2b^2u_{q-4}u_{q-5} + b^2u_{q-3}u_{q-4} + b^3u_{q-4}u_{q-5})u_{q-3}\\
&+ (bu_{q-2}u_{q-3} + ab^2u_{q-4}^2)u_{q-3} + (b^2u_{q-2}u_{q-4} + ab^3u_{q-4}u_{q-5})u_{q-4}.
\end{aligned}$$

In the next part, we prove that relations (1.5)–(1.7) hold. Firstly, we insert α_{q+2}, α_{q+1} and α_q into (1.5) to have

$$abu_{q-3} + u_q = a(abu_{q-4} + u_{q-1}) + b(abu_{q-5} + u_{q-2}). \tag{2.3}$$

Apply (1.2) consecutively with $n = q, q-1, \dots$ as follows. First plug u_q into the equation (2.3), then substitute u_{q-1} in the new equation, and so an. Finally, when $n = q-3$, we find that (2.3) is an identity, so (1.5) holds. If $q = 4$ and $q = 5$, then α_q provides the initial values. The proofs of (1.6) and (1.7) go similarly.

Finally, we show that $\delta_q = -b^{2(q-2)}$. For $q = 4$ we immediately obtain $\delta_4 = -b^4(u_{4-5}^2u_{4-3}^2 - 2u_{4-5}u_{4-4}^2u_{4-3} + u_{4-4}^4) = -b^{2\cdot 2}$. Then we consider the recurrence relation ($q \geq 4$)

$$x^{q+1} = b^2x^q. \tag{2.4}$$

Some calculations show that both expressions (δ_q and $-b^{2(q-2)}$) satisfies recursion (2.4), which implies the equality.

We express the values by u_{q-4} and u_{q-5} by using relation (1.2). Thus we have

$$\begin{aligned}
\alpha_q &= (a^2 + b)u_{q-4} + 2abu_{q-5},\\
\beta_q &= (2a^2 + 2b)bu_{q-4}^2 + (-a^3 + 2ab)bu_{q-4}u_{q-5} + (-a^2b + 2b^2)bu_{q-5}^2,\\
\gamma_q &= (a^2 - b)b^2u_{q-4}^3 - (a^3 - 3ab)b^2u_{q-4}^2u_{q-5} - (3a^2b - b^2)b^2u_{q-4}u_{q-5}^2 - 2ab^4u_{q-5}^3,\\
\delta_q &= -b^{2(q-2)}.
\end{aligned}$$

As $F_n^2 - F_nF_{n-1} - F_{n-1}^2 = (-1)^{n-1}$, if $a = b = 1$, then we obtain

$$\begin{aligned}
\alpha_q &= 2f_{q-4} + 2f_{q-5} = 2f_{q-3},\\
\beta_q &= 4f_{q-4}^2 + f_{q-4}f_{q-5} + f_{q-5}^2 = 5f_{q-4}^2 + (-1)^{q-1},\\
\gamma_q &= 2f_{q-4}^2f_{q-5} - 2f_{q-4}f_{q-5}^2 - 2f_{q-5}^3 = 2(-1)^qf_{q-5},\\
\delta_q &= -1.
\end{aligned}$$

Now the initial values R_i lead to the initial values r_i ($i = 1, 2, 3$).

2.2 Unbreakable tilings

In this subsection we determine the number of unbreakable tilings. Let $\widetilde{r}_n$ (and $\widetilde{R}_n$) be the number of different unbreakable tilings with (colored) squares and dominoes of $(2 \times n)$-board of $\{4, q\}$. Moreover, let $\widetilde{A}_i$, $\widetilde{B}_i$ and $\widetilde{C}_i$ denote the number of the different unbreakable colored tilings of $\mathcal{A}_i$, $\mathcal{B}_i$ and $\mathcal{C}_i$, respectively.

Theorem 2.1. *The sequence $\{\widetilde{R}_n\}$ can be described by the binary recurrence relation*

$$\widetilde{R}_n = abu_{q-5}\widetilde{R}_{n-1} + b^2\left(u_{q-4}^2 + bu_{q-5}^2\right)\widetilde{R}_{n-2}, \quad (n \geq 3)$$

where the initial values are $\widetilde{R}_1 = u_{q-2}$ and $\widetilde{R}_2 = abu_{q-3}u_{q-4} + bu_{q-3}^2 + b^2u_{q-4}^2$.

Proof. The proof is similar to the proof of the first theorem. By deleting the breakable tilings from Figure 5 (the second column) we gain the system of recurrence sequences $(n \geq 2)$

$$\begin{aligned}
\widetilde{R}_n &= abu_{q-4}\widetilde{A}_{n-1} + bu_{q-3}\widetilde{B}_{n-1} + b^2u_{q-4}\widetilde{C}_{n-1}\\
\widetilde{A}_n &= abu_{q-5}\widetilde{A}_{n-1} + bu_{q-4}\widetilde{B}_{n-1} + b^2u_{q-5}\widetilde{C}_{n-1}\\
\widetilde{B}_n &= bu_{q-4}\widetilde{A}_{n-1}\\
\widetilde{C}_n &= bu_{q-5}\widetilde{A}_{n-1}
\end{aligned}$$

with initial values $\widetilde{R}_1 = u_{q-2}$, $\widetilde{A}_1 = u_{q-3}$, $\widetilde{B}_1 = u_{q-3}$, $\widetilde{C}_1 = u_{q-4}$. The characteristic equation of its coefficients matrix gives the recurrence for $\widetilde{R}_n$. From the system of recurrence sequences we gain $\widetilde{R}_2$. □

Supposing $a = b = 1$, together with

$$\begin{aligned}
\widetilde{r}_2 &= 3f_{q-4}^2 + 3f_{q-4}f_{q-5} + f_{q-5}^2 = 4f_{q-4}^2 + 2f_{q-4}f_{q-5} + (-1)^{q-1}\\
&= 2f_{q-4}(2f_{q-4} + f_{q-5}) + (-1)^{q-1},
\end{aligned}$$

we obtain the following corollary.

Corollary 2.2. *The sequence $\{\widetilde{r}_n\}$ satisfies the binary recurrence relation*

$$\widetilde{r}_n = f_{q-5}\widetilde{r}_{n-1} + \left(f_{q-4}^2 + f_{q-5}^2\right)\widetilde{r}_{n-2}, \quad (n \geq 3)$$

with coefficients linked to Fibonacci numbers, where the initial values are $\widetilde{r}_1 = f_{q-2}$ and $\widetilde{r}_2 = 2f_{q-4}f_{q-2} + (-1)^{q-1}$.

3 Some identities

In the sequel, we give certain identities related to the sequences $\{R_n\}$ and $\{\widetilde{R}_n\}$. The proofs are based on the tilings, not on the recursive formulae.

Identity 3.1. If $n \geq 1$, then

$$R_n = \sum_{i=0}^{n-1} R_i\widetilde{R}_{n-i}.$$

Proof. Let us consider the breakable colored tilings in position i $(0 \le i < n)$ of $(2 \times n)$-board, where the tilings on the right $\big(2 \times (n-i)\big)$-subboard are unbreakable (see Figure 6). The number of this tilings is $R_i \widetilde{R}_{n-i}$. If $i = 0$, then the tilings are unbreakable on the whole $(2 \times n)$-board. Clearly, when i goes from 1 to $n-1$, we have different tilings and we consider all of them exhaustedly. □

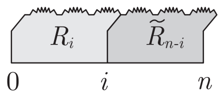

Figure 6: Breakable tilings in position i in case of Identity 3.1.

An equivalent form of Identity 3.1 is

Identity 3.2. If $n \ge 1$, then

$$R_n = \sum_{i=1}^{n} R_{n-i} \widetilde{R}_i .$$

The next statement gives another rule of summation.

Identity 3.3. If $m \ge 1$ and $n \ge 1$, then

$$R_{n+m} = R_n R_m + \sum_{i=1}^{n} \sum_{j=1}^{m} R_{n-i} R_{m-j} \widetilde{R}_{i+j} .$$

Proof. Let us consider a $\big(2 \times (n+m)\big)$-board as the concatenation of $(2 \times n)$-board and $(2 \times m)$-board (in other words, tilings are breakable in position n). First we take the breakable tilings in position n, their cardinality is $R_n R_m$. Then we examine the unbreakable tilings in this position. We cover the position n by $i + j$ long unbreakable tilings from position $n - i$ to $n + j$. They give the rest tilings. Figure 7 illustrates these two cases. □

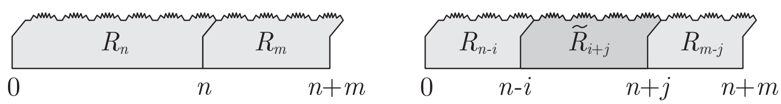

Figure 7: Tilings in case of Identity 3.3.

Identity 3.3 admits the following remarkable specific cases by the choice of $m = 1$, $m = (k-1)n$ and $n = n - k$, $m = n + k$, respectively.

Identity 3.4. If $n \ge 1$, then

$$R_{n+1} = R_n R_1 + \sum_{i=1}^{n} R_{n-i} \widetilde{R}_{i+1} .$$

Identity 3.5. If $n \ge 1$ and $k \ge 2$, then

$$R_{kn} = R_n R_{(k-1)n} + \sum_{i=1}^{n} \sum_{j=1}^{(k-1)n} R_{n-i} R_{(k-1)n-j} \widetilde{R}_{i+j} .$$

Identity 3.6. If $n > k \geq 0$ then

$$R_{2n} = R_{n-k}R_{n+k} + \sum_{i=1}^{n-k}\sum_{j=1}^{n+k} R_{n-k-i}R_{n+k-j}\widetilde{R}_{i+j}.$$

Finally, we give an identity about the product of two arbitrary terms of the sequence $\{R_n\}$.

Identity 3.7. If $n, m \geq 1$, then

$$R_nR_m = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} R_iR_j\widetilde{R}_{n-i}\widetilde{R}_{m-j}.$$

Proof. Consider a $\big(2 \times (n+m)\big)$-board as a concatenation of $(2 \times n)$-board and $(2 \times m)$-board. The result is derived in a direct manner from the number of the breakable tilings in position n. See Figure 8. □

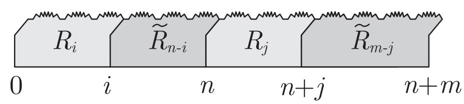

Figure 8: Tilings in case of Identity 3.7.

4 Conclusion and future work

In this article, we introduced a generalization of the square boards on the hyperbolic regular square mosaics and examined the combinatorial properties of tilings on these mosaics with colored squares and dominoes. As there are the infinite number of regular mosaics in the hyperbolic plane we hope that the examinations of the combinatorial properties of other tilings give some useful results. Moreover, we are informed on two additional timely articles about hyperbolic space tilings [8, 11].

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Mirrors of reflections of regular maps

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Abstract

A regular map $\mathcal{M}$ is an embedding of a finite connected graph into a compact surface S such that its automorphism group $\mathrm{Aut}^+(\mathcal{M})$ acts transitively on the directed edges. A reflection of $\mathcal{M}$ fixes a number of simple closed geodesics on S, which are called mirrors. In this paper, we prove two theorems which enable us to calculate the total number of mirrors fixed by the reflections of a regular map and the lengths of these mirrors. Furthermore, by applying these theorems to Hurwitz maps, we obtain some interesting results. In particular, we find an upper bound for the number of mirrors on Hurwitz surfaces.

Keywords: Riemann surface, regular map, Hurwitz map, reflection, mirror.

Math. Subj. Class.: 05C10, 30F10

1 Introduction

Let S be a compact Riemann surface of genus g. It is known that S can be expressed in the form $\mathbb{U}/\Lambda$, where $\mathbb{U}$ is the Riemann sphere Σ, the Euclidean plane $\mathbb{C}$, or the hyperbolic plane $\mathbb{H}$, depending on whether g is 0, 1 or > 1, respectively, and Λ is a discrete group of isometries of $\mathbb{U}$. A conformal or anti-conformal homeomorphism $f\colon S \to S$ is called an *automorphism* of S. If S admits an anti-conformal involution $r\colon S \to S$, then it is called *symmetric* and r is called a *symmetry* of S. The fixed-point set of r is either empty, or consists of disjoint simple closed geodesics on S. These geodesics are called the *mirrors* of r and their number cannot exceed $g + 1$ by a classical theorem of Harnack [10]. All automorphisms of S form a group under composition and it is denoted by $\mathrm{Aut}^{\pm}(S)$. The subgroup of $\mathrm{Aut}^{\pm}(S)$ consisting of orientation-preserving automorphisms is denoted by $\mathrm{Aut}^+(S)$.

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Let T be a triangle in $\mathbb{U}$, with angles $\pi/2$, π/m and π/n, where m and n are integers greater than one and $\frac{1}{m}+\frac{1}{n}$ is greater than, equal to or less than $\frac{1}{2}$ depending on whether $\mathbb{U}$ is Σ, $\mathbb{C}$ or $\mathbb{H}$, respectively. Such a triangle is said to be a $(2,m,n)$*-triangle*. Let Γ be the group generated by the rotations about the corners of T. Then it is called the *ordinary triangle group* $\Gamma[2,m,n]$ and it has a presentation

$$\langle x,y,z \mid x^2=y^m=z^n=xyz=1\rangle.$$

If Γ is the group generated by the reflections in the sides of T, then it is called the *extended triangle group* $\Gamma(2,m,n)$, which has a presentation

$$\langle a,b,c \mid a^2=b^2=c^2=(ab)^2=(bc)^m=(ca)^n=1\rangle.$$

A *map* $\mathcal{M}$ on S is an embedding of a finite connected graph G into S such that the interior of each *face* (a component of $S\setminus G$) is homeomorphic to an open disc. The *genus* of $\mathcal{M}$ is defined to be the genus of S. A directed edge of $\mathcal{M}$ is called a *dart* and $\mathcal{M}$ is said to be of *type* $\{m,n\}$ if every face of $\mathcal{M}$ has m sides and n darts meet at every vertex. An automorphism of S that leaves $\mathcal{M}$ invariant and preserves incidence is called an *automorphism* of $\mathcal{M}$. All automorphisms of $\mathcal{M}$ form a group under composition and this group is denoted by $\mathrm{Aut}^{\pm}(\mathcal{M})$. The subgroup of $\mathrm{Aut}^{\pm}(\mathcal{M})$ consisting of orientation-preserving automorphisms is denoted by $\mathrm{Aut}^{+}(\mathcal{M})$. If $\mathrm{Aut}^{+}(\mathcal{M})$ is transitive on the darts, then $\mathcal{M}$ is called *regular*. It is clear that if $\mathcal{M}$ is regular, then the number of darts is equal to $|\mathrm{Aut}^{+}(\mathcal{M})|$ and $\mathcal{M}$ has $|\mathrm{Aut}^{+}(\mathcal{M})|/2$ edges, $|\mathrm{Aut}^{+}(\mathcal{M})|/m$ faces and $|\mathrm{Aut}^{+}(\mathcal{M})|/n$ vertices.

If $\mathcal{M}$ is a regular map of type $\{m,n\}$ and $S=\mathbb{U}/\Lambda$ is the underlying Riemann surface, then by [12], Λ is normal in the ordinary triangle group $\Gamma[2,m,n]$. If Λ is also normal in the extended triangle group $\Gamma(2,m,n)$, then $\mathcal{M}$ is called *reflexible*. In that case $\mathcal{M}$ admits an anti-conformal involution r, which is a symmetry of S with fixed-points, called a *reflection* of $\mathcal{M}$.

In this paper, we prove two theorems which enable us to calculate the total number of mirrors fixed by the reflections of a regular map and the lengths of these mirrors. Furthermore, we use these theorems to obtain an upper bound for the total number of mirrors in Hurwitz maps.

Throughout this paper, we assume that the maps we deal with are regular and reflexible.

2 Patterns and mirror automorphisms

Let $\mathcal{M}$ be a regular map of type $\{m,n\}$ on a compact Riemann surface S of genus g. By joining the centers of the faces of $\mathcal{M}$ to the midpoints of the neighboring edges and vertices by geodesic arcs, we can divide S into $|\mathrm{Aut}^{\pm}(\mathcal{M})|$ $(2,m,n)$-triangles. If T is one of these triangles, then the group $\mathrm{Aut}^{\pm}(\mathcal{M})$ can be generated by the reflections in the sides of T and it has a presentation of the form

$$\langle A,B,C \mid A^2=B^2=C^2=(AB)^2=(BC)^m=(CA)^n=\cdots=1\rangle. \qquad (2.1)$$

Similarly, the group $\mathrm{Aut}^{+}(\mathcal{M})$ can be generated by the rotations about the corners of T and it has a presentation of the form

$$\langle X,Y,Z \mid X^2=Y^m=Z^n=XYZ=\cdots=1\rangle. \qquad (2.2)$$

Note that if $g = 0$, then the groups $\mathrm{Aut}^{\pm}(\mathcal{M})$ and $\mathrm{Aut}^{+}(\mathcal{M})$ are finite and the explicitly listed relations in (2.1) and (2.2) give presentations for these groups, respectively. If $g \geq 1$, then these presentations must contain at least one more relation.

Following [7], we label the vertices, edge-centers and face-centers of $\mathcal{M}$ with $\mathbf{0}$, $\mathbf{1}$ and $\mathbf{2}$, respectively. They are called the *geometric points* of $\mathcal{M}$. As an automorphism preserves the geometric points, it follows that a mirror of a reflection of $\mathcal{M}$ passes through some geometric points of $\mathcal{M}$ and these geometric points form a periodic sequence. Since S is compact, this sequence is finite and it is called the *pattern* of the mirror. As an example, consider the icosahedral map on the sphere, which has type $\{3, 5\}$. Each reflection of this map fixes a mirror with pattern $\mathbf{010212010212}$, which is abbreviated as $(\mathbf{010212})^2$; see [7]. Each repeated part of a pattern is called a *link*, and the number of links is called the *link index*. So in this example, $\mathbf{010212}$ is a link and the link index is 2.

In [15], it has been shown that the pattern of a mirror is always obtained from one of the six links $\mathbf{01}$, $\mathbf{02}$, $\mathbf{12}$, $\mathbf{0102}$, $\mathbf{0212}$, $\mathbf{010212}$, and there cannot be more than three mirrors with different patterns on the same Riemann surface. (See Figures 1 and 2, which represent regular maps admitting two and three different patterns, respectively.) The following theorem expresses this idea and it can be deduced from [15].

Theorem 2.1. *Let $\mathcal{M}$ be a regular map of type $\{m, n\}$ on a compact Riemann surface S and let M be a mirror of a reflection of $\mathcal{M}$. Then:*

(i) *If m and n are odd, then M has pattern of the form $(\mathbf{010212})^{\ell}$;*

(ii) *If m is even and n is odd, then M has pattern of the form $(\mathbf{0102})^{\ell_1}$ or $(\mathbf{12})^{\ell_2}$;*

(iii) *If m is odd and n is even, then M has pattern of the form $(\mathbf{0212})^{\ell_1}$ or $(\mathbf{01})^{\ell_2}$;*

(iv) *If m and n are even, then M has pattern of the form $(\mathbf{01})^{\ell_1}$, $(\mathbf{02})^{\ell_2}$ or $(\mathbf{02})^{\ell_3}$.*

Here ℓ, ℓ_1, ℓ_2 and ℓ_3 are positive integers, which depend only on $\mathcal{M}$, not on M. Furthermore, ℓ_is in different lines need not be equal.

Note that all the patterns listed in each part of Theorem 2.1 do occur. For example, in part (ii) the surface S contains two classes of mirrors such that the mirrors in different classes have different patterns, namely $(\mathbf{0102})^{\ell_1}$ and $(\mathbf{12})^{\ell_2}$. The same argument applies to all parts of Theorem 2.1.

Now let $\mathcal{M}$ be a regular map on a compact Riemann surface S and let M be a mirror of a reflection of $\mathcal{M}$. Suppose that ℓ is the link index of the pattern of M. If $\ell > 2$, then there exist two orientation-preserving automorphisms of $\mathcal{M}$ of order ℓ, which fix M setwise and have no fixed points on M. They rotate M in opposite directions and cyclically permute the links of the pattern of M. These automorphisms are inverses of each other and they are called the *mirror automorphisms* of M. Note that if $\ell = 2$, then M has a unique mirror automorphism. If $\ell = 1$, then we assume that the mirror automorphism of M is the identity. Associated to each pattern, there is a conjugacy class of mirror automorphisms such that the order of each mirror automorphism in this conjugacy class is equal to the link index of the pattern; see [15, Lemma 1]. In Table 1, for each pattern, a representative mirror automorphism is displayed in terms of the generators of $\mathrm{Aut}^{+}(\mathcal{M})$ in (2.2). Note that in the table, for each pattern, only one link is displayed. See [15] for details.

Table 1: Patterns and mirror automorphisms.

Case	Link	Mirror automorphism
1	**01**	$Z^{\frac{n}{2}}X$
2	**02**	$Y^{\frac{m}{2}}Z^{\frac{n}{2}}$
3	**12**	$Y^{\frac{m}{2}}X$
4	**0102**	$Z^{\frac{n+1}{2}}YZ^{\frac{n+1}{2}}Y^{\frac{m}{2}}$
5	**0212**	$Z^{\frac{n}{2}}Y^{\frac{m+1}{2}}ZY^{\frac{m+1}{2}}$
6	**010212**	$Y^{\frac{m+1}{2}}ZY^{\frac{m+1}{2}}Z^{\frac{n+1}{2}}YZ^{\frac{n+1}{2}}$

3 Number of mirrors

From now on, $\|\mathcal{M}\|$ will denote the total number of mirrors fixed by the reflections of a regular map $\mathcal{M}$.

Theorem 3.1. *Let $\mathcal{M}$ be a regular map of type $\{m, n\}$ on a compact Riemann surface, and let ℓ, ℓ_1, ℓ_2, ℓ_3 be as in Theorem* 2.1. *Then:*

(i) *If m and n are odd, then $\|\mathcal{M}\| = \frac{|\mathrm{Aut}^+(\mathcal{M})|}{2\ell}$;*

(ii) *If m and n have different parities, then $\|\mathcal{M}\| = \frac{|\mathrm{Aut}^+(\mathcal{M})|}{2}(\frac{1}{\ell_1} + \frac{1}{\ell_2})$;*

(iii) *If m and n are even, then $\|\mathcal{M}\| = \frac{|\mathrm{Aut}^+(\mathcal{M})|}{2}(\frac{1}{\ell_1} + \frac{1}{\ell_2} + \frac{1}{\ell_3})$.*

Proof. (i) By Theorem 2.1, every mirror of a reflection of $\mathcal{M}$ has pattern $(\mathbf{010212})^{\ell}$. It is clear that each of these mirrors contains ℓ edges of $\mathcal{M}$. Since $\mathcal{M}$ has $\frac{|\mathrm{Aut}^+(\mathcal{M})|}{2}$ edges, we find that $\|\mathcal{M}\| = \frac{|\mathrm{Aut}^+(\mathcal{M})|}{2\ell}$.

(ii) Suppose that m is even and n is odd. It follows from Theorem 2.1 that the pattern of a mirror of a reflection of $\mathcal{M}$ is either $(\mathbf{0102})^{\ell_1}$ or $(\mathbf{12})^{\ell_2}$. It is known that $\mathcal{M}$ has $\frac{|\mathrm{Aut}^+(\mathcal{M})|}{m}$ faces and a mirror with pattern $(\mathbf{0102})^{\ell_1}$ passes through the centers of ℓ_1 faces of $\mathcal{M}$. Also, the number of mirrors with pattern $(\mathbf{0102})^{\ell_1}$ passing through the center of a face F of $\mathcal{M}$ is $m/2$. (See Figure 1, where $m = 6$ and $n = 3$. The dashed lines denote the mirrors that have pattern $(\mathbf{0102})^{\ell_1}$ and pass through the center of F.) Therefore, there are

$$\frac{|\mathrm{Aut}^+(\mathcal{M})|}{m}\frac{1}{\ell_1}\frac{m}{2} = \frac{|\mathrm{Aut}^+(\mathcal{M})|}{2\ell_1}$$

mirrors with pattern $(\mathbf{0102})^{\ell_1}$. A similar argument shows that there are

$$\frac{|\mathrm{Aut}^+(\mathcal{M})|}{m}\frac{1}{\ell_2}\frac{m}{2} = \frac{|\mathrm{Aut}^+(\mathcal{M})|}{2\ell_2}$$

mirrors with pattern $(\mathbf{12})^{\ell_2}$. As a result, we find that

$$\|\mathcal{M}\| = \frac{|\mathrm{Aut}^+(\mathcal{M})|}{2}\left(\frac{1}{\ell_1} + \frac{1}{\ell_2}\right).$$

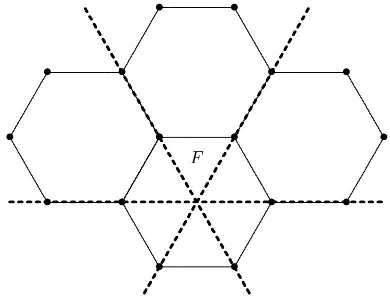

Figure 1: Mirrors with pattern $(\mathbf{0102})^{\ell_1}$, passing through a face-center.

The case where m is odd and n is even is similar.

(iii) In this case, by Theorem 2.1, the pattern of a mirror is either $(\mathbf{01})^{\ell_1}$, $(\mathbf{12})^{\ell_2}$ or $(\mathbf{02})^{\ell_3}$. We know that $\mathcal{M}$ has $\frac{|\mathrm{Aut}^+(\mathcal{M})|}{n}$ vertices and a mirror with pattern $(\mathbf{01})^{\ell_1}$ passes through ℓ_1 vertices of $\mathcal{M}$. Moreover, the number of mirrors with pattern $(\mathbf{01})^{\ell_1}$ passing through a vertex of $\mathcal{M}$ is $n/2$. (See Figure 2, where $m = n = 4$. The dashed lines denote the mirrors that have pattern $(\mathbf{01})^{\ell_1}$ and pass through a vertex v of $\mathcal{M}$.) Thus, there are

$$\frac{|\mathrm{Aut}^+(\mathcal{M})|}{n}\frac{1}{\ell_1}\frac{n}{2} = \frac{|\mathrm{Aut}^+(\mathcal{M})|}{2\ell_1}$$

mirrors with pattern $(\mathbf{01})^{\ell_1}$. Similar arguments show that there are $\frac{|\mathrm{Aut}^+(\mathcal{M})|}{2\ell_2}$ mirrors with pattern $(\mathbf{12})^{\ell_2}$ and $\frac{|\mathrm{Aut}^+(\mathcal{M})|}{2\ell_3}$ mirrors with pattern $(\mathbf{02})^{\ell_3}$. Consequently, we find that $\|\mathcal{M}\| = \frac{|\mathrm{Aut}^+(\mathcal{M})|}{2}(\frac{1}{\ell_1} + \frac{1}{\ell_2} + \frac{1}{\ell_3})$. □

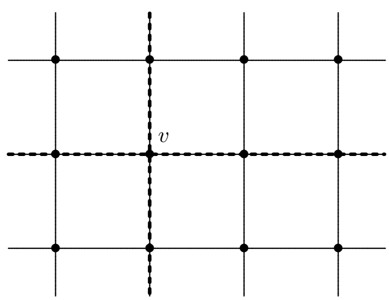

Figure 2: Mirrors with pattern $(\mathbf{01})^{\ell_1}$, passing through a vertex.

Note that if $\mathcal{M}$ is a reflexible regular map and if we are given a presentation for $\mathrm{Aut}^+(\mathcal{M})$ as in (2.2), then we can easily determine the link indices by using Table 1 and MAGMA [1]. This is because the link indices are the orders of the mirror automorphisms (see [15, Lemma 1]), and the latter are explicitly known (see Table 1). Then by using Theorem 3.1 we can easily calculate $\|\mathcal{M}\|$.

Example 3.2. Let S be the Riemann surface of genus 7 admitting 504 conformal automorphisms. This surface is known as the *Fricke-Macbeath surface*; see [8, 14]. It is known that S underlies a regular map $\mathcal{M}$ of type $\{3, 7\}$, which is called the *Fricke-Macbeath map*. It follows from [15] that $\mathrm{Aut}^+(\mathcal{M})$ has a presentation

$$\langle X, Y, Z \mid X^2 = Y^3 = Z^7 = XYZ = (Y^2ZY^2Z^4YZ^4)^2 = 1\rangle,$$

and $Y^2ZY^2Z^4YZ^4$ is a mirror automorphism. Since this automorphism has order 2, by Theorem 2.1 every mirror on S has pattern $(\mathbf{010212})^2$. Thus, by using Theorem 3.1 we find that $\|\mathcal{M}\| = \frac{504}{4} = 126$.

Remark 3.3. Let $\mathcal{M}$ be a regular map and $\mathcal{M}^*$ be its dual. Since the reflections of $\mathcal{M}$ and $\mathcal{M}^*$ coincide, the mirrors of $\mathcal{M}^*$ are the same as those of $\mathcal{M}$. So $\|\mathcal{M}\| = \|\mathcal{M}^*\|$.

Remark 3.4. Let $\mathcal{M}$ be a regular map on a compact Riemann surface S and let M be a mirror of a reflection of $\mathcal{M}$. If ℓ is the link index corresponding to the pattern of M, then the stabilizer of M in $\mathrm{Aut}^+(\mathcal{M})$ is the dihedral group D_ℓ. Here D_ℓ is generated by a mirror automorphism of M and an involution fixing two antipodal points of M. Since $\mathrm{Aut}^+(\mathcal{M})$ is transitive on the mirrors with the same pattern, the orbit of M consists of the mirrors on S which have the same pattern as M. So by the Orbit-Stabilizer theorem, we find that there are $|\mathrm{Aut}^+(\mathcal{M})|/2\ell$ mirrors in the orbit of M. By Theorem 2.1, there are at most three orbits, and their sizes can be determined in the same way. Therefore, we obtain an alternative proof of Theorem 3.1.

4 Lengths of mirrors

Let $\mathcal{M}$ be a regular map of type $\{m, n\}$ on a compact Riemann surface S of genus g and let M be a mirror of a reflection of $\mathcal{M}$. As pointed out in Section 2, S can be divided into $|\mathrm{Aut}^{\pm}(\mathcal{M})|$ $(2, m, n)$-triangles and M is a combination of the sides of $(2, m, n)$-triangles. Let a, b and c be the lengths of the sides of a $(2, m, n)$-triangle as indicated in Figure 3. If $g > 1$, then by using sine and cosine rules for hyperbolic triangles, we can calculate a, b and c. So the length of M can be calculated as described below.

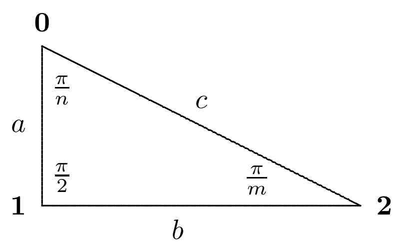

Figure 3: A $(2, m, n)$-triangle with side lengths a, b, c.

Let m and n be odd. Then M will have pattern of the form $(\mathbf{010212})^\ell$. Now every link corresponds to a segment of M, which has length $2(a + b + c)$. Thus, M has length $2\ell(a + b + c)$. Clearly, every mirror has the same length as M in this case. If m and n have different parities, then there are two classes of mirrors on S. If m and n are both even, then there are three classes of mirrors on S. In both cases the mirrors in each class have the same length and pattern. In each case, the lengths of the mirrors can be calculated in the same way. Note that if $g = 0$ or 1, then it is not difficult to find the length of the mirrors explicitly. So we have the following result:

Theorem 4.1. *Let $\mathcal{M}$ be a regular map of type $\{m, n\}$ on a compact Riemann surface S and let the lengths of the sides of a $(2, m, n)$-triangle be a, b and c as indicated in Figure 3. Then the lengths of the mirrors of the reflections of $\mathcal{M}$ can be determined by the formulae in Table 2, where ℓ, ℓ_1, ℓ_2 and ℓ_3 are the link indices and ℓ_is in different lines need not be equal.*

Table 2: Lengths of mirrors.

Case	Pattern	Length of mirror
m and n are odd	$(\mathbf{010212})^{\ell}$	$2\ell(a+b+c)$
m odd n even	$(\mathbf{01})^{\ell_1}$	$2\ell_1 a$
m odd n even	$(\mathbf{0212})^{\ell_2}$	$2\ell_2(b+c)$
m even n odd	$(\mathbf{12})^{\ell_1}$	$2\ell_1 b$
m even n odd	$(\mathbf{0102})^{\ell_2}$	$2\ell_2(a+c)$
m and n are even	$(\mathbf{01})^{\ell_1}$	$2\ell_1 a$
m and n are even	$(\mathbf{12})^{\ell_2}$	$2\ell_2 b$
m and n are even	$(\mathbf{02})^{\ell_3}$	$2\ell_3 c$

5 Application to Hurwitz maps

By a classical theorem of Hurwitz [11], a compact Riemann surface of genus $g > 1$ has at most $84(g-1)$ conformal automorphisms. Any such surface $S = \mathbb{H}/\Lambda$ is called a *Hurwitz surface*, and in that case $\mathrm{Aut}^+(S)$ is called a *Hurwitz group*. It is known that if S is a Hurwitz surface, then Λ is normal in the ordinary triangle group $\Gamma[2, 3, 7]$. Thus, every Hurwitz surface underlies a regular map of type $\{3, 7\}$, which is called a *Hurwitz map*. Furthermore, $\mathrm{Aut}^+(\mathcal{M})$ is isomorphic to $\mathrm{Aut}^+(S)$ and has a presentation of the form

$$\langle X, Y, Z \mid X^2 = Y^3 = Z^7 = XYZ = \cdots = 1\rangle.$$

It has been shown by [13] that the upper bound in Hurwitz's theorem is attained for infinitely many values of the genus g. Thus, there exist infinitely many Hurwitz maps and surfaces. See [2, 3, 4, 5, 13] for further details.

Theorem 5.1. *Let $\mathcal{M}$ be a Hurwitz map of genus g and let S be the underlying surface. Then $\|\mathcal{M}\| \leq 21(g-1)$, where equality holds if and only if S is the Fricke-Macbeath surface.*

Proof. Let ℓ be the link index of $\mathcal{M}$. By Theorem 3.1, we find that

$$\|\mathcal{M}\| = \frac{|\mathrm{Aut}^+(\mathcal{M})|}{2\ell} = \frac{84(g-1)}{2\ell} = \frac{42(g-1)}{\ell}.$$

It follows from [15, Theorem 5] that $\ell \geq 2$ and hence $\|\mathcal{M}\| \leq 21(g-1)$ and that equality holds if and only if S is the Fricke-Macbeath surface. See also [9, Theorem 4.1]. □

It follows from Theorem 5.1 that if $\mathcal{M}$ is a Hurwitz map of genus g with link index ℓ, then $\|\mathcal{M}\|$ is bounded above by $21(g-1)$. When this upper bound is attained, $\ell = 2$ and the underlying surface is the Fricke-Macbeath surface. However, $\|\mathcal{M}\|$ cannot have a lower bound in terms of g. This follows from the theorem below, which was given in [6].

Theorem 5.2. *For every positive integer n, there exist Hurwitz maps with link indices $2n$ and $3n$. In particular, the link index of a Hurwitz map can be any even positive integer.*

Let L be the sum of the lengths of the sides a $(2,3,7)$-triangle. Then by using the sine and cosine rules for hyperbolic triangles we find that $L \simeq 1.4490747226$. It follows from Theorem 4.1 that the length of a mirror on a Hurwitz surface is $2\ell L$, where ℓ is the link index. Also, the minimum possible length of a mirror on a Hurwitz surface is $4L \simeq 5.7962988904$, and in that case the underlying surface is the Fricke-Macbeath surface; see [15, Theorem 6]. However, by Theorem 5.2, there is no upper bound on the lengths of mirrors on Hurwitz surfaces.

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The thickness of $K_{1,n,n}$ and $K_{2,n,n}$*

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Abstract

The thickness of a graph G is the minimum number of planar subgraphs whose union is G. In this paper, we obtain the thickness of complete 3-partite graph $K_{1,n,n}$, $K_{2,n,n}$ and complete 4-partite graph $K_{1,1,n,n}$.

Keywords: Thickness, complete 3-partite graph, complete 4-partite graph.

Math. Subj. Class.: 05C10

1 Introduction

The *thickness* $\theta(G)$ of a graph G is the minimum number of planar subgraphs whose union is G. It was first defined by W. T. Tutte [7] in 1963, then a few authors obtained the thickness of hypercubes [5], complete graphs [1, 2, 8] and complete bipartite graphs [3]. Naturally, people wonder about the thickness of the complete multipartite graphs.

A *complete k-partite graph* is a graph whose vertex set can be partitioned into k parts, such that every edge has its ends in different parts and every two vertices in different parts are adjacent. Let $K_{p_1,p_2,\ldots,p_k}$ denote a complete k-partite graph in which the ith part contains p_i $(1 \leq i \leq k)$ vertices. For the complete 3-partite graph, Poranen proved $\theta(K_{n,n,n}) \leq \lceil \frac{n}{2} \rceil$ in [6], then Yang [10] gave a new upper bound for $\theta(K_{n,n,n})$, i.e., $\theta(K_{n,n,n}) \leq \lceil \frac{n+1}{3} \rceil + 1$ and obtained $\theta(K_{n,n,n}) = \lceil \frac{n+1}{3} \rceil$, when $n \equiv 3 \pmod 6$. And also Yang [9] gave the thickness number of $K_{l,m,n}(l \leq m \leq n)$ when $l + m \leq 5$ and showed that $\theta(K_{l,m,n}) = \lceil \frac{l+m}{2} \rceil$ when $l + m$ is even and $n > \frac{1}{2}(l + m - 2)^2$; or $l + m$ is odd and $n > (l + m - 2)(l + m - 1)$.

In this paper, we obtain the thickness of complete 3-partite graph $K_{1,n,n}$ and $K_{2,n,n}$, and we also deduce the thickness of complete 4-partite graph $K_{1,1,n,n}$ from that of $K_{2,n,n}$.

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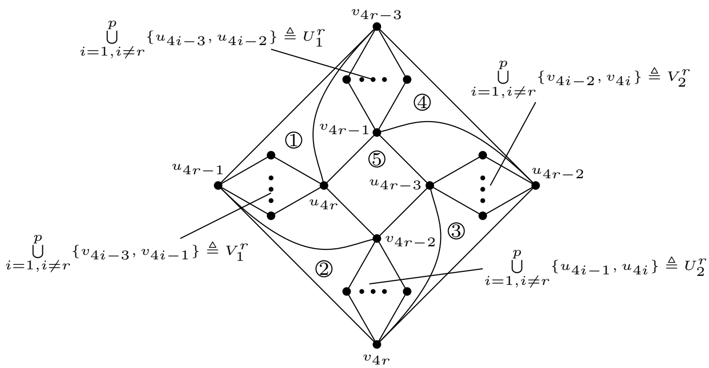
2 The thickness of $K_{1,n,n}$

In [3], Beineke, Harary and Moon gave the thickness of complete bipartite graphs $K_{m,n}$ for most value of m and n, and their theorem implies the following result immediately.

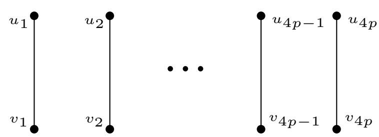
Lemma 2.1 ([3])**.** *The thickness of the complete bipartite graph $K_{n,n}$ is*

$$\theta(K_{n,n}) = \left\lceil \frac{n+2}{4} \right\rceil .$$

In [4], Chen and Yin gave a planar decomposition of the complete bipartite graph $K_{4p,4p}$ with $p+1$ planar subgraphs. Figure 1 shows their planar decomposition of $K_{4p,4p}$, in which $\{u_1, \ldots, u_{4p}\} = U$ and $\{v_1, \ldots, v_{4p}\} = V$ are the 2-partite vertex sets of it. Based on their decomposition, we give a planar decomposition of $K_{2,n,n}$ with $p+1$ subgraphs when $n \equiv 0$ or $3 \pmod 4$ and prove the following lemma.



(a) The graph G_r $(1 \leq r \leq p)$.



(b) The graph G_{p+1}.

Figure 1: A planar decomposition of $K_{4p,4p}$.

Lemma 2.2. *The thickness of the complete* 3*-partite graph $K_{1,n,n}$ and $K_{2,n,n}$ is*

$$\theta(K_{1,n,n}) = \theta(K_{2,n,n}) = \left\lceil \frac{n+2}{4} \right\rceil ,$$

when $n \equiv 0$ or $3 \pmod 4$.

Proof. Let the vertex partition of $K_{2,n,n}$ be (X, U, V), where $X = \{x_1, x_2\}$, $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$.

When $n \equiv 0 \pmod 4$, let $n = 4p$ $(p \geq 1)$. Let $\{G_1, \ldots, G_{p+1}\}$ be the planar decomposition of $K_{n,n}$ constructed by Chen and Yin in [4]. As shown in Figure 1, the graph G_{p+1} consists of n paths of length one. We put all the n paths in a row, place vertex x_1 on one side of the row and the vertex x_2 on the other side of the row, join both x_1 and x_2 to all vertices in G_{p+1}. Then we get a planar graph, denote it by $\widehat{G}_{p+1}$. It is easy to see that $\{G_1, \ldots, G_p, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{2,n,n}$. Therefore, we have $\theta(K_{2,n,n}) \leq p+1$. Since $K_{n,n} \subset K_{1,n,n} \subset K_{2,n,n}$, combining it with Lemma 2.1, we have

$$p+1 = \theta(K_{n,n}) \leq \theta(K_{1,n,n}) \leq \theta(K_{2,n,n}) \leq p+1,$$

that is, $\theta(K_{1,n,n}) = \theta(K_{2,n,n}) = p+1$ when $n \equiv 0 \pmod 4$.

When $n \equiv 3 \pmod 4$, then $n = 4p+3$ $(p \geq 0)$. When $p = 0$, from [9], we have $\theta(K_{1,3,3}) = \theta(K_{2,3,3}) = 2$. When $p \geq 1$, since $K_{n,n} \subset K_{1,n,n} \subset K_{2,n,n} \subset K_{2,n+1,n+1}$, according to Lemma 2.1 and $\theta(K_{2,4p,4p}) = p+1$, we have

$$p+2 = \theta(K_{n,n}) \leq \theta(K_{1,n,n}) \leq \theta(K_{2,n,n}) \leq \theta(K_{2,n+1,n+1}) = p+2.$$

Then, we get $\theta(K_{1,n,n}) = \theta(K_{2,n,n}) = p+2$ when $n \equiv 3 \pmod 4$.

Summarizing the above, the lemma is obtained. □

Lemma 2.3. *There exists a planar decomposition of the complete 3-partite graph $K_{1,4p+2,4p+2}$ $(p \geq 0)$ with $p+1$ subgraphs.*

Proof. Suppose the vertex partition of the complete 3-partite graph $K_{1,n,n}$ is (X, U, V), where $X = \{x\}$, $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$. When $n = 4p+2$, we will construct a planar decomposition of $K_{1,4p+2,4p+2}$ with $p+1$ planar subgraphs to complete the proof. Our construction is based on the planar decomposition $\{G_1, G_2, \ldots, G_{p+1}\}$ of $K_{4p,4p}$ given in [4], as shown in Figure 1 and the reader is referred to [4] for more details about this decomposition. For convenience, we denote the vertex set $\bigcup_{i=1,i\neq r}^{p}\{u_{4i-3}, u_{4i-2}\}$, $\bigcup_{i=1,i\neq r}^{p}\{u_{4i-1}, u_{4i}\}$, $\bigcup_{i=1,i\neq r}^{p}\{v_{4i-3}, v_{4i-1}\}$ and $\bigcup_{i=1,i\neq r}^{p}\{v_{4i-2}, v_{4i}\}$ by U_1^r, U_2^r, V_1^r and V_2^r respectively. We also label some faces of G_r $(1 \leq r \leq p)$, as indicated in Figure 1, for example, the face 1 is bounded by $v_{4r-3}u_{4r}v_ju_{4r-1}$ in which v_j is some vertex from V_1^r.

In the following, for $1 \leq r \leq p+1$, by adding vertices $x, u_{4p+1}, u_{4p+2}, v_{4p+1}, v_{4p+2}$ and some edges to G_r, and deleting some edges from G_r such edges will be added to the graph G_{p+1}, we will get a new planar graph $\widehat{G}_r$ such that $\{\widehat{G}_1, \ldots, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{1,4p+2,4p+2}$. Because v_{4r-3} and v_{4r-1} in G_r $(1 \leq r \leq p)$ is joined by $2p-2$ edge-disjoint paths of length two that we call parallel paths, we can change the order of these parallel paths without changing the planarity of G_r. For the same reason, we can do changes like this for parallel paths between u_{4r-1} and u_{4r}, v_{4r-2} and v_{4r}, u_{4r-3} and u_{4r-2}. We call this change by parallel paths modification for simplicity. All the subscripts of vertices are taken modulo $4p$, except that of $v_{4p+1}, v_{4p+2}, u_{4p+1}$ and u_{4p+2} (the vertices we added to G_r).

Case 1. When p is even and $p > 2$.

(a) The construction for $\widehat{G}_r$, $1 \leq r \leq p$, and r is odd.
Step 1: Place the vertex x in the face 1 of G_r, delete edges $v_{4r-3}u_{4r}$ and $u_{4r}v_{4r-1}$ from G_r. Do parallel paths modification, such that $u_{4r+6} \in U_1^r$, $v_{4r+1} \in V_1^r$ and u_{4r-3}, u_{4r-1},

$u_{4r}, v_{4r-3}, v_{4r-2}, v_{4r-1}$ are incident with a common face which the vertex x is in. Join x to $u_{4r-3}, u_{4r-1}, u_{4r}, v_{4r-3}, v_{4r-2}, v_{4r-1}$ and u_{4r+6}, v_{4r+1}.

Step 2: Do parallel paths modification, such that $u_{4r+11}, u_{4r+12} \in U_2^r$ are incident with a common face. Place the vertex v_{4p+1} in the face, and join it to both u_{4r+11} and u_{4r+12}.

Step 3: Do parallel paths modification, such that $u_{4r+7}, u_{4r+8} \in U_2^r$ are incident with a common face. Place the vertex v_{4p+2} in the face, and join it to both u_{4r+7} and u_{4r+8}.

Step 4: Do parallel paths modification, such that $v_{4r+10}, v_{4r+12} \in V_2^r$ are incident with a common face. Place the vertex u_{4p+1} in the face, and join it to both v_{4r+10} and v_{4r+12}.

Step 5: Do parallel paths modification, such that $v_{4r+6}, v_{4r+8} \in V_2^r$ are incident with a common face. Place the vertex u_{4p+2} in the face, and join it to both v_{4r+6} and v_{4r+8}.

(b) The construction for $\widehat{G}_r$, $1 \leq r \leq p$, and r is even.

Step 1: Place the vertex x in the face 3 of G_r, delete edges $v_{4r}u_{4r-3}$ and $u_{4r-3}v_{4r-2}$ from G_r. Do parallel paths modification, such that $u_{4r+7} \in U_2^r$, $v_{4r+4} \in V_2^r$ and $u_{4r-3}, u_{4r-2}, u_{4r}, v_{4r-2}, v_{4r-1}, v_{4r}$ are incident with a common face which the vertex x is in. Join x to $u_{4r-3}, u_{4r-2}, u_{4r}, v_{4r-2}, v_{4r-1}, v_{4r}$ and u_{4r+7}, v_{4r+4}.

Step 2: Do parallel paths modifications, such that $u_{4r+5}, u_{4r+6} \in U_1^r$, $u_{4r+1}, u_{4r+2} \in U_1^r$, $v_{4r+5}, v_{4r+7} \in V_1^r$, $v_{4r+1}, v_{4r+3} \in V_1^r$ are incident with a common face, respectively. Join v_{4p+1} to both u_{4r+5} and u_{4r+6}, join v_{4p+2} to both u_{4r+1} and u_{4r+2}, join u_{4p+1} to both v_{4r+5} and v_{4r+7}, join u_{4p+2} to both v_{4r+1} and v_{4r+3}.

Table 1 shows how we add edges to G_r $(1 \leq r \leq p)$ in Case 1. The first column lists the edges we added, the second and third column lists the subscript of vertices, and we also indicate the vertex set which they belong to in brackets.

Table 1: The edges we add to $G_r(1 \leq r \leq p)$ in Case 1.

subscript \ case / edge	r is odd		r is even	
xu_j	$4r-3, 4r-1, 4r$	$4r+6$ (U_1^r)	$4r-3, 4r-2, 4r$	$4r+7$ (U_2^r)
xv_j	$4r-3, 4r-2, 4r-1$	$4r+1$ (V_1^r)	$4r-2, 4r-1, 4r$	$4r+4$ (V_2^r)
$v_{4p+1}u_j$	$4r+11, 4r+12$ (U_2^r)		$4r+5, 4r+6$ (U_1^r)	
$v_{4p+2}u_j$	$4r+7, 4r+8$ (U_2^r)		$4r+1, 4r+2$ (U_1^r)	
$u_{4p+1}v_j$	$4r+10, 4r+12$ (V_2^r)		$4r+5, 4r+7$ (V_1^r)	
$u_{4p+2}v_j$	$4r+6, 4r+8$ (V_2^r)		$4r+1, 4r+3$ (V_1^r)	

(c) The construction for $\widehat{G}_{p+1}$.

From the construction in (a) and (b), the subscript set of u_j that xu_j is an edge in $\widehat{G}_r$ for some $r \in \{1, \ldots, p\}$ is

$$\{4r-3, 4r-1, 4r, 4r+6 \pmod{4p} \mid 1 \leq r \leq p, \text{and } r \text{ is odd}\}$$
$$\cup \{4r-3, 4r-2, 4r, 4r+7 \pmod{4p} \mid 1 \leq r \leq p, \text{and } r \text{ is even}\} = \{1, \ldots, p\}.$$

The subscript set of u_j that $v_{4p+1}u_j$ is an edge in $\widehat{G}_r$ for some $r \in \{1, \dots, p\}$ is

$$\begin{aligned}\{4r+11, 4r+12 \pmod{4p} \mid 1 \le r \le p, \text{and } r \text{ is odd}\}\\ \cup \{4r+5, 4r+6 \pmod{4p} \mid 1 \le r \le p, \text{and } r \text{ is even}\}\\ = \{4r-3, 4r-2, 4r-1, 4r \mid 1 \le r \le p, \text{and } r \text{ is even}\}.\end{aligned}$$

Using the same procedure, we can list all the edges incident with x, v_{4p+1}, v_{4p+2}, u_{4p+1} and u_{4p+2} in $\widehat{G}_r$ $(1 \le r \le p)$, so we can also list the edges that are incident with x, v_{4p+1}, v_{4p+2}, u_{4p+1} in $K_{1,4p+2,4p+2}$ but not in any $\widehat{G}_r$ $(1 \le r \le p)$. Table 2 shows the edges that belong to $K_{1,4p+2,4p+2}$ but not to any $\widehat{G}_r$, $1 \le r \le p$, in which the the fourth and fifth rows list the edges deleted form G_r $(1 \le r \le p)$ in step one of (a) and (b), and the sixth row lists the edges of G_{p+1}. The $\widehat{G}_{p+1}$ is the graph consists of the edges in Table 2, Figure 2 shows $\widehat{G}_{p+1}$ is a planar graph.

Table 2: The edges of $\widehat{G}_{p+1}$ in Case 1.

edges	subscript
$xv_{4p+1}, xu_{4p+1}, v_{4p+1}u_j, u_{4p+1}v_j$	$j = 4r-3, 4r-2, 4r-1, 4r, 4p+2$ $(r = 1, 3, \dots, p-1)$
$xv_{4p+2}, xu_{4p+2}, v_{4p+2}u_j, u_{4p+2}v_j$	$j = 4r-3, 4r-2, 4r-1, 4r, 4p+1$ $(r = 2, 4, \dots, p)$
$v_{4r-3}u_{4r}, u_{4r}v_{4r-1}$	$r = 1, 3, \dots, p-1$
$v_{4r}u_{4r-3}, u_{4r-3}v_{4r-2}$	$r = 2, 4, \dots, p$
u_jv_j	$j = 1, \dots, 4p+2$

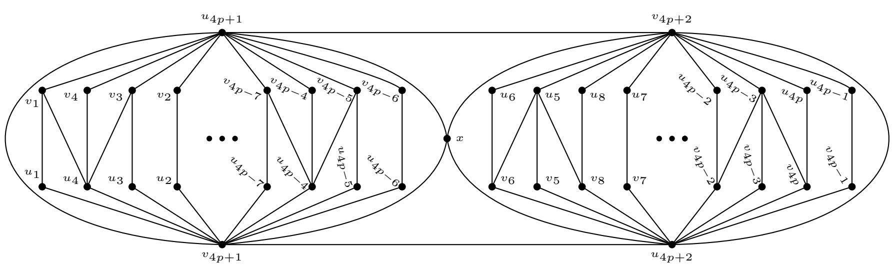
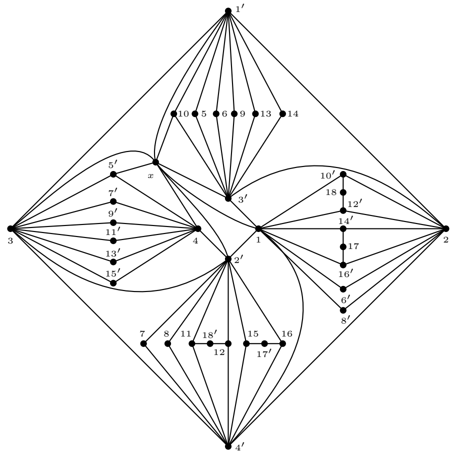


Figure 2: The graph $\widehat{G}_{p+1}$ in Case 1.

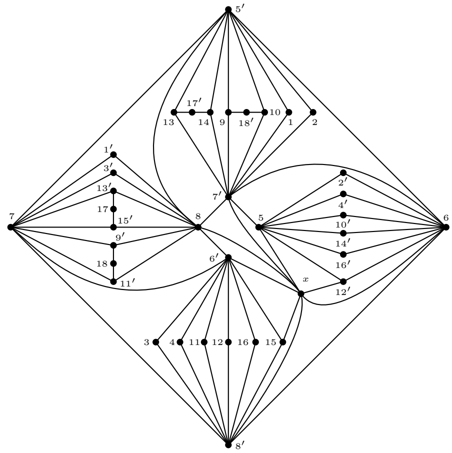
A planar decomposition $\{\widehat{G}_1, \dots, \widehat{G}_{p+1}\}$ of $K_{1,4p+2,4p+2}$ is obtained as above in this case. In Figure 3, we draw the planar decomposition of $K_{1,18,18}$, it is the smallest example for the Case 1. We denote vertex u_i and v_i by i and i' respectively in this figure.

Case 2. When p is odd and $p > 3$. The process is similar to that in Case 1.

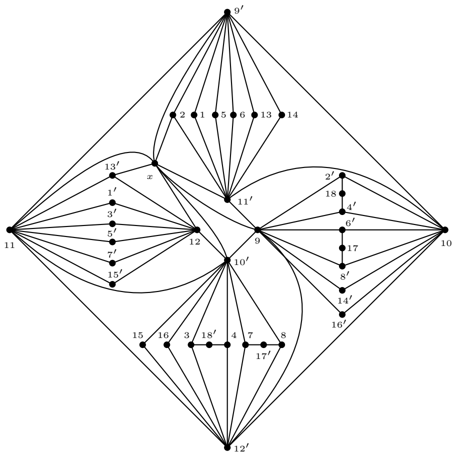
(a) The construction for $\widehat{G}_r$, $1 \le r \le p$, and r is odd.
Step 1: Place the vertex x in the face 1 of G_r, delete edges $v_{4r-3}u_{4r}$ and $u_{4r}v_{4r-1}$ from G_r, for $1 \le r \le p$, and delete v_2u_1 from G_1 additionally.



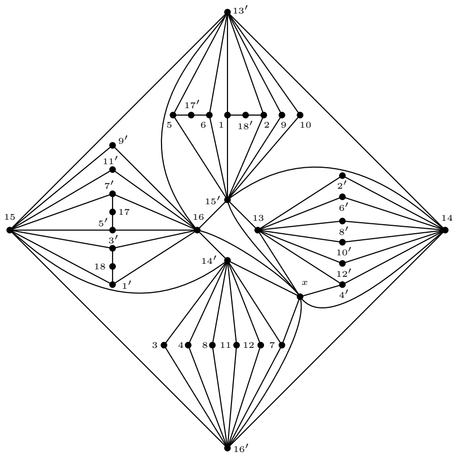
(a) The graph $\widehat{G}_1$.



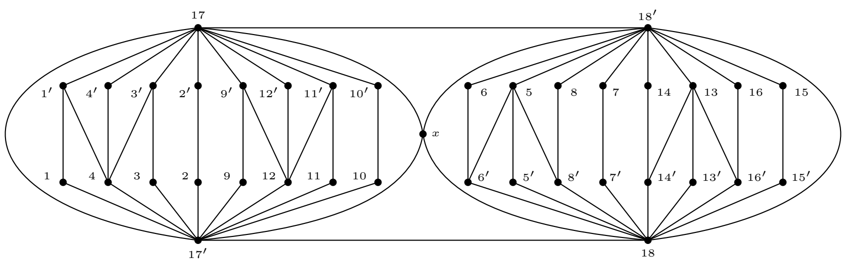
(b) The graph $\widehat{G}_2$.



(c) The graph $\widehat{G}_3$.



(d) The graph $\widehat{G}_4$.



(e) The graph $\widehat{G}_5$.

Figure 3: A planar decomposition of $K_{1,18,18}$.

For $1 < r < p$, do parallel paths modification to G_r, such that $u_{4r+6} \in U_1^r$, $v_{4r+1} \in V_1^r$ and $u_{4r-3}, u_{4r-1}, u_{4r}, v_{4r-3}, v_{4r-2}, v_{4r-1}$ are incident with a common face which the vertex x is in. Join x to $u_{4r-3}, u_{4r-1}, u_{4r}, v_{4r-3}, v_{4r-2}, v_{4r-1}$ and u_{4r+6}, v_{4r+1}.

Similarly, in G_1, join x to $u_1, u_3, u_4, v_1, v_2, v_3, v_4$ and $u_{10} \in U_1^1$, $v_5 \in V_1^1$. In G_p, join x to u_{4p-3},u_{4p-1},u_{4p},v_{4p-3},v_{4p-2},v_{4p-1} and $u_2 \in U_1^p$.

Step 2: For $1 \leq r < p$, do parallel paths modification to G_r, such that u_{4r+11}, $u_{4r+12} \in U_2^r$, $u_{4r+7}, u_{4r+8} \in U_2^r$, $v_{4r+10}, v_{4r+12} \in V_2^r$ and $v_{4r+6}, v_{4r+8} \in V_2^r$ are incident with a common face, respectively. Join v_{4p+1} to both u_{4r+11} and u_{4r+12}, join v_{4p+2} to both u_{4r+7} and u_{4r+8}, join u_{4p+1} to both v_{4r+10} and v_{4r+12}, join u_{4p+2} to both v_{4r+6} and v_{4r+8}.

Similarly, in G_p, join v_{4p+1} to $u_5, u_6 \in U_1^p$, join v_{4p+2} to $u_7, u_8 \in U_2^p$, join u_{4p+1} to $v_6, v_8 \in V_2^p$, join u_{4p+2} to $v_5, v_7 \in V_1^p$.

(b) The construction for $\widehat{G}_r$, $1 \leq r \leq p$, and r is even.

Step 1: Place the vertex x in the face 3 of G_r, delete edges $v_{4r}u_{4r-3}$ and $u_{4r-3}v_{4r-2}$ from G_r, $1 \leq r \leq p-1$.

Do parallel paths modification to G_r, $1 \leq r < p-1$, such that $u_{4r+7} \in U_2^r$, $v_{4r+4} \in V_2^r$ and $u_{4r-3}, u_{4r-2}, u_{4r}, v_{4r-2}, v_{4r-1}, v_{4r}$ are incident with a common face which the vertex x is in. Join x to $u_{4r-3}, u_{4r-2}, u_{4r}, v_{4r-2}, v_{4r-1}, v_{4r}$ and u_{4r+7}, v_{4r+4}. Similarly, in G_{p-1}, join x to $u_{4p-7}, u_{4p-6}, u_{4p-4}, v_{4p-6}, v_{4p-5}, v_{4p-4}$ and $u_7 \in U_2^{p-1}$, $v_{4p} \in V_2^{p-1}$.

Step 2: Do parallel paths modifications, such that $u_{4r+5}, u_{4r+6} \in U_1^r$, $u_{4r+1}, u_{4r+2} \in U_1^r$, $v_{4r+5}, v_{4r+7} \in V_1^r$, $v_{4r+1}, v_{4r+3} \in V_1^r$ are incident with a common face, respectively. Join v_{4p+1} to both u_{4r+5} and u_{4r+6}, join v_{4p+2} to both u_{4r+1} and u_{4r+2}, join u_{4p+1} to both v_{4r+5} and v_{4r+7}, join u_{4p+2} to both v_{4r+1} and v_{4r+3}.

Table 3 shows how we add edges to G_r $(1 \leq r \leq p)$ in Case 2.

(c) The construction for $\widehat{G}_{p+1}$.

With a similar argument to that in Case 1, we can list the edges that belong to $K_{1,4p+2,4p+2}$ but not to any $\widehat{G}_r$, $1 \leq r \leq p$, in this case, as shown in Table 4. Then $\widehat{G}_{p+1}$ is the graph that consists of the edges in Table 4. Figure 4 shows $\widehat{G}_{p+1}$ is a planar graph.

Therefore, $\{\widehat{G}_1, \ldots, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{1,4p+2,4p+2}$ in this case.

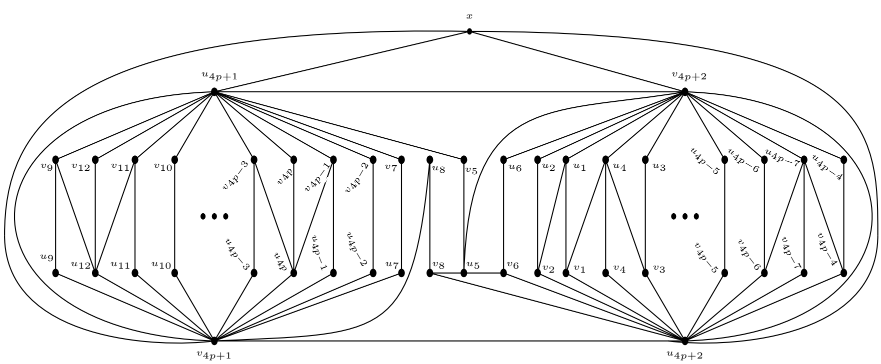


Figure 4: The graph $\widehat{G}_{p+1}$ in Case 2.

Table 3: The edges we add to G_r ($1 \leq r \leq p$) in Case 2.

subscript / case / edge	r is odd		r is even	
xu_j	$4r-3$, $4r-1$, $4r$	$4r+6, r \neq p$ (U_1^r) $2, r = p$ (U_1^r)	$4r-3$, $4r-2$, $4r$	$4r+7, r \neq p-1$ (U_2^r) $7, r = p-1$ (U_2^r)
xv_j	$4r-3$, $4r-2$, $4r-1$	$4, 5, r = 1$ $4r+1, r \neq 1, p$ (V_1^r)	$4r-2$, $4r-1$, $4r$	$4r+4$ (V_2^r)
$v_{4p+1}u_j$	$4r+11, 4r+12, r \neq p$ (U_2^r) $5, 6, r = p$ (U_1^r)		$4r+5, 4r+6$ (U_1^r)	
$v_{4p+2}u_j$	$4r+7, 4r+8$ (U_2^r)		$4r+1, 4r+2$ (U_1^r)	
$u_{4p+1}v_j$	$4r+10, 4r+12, r \neq p$ (V_2^r) $6, 8, r = p$ (V_2^r)		$4r+5, 4r+7$ (V_1^r)	
$u_{4p+2}v_j$	$4r+6, 4r+8, r \neq p$ (V_2^r) $5, 7, r = p$ (V_1^r)		$4r+1, 4r+3$ (V_1^r)	

Table 4: The edges of $\widehat{G}_{p+1}$ in Case 2.

edges	subscript
$xv_{4p+1}, v_{4p+1}u_j$	$j = 4r-3, 4r-2, 4r-1, 4r, 7, 8, 4p+2$ $(r = 3, 5, 7, \ldots, p)$
$xu_{4p+1}, u_{4p+1}v_j$	$j = 4r-3, 4r-2, 4r-1, 4r, 5, 7, 4p+2$ $(r = 3, 5, 7, \ldots, p)$
$xv_{4p+2}, v_{4p+2}u_j$	$j = 4r-3, 4r-2, 4r-1, 4r, 5, 6, 4p+1$ $(r = 1, 4, 6, 8, \ldots, p-1)$
$xu_{4p+2}, u_{4p+2}v_j$	$j = 4r-3, 4r-2, 4r-1, 4r, 6, 8, 4p+1$ $(r = 1, 4, 6, 8, \ldots, p-1)$
$u_1v_2, v_{4r-3}u_{4r}, u_{4r}v_{4r-1}$	$r = 1, 3, \ldots, p$
$v_{4r}u_{4r-3}, u_{4r-3}v_{4r-2}$	$r = 2, 4, \ldots, p-1$
u_jv_j	$j = 1, \ldots, 4p+2$

Case 3. When $p \leq 3$.

When $p = 0$, $K_{1,2,2}$ is a planar graph. When $p = 1, 2, 3$, we give a planar decomposition for $K_{1,6,6}$, $K_{1,10,10}$ and $K_{1,14,14}$ with 2, 3 and 4 subgraphs respectively, as shown in Figure 5, Figure 6 and Figure 7.

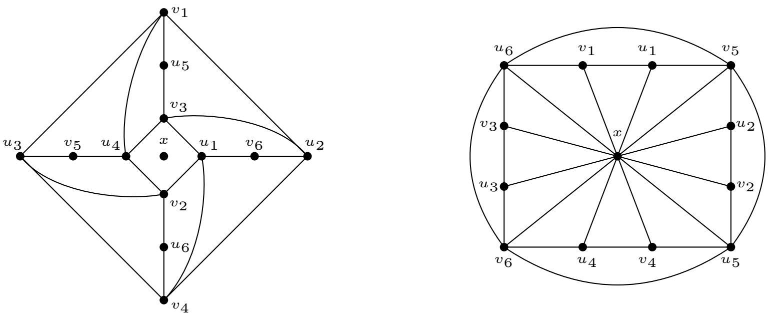


Figure 5: A planar decomposition of $K_{1,6,6}$.

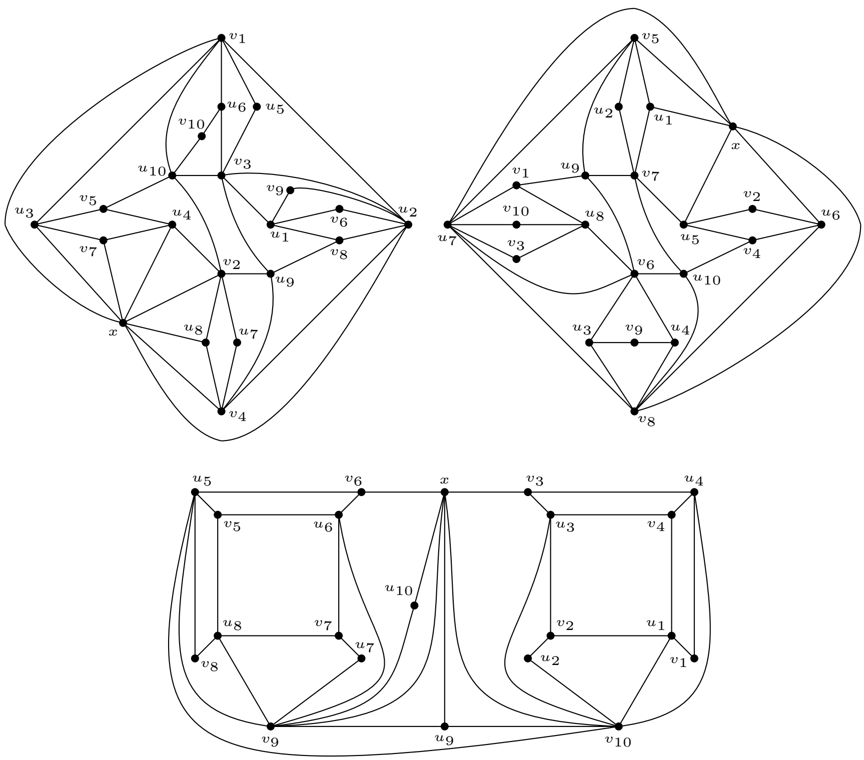


Figure 6: A planar decomposition of $K_{1,10,10}$.

Lemma follows from Cases 1, 2 and 3. □

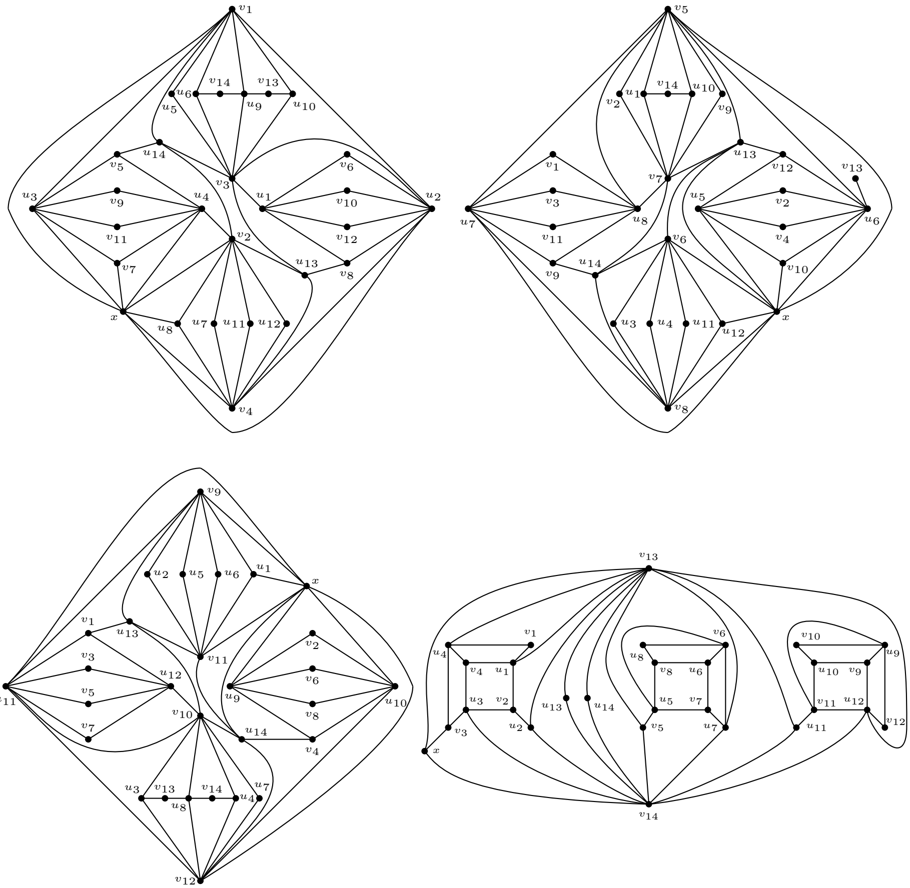


Figure 7: A planar decomposition of $K_{1,14,14}$.

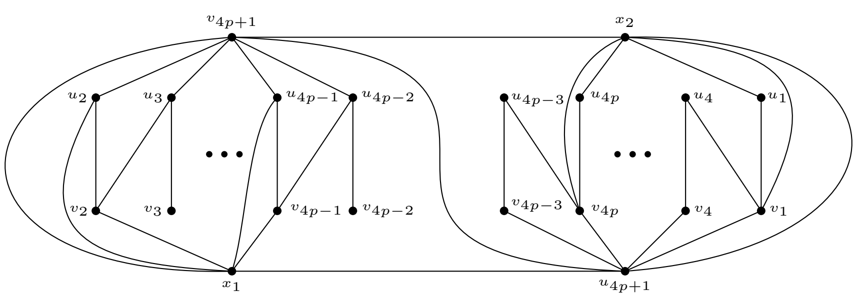


Figure 8: The graph $\widehat{G}_{p+1}$ in Case 1.

Theorem 2.4. *The thickness of the complete* 3*-partite graph* $K_{1,n,n}$ *is*

$$\theta(K_{1,n,n}) = \left\lceil \frac{n+2}{4} \right\rceil.$$

Proof. When $n = 4p, 4p+3$, the theorem follows from Lemma 2.2.

When $n = 4p+1, n = 4p+2$, from Lemma 2.3, we have $\theta(K_{1,4p+2,4p+2}) \leq p+1$. Since $\theta(K_{4p,4p}) = p+1$ and $K_{4p,4p} \subset K_{1,4p+1,4p+1} \subset K_{1,4p+2,4p+2}$, we obtain

$$p+1 \leq \theta(K_{1,4p+1,4p+1}) \leq \theta(K_{1,4p+2,4p+2}) \leq p+1.$$

Therefore, $\theta(K_{1,4p+1,4p+1}) = \theta(K_{1,4p+2,4p+2}) = p+1$.

Summarizing the above, the theorem is obtained. □

3 The thickness of $K_{2,n,n}$

Lemma 3.1. *There exists a planar decomposition of the complete* 3*-partite graph* $K_{2,4p+1,4p+1}$ *(*$p \geq 0$*) with* $p+1$ *subgraphs.*

Proof. Let (X, U, V) be the vertex partition of the complete 3-partite graph $K_{2,n,n}$, in which $X = \{x_1, x_2\}$, $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$. When $n = 4p+1$, we will construct a planar decomposition of $K_{2,4p+1,4p+1}$ with $p+1$ planar subgraphs.

The construction is analogous to that in Lemma 2.3. Let $\{G_1, G_2, \ldots, G_{p+1}\}$ be a planar decomposition of $K_{4p,4p}$ given in [4]. In the following, for $1 \leq r \leq p+1$, by adding vertices $x_1, x_2, u_{4p+1}, v_{4p+1}$ to G_r, deleting some edges from G_r and adding some edges to G_r, we will get a new planar graph $\widehat{G}_r$ such that $\{\widehat{G}_1, \ldots, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{2,4p+1,4p+1}$. All the subscripts of vertices are taken modulo $4p$, except that of u_{4p+1} and v_{4p+1} (the vertices we added to G_r).

Case 1. When p is even and $p > 2$.

(a) The construction for $\widehat{G}_r$, $1 \leq r \leq p$.

Step 1: When r is odd, place the vertex x_1, x_2 and u_{4p+1} in the face 1, 2 and 5 of G_r respectively. Delete edges $v_{4r-3}u_{4r}$ and $u_{4r-1}v_{4r-2}$ from G_r.

When r is even, place the vertex x_1, x_2 and u_{4p+1} in the face 3, 4 and 5 of G_r, respectively. Delete edge $v_{4r}u_{4r-3}$ and $u_{4r-2}v_{4r-1}$ from G_r.

Step 2: Do parallel paths modifications, then join x_1, x_2, u_{4p+1} and v_{4p+1} to some u_j and v_j, as shown in Table 5.

(b) The construction for $\widehat{G}_{p+1}$.

We list the edges that belong to $K_{2,4p+1,4p+1}$ but not to any $\widehat{G}_r$, $1 \leq r \leq p$, as shown in Table 6. Then $\widehat{G}_{p+1}$ is the graph that consists of the edges in Table 6. Figure 8 shows $\widehat{G}_{p+1}$ is a planar graph.

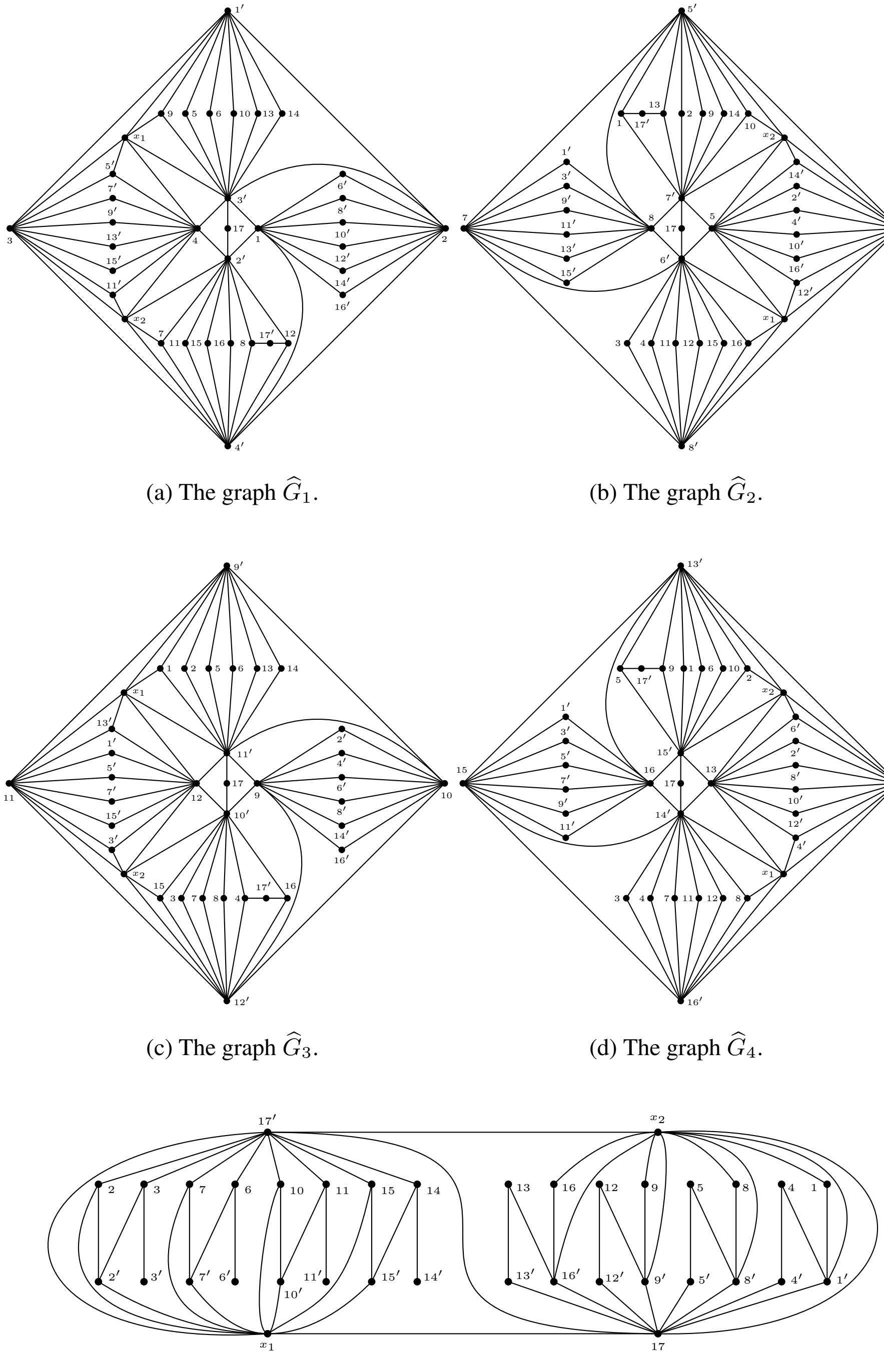
Therefore, $\{\widehat{G}_1, \ldots, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{2,4p+1,4p+1}$ in this case. In Figure 9, we draw the planar decomposition of $K_{2,17,17}$ it is the smallest example for the Case 1. We denote vertex u_i and v_i by i and i' respectively in this figure.

Table 5: The edges we add to G_r $(1 \le r \le p)$ in Case 1.

subscript \ case / edge	r is odd		r is even	
x_1u_j	$4r-1, 4r$	$4r+5$ (U_1^r)	$4r-3, 4r-2$	$4r+8$ (U_2^r)
x_1v_j	$4r-3, 4r-1$	$4r+1$ (V_1^r)	$4r-2, 4r$	$4r+4$ (V_2^r)
x_2u_j	$4r-1, 4r$	$4r+3$ (U_2^r)	$4r-3, 4r-2$	$4r+2$ (U_1^r)
x_2v_j	$4r-2, 4r$	$4r+7$ (V_1^r)	$4r-3, 4r-1$	$4r+6$ (V_2^r)
$u_{4p+1}v_j$	$4r-2, 4r-1$			
$v_{4p+1}u_j$	$4r+4, 4r+8$ (U_2^r)		$4r-11, 4r-7$ (U_1^r)	

Table 6: The edges of $\widehat{G}_{p+1}$ in Case 1.

edges	subscript
x_1u_j	$j = 4r-2, 4r+3, 4p+1 \quad (r = 1, 3, \ldots, p-1)$
x_1v_j	
x_2u_j	$j = 4r-7, 4r, 4p+1 \quad (r = 2, 4, \ldots, p)$
x_2v_j	
$u_{4p+1}v_j$	$j = 4r-3, 4r \quad (r = 1, 2, \ldots, p)$
$v_{4p+1}u_j$	$j = 4r-2, 4r-1 \quad (r = 1, 2, \ldots, p)$
$v_{4r-3}u_{4r}, v_{4r-2}u_{4r-1}$	$r = 1, 3, \ldots, p-1$
$u_{4r-3}v_{4r}, u_{4r-2}v_{4r-1}$	$r = 2, 4, \ldots, p$
u_jv_j	$j = 1, \ldots, 4p+1$



(a) The graph $\widehat{G}_1$.

(b) The graph $\widehat{G}_2$.

(c) The graph $\widehat{G}_3$.

(d) The graph $\widehat{G}_4$.

(e) The graph $\widehat{G}_5$.

Figure 9: A planar decomposition of $K_{2,17,17}$.

Case 2. When p is odd and $p > 3$.

(a) The construction for $\widehat{G}_r$, $1 \le r \le p$.

Step 1: When r is odd, place the vertex x_1, x_2 and u_{4p+1} in the face 1, 2 and 5 of G_r respectively. Delete edges $v_{4r-3}u_{4r}$ and $u_{4r-1}v_{4r-2}$ from G_r.

When r is even, place the vertex x_1, x_2 and u_{4p+1} in the face 3, 4 and 5 of G_r, respectively. Delete edge $v_{4r}u_{4r-3}$ and $u_{4r-2}v_{4r-1}$ from G_r.

Step 2: Do parallel paths modifications, then join x_1, x_2, u_{4p+1} and v_{4p+1} to some u_j and v_j, as shown in Table 7.

Table 7: The edges we add to G_r ($1 \le r \le p$) in Case 2.

edge \ case (subscript)	r is odd		r is even	
x_1u_j	$4r-1$, $4r$	$4r+5, r \neq p$ (U_1^r) $1, r = p$ (U_1^r)	$4r-3$, $4r-2$	$4r+8, r \neq p-1$ (U_2^r) $8, r = p-1$ (U_2^r)
x_1v_j	$4r-3$, $4r-1$	$4r+1, r \neq p$ (V_1^r)	$4r-2$, $4r$	$4r+4$ (V_2^r)
x_2u_j	$4r-1$, $4r$	$4r+3, r \neq p$ (U_2^r) $8, r = p$ (U_2^r)	$4r-3$, $4r-2$	$4r+2$ (U_1^r)
x_2v_j	$4r-2$, $4r$	$4r+7, r \neq p$ (V_1^r) $3, r = p$ (V_1^r)	$4r-3$, $4r-1$	$4r+6, r \neq p-1$ (V_2^r) $6, r = p-1$ (V_2^r)
$u_{4p+1}v_j$	$4r-2, 4r-1$			
$v_{4p+1}u_j$	$4r+4, 4r+8, r \neq p$ (U_2^r) $4, r = p$ (U_2^r)		$4r-11, 4r-7$ (U_1^r)	

(b) The construction for $\widehat{G}_{p+1}$.

We list the edges that belong to $K_{2,4p+1,4p+1}$ but not to any $\widehat{G}_r$, $1 \le r \le p$, as shown in Table 8. Then $\widehat{G}_{p+1}$ is the graph that consists of the edges in Table 8. Figure 10 shows $\widehat{G}_{p+1}$ is a planar graph.

Therefore, $\{\widehat{G}_1, \dots, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{2,4p+1,4p+1}$ in this case.

Case 3. When $p \le 3$.

When $p = 0$, $K_{2,1,1}$ is a planar graph. When $p = 1, 2, 3$, we give a planar decomposition for $K_{2,5,5}$, $K_{2,9,9}$ and $K_{2,13,13}$ with 2, 3 and 4 subgraphs respectively, as shown in Figure 11, Figure 12 and Figure 13.

Summarizing Cases 1, 2 and 3, the lemma follows. □

Table 8: The edges of $\widehat{G}_{p+1}$ in Case 2.

edges	subscript
x_1u_j	$j = 2, 4r+3, 4r+6, 4p+1 \quad (r = 1, 3, \ldots, p-2)$
x_1v_j	$j = 2, 4, 4r+3, 4r+6, 4p+1 \quad (r = 1, 3, \ldots, p-2)$
x_2u_j	$j = 1, 2, 9, 4r, 4r+1, 4p+1 \quad (r = 4, \ldots, p-1)$
x_2v_j	$j = 1, 8, 9, 4r, 4r+1, 4p+1 \quad (r = 4, \ldots, p-1)$
$u_{4p+1}v_j$	$j = 4r-3, 4r \quad (r = 1, 2, \ldots, p)$
$v_{4p+1}u_j$	$j = 4r-2, 4r-1, 4p-7 \quad (r = 1, 2, \ldots, p)$
$v_{4r-3}u_{4r}, v_{4r-2}u_{4r-1}$	$r = 1, 3, \ldots, p$
$u_{4r-3}v_{4r}, u_{4r-2}v_{4r-1}$	$r = 2, 4, \ldots, p-1$
u_jv_j	$j = 1, \ldots, 4p+1$

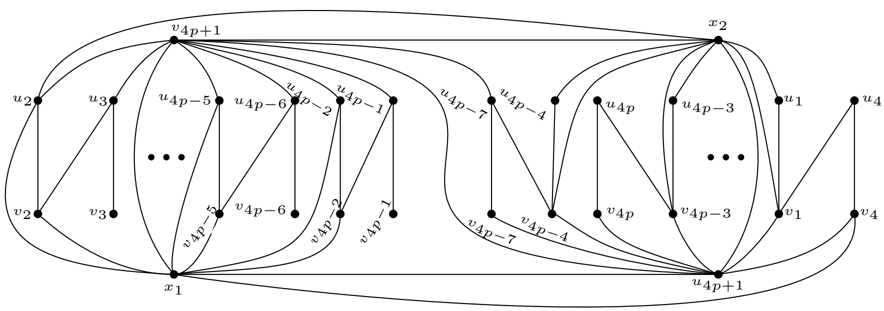


Figure 10: The graph $\widehat{G}_{p+1}$ in Case 2.

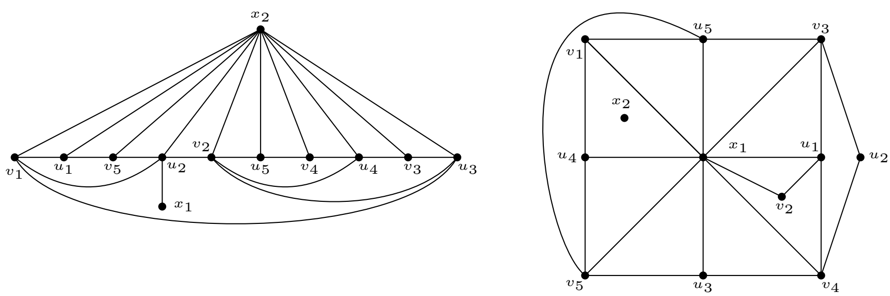


Figure 11: A planar decomposition $K_{2,5,5}$.

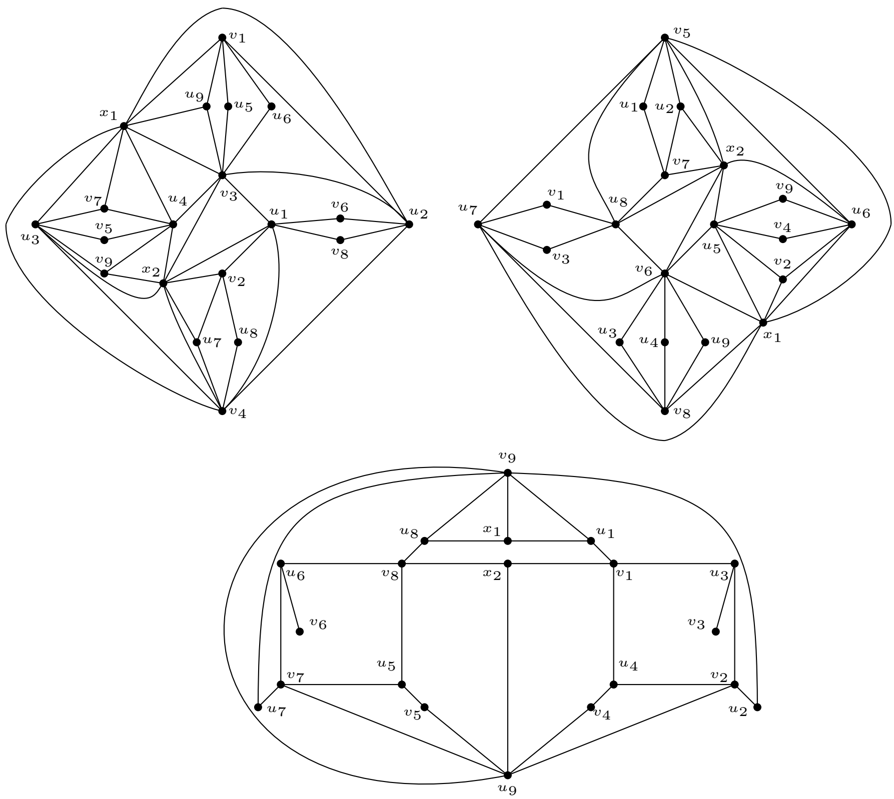


Figure 12: A planar decomposition $K_{2,9,9}$.

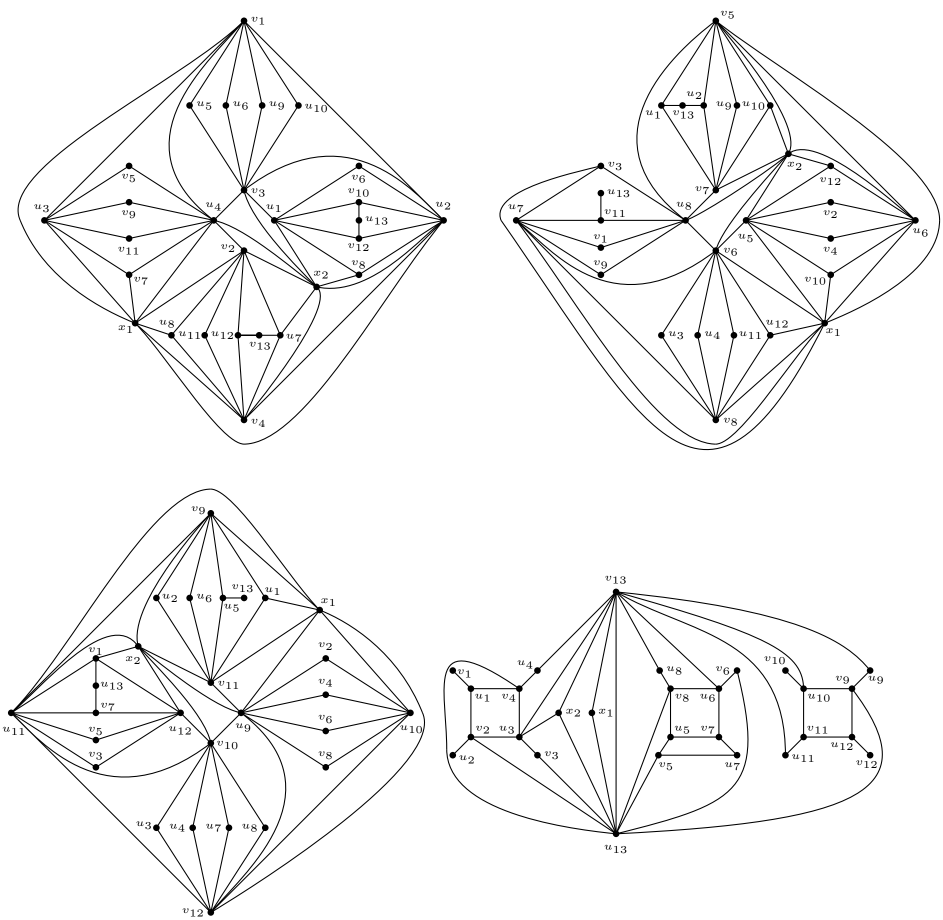


Figure 13: A planar decomposition of $K_{2,13,13}$.

Theorem 3.2. *The thickness of the complete* 3*-partite graph* $K_{2,n,n}$ *is*

$$\theta(K_{2,n,n}) = \left\lceil \frac{n+3}{4} \right\rceil.$$

Proof. When $n = 4p, 4p+3$, from Lemma 2.2, the theorem holds.

When $n = 4p+1$, from Lemma 3.1, we have $\theta(K_{2,4p+1,4p+1}) \le p+1$. Since $\theta(K_{4p,4p}) = p+1$ and $K_{4p,4p} \subset K_{2,4p+1,4p+1}$, we have

$$p+1 = \theta(K_{4p,4p}) \le \theta(K_{2,4p+1,4p+1}) \le p+1.$$

Therefore, $\theta(K_{2,4p+1,4p+1}) = p+1$.

When $n = 4p+2$, since $K_{4p+3,4p+3} \subset K_{2,4p+2,4p+2}$, from Lemma 2.1, we have $p+2 = \theta(K_{4p+3,4p+3}) \le \theta(K_{2,4p+2,4p+2})$. On the other hand, it is easy to see $\theta(K_{2,4p+2,4p+2}) \le \theta(K_{2,4p+1,4p+1}) + 1 = p+2$, so we have $\theta(K_{2,4p+2,4p+2}) = p+2$.

Summarizing the above, the theorem is obtained. □

4 The thickness of $K_{1,1,n,n}$

Theorem 4.1. *The thickness of the complete* 4*-partite graph* $K_{1,1,n,n}$ *is*

$$\theta(K_{1,1,n,n}) = \left\lceil \frac{n+3}{4} \right\rceil.$$

Proof. When $n = 4p+1$, we can get a planar decomposition for $K_{1,1,4p+1,4p+1}$ from that of $K_{2,4p+1,4p+1}$ as follows.

(1) When $p = 0$, $K_{1,1,1,1}$ is a planar graph, $\theta(K_{1,1,1,1}) = 1$. When $p = 1, 2$ and 3, we join the vertex x_1 to x_2 in the last planar subgraph in the planar decomposition for $K_{2,5,5}$, $K_{2,9,9}$ and $K_{2,13,13}$ which was shown in Figure 11, 12 and 13. Then we get the planar decomposition for $K_{1,1,5,5}$, $K_{1,1,9,9}$ and $K_{1,1,13,13}$ with 2, 3 and 4 planar subgraphs respectively.

(2) When $p \ge 4$, we join the vertex x_1 to x_2 in $\widehat{G}_{p+1}$ in the planar decomposition for $K_{2,4p+1,4p+1}$ which was constructed in Lemma 3.1. The $\widehat{G}_{p+1}$ is shown in Figure 8 or 10 according to p is even or odd. Because x_1 and x_2 lie on the boundary of the same face, we will get a planar graph by adding edge x_1x_2 to $\widehat{G}_{p+1}$. Then a planar decomposition for $K_{1,1,4p+1,4p+1}$ with $p+1$ planar subgraphs can be obtained.

Summarizing (1) and (2), we have $K_{1,1,4p+1,4p+1} \le p+1$.

On the other hand, from Lemma 2.1, we have $\theta(K_{4p+1,4p+1}) = p+1$. Due to $K_{4p+1,4p+1} \subset K_{1,1,4p,4p} \subset K_{1,1,4p+1,4p+1}$, we get

$$p+1 \le \theta(K_{1,1,4p,4p}) \le \theta(K_{1,1,4p+1,4p+1}).$$

So we have

$$\theta(K_{1,1,4p,4p}) = \theta(K_{1,1,4p+1,4p+1}) = p+1.$$

When $n = 4p+3$, from Theorem 3.2 , we have $\theta(K_{2,4p+2,4p+2}) = p+2$. Since $K_{2,4p+2,4p+2} \subset K_{1,1,4p+2,4p+2} \subset K_{1,1,4p+3,4p+3} \subset K_{1,1,4(p+1),4(p+1)}$, and the ideas from the previous case establish, we have

$$p+2 \le \theta(K_{1,1,4p+2,4p+2}) \le \theta(K_{1,1,4p+3,4p+3}) \le \theta(K_{1,1,4(p+1),4(p+1)}) = p+2,$$

which shows

$$\theta(K_{1,1,4p+2,4p+2}) = \theta(K_{1,1,4p+3,4p+3}) = p+2.$$

Summarizing the above, the theorem follows. □

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Touching perfect matchings and halving lines

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Abstract

Let V be a set of $2m$ $(1 \leq m < \infty)$ points in the plane. Two segments I, J with endpoints in V *cross* if $\operatorname{relint} I \cap \operatorname{relint} J$ is a singleton. A (perfect) *cross-matching* M on V is a set of m segments with endpoints in V such that every two segments in M cross. A *halving line* of V is a line l spanned by two points of V such that each one of the two open half planes bounded by l contains fewer than m points of V. Pach and Solymosi proved that if V is in general position, then V admits a perfect cross-matching iff V has exactly m halving lines. The aim of this note is to extend this result to the general case (where V is unrestricted).

Keywords: Bigraphs, cross-matching, halving lines, perfect matchings.

Math. Subj. Class.: 05C62, 68R10, 52C35

1 Introduction, notions and main results

Let V be a set of $2m$ distinct points in the plane $\mathbb{R}^2$ $(1 \leq m < \infty)$. By a (perfect geometric) *matching* of V we mean a set $M = \{I_1, \ldots, I_m\}$ of m non-degenerate closed line segments whose endpoints are (all) the points of V. The number of matchings of V is

$$(2m-1)!! = \prod_{i=1}^{m}(2i-1) = \frac{(2m)!}{2^m \cdot m!}.$$

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If V is in general position (no three points on a line), then two distinct segments $I, J \in M$ may be

(a) disjoint ($I \cap J = \emptyset$),

(b) or they may *cross*, i.e., share a unique point that lies in the relative interior of both I and J.

When V is unrestricted, two more possibilities arise.

(c) The unique common point of I and J maybe an interior point of I and an endpoint of J (or vice versa).

(d) If the four endpoints of I and J are collinear, then I and J may share a line segment. (This includes the possibility that $I \subset \operatorname{relint} J$, or vice versa.)

We shall say that two segments I, J *touch* if they have at least one point in common ($I \cap J \neq \emptyset$). We call M a *simple matching* (SM) if the segments of M are pairwise disjoint.

It is well known and quite easy to show (see [2, Theorem 4.2]) that if V is in general position, then the number $\operatorname{sm}(V)$ of simple matchings on V is bounded from below by the m-th Catalan number C_m, i.e.,

$$\operatorname{sm}(V) \geq C_m = \frac{1}{m+1}\binom{2m}{m}. \tag{1.1}$$

Equality holds for $m = 1$ or when V is the set of vertices of a convex $2m$-gon. (It can be shown that if V is in general position but not in convex position, then $\operatorname{sm}(V) > C_m$, with only one exception: when $m = 3$ and V consists of the vertices of a convex pentagon P plus a sixth point that lies in the interior of the pentagon formed by the diagonals of P.)

Call M a *cross-matching* (CM) if each two distinct segments of M cross. Let us call M a *touching matching* (TM) if every two segments of M touch.

1.1 Halving lines

Definition 1.1. A line L is a *halving line of* V if each of the two open half-planes L^+, L^- bounded by L contains fewer than m points of V.

This clearly implies that $|L \cap V| \geq 2$, i.e., that the line L is spanned by V. When V is in general position, then necessarily $|L \cap V| = 2$, and $|L^- \cap V| = |L^+ \cap V| = m - 1$. When V is unrestricted we call L a *halving line of order* k if $\max(|L^- \cap V|, |L^+ \cap V|) = m - k$ ($1 \leq k \leq m$). In that case we may assume that, say, $|L^+ \cap V| = m - k$, $|L^- \cap V| = m - k - \varepsilon$, and $|L \cap V| = 2k + \varepsilon$, for some ε, $0 \leq \varepsilon \leq m - k$. (See Figure 1.)

1.2 Halving lines and TMs

If M is a TM on V, I is a segment of M, and $L = \operatorname{aff} I$ is the line spanned by I, then L is a halving line. Indeed, an open half-plane bounded by L contains no endpoint of I, and at most one endpoint of each other segment of M.

The connection between the number $h(V)$ of halving lines of V, and the existence of a cross-matching on V, *in the case where V is in general position*, was established by Pach and Solymosi in [3] as follows: They observed that each point of V lies on at least one halving line, hence $h(V) \geq m$. Then they found that *either* each point of V lies on just one halving line, $h(V) = m$ and V admits a unique CM, *or* at least one point of V lies on more than one halving line, $h(V) > m$, and V admits no CM at all. This result was generalized in [1] (see Theorem 1 and Corollary 3 there). In [4] we prove an extremal property of CMs, namely that if V admits a CM M, and M' is another (perfect) geometric matching on V, then the sum of the (Euclidean) lengths of the edges of M' is strictly less than the sum of the lengths of the edges of M. An analogous result holds for TMs. The geometric graph whose edges span (all) the halving lines of its vertex set V (with $|V|$ even and V in general position) is said to be a *bigraph*. We refer to [5] regarding results on bigraphs.

The aim of this note is to extend the result of [3] to arbitrary, unrestricted $2m$-subsets V of $\mathbb{R}^2$.

In the next section we define the notion of "a halving line at a point $p \in V$", and show that a halving line of order k is a halving line at exactly $2k$ points. We also show that the number of halving lines at any point $p \in V$ is odd, hence ≥ 1. The main results can be summarized as follows:

Theorem 1.2. *Suppose $L_1, \ldots, L_t$ ($t = h(V)$) are all the halving lines of V, with L_i of order k_i ($1 \leq k_i \leq m$, $i = 1, \ldots, t$). If for each $p \in V$ there is just one halving line at p, then*

$$\sum_{i=1}^{t} k_i = m,$$

and the number of TMs of V is precisely

$$\prod_{i=1}^{t} (k_i!).$$

If, for some $p \in V$, there is more than one halving line at p, then

$$\sum_{i=1}^{t} k_i > m,$$

and V has no TM.

In particular we have

Corollary 1.3. *The set V has a unique TM iff V has exactly m halving lines, each of order 1. The unique TM is a CM if each of the m halving lines contains just two points of V.*

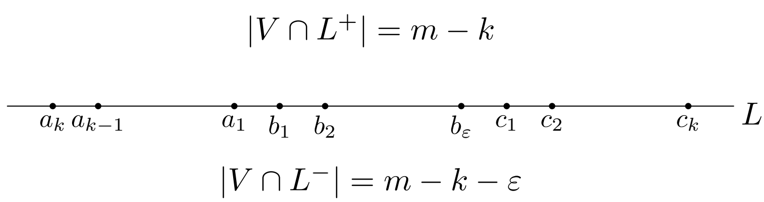

Figure 1: A halving line of order k.

2 Proofs

We start with the definition of "a halving line of V at p", where V is a set of $2m$ points in $\mathbb{R}^2$, and $p \in V$. For a point $p \in V$ and a unit vector $\underline{u} = (u_1, u_2)$, denote by $L(p, \underline{u})$ the directed line $\{p + \lambda\underline{u} : \lambda \in \mathbb{R}\}$. (The direction is from small λ to larger λ.) Note that $L(p, -\underline{u})$ is the same line, directed backwards. Define $\underline{u}_+ = (-u_2, u_1)$,

$$L(p, \underline{u})_F = L(p, \underline{u}) + \{\mu\underline{u}_+ : \mu > 0\}, \text{ and}$$
$$L(p, \underline{u})_B = L(p, \underline{u}) + \{\mu\underline{u}_+ : \mu < 0\}.$$

F and B stand for "Front" and "Back", respectively.

$L(p, \underline{u})_F$ and $L(p, \underline{u})_B$ are the two open half-planes bounded by $L(p, \underline{u})$. Now move the unit vector $\underline{u}$ continuously on the unit circle in counterclockwise direction. Note that $L(p, \underline{u})_F$ and $L(p, \underline{u})_B$ switch when $\underline{u}$ is replaced by $-\underline{u}$. As long as $L(p, \underline{u})$ does not meet $V \setminus \{p\}$, we find that

$$|V \cap L(p, \underline{u})_F| + |V \cap L(p, \underline{u})_B| = |V - \{p\}| = 2m - 1,$$

and therefore one side of $L(p, \underline{u})$ (the "major" side) contains at least m points of V, whereas the other side (the "minor" side) contains at most $m - 1$ points of V.

As we change the direction $\underline{u}$, the major side of $L(p, \underline{u})$ will remain (Front or Back) as long as the rotating line $L(p, \underline{u})$ does not meet $V \setminus \{p\}$. We call $L(p, \underline{u}_o)$ *a halving line of V at p* if the major side of $L(p, \underline{u})$ switches (from B to F or vice versa) as $\underline{u}$ passes through $\underline{u}_o$.

Proposition 2.1. *If $L = L(p, \underline{u}_o)$ is a halving line of V at p, then L is a halving line of V.*

Proof. We must show that both open sides of L, $L(p, \underline{u}_o)_F$ and $L(p, \underline{u}_o)_B$, contain fewer than m points of V each. If, say, $|V \cap L(p, \underline{u}_o)_F| \geq m$, then $V \cap L(p, \underline{u})_F \supset V \cap L(p, \underline{u}_o)_F$, and therefore $|V \cap L(p, \underline{u})_F| \geq m$, for all unit vectors $\underline{u}$ sufficiently close to $\underline{u}_o$, on both sides of $\underline{u}_o$, so the major side of $L(p, \underline{u})$ does not switch at $\underline{u} = \underline{u}_o$. □

Proposition 2.2. *For each point $p \in V$, the number of halving lines of V at p is odd (hence ≥ 1).*

Proof. Choose an initial direction $\underline{u}_o$, such that $V \cap L(p, \underline{u}_o) = \{p\}$. Suppose the major side of $L(p, \underline{u}_o)$ is, say, $L(p, \underline{u}_o)_F$. Rotate the line through p counterclockwise by 180°, i.e., move $\underline{u}$ along a semicircle, until we reach $L(p, -\underline{u}_o)$. Now the major side is $L(p, -\underline{u}_o)_B$ $(= L(p, \underline{u}_o)_F)$. We conclude that on the way the major side switched (from F to B or vice versa) an odd number of times. □

Proposition 2.3. *Suppose L is a halving line of V of order k $(1 \leq k \leq m)$. Then L is a halving line of V at p for exactly $2k$ points of V.*

Proof. Assume, w.l.o.g., that

$$|V \cap L^-| = m - k - \varepsilon,\ |V \cap L^+| = m - k, \text{ and } |V \cap L| = 2k + \varepsilon,$$

for some $0 \leq \varepsilon \leq m - k$. Label the points of $V \cap L$ in order

$$a_k, a_{k-1}, \ldots, a_1, b_1, \ldots, b_\varepsilon, c_1, \ldots, c_k,$$

as in Figure 1. Fix a point $p \in V \cap L$, and consider a line that rotates counterclockwise through p. As the rotating line passes through the horizontal position (see Figure 1), the major side switches from Above to Below if p is one of the a_i's, and from Below to Above if p is one of the c_i's. But if p is one of the b_i's, then the major side remains Above (at least in a small neighborhood on both sides of the horizontal position). □

Next we show that if L is a halving line of V of order k, as in Figure 1, and M is a TM on V, then M matches the a_i's with the c_i's (and vice versa).

Proposition 2.4. *Suppose $V = S \cup T$ is a partition of V into two sets of equal size ($|S| = |T| = m$), and $\operatorname{conv} S \cap \operatorname{conv} T = \emptyset$. If M is a TM of V, then each segment $I \in M$ connects a point of S with a point of T.*

Proof. Assume, on the contrary, that some segment $I \in M$ has both endpoints in S. This leaves (at most) $m - 2$ points of S to be matched to points of T, and thus some other segment $J \in M$ has both endpoints in T. But then $I \cap J \subset \operatorname{conv} S \cap \operatorname{conv} T = \emptyset$. □

Now look again at the halving line L in Figure 1. Define $A = \{a_1, \dots, a_k\}$, $B = \{b_1, \dots, b_\varepsilon\}$, $C = \{c_1, \dots, c_k\}$, $D_- = B \cup (V \cap L^-)$ and $D_+ = V \cap L^+$ ($|D_-| = |D_+| = m - k$). Applying Proposition 2.4 twice, first with $S = A \cup D_-$, $T = C \cup D_+$, and then with $S' = C \cup D_-$, $T' = A \cup D_+$, we find:

Proposition 2.5. *If M is a TM of V, then each segment $I \in M$ with one endpoint in A has its other endpoint in C (and vice versa), and each segment $J \in M$ with one endpoint in D_- has its other endpoint in D_+ (and vice versa).*

Note also that for *any* permutation θ of $\{1, 2, \dots, k\}$, the intersection of the k segments $[a_i, c_{\theta(i)}]$ ($i = 1, \dots, k$) is the segment $[a_1, c_1]$, that connects the k'th point of $V \cap L$ from the right with the k'th point of $V \cap L$ from the left. We call this segment $[a_1, c_1]$ the *central segment* of the halving line L.

Suppose $L_1, \dots, L_t$ ($t = h(V)$) are all the halving lines of V, with L_i of order k_i for $i = 1, \dots, t$. For $p \in V$, denote by $h(p)$ the number of halving lines at p. In view of Propositions 2.1 – 2.3, we have

$$\sum_{i=1}^{t} k_i = \frac{1}{2} \sum_{p \in V} h(p) \geq m,$$

with equality ($= m$) iff $h(p) = 1$ for all $p \in V$.

Proposition 2.6. *If $h(p) > 1$ for some $p \in V$, then there is no TM on V.*

Proof. Suppose, on the contrary, that V admits a TM M. Let $I = [p, q]$ be a segment in M with one endpoint p. Let L, L' be two different halving lines of V at p ($h(p) > 1$). By Proposition 2.5 we have $q \in L \cap L'$. But $L \cap L' = \{p\}$. □

Assume, from now on, that $h(p) = 1$ for all $p \in V$. Thus $\sum_{i=1}^{t} k_i = m$. In other words, on each line L_i we can match two disjoint subsets of $V \cap L_i$, each of order k_i, A_i (the k_i "leftmost" points of $V \cap L_i$) and C_i (the k_i "rightmost" points of $V \cap L_i$). L_i is a halving line of V at p iff $p \in A_i \cup C_i$. The sets $A_1, C_1, \dots, A_t, C_t$ form a partition of V. As we have seen in Proposition 2.5, any TM of V will match the points of A_i with those of C_i.

There are $k_i!$ ways to match A_i with C_i, and in each of these matchings, the intersection of the connecting segments is the "central segment" of the halving line L_i. To show that the individual TM's of $A_i \cup C_i$ on L_i ($i = 1, \dots, t$) yield a TM of V, it suffices to show that the central segments of different halving lines L_i and L_j do meet (assuming, of course, that $h(p) = 1$ for all $p \in V$). This will be done in the next proposition.

Proposition 2.7. *Suppose L is a halving line of V of order k, with $V \cap L$ labelled $a_k, \dots, a_1, b_1, \dots, b_\varepsilon, c_1, \dots, c_k$ as in Figure 1, $A = \{a_k, \dots, a_1\}$, $C = \{c_1, \dots, c_k\}$, and let L' be another halving line of V, of order k', with $V \cap L'$ labelled similarly: $a'_{k'}, \dots, a'_1, b'_1, \dots, b'_{\varepsilon'}, c'_1, \dots, c'_{k'}$, $A' = \{a'_{k'}, \dots, a'_1\}$, $C' = \{c'_1, \dots, c'_{k'}\}$. If the central segments $[a_1, c_1]$ (of L) and $[a'_1, c'_1]$ (of L') do not meet, then $h(p) > 1$ for some $p \in \{a_1, c_1, a'_1, c'_1\}$.*

Proof. The two distinct lines L, L' cannot be parallel. If they are, and L' lies, say, above L, then the open side L^+ of L includes the closed side $\operatorname{cl} L'^+$ of L', and therefore $|V \cap L^+| \geq |V \cap \operatorname{cl} L'^+| > m$, which is impossible. Let z be the crossing point of L and L', and suppose, w.l.o.g., that z misses the central segment $[a_1, c_1]$ of L, and lies to the left of a_1 on L, see Figure 2.

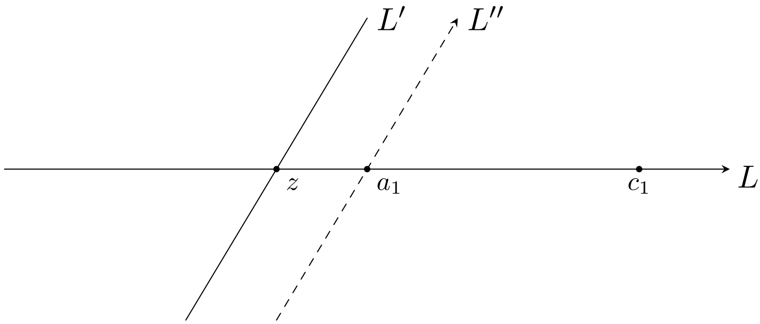

Figure 2: Proof of Proposition 2.7.

Consider a directed line that rotates counter-clockwise through a_1. As it passes through L (directed from left to right), the major side of V switches from Front to Back. As it reaches L'' (parallel to L'), or any direction sufficiently close to that of L', the major side of V is again Front, since the open half-plane to the left of L'' includes the closed half-plane to the left of L', which in turn contains at least $m + k'$ points of V. Thus, there must have been another switch from Back to Front on the way, or, in other words, $h(a_1) > 1$. □

3 Algorithmic aspects

The insights gained in the earlier sections of this note can be used to device an algorithm that decides whether a set $P \subset \mathbb{R}^2$ ($|P| = 2m$) admits a TM, and to find a TM (or all TMs) if one exists. The algorithm is conceptually simple, and seems to be also computationally quite effective, though not as efficient as the one proposed in [3] (m^2 vs. $m \log m$).

Step 1: Find the point $p_0 = (x_0, y_0) \in P$ that is the first in P with respect to the lexicographic order of points $(x, y) \in \mathbb{R}^2$. p_0 is a vertex of the convex hull $[P] = \operatorname{conv} P$.

Step 2: Calculate the slopes of the $2m - 1$ segments $[p_0, p]$ ($p \in P \setminus \{p_0\}$), arrange them in non-decreasing order and find the median slope (this can be shared by several segments, of course). This slope determines the (unique) halving line L of P at p_0. Find the number of points of P that lie below L, on L and above L, and order the points of $P \cap L$ lexicographically. This enables us to determine the order k of the halving line L, and the sets A, C consisting of the first (resp. last) k points of $P \cap L$. These are the $2k$ points $p \in P \cap L$ such that L is a halving line at p. Erase these $2k$ points, and call the remaining set P' ($|P'| = 2(m-k)$). If $P' = \emptyset$, stop. Otherwise, return to Step 1 with P replaced by P'.

To see that this really works, we make the following observations:

(A) If P admits a TM M, then M contains k segments (on L) that connect points of A with points of C. The rest of M is a TM of P' ($= P \setminus (A \cup C)$). Moreover, if $\tilde{L}$ is any halving line of P other than L, of order $\tilde{k}$, then removal of $A \cup C$ leaves $\tilde{L}$ a halving line of P' of the same order $\tilde{k}$. This is clear when the central segments of L and of $\tilde{L}$ meet at a point that is interior to the central segment $[a_1, c_1]$ of L. In that case we lose k points on each side of $\tilde{L}$.

The case when the common point of these two central segments is an endpoint, say a_1, of $[a_1, c_1]$, is shown in Figure 3. (The reason why C is included in $\tilde{L}^+$ and not in $\tilde{L}^-$, is given below.)

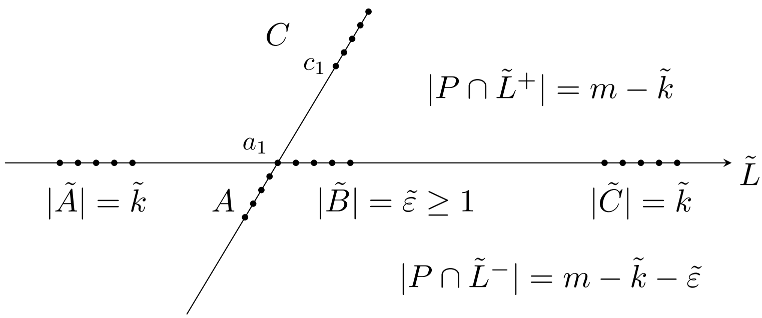

Figure 3: Two central segments whose common point is an endpoint in one of them.

Since M matches $P \cap \tilde{L}^+$ with $(P \cap \tilde{L}^-) \cup \tilde{B}$ and a_1 ($a_1 \in \tilde{B}$ and $a_1 \in A \subset P \cap L$) with some point of C (Proposition 2.5), $C \subset P \cap \tilde{L}^+$ (as in Figure 3). Thus, removing $A \cup C$ will reduce $|P \cap \tilde{L}^+|$ by k to $(m-k) - \tilde{k}$, $P \cap \tilde{B}$ by 1 to $\tilde{\varepsilon} - 1$ (≥ 0, since $a_1 \in \tilde{B}_1$), and $|P \cap \tilde{L}^-|$ by $k-1$ to $(m-k) - \tilde{k} - (\tilde{\varepsilon} - 1)$.

(B) If M' is a TM of P', and N is a matching of A to C (on $P \cap L$), then $M = M' \cup N$ is a TM of P iff the central segment $[a_1, c_1]$ of L meets the central segment of each halving line of P'. Thus, if applying our algorithm to P' we find that P' has no TM, then the same holds for P. If P' does admit a TM, then P has a TM iff the central segment of L meets the central segment of each halving line of P'. To check this, we may need $O(m^2)$ operations.

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Characterizing all graphs with 2-exceptional edges*

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Abstract

Dirac and Shuster in 1954 exhibited a simple proof of Kuratowski theorem by showing that any 1-crossing-critical edge of G belongs to a Kuratowski subdivision of G. In 1983, Širáň extended this result to any 2-crossing-critical edge e with endvertices b and c of a graph G with crossing number at least two, whenever no two blocks of $G - b - c$ contain all its vertices. Calling an edge f of G k-*exceptional* whenever f is k-crossing-critical and it does not belong to any Kuratowski subgraph of G, he showed that simple 3-connected graphs with k-exceptional edges exist for any $k \geq 6$, and they exist even for arbitrarily large difference of $\mathrm{cr}(G) - \mathrm{cr}(G - f)$. In 1991, Kochol constructed such examples for any $k \geq 4$, and commented that Širáň's result holds for any simple graph.

Examining the case when two blocks contain all the vertices of $G - b - c$, we show that graphs with k-exceptional edges exist for any $k \geq 2$, albeit not necessarily simple. We confirm that no such simple graphs with 2-exceptional edges exist by applying the techniques of the recent characterization of 2-crossing-critical graphs to explicitly describe the set of all graphs with 2-exceptional edges and noting they all contain parallel edges. In this context, the paper can be read as an accessible prelude to the characterization of 2-crossing-critical graphs.

Keywords: Kuratowski subgraphs, crossing number, exceptional edges.

Math. Subj. Class.: 05C10, 05C62

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1 Introduction

The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise crossings of edges in a drawing of G in the plane. An edge e of a graph G is said to be *k-crossing-critical*, if $\mathrm{cr}(G) \geq k > \mathrm{cr}(G-e)$, and a graph is k-crossing-critical, if each its edge is k-crossing-critical. Therefore $K_{3,3}$ and K_5 are the only 3-connected 1-crossing-critical graphs. Any subdivision of $K_{3,3}$ or K_5 in G is called a *Kuratowski subgraph* of G and an edge e is a *Kuratowski edge*, if e belongs to a Kuratowski subgraph of G. Any edge of G which is not a Kuratowski edge, will be called a *non-Kuratowski edge*. Following [12], we call an edge e of G *k-exceptional* if e is k-crossing-critical and e is a non-Kuratowski edge. Note that the existence of a k-exceptional edge in G for $k > 0$ implies the existence of a Kuratowski subgraph, and hence that G is non-planar. Since loops are irrelevant for crossing number purposes, all graphs in this paper are loopless, but they may have multiple edges.

In their simple proof of Kuratowski theorem from 1954, Dirac and Shuster established that any 1-crossing-critical edge e of a graph G belongs to a Kuratowski subdivision of G [6]. In 1983, Širáň showed that the number of non-Kuratowski edges (and hence the number of exceptional edges) of a 3-connected simple non-planar graph of order at least 6 is at most 4 [13]. The following statement was exhibited in the same year.

Statement 1.1 (Theorem 2 in [12])**.** *Let e be a crossing-critical edge of a graph G, for which $\mathrm{cr}(G-e) \leq 1$. Then e belongs to a Kuratowski subgraph of G.*

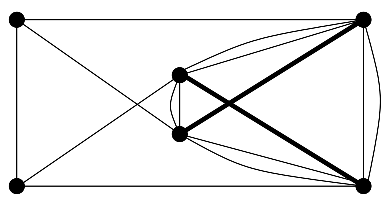

Figure 1: A minimal graph with two 2-exceptional edges.

We have found a family of exceptions (see Figure 1) to Statement 1.1, i.e. a family of graphs with 2-exceptional edges. That such graphs exist was already exhibited by Kochol [8], who noted without proof that Širáň's result may only be true for simple graphs. Closely investigating Širáň's proof, it establishes [12] the following:

Theorem 1.2 (Theorem 2 in [12])**.** *Let e with endvertices b and c be a crossing-critical edge of a graph G for which $\mathrm{cr}(G-e) \leq 1$. If no two blocks of $G-b-c$ contain all its vertices, then e belongs to a Kuratowski subgraph of G.*

The correct statement indicates that the structure of graphs with 2-exceptional edges is limited, and the aim of the present paper is to characterize these graphs, i.e. to explicitly describe the family $\mathcal{E}$ of graphs with 2-exceptional edges.

The rest of this paper is organized as follows. In the following section, we exhibit some known and new properties of Kuratowski edges in graphs, and offer their characterization in Theorem 2.6, as well as introduce our main result. In Section 3, we sketch our overall approach, which follows a simplified version of the recent characterizatizaton of 2-crossing-critical graphs [4]. The description of all 3-connected graphs with 2-exceptional edges is given in Section 4, along with the proof of the sufficiency direction of the characterization and some other properties of graphs with 2-exceptional edges.

The remainder of the paper is devoted to proving necessity of the characterization of 3-connected graphs with 2-exceptional edges. A skeleton graph, the basic subgraph that is used to describe 3-connected graphs with exceptional 2-edges, is studied in Section 5. Bridges of the skeleton graph are studied in Section 6. Also the necessity of characterization is established there. We conclude with some corollaries bearing upon existence of k-exceptional edges and some open problems in Section 7.

2 Kuratowski edges

First we introduce some notation, aligned with the notation of [4]. Any vertex of a graph G of degree at least 3 is called a *node* of G. A *branch* is a maximal path with no internal nodes connecting two nodes of G. Two distinct nodes u, v of a subdivision K of $K_{3,3}$ are said to be *independent* if any u, v-path in K contains a node of K different from u and v, i.e. if there is no branch between them. Let A, B be either two subsets of $V(G)$ or two subgraphs of G. Then, an A, B-path is a path with first end in A and last end in B that is internally disjoint from both A and B.

When $A = \{s\}$ and $B = \{t\}$ are just vertices, we shorten the notation to just s, t-path. When the ends need to be emphasized, we write $P = sPt = [sPt]$, the former emphasizing ends of P and the latter emphasizing that the complete path with ends is considered. When either end or both ends of P are removed from the path, we use $P - t = [sPt)$, $P - s = (sPt]$, and $P - s - t = (sPt)$. We refer to these paths as the *(semi) open* paths P.

Let G be a graph and H its subgraph. A path P is H-*avoiding*, if all the non-end vertices and all the edges of P are not in H. The ends of an H-avoiding path are allowed to be in H.

In [13], J. Širáň gave a characterization of Kuratowski edges in k-connected graphs with $k \geq 3$, cf. Lemmas 2.1 and 2.2. In this section, we extend his a characterization to any graph. Next two lemmas were proved in [13] and will be very useful for our purposes.

Lemma 2.1 (Lemma 1 in [13])**.** *Let K be a subdivision of $K_{3,3}$ and let u, v be distinct vertices of K. Let $K' := K + P$ be a graph obtained by joining u, v with a path P internally disjoint of K. Then any edge of P is Kuratowski edge of K' if and only if u, v are not independent nodes of K.*

Lemma 2.2 (Lemma 2 in [13])**.** *Let G be a 3-connected non-planar graph. Let $e = uv$ be an edge of G which belongs to no subdivision of $K_{3,3}$ in G. Then u, v are independent vertices of any subdivision of $K_{3,3}$ in G.*

Although [13] considered only simple graphs, it is easy to see that the two lemmas apply to multigraphs as well. Our first statement is an easy exercise.

Lemma 2.3. *Let G be a graph and let G' be a subdivision of G. Let e be an edge of G and let P be the path of G' obtained by subdividing e. The following are equivalent:*

(i) e is a Kuratowski edge of G,

(ii) every edge of P is a Kuratowski edge of G',

(iii) some edge of P is a Kuratowski edge of G'.

The proof of our next result is essentially the same as that of Lemma 1 in [13].

Lemma 2.4.

(i) Let K be a subdivision of K_5 and let u, v be distinct vertices of K. Let $K' := K + P$ be a graph obtained by joining u, v with a path P internally disjoint of K. Then any edge of P is a Kuratowski edge of K'.

(ii) Let G be a 2-connected graph that contains a subdivision of K_5 as a subgraph. Then every edge of G is a Kuratowski edge.

Proof. For (i), one can easily check using the symmetry of K_5 that there are exactly six homeomorphism classes of graphs to which K' can belong. In all of them, it is easy to find the required subdivision of K_5 or $K_{3,3}$ in K': if both u, v are on the same branch of K, a slight rerouting establishes the claim. In other cases we find a suitable $K_{3,3}$ subdivision including a degree-three vertex in one part, and its nearest degree at least 3 vertices in the other part.

For (ii), let e be an edge of G with ends u and v and let K be a subgraph of G, which is isomorphic to a subdivision of K_5. If $e \in K$, we are done. Thus, we may assume that e is not an edge of K. By Menger's theorem, G contains two disjoint paths $P_1 := uP_1x_1$ and $P_2 := vP_2x_2$ with $x_1, x_2 \in V(K)$ such that $V(K) \cap P_i = \{x_i\}$ for $i = 1, 2$. Now by applying (i) to $P := x_1P_1uevP_2x_2$ and K, we have that e is a Kuratowski edge. □

The following is immediate from the definition of exceptional edge, Kuratowski's theorem and Lemma 2.4.

Lemma 2.5. *If G is a 2-connected graph containing exceptional edges, then G contains at least one Kuratowski subgraph and every Kuratowski subgraph of G is a subdivision of $K_{3,3}$.*

The above gives the following characterization of Kuratowski edges in general graphs:

Theorem 2.6. *Let G be a graph and let e be an edge of G. Then e is a Kuratowski edge of G if and only if G contains a Kuratowski subgraph K and a path P such that:*

(i) P contains e,

(ii) P joins distinct vertices u, v of K,

(iii) u, v are not independent nodes of K, and

(iv) either P is contained in K or P is internally disjoint from K.

Proof. The necessity part is immediate: if e is a Kuratowski edge of G, then e belongs to a Kuratowski subgraph K_1 and $K := K_1$, $P := e$ satisfy conditions (i)–(iv).

The sufficiency follows from previous lemmas: assume that G contains subgraphs K and P satisfying (i)–(iv), and apply Lemma 2.1 or 2.4 to the pair K, P depending on whether K is a subdivision of $K_{3,3}$ or K_5. □

This characterization is important due to the following corollary, which allows us to restrict ourselves to 2-connected graphs when characterizing all graphs with 2-exceptional edges:

Corollary 2.7. *Let G be a graph, let e be an edge of G and let B be the block of G containing e. Then e is a Kuratowski edge of G if and only if e is a Kuratowski edge of B.*

Following [12], for a (possibly empty) subset S of vertices of G, a pair (H, K) of subgraphs of G is an S-decomposition, if

(i) each edge of G belongs to precisely one of H, K and

(ii) $H \cap K = S$.

For $|S| = 0, 1$, let $H^+ = H$, $K^+ = K$, and for $|S| = 2$, let H^+ (respectively K^+) be obtained from H (respectively, K) by adding an edge between the two vertices of S.

Lemma 2.8. *Let G be a graph with a 2-exceptional edge and let (H, K) be an S-decomposition of G with $|S| \leq 2$. Then, precisely one of H^+, K^+ is non-planar.*

Proof. If both are planar, so is G, a contradiction, so at least one is non-planar.

Suppose that both are non-planar. As e is not on any Kuratowski graph of G, $G - e$ has the same Kuratowski graphs as G. Let K_H be a Kuratowski graph of H^+ and K_K a Kuratowski graph of K^+. Since at most one branch of K_K and K_H contains S, for each $K \in \{K_K, K_H\}$, every drawing of K has a crossed edge that is not an edge of the graph in $\{K_K, K_H\} \setminus \{K\}$. This shows that any drawing of $K_K \cup K_H$ has at least two crossings, implying that $\mathrm{cr}(G - e) \geq 2$, a contradiction. □

For $k = 0, 1, 2, 3$, let $\mathcal{E}_k$ be the family of k-connected graphs that contain 2-exceptional edges. Our main theorem describes these sets, but recursive description of $\mathcal{E}_2 \setminus \mathcal{E}_3$ requires an additional lemma:

Lemma 2.9 ([12])**.** *Let (H, K) be an $\{u, v\}$-decomposition of G and suppose that $\mathrm{cr}(H) = \mathrm{cr}(H^+)$. Then, $\mathrm{cr}(G) = \mathrm{cr}(K + \lambda uv) + \mathrm{cr}(H)$, where λ is the maximum number of edge-disjoint paths from u to v in H.*

Theorem 2.10. *Let G be a graph that has a 2-exceptional edge e with endvertices b and c. Then*

1. *$G \in \mathcal{E}_0 \setminus \mathcal{E}_1$ is disconnected, all but one of its components are planar, and the non-planar component belongs to $\mathcal{E}_1$.*

2. *$G \in \mathcal{E}_1 \setminus \mathcal{E}_2$ is connected, but not 2-connected, all but one of its blocks are planar, and the non-planar block belongs to $\mathcal{E}_2$.*

3. *$G \in \mathcal{E}_2 \setminus \mathcal{E}_3$ is obtained from a subdivision of $G' \in \mathcal{E}_3$ by replacing its edge st of multiplicity μ with a planar graph H containing vertices s and t, such that $H + st$ is 2-connected and there are at least μ edge-disjoint s, t-paths in H.*

4. *$G \in \mathcal{E}_3$ is a cyclization of four tiles, as described in Theorem 4.2.*

Proof. Claims 1 and 2 follow from applying Lemma 2.8, with $|S| = 0, 1$, respectively. As we defer the proof of Theorem 4.2 to the next sections, we only need to prove Claim 3.

Suppose that (H, K) is a $\{u, v\}$-decomposition of G that has exceptional edges. By Lemma 2.8, we may assume H^+ is planar. Then by Lemma 2.9, $\mathrm{cr}(G) = \mathrm{cr}(K + \lambda uv)$ and $\mathrm{cr}(G - e) = \mathrm{cr}((K - e) + \lambda uv)$. Therefore, e is exceptional in $K + \lambda uv$. Applying this reduction to any (H, K) decomposition in which H has vertices not in K reduces G to a subdivision of a graph in $\mathcal{E}_3$. This reduction has a constructive counterpart: any edge f of a graph in $\mathcal{E}_2$ can be subdivided, yielding a graph of $\mathcal{E}_2$. If the original edge was exceptional, so are both new edges. Furthermore, if $e = uv$ is not the only exceptional edge of G, then e can be replaced by any planar 2-connected H, for which $H + uv$ is also planar. The resulting graph is again in $\mathcal{E}_2$. Moreover, if uv has multiplicity λ, some of its edges can be replaced by H simultaneously, provided H has at least that many edge-disjoint u, v-paths. Thus, any graph of $\mathcal{E}_2$ can be obtained starting with a graph in $\mathcal{E}_3$, applying subdivisions and replacing edges by described planar graphs, proving Claim 3. □

3 Tile decomposition method

For clarity, we describe the structure of our characterization of graphs with 2-exceptional edges in this section. It will follow the ideas of recent characterization of 2-crossing-critical graphs [4]. The approach can be abstracted into the following steps, which allow to decompose an abstract graph with properties of interest into smaller pieces called tiles. The tiles are a tool often applied in the investigation of crossing critical graphs [2, 3, 4, 7, 9, 10, 11]. The method structures arguments as follows:

1. *Limit connectivity of graphs of interest.* In both instances, we focus on 3-connected graphs, showing how to obtain graphs of lower connectivity from these and identifying exceptional less connected instances. For us, this step is simple as all graphs fit the pattern; in characterization of 2-crossing-critical graphs, it involved analyzing the exceptional graphs.

2. *Identify a skeleton graph K.* In the case of 2-crossing-critical graphs, the skeleton graph K is V_{10}. In our case, it is the graph K in which contracting the edge incident with two degree three vertices produces K_5.

3. *Study drawings or embeddings of the skeleton graph.* In the case of 2-crossing-critical graphs, V_{10} has two nonhomeomorphic drawings in the plane, which turned to be better analyzed as two essentially different projective-planar embeddings. For a graph G with 2-exceptional edge e and the skeleton graph K, we show that in any optimal drawing of $G - e$, the subdrawing of $K - e$ is determined up to homeomorphism.

4. *Find a skeleton graph $H \cong K$ and its drawing/embedding that offers sufficient structure for finding tiles.* Usually this amounts to a skeleton graph, for which in the selected embedding, all bridges lie in well-controlled faces. For 2-crossing-critical graphs, there were three steps (friendly embedding, pre-tidy V_{10}, tidy V_{10}). Showing existence of such embeddings and skeleton graphs turned to be an important step in both cases. After this step, a *standard quadruple* was introduced in both cases to carry the information about the investigated graph G, its selected drawing or embedding Π, and the tidy skeleton graph $K \cong H \subseteq G$ that were required for subsequent proofs. Once a special skeleton graph and its drawing are defined, introduce a standard labelling with respect to that skeleton and its drawing.

5. *Restrict bridges of (parts of) the skeleton graph.* In the case of 2-crossing-critical graphs, bridges of V_{10} are shown to be either edges or small stars, and their attachments are near in the V_{10} subdivision. In our case, we show that there exists a

skeleton graph K, such that all its bridges that are not edges lie in the infinite face of some of its optimal drawings, and that after removing the K_4 subgraph of the skeleton with its parallel edges, we (roughly) obtain a join of two 3-connected planar graphs.

6. *Combine bridges into tiles.* In the case of 2-crossing-critical graphs, this analysis relies on identifying types of edges in the V_{10} subdivision and then splitting pieces between two consecutive edges of a specific type. Our instance is simpler: we show that our bridges constitute a sequence of four tiles, whose cyclization yields a graph of interest.

7. *Prove that every tiled structure yields a graph of interest.* Once the structure is determined, this is usually an easy task, and to confirm intuition about the listed steps, it can even be done as soon as the tiles are conjectured.

We conclude this section by introducing the needed notation related to tiles. As in our approach we do not need the gadgets limiting crossing numbers of tiles, only the very basics are needed. For most recent developments on the theory of tiles, see [3, 4].

A *tile* is a triple $T = (G, \lambda, \rho)$, where G is a graph and λ, ρ are two disjoint sequences of distinct vertices of G, called the *left and right wall* of T, respectively. Two tiles $T = (G, \lambda, \rho)$ and $T' = (G', \lambda', \rho')$ are *compatible*, if $|\rho| = |\lambda'|$. The *join of two compatible tiles* T and T' with $\rho = (\rho_1, \dots, \rho_w)$ and $\lambda' = (\lambda'_1 \dots, \lambda'_w)$ is defined as the tile $T \otimes T' := (G'', \lambda, \rho')$, where G'' is the graph obtained from the disjoint union of G and G' by identifying ρ_i with λ'_i, for $i = 1, \dots, w$. Specially, if $\rho_i = \lambda'_i$ is a vertex with precisely two neighbors (after the identification), we replace it with a single edge in G'' of multiplicity equal to the smaller of the multiplicities of the edges incident with $\rho_i = \lambda'_i$. This technical detail is important when considering 3-connected graphs. Since the operation $\otimes$ is associative, we can safely define the *join of a compatible sequence of tiles* $\mathcal{T} = (T_0, T_1, \dots, T_m)$ as $\otimes\mathcal{T} = T_0 \otimes T_1 \otimes \dots \otimes T_m$. The *cyclization* of a self-compatible tile $T = (G, \lambda, \rho)$, denoted by $\circ T$, is the ordinary graph obtained from G by identifying λ_i with ρ_i for $i = 1, \dots, w$. The *cyclization of a self-compatible sequence of tiles* $\mathcal{T} = (T_0, T_1, \dots, T_m)$ is $\circ\mathcal{T} := \circ(\otimes\mathcal{T})$. Again, possible vertices with two neighbors are replaced with an edge maintaining smaller edge multiplicity, as above. We will also need the concept of a *reversed* tile of T, which is the tile with the two walls exchanged, $T^{\leftrightarrow} = (G, \rho, \lambda)$.

4 3-connected graphs with 2-exceptional edges

In this section, we describe the class of graphs whose members are precisely all the 3-connected graphs containing at least one 2-exceptional edge. In particular, we define such a class and show that all its elements are 3-connected and have 2-exceptional edges. In the following sections, we show that any graph with 2-exceptional edges belongs to this class.

For $i \geq 1$, let $G_1^i = u_i v_i$ be an edge of multiplicity i, and $O_i = (G_1^i, (u_i), (v_i))$ a corresponding tile. Let $\mathcal{O}$ be the family of all tiles O_i, $i \geq 1$.

For $i, j \geq 1$, let ${}^iG_2^j$ be the graph obtained by identifying the vertices v_i and u_j of O_i and O_j, respectively. Then ${}^iG_2^j$ has a vertex w of degree $i + j$ and two vertices u_i, v_j of degree i, j, respectively, see Figure 2. By ${}^iQ^j = ({}^iG_2^j, (u_i), (v_j, w))$, we denote the tile constructed using ${}^iG_2^j$.

Let s_1, s_2, s_3 and s_4 be the vertices of K_4. We use $H = {}^0H^0_{1,1}$ to denote the graph obtained from such a K_4 by doubling the edges of the path $s_4s_1s_2s_3$ and adding to s_3 and s_4 a new edge leading to a new vertex w_2 and w_1, respectively (see Figure 2). The graph $H^i = {}^0H^i_{1,1}$ is obtained from the disjoint union of H and G^i_1 by identifying s_1 of H with v_i of G^i_1, and let ${}^jH^i = {}^jH^i_{1,1}$ be the graph obtained from the disjoint union of G^j_1 and H^i by identifying the vertex u_j of G^j_1 with s_2 of H^i. Note that for $i, j = 0$ the graph ${}^iH^j$ is defined independently of G^i_1, G^j_1, which only exist for $i, j \geq 1$.

For k a positive integer, we denote by ${}^jH^i_k$ the graph obtained from ${}^jH^i$ by increasing multiplicity of one of the edges s_1s_3 or s_2s_4 (but not both) to k. Finally, for l a positive integer, the graph ${}^jH^i_{k,l}$ is obtained from ${}^jH^i_k$ by increasing the multiplicity of the edge s_3s_4 to l.

For any integers i, j, k and l such that $i, j \geq 0$, and $k, l \geq 1$, we define a tile ${}^jR^i_{k,l} = ({}^jH^i_{k,l}, (u_i, w_2), (v_j, w_1))$; for $i, j = 0$, we set $u_0 = s_1$ and $v_0 = s_2$, respectively.

We use $\mathcal{R}$ to denote the family consisting of all the tiles ${}^jR^i_{k,l}$ and all the tiles that can be obtained from these by arbitrarily increasing multiplicity of each edge on the path $s_4s_1s_2s_3$ (which must, however, remain at least two).

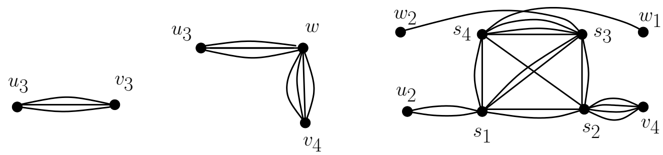

Figure 2: From left to right: O_3, ${}^3Q^4$, and ${}^2R^4_{2,3}$.

Let $\mathcal{P}$ be the family that contains each tile T that can be obtained from any 3-connected planar G containing a degree three vertex x with neighbors u, v, w as $T = (G - x, (u), (v, w))$. In addition to these, let us assume that $\mathcal{P}$ also contain each tile ${}^iQ^j$, with $i, j \geq 1$.

A *pre-exceptional sequence* $\mathcal{T}$ of tiles has four tiles (T_1, T_2, T_3, T_4), such that:

(C_1) $T_1 = O_{i_1} \in \mathcal{O}$,

(C_2) $T_2 \in \mathcal{P}$,

(C_3) $T_3 = {}^{i_3}R^{j_3}_{k_3,l_3} \in \mathcal{R}$,

(C_4) $T_4^{\leftrightarrow} \in \mathcal{P}$, and

(C_5) if $T_2 = {}^{i_2}Q^{j_2}$, then $i_3 \geq 1$ (respectively, if $T_4^{\leftrightarrow} = {}^{i_4}Q^{j_4}$ then $j_3 \geq 1$).

Then there are exactly six types of pre-exceptional sequences: depending on whether T_2 (respectively, T_4) comes from a 3-connected planar graph, or $T_2 = {}^{i_2}Q^{j_2}$ (respectively, if $T_4^{\leftrightarrow} = {}^{i4}Q^{j_4}$), we have sixteen types of $\mathcal{T}$'s, which are reduced to six by considering (C_5) and the symmetry. Such six types of $\mathcal{T}$'s are shown in Figure 3.

The *signature* of a pre-exceptional sequence $\mathcal{T} = (T_1, T_2, T_3, T_4)$ is an ordered list of integers $\sigma(\mathcal{T}) = (i_1, i_2, j_2, i_3, j_3, k_3, l_3, i_4, j_4)$ which is obtained from $\mathcal{T}$ as follows: the integer i_1 is taken from $T_1 = O_{i_1}$. If T_2 comes from a 3-connected planar graph G, then i_2 and j_2 are both equal to the number of edge-disjoint u, v-paths in G, and in the other case, we take them from $T_2 = {}^{i_2}Q^{j_2}$. The integers i_3, j_3, k_3, l_3 are taken from

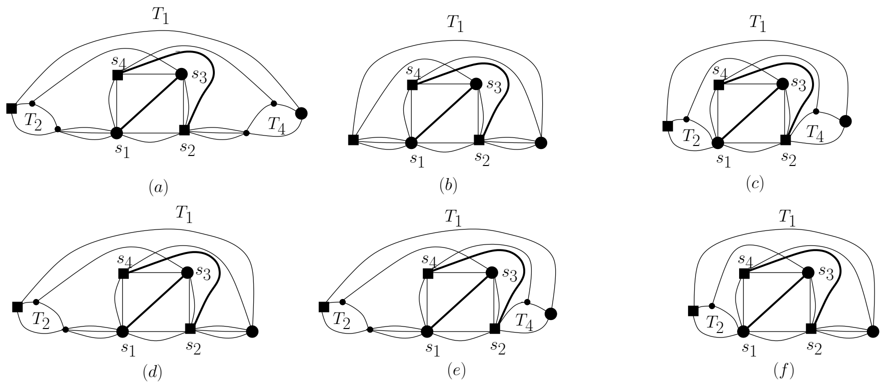

Figure 3: The six types of pre-exceptional sequences.

$T_3 = {}^{i_3}R^{j_3}_{k_3,l_3}$. Finally, if $T_4 \in \mathcal{P}$ comes from a 3-connected planar graph G, then i_4 and j_4 are both equal to the number of edge-disjoint u,v-paths in G, and in the other case, we take them from $T_4^{\leftrightarrow} = {}^{i_4}Q^{j_4}$. The *relevant signature* $\rho(\mathcal{T})$ is the set of positive values in $\{i_1, i_2, j_2, i_3, j_3, i_4, j_4\}$. An pre-exceptional sequence $\mathcal{T}$ is *exceptional*, if either

(i) $\min \rho(\mathcal{T}) = 2$ or

(ii) $\min \rho(\mathcal{T}) = 1$ and $l_3 \geq 2$.

The following technical observation is also needed in our proof.

Lemma 4.1. *Let G be a graph, and let $\{u, v, w\}$ be a vertex cut of G such that $\{u, v, w\}$ is not the neighborhood of a vertex in G. Let G_i be a non-trivial bridge of $\{u, v, w\}$ in G, and obtain G'_i by connecting a new vertex t_i to each of u, v, w. Then, G is 3-connected, if and only if each of G'_i is 3-connected.*

Proof. First we assume that G is 3-connected. Let p, q be any two vertices in G'_i. If $p = t_i$, choose as p any vertex of G_j for $j \neq i$. The three internally disjoint paths in G connecting p and q can be easily converted to three internally disjoint paths connecting t_i and q. If both p and q are distinct from t_i, there are three internally disjoint paths pP_1q, pP_2q, pP_3q in G. At least two of these paths are in G_i, and if the third one is not, it uses two of the vertices u, v, w. We may assume it is $pP_uuP'_3vP_vq$. But then, $pP_uut_ivP_vq$ is a path of G_i, internally disjoint from P_1, P_2, completing the necessity direction.

For sufficiency direction, let p, q be two arbitrary vertices of G. If they are in the same G_i, there are three internally disjoint paths in G'_i that connect them, which can easily be augmented to paths in G. If $p \in G_i$, $q \in G_j$, $i \neq j$, then let $pP^p_uut_i$, $pP^p_vvt_i$, $pP^p_wwt_i$ be three internally disjoint paths between p and t_i in G_i, and similarly $qP^q_uut_j$, $qP^q_vvt_j$, $qP^q_wwt_j$ be three internally disjoint paths between q and t_j in G_j. Then, $pP^p_uuP^q_uq$, $pP^p_vvP^q_vq$, $pP^p_wwP^q_wq$ are three internally disjoint paths between p and q in G. □

Theorem 4.2.

(I) A graph G is in $\mathcal{E}_3$, if and only if G can be described as a cyclization of an exceptional sequence $\mathcal{T} = (T_1, T_2, T_3, T_4)$.

(II) G has two 2-exceptional edges if and only if $k_3 = 1$ in $T_3 = {}^{i_3}R^{j_3}_{k_3,l_3}$.

(III) If $G \in \mathcal{E}_0$, then $\mathrm{cr}(G) = 2$.

Proof. We show Claims (II) and (III), and sufficiency of Claim (I). The necessity of Claim (I) is covered in the subsequent sections.

Let G be a cyclization of $\mathcal{T} = (T_1, T_2, T_3, T_4)$ as above. In this proof, we use the notation and drawings provided in Figure 3. Without loss of generality, we may assume that the edge of T_3 with endvertices s_1 and s_3 has multiplicity k_3.

In order to show that $G \in \mathcal{E}_3$, we need verify that it is 3-connected and that it contains 2-exceptional edges. From the construction of G and Lemma 4.1 it is not difficult to see that G is 3-connected. Thus it is enough to show that the edge $e = s_2s_4$ is a 2-exceptional edge of G. In other words, we need to show that:

(I1) $\mathrm{cr}(G - e) \leq 1$,

(I2) e is not a Kuratowski edge of G, and

(I3) $\mathrm{cr}(G) \geq 2$.

For $r \in \{2, 4\}$, we assume that in any drawing of G under consideration, the restriction of such a drawing to T_r is a plane graph. Indeed, since there exists a drawing of T_r that has all the wall vertices on the same face, such a face can be made the infinite face by inversion and the resulting drawing or its mirror can be used to form the required drawing of G.

Then, regardless the multiplicities of the edges in T_1 or T_3, the drawings in Figure 3 imply that $\mathrm{cr}(G) \leq 2$ and $\mathrm{cr}(G - e) \leq 1$, which reduces (I1) to (I3).

For (I2), seeking a contradiction, assume that e lies on some Kuratowski subgraph K of G. As G has exactly 4 vertices not in $T_2 \cup T_4$ (namely, s_1, s_2, s_3 and s_4), then for some $r \in \{2, 4\}$, T_r contains at least one node of K. On the other hand, the planarity of T_r and the fact that T_r contains only three wall vertices imply that at least one node of K is not in T_r. Because T_r is joined to exactly three vertices of $G - T_r$, K is not homeomorphic to K_5. Then K is homeomorphic to $K_{3,3}$. Since any set of four edges with an end in T_r and the other in $G - T_r$ contain at least one pair of parallel edges, the number of nodes of K in T_r must be exactly one. In particular, this implies that s_1, s_2, s_3, and s_4 are nodes of K, and that each of T_2 and T_4 contains exactly one node of K. Since the node of K in T_4 is joined to s_2 and s_4, these vertices belong to the same chromatic class in K, however, as $e = s_2s_4$ is an edge in K, s_2 and s_4 belong to distinct chromatic classes, a contradiction.

Now we show that (I3) $\mathrm{cr}(G) \geq 2$. We analyze separately two cases, depending on whether $\min \rho(\mathcal{T}) = 2$ or $\min \rho(\mathcal{T}) = 1$.

Case 1. $\min \rho(\mathcal{T}) \geq 2$. Let H be the graph that results by deleting from G all the vertices of T_2 and T_4 that are not in the face containing the wall vertices. Note that if T_2 (respectively, T_4) comes from a 3-connected planar graph, then $T_2 \cap H$ (respectively, $T_4 \cap H$) is a cycle of length at least 3, and in the other case, $T_2 \cap H = T_2 = {}^{i_2}Q^{j_2}$ (respectively, $T_4 \cap H = T_4 = {}^{i_4}Q^{j_4}$). Clearly, $\mathrm{cr}(G) \geq \mathrm{cr}(H)$. Now we verify that $\mathrm{cr}(G) \geq 2$ by showing that $\mathrm{cr}(H) \geq 2$.

Let D be an optimal drawing of H. As usual, we assume that parallel edges are placed very closely to each other in D, and hence, that they have the same number of crossings. Then if any edge of the path $P := s_4s_1s_2s_3$ is crossed in D, we are done.

Let h be an edge of H with endvertices s_1 and s_3. From Figure 3, we can see that H contains a subgraph J homeomorphic to $K_{3,3}$ that avoids e and h (the thick edges). Indeed, the nodes of J are s_1, s_2, s_3, s_4 and p, q the endvertices of T_1. If the restriction $D[J]$ of D to J has at least two crossings, or at least one of e, h is crossed in D, we are done. Thus we assume that $\mathrm{cr}(D[J]) = 1$ and that both e, h are clean in D. In particular, note that the restriction D' of D to the subgraph H' of H induced by s_1, s_2, s_3 and s_4 is a plane graph and that H' contains to K_4 as subgraph.

On the other hand, $\min \rho(\mathcal{T}) \geq 2$ implies that the number of parallel edges between p and q is at least 2, and hence both p and q are in the same face of D', or we are done. By using stereographic projection if necessary, we may assume that such a face is the infinite face of D'. Then exactly one vertex of H', say s', is in the triangular finite face formed by the other three vertices. Moreover, from the definition of J it follows that at least one of p or q is joined with s' by a path P', which is internally disjoint of H'.

Since, for $r = 1, 2$, H contains at least two p, s_r-paths edge disjoint and internally disjoint from H', then $s' \notin \{s_1, s_2\}$, or such p, s_r-paths provide the required crossings. If $s' \in \{s_3, s_4\}$, then P' crosses at least one edge of $E(P) \cup \{e, h\}$, which is impossible.

Case 2. $\min \rho(\mathcal{T}) = 1$ and $l_3 \geq 2$. Since T_2 and T_4 are connected and $l_3 \geq 2$, then G contains a subgraph H which is homeomorphic to the graph shown in Figure 4. As before, we verify $\mathrm{cr}(G) \geq 2$ by showing that $\mathrm{cr}(H) \geq 2$.

Let C be the double cycle of H whose vertices are s_1, s_2, s_3, and s_4, and let D be an optimal drawing of H. If any edge of C is crossed, we are done. Then we may assume that the restriction $D[C]$ of D looks like in Figure 4. Then u and v must be in the same face of $D[C]$: otherwise, at least one edge of C is crossed by the edge with endvertices u and v and we are done. Without loss of generality, we assume that both are in the infinite face of $D[C]$, as shown in Figure 4. Note that the paths s_1us_3 and s_2vs_4 cross each other because they have alternating ends in C. Similarly, if the edges s_1s_3 and s_2s_4 are in the same face of C, we have the required crossing. Then at least one of them is in the infinite face of C and such an edge must cross with some of s_1us_3 or s_2vs_4 providing the required crossing. This proves (I3) and hence sufficiency of (I).

The inequality in (I3) was independently checked with the crossing number computing tool of Chimani et al. [5]

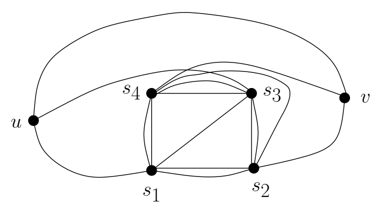

Figure 4: A drawing of H.

Now we show (II) that G has two 2-exceptional edges if and only if $k_3 = 1$ in $T_3 = {}^{i_3}R^{j_3}_{k_3,l_3}$. Note that, by symmetry, the argument used in (I2) also shows that any edge of G with endvetices s_1 and s_3 is a not a Kuratowski edge. Let us denote by K_3 the set of edges of G with ends s_1 and s_3.

On the other hand, from the definition of ${}^{i_3}R^{j_3}_{k_3,l_3}$, we know that e is the only edge of G with ends s_2 and s_4. Then Lemmas 2.2 and 2.5 imply that s_1 and s_3 (respectively, s_2 and

s_4) are in the same chromatic class of nodes of a subdivision K of $K_{3,3}$ in G.

We derive a contradiction from the assumption that G contains a non-Kuratowski edge $e' \notin K_3 \cup \{e\}$. Then Lemma 2.2 implies that e' joins two nodes in the same chromatic class of nodes of K. Furthermore, since $e' \notin K_3 \cup \{e\}$, then it must have an end in $\{s_1, s_2, s_3, s_4\}$ and the other in a node of $K \setminus \{s_1, s_2, s_3, s_4\}$. The existence of such an e' implies that $K \cup K_3 \cup \{e, e'\} \subseteq G$ contains K_5 as subdivision. This and Lemma 2.5 imply that all the edges of G are Kuratowski edges, a contradiction.

Let us assume that $k_3 = 1$ in $T_3 = {}^{i_3}R^{j_3}_{k_3,l_3}$. Then K_3 consists of an edge h. From (I1) and (I3) we have $\mathrm{cr}(G) = 2$. Now, if we draw e inside of the square $s_1s_2s_3s_4$ in Figure 3, we get, in all the cases, an optimal drawing of G in which h is crossed by e. This proves that h is 2-crossing-critical, and hence e and h are both 2-exceptional edges.

On the other hand, since $\mathrm{cr}(G) = 2$ for any $k_3 \geq 1$, then if $k_3 \geq 2$ we have that no edge in K_3 is 2-crossing-critical and since $K_3 \cup \{e\}$ are the only non-Kuratowski edges of G, then $k_3 > 1$ implies that e is the only 2-exceptional edge of G. This proves (II).

Finally, we show (III) that if $G \in \mathcal{E}_0$, then $\mathrm{cr}(G) = 2$.

(1) If $G \in \mathcal{E}_3$ we are done by (I1) and (I3).

(2) If $G \in \mathcal{E}_2 \setminus \mathcal{E}_3$ then, by Theorem 2.10(3), there exists $G' \in \mathcal{E}_3$ such that $\mathrm{cr}(G) = \mathrm{cr}(G')$. Since $\mathrm{cr}(G') = 2$, we are done.

(3) If $G \in \mathcal{E}_1 \setminus \mathcal{E}_2$ then, by Theorem 2.10(2), all but one of blocks of G, say B, are planar and $B \in \mathcal{E}_2$. Then $\mathrm{cr}(G) = \mathrm{cr}(B)$. If $B \in \mathcal{E}_3$ (respectively $B \in \mathcal{E}_2 \setminus \mathcal{E}_3$) we are done by (1) (respectively (2)).

(4) If $G \in \mathcal{E}_0 \setminus \mathcal{E}_1$ then, by Theorem 2.10(1), all but one of the components of G, say C, are planar and $C \in \mathcal{E}_1$. Then $\mathrm{cr}(G) = \mathrm{cr}(C)$. Clearly, exactly one of the following is true: $C \in \mathcal{E}_3$, $C \in \mathcal{E}_2 \setminus \mathcal{E}_3$, or $C \in \mathcal{E}_1 \setminus \mathcal{E}_2$. Note that these three cases have been studied, respectively, in (1), (2), and (3), and in all of them the conclusion is $\mathrm{cr}(C) = 2$, as required. □

5 The skeleton graph

In this section, we present the skeleton graph, which is the essential structure of 3-connected graphs with 2-exceptional edges.

First, we introduce some notation, aligned with the notation of [4]. Let H be a subdivision of a graph G and let e be an edge of G. If s and t are the ends of e, then we denote by sHt the s,t-path of H which results from subdividing e. We use v_{st} to denote an arbitrary, but fixed, vertex of (sHt).

Following this general notation, we turn our attention to the specific graph K'', which we show to constitute the skeleton of graphs in $\mathcal{E}_3$. It is depicted in Figure 5. We always use the labelling from the figure (and we call it *standard labelling*), so $\{\{a, b, c\}, \{x, y, z\}\}$ constitute the bipartition of a subdivision $K \cong K_{3,3}$, and bc, yz are the exceptional edges of K''. We will use K' for $K + bc$, and refer to it as a *pre-skeleton*.

A *bypass* of a non-Kuratowski edge e of K'' is the union of any two $K_{3,3}$ branches that together with e form a cycle containing exactly 3 nodes of K''. A bypass is *open*, if the endvertices of e are removed from it and *closed* if they are contained in it. The common node $t \in \{a, b, c, x, y, z\}$ of the K'' branches used in the bypass is the *peak* of the bypass, and we denote the bypass by K_t. For instance, $K_x = bK''xK''c$ and $K_b = yK''bK''z$.

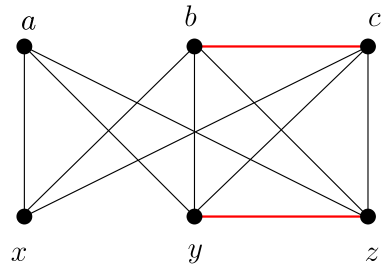

Figure 5: The skeleton graph K''.

We will be vague by using K_t both for open and closed bypass, but where distinction will be required, (K_t) is open and $[K_t]$ is closed.

Besides bypasses, *claws* at a and x will play a significant role. We define them by $D_a := aK'x \cup aK'y \cup aK'z$ and $D_x := xK'a \cup xK'b \cup xK'c$. A *talon* of a claw is its one degree vertex. A claw is *open*, if we remove its talons. Again, we will use $[D_a]$ and $[D_x]$ for closed, and (D_a), (D_x) for open claws. The graph $K''_4 := K'' \setminus ((D_a) \cup (D_x))$ is a subdivision of K_4. When $H \cong K''$ is a subdivision of K'', we extend the definition of bypasses and claws naturally to H.

The next lemmas restrict the possible bridges of a skeleton graph in G. The first one shows that graphs in $\mathcal{E}_3$ do not contain a subdivision of a graph, obtained from K' by adding a path with ends in two distinct bypasses, except for three exceptions.

Lemma 5.1. *Let $H := K' + P$, where P is a path joining two distinct elements of $\{(K_x), (K_y), (K_z)\}$ and internally disjoint from K'. Then every edge of H is a Kuratowski edge, or P joins distinct vertices of $\{x, y, z\}$.*

Proof. By Lemma 2.3, we may assume that H has no vertices of degree 2. In particular, P is an edge. Assume that P does not join distinct vertices of $\{x, y, z\}$. Let q and r be the endvertices of P. As (K_x), (K_y), and (K_z) are open, P does not join two vertices of $\{a, b, c\}$. By Lemma 2.1, we have that all the edges in H except bc are Kuratowski edges. It remains to show that in each case, bc belongs to a subdivision of $K_{3,3}$. By the symmetry of K', we need only analyze the cases in which $q \in \{y, v_{yb}\}$ and $r \in \{v_{bz}, v_{cz}\}$.

If $q = y$ and $r = v_{bz}$, then $H \setminus \{by, cx\}$ is the required subdivision.

If $q = y$ and $r = v_{cz}$, then $H \setminus \{bx, cy\}$ is the required subdivision.

If $q = v_{yb}$ and $r = v_{bz}$, then $H \setminus \{cx, bv_{bz}\}$ is the required subdivision.

If $q = v_{yb}$ and $r = v_{cz}$, then $H \setminus \{x\}$ is the required subdivision. □

The next lemma restricts paths adjacent to paths linking two nodes of K.

Lemma 5.2. *Let $H := K' + P + Q$, where P is a path joining two distinct elements of $\{x, y, z\}$ and internally disjoint from K', and Q is a path joining an inner vertex p of P with a vertex $q \in V(K') \setminus V((D_a))$ and internally disjoint from $K' + P$. Then every edge of H is a Kuratowski edge, or* $\mathrm{cr}(H - bc) \geq 2$*, or $q \in P$.*

Proof. Without loss of generality, we may assume $P = yPz$. By Lemma 2.3, we may assume that H has no vertices of degree 2. Lemma 2.1 implies that all the edges in H except bc and possibly edges of $P \cup Q$ are Kuratowski edges.

Since $q \notin V((D_a))$ and H has no vertices of degree 2, then q is a node of H distinct from a. If $q \in \{y, z\}$, we are done. If $q = x$, then $(K' - bc) \cup P \cup Q$ is a subdivision of $K_{3,4}$, and $\mathrm{cr}(H) \geq 2$. So we assume that $q \in \{b, c\}$. By symmetry, we may assume that $q = b$. In this case, $H \setminus \{cx, by, bz\}$ is a subdivision of $K_{3,3}$ that uses the edge bc and all edges of $P \cup Q$, concluding the proof. □

Lemma 5.1 implies the following useful structure of optimal drawings of $G - e$:

Lemma 5.3. *Let $G \in \mathcal{E}_3$ and let e be its 2-exceptional edge with endvertices b and c, and let $K_{3,3} \cong K \subseteq G$. If D is an optimal drawing of $G - e$ and D_K is the induced subdrawing of K, then the ends of e lie on a face of D_K that is not incident with its crossing.*

Proof. By Lemma 2.2, b, c are independent nodes of K, so they are on the boundary of some (possibly different) face(s) of D_K. Up to homeomorphism, D_K is drawn in Figure 6. The parts in the bipartition of $K_{3,3}$ are $\{1, 3, 5\}$ and $\{2, 4, 6\}$. Any pair of independent nodes of K lies on a common face of D_K, and E_1, E_2 are the only faces contradicting the conclusion of Lemma 5.3.

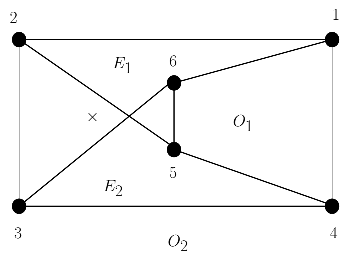

Figure 6: The unique drawing of $K_{3,3}$, up to homeomorphisms.

By symmetry, we may assume b, c lie in E_1, implying $\{b, c\} = \{2, 6\}$. As $\mathrm{cr}(G - e) \leq 1$, the crossing of D_K is the only crossing of D. As $\mathrm{cr}(G) \geq 2$, there is an arc in D connecting the two segments of the boundary of E_1 having b, c as ends. As this path avoids the only crossing of D, it is a path in $G - e$ that connects two distinct open bc-bypasses, and at least one of its endvertices is not a node of K, contradicting Lemma 5.1.

Therefore, b, c lie either in O_1 or O_2, and neither of these is incident with the crossing of D, as claimed. □

In the analysis, we use the following result from [12]. We also repeat some notation.

Lemma 5.4 (Lemma 3 in [12])**.** *Let G be a 3-connected non-planar graph, and let e be a non-Kuratowski edge of G with endvertices b and c. Then the graph G/e is 2-connected but not 3-connected, and the graph $G - b - c$ is connected, but not 2-connected.*

Let $H := G - b - c$ and let $T(H)$ be the block-cutpoint tree of the graph H. Lemma 5.4 implies that $T(H)$ is a non-trivial tree. According to Theorem 1.2, $G - b - c$ has all vertices in two blocks for any graph G with 2-exceptional edge bc.

By now, we are ready to establish that the pre-skeleton graph is a subdivision contained in any graph with a 2-exceptional edge.

Theorem 5.5. *Let $G \in \mathcal{E}_3$ with e its 2-exceptional edge with endvertices b and c. There exist a pre-skeleton subgraph H with $K' \cong H \subseteq G$ and an edge f of G, such that $H + f$ is a subdivision of the skeleton graph K''.*

Proof. By Lemma 2.5, G has a subdivision $K \cong K_{3,3}$. As e is not a Kuratowski edge, e is not in K. Let uPv be any maximal K-avoiding path containing e. As G is 3-connected, Theorem 2.6 implies that u, v are distinct nodes of K; we choose the standard labelling of K such that $\{u, v\} = \{b, c\}$. As G is 3-connected, P is either an edge, or there exists a $K + P$-avoiding path pQq connecting a vertex of $p \in (P)$ with a vertex $q \in G \setminus V(P)$. One of the paths $bPQq$ and $cPQq$ contains e, hence Theorem 2.6 applied to it implies $q = a$. Then, $K \cup P \cup Q$ is a subdivision of $K_{3,4}$ containing e, a contradiction to e not being a Kuratowski edge. Hence Q does not exist and bPc is just a single edge, showing that $H := K + e$ is a pre-skeleton in G.

Next we prove that there exists a K'-avoiding path Q, connecting two nodes from $\{x, y, z\}$. We may be forced to change K' for this.

Claim 5.6. *There exists a pre-skeleton subdivision $\bar{K}'$ in G, such that $K' \cap \bar{K}'$ contains the closed bypasses of e (in particular, b, c, x, y, z are nodes of $\bar{K}'$), and there is an $\bar{K}'$-avoiding path P of G connecting $q, r \in \{x, y, z\}$.*

Proof. As there are two blocks of $H = G - b - c$ containing all its vertices, at least two vertices p, q of $\{x, y, z\}$ are in the same block B of H. We may assume without loss of generality that they are y and z. As B is 2-connected, there are two internally disjoint paths yP_1z and yP_2z in B. By Lemma 5.1, the intersection of $P_1 \cup P_2$ with $K_x \cup K_y \cup K_z$ is contained in $\{x, y, z\}$.

Suppose that $x \notin P_1 \cup P_2$. If either $P_1 \cap (D_a)$ or $P_2 \cap (D_a)$ is empty, it is the required path and $\bar{K}' = K'$. So we may assume they are both non-empty. Let a' be a vertex of $P_1 \cup P_2$, such that xD_aa' has no vertex of $P_1 \cup P_2$. We may assume $a' \in P_1$. As $(P_1 \cup a'D_ax) \cap K'$ is contained in $[D_a]$, $\bar{K}' := (K' - (D_a)) \cup (P_1 \cup a'D_aaD_ax)$ is the required skeleton and P_2 is the required $\bar{K}'$-avoiding y, z-path.

Now we may assume $x \in P_1 \cup P_2$. Then x, y, z split $C := P_1 \cup P_2$ into three arcs $C_{xy} := xCy$, $C_{yz} := yCz$, and $C_{zx} := zCx$, such that $C = xCyyCzzCx$. Let a_x, a_y, a_z be the a-closest vertices of $P_1 \cup P_2$ in xD_aa, yD_aa, zD_aa, respectively; they may all be equal to a.

If each segment of C_{xy}, C_{yz}, C_{zx} contains a vertex of a_x, a_y, a_z, then let a'' be the one of a_x, a_y, a_z in C_{yz} and let a' be any other one. Then, $C \cup (([a_xD_aa] \cup [a_yD_aa] \cup [a_zD_aa]) - (aD_aa'') - (C_{yz}))$ contains an x, y, z-claw T with center a' and is internally disjoint from C_{yz}. Hence, $\bar{K}' = (K' - (D_a)) \cup T$ is the required pre-skeleton and C_{yz} the y, z-path internally disjoint from $\bar{K}'$.

If a segment C_{xy}, C_{yz}, C_{zx} contains two vertices of a_x, a_y, a_z, and a segment C_0 contains none, we relabel $\{x, y, z\}$, so that $C_{yz} = C_0$. Then $D_a \cup P_1 \cup P_2 - (C_{yz})$ contains a claw T with center a and talons x, y, z so that C_{yz} is internally disjoint from it; again, $\bar{K}' = (K' - (D_a)) \cup T$ is the required pre-skeleton and C_{yz} the y, z-path internally disjoint from $\bar{K}'$.

If a segment C_3 of C_{xy}, C_{yz}, C_{zx} contains all three vertices of a_x, a_y, a_z, then in $C \cup D_a$, there is a C-avoiding path pRq from (C_3) to $C - [C_3]$. We relabel $\{x, y, z\}$, so that $p \in C_{xy}$ and $q \in C_{xz}$. Then, $(C \cup pRq) - (C_{yz})$ contains an x, y, z-claw T, with center in p, so that C_{yz} is internally disjoint from it; again, $\bar{K}' = (K' - (D_a)) \cup T$ is the required pre-skeleton and C_{yz} the y, z-path internally disjoint from $\bar{K}'$. □

Without loss of generality, we label the nodes of K so that $P = yPz$.

Claim 5.7. *The path yPz from Claim 5.6 is an edge.*

Proof. Seeking a contradiction, assume that P has an internal vertex v. Consider an optimal drawing D of $G - e$. Since $K \subset G - e$ and $\operatorname{cr}(G - e) \leq 1$, we have $\operatorname{cr}(G - e) = 1$. Thus the drawing D restricted to K is homeomorphic to the drawing D_K in Figure 6. Because G is 3-connected, $G - y - z$ contains a path Q from v to $K - y - z$, which is internally disjoint from $K \cup P$. If q is the endvertex of Q in K, Lemma 5.2 implies $q \in V(D_a)$.

Since the crossing d of D_K is the only crossing of D, no edge of $P \cup Q$ is crossed in D. Hence P is drawn in a face of D_K incident with two independent nodes. By the symmetry of D_K, we may assume that $v \in E_1$ or $v \in O_1$. See Figure 6.

If $v \in O_1$, then $\{y, z\} = \{4, 6\}$ or $\{y, z\} = \{1, 5\}$. By the symmetry of D_K, we may assume $\{y, z\} = \{4, 6\}$, and hence $x = 2$. This implies that $a = 1$ or $a = 5$. If $a = 5$, then $\{b, c\} = \{1, 3\}$. Since $\operatorname{cr}(G - e) < \operatorname{cr}(G)$, then there must be a simple arc α of D contained in O_2, with endpoints on its boundary and separating b from c (1 from 3 in Figure 6). Since d is the only crossing of D, α corresponds to a path R of G which joins two vertices of $V(K') \setminus V(D_a)$. Lemma 5.1 implies R joins x and z. Now it is easy to see that $K' \cup P \cup R$ contains a subdivision of K_5, contradicting Lemma 2.4.

For the final case, $v \in E_1$. Then, without loss of generality, P connects $y = 2$ and $z = 6$, implying $x = 4$. As q is on some path in the boundary of E_1, a can be any of the vertices 1, 3, or 5. Suppose $a = 1$. This implies $bc = 35$, contradicting Lemma 5.3. Suppose next $a = 3$, implying $bc = 15$. As $\operatorname{cr}(G) \geq 2$, there is an arc in D separating 1 from 5 in O_1. By Lemma 5.1 and as there is only one crossing in D, this arc is a path R from 4 to 6. As $K' \cup P \cup R$ has a subdivision of K_5, it contradicts Lemma 2.4. The subcase $a = 5$ is similar, with $bc = 13$ and $2R4$. □

Thus f is an edge connecting y and z, and $K' \cup f$ is a subdivision of K'', as claimed. □

Proposition 5.8. *Let $G \in \mathcal{E}_3$ and let $K'' \cong H \subseteq G$ be its skeleton subgraph with standard labelling. Then G does not contain a path P internally disjoint from K'' with endvertices in any of the pairs $\{a, b\}, \{a, c\}, \{x, y\}, \{x, z\}$.*

Proof. Let u, v be the endvertices of P. If $\{u, v\} \in \{\{a, b\}, \{a, c\}, \{x, y\}, \{x, z\}\}$ then the subgraph $(K'' + P) \subset G$ contains K_5 as subdivision. This and Lemma 2.4 imply that G has no exceptional edges, a contradiction. □

Corollary 5.9. *Let $G \in \mathcal{E}_3$. Any non-Kuratowski edge g of G is parallel to e or f.*

Proof. Let u, v be the endvertices of g. By Lemma 2.2, we know that u, v are independent vertices of K and by Proposition 5.8, we have that $\{u, v\} = \{b, c\}$ or $\{u, v\} = \{y, z\}$. □

Let e and f be the exceptional edges of $G \in \mathcal{E}_3$. The graph obtained from G by adding a parallel edge f' to f is also an exception to Statement 1.1, but such a graph contains only e as an exceptional edge, because both edges f and f' are non-critical. These observations yield the following corollary to Theorem 4.2:

Corollary 5.10. *Let $G \in \mathcal{E}_3$. The number of 2-exceptional edges of G is at most two.*

Proof. By Corollary 5.9, it is enough to show that if f and f' are parallel edges of G, then they are not critical.

Suppose an arbitrary of them is and let it be f. Then $\mathrm{cr}(G - f) \leq 1$. As $K \subset G - f$, there exists an optimal drawing D of $G - f$ in which $f' \notin K$ has no crossings. By drawing f very close to f' in D, we get a drawing of G with exactly one crossing, a contradiction implying that f is not critical and hence not exceptional. □

6 Bridges of the skeleton graph

Let H be a subgraph of a graph G. An H*-bridge* is either an edge not in H together with its two incident vertices that are in H or is obtained from a component J of $G - V(H)$ by adding all edges incident with a vertex of J together with their incident vertices in H. This concept will be helpful for the remainder of this section. A bridge is trivial, if it is just an edge, and non-trivial otherwise. For a graph H and its bridge B, any vertex of $att(B) := V(H) \cap V(B)$ is an *attachment* of B.

First we exhibit the structure of an optimal drawing of $G - e$.

Lemma 6.1. *Let $G \in \mathcal{E}_3$, let K'' be its skeleton graph, let e be its 2-exceptional edge with endvertices b and c, and let f be a non-Kuratowski edge not parallel to e. If D is an optimal drawing of $G - e$, then the drawing D restricted to $K + f$ is homeomorphic to the drawing in Figure 7 (right) and b, c are the ends of e.*

Proof. Let D be an optimal drawing of $G - e$ and K the $K_{3,3}$ subdivision in K''. As e is a 2-exceptional edge, D has a unique crossing and D restricted to K is homeomorphic to the drawing D_K in Figure 7 (left). Using symmetry, stereographic projection, and Lemma 5.3, we may assume that the ends of e are b and c. Hence, $f \in \{xy, yz, xz\}$.

If $f \in \{xy, xz\}$, $\mathrm{cr}(G) \geq 2$ implies there is a path P of $G - e$ that is by Lemma 5.1 drawn from y to z in E, yielding a K_5 subdivision in $K'' \cup P$, contradicting Lemma 2.4. Thus y and z are the ends of f and the drawing D restricted to $K + f$ is homeomorphic to the drawing in Figure 7 (right), as required. □

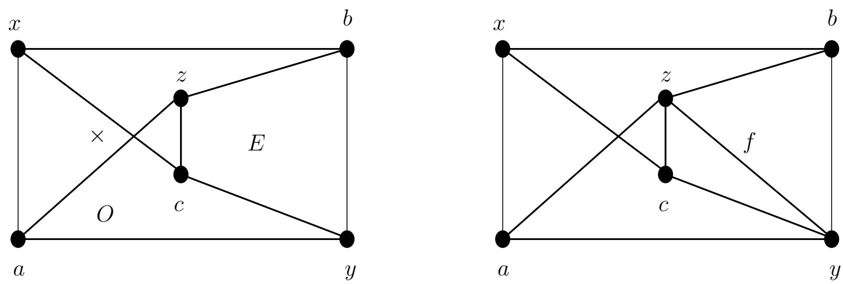

Figure 7: If D is an optimal drawing of $G - e$, then the drawing D restricted to $K + f$ is homeomorphic to the right drawing.

In what follows, we call (G, H, e, D) a *standard quadruple*, abbreviated *sq*, if $G \in \mathcal{E}_3$, $K'' \cong H \subseteq G$, such that H has standard labelling, e is a 2-exceptional edge of G, and D is an optimal drawing of $G - e$, with the induced subdrawing of $H - e$ drawn as in Figure 7.

Lemma 6.2. *Let $G \in \mathcal{E}_3$. Then, there exists a standard quadruple (G, H, e, D) containing G.*

Proof. As $G \in \mathcal{E}_3$, there exists a 2-exceptional edge e of G. Theorem 5.5 guarantees existence of the skeleton graph H in G, that is a subdivision of K''. Finally, Lemma 6.1 yields existence of the desired drawing D of $G - e$. □

Lemma 6.3. *Let $G \in \mathcal{E}_3$, let $K'' \cong H \subseteq G$ with the standard labelling. If B is a bridge of H, $v \in att(B)$, and v is not a node of H, then $att(B) \subseteq [D_a] \cup [D_x]$.*

Proof. Let v be as in the statement. Seeking a contradiction, suppose that there exists $u \in att(B) \setminus ([D_a] \cup [D_x])$. Then $u \in (K_y) \cup (K_z)$ (or equivalently, $u \in (K_b) \cup (K_c)$). Since B is connected, it contains an u, v-path, say P. From Lemma 5.1 and the hypothesis that v is not a node of H, we have that $v \in (aHx)$. By the symmetry of K_4'', we need only analyze the case in which $u = v_{by}$. But in such a case, $(H \cup P) \setminus \{ay, bx, cz\}$ is a subdivision of $K_{3,3}$ containing both bc and yz as edges, which contradicts that bc is a 2-exceptional edge. □

Let $G \in \mathcal{E}_3$. In what follows, we will denote with H_4 as the subgraph of G induced by the four vertices that are ends of the non-Kuratowski edges of G. By Corollary 5.9, H_4 is well-defined for any G, i.e. it is independent of the choice of H.

Lemma 6.4. *Let (G, H, e, D) be a standard quadruple of a graph $G \in \mathcal{E}_3$. The subgraph H_4 of G is isomorphic to K_4 with some multiple edges, and it has only one bridge that contains both a and x and the only crossing of D.*

Proof. As G is 3-connected and H_4 is induced in G, H_4 has no trivial bridges.

As there exists an H_4-avoiding path aHx, a and x are in the same H_4-bridge B, and that bridge is crossed in D. If B is the only bridge, we are done, otherwise let B' be any other bridge.

As G is 3-connected, each of B, B' has at least three attachments. As $e \notin B'$ and the only crossing of D is in B, B' is drawn planarly in D. Then D implies the attachments of B' are either b, y, z or c, y, z, both contradicting Lemma 5.2.

Let $\{u, v\} \in \{\{c, z\}, \{b, z\}, \{b, y\}, \{c, y\}\}$. Now consider the branch of H connecting u to v. If such branch is not an edge, then it has one internal vertex, say w. Using the 3-connectivity, the drawing D, and the fact that B is the only bridge in H_4, we know that there is a path, that is internally disjoint from H, connecting w to a vertex in $(D_a) \cup (D_x)$. However, no such path exists by Lemma 6.3. □

For a graph $G \in \mathcal{E}_3$, we will denote the only bridge of its graph H_4 by B_4.

Lemma 6.5. *Let $K'' \cong H \subseteq G$ and let P be a path from $u \in (K_a)$ to $v \in [K_x]$ with $\{u, v\} \neq \{a, x\}$, and internally disjoint from H. Then every edge of $H + P$ is a Kuratowski edge. The claim also holds with the role of a and x exchanged.*

Proof. By Lemma 2.3, we may assume that H has no vertices of degree 2. It is enough to show that in each case H has a subdivision of $K_{3,3}$ or K_5 containing either bc or yz. By the symmetry of H, we need only analyze the following cases; the same arguments also show the claim with the role of a and x interchanged:

If $u = v_{az}$ and $v = v_{cx}$, $(H \cup P) \setminus \{by, cz, xHv\}$ is a subdivision of $K_{3,3}$.
If $u = v_{az}$ and $v = c$, $(H \cup P) \setminus \{xc, cz, by\}$ is a subdivision of $K_{3,3}$.
If $u = a$ and $v = v_{cx}$, $(H \cup P) \setminus \{xv\}$ is a subdivision of K_5.
If $u = a$ and $v = c$, $(H \cup P) - xc$ is a subdivision of K_5. □

Lemma 6.6. *Let (G, H, e, D) be a standard quadruple of a graph $G \in \mathcal{E}_3$. If P is a b, y-path of length at least 2 contained in B_4, then P intersects $[aHx]$.*

Proof. Seeking a contradiction suppose that $P \cap [aHx] = \emptyset$.

By Lemma 6.5 and our hypothesis, at least one of $P \cap (K_a)$ or $P \cap (K_x)$ is empty. By symmetry, we may assume that $P \cap (K_a) = \emptyset$.

If $P \cap (K_x)$ is also empty, then P is internally disjoint from H. Then, $(H - (by)) \cup P$ is a different choice of H for G whose structure contradicts Lemma 6.4, as the H_4 produced by this H has a subdivided edge by. Hence, $P \cap (K_x)$ is nonempty.

Lemma 6.1 and disjointness of P from $(D_a) + x$ imply that there is a H-bridge B with attachments in y and $(xHb]$. By the previous paragraph, at least one attachment is in (xHb). However, any path in B from y to (xHb) contradicts Lemma 6.5, concluding the proof. □

Lemma 6.7. *Let (G, H, e, D) be a standard quadruple of a graph $G \in \mathcal{E}_3$. Then, there exist vertices v_c and v_z in B_4, such that $v_c c$ and $v_z z$ are the only attaching edges of B_4 at c and z. Moreover, these edges are crossed in D.*

Proof. We show the claim for $v_c c$, the claim for $v_z z$ is analogous. Let $\times$ be the crossing of D. By Lemma 6.5, there is no H-avoiding path in B_4 from $(\times Hc)$ to $[D_a]$ avoiding $\times$. Let F be any face of D incident with the segment $(\times Hc)$.

As $V(\partial F) \subset V([D_a]) \cup V((\times Hc))$, existence of a vertex in $(\times Hc)$ would contradict 3-connectivity of G. Hence, $(\times Hc)$ lies on some edge cv_c of B_4, and cv_c is crossed in D. Analogously, we can conclude that $(\times Hz)$ lies on some edge $v_z z$ of B_4, and hence cv_c and $v_z z$ are the only two crossing edges of D.

By Lemma 6.5 we know that G does not have a path internally disjoint from H, with an end in c and the other end belonging to (K_a). Thus, the existence of any other edge of B_4 attaching at c, together with the location of c in D imply that at least one of $v_z z$ or zy is crossed by some edge in $B_4 - cv_c$, contradicting that $\mathrm{cr}(D) = 1$. □

Lemma 6.8. *Let (G, H, e, D) be a standard quadruple of a graph $G \in \mathcal{E}_3$. There exists $K'' \cong H' \subseteq G$ and an optimal drawing D' of $G - e$, such that $a'z$ and cx' are edges of H', and any face incident with the crossing of D' has no bridges of H' drawn in it.*

Proof. By Lemma 6.7, there exist vertices v_c and v_z, such that cv_c and zv_z are edges of G. Moreover, cv_c crosses zv_z in D and such a crossing $\times$ is the only crossing of D.

Let D'' be the subdrawing of D, induced by $G - c - z$. Since $\times$ is the only crossing of D, then b, v_c, v_z and y lie in the same face F of D''. Note that F contains (in the interior) vertices c and z. By Lemma 6.1, the boundary walk ∂F can be decomposed into $bP_1 v_c P_2 v_z P_3 yb$.

Note that if some $P \in \{P_1, P_2, P_3\}$ is not a path, then P must have a cut vertex, say w. By Lemma 6.7 we know that v_c, y and v_z, b are the only vertices of B_4 adjacent to c and z, respectively. From this and the supposition that w is a cut vertex of P it follows that w is a cut vertex of $G - e$, which contradicts the connectivity of $G - e$. Thus we can assume that P_1, P_2 and P_3 are paths. Define $H' = H - (D_a) - (D_x) \cup [bP_1 v_c P_2 v_z P_3 y] \cup cv_c \cup zv_z$. We relabel $a' := v_z$ and $x' := v_c$. Observing how H' is drawn in D, the claim follows with $D = D'$. □

A standard quadruple (G, H', e, D') from Lemma 6.8 is called a *tidy* standard quadruple, abbreviated *tsq*.

Lemma 6.9. *Let (G, H, e, D) be a tsq of a graph $G \in \mathcal{E}_3$. Then, $B = B_4 - c - z$ has two cut vertices u, v in $[aHx]$, and uHv is an edge of G, and any u, v-path in G avoiding H is an edge.*

Proof. Let F be the face of the subdrawing of D of the cycle $C = bHxHaHyb$ not containing the crossing of D. As C is clean in D, $\partial F = C$. Tidiness implies that $D[B]$ is contained in F and that $D[B + by]$ is planar. Now, let F' be the face of the subdrawing $D[B]$ containing the edge by. Note that $D[B]$ is a drawing of $B_4 - c - z$ and hence contains no edges of H_4. We decompose the boundary of F' into two paths, $bHxHaHy$ and P as follows:

As H is tidy, $bHxHaHy$ is on the boundary of F', and let P be the remaining part of the boundary, i.e. $\partial F' = yPbHxHaHy$. As P is a b, y-path in B, P intersects $[aHx]$ in a vertex v by Lemma 6.6. As v appears twice in the boundary of F', it is a cut-vertex of B.

Let $P = yP_1vP_2b$ and assume that $P_1 - v$, $P_2 - v$ do not intersect $[aHx]$. Then, $H_4 + P_1 + P_2 + vHxc + vHaz$ is a subdivision of K_5, a contradiction to Lemma 2.4. Thus there is a vertex $u \in P \cap [aHx]$, $u \neq v$, and u is another cut-vertex of B.

Now consider any $\{u, v\}$-bridge B' in B with attachments in both u and v that has a vertex w distinct from u, v. As G is 3-connected, either B' contains (i) b or (ii) y, or (iii) an attachment of B on $H + cx + az$. The latter option (iii) is dismissed by tidiness: az, cx are the only edges from c, z to B in a tsq, and the vertices b, y can be interpreted as (i) or (ii). The former two options (i) and (ii) both contradict the claim that both u, v are cut vertices of B.

As any H-avoiding u, v-path is either an edge uv or contained in a B-bridge with attachments u, v, the lemma follows. □

Lemma 6.10. *Let (G, H, e, D) be a tsq. If there are no two internally disjoint b, x- (respectively a, y-) paths in B_4, then there is a cut vertex $v \in bHx$ (respectively, $v' \in aHy$), such that bv (yv') is an edge and any H-bridge $B' \subseteq B_4$ attaching at b (y) is an edge bv (yv').*

Proof. We prove the claim for bHx, the proof for aHy is analogous.

Suppose there are no two disjoint b, x-paths, implying there is a cut-vertex $v \in bHx$. Let u be any vertex of bHv. As G is 3-connected, there is a path from u to $G - (bHv)$ in $G - b - v$. This path is in B, and by Lemma 5.1, it does not attach to H_4. Therefore it attaches in $(D_a) \cup (D_x) - bHv$, a contradiction to v being a cut. Therefore if there is a cut vertex v, then bv is an edge.

The same argument implies that any H-bridge $B' \subseteq B$ attaching at b has attachments only at b and v, and is therefore a trivial bridge. □

Lemma 6.11. *Let (G, H, e, D) be a tsq. The edges by, bz, cy have multiplicity at least two.*

Proof. Suppose on the contrary that at least one of the mentioned edges has multiplicity one. First we handle bz and cy with a slightly modified drawing with the existing crossing of xc, az replaced by a crossing of an edge with assumed multiplicity one, and for by we also twist the modified drawing:

Augment the sub-drawing $D[B_4]$ by contracting the edges xc, az slightly with c, z following their edge's drawings past the crossing, so that xc, az no longer cross, and call this new drawing D' (cf. Figure 8 left). As B_4 contains all the vertices of G and all edges not in B_4 are connecting nodes of H_4, it is a routine exercise to extend the drawing D' to a drawing of G with just one crossing, in which either bz is crossing cx or cy is crossing az, contradicting criticality of bc whenever either bz or cy have multiplicity one.

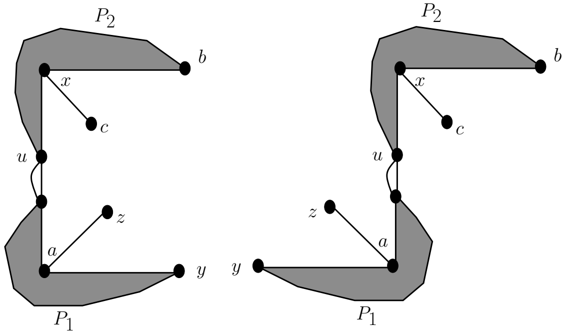

Figure 8: A twist in a drawing demonstrating multiplicity of certain edges.

For by, we need to twist D': As u is a cut-vertex of the bridge B by Lemma 6.9, the outer face of D' has the following boundary: $bHxcxHuvHazaHyP_1vuP_2b$. Let D'' be obtained by twisting D' at u, so that D'' is a drawing of B with the outer face $bHxcxHuP_1yHazaHuP_2b$ (cf. Figure 8, right). Then D'' can be augmented to a drawing of G in which the only crossing is between by and az, a contradiction. □

Lemma 6.12. *Let (G, H, e, D) be a tsq. If there do not exist two edge disjoint p, q-paths in B_4 for any of $\{p, q\} \in \{\{a, x\}, \{a, y\}, \{b, x\}\}$, then cz has multiplicity at least two.*

Proof. In the proof, we use the drawings D' and D'' defined in the proof of Lemma 6.11.

For $\{p, q\} = \{b, x\}$, suppose there are no two edge disjoint b, x-paths. Therefore, there are no internally disjoint b, x-paths. By Lemma 6.10, there is a cut-vertex v in bHx and bv is an edge, and any other H-bridge attaching at b is a trivial edge bv. If the edge bv has multiplicity two, then there are two edge-disjoint b, x paths in B_4: vP_2uHx and vHx (they are edge disjoint, as them sharing an edge would imply G is not 3-connected). Hence the edge b, v has multiplicity one.

If cz is a single edge, then the drawing D'' can be modified to a drawing of G in which the only crossing is of cz with bv, a contradiction establishing cz is a double edge. Symmetric arguments apply to the case $\{p, q\} = \{y, a\}$.

In the final case of $\{p, q\} = \{a, x\}$, Lemma 6.9 implies there is an edge uv in G, such that u, v are cut-vertices, uv is a single edge and D'' is obtained from $D[B]$ by twisting at the vertex u. If cz is a single edge, D'' can be augmented to a drawing of G by cz crossing uv as its only crossing, the final contradiction establishing the claim. □

Now we have all the ingredients to establish necessity in Theorem 4.2.

Proof of necessity in Theorem 4.2, (i). Let $G \in \mathcal{E}_3$, and let (G, H, e, D) be its tidy standard quadruple, whose existence is guaranteed by Lemma 6.8. By Lemma 6.9, there exist two vertices p, q in $[aHx]$, such that G contains an edge h with endvertices p and q and any H-avoiding p, q-path is an edge. Let O be the union of all these edges; then $T_1 := (O, (p), (q)) \in \mathcal{O}$. Without loss of generality, we may assume that $aHx = aHqhpHx$.

By definition of the tidy standard quadruple, az and cx are edges of G. Furthermore, Lemma 6.10 asserts that either there is an edge g with endvertices b and v such that any

H-bridge $B' \subseteq B_4$ attaching at b is an edge parallel to g, or there are two internally disjoint b, x-paths (in this case, we let $b = v$) in the bridge B_4. Symmetrically by the same lemma, there is an edge g' with endvertices u, y with H-bridges within B_4 restricted to edges parallel to g', or there are two internally-disjoint a, y-paths in B_4 (in this case, we let $y = u$).

Let H_4 be the subgraph of G, induced by the vertices $\{b, c, y, z\}$. We let R to be obtained from H_4 by adding the two edges az and cx, and, if $b \neq v$, all the edges which are parallel to g, and, if $y \neq u$, all the parallel edges to g'. Note that when T_2 (respectively, T_4) is a Q-tile, we have $v = x$ (respectively $u = a$) in G due to suppression of vertices with two neighbors when joining tiles, but in R, we always have $u \neq a$ and $v \neq x$. Lemma 6.11 implies that the edges by, bz, and cy have multiplicity at least 2. As bc is a single edge, we have that $T_3 = (R, (a, u), (x, v))$ is a tile in $\mathcal{R}$.

Now consider the vertices p, v, and x. As cx is an edge, v is a vertex-cut in B_4, which disconnects b from x, and x, v are two attachments of an R-bridge B'. As p is a vertex-cut in B_4 disconnecting a from x, then $\{p, x, v\}$ form a cut in G, or they are all equal. If they are a cut, then they are all three distinct as G is 3-connected.

We first analyze the case when they are all distinct. Let P be a bridge of $\{p, x, v\}$ disjoint from R, and let P' be the graph obtained from P by adding a vertex t adjacent to precisely its three attachments. As G is 3-connected, Lemma 4.1 implies that P' is 3-connected, so the tile $T_2 = (P, (p), (x, v))$ is a tile in $\mathcal{P}$.

Suppose now that $p = x = v$. Then $v \neq b$. Let i be the multiplicity of the edge pq, and let j be the multiplicity of the edge vb. We set $T_2 = ({}^iQ^j, (q), (p, b))$, which is a tile in $\mathcal{P}$.

Symmetric arguments applied to y, a, q, and u imply that there is a tile T_4, such that $T_4^{\leftrightarrow} \in \mathcal{P}$. As vertices with just two neighbors are suppressed when joining tiles, we have that G is a join $\circ\mathcal{T}$ of a pre-exceptional sequence $\mathcal{T} = (T_1, T_2, T_3, T_4)$.

To see that $\mathcal{T}$ is exceptional, assume that $\min \rho(\mathcal{T}) = 1$. Thus, after joining the tiles, one of the edges pq, bv, or yu is a single edge. This implies that in G, there are no two internally disjoint w, s-paths for one of $\{w, s\} \in \{\{a, x\}, \{a, y\}, \{b, x\}\}$, and Lemma 6.12 implies cz has multiplicity at least two. In terms of $\sigma(\mathcal{T})$, $l_3 \geq 2$ as desired. □

7 Conclusions

We conclude with some comments regarding the existence of k-exceptional edges. Theorem 4.2 immediately gives the following corollary, claimed by Širáň [12] and Kochol [8]:

Corollary 7.1. *Let G be a simple graph and e its crossing-critical edge with* $\mathrm{cr}(G - e) \leq 1$. *Then, e is a Kuratowski edge of G.*

It is also easy to obtain the following:

Corollary 7.2. *For any integer $k \geq 2$, there exist infinitely many 3-connected graphs with k-exceptional edges.*

Proof. For $k = 2$, the claim follows from Theorem 4.2. For higher k, we only sketch the proof by induction; an attentative reader will be able to provide the technical details. Let $\mathcal{F}_k$ be the family of 3-connected graphs with k-exceptional edges containing a tidy skeleton subdivision H, such that $G - az - bc - cz$ is planar. By induction and Theorem 4.2, we assume that $\mathcal{F}_{k-1}$ is infinite for $k \geq 3$. Let $G \in \mathcal{F}_{k-1}$ be arbitrary. Assuming the standard labelling of H, we produce a graph $G' \in \mathcal{F}_k$ as follows:

i) For G_1, we make any edge of $G - az - bc - cx$ have multiplicity at least k. Note that G_1 is still planar.

ii) For G_2, we add to G_1 single edges az, cx; this graph has crossing number 1 and any optimal drawing of G_2 has az crossing cx.

iii) For G_3, we add the edge bc with multiplicity $k - 1$. In any optimal drawing, bc edges cross the edge az, implying the crossing number of the graph G to be at most k. Should any other edge be crossed, that would add at least k crossings, implying crossing number $\geq k$. As the edges of G need to cross at least twice (this is the technical detail we omit, but it is true due to the construction of graphs in $\mathcal{F}_2$), $\mathrm{cr}(G_3) = k$ and $\mathrm{cr}(G_3 - e) < k$ for any edge parallel to bc, hence $G' = G_3$ has k-exceptional edges and the claim follows. □

Note that the graphs of Corollary 7.2 cannot be made simple by subdividing edges and connecting the new vertices in a cycle (the operation is called π-subdivision in [1]), as was done in [8]: then the new vertices introduced in π-paths would violate Lemma 2.5 and a K_5 would be introduced in the new graph. Hence, the following remain open:

Problem 7.3 ([8])**.** What is the smallest k, for which simple 3-connected graphs with k-exceptional edges exist? Clearly, $3 \leq k \leq 4$.

Hence, the simple graphs obtained by Kochol do not follow our tile structure, but as all the $K_{3,3}$'s of G need to share the endvertices of the exceptional edges, there may still exist an explicit description of graphs with k-exceptional edges. We therefore conclude with the following:

Problem 7.4. Is there a descriptive characterization (i.e. a tile description) of 3-connected graphs with k-exceptional edges?

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Isomorphisms of generalized Cayley graphs*

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Abstract

In this paper, we investigate the isomorphism problems of the generalized Cayley graphs, which are generalizations of the traditional Cayley graphs. We find that there are two types of natural isomorphisms for the generalized Cayley graphs. We also study the GCI-groups among the generalized Cayley graphs, and the Cayley regressions of some groups. We mainly showed that, for an odd prime power n, Z_{2n} (resp. D_{2n}) is a restricted GCI-group if D_{2n} (resp. Z_{2n}) is a CI-group. We also obtain that the cyclic group of order 2^n is a 4-quasi-Cayley regression if and only if $n = 3$.

Keywords: Generalized Cayley graph, natural isomorphism, GCI-group, Cayley regression.

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1 Introduction

Let G be a finite group, $S \subseteq G$ be a subset and $\alpha \in \mathrm{Aut}(G)$. If G, S, α satisfy the following three conditions:

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(i) $\alpha^2 = 1$;

(ii) if $g \in G$, then $\alpha(g^{-1})g \notin S$;

(iii) if $g, h \in G$ and $\alpha(g^{-1})h \in S$, then $\alpha(h^{-1})g \in S$,

then the structure $\Gamma = \mathrm{GC}(G, S, \alpha)$ is called a *generalized Cayley graph* with $V(\Gamma) = G$, $E(\Gamma) = \{\{g, h\} \mid \alpha(g^{-1})h \in S\}$. The *neighborhood* of a vertex $g \in G$ is the set of vertices adjacent to g, denoted by $N(g)$. Then $N(g) = \{\alpha(g)s \mid s \in S\}$.

According to condition (i), α is either the identity of $\mathrm{Aut}(G)$ or an involution. When α is the identity, then the definition of $\mathrm{GC}(G, S, \alpha)$ is just the same as that of Cayley graphs, and thus $\mathrm{GC}(G, S, \alpha) = \mathrm{Cay}(G, S)$. In this case, S is *symmetrical*, i.e., $S = S^{-1} = \{s^{-1} \mid s \in S\}$ and for $\sigma \in \mathrm{Aut}(G)$, we have that σ acts on $V(\Gamma)$ naturally as $V(\Gamma) = G$. Also, if $T = S^\sigma$, then there is a bijection from Γ to $\Gamma^\sigma = \mathrm{Cay}(G, T)$ induced by σ, defined as $\sigma\colon V(\Gamma) \to V(\Gamma^\sigma)$, $g \mapsto g^\sigma$. It follows $\Gamma \cong \Gamma^\sigma$. This kind of isomorphism between Cayley graphs induced by the automorphisms of G is called the *Cayley isomorphism*. It should be mentioned that not all isomorphisms between Cayley graphs are Cayley isomorphisms. In fact, there are pairs of isomorphic Cayley graphs with no Cayley isomorphism between them. This encourages us to investigate the so-called CI-graphs and CI-groups defined below.

Definition 1.1. A Cayley graph $\mathrm{Cay}(G, S)$ is called a CI-graph of G, if for any Cayley graph $\mathrm{Cay}(G, T)$, $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$ implies $S^\sigma = T$ for some $\sigma \in \mathrm{Aut}(G)$. In this case, S is called a CI-subset. Furthermore, G is called a CI-group if any symmetrical subset not containing the identity is a CI-subset.

For those graphs having particular transitive properties, such as Cayley graphs and bi-Cayley graphs, their isomorphism problems are well studied in the literature (recall that a bi-Cayley graph is a graph which admits a semiregular group of automorphisms with two orbits on the vertices). The isomorphism problem for Cayley graphs was proposed decades ago and has been investigated deeply up to now. It was initiated by Ádám in 1967 who conjectured that any cyclic group is a DCI-group, where a DCI-group satisfies that any subset not containing the identity and not necessarily symmetrical is a CI-subset. Although this conjecture was soon denied by Elspas and Turner [4], it stimulated the study of CI- and DCI-groups. Alspach, Parsons [1] and Babai [3] presented a criteria for CI-graphs. Muzychuk [18, 19] obtained a complete classification of the CI-groups in finite cyclic groups. Li [14] showed that all finite CI-groups are solvable. The isomorphism problem and the automorphism groups for bi-Cayley graphs have also been studied flourishingly; one may refer to [10, 11, 28]. Other related results could be found in [15, 16, 23, 24, 26, 27].

The concept of generalized Cayley graphs was introduced by Marušič et al. [17] when they dealt with the double covering of graphs. Answering a question in [17], the authors in [8] found some vertex-transitive generalized Cayley graphs which are not Cayley graphs. Further, the authors in [25] studied the isomorphism problems of generalized Cayley graphs and found that the alternating group A_n is a restricted GCI-group if and only if $n = 4$.

The present paper can be regarded as the continuance of the above work, and also provides support to the question at the end of [8], where the authors asked for the classification of all generalized Cayley graphs arising from cyclic groups. The structure of this paper is as follows. In Section 2, we give several properties of the generalized Cayely graphs and some lemmas which will be used later. In Section 3, we introduce two types of natural isomorphisms for any generalized Cayley graph. In Section 5, we study the GCI-groups in cyclic groups. We show that when G is a dihedral group of order $2n$ with n an odd prime power, if G is a CI-group, then Z_{2n} is a restricted GCI-group. In Section 6, we study the GCI-groups in dihedral groups. We show that when G is a cyclic group of order $2n$ with n an odd prime power, if G is a CI-group, then D_{2n} is a restricted GCI-group. In Section 7, we study the Cayley regressions, a concept relating to both Cayley graphs and generalized Cayley graphs. We show that the cyclic group Z_{2^n} is a 4-quasi-Cayley regression if and only if $n = 3$. Finally, we propose some questions for future research.

2 Preliminaries

All graphs considered in the paper are simple, finite and undirected. All the automorphisms in the paper that induce generalized Cayley graphs are assumed to be some involutions.

Let G be a finite group that admits an automorphism α of order two. For $g = 1$, we have $\alpha(h^{-1}) \in S$ whenever $h \in S$, by condition (iii), implying $\alpha(S) = S^{-1}$. Let $\omega_\alpha\colon G \to G$ be the mapping defined by $\omega_\alpha(g) = \alpha(g^{-1})g$ for any $g \in G$. Note that ω_α is not necessarily a bijection. Let $\omega_\alpha(G) = \{\omega_\alpha(g) \mid g \in G\}$. We use the same notation and terminology as in [8]. Suppose $s \in S$, then $\alpha(s) \in \alpha(S)$, and thus $\alpha(s) \in S^{-1}$. Therefore $s \in S$ if and only if $\alpha(s^{-1}) \in S$. Let Ω_α be the set containing all elements satisfying $\alpha(g) = g^{-1}$ in $G \setminus \omega_\alpha(G)$, and $\mho_\alpha$ be the set containing all elements in G satisfying $\alpha(g) \neq g^{-1}$. Let $K_\alpha = \{g \in G \mid \alpha(g)g = 1\}$. Then we have

Proposition 2.1 ([25]). *Let* $\mathrm{GC}(G, S, \alpha)$ *be a generalized Cayley graph of* G*. Then*

(1) $S \cap \omega_\alpha(G) = \emptyset$*. Conversely, if* $S \cap \omega_\alpha(G) = \emptyset$*,* α *is an involution in* $\mathrm{Aut}(G)$ *and* $\alpha(S) = S^{-1}$*, then* G, S, α *can induce a generalized Cayley graph.*

(2) $G = K_\alpha \cup \mho_\alpha$ *and* $K_\alpha = \omega_\alpha(G) \cup \Omega_\alpha$*. Furthermore,* $\omega_\alpha(G), \Omega_\alpha, \mho_\alpha$ *are all symmetrical.*

(3) $S = S_1 \cup S_2$*, where* $S_1 \subseteq \Omega_\alpha$ *and* $S_2 \subseteq \mho_\alpha$*.*

Proposition 2.2. *Let* G *be a finite group admitting two automorphisms* α, β *of order two. If* α, β *are conjugate in* $\mathrm{Aut}(G)$*, then* $\mathrm{Cay}(G, \omega_\alpha(G) \setminus \{1\}) \cong \mathrm{Cay}(G, \omega_\beta(G) \setminus \{1\})$*.*

Proof. By Proposition 2.1, we have $\omega_\alpha(G) = \omega_\alpha(G)^{-1}$ and $\omega_\beta(G) = \omega_\beta(G)^{-1}$. Since α, β are conjugate, there exists some $\gamma \in \mathrm{Aut}(G)$ such that $\beta = \gamma\alpha\gamma^{-1} = \alpha^\gamma$. Therefore

$$\begin{aligned}
\gamma(\omega_\alpha(G)) &= \{\gamma(\alpha(g^{-1})g) \mid g \in G\} \\
&= \{\gamma\alpha\gamma^{-1}\gamma(g^{-1})\gamma(g) \mid g \in G\} \\
&= \{\beta(\gamma(g)^{-1})\gamma(g) \mid \gamma(g) \in G\} \\
&= \omega_\beta(G).
\end{aligned}$$

It follows that $\gamma(\omega_\alpha(G) \setminus \{1\}) = \omega_\beta(G) \setminus \{1\}$. Hence the result follows. □

Theorem 2.3. *Let* G *be a finite group admitting an automorphism* α *of order two,* $S \subseteq G$ *such that* $S \cap \omega_\alpha(G) = \emptyset$*. Let* $\Phi(g) = \alpha(g)Sg^{-1}$*. If* S *is symmetrical and* $\Phi(g) = S$ *for any* $g \in G$*, then* $\mathrm{GC}(G, S, \alpha) \cong \mathrm{Cay}(G, S)$*.*

Proof. Let $\Gamma_1 = \mathrm{GC}(G, S, \alpha)$ and $\Gamma_2 = \mathrm{Cay}(G, S)$. Let $\phi\colon V(\Gamma_1) \to V(\Gamma_2)$, $x \mapsto x^{-1}$ be a bijection between these two graphs. For any $\{g, h\} \in E(\Gamma_1)$, there exists some $s \in S$ such that $h = \alpha(g)s$. $\{g, h\}^\phi = \{g^{-1}, h^{-1}\}$. Note that $gh^{-1} = gs^{-1}\alpha(g)^{-1} = \alpha(\alpha(g))s^{-1}\alpha(g)^{-1}$. Since S is symmetrical and $\Phi(g) = S$ for any $g \in G$, we have $\alpha(\alpha(g))s^{-1}\alpha(g)^{-1} \in S$. This implies $\{g, h\}^\phi \in E(\Gamma_2)$, and thus $\mathrm{GC}(G, S, \alpha) \cong \mathrm{Cay}(G, S)$. □

Theorem 2.3 can be regarded as a criteria to judge whether some generalized Cayley graphs are Cayley graphs or not.

Theorem 2.4. *Let G be any finite group admitting an automorphism α of order 2. Then we have $\mathrm{GC}(G, S, \alpha) \cong \mathrm{Cay}(G, S)$, where $S = \mho_\alpha, \Omega_\alpha$ or $G \setminus \omega_\alpha(G)$.*

Proof. By proposition 2.1, $G = \omega_\alpha(G) \cup \Omega_\alpha \cup \mho_\alpha$. For any $g \in G$, $G = \alpha(g)Gg^{-1}$. For any $x \in \omega_\alpha(G)$, there exists some $h \in G$ such that $x = \alpha(h^{-1})h$. So $\alpha(g)xg^{-1} = \alpha(g)\alpha(h^{-1})hg^{-1} = \alpha((hg^{-1})^{-1})hg^{-1}$, and hence $\omega_\alpha(G) = \alpha(g)\omega_\alpha(G)g^{-1}$. As a result, $\Omega_\alpha \cup \mho_\alpha = \alpha(g)\Omega_\alpha g^{-1} \cup \alpha(g)\mho_\alpha g^{-1}$.

For any $s \in \Omega_\alpha$, assume that $\alpha(g)sg^{-1} \in \mho_\alpha$, then $\alpha(\alpha(g)sg^{-1})^{-1} \in \mho_\alpha$ and $\alpha(g)sg^{-1} \neq \alpha(\alpha(g)sg^{-1})^{-1}$. Since $\alpha(\alpha(g)sg^{-1})^{-1} = \alpha(g)\alpha(s^{-1})g^{-1}$, we have that $s \neq \alpha(s^{-1})$, which is a contradiction as $s \in \Omega_\alpha$. This means $\alpha(g)sg^{-1} \in \Omega_\alpha$. Thus, $\Omega_\alpha = \alpha(g)\Omega_\alpha g^{-1}$ and $\mho_\alpha = \alpha(g)\mho_\alpha g^{-1}$. By Theorem 2.3, we get the result. □

Let $\mathrm{Fix}(\alpha) = \{g \in G \mid \alpha(g) = g\}$. So $\mathrm{Fix}(\alpha) \leq G$ and we have the following lemma.

Lemma 2.5 ([8]). $|\omega_\alpha(G)| = \frac{|G|}{|\mathrm{Fix}(\alpha)|}$.

Note that some references also use $C_G(\alpha)$ to denote $\mathrm{Fix}(\alpha)$. Those papers mainly investigate the properties of the finite groups which admit involutory automorphisms; one can refer to [2, 13, 21, 22]. Although those problems are not considered in this paper, we borrow the following well-known result.

Lemma 2.6 ([7]). *Let G be a finite group of odd order admitting an automorphism ϕ of order two. Then the following statements hold.*

(1) $G = FK = KF$, $F \cap K = 1$, and $|K| = |G : F|$, where $F = C_G(\phi)$ and $K = K_\phi$;

(2) Two elements of K conjugate in G are conjugate by an element of F;

(3) If H is a subgroup of F, then $N_G(H) = C_G(H)N_F(K)$.

By Lemmas 2.5 and 2.6, we get

Proposition 2.7. *Let G be a group of odd order admitting an automorphism α of order two. Then $\Omega_\alpha = \emptyset$.*

Proof. By Lemmas 2.5 and 2.6, $|K_\alpha| = |\omega_\alpha(G)| = \frac{|G|}{|\mathrm{Fix}(\alpha)|}$. As $K_\alpha = \omega_\alpha(G) \cup \Omega_\alpha$, we obtain $\Omega_\alpha = \emptyset$. □

Remark 2.8. By Proposition 2.7, for any generalized Cayley graph $\mathrm{GC}(G, S, \alpha)$, if $|G|$ is odd, $S \subseteq \mho_\alpha$. We present an alternative proof avoiding Lemmas 2.5 and 2.6. If $\Omega_\alpha \neq \emptyset$, assume that $\mho_\alpha = \emptyset$. Then G is an abelian group of odd order by Proposition 2.1. Thus α is a fixed-point-free automorphism of G. Then $K_\alpha = \omega_\alpha(G) = G$ according to [7, Lemma 10.1.1], which is a contradiction. This implies that $\mho_\alpha \neq \emptyset$. Since the S in $\mathrm{GC}(G, S, \alpha)$ are choosen from Ω_α and $\mho_\alpha$. Therefore $|S|$ must be odd, which is a contradiction as, there are no regular graphs of odd order with odd valency. This implies $\Omega_\alpha = \emptyset$.

It is well known that a finite group G of odd order is solvable by Feit-Thompson Theorem [5]. From above, we can see that the classification of $\mathrm{GC}(G, S, \alpha)$ of finite group G of odd order seems to be more clear as the elements of S can only be chosen from $\mho_\alpha$ since $\Omega_\alpha = \emptyset$.

In [8], Hujdurović et al. defined the following set

$$\mathrm{Aut}(G, S, \alpha) = \{\varphi \in \mathrm{Aut}(G) \mid \varphi(S) = S, \alpha\varphi = \varphi\alpha\}.$$

Moreover, one sees that $\mathrm{Aut}(G, S, \alpha) = \mathrm{Aut}(G, S) \cap C_{\mathrm{Aut}(G)}(\alpha)$, where $\mathrm{Aut}(G, S) = \mathrm{Aut}(G, S, 1)$.

Proposition 2.9. *Let S be the set as in (3) of Proposition 2.1. Then* $\mathrm{Aut}(G, S, \alpha) = \mathrm{Aut}(G, S_1, \alpha) \cap \mathrm{Aut}(G, S_2, \alpha) = \mathrm{Aut}(G, S_1) \cap \mathrm{Aut}(G, S_2) \cap C_{\mathrm{Aut}(G)}(\alpha)$. *Furthermore, the couples of the form like $\{s, \alpha(s^{-1})\}$ are imprimitive blocks of* $\mathrm{Aut}(G, S, \alpha)$.

Proof. For any $s \in S_1$ and $s' \in S_2$, if there exists some $\varphi \in \mathrm{Aut}(G, S, \alpha)$ such that $s = \varphi(s')$, then $\alpha\varphi(s') = \alpha(s)$. Since $\alpha\varphi = \varphi\alpha$ and $s = \alpha(s^{-1})$, $\varphi\alpha(s'^{-1}) = s$. This implies $\alpha(s') = s'^{-1}$, which is a contradiction as $s' \in S_2$. Hence $\varphi(S_1) = S_1$ and $\varphi(S_2) = S_2$ for any $\varphi \in \mathrm{Aut}(G, S, \alpha)$.

Let $\Delta = \{s, \alpha(s^{-1})\}$ be a couple in S_2. For any $\varphi \in \mathrm{Aut}(G, S, \alpha)$, $\Delta^\varphi \subseteq S_2$. If $\Delta \cap \Delta^\varphi \neq \emptyset$, then $s = \varphi(s)$ or $s = \varphi\alpha(s^{-1})$. If $s = \varphi(s)$, then $\alpha(s^{-1}) = \varphi\alpha(s^{-1})$. If $s = \varphi\alpha(s^{-1})$, then $\alpha(s^{-1}) = \varphi(s)$. This implies that $\Delta = \Delta^\varphi$. Thus Δ is an imprimitive block. □

Let $\mathrm{GC}(G, S, \alpha)$ be a generalized Cayley graph of G. Under the condition of Proposition 2.9, $S \cap S^{-1} = (S_1 \cup S_2) \cap (S_1 \cup S_2)^{-1} = (S_1 \cap S_1^{-1}) \cup (S_1 \cap S_2^{-1}) \cup (S_2 \cap S_1^{-1}) \cup (S_2 \cap S_2^{-1})$. Note that $S_1 \cap S_2^{-1} = S_2 \cap S_1^{-1} = \emptyset$, it follows that $S \cap S^{-1} = (S_1 \cap S_1^{-1}) \cup (S_2 \cap S_2^{-1})$. Since $S_1 \subseteq \Omega_\alpha$, and Ω_α is symmetrical, so $S_1 \cap S_1^{-1} \subseteq \Omega_\alpha$. Similarly, $S_2 \cap S_2^{-1} \subseteq \mho_\alpha$. Let $T = S \cap S^{-1}$. It follows that $\mathrm{GC}(G, T, \alpha)$ is still a generalized Cayley graph of G. We call $\mathrm{GC}(G, T, \alpha)$ the *induced* generalized Cayley graph of $\mathrm{GC}(G, S, \alpha)$. Note that $T^{-1} = T$, this encourages us to consider the Cayley graph $\mathrm{Cay}(G, T)$, called the *induced* Cayley graph of $\mathrm{GC}(G, S, \alpha)$. Next we consider $\mathrm{Aut}(G, S, \alpha)$, $\mathrm{Aut}(G, T, \alpha)$ and $\mathrm{Aut}(G, T)$.

Proposition 2.10. $\mathrm{Aut}(G, S, \alpha) \leq \mathrm{Aut}(G, T, \alpha) \leq \mathrm{Aut}(G, T)$. *Furthermore,* $\mathrm{Aut}(G, S, \alpha) < \mathrm{Aut}(G, T, \alpha)$ *if S is not symmetrical;* $\mathrm{Aut}(G, T, \alpha) = \mathrm{Aut}(G, T)$ *if* $\alpha \in Z(\mathrm{Aut}(G))$.

Proof. For any $\varphi \in \mathrm{Aut}(G, S, \alpha)$, we have $\varphi(S) = S$ and $\varphi(S^{-1}) = S^{-1}$, thus $\varphi(T) = T$, $\varphi \in \mathrm{Aut}(G, T, \alpha)$. If S is not symmetrical, we have $\alpha \notin \mathrm{Aut}(G, S, \alpha)$ as $\alpha(S) = S^{-1} \neq S$, but $\alpha \in \mathrm{Aut}(G, T, \alpha)$ as $\alpha(T) = T$. $\mathrm{Aut}(G, T, \alpha) \leq \mathrm{Aut}(G, T)$ is obvious by the definition. Since $\mathrm{Aut}(G, T, \alpha) = \mathrm{Aut}(G, T) \cap C_{\mathrm{Aut}(G)}(\alpha)$, we get the result. □

Finally, we introduce a lemma about the connectivity of the generalized Cayley graph.

Lemma 2.11. *Let G be a group, $A \subseteq G$ and $\alpha \in \mathrm{Aut}(G)$ of order 2. The generalized Cayley graph $X = \mathrm{GC}(G, A, \alpha)$ is connected if and only if A is a left generating set for $(G, *)$, where $f * g = \alpha(f)g$ for all $f, g \in G$.*

3 Two basic types of isomorphisms

In this section, we will introduce two types of natural isomorphisms of generalized Cayley graphs for any finite group. First, we introduce the first type of natural isomorphism found by A. Hujdurović et al.

Theorem 3.1 ([9])**.** $\mathrm{GC}(G, S, \alpha) \cong \mathrm{GC}(G, S^\beta, \alpha^\beta)$ *for any* $\beta \in \mathrm{Aut}(G)$*, where* $\alpha^\beta = \beta\alpha\beta^{-1}$.

Remark 3.2. From Theorem 3.1, one can see that if α, γ are conjugate, then there is a generalized Cayley graph $\mathrm{GC}(G, S, \alpha)$ if and only if there is a generalized Cayley graph $\mathrm{GC}(G, S^\beta, \gamma)$ with $\gamma = \alpha^\beta$ such that these two graphs are isomorphic. Hence, if we intend to study all the generalized Cayley graphs of some group G, we only need to study the generalized Cayley graphs related to the representatives of the conjugacy classes of elements in $\mathrm{Aut}(G)$.

Corollary 3.3. $\mathrm{GC}(G, S, \alpha) \cong \mathrm{GC}(G, S^{-1}, \alpha)$.

Proof. Let $\beta = \alpha$. Then $\mathrm{GC}(G, S, \alpha) \cong \mathrm{GC}(G, \alpha(S), \alpha^\alpha)$ by Theorem 3.1. Note that $\alpha(S) = S^{-1}$, this completes the proof. □

Next, we introduce the second type of natural isomorphism.

Theorem 3.4. *Let $\mathrm{GC}(G, S, \alpha)$ be a generalized Cayley graph. Then $\mathrm{GC}(G, \alpha(g)Sg^{-1}, \alpha)$ is also a generalized Cayley graph of G for any $g \in G$. Furthermore, $\mathrm{GC}(G, S, \alpha) \cong \mathrm{GC}(G, \alpha(g)Sg^{-1}, \alpha)$.*

Proof. For any $x \in G$, if $\alpha(x^{-1})x \in \alpha(g)Sg^{-1}$, $\alpha(g^{-1})\alpha(x^{-1})xg \in S$, that is, $\alpha((xg)^{-1})xg \in S$, which conflicts with condition (ii). If $\alpha(x^{-1})y \in \alpha(g)Sg^{-1}$, then we have $\alpha((xg)^{-1})yg \in S$. Thus $\alpha((yg)^{-1})xg \in S$ by condition (iii). It follows that $\alpha(y^{-1})x \in \alpha(g)Sg^{-1}$. Therefore, $\mathrm{GC}(G, \alpha(g)Sg^{-1}, \alpha)$ is also a generalized Cayley graph of G for any $g \in G$.

Let $\Gamma = \mathrm{GC}(G, S, \alpha)$ and $\Gamma_g = \mathrm{GC}(G, \alpha(g)Sg^{-1}, \alpha)$. Let $\theta \colon V(\Gamma) \to V(\Gamma_g)$, $a \mapsto ag^{-1}$. So θ is a bijection. For any $\{a, b\} \in E(\Gamma)$, $\alpha(a^{-1})b \in S$. Since

$$\alpha((ag^{-1})^{-1})(bg^{-1}) = \alpha(g)(\alpha(a^{-1})b)g^{-1} \in \alpha(g)Sg^{-1},$$

we have $\{ag^{-1}, bg^{-1}\} \in E(\Gamma_g)$. Therefore $\{a, b\} \in E(\Gamma)$ if and only if $\{a, b\}^\theta \in E(\Gamma_\alpha)$. Thus they are isomorphic. □

According to Theorem 3.1, $\Gamma \cong \Gamma^\beta$ for any $\beta \in \mathrm{Aut}(G)$, we call the mapping $x \mapsto x^\beta$ the *the first basic type of isomorphism* of Γ. By Theorem 3.4, $\Gamma \cong \Gamma_g$ for any $g \in G$, we call the mapping $x \mapsto xg^{-1}$ *the second basic type of isomorphism* of Γ.

For any $g \in G$, $R(g) \colon x \mapsto xg$ is a permutation of G. Set $R(H) = \{R(h) \mid S = \alpha(h)Sh^{-1}\}$.

Theorem 3.5. *Let $\Gamma = \mathrm{GC}(G, S, \alpha)$ be a generalized Cayley graph. Then $R(H) \leq \mathrm{Aut}(\Gamma)$.*

Proof. For any $\{a,b\} \in E(\Gamma)$, it suffices to show that $\{a,b\}^{R(h)} \in E(\Gamma)$ for any $R(h) \in R(H)$. Since $\{a,b\} \in E(\Gamma)$, $\alpha(a^{-1})b \in S = \alpha(h)Sh^{-1}$. It follows that $\alpha((ah)^{-1})bh \in S$, which implies that $\{ah, bh\} \in E(\Gamma)$. Thus $R(h) \in \mathrm{Aut}(\Gamma)$. For any $R(h), R(h') \in R(H)$, $S = \alpha(h)Sh^{-1}$ and $S = \alpha(h')Sh'^{-1}$. Therefore $S = \alpha(h'^{-1}h)S(h'^{-1}h)^{-1}$, thus $R(h'^{-1}h) \in R(H)$. This implies that $R(H) \leq \mathrm{Aut}(\Gamma)$. □

4 GCI, restricted GCI and strongly GCI groups

Similarly to the CI-groups in Cayley graphs and BCI-groups in bi-Cayley graphs, we propose the following definitions relating to generalized Cayley graphs.

Definition 4.1. Let G be a finite group. Let M be the set of all Cayley graphs and N be the set of all generalized Cayley graphs constructed by automorphisms of order two. Then

1. G is called a GCI-group if both of the following are satisfied:
   (i) for any two nontrivial generalized Cayley graphs $\mathrm{GC}(G,S,1)$ and $\mathrm{GC}(G,T,1)$ in M, whenever $\mathrm{GC}(G,S,1) \cong \mathrm{GC}(G,T,1)$, there exists $\delta \in \mathrm{Aut}(G)$ such that $S^\delta = T$.
   (ii) for any two nontrivial generalized Cayley graphs $\mathrm{GC}(G,S,\alpha)$ and $\mathrm{GC}(G,T,\beta)$ in N, whenever $\mathrm{GC}(G,S,\alpha) \cong \mathrm{GC}(G,T,\beta)$, there exists $\delta \in \mathrm{Aut}(G)$ such that $\beta = \alpha^\delta = \delta\alpha\delta^{-1}$ and $T = \alpha^\delta(g)S^\delta g^{-1}$.
2. G is called a restricted GCI-group if (ii) is satisfied.
3. G is called a strongly GCI-group if for any nontrivial $\mathrm{GC}(G,S,\alpha)$, whenever $\mathrm{GC}(G,S,\alpha) \cong \mathrm{GC}(G,T,\beta)$, there exists $\delta \in \mathrm{Aut}(G)$ such that $\beta = \alpha^\delta = \delta\alpha\delta^{-1}$ and $T = \alpha^\delta(g)S^\delta g^{-1}$.

Remark 4.2.

1. The definition is based on Theorems 3.1 and 3.4 and Definition 1.1. The two basic types of isomorphisms and their compositions are called the *natural isomorphisms* of generalized Cayley graphs. For instance, $\mathrm{GC}(G,S,\alpha) \cong \mathrm{GC}(G,S^\gamma,\alpha^\gamma)$ by Theorem 3.1, $\mathrm{GC}(G,S^\gamma,\alpha^\gamma) \cong \mathrm{GC}(G,\alpha^\gamma(g)S^\gamma g^{-1},\alpha^\gamma)$ by Theorem 3.4, then we have $\mathrm{GC}(G,S,\alpha) \cong \mathrm{GC}(G,\alpha^\gamma(g)S^\gamma g^{-1},\alpha^\gamma)$.
2. The word 'nontrivial' in the definition means that the null graph is not considered. In fact, if it is included, for a finite group G which has an automorphism α of order 2, $\mathrm{GC}(G,\emptyset,1)$ and $\mathrm{GC}(G,\emptyset,\alpha)$ are both isomorphic to the null graph. By the definition, G cannot be a strongly GCI-group, otherwise it will make the definition meaningless, thus the null graph is not considered in the definition.
3. If a finite group G has no automorphisms of order two, then we still consider that (ii) is satisfied for G.
4. By definition, strongly GCI-group implies GCI-group, GCI-group implies CI-group and restricted GCI-group. However, restricted GCI does not imply GCI and does not imply CI either. If G is not a restricted GCI-group or a CI-group, then it is not a GCI-group either.

Next we will give some examples of finite groups satisfying special conditions:

Example 4.3. Let $G = Z_4$. Then G is a GCI group by Theorem 5.2. However, let $\alpha\colon x \mapsto -x$ be an involution. Thus $\mathrm{GC}(G, \{1\}, \alpha)$ is a generalized Cayley graph of G. Also, $\mathrm{GC}(G, \{2\}, 1)$ is a generalized Cayley graph of G. Although $\mathrm{GC}(G, \{1\}, \alpha) \cong \mathrm{GC}(G, \{2\}, 1)$ but, α is not conjugate to 1, that means G is not a strongly GCI group. Therefore Z_4 is a GCI but not strongly GCI group.

Let $G = Z_8$. Then G is a CI group [19]. However, Z_{2^n} is a GCI group if and only if it is Z_2 or Z_4 by Theorem 5.2. It follows that G is not a GCI group. Thus Z_8 is a CI but not GCI group.

Though we find example of CI but not restricted GCI groups, like Z_8, we have not found out the example of restricted GCI but not CI groups up to now. Thus we propose the following question:

Question 4.4. Is every restricted GCI group a CI group?

The next theorem is useful to determine whether a group is a restricted GCI-group or not.

Theorem 4.5. *Let G be a finite group admitting two automorphisms α, β of order two. If α, β satisfy the following three conditions:*

(1) α and β are not conjugate;

(2) $|\omega_\alpha(G)| \neq |K_\alpha|$;

(3) $|\omega_\beta(G)| \neq |K_\beta|$,

then G is not a restricted GCI-group.

Proof. Assume $|G| = n$. If these three conditions are satisfied, then n is even by Proposition 2.7. Furthermore, there must exist two generalized Cayley graphs, say $\mathrm{GC}(G, \{s\}, \alpha)$ and $\mathrm{GC}(G, \{s'\}, \alpha)$, which are both isomorphic to $\frac{n}{2}K_2$. But there is no natural automorphism as α and β are not conjugate. Hence G is not a restricted GCI-group. □

To conclude, we give the characterization of strongly GCI-groups.

Theorem 4.6. *A finite group G is a strongly GCI-groups if and only if G is a CI-group and one of the following is true for G:*

(1) G has no involutory automorphisms;

(2) all involutory automorphisms are fixed-point-free.

Proof. First we show the necessity. If G is a strongly GCI-groups, then G must be a CI-group. If not all involutory automorphisms of G are fixed-point-free automorphisms or, as we will show that G has no automorphisms of order two. If there exists some involutory automorphism which is not fixed-point-free, say α, this means $|\operatorname{Fix}(\alpha)| \neq 1$. By Lemma 2.5, we get $\omega_\alpha(G) \neq G$. Since $G = \omega_\alpha(G) \cup \Omega_\alpha \cup \mho_\alpha$ by Proposition 2.1, it follows that $\Omega_\alpha \cup \mho_\alpha \neq \emptyset$. Thus at least one of Ω_α and $\mho_\alpha$, say Ω_α, is not an empty set. According to Theorem 2.4, $\mathrm{GC}(G, \Omega_\alpha, \alpha) \cong \mathrm{GC}(G, \Omega_\alpha, 1)$ which is not a null graph. This is a contradiction to the fact that G is a strongly GCI-group. Therefore G has no automorphisms of order two since otherwise all automorphisms of order two of G are fixed-point-free automorphisms. If G has no automorphisms of order two, then G must be a CI-group as G is a strongly GCI-group.

Next we show the sufficiency. Suppose that all automorphisms of order two of G are fixed-point-free. Let $\alpha \in \mathrm{Aut}(G)$ be such an involution. Then $G = \omega_\alpha(G)$ by Lemma 2.5, so any generalized Cayley graph induced by involutory automorphism is a null graph. □

5 The cyclic GCI groups

Theorem 5.1. *The cyclic group of order p^n with p an odd prime is a GCI-group if and only if it is a CI-group.*

Proof. Let $G = Z_{p^n}$. Then G has only one automorphism of order two, that is $\alpha\colon x \mapsto -x$. Note that $\omega_\alpha(G) = \{\alpha(g^{-1})g \mid g \in G\} = \{2g \mid g \in G\}$, it follows that $S = \emptyset$ as any non-identity of G is a square since $|G|$ is odd. Thus the only generalized Cayley graph of G induced by automorphisms of order two is $\mathrm{GC}(G, \emptyset, \alpha) \cong p^n K_1$. □

Babai [3] classified the CI-groups of cyclic groups of order $2p$ with p a prime. Godsil [6] classified the CI-groups of cyclic groups of order $4p$. Next we will classify the GCI-groups of cyclic groups of even order. We will deal with the problem step by step in this section.

Theorem 5.2. *Let G be a finite cyclic group of order 2^n. Then G is a GCI-group if and only if $n = 1, 2$.*

Proof. Let $G = Z_{2^n} = \{0, 1, \ldots, 2^n - 1\}$. When $n = 1$, $\mathrm{Aut}(G) = 1$, there are no automorphisms of order two in $\mathrm{Aut}(G)$. Therefore G is a GCI-group by Definition 4.1. When $n = 2$, then $\mathrm{Aut}(G) \cong Z_2$, there is a unique element of order two in $\mathrm{Aut}(G)$ since $\mathrm{Aut}(G)$ is cyclic, say $\alpha\colon x \mapsto -x$. If $g \in G$, then $\alpha(g^{-1})g = 2g \notin S$. Hence $S \subseteq \{1, 3\}$. Therefore there are only three generalized Cayley graphs of G, with S being $\{1\}$, $\{3\}$ and $\{1, 3\}$, respectively. Let $\Gamma_1 = \mathrm{GC}(G, \{1\}, \alpha)$, $\Gamma_2 = \mathrm{GC}(G, \{3\}, \alpha)$. Note that $-1 \equiv 3 \pmod 4$, and so $\Gamma_1 \cong \Gamma_2$ by Corollary 3.3.

When $n \geq 3$, then $\mathrm{Aut}(G) \cong Z_2 \times Z_{2^{n-2}}$, and there are only three automorphisms of order two in $\mathrm{Aut}(G)$, say,

$$\alpha\colon x \mapsto -x, \qquad \beta\colon x \mapsto (2^{n-1} - 1)x, \qquad \gamma\colon x \mapsto (2^{n-1} + 1)x.$$

Let $S = \{1, 2^{n-1} + 1\}$. Since $1 \not\equiv 2^{n-1} + 1 \pmod{2^n}$ and they are both odd, we have $S \cap \omega_\alpha(G) = \emptyset$ as $\omega_\alpha(g) = \alpha(g^{-1})g = 2g$ is even for any $g \in G$. Further, $S \cap \omega_\beta(G) = \emptyset$ as $\omega_\beta(g) = \beta(g^{-1})g \equiv 2^{n-1}g + 2g \pmod{2^n}$ is also even for any $g \in G$. Recall that $\beta(-1) = 2^{n-1} + 1$, $\alpha(-1) = 1$ and $\alpha(-(2^{n-1} + 1)) = 2^{n-1} + 1$, hence $\alpha(S) = S^{-1}$ and $\beta(S) = S^{-1}$. Therefore both $\mathrm{GC}(G, S, \alpha)$ and $\mathrm{GC}(G, S, \beta)$ are generalized Cayley graphs of G.

Let $\Gamma_1 = \mathrm{GC}(G, S, \alpha)$. Since $|S| = 2$, the valency of Γ_1 is two. For any $x \in V(\Gamma_1)$, $N(x) = \{\alpha(x) + y \mid y \in S\} = \{-x + 1, -x + 2^{n-1} + 1\}$. Consider the vertex $2^{n-1} + x \pmod{2^n}$, it follows that $x \not\equiv 2^{n-1} + x \pmod{2^n}$. $N(2^{n-1} + x) = \{\alpha(2^{n-1} + x) + y \mid y \in S\} = \{2^{n-1} - x + 1, -x + 1\}$. Thus '$x \to (-x + 1) \to (2^{n-1} + x) \to (2^{n-1} - x + 1) \to x$' is a 4-cycle in Γ_1. Therefore $\Gamma_1 \cong 2^{n-2}C_4$.

Let $\Gamma_2 = \mathrm{GC}(G, S, \beta)$. Since $|S| = 2$, the valency of Γ_2 is two. For any $x \in V(\Gamma_2)$, $N(x) = \{\beta(x) + y \mid y \in S\} = \{(2^{n-1} - 1)x + 1, (2^{n-1} - 1)(x - 1)\}$. We consider the vertex $2^{n-1} + x \pmod{2^n}$. Then $N(2^{n-1} + x) = \{\beta(2^{n-1} + x) + y \mid y \in S\} = \{(2^{n-1} - 1)x - 2^{n-1} + 1, (2^{n-1} - 1)x + 1\}$. Thus '$x \to (2^{n-1} - 1)x + 1 \to (2^{n-1} + x) \to (2^{n-1} - 1)(x - 1) \to x$' is a 4-cycle in Γ_1. Therefore $\Gamma_2 \cong 2^{n-2}C_4$.

From above, $\mathrm{GC}(G,S,\alpha) \cong \mathrm{GC}(G,S,\beta) \cong 2^{n-2}C_4$, but α and β are not conjugate in $\mathrm{Aut}(G)$ as $\mathrm{Aut}(G)$ is abelian, hence G is not a restricted GCI-group by Definition 4.1. □

Theorem 5.3. *Let G be a finite cyclic group of order $2^a p^b$ with p an odd prime and $a, b > 0$. If G is a restricted GCI-group, then $a = 1$.*

Proof. Since G is a finite cyclic group of order $2^a p^b$, let $G = G_1 \times G_2$, where $G_1 = Z_{2^a}$ and $G_2 = Z_{p^b}$.

We claim that $a \le 2$. Now we suppose $a \ge 3$. By Theorem 5.2, $\alpha\colon (g_1, g_2) \mapsto (-g_1, g_2)$ and $\beta\colon (g_1, g_2) \mapsto ((2^{n-1}-1)g_1, g_2)$ are two different automorphisms of G with order two when $a \ge 3$. Let $S = \{(1,0),(2^{n-1},0)\}$. Then $\mathrm{GC}(G,S,\alpha)$ and $\mathrm{GC}(G,S,\beta)$ are two generalized Cayley graphs of G. According to Theorem 5.2, we have $\mathrm{GC}(G,S,\alpha) \cong \mathrm{GC}(G,S,\beta) \cong 2^{n-2}p^b C_4$. Note that α and β are not conjugate in $\mathrm{Aut}(G)$, it follows that $a \le 2$.

Assume $a = 2$. Note that $\alpha\colon (g_1,g_2) \mapsto (-g_1, g_2), \beta\colon (g_1,g_2) \mapsto (g_1,-g_2)$ are two automorphisms of G. Furthermore, $\omega_\alpha(G) = \{(0,0),(2,0)\}$, $K_\alpha = \{(g_1,0) \mid g_1 \in G_1\}$. Therefore $\Omega_\alpha = \{(1,0),(3,0)\}$. $\omega_\beta(G) = \{(0,g_2) \mid g_2 \in G_2\}$, $K_\beta = \{(0,g_2),(2,g_2) \mid g_2 \in G_2\}$. Thus $\Omega_\beta = \{(2,g_2) \mid g_2 \in G_2\}$. Let $S_1 = \{(1,0)\}$ and $S_2 = \{(2,0)\}$. We can see that $\mathrm{GC}(G,S_1,\alpha) \cong \mathrm{GC}(G,S_2,\beta) \cong 2p^b K_2$. However α and β are not conjugate as $\mathrm{Aut}(G)$ is abelian. It follows from above discussion that $a = 1$. □

Theorem 5.4. *Let G be a finite cyclic group of order n, where n is even with at least two different odd prime divisors. Then G is not a restricted GCI-group.*

Proof. Suppose that $n = p_0^{s_0} \cdot p_1^{s_1} \cdots p_k^{s_k}$, where $p_0 = 2$, p_i, p_j are different odd primes for any $i, j \in \{1,\dots,k\}$ and $s_t \ge 1$ is an integer for any $t \in \{0,1,\dots,k\}$, $k \ge 2$. It follows that G can be decomposed into the direct product of some cyclic groups, say

$$G = G_0 \times \cdots \times G_k = Z_{2^{s_0}} \times Z_{p_1^{s_1}} \times \cdots \times Z_{p_k^{s_k}}, \text{ where } G_i = Z_{p_i^{s_i}},\ i = 0,1,\dots,k.$$

Let

$$\alpha\colon (x_0, x_1, \dots, x_k) \mapsto (x_0, -x_1, \dots, x_k)$$

and

$$\beta\colon (x_0, x_1, x_2, \dots, x_k) \mapsto (x_0, x_1, -x_2, \dots, x_k).$$

Since $k \ge 2$, then such α, β can not appear in $\mathrm{Aut}(G)$. Obviously $\omega_\alpha(G) = \{(0,x_1,0,\dots,0) \mid x_1 \in G_1\}$ and $\omega_\beta(G) = \{(0,0,x_2,\dots,0) \mid x_2 \in G_2\}$. Let $g_i \in G_i$, $i \in \{0,1,\dots,k\}$ and g_0 the element of order two. Then $(g_0,g_1,0,\dots,0) \in \Omega_\alpha$ and $(g_0,0,g_2,0,\dots,0) \in \Omega_\beta$. Therefore $\mathrm{GC}(G,\{(g_0,g_1,0,\dots,0)\},\alpha)$ and $\mathrm{GC}(G,\{(g_0,0,g_2,0,\dots,0)\},\beta)$ are both generalized Cayley graphs of G. In fact, they are both isomorphic to $\frac{n}{2}K_2$, but α and β are not conjugate in $\mathrm{Aut}(G)$. Thus G is not a restricted GCI-group by Theorem 4.5. □

Theorem 5.5. *Let $G = Z_{2n}$, where n is an odd prime power. Then G is not a strongly GCI-group.*

Proof. Let $G = \langle a, b \mid a^n = b^2 = 1, ab = ba\rangle$. It can be checked that the mapping $\alpha\colon a \mapsto a^{-1}, b \mapsto b$ is the only automorphism of G of order two. Also $\Omega_\alpha = \{a^i b \mid i \in \{1,\dots,n\}\}$ and $\mho_\alpha = \emptyset$ by direct computation. Let $\mathrm{GC}(G,S,\alpha)$ be any generalized Cayley graph of G. Then $S \subseteq \Omega_\alpha$. Let $H = \langle a', b' \mid a'^n = b'^2 = 1, b'a'b' = a'^{-1}\rangle$ and $\varphi\colon a^s \mapsto a'^s, a^t b \mapsto a'^{-t} b$. It follows that φ is a bijection from G to H. Furthermore,

$\mathrm{GC}(G,S,\alpha) \cong \mathrm{Cay}(H,\varphi(S))$. Let $S = \{ab, a^2b\}$. Then $\varphi(S) = \{a^{-1}b, a^{-2}b\}$, this implies $\mathrm{Cay}(H,\varphi(S)) \cong C_{2n}$ as $\langle\varphi(S)\rangle = H$. Let $T = \{ab, a^{-1}b\}$. Then $\langle T\rangle = G$, therefore $\mathrm{Cay}(G,T) \cong C_{2n}$. Thus $\mathrm{GC}(G,S,\alpha) \cong \mathrm{GC}(G,T,1) \cong C_{2n}$, which means that G is not a strongly GCI-group from Definition 4.1. □

Theorem 5.6. *Let $G = Z_{2n}$, $H = D_{2n}$, where n is an odd prime power. Then G is a restricted GCI-group if H is a CI-group.*

Proof. Let $G = \langle a, b \mid a^n = b^2 = 1, ab = ba\rangle$ and $H = \langle a', b' \mid a'^n = b'^2 = 1, b'a'b' = a'^{-1}\rangle$. It is easy to see that $\alpha\colon a \mapsto a^{-1}, b \mapsto b$ is the only automorphism of G of order two. Then we have $\Omega_\alpha = \{a^ib \mid i \in \{1,\dots,n\}\}$ and $\mho_\alpha = \emptyset$ by direct computation. Let $\mathrm{GC}(G,S,\alpha)$ be any generalized Cayley graph of G. Then $S \subseteq \Omega_\alpha$. Let $\varphi\colon a^s \mapsto a'^s, a^tb \mapsto a'^{-t}b$. Obviously φ is a bijection from G to H. Furthermore, $\mathrm{GC}(G,S,\alpha) \cong \mathrm{Cay}(H,\varphi(S))$. Assume that $\mathrm{GC}(G,S_1,\alpha) \cong \mathrm{GC}(G,S_2,\alpha)$, then $\mathrm{Cay}(H,\varphi(S_1)) \cong \mathrm{Cay}(H,\varphi(S_2))$. Since H is a CI-group, there exists some $\gamma \in \mathrm{Aut}(H)$ such that $\gamma(\varphi(S_1)) = \varphi(S_2)$. Without loss of generality, assume that there exist k, l satisfying $(k,n) = 1$ and $1 \le l \le n$ such that γ is the mapping $a' \mapsto a'^k, b' \mapsto a'^lb'$. Let $\delta\colon a \mapsto a^k, b \mapsto b$. Then $\delta \in \mathrm{Aut}(G)$. Since n is some odd prime power, there must exist some $1 \le m \le n$ such that $a^l = a^{2m}$. Therefore there exist $\delta \in \mathrm{Aut}(G)$ and $g = a^{-m}$ such that $S_2 = \alpha(g)S_1^\delta g^{-1}$. Hence G is a restricted GCI-group. □

Theorem 5.7. *Let G be a finite cyclic group of odd order n, where n has at least two different prime divisors. Then G is not a strongly GCI-group.*

Proof. Let $G = G_1 \times G_2 \times \cdots \times G_s$ where $G_i = Z_{p_i^{k_i}}$, p_i is some odd prime. Let $\alpha\colon (g_1, g_2, \dots, g_s) \mapsto (-g_1, g_2, \dots, g_s)$. Then $\omega_\alpha(G) = G_1$, and thus $G \setminus \omega_\alpha(G) \ne \emptyset$. By Theorem 2.4, $\mathrm{GC}(G,S,\alpha) \cong \mathrm{GC}(G,S,1)$, where $S = G \setminus \omega_\alpha(G)$. It follows that G is not a GCI-group. □

6 The GCI-groups in dihedral groups

Theorem 6.1. *Let $G = D_{2n}$ ($n \ge 3$) be a dihedral group. If G is a restricted GCI-group, then n is some odd prime power.*

Proof. Let $G = D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab^{-1} = a^{-1}\rangle$ be a GCI-group. Assume first that n is even. Let $\alpha\colon a \mapsto a^{-1}, b \mapsto b$. Then $\alpha \in \mathrm{Aut}(G)$ is of order two. $\omega_\alpha(G) = \{\alpha(g^{-1})g \mid g \in G\} = \{a^i \in G \mid i \text{ is even}\}$. $K_\alpha = \{\alpha(g)g = 1 \mid g \in G\} = \{a^i, b, a^{\frac{n}{2}}b \mid a^i \in G\}$. It follows that $\Omega_\alpha(G) = \{a^i, b, a^{\frac{n}{2}}b \mid i \text{ is odd}\}$. This implies $\mathrm{GC}(G,\{a\},\alpha)$ and $\mathrm{GC}(G,\{b\},\alpha)$ are always generalized Cayley graphs of G. Note that they are both isomorphic to nK_2. Furthermore, $\alpha(g)a^\gamma g^{-1} = a^{-2i+j}$ if $g = a^i$ and $a^\gamma = a^j$, $\alpha(g)a^\gamma g^{-1} = a^{-2i-j}$ if $g = a^ib$ and $a^\gamma = a^j$. It follows that $\alpha(g)a^\gamma g^{-1} \in \langle a\rangle$. Since G is a GCI-group, $\alpha(g)a^\gamma g^{-1} = b$ for some $g \in G$ and $\gamma \in \mathrm{Aut}(G)$, but this is impossible. Thus n is not even.

Assume n is odd and has at least two different prime factors, say $n = p_1^{r_1} \cdots p_t^{r_t}$ is the prime decomposition and $t \ge 2$. By [20, Lemma 3.4], $\mathrm{Aut}(G) = \mathrm{Aut}(G_1) \times \cdots \times \mathrm{Aut}(G_t)$, where $G_i = \langle a_i, b\rangle$ and $\langle a\rangle = \langle a_1\rangle \times \cdots \times \langle a_t\rangle$. It can be checked that there must exist two automorphisms $\alpha\colon a_1 \mapsto a_1^{-1}, a_i \mapsto a_i, b \mapsto b$ and $\beta\colon a_2 \mapsto a_2^{-1}, a_j \mapsto a_j$, $b \mapsto b$ in $\mathrm{Aut}(G)$. Notice that each is of order two, and they are not conjugate in $\mathrm{Aut}(G)$ as they belong to $\mathrm{Aut}(G_1)$ and $\mathrm{Aut}(G_2)$ respectively, and $\mathrm{Aut}(G)$ is the direct product of these $\mathrm{Aut}(G_i)$. Furthermore, $b \in \Omega_\alpha(G) \cap \Omega_\beta(G)$. Thus $\mathrm{GC}(G,\{b\},\alpha)$ and

$\mathrm{GC}(G, \{b\}, \beta)$ are two generalized Cayley graphs of G which are isomorphic to nK_2. However α and β are not conjugate. Thus n is some odd prime power. □

Theorem 6.2. *Let $G = D_{2n}$, $H = Z_{2n}$, with n odd prime power. Then G is a restricted GCI-group if H is a CI-group.*

Proof. Let $G = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ and $H = Z_{2n} = \{0, 1, \ldots, 2n-1\}$. We will show first that any two automorphisms of G of order two are conjugate. Let $\alpha\colon a \mapsto a^i, b \mapsto a^j b$ and $\beta\colon a \mapsto a^k, b \mapsto a^l b$ be two automorphisms of order two. Then $i, k = -1$. Let $\gamma\colon a \mapsto a^s, b \mapsto a^t b$ with $(n, s) = 1$. Then $\gamma \in \mathrm{Aut}(G)$. We can see that $\gamma^{-1}\colon a \mapsto a^r, b \mapsto a^{-rt} b$ with $rs \equiv 1 \pmod{n}$. It follows that $\gamma\alpha\gamma^{-1}\colon a \mapsto a^{-1}$, $b \mapsto a^{2t+sj} b$. For any j, l, the equation $2t + sj \equiv l \pmod{n}$ has a solution. It follows that α, β are conjugate.

According to the Remark 3.2, it suffices to consider the isomorphisms of the generalized Cayley graphs induced by the same automorphisms. Without loss of generality, we consider $\alpha\colon a \mapsto a^{-1}, b \mapsto b$. Let $s = \frac{n-1}{2}$ and $I = \{1, \ldots, s\}$. Then $\omega_\alpha(G) = \{\alpha(g^{-1})g \mid g \in G\} = \langle a\rangle$. $K_\alpha = \{b\} \cup \langle a\rangle$. Thus $\Omega_\alpha = \{b\}$ and $\mho_\alpha = \cup_{i\in I}\{a^i b, a^{-i} b\}$.

Let $\mathrm{GC}(G, S, \alpha)$ and $\mathrm{GC}(G, T, \alpha)$ be any two isomorphic generalized Cayley graphs. We divide the proof into two cases.

Case 1: $\Omega_\alpha \subseteq S$.

If $\Omega_\alpha \subseteq S$, then $\Omega_\alpha \subseteq T$. Suppose $S = \cup_{i\in I_1\subseteq I}\{a^i b, a^{-i} b\} \cup \Omega_\alpha$ and $T = \cup_{i\in I_2\subseteq I}\{a^i b, a^{-i} b\} \cup \Omega_\alpha$. Let $\varphi\colon G \to H, a^s \mapsto 2s, a^t b \mapsto n - 2t$. Then φ is a bijection from G to H. Furthermore, $\mathrm{GC}(G, S, \alpha) \cong \mathrm{Cay}(H, \varphi(S))$ and $\mathrm{GC}(G, T, \alpha) \cong \mathrm{Cay}(H, \varphi(T))$, where $\varphi(S) = \cup_{i\in I_1\subseteq I}\{n - 2i, n + 2i\} \cup \{n\}$ and $\varphi(T) = \cup_{i\in I_2\subseteq I}\{n - 2i, n + 2i\} \cup \{n\}$. Since H is a CI-group, there exists some automorphism $\phi \in \mathrm{Aut}(H)$ such that $\varphi(T) = \phi(\varphi(S))$. Since n is the unique involution in H, $\phi(n) = n$ and $\phi(n - 2i) = \phi(n) - 2\phi(i) = n - 2\phi(i)$ for any $i \in I_1 \subseteq I$. This implies that ϕ can induce an automorphism $\overline{\phi}$ of G with rules $a^i \mapsto a^{\phi(i)}, b \mapsto b$. As $\overline{\phi}\alpha = \alpha\overline{\phi}$, there exist $\overline{\phi}$ and $g = 1 \in G$ such that the isomorphism between $\mathrm{GC}(G, S, \alpha)$ and $\mathrm{GC}(G, T, \alpha)$ is a natural isomorphism.

Case 2: $\Omega_\alpha \nsubseteq S$.

If $\Omega_\alpha \nsubseteq S$, then $\Omega_\alpha \nsubseteq T$. The rest of the proof is similar to that of Case 1. □

The next result is about the graph structure. Recall that a graph Γ is Hamiltonian if it contains a cycle passing through all vertices of Γ.

Theorem 6.3. *Let $G = D_{2n}$ with n odd prime power. Then any connected generalized Cayley graph of G is Hamiltonian.*

Proof. Let $H = Z_{2n}$ and $\varphi\colon G \to H, a^s \mapsto 2s, a^t b \mapsto n - 2t$ be the bijection from G to H. Then any generalized Cayley graph $\mathrm{GC}(G, S, \alpha)$ of G is isomorphic to the Cayley graph $\mathrm{Cay}(H, \varphi(S))$ of H. Therefore $\mathrm{GC}(G, S, \alpha)$ is connected if and only if $\mathrm{Cay}(H, \delta(S))$ is connected. It is well known that $\mathrm{Cay}(H, \varphi(S))$ is connected if and only if $\langle\varphi(S)\rangle = H$. $\langle\varphi(S)\rangle = H$ if and only if there exist some $a^i b, a^{-i} b \in S$ satisfying $(i, n) = 1$ as $\varphi(a^i b) = n - 2i$. Then there always exists a Hamilton cycle $\mathrm{GC}(G, \{a^i b, a^{-i} b\}, S)$ in a connected generalized Cayley graph of G. This completes the proof. □

7 Cayley regression

First of all, we give the following related definitions.

Definition 7.1. Let G be a finite group.

(1) G is called a *Cayley regression* if any generalized Cayley graph of G is isomorphic to some Cayley graph of G.

(2) G is called an *α-Cayley regression* if any generalized Cayley graph of G induced by $\alpha \in \mathrm{Aut}(G)$ is isomorphic to some Cayley graph of G.

(3) G is called a *quasi-Cayley regression* if any generalized Cayley graph not induced by the automorphism $x \mapsto x^{-1}$ is isomorphic to some Cayley graph of G.

(4) G is called an *m-Cayley regression* if any generalized Cayley graph of G with valency at most m is isomorphic to some Cayley graph of G.

(5) G is called an *m-quasi-Cayley regression* if any generalized Cayley graph not induced by the automorphism $x \mapsto x^{-1}$ of G with valency at most m is isomorphic to some Cayley graph of G.

(6) G is called a *skew Cayley regression* if any generalized Cayley graph of G is isomorphic to some generalized Cayley graph of another finite group.

It is well known that every Cayley graph is also a generalized Cayley graph. But many examples, see [8] for instance, reveal that the converse is not true. Therefore a natural question arises.

Question 7.2. Characterize Cayley regressions.

Remark 7.3. If $\alpha\colon x \mapsto x^{-1}$ is an automorphism of G, then G is abelian. This case is very special as $K_\alpha = G$ and $\mho_\alpha = \emptyset$. In fact, Hujdurović et al. in [9] had already noticed this situation. They studied the relationship between the generalized Cayley graphs induced by involutory automorphism and Cayley graphs. They obtained two families of generalized Cayley graphs induced by involutory automorphisms on $Z_{2^m} \times Z_{2^n}$ and $Z_2 \times Z_2 \times Z_{2k+1}$ respectively (where $m \geq 1, n \geq 2, k \geq 1$) are not vertex-transitive. Therefore we propose the definition of 'quasi-Cayley regression' and 'm-quasi-Cayley regression'. Also, we propose the following problem: Are there finite groups which are quasi-Cayley regressions but not Cayley regressions?

When G is an abelian simple group, then G is a cyclic group of prime order and $\mathrm{Aut}(G)$ is not necessarily a Cayley regression.

Example 7.4. For the prime $p = 61$, obviously, $\mathrm{Aut}(Z_p) \cong Z_{p-1} = Z_{60}$. Let $G = \mathrm{Aut}(Z_p)$, $S = \{\pm 6, \pm 12, 5, 25\}$ and $\alpha(x) = 31x$. By [17, Theorem 4.4], we have $\mathrm{GC}(G, S, \alpha)$ is not a Cayley graph. Thus G is not a Cayley regression.

Theorem 7.5. *Let G be a finite cyclic group of odd order n. Then G is a Cayley regression if and only if n is some odd prime power.*

Proof. The sufficiency is obvious by Theorem 5.1, it suffices to show the necessity. Assume on the contrary that n has at least two different odd prime divisors, say $n = q_1 q_2 m$, where q_1 and q_2 are different prime powers and $(q_1, q_2) = 1$, then we have $G = G_1 \times G_2 \times G_3$, where $|G_1| = q_1$, $|G_2| = q_2$ and $|G_3| = m$. Let $\alpha\colon G \to G$, $(g_1, g_2, g_3) \mapsto (-g_1, g_2, g_3)$. It is easy to see that the order of α is two, so α can induce some generalized Cayley graphs of G. Note that $\omega_\alpha(G) = \{(g_1, 0, 0) \mid g_1 \in G_1\}$. Let $S = \{(0, 1, 0), (0, q_2 - 1, 0)\}$. Then $\Gamma = \mathrm{GC}(G, S, \alpha)$ is a generalized Cayley graph of G. Consider the vertex of the form $(0, g_2, g_3)$ in Γ for any $g_2 \in G_2$, $g_3 \in G_3$. For any fixed g_3, there are q_2 vertices of the form $\{(0, g_2, g_3) \mid g_2 \in G_2\}$ which induce a cycle of length q_2. For any other vertex of the form (g_1, g_2, g_3) with $g_1 \neq 0$, there are $2q_2$ vertices $\{(g_1, g_2, g_3), (-g_1, g_2, g_3) \mid g_2 \in G\}$ which induce a cycle of length $2q_2$. Therefore $\Gamma_1 = mC_{q_2} \cup \frac{(q_1-1)m}{2} C_{2q_2}$, which is not vertex-transitive. Thus $\mathrm{GC}(G, S, \alpha)$ is not a Cayley graph, and hence G is not a Cayley regression. □

Theorem 7.6. *Let* $G = \underbrace{Z_n \times \cdots \times Z_n}_{s}$ *with* n *odd,* $s \geq 2$*. Then* G *is not a Cayley regression.*

Proof. Let $\alpha\colon (i_1, i_2, i_3 \ldots, i_s) \mapsto (i_2, i_1, i_3, \ldots, i_s)$ for all $i_t \in Z_n$. So $\alpha \in \mathrm{Aut}(G)$ and $o(\alpha) = 2$. Therefore α can induce generalized Cayley graphs of G. Let $S = \{(1, 0, \ldots, 0), (0, n-1, 0, \ldots, 0)\}$. It follows that $\mathrm{GC}(G, S, \alpha)$ is a generalized Cayley graph of G.

Consider vertex $(0, \ldots, 0)$, then vertices like $(i, i, 0, \ldots, 0)$ and $(i, i-1, 0, \ldots, 0)$ are in the same cycle with $(0, \ldots, 0)$. Thus $(0, \ldots, 0)$ is in a cycle of length $2n$.

Consider vertex $(0, \frac{n-1}{2}, 0, \ldots, 0)$, then vertices like $(i, \frac{n-1}{2} + i, 0, \ldots, 0)$ and $(\frac{n+1}{2} + i, i, \ldots, 0)$ are in the same cycle with $(0, \frac{n-1}{2}, 0, \ldots, 0)$. It follows that $(0, \frac{n-1}{2}, 0, \ldots, 0)$ is in a cycle of length n.

Therefore $\mathrm{GC}(G, S, \alpha)$ is not vertex-transitive, and thus $\mathrm{GC}(G, S, \alpha)$ is not a Cayley graph. That completes the proof. □

From Theorem 7.5, we see that the cyclic group of odd non prime power order is not an m-Cayley regression for any $m > 0$. So we only consider the cyclic groups of even order in the rest of this section.

Corollary 7.7 ([9])**.** *Let* $G = Z_{2n}$*. Then* $\mathrm{GC}(G, S, \alpha)$ *is isomorphic to a Cayley graph on* D_{2n}*, where* $\alpha\colon x \mapsto -x$*.*

According to Corollary 7.7, we can see that Z_{2p^n} (with p an odd prime) is a skew-Cayley regression since $\alpha\colon x \mapsto -x$ is the only automorphism of G of order two.

Theorem 7.8. *Let* G *be a finite cyclic group of order* 2^n *with* $n \geq 3$*. Then*

(1) G *is a* 3*-quasi-Cayley regression;*

(2) G *is a* 4*-quasi-Cayley regression if and only if* $n = 3$*.*

Proof. Assume that $G = \{0, 1, \ldots, 2^n - 1\}$. By Theorem 5.2, we have that $\alpha\colon x \mapsto (2^{n-1} - 1)x$ and $\beta\colon x \mapsto (2^{n-1} + 1)x$ are the only two automorphisms of G of order two except the automorphism $x \mapsto -x$. Also, the valency of the generalized Cayley of G induced by α or β are even as $x + \alpha(x) \neq 0$ and $x + \beta(x) \neq 0$ for any $x \in G$.

(1) Consider the generalized Cayley graphs induced by α, β respectively. For any $g \in G$,

$$\begin{aligned}\omega_\alpha(G) &= \{\alpha(-g)g \mid g \in G\} = \{(2^{n-1}-1)(-g)+g \mid g \in G\} \\ &= \{2^{n-1}g + 2g \pmod{2^n} \mid g \in G\} = \{g \mid g \equiv 0 \pmod 2\} = K_\alpha.\end{aligned}$$

$$\begin{aligned}\omega_\beta(G) &= \{\beta(-g)g \mid g \in G\} = \{(2^{n-1}+1)(-g)+g \pmod{2^n} \mid g \in G\} \\ &= \{-2^{n-1}g \pmod{2^n} \mid g \in G\} = \{0, 2^{n-1}\} = K_\beta.\end{aligned}$$

It follows that for any generalized Cayley graph $\mathrm{GC}(G, S, \alpha)$, S contains no even integers and, for any generalized Cayley graph $\mathrm{GC}(G, S, \beta)$, 0 and 2^{n-1} are not contained in S.

For any $g \in G$ with g odd,

$$\begin{aligned}\alpha(-g) &= (2^{n-1}-1)(-g) \pmod{2^n} \\ &= g - 2^{n-1}g \pmod{2^n} \\ &= 2^n g + g - 2^{n-1}g \pmod{2^n} \\ &= 2^{n-1}g + g \pmod{2^n} \\ &= 2^{n-1} + g \pmod{2^n}.\end{aligned}$$

This implies that there are 2^{n-2} couples can be included in S, that is, $S_1 = \{1, 2^{n-1}+1\}$, $S_3 = \{3, 2^{n-1}+3\}, \ldots, S_{2^{n-1}-1} = \{2^{n-1}-1, 2^n-1\}$.

For any $g \in G \setminus \omega_\beta(G)$,

$$\begin{aligned}\beta(-g) &= (2^{n-1}+1)(-g) \pmod{2^n} = -g - 2^{n-1}g \pmod{2^n} \\ &= 2^n g - g - 2^{n-1}g \pmod{2^n} = 2^{n-1}g - g \pmod{2^n}.\end{aligned}$$

Then we have

$$\beta(-g) = \begin{cases} 2^{n-1} - g, & \text{if } g \text{ is odd;} \\ 2^n - g, & \text{if } g \text{ is even.} \end{cases}$$

This implies that there are $(2^{n-1}-1)$ couples which could be included in S, they are:

$$\begin{aligned}S_1 &= \{1, 2^{n-1}-1\}, \\ S_3 &= \{3, 2^{n-1}-3\}, \\ &\cdots \\ S_{2^{n-2}-1} &= \{2^{n-2}-1, 2^{n-2}+1\}, \\ S_{2^{n-1}+1} &= \{2^{n-1}+1, 2^n-1\}, \\ &\cdots \\ S_{2^{n-1}+2^{n-2}-1} &= \{2^{n-1}+2^{n-2}-1, 2^{n-1}+2^{n-2}+1\}, \\ T_2 &= \{2, 2^n-2\}, \\ &\cdots \\ T_{2^{n-1}-2} &= \{2^{n-1}-2, 2^{n-1}+2\}.\end{aligned}$$

Let $\Gamma = \mathrm{GC}(G, S, \alpha)$, where $S = S_i$, then $\Gamma \cong 2^{n-2}C_4$ by Theorem 5.2. Let $\Gamma = \mathrm{GC}(G, S, \beta)$. If $S = S_i$, we have $\mathrm{GC}(G, S, \beta) \cong C_{2^n}$ as S_i is the left generating set for $(G, *)$ by Lemma 2.11. If $S = T_i$, then $\mathrm{GC}(G, S, \beta)$ is isomorphic to $2^{n-k}C_{2^k}$

if $2^k i \equiv 0 \pmod{2^n}$; and isomorphic to C_{2^n} otherwise. In conclude, all of the 2-valent generalized Cayley graphs of G induced by α or β are Cayley graphs and, this implies that G is a 3-quasi-Cayley regression.

(2) When $n = 3$, it is easy to check that those 4-valent generalized Cayley graphs induced by α and β, respectively, are all Cayley graphs. Next we construct a family of generalized Cayley graphs which is not vertex-transitive to show the necessity.

Let $S = S_i \cup T_j$, where $i \in \{1, \dots, 2^{n-2} - 1\} \cup \{2^{n-1} + 1, \dots, 2^{n-1} + 2^{n-2} - 1\}$ is odd and $j \in \{2, \dots, 2^{n-1} - 2\}$ is even. If x is odd, then $N(x) = \{2^{n-1} + x + i, x - i, 2^{n-1} + x + j, 2^{n-1} + x - j\}$. If x is even, then $N(x) = \{x + i, 2^{n-1} + x - i, x + j, x - j\}$. Suppose X is the bicirculant such that the vertex set $V(X)$ can be partitioned into to subsets $U = \{u_k \mid k \in Z_{2^{n-1}}\}$ and $V = \{v_k \mid k \in Z_{2^{n-1}}\}$, and there is an automorphism of X such that $\rho(u_k) = u_{k+1}$ and $\rho(v_k) = v_{k+1}$, $k \in Z_{2^{n-1}}$. The edge set $E(X)$ can be partitioned into three subsets:

$$L = \cup_{k \in Z_{2^{n-1}}} \{u_k, u_{k+l} \mid l \in L\},$$
$$M = \cup_{k \in Z_{2^{n-1}}} \{u_k, v_{k+m} \mid m \in M\},$$
$$R = \cup_{k \in Z_{2^{n-1}}} \{v_k, v_{k+r} \mid r \in R\},$$

so we have $L = \{\pm(2^{n-2} + \frac{j}{2})\}$, $M = \{2^{n-2} + \frac{i+1}{2}, -\frac{i-1}{2}\}$, $R = \{\pm\frac{j}{2}\}$. Then $X = BC_{2^{n-1}}[L, M, R]$. Let γ be the mapping as follows:

$$\gamma := \begin{cases} x \mapsto u_{\frac{x-1}{2}}, & \text{if } x \text{ is odd;} \\ x \mapsto v_{\frac{x}{2}}, & \text{if } x \text{ is even.} \end{cases}$$

It follows that $\Gamma \cong X$. Note that $BC_{2^{n-1}}[L, M, R] \cong BC_{2^{n-1}}[aL, aM + b, aR]$ with $a, b \in Z_{2^{n-1}}$ and a invertible [12]. Then $\Gamma \cong BC_{2^{n-1}}[L, M', R]$ with $M' = M + \frac{i-1}{2} = \{0, 2^{n-2} + i\}$. In particular, Γ is connected since $\langle L, M', R\rangle = Z_{2^{n-1}}$. When $j = 2i$, there are no triangles with three vertices of the form $\{u_k, v_{k+2^{n-2}+\frac{i+1}{2}}, v_{k-\frac{i-1}{2}}\}$, but there is a triangle with three vertices as $\{v_{k'}, u_{k'+\frac{i-1}{2}}, u_{k'-2^{n-2}-\frac{i+1}{2}}\}$ since for $n > 3$,

$$k + 2^{n-2} + \frac{i+1}{2} \pm \frac{j}{2} \not\equiv k - \frac{i-1}{2} \pmod{2^{n-1}}$$
$$k' + \frac{i-1}{2} - \left(2^{n-2} + \frac{j}{2}\right) \equiv k' - 2^{n-2} - \frac{i+1}{2} \pmod{2^{n-1}}.$$

This implies that there is no automorphism of X which permutates u_k and $v_{k'}$. So X is not vertex-transitive when $n > 3$. This completes the proof. □

At last, we propose the following questions for further research.

Question 7.9. Classify finite GCI-groups, such as Z_m where m is odd with at least two different prime divisors, abelian groups, dihedral groups and some classes of finite simple groups.

Question 7.10. Characterize the structure of the automorphism group of any generalized Cayley graph.

Question 7.11. Classify Cayley regressions for certain types of group, such as the cyclic groups and the dihedral groups.

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On constructing expander families of G-graphs*

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Abstract

Like Cayley graphs, G-graphs are graphs that are constructed from groups. A method for constructing expander families of G-graphs is presented and is used to construct new expander families of irregular graphs. This technique depends on a relation between some known expander families of Cayley graphs and certain expander families of G-graphs. Several other properties of expander families of G-graphs are presented.

Keywords: Cayley graph, diameter of a graph, abelian group, G-graph, expander family.

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1 Introduction

Expander graphs are sparse graphs that have strong connectivity properties. Expander constructions have found extensive applications in computer science [13, 16], in constructing of algorithms, error correcting codes [12], random walks [23], and sorting networks [1]. If one chooses at random a family of d-regular graphs, it is almost certain to be an expander graph [10]. Nevertheless, constructing expander families is not an easy task. Most constructions use deep algebraic and combinatorial techniques; mainly through Cayley graphs and the Zig-Zag product (see for example [15, 19]).

Like Cayley graphs, G-graphs are defined from groups, but they correspond to an alternative construction. These graphs, introduced in [6], have highly regular properties. In particular, because the algorithm for constructing G-graphs is simple, it appears to be a useful tool to construct new symmetric and semi-symmetric graphs [7]. Several extensively studied problems in graph theory such as the hamiltonicity of Cayley graphs (see e.g. [3, 18] for the latest development on this problem) may as well be approached using these objects. For instance, G-graphs are used to characterize new classes of Hamiltonian Cayley graphs [4], and to improve some upper bounds in the cage graphs problem [6]. Recently in [9], the authors studied some robustness properties of G-graphs such as edge/vertex-connectivity and vertex/edge-transitivity. It turns out, that several families of G-graphs are optimally connected where an optimally connected graph can be thought of as a graph whose vertex-connectivity is equal to its minimum degree. Because of their nice properties, it is natural to consider the problem of constructing an expander family of G-graphs.

One of the chief tools for constructing a family of expander graphs is the concept of Cayley graphs. The main advantage for using such graphs is that at first it enables us when fixing the size of the generating set, to construct a large family of sparse graphs in an effective and concise way. Additionally, the underlying properties of a group G and its generating set S can give us an insightful gaze on the expansion properties of its corresponding Cayley graph $\mathrm{Cay}(G, S)$. Generally speaking, it is hard to prove that a certain family of Cayley graphs is an expander family. Concerning this, a huge amount of research in the last few decades has been devoted to dealing with the following question: which sequence of groups corresponds to an expander family of Cayley graphs? Using some algebraic techniques that depend mainly on Kazhdan constant, many partial results were obtained. In fact, most of these results gave negative answers to this question for certain groups (see [14] and [17], see also Example 3.2 below). The purpose of this article is to present a technique for constructing such families. Our construction is based on a relation between some known expander families of Cayley graphs and certain expander families of G-graphs. More precisely, for a group G and a subset S of G with $S^* = \bigcup_{s\in S} \langle s\rangle \setminus \{e\}$ (i.e. if $S = \{s_1, \dots, s_k\}$, then $S^* = \{s_1, \dots, s_1^{o(s_1)-1}, \dots, s_k, \dots, s_k^{o(s_k)-1}\}$, where $o(s_i)$ denotes the order of s_i), we prove the following main result (see below for terminology).

Theorem 1.1. *If $\{\mathrm{Cay}(G_n, S_n^*),\ n \in \mathbb{N}^+\}$ is an expander family, then $\{\tilde{\Phi}(G_n, S_n),\ n \in \mathbb{N}^+\}$ is also an expander family.*

The rest of the paper is organized as follows. In Section 2, we give a review of some basic facts concerning groups, multigraphs, G-graphs and expander graphs that are needed for our purposes. In Section 3, we shall prove the preceding theorem. In addition, just like in the case of Cayley graphs, we prove that abelian groups can not yield an expander family of G-graphs. In Sections 4 and 5, we first identify a new method for generating an infinite regular family of Cayley graphs from another one by switching specific edges. This leads to a new infinite expander family of Cayley graphs on the projective special linear group $\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z})$. Consequently, we construct several new infinite families of expander G-graphs on the special linear group $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ and projective special linear group $\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z})$. These families are formed of irregular graphs, in particular semi-regular, which are of the very few ones.

2 Preliminaries

This section has been designed to give a general introduction to some of the basic facts needed from the theory of groups, multigraphs, expanders and G-graphs. This introduction is given here to provide a convenient repository for all readers. We discuss briefly the material we shall require from these theories and for more details on any of these subjects, see for example [2, 11, 14, 17, 20].

2.1 Groups

Throughout this paper, all groups are assumed to be finite. Let $(G, ., e)$ be a group, where e denotes the identity element of G and "." denotes the group operation (multiplicative notation). For every g in G we define the order of g, denoted by $o(g)$, as the smallest integer l such that $g^l = e$. Let $S = \{s_1, \ldots, s_k\}$ be a non-empty subset of G, and let $O_{\max}(S)$ and $O_{\min}(S)$ be respectively the maximum and the minimum $o(s_i)$ for all $i \in \{1, \ldots, k\}$. A subset S of G is said to be *symmetric* if every element in S has its inverse in S. We define $\langle S \rangle$ to be the smallest subgroup of G which contains S. If $\langle S \rangle = G$, then set S is said to be a *generating set* of G, or G is *generated* by S. If H is a subgroup of G then the set Hx is called right coset of H in G, and we denote by G/H to be the set of all right cosets of H in G. A subset T_H of G is said to be a *right transversal* for H if T_H contains exactly one element from each right coset of H in G. Let A and B be subsets of a set U, then we denote $B \setminus A = \{x \in B \text{ and } x \notin A\}$ and $\overline{A} = U \setminus A$. The special linear group $\mathrm{SL}(2, \mathbb{Z}/q\mathbb{Z})$ is defined as follows:

$$\mathrm{SL}(2, \mathbb{Z}/q\mathbb{Z}) = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} ;\ a_1, a_2, a_3, a_4 \in \mathbb{Z}/q\mathbb{Z} \text{ and } a_1a_4 - a_2a_3 = 1 \right\}.$$

The projective special linear group $\mathrm{PSL}(2, \mathbb{Z}/q\mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z}/q\mathbb{Z})/\{\pm I_2\}$, where I_2 is the 2×2 identity matrix.

2.2 Multigraphs

All multigraphs considered in this paper are undirected and finite. Generally, we define an *undirected multigraph* Γ as the triple $(V(\Gamma), E(\Gamma), \xi_\Gamma)$, where $V(\Gamma)$ is the *set of vertices*, $E(\Gamma)$ is the *set of edges*, and ξ_Γ is an incidence function that associates with each edge of Γ an unordered pair of vertices of Γ. In addition, we denote by $\{u, v\}$ the multi-edge that links vertices u and v. The multiplicity of the multi-edge $\{u, v\}$ is the cardinality of the set of edges that links u and v. A multi-edge with identical end-points is called a *loop*. A multigraph is a *simple graph* if it has neither loops nor multi-edges with multiplicity greater than or equal to 2.

The *neighborhood* of vertex u denoted by $N(u)$ is the set of all vertices that are adjacent to u. The *degree of a vertex* v in a multigraph Γ, denoted by $d(v)$ is the number of edges of Γ incident to v where each loop counts as two edges. The *maximum* and *minimum degree* of a multigraph Γ are denoted by $\Delta(\Gamma)$ and $\delta(\Gamma)$ respectively. A multigraph Γ is d-regular if $d(u) = d$ for all $u \in V(\Gamma)$. A family of d-regular multigraphs is formed of regular multigraphs where each has degree d, while a family of regular multigraphs is formed of regular multigraphs with possibly varying degrees. The *distance* $d(u, v)$ between two vertices u and v is the number of edges in a shortest path that connects u and v. The *diameter* $\mathrm{diam}(\Gamma)$ of a multigraph Γ is defined by:

$$\mathrm{diam}(\Gamma) = \max\{d(u, v); u, v \in V(\Gamma)\}.$$

Let $\Gamma_1 = (V_1, E_1, \xi_1)$ and $\Gamma_2 = (V_2, E_2, \xi_2)$ be two multigraphs, a homomorphism from Γ_1 to Γ_2 is a couple $(f, f^{\#})$ where $f\colon V_1 \to V_2$ and $f^{\#}\colon E_1 \to E_2$ such that if $\xi_1(a) = \{u, v\}$ then $\xi_2(f^{\#}(a)) = \{f(u), f(v)\}$. A graph isomorphism is the couple $(f, f^{\#})$ where f and $f^{\#}$ are bijective. We say that Γ_1 is isomorphic to Γ_2 if there exists an isomorphism between Γ_1 and Γ_2. In such a case, we write $\Gamma \simeq \Gamma'$.

A multigraph $\Gamma = (V, E, \xi_\Gamma)$ is *k-partite* if there is a partition of V into k parts such that each part is a stable set. We will write $\Gamma = (\bigsqcup_{i \in I} V_i; E)$ where $I = \{1, \dots, k\}$. A multigraph is *minimum k-partite* ($k \geq 1$) if it is k-partite and not $(k-1)$-partite. It is easy to verify that for any multigraph Γ, there exists k such that Γ is minimum k-partite. If a multigraph Γ is k-partite, then we will say that $(V_i)_{i \in \{1,2,\dots,k\}}$ is a k-representation of Γ.

Cayley graphs offer a combinatorial depiction of groups and their generators. More precisely, the Cayley graph $\mathrm{Cay}(G, S)$ is the multigraph with vertex set G and two elements x and y of G are adjacent if and only if $y = s.x$ for some $s \in S$. It is well-known that $\mathrm{Cay}(G, S)$ is connected if and only if $G = \langle S \rangle$ (see for example [14]).

2.3 *G*-graphs

Definition 2.1. Let $(G, ., e)$ be a finite group. Let S be a nonempty subset of G. For $s \in S$, consider the cycles $(s)x = (x, sx, \dots, s^{o(s)-1}x)$ of permutation $g_s\colon x \mapsto sx$. Note that $\langle s \rangle x$ is the set $\{x, sx, \dots, s^{o(s)-1}x\}$. We define the G-graph $\Phi(G, S)$ in the following way:

1. The vertex set of $\Phi(G, S)$ is $V = \bigsqcup_{s \in S} V_s$ where $V_s = \{(s)x,\ x \in T_{\langle s \rangle}\}$ where $T_{\langle s \rangle}$ is a right transversal for the subgroup $\langle s \rangle$.
2. For each $(s)x, (t)y \in V$, there exists edge between $(s)x$ and $(t)y$ labeled g for each $g \in \langle s \rangle x \cap \langle t \rangle y$, such an edge will be denoted by $(\{(s)x, (t)y\}, g)$. If $\mathrm{card}(\langle s \rangle x \cap \langle t \rangle y) = p$, $p \geq 1$ then there exists p labeled edges between $(s)x$ and $(t)y$, or $\{(s)x, (t)y\}$ is a multi-edge with multiplicity p.

Since the cosets of $\langle s \rangle$ form a partition of G, $(V_s)_{s \in S}$ is a $|S|$-representation of $\Phi(G, S)$. Every vertex $(s)x$ has $o(s)$ loops. We denote by $\tilde{\Phi}(G, S)$ the multigraph $\Phi(G, S)$ without loops. The multigraph $\tilde{\tilde{\Phi}}(G, S)$ is the simple graph underlying $\Phi(G, S)$, that is, the vertices $(s)x$ and $(t)y$ in $V(\tilde{\tilde{\Phi}}(G, S))$ are connected by a single edge if $\langle s \rangle x \cap \langle t \rangle y$ is non-empty. If $S = \{s_1, \dots, s_k\}$ then the *level* of any s_i, noted V_{s_i} (or simply V_i), is the stable set of $\tilde{\Phi}(G, S)$ which comprises all the vertices of the form $(s_i)x$ where $x \in G$. Note that each level V_s contains $\frac{|G|}{o(s)}$ vertices, therefore we have the following relation:

$$|V(\tilde{\Phi}(G, S))| = |G| \sum_{s \in S} \frac{1}{o(s)}.$$

The *principal clique*[1] of $x \in G$, denoted by C_x, is the subgraph of $\tilde{\Phi}(G,S)$ induced by the set of vertices which contain x. In $\tilde{\Phi}(G,S)$ there are $|G|$ principal cliques; each contains $|S|$ vertices.

Example 2.2. Let G be the cyclic group of order 6, i.e. $G = \{e, a, a^2, a^3, a^4, a^5\}$. Clearly G can be generated by an element of order 3 and another of order 2. Let S be $\{a^2, a^3\}$. Then the vertices of the corresponding G-graph without loops $\tilde{\Phi}(G,S)$ are

$$(a^2)e = (e, a^2e, a^4e) = (e, a^2, a^4), \quad (a^2)a = (a, a^2a, a^4a) = (a, a^3, a^5)$$

which are the 3-cycles and

$$(a^3)e = (e, a^3), \quad (a^3)a = (a, a^3a) = (a, a^4), \quad (a^3)a^2 = (a^2, a^3a^2) = (a^2, a^5)$$

which are the 2-cycles. Obviously, in this case the multigraph $\tilde{\Phi}(G,S)$ is isomorphic to $K_{2,3}$ (Figure 1). The levels V_{a^2} and V_{a^3} are respectively $\{(a^2)e, (a^2)a\}$ and $\{(a^3)e, (a^3)a, (a^3)a^2\}$. There are 6 principal cliques each of size $|S| = 2$. For instance, the principal cliques C_e and C_a are the induced subgraphs of $\tilde{\Phi}(G,S)$ with vertex set $\{(a^2)e, (a^3)e\}$ and $\{(a^2)a, (a^3)a\}$ respectively.

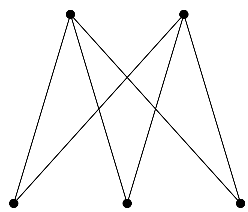

Figure 1: The bipartite multigraph $K_{2,3}$.

The next 3 propositions can be found in [5].

Proposition 2.3 ([5]). *$\Phi(G,S)$ and $\tilde{\Phi}(G,S)$ are minimum $|S|$-partite graphs.*

Proposition 2.4 ([5]). *$\tilde{\Phi}(G,S)$ is connected if and only if S is a generating set of G.*

Proposition 2.5 ([5]). *Let $\tilde{\Phi}(G,S) = (\bigsqcup_{s\in S} V_s; E)$ be a G-graph with $|G| = n$ and $|S| = k$. Then the following holds.*

$$d((s)x) = o(s)(k-1), \quad \textit{for all } (s)x \in V_s,$$

$$\sum_{(s)x\in V_s} d((s)x) = n(k-1), \quad \textit{for all } s \in S,$$

$$|E(\tilde{\Phi}(G,S))| = \frac{nk(k-1)}{2}.$$

2.3.1 New results on G-graphs

Proposition 2.6. *Let $\tilde{\Phi}(G,S)$ be any G-graph such that $|S| = \{s_1, \ldots, s_k\}$. Then the following are equivalent:*

[1] This definition is due to [4].

i. $\tilde{\Phi}(G,S)$ *is d-regular,*

ii. $o(s_i) = \frac{d}{k-1}$, *for all* $i \in \{1,\dots,k\}$,

iii. $|V_{s_i}| = |V_{s_j}|$, *for all* $i,j \in \{1,\dots,k\}$.

Proof. Let $(s)x \in V_s$, where $s \in S$. From Proposition 2.5, we have

$$d((s)x) = o(s)(k-1) \quad \text{or} \quad o(s) = \frac{d((s)x)}{k-1},$$

and then

$$|V_s| = \frac{|G|}{o(s)} = \frac{|G|(k-1)}{d((s)x)}.$$

Therefore $o(s_i) = o(s_j)$ if and only if $|V_{s_i}| = |V_{s_j}|$, for all $i,j \in \{1,\dots,k\}$. □

Remark 2.7. When $\tilde{\Phi}(G,S)$ is a regular multigraph, we use the notation O instead of $o(s)$ for any $s \in S$.

The following lemma can be found in [22].

Lemma 2.8 ([22])**.** *Let* $\Phi(G,S)$ *be a G-graph with* $S = \{s_1,\dots,s_k\}$ *a generating set of G, then all the multi-edges between levels* V_{s_i} *and* V_{s_j} *have the same multiplicity* $|\langle s_i\rangle \cap \langle s_j\rangle|$.

As a result, we have the following corollary.

Corollary 2.9. *Let* $\tilde{\Phi}(G,S)$ *be a G-graph with* $S = \{s_1,\dots,s_k\}$*. Then* $\tilde{\Phi}(G,S)$ *is a simple graph if and only if* $\langle s_i\rangle \cap \langle s_j\rangle = \{e\}$ *for all* $i,j \in \{1,\dots,k\}$ *with* $i \neq j$.

2.4 Expanders

Before we define expander graphs, we need to define some expansion parameters. Let $\Gamma = (V, E, \xi_\Gamma)$ be a non-oriented multigraph with $|V| \geq 2$ and V' be a subset of V. The *edge boundary* of V' in Γ denoted by $\partial V'(\Gamma)$ (or simply $\partial V'$ when no ambiguity occurs) is defined as follows:

$$\partial V'(\Gamma) = \{\alpha \in E;\ \xi_\Gamma(\alpha) \in V' \times \bar{V}'\}.$$

In other words, this is the set of edges emanating from the set V' to its complement. The *rate of expansion* of Γ is then defined as follows:

$$h(\Gamma) = \min_{0<|V'|\leq\frac{|V|}{2}} \frac{|\partial V'|}{|V'|}.$$

Definition 2.10. For $\epsilon \in \mathbb{R}_+^*$, a multigraph Γ is said to be an ϵ-*expander* if

$$\epsilon \leq h(\Gamma).$$

Definition 2.11. If a family of multigraphs $\{\Gamma_i = (V_i, E_i, \xi_i),\ i \in \mathbb{N}^+\}$ satisfies the following three conditions:

i. $|V_i| \to \infty$ as $i \to \infty$,

ii. There exists $r \in \mathbb{N}^+$ such that $\Delta(\Gamma_i) \leq r$ for all $i \in \mathbb{N}^+$. That is $\{\Gamma_i,\ i \in \mathbb{N}^+\}$ is a sequence of bounded degree multigraphs,

iii. There exists $\epsilon \in \mathbb{R}_+^*$ such that Γ_i is an ϵ-expander for all $i \in \mathbb{N}^+$,

then this family is called an *expander family* and an element of this family is an *expander graph*.

If Γ is a d-regular multigraph, then in [14] it is proved that $\log_d |V(\Gamma)| \leq \mathrm{diam}(\Gamma)$. The next proposition is a simple generalization of this result.

Proposition 2.12. *Let Γ be a connected multigraph such that $\Delta(\Gamma) \leq r \in \mathbb{N}^+$. Then*

$$\log_r |V(\Gamma)| \leq \mathrm{diam}(\Gamma).$$

Proof. Consider $v \in V(\Gamma)$ and define $B_l(v) = \{u \in V(\Gamma);\ d(v,u) \leq l\}$. We show by induction that $|B_l(v)| \leq r^l$. The result is trivial for $l = 0$. Suppose it is true up to $l-1$ and let's prove it for l. Since every vertex in $B_{l-1}(v)$ has at most $r-1$ neighbors in $\overline{B_{l-1}}(v)$, then $|B_l(v)| \leq (r-1)|B_{l-1}(v)| + |B_{l-1}(v)| = r|B_{l-1}(v)| \leq rr^{l-1} = r^l$. If $l = \mathrm{diam}(\Gamma)$, then $B_l(v) = V(\Gamma)$ and therefore $|V(\Gamma)| \leq r^{\mathrm{diam}(\Gamma)}$. □

3 $\mathbb{C}\mathrm{ay}$-expanders and $\mathbb{G}$-expanders

In this section, we are mainly concerned with proving Theorem 1.1. First, we need to introduce more auxiliary materials. We start with the following definition which is virtually an interpretation of Definition 2.11 for the G-graph and Cayley graph cases.

Definition 3.1. Let $\{G_i,\ i \in \mathbb{N}^+\}$ be a family of finite groups. We say that $\{G_i,\ i \in \mathbb{N}^+\}$ is a $\mathbb{G}$-*expander family*, if for every $i \in \mathbb{N}^+$ there exists a generating subset S_i of G_i such that $\{\tilde{\Phi}(G_i, S_i),\ i \in \mathbb{N}^+\}$ is an expander family. More precisely, $\{G_i,\ i \in \mathbb{N}^+\}$ is a $\mathbb{G}$-*expander family* if the following 3 conditions are satisfied:

i. $|V(\tilde{\Phi}(G_i, S_i))| = |G_i| \sum_{s \in S_i} \frac{1}{o(s)} \to \infty$ as $i \to \infty$.

ii. There exists a positive integer r such that $\Delta(\tilde{\Phi}(G_i, S_i)) \leq r$ for all $i \in \mathbb{N}^+$ which by Proposition 2.5 means that for every $(s)x \in V_s$ we have $d((s)x) = (|S_i| - 1)o(s) \leq \Delta(\tilde{\Phi}(G_i, S_i)) \leq r \in \mathbb{N}^+$ for all $i \in \mathbb{N}^+$. This in turn means that there exists $r_1, r_2 \in \mathbb{N}^+$ such that $2 \leq |S_i| \leq r_1$ and $o(s) \leq r_2$ for all $s \in S_i$ and for all $i \in \mathbb{N}^+$. In addition, since $\Delta(\tilde{\Phi}(G_i, S_i)) \leq r$ for all $i \in \mathbb{N}^+$, then clearly Condition i. is equivalent to saying that $|G_i| \to \infty$ as $i \to \infty$.

iii. There exists an $\epsilon \in \mathbb{R}_+^*$ such that $\epsilon \leq h(\tilde{\Phi}(G_i, S_i))$ for all $i \in \mathbb{N}^+$.

Note that $2 \leq |S_i|$ since otherwise $\tilde{\Phi}(G_i, S_i)$ will be a disconnected multigraph so that $h(\tilde{\Phi}(G_i, S_i)) = 0$, and so it is clear that $\max\{r_1, r_2\} \leq r$.

On the other hand, we say that $\{G_i,\ i \in \mathbb{N}^+\}$ is a $\mathbb{C}\mathrm{ay}$-*expander family*, if for every $i \in \mathbb{N}^+$ there exists a symmetric generating subset S_i of G_i such that $\{|S_i| : i \in \mathbb{N}^+\}$ is uniformly bounded and provided that $\{\mathrm{Cay}(G_i, S_i),\ i \in \mathbb{N}^+\}$ is an expander family. More explicitly, $\{G_i, i \in \mathbb{N}^+\}$ is a $\mathbb{C}\mathrm{ay}$-*expander family* if the following 2 conditions are satisfied:

i. $|V(\mathrm{Cay}(G_i, S_i))| = |G_i| \to \infty$ as $i \to \infty$,

ii. There exists an $\epsilon \in \mathbb{R}_+^*$ such that $\epsilon \leq h(\mathrm{Cay}(G_i, S_i))$ for all $i \in \mathbb{N}^+$.

Example 3.2. For every $i \in \mathbb{N}^+$, let D_{2i} be the dihedral group:

$$D_{2i} = \left\langle s, f \mid s^2 = f^i = e, sf = f^{-1}s \right\rangle.$$

In 2002, Rosenhouse [21] showed that $h(\mathrm{Cay}(D_{2i}, \{f, f^{-1}, s\})) = \frac{4}{i}$. Hence, $h(\mathrm{Cay}(D_{2i}, \{f, f^{-1}, s\})) \to 0$ as $i \to \infty$. Thus $\{\mathrm{Cay}(D_{2i}, \{f, f^{-1}, s\}),\ i \in \mathbb{N}^+\}$ is not an expander family. In fact, it was shown later (see [14]) that for any set of generator S_i of D_{2i}, $\{\mathrm{Cay}(D_{2i}, S_i),\ i \in \mathbb{N}^+\}$ is not an expander family. Thus $\{D_{2i},\ i \in \mathbb{N}^+\}$ is not a $\mathbb{C}\mathrm{ay}$-expander family.

It is well-known that no family of abelian groups is a $\mathbb{C}\mathrm{ay}$-expander [14]. Before we prove the same result for the $\mathbb{G}$-expander case, we need the following lemma.

Lemma 3.3. *Let G be an abelian group generated by $S = \{s_1, \dots, s_k\}$ and let $\tilde{\Phi}(G, S)$ be the corresponding G-graph, then*

$$\mathrm{diam}(\tilde{\Phi}(G, S)) \leq |S|.$$

Proof. Let $(s_p)x, (s_q)y \in V(\tilde{\Phi}(G, S))$, where $x, y \in G$ and $1 \leq p, q \leq |S| = k$. Since $G = \langle S \rangle$ is an abelian group, then

$$x = s_1^{i_1} \dots s_p^{i_p} \dots s_q^{i_q} \dots s_k^{i_k} y = s_1^{i_1} \dots s_p^{i_p} \dots s_{q-1}^{i_{q-1}} s_{q+1}^{i_{q+1}} \dots s_k^{i_k} s_q^{i_q} y,$$

where $1 \leq i_l \leq o(s_l)$ for all $1 \leq l \leq k$. It is easy to see that $(s_p)x$ is adjacent to $(s_1)s_2^{i_2} \dots s_k^{i_k} y$ which is in turn connected to $(s_2)s_3^{i_3} \dots s_k^{i_k} y$ and so on up to $(s_k)s_q^{i_q} y$ which is connected to $(s_q)y$. Thus $d((s_p)x, (s_q)y) \leq |S|$. □

Corollary 3.4. *No family of abelian groups is a $\mathbb{G}$-expander.*

Proof. Suppose that $\{G_i,\ i \in \mathbb{N}^+\}$ is a family of finite abelian groups and that $\{\tilde{\Phi}(G_i, S_i), i \in \mathbb{N}^+\}$ is an expander family. Then there exists $r \in \mathbb{N}^+$ such that $|S_i| \leq r$ for all $i \in \mathbb{N}^+$. But then by the preceding lemma $\mathrm{diam}(\tilde{\Phi}(G_i, S_i)) \leq |S_i| \leq r \in \mathbb{N}^+$, and that contradicts Proposition 2.12. □

Now we are ready to prove the main result of this paper which is Theorem 1.1.

Proof of Theorem 1.1. Since $\mathrm{Cay}(G_n, S_n^*)$ is an expander family, then $|G_n| \to \infty$ as $n \to \infty$ and there is an $r \in \mathbb{N}^+$ such that $|S_n^*| \leq r$ for all $n \in \mathbb{N}^+$. Hence $|S_n| \leq r$ and $O_{\max}(S_n) \leq r$ for every $n \in \mathbb{N}^+$. Then $|V(\tilde{\Phi}(G_n, S_n))| \to \infty$ as $n \to \infty$ and $\Delta(\tilde{\Phi}(G_n, S_n)) < r^2$ for all $n \in \mathbb{N}^+$.

Suppose that $H \subset V(\tilde{\Phi}(G_n, S_n))$ where $0 < |H| < \frac{|V(\tilde{\Phi}(G_n, S_n))|}{2}$, and $H_i = H \cap V_i$ for every $1 \leq i \leq |S_n|$. Then, clearly we have

$$H = \bigsqcup_i H_i.$$

Let $W = \bigcap_i \bigcup_{(s)x \in H_i} \langle s \rangle x \subset G$. Since $|H| \leq \frac{|V(\tilde{\Phi}(G_n, S_n))|}{2}$, we have $|W| \leq \frac{|G|}{2}$. Now let $X_i = \{(s_i)x \in H_i \mid \langle s_i \rangle x \subset W\}$, then $|X_i| \leq |W|$. Denote by X and Y the following sets of vertices,

$$X = \bigsqcup_{i=1}^{|S_n|} X_i, \quad \text{and} \quad Y = H \setminus X.$$

If $(s)x \in Y$, there is an edge between $(s)x$ and a vertex in $V(\tilde{\Phi}(G_n, S_n)) \setminus H$. Hence

$$|\partial H| \geq |Y|.$$

In $\mathrm{Cay}(G_n, S_n^*)$, we have $|\partial W| \geq \epsilon|W|$. Let $f\colon \partial W \to \partial H$, $\{x, y\} \mapsto (\{(s_i)x, (s_j)y\}, y)$, where $x \in W$, $y \notin W$, i and j are chosen so that $xy^{-1} \in \langle s_i \rangle$ and $y \notin \bigsqcup_{(s)x \in H_j} \langle s \rangle x$ (note here that there may be several possible choices for i and j). Now observe that if $f(x, y) = f(x', y')$, then $xx'^{-1} \in \langle s_i \rangle$ and $y = y'$. So for all $\alpha \in \partial H$, $|f^{-1}(\alpha)| \leq O_{\max}(S_n)$. Hence,

$$|\partial H| \geq \frac{|\partial W|}{O_{\max}(S_n)} \geq \frac{\epsilon|W|}{O_{\max}(S_n)} \geq \frac{\epsilon \max_i |X_i|}{O_{\max}(S_n)} \geq \frac{\epsilon|X|}{O_{\max}(S_n)|S_n|}.$$

Using $|\partial H| \geq |Y|$ and $|H| = |X| + |Y|$, we obtain

$$|\partial H| \geq \frac{1}{2} \min\left\{\frac{\epsilon}{O_{\max}(S_n)|S_n|}, 1\right\} |H| \geq \frac{1}{2} \min\left\{\frac{\epsilon}{r^2}, 1\right\} |H|.$$

This completes the proof. □

The following results are obvious consequences of Theorem 1.1.

Corollary 3.5. *If $\{G_n,\ n \in \mathbb{N}^+\}$ is a $\mathbb{C}\mathrm{ay}$-expander family, then it is also a $\mathbb{G}$-expander family.*

Corollary 3.6. *If $\{\mathrm{Cay}(G_i, S_i^*),\ i \in \mathbb{N}^+\}$ is an expander family, then $\{\tilde{\tilde{\Phi}}(G_i, S_i),\ i \in \mathbb{N}^+\}$ is also an expander family.*

Proof. By Theorem 1.1, $\{\tilde{\Phi}(G_i, S_i),\ i \in \mathbb{N}^+\}$ is an expander family. By Definition 3.1, there exists $r \in \mathbb{N}^+$ such that $o(s_j) \leq r$, for every $s_j \in S_i$. Then $|\langle s_{j_1} \rangle \cap \langle s_{j_2} \rangle| \leq r$ for all $s_{j_1}, s_{j_2} \in S_i$. Thus $\frac{h(\tilde{\Phi}(G_i, S_i))}{r} \leq h(\tilde{\tilde{\Phi}}(G_i, S_i))$. □

Remark 3.7.

1. Unlike most constructed expander families which are d-regular, our construction produces expander families that may be d-regular, regular, or irregular. More specifically, by Proposition 2.6, if the order of all elements in the generating set S_i is the same, then the constructed family is either a d-regular or regular family depending on whether there exist $s_i \in S_i$ and $s_j \in S_j$ such that $o(s_i) \neq o(s_j)$. Otherwise, it will be an irregular family.

2. By Corollary 2.9, if $\langle s_{j_1} \rangle \cap \langle s_{j_2} \rangle = \{e\}$ for all $s_{j_1} \in S_i$, $s_{j_2} \in S_i \setminus s_{j_1}$, and for every $i \in \mathbb{N}^+$, then the constructed expander family $\{\tilde{\Phi}(G_i, S_i),\ i \in \mathbb{N}^+\}$ is formed of simple graphs. Note that $\{\tilde{\tilde{\Phi}}(G_i, S_i),\ i \in \mathbb{N}^+\}$ is always an expander family of simple graphs.

3. In Table 1, we compare some graph invariants for the Cayley graph $\mathrm{Cay}(G, S^*)$ and the G-graph $\tilde{\Phi}(G, S)$.

   It is worthy to note that $|S^*| = \sum_{s \in S} o(s) - |S|$ and every vertex in level V_s of $\tilde{\Phi}(G, S)$ has degree equal to $o(s)(|S| - 1)$ where $|V_s| = \frac{|G|}{o(s)}$. Thus, the degree

Table 1: Some graph invariants of $\mathrm{Cay}(G, S^*)$ and $\tilde{\Phi}(G, S)$.

	$\mathrm{Cay}(G, S^*)$	$\tilde{\Phi}(G, S)$
Order	$\|G\|$	$\sum_{s \in S} \frac{\|G\|}{o(s)}$
Degree	$\|S^*\|$-regular multigraph	$d(u) = o(s)(\|S\| - 1)$ for all $u \in V_s$ and $s \in S$
Size	$\frac{1}{2}\|G\|\|S^*\| = \frac{1}{2}\|G\|(\sum_{s \in S} o(s) - \|S\|)$	$\frac{1}{2}\|G\|\|S\|(\|S\| - 1)$

of most vertices of $\tilde{\Phi}(G, S)$ is smaller than $|S^*|$ (see also the remark after Theorem 5.10). In other words, this means that G-graphs enable us to construct sparser multigraphs than those which can be constructed using the family $\mathrm{Cay}(G, S^*)$, and in some cases even sparser than the ones constructed from the family $\mathrm{Cay}(G, S)$, with possibly smaller expansion ratios (see the proof of Theorem 1.1).

4 Applications

In this section, we present some direct results of Theorem 1.1. But first we start with some auxiliary materials.

Proposition 4.1. *Let $x_i \in G_i \setminus S_i$. If $\{\mathrm{Cay}(G_i, S_i),\ i \in \mathbb{N}^+\}$ is an expander family, then $\{\mathrm{Cay}(G_i, S_i \cup {x_i}^{\pm 1}),\ i \in \mathbb{N}^+\}$ is also an expander family.*

Proof. Since $\{\mathrm{Cay}(G_i, S_i), i \in \mathbb{N}^+\}$ is an expander family, then there exists $r \in \mathbb{N}^+$ such that $|S_i| \le r$, for all $i \in \mathbb{N}^+$. Thus $|S_i \cup {x_i}^{\pm 1}| \le r + 2$ for all $i \in \mathbb{N}^+$, so the second condition of Definition 2.11 is satisfied. Note that $\mathrm{Cay}(G_i, S_i)$ is a spanning subgraph of $\mathrm{Cay}(G_i, S_i \cup {x_i}^{\pm 1})$, hence

$$0 < \epsilon \le h(\mathrm{Cay}(G_i, S_i)) \le h(\mathrm{Cay}(G_i, S_i \cup {x_i}^{\pm 1})). \qquad \square$$

A direct consequence of the preceding proposition is the following.

Corollary 4.2. *Let $\{\mathrm{Cay}(G_i, S_i), i \in \mathbb{N}^+\}$ be an expander family. If there exists $l \in \mathbb{N}^+$ such that $|S_i^*| \le l$ for all $i \in \mathbb{N}^+$, then $\{\mathrm{Cay}(G_i, S_i^*), i \in \mathbb{N}^+\}$ is also an expander family.*

The following theorem was proved by Breuillard and Gamburd in [8].

Theorem 4.3 ([8]). *There exists $\epsilon \in \mathbb{R}_+^*$ and an infinite set of prime numbers $\mathbb{P}'$ such that for every $p \in \mathbb{P}'$ and every generating set $\{x, y\}$ of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$, the family*

$$\mathrm{Cay}(\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z}),\ \{x^{\pm 1}, y^{\pm 1}\})$$

is an ϵ-expander.

Let

$$S_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad S_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is well-known that $\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z}) = \langle S_1, S_3\rangle = \langle S_2, S_3\rangle$. The order of S_1, S_2 is 4, while the order of S_3 in $\mathbb{Z}/p\mathbb{Z}$ is p. Thus $\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z})$ is also generated by one of the following sets: $\{S_1, S_1S_3\}$, $\{S_1, S_3S_1\}$, $\{S_2, S_2S_3\}$, $\{S_2, S_3S_2\}$. Where

$$S_1S_3 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \qquad S_3S_1 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

$$S_2S_3 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \text{ and} \qquad S_3S_2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that the orders of S_1S_3, S_3S_1, S_2S_3, and S_3S_2 are respectively 6, 6, 3 and 3. With the above notation, we have the following conclusion.

Corollary 4.4. *Let*

$$A_1 = \{S_1^{\pm 1}, S_1S_3, S_3^{-1}S_1^{-1}\}, \qquad A_2 = \{S_1^{\pm 1}, S_3S_1, S_1^{-1}S_3^{-1}\},$$

$$A_3 = \{S_2^{\pm 1}, S_2S_3, S_3^{-1}S_2^{-1}\}, \textit{ and} \qquad A_4 = \{S_2^{\pm 1}, S_3S_2, S_2^{-1}S_3^{-1}\}.$$

There exist sets $\mathbb{P}_i^a$ of prime numbers such that $\{\mathrm{Cay}(\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z}), A_i);\ p \in \mathbb{P}_i^a\}$ is an expander family for all $1 \le i \le 4$.

Let $B_1 = \{S_1, S_1S_3\}$, by Corollaries 4.2 and 4.4 we directly deduce that there exists a set $\mathbb{P}'$ of prime numbers such that $\{\mathrm{Cay}(\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z}), B_1^*);\ p \in \mathbb{P}'\}$ is an expander family. Using Theorem 1.1, we can easily deduce that $\{\tilde{\Phi}(\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z}), B_1);\ p \in \mathbb{P}'\}$ is an expander family. By the same analogy, we obtain the following.

Corollary 4.5. *Let*

$$B_1 = \{S_1, S_1S_3\}, \qquad B_2 = \{S_1, S_3S_1\},$$

$$B_3 = \{S_2, S_2S_3\}, \textit{ and} \qquad B_4 = \{S_2, S_3S_2\}.$$

There exist sets $\mathbb{P}_i^b$ of prime numbers such that $\{\tilde{\Phi}(\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z}), B_i);\ p \in \mathbb{P}_i^b\}$ is an expander family for all $1 \le i \le 4$.

In a similar fashion, many other G-graph families on the special linear group $\mathrm{SL}(2,\mathbb{Z}/q\mathbb{Z})$ can be constructed.

5 New expander families of G-graphs

In this section, we present a method for constructing a family of Cayley graphs from another given family by rearrangement of edges in such a way to almost maintain the same expansion ratio. Consequently, we prove that if the family of Cayley graphs $\{\mathrm{Cay}(G_i, \{s_1^{\pm 1}, s_2^{\pm 1}\});\ i \in \mathbb{N}^+\}$ is an expander, then so is the family of Cayley graphs $\{\mathrm{Cay}(G_i, \{s_1^{\pm 1}, s_1s_2, s_2^{-1}s_1^{-1}\});\ i \in \mathbb{N}^+\}$. Then using Theorem 1.1, several expander families of G-graphs are constructed. But first we need to introduce more notation.

Remark 5.1. Let $\mathrm{Cay}(G, S)$ be a Cayley graph and let $H' \subseteq H \subseteq G$. Let $s \in S$, we denote by $N_s(H)$ and $N_s(H)(H')$ the set of vertices of $\mathrm{Cay}(G, S)$ that are defined in the following way:

i. $N_s(H) = sH \cap \bar{H}$,

ii. $N_s(H)(H') = sH' \cap \bar{H}$.

Next, we start by the following simple lemma.

Lemma 5.2. *Let* $\mathrm{Cay}(G,S)$ *be a Cayley graph, where* $S = \{s_1^{\pm 1}, \dots, s_k^{\pm 1}\}$. *Let* $H \subseteq G$, *then*

$$|\partial H(\mathrm{Cay}(G,S))| = 2 \sum_{i|o(s_i)>2} |N_{s_i}(H)| + \sum_{i|o(s_i)=2} |N_{s_i}(H)| = \sum_{1\le i\le k} |N_{s_i^{\pm 1}}(H)|.$$

Proof. Let $x, y \in H$ such that $y = s_i x$ for some $s_i \in S$, then $x = s_i^{-1} y$. Thus the number of edges in the subgraph H of G that corresponds to s_i is equal to that of s_i^{-1} and $|N_{s_i}| = |N_{s_i^{-1}}|$. It is easy to see that: $|\partial H(\mathrm{Cay}(G,S))| = \sum_{1\le i\le k} |N_{s_i^{\pm 1}}(H)|$ and the proof is complete. □

Example 5.3. Let $(\mathbb{Z}/n\mathbb{Z}, +, 0)$, $n \ge 10$ and $S = \{\pm 1, \pm 2\}$. Then $\mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, S)$ is 4-regular multigraph on n vertices. Let H be a subgraph of $\mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, S)$ such that $V(H) = \{1, 2, 3, 7\}$. Let $s_1 = +1$ and $s_2 = +2$. Then $N_{s_1}(H) = \{4, 8\}$, $N_{s_1^{-1}}(H) = \{0, 6\}$, $N_{s_2}(H) = \{4, 5, 9\}$, and $N_{s_2^{-1}}(H) = \{0, 5, n-1\}$. Thus

$$|\partial H(\mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, S))| = 2\big(|N_{s_1}(H)| + |N_{s_2}(H)|\big) = 10.$$

Next, we shall show that it is possible to construct an expander family of Cayley graphs from another one by switching some of its edges.

Corollary 5.4. *Let* $\{\mathrm{Cay}(G_i, \{s_1^{\pm 1}, s_2^{\pm 1}\});\ i \in \mathbb{N}^+\}$ *be an expander family. If* $o(s_1)$, $o(s_2)$, *and* $o(s_1 s_2) > 2$, *then* $\{\mathrm{Cay}(G_i, \{s_1^{\pm 1}, s_1 s_2, s_2^{-1} s_1^{-1}\});\ i \in \mathbb{N}^+\}$ *is also an expander family.*

Proof. Let $V(H) = \{x_1, \dots, x_t\} \in G$. Define $\partial' H$, $\partial'' H$ to be the sets of emanating edges from $V(H)$ in the multigraphs $\mathrm{Cay}(G_i, \{s_1^{\pm 1}, s_2^{\pm 1}\})$ and $\mathrm{Cay}(G_i, \{s_1^{\pm 1}, s_1 s_2, s_2^{-1} s_1^{-1}\})$ respectively. By Lemma 5.2, we have:

$$|\partial' H| = 2|N_{s_1}(H)| + 2|N_{s_2}(H)|, \text{ and} \tag{1}$$
$$|\partial'' H| = 2|N_{s_1}(H)| + 2|N_{s_1 s_2}(H)|. \tag{2}$$

Let $y \in N_{s_2}(H), y = s_2 x$ for some $x \in H$.

i. If $s_1 y \notin H$, then $s_1 s_2 x \notin H$ and $s_1 s_2 x \in N_{s_1 s_2}(H)$.

ii. And if $s_1 y \in H$, then $s_1 s_2 x \in H$.

Let H_1 and H_2 be the set of vertices of H defined as follows:

$$H_1 = \{x \in H / s_2 x \notin H \text{ and } s_1 s_2 x \notin H\},$$
$$H_2 = \{x \in H / s_2 x \notin H \text{ and } s_1 s_2 x \in H\}.$$

From equalities (1) and (2), we have

$$2|N_{s_1}(H)| + 2|N_{s_2}(H)(H_1)| + 2|N_{s_2}(H)(H_2)| = |\partial' H|,$$
$$2|N_{s_1}(H)| + 2|N_{s_1 s_2}(H)(H_1)| + 2|N_{s_1 s_2}(H)(H_2)| \le |\partial'' H|.$$

From the definition of H_2, we have $|N_{s_1s_2}(H)(H_2)| = 0$, then

$$2|N_{s_1}(H)| + 2|N_{s_1s_2}(H)(H_1)| \leq |\partial'' H|.$$

Therefore, it holds that

$$2|N_{s_1}(H)| + 4|N_{s_1s_2}(H)(H_1)| - 2|N_{s_2}(H)(H_1)| - 2|N_{s_2}(H)(H_2)| \leq 2|\partial'' H| - |\partial' H|.$$

From the definition of H_1, we have $|N_{s_1s_2}(H)(H_1)| = |N_{s_2}(H)(H_1)|$ and similarly from the definition of H_2, we have $|N_{s_2}(H)(H_2)| = |N_{s_1^{-1}}(H) \cap N_{s_2}(H)|$. Thus,

$$2|N_{s_1}(H)| + 2|N_{s_2}(H)(H_1)| - 2|N_{s_1^{-1}}(H) \cap N_{s_2}(H)| \leq 2|\partial'' H| - |\partial' H|.$$

Noticing that

$$|N_{s_1^{-1}}(H) \cap N_{s_2}(H)| \leq |N_{s_1^{-1}}(H)| = |N_{s_1}(H)|,$$

then

$$2|N_{s_2}(H)(H_1)| \leq 2|\partial'' H| - |\partial' H|.$$

Finally, we obtain

$$0 < \epsilon \leq \frac{|\partial' H|}{2|H|} \leq \frac{|\partial'' H|}{|H|}.$$ □

Remark 5.5. Note that in general $\{\mathrm{Cay}(G_i, \{s_1^{\pm 1}, s_2^{\pm 1}\});\ i \in \mathbb{N}^+\}$ and $\{\mathrm{Cay}(G_i, \{s_1^{\pm 1}, s_1s_2, s_2^{-1}s_1^{-1}\});\ i \in \mathbb{N}^+\}$ may be not isomorphic. An example of this situation is given by the dihedral group D_{2i} which is defined earlier as follows:

$$D_{2i} = \left\langle s, f \mid s^2 = f^i = e, sf = f^{-1}s \right\rangle.$$

Let $s_1 = s$ and $s_2 = f$, then $S = \{s_1^{\pm 1}, s_2^{\pm 1}\} = \{s, f^{\pm 1}\}$ and $L = \{s_1^{\pm 1}, s_1s_2, s_2^{-1}s_1^{-1}\} = \{s, sf\}$. Clearly, the 3-regular $\mathrm{Cay}(D_{2i}, \{s_1^{\pm 1}, s_2^{\pm 1}\})$ is not isomorphic to the 2-regular multigraph $\mathrm{Cay}(D_{2i}, \{s_1^{\pm 1}, s_1s_2, s_2^{-1}s_1^{-1}\})$.

The next theorem is Theorem 4.4.2 in [16].

Theorem 5.6 ([16]). *Let $\mathbb{P}$ be the set of all prime numbers, then $\{\mathrm{Cay}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2^{\pm 1}, S_3^{\pm 1}\});\ p \in \mathbb{P}\}$ is an expander family.*

As a consequence, we have the following.

Corollary 5.7. *Let $\mathbb{P}$ be the set of all prime numbers, then $\{\mathrm{Cay}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z});\ \{S_2^{\pm 1}, S_2S_3, S_3^{-1}S_2^{-1}\});\ p \in \mathbb{P}\}$ is an expander family.*

Corollary 5.8. *Let $\mathbb{P}$ be the set of all prime numbers, then $\{\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z});\ p \in \mathbb{P}\}$ is a $\mathbb{C}\mathrm{ay}$-expander family.*

Remark 5.9. The order of S_2 and S_2S_3 are 4 and 3 respectively. Let $L = \{S_2, S_2S_3\}$ and $W = \{S_2, S_2^2, S_2S_3\}$, then we see that $\max\{|L^*|, |W^*|\} \leq 7$. Using Corollaries 4.2 and 5.7, we deduce that $\{\mathrm{Cay}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), L^*\});\ p \in \mathbb{P}\}$ and $\{\mathrm{Cay}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), W^*\});\ p \in \mathbb{P}\}$ are all expander families. Now by Theorem 1.1, we are able to directly construct several expander families of G-graphs.

Thus, we can conclude the following.

Theorem 5.10. *Let $\mathbb{P}$ be the set of all prime numbers. Then the G-graphs families given by*

1. $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2, S_2S_3\});\ p \in \mathbb{P}\}$,
2. $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2, S_2^2, S_2S_3\});\ p \in \mathbb{P}\}$.

are expanders.

Remark 5.11.

1. Using Corollary 2.9, it is easy to check that the first expander family given in Theorem 5.10 is formed of simple graphs, while the second one is not. By Proposition 2.6, we also deduce that the multigraphs in both families are semiregular; in otherwords the above two expander families are irregular.

2. Each $\{\mathrm{Cay}(G_i, S_i^*);\ i \in \mathbb{N}^+\}$ expander family enables us to construct several expander families of G-graphs depending on the choice of S_i in S_i^* with the possibility that some of these families may be isomorphic. For example, the following expander families $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2, S_3^{-1}S_2^{-1}\});\ p \in \mathbb{P}\}$, $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2^{-1}, S_2S_3\});\ p \in \mathbb{P}\}$, and $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2^{-1}, S_3^{-1}S_2^{-1}\});\ p \in \mathbb{P}\}$ are all isomorphic to $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2, S_2S_3\});\ p \in \mathbb{P}\}$. Similarly, the expander families $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2, S_2^2, S_3^{-1}S_2^{-1}\});\ p \in \mathbb{P}\}$, $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2^{-1}, S_2^2, S_2S_3\});\ p \in \mathbb{P}\}$, and $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2^{-1}, S_2^2, S_3^{-1}S_2^{-1}\});\ p \in \mathbb{P}\}$ are all isomorphic to $\{\tilde{\Phi}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2, S_2^2, S_2S_3\});\ p \in \mathbb{P}\}$.

Table 2: Comparison of some graph invariants between $\mathrm{Cay}(G, S^*)$ and $\tilde{\Phi}(G, S)$ for $S = L$ and $S = W$.

	$\mathrm{Cay}(G, L^*)$	$\tilde{\Phi}(G, L)$
Order	$\lvert G\rvert$	$\sum_{s\in S} \frac{\lvert G\rvert}{o(s)} = \frac{7}{12}\lvert G\rvert$
Degree	5-regular multigraph	$d(u) = 4$ for all $u \in V_{S_2}$ $d(v) = 3$ for all $v \in V_{S_2S_3}$
Size	$\frac{5}{2}\lvert G\rvert$	$\lvert G\rvert$
	$\mathrm{Cay}(G, W^*)$	$\tilde{\Phi}(G, W)$
Order	$\lvert G\rvert$	$\frac{13}{12}\lvert G\rvert$
Degree	6-regular multigraph	$d(u) = 8$ for all $u \in V_{S_2}$ $d(v) = 6$ for all $v \in V_{S_2S_3}$ $d(w) = 4$ for all $w \in V_{S_2^2}$
Size	$3\lvert G\rvert$	$3\lvert G\rvert$

3. Let G be the projective special linear group, that is $G = \mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z})$. In Table 2, we compare the order, the degree, and the size of the following expander family of

Cayley graphs $\{\mathrm{Cay}(G, L^*), p \in \mathbb{P}\}$ (resp. $\{\mathrm{Cay}(G, W^*), p \in \mathbb{P}\}$) with their corresponding ones in the G-graphs family $\{\tilde{\Phi}(G, L);\ p \in \mathbb{P}\}$ (resp. $\{\tilde{\Phi}(G, W);\ p \in \mathbb{P}\}$) (see Theorem 5.10). Form the preceding table, it is easy to see that the infinite expander family of G-graphs $\tilde{\Phi}(G, L)$ is sparser than the original expander family of the 4-regular graphs $\mathrm{Cay}(G, L^{\pm 1})$ and the 5-regular graphs $\mathrm{Cay}(G, L^*)$. The same can be said concerning the infinite expander family of G-graphs $\tilde{\Phi}(G, W)$ and the Cayley graph one $\mathrm{Cay}(G, W^*)$.

We close this section by the following corollary which can be easily obtained by using Theorem 5.10 and Corollary 3.6.

Corollary 5.12. *Let $\mathbb{P}$ be the set of all prime numbers. Then the family of G-graphs given by*

$$\{\tilde{\tilde{\Phi}}(\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \{S_2, S_2^2, S_2S_3\});\ p \in \mathbb{P}\}$$

is an expander family.

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The Hosoya polynomial of double weighted graphs*

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Abstract

The modified Hosoya polynomial of double weighted graphs, i.e. edge and vertex weighted graphs, is introduced that enables derivation of closed expressions for Hosoya polynomial of some special graphs including unicyclic graphs. Furthermore, the Hosoya polynomial is given as a sum of edge contributions generalizing well known analogous results for the Wiener number. A linear algorithm for computing the Hosoya polynomial on cactus graphs is provided. Hosoya polynomial is extensively studied in chemical graph theory, and in particular its weighted versions have interesting applications in theory of communication networks.

Keywords: Wiener number, Hosoya polynomial, Wiener polynomial, edge contributions, communication network, cactus graph, linear algorithm.

Math. Subj. Class.: 05C12, 92E10, 68R10

1 Introduction

The Hosoya polynomial was first studied by Hosoya [12], and later introduced independently under the name Wiener polynomial [20], perhaps because of its property that the first derivative of the polynomial evaluated at $x = 1$ equals the Wiener number. The name Hosoya-Wiener polynomial may be a good compromise [25, 26], however the majority of researchers nowadays use the term Hosoya polynomial. The main advantage of the Hosoya polynomial is that it contains a wealth of information about distance based invariants. Besides the above mentioned relation to the Wiener number, natural relations to other

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indices are known including the hyper-Wiener index [3] and Tratch-Stankovich-Zefirov index [1, 10]. The Hosoya polynomial has been investigated on many special classes of graphs, and vast literature includes some very recent studies, for example [4, 7, 14, 16].

The Hosoya polynomial generalizes nicely to the weighted graphs, as noted already in [25]. It seems that many properties of the polynomial remain valid in the weighted case, starting with the relations to the weighted Wiener number and hyper-Wiener index [25]. It may be interesting to note that the Wiener number, the polynomial and the corresponding generalizations also have natural applications in theory of communication networks, because the distance properties of a graph are of central importance there [6, 8, 18, 24]. In [26], a recursive formula for the Hosoya polynomial is derived yielding a linear time algorithm for computing the polynomial on trees. A generalization to cactus graphs was left as an open problem in [26]. The main motivation for this work was to design a linear algorithm for the Hosoya polynomial on double weighted cactus graphs. To achieve the main objective, we provide some auxiliary structural results that may be of independent interest, for example we show how the polynomial of a graph with cut edge can be expressed in terms of certain polynomials of the subgraphs (Lemma 4.1). Analogous results are given for graphs with cut vertex (Lemma 4.4) and for graphs generalizing unicyclic graphs (Lemma 5.1). As a related result that may be of independent interest, we show how the Hosoya polynomial can be expressed in terms of edge-contributions. Again, this may be of particular importance in theory of communications, as the edge contributions are under some natural assumptions directly related to communication load of edges (i.e. communication links). Finally, we outline a linear algorithm for computing the Hosoya polynomial on double weighted cactus graphs. More precisely, our algorithm is linear in the number of basic operations on polynomials i.e. addition and multiplication of polynomials.

The rest of the paper is organized as follows. In the next two sections some basic definitions including the definition of Hosoya polynomial are recalled. In Section 4, the Hosoya polynomials of some special graphs are calculated. In Section 5, cycle-like graphs are considered. For later use, the "modified Hosoya polynomial" of two variables is introduced. In Section 6, the Hosoya polynomial is expressed in terms of edge contributions. In Section 7, the algorithm for calculating the Hosoya polynomial is outlined in some details and its linear time complexity is shown.

2 Definitions

A *double weighted graph* $G = (V, E, w, \lambda)$ is a combinatorial object consisting of an arbitrary set $V = V(G)$ of *vertices*, a set $E = E(G)$ of unordered pairs $\{u, v\} = uv = e$ of distinct vertices of G called *edges*, and two *weighting functions*, w and λ. The weight function $w\colon V(G) \mapsto \mathbb{R}^+$ assigns positive real numbers (weights) to vertices and the distance function $\lambda\colon E(G) \mapsto \mathbb{R}^+$ assigns positive real numbers (lengths) to edges. The order and size of G are $n = |V(G)|$ and $m = |E(G)|$, respectively.

A *path* P between u and v is a sequence of distinct vertices $u = v_i, v_{i+1}, \ldots, v_{k-1}, v_k = v$ such that each pair $v_l v_{l+1}$ is connected by an edge. The *length* of the path P is the sum of the lengths of its edges,

$$\ell(P) = \sum_{l=i}^{k-1} \lambda(v_l v_{l+1}).$$

The *distance* $d_G(u, v)$, or simpler $d(u, v)$, between vertices u and v in graph G is the length of a shortest path between u and v. If there is no such path, we write $d(u,v) = \infty$. The *diameter* of a graph G is the maximal distance in G, $\mathcal{D}(G) = \max_{u,v\in V(G)} d_G(u,v)$.

A graph G is *connected* if $d(u,v) < \infty$ for any pair of vertices $u, v \in V(G)$. A vertex v is a *cut vertex* if after removing v and all edges incident to it the resulting graph is not connected. A graph without a cut vertex is called *nonseparable*. A *block* is a maximal nonseparable graph. Here, a *cycle* is an induced subgraph which is connected and every vertex is of degree two. A *cactus* is a graph in which every block of three or more vertices is a cycle. We also can say that cactus is a graph in which every edge is a part of at most one cycle.

The *weighted Wiener number* of a weighted connected graph G is defined as

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} w(u)w(v)d(u,v).$$

This is obviously a generalization of the usual definition of (unweighted) Wiener number, the sum of distances over all unordered pairs of vertices of G. The definition also generalizes the definition of the Wiener number for vertex-weighted graphs as used in [13]. Let us only mention that the Gutman index or the Schultz index of the second kind [9], where the weights of vertices are their degrees, is listed in [15] as an example of weighted versions of the Wiener index.

3 The Hosoya polynomial

A notion closely related to the Wiener number is the Hosoya polynomial of a graph G which is defined as

$$H(G,x) = \sum_{u,v\in V(G)} x^{d(u,v)}, \tag{3.1}$$

where the sum runs over all unordered pairs of vertices $u, v \in V(G)$. This definition, which is used for example in [11], slightly differs from the definition used by Hosoya [12]:

$$\hat{H}(G,x) = \sum_{\{u,v\}\subseteq V(G)} x^{d(u,v)}. \tag{3.2}$$

Obviously, $H(G,x) = \hat{H}(G,x) + |V(G)|$, because in (3.2) the sum runs over all unordered pairs $\{u,v\}$ of distinct vertices ($u,v \in V(G),\ u \neq v$), while in (3.1) u and v need not be distinct.

The Hosoya polynomial of a weighted graph G is defined as

$$\hat{H}(G,x) = \sum_{\{u,v\}\subseteq V(G)} w(u)w(v)\, x^{d(u,v)}, \tag{3.3}$$

where the sum runs over all unordered pairs $\{u,v\}$ of distinct vertices ($u,v \in V(G)$, $u \neq v$), as in definition (3.2). In the case when all weights of edges and vertices equal 1, we get the Hosoya polynomial as usually defined for unweighted graphs.

Remark 3.1. If G is a graph with one vertex and no edges, $G = \{v\}$, we define

$$\hat{H}(\{v\},x) := 0. \tag{3.4}$$

Note that $\hat{H}(G,x)$ may not be a polynomial if edge lengths are allowed to be arbitrary real numbers. Obviously, if positive integers are used for edge lengths, the function $\hat{H}(G,x)$ is a polynomial. Hence, with appropriate scaling factor, we can always consider $\hat{H}(G,x)$ to be a polynomial, for any model using rational edge lengths.

The Hosoya polynomial has many interesting properties [10, 12, 25], perhaps the most interesting of them is that its derivative at 1 equals the Wiener number.

For a connected graph G with more than one vertex, denote the **modified Hosoya polynomial** by

$$M(G,x) = \sum_{\{u,v\}\subseteq V(G)} d(u,v)w(u)w(v)x^{d(u,v)}. \tag{3.5}$$

Then clearly,

$$M(G,x) = x \cdot \frac{d}{dx}\hat{H}(G,x), \qquad \text{and} \qquad \hat{H}(G,x) = \int_0^x \frac{M(G,t)}{t}\,dt.$$

Later we will use the contributions of a vertex to the Hosoya polynomial. More precisely, we denote the contribution of all paths from a fixed vertex a to all vertices of some subgraph H of G (it is also possible $a \in H$) by

$$\hat{H}_a(H,x) = \sum_{v\in V(H)} w(a)w(v)\,x^{d(a,v)} = w(a)\sum_{v\in V(H)} w(v)\,x^{d(a,v)} \tag{3.6}$$

and

$$M_a(H,x) = \sum_{v\in V(H)} d(a,v)w(a)w(v)\,x^{d(a,v)} = w(a)\sum_{v\in V(H)} d(a,v)w(v)\,x^{d(a,v)}.$$

Obviously,

$$\hat{H}_a(\{a\},x) = w(a)^2, \qquad M_a(\{a\},x) = 0$$

and

$$\hat{H}(G,x) = \frac{1}{2}\sum_{a\in V(G)} \left(\hat{H}_a(G,x) - w(a)^2\right).$$

Remark 3.2. Note that $\hat{H}_a(G,x)$ may be regarded as a natural generalization of “partial Wiener polynomial” $H_a(G,x)$ used by Došlić [5] on unweighted graphs. More precisely,

$$H_a(G,x) = \sum_{v\in V(G),v\neq a} w(a)w(v)\,x^{d(a,v)} = \hat{H}_a(G,x) - w(a)^2\,. \tag{3.7}$$

4 Auxiliary results

Following the idea of [17], we calculate the Hosoya polynomial of some special examples of weighted graphs from the Hosoya polynomials of the given subgraphs. For later reference, auxiliary polynomials $\hat{H}_a(G,x)$, i.e. the contributions of all paths from fixed vertex a to all vertices of G, are also explicitly evaluated. Until further notice, the subgraphs considered are assumed to have at least one edge.

Lemma 4.1. *Let G_a and G_b be disjoint rooted graphs with roots a and b respectively, and let G be a disjoint union of G_a and G_b by the edge $e = ab$, see Figure 1. Then the Hosoya polynomial of G equals to*

$$\hat{H}(G,x) = \hat{H}(G_a,x) + \hat{H}(G_b,x) + \frac{1}{w(a)w(b)}\,\hat{H}_a(G_a,x)\,\hat{H}_b(G_b,x)\,x^{\lambda(e)}$$

and

$$\hat{H}_a(G,x) = \hat{H}_a(G_a,x) + \frac{w(a)}{w(b)}\,\hat{H}_b(G_b,x)\,x^{\lambda(e)}.$$

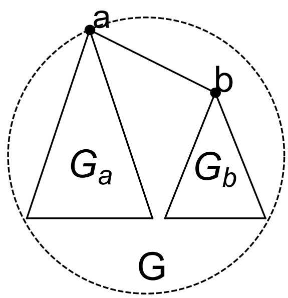

Figure 1: A graph with a bridge.

Proof.

$$\hat{H}(G,x) = \sum_{\{u,v\}\subseteq V(G)} w(u)w(v)x^{d(u,v)} = \sum_{\{u,v\}\subseteq V(G_a)} w(u)w(v)x^{d(u,v)} + \\ + \sum_{\{u,v\}\subseteq V(G_b)} w(u)w(v)x^{d(u,v)} + \sum_{\substack{u\in V(G_a)\\ v\in V(G_b)}} w(u)w(v)x^{d(u,v)}$$

It is obvious that first two sums equal to $\hat{H}(G_a,x)$ and $\hat{H}(G_b,x)$, respectively. Furthermore, observe that

$$\frac{1}{w(a)}\hat{H}_a(G_a,x) = \sum_{u\in V(G_a)} w(u)x^{d(u,a)},$$
$$\frac{1}{w(b)}\hat{H}_b(G_b,x) = \sum_{v\in V(G_b)} w(v)x^{d(b,v)},$$

and

$$\frac{1}{w(a)}\hat{H}_a(G_a,x)\cdot\frac{1}{w(b)}\hat{H}_b(G_b,x) = \sum_{\substack{u\in V(G_a)\\ v\in V(G_b)}} w(u)w(v)x^{d(u,a)+d(b,v)}.$$

Thus

$$x^{\lambda(e)} \cdot \frac{1}{w(a)} \hat{H}_a(G_a, x) \cdot \frac{1}{w(b)} \hat{H}_b(G_b, x) = \sum_{\substack{u \in V(G_a) \\ v \in V(G_b)}} w(u) w(v) x^{d(u,v)},$$

since

$$d(u, a) + \lambda(e) + d(b, v) = d(u, v).$$

Similarly,

$$\begin{aligned} \hat{H}_a(G, x) &= w(a) \sum_{v \in G} w(v) x^{d(a,v)} = \\ &= w(a) \sum_{v \in G_a} w(v) x^{d(a,v)} + w(a) \sum_{v \in G_b} w(v) x^{d(a,v)} = \\ &= \hat{H}_a(G_a, x) + w(a) \frac{1}{w(b)} \hat{H}_b(G_b, x)\, x^{\lambda(e)}. \end{aligned}$$

□

Example 4.2. Let $G_b = \{b\}$, $G_a = \{a\}$ and $G = \{a, b\} \cup ab$ (see Figure 2, left). From the definition (3.3) of the Hosoya polynomial it follows $\hat{H}(G, x) = w(a)w(b)x^{\lambda(ab)}$. On the other hand, using Lemma 4.1 we get (the initial values for vertices a and b are determined as in (3.4)):

$$\begin{aligned} \hat{H}(G, x) = \hat{H}(G_a, x) + \hat{H}(G_b, x) + \frac{1}{w(a)w(b)}\, \hat{H}_a(G_a, x)\, \hat{H}_b(G_b, x)\, x^{\lambda(ab)} = \\ = 0 + 0 + \frac{1}{w(a)w(b)}\, w(a)^2 w(b)^2 x^{\lambda(ab)}. \end{aligned}$$

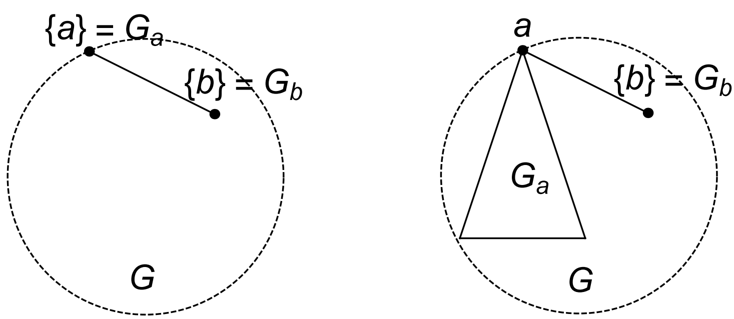

Figure 2: A graph G with two vertices (left), a rooted graph G with a bridge (right).

Example 4.3. Let $G_b = \{b\}$ and G_a be an arbitrary rooted graph, such that $b \notin V(G_a)$. Let G be the rooted graph $G = G_a \cup \{b\} \cup ab$ (see Figure 2, right). Using Lemma 4.1 and equation (3.4)

$$\begin{aligned}\hat{H}(G,x) &= \hat{H}(G_a,x) + \hat{H}(G_b,x) + \frac{1}{w(a)w(b)}\,\hat{H}_a(G_a,x)\,\hat{H}_b(G_b,x)\,x^{\lambda(ab)} = \\ &= \hat{H}(G_a,x) + 0 + \frac{1}{w(a)w(b)}\,\hat{H}_a(G_a,x)\,w(b)^2\,x^{\lambda(ab)} = \\ &= \hat{H}(G_a,x) + w(a)w(b)\,x^{\lambda(ab)} + w(b)\sum_{\substack{v\in V(G_a)\\ v\neq a}} w(v)x^{d(v,a)+\lambda(ab)}.\end{aligned}$$

Lemma 4.4. *Let G_1 and G_2 be graphs with one common vertex a and let $G_1 - a$ and $G_2 - a$ be disjoint. If $G = G_1 \cup G_2$ (see Figure 3), the Hosoya polynomial of the graph G equals to*

$$\begin{aligned}\hat{H}(G,x) = \hat{H}(G_1,x) + \hat{H}(G_2,x) + \\ + \frac{1}{w(a)^2}\left(\hat{H}_a(G_1,x) - w(a)^2\right)\left(\hat{H}_a(G_2,x) - w(a)^2\right)\end{aligned}$$

and

$$\hat{H}_a(G,x) = \hat{H}_a(G_1,x) + \hat{H}_a(G_2,x) - w(a)^2.$$

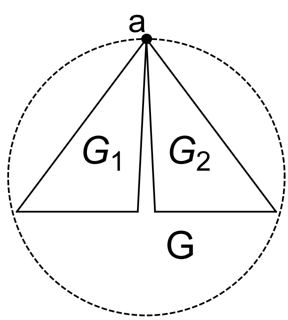

Figure 3: Two graphs with a common vertex.

Proof. Similarly to the proof of Lemma 4.1, we have

$$\begin{aligned}\hat{H}(G,x) &= \sum_{\{u,v\}\subseteq V(G)} w(u)w(v)x^{d(u,v)} = \\ &= \sum_{\{u,v\}\subseteq V(G_1)} w(u)w(v)x^{d(u,v)} + \sum_{\{u,v\}\subseteq V(G_2)} w(u)w(v)x^{d(u,v)} + \\ &+ \sum_{\substack{u\in V(G_1)\\ v\in V(G_2)}} w(u)w(v)x^{d(u,a)+d(a,v)} - \sum_{u\in V(G_1)} w(u)w(a)x^{d(u,a)} - \\ &- \sum_{v\in V(G_2)} w(a)w(v)x^{d(a,v)} + w(a)^2 =\end{aligned}$$

$$= \hat{H}(G_1, x) + \hat{H}(G_2, x) + \\ + \frac{\hat{H}_a(G_1, x)}{w(a)} \cdot \frac{\hat{H}_a(G_2, x)}{w(a)} - \hat{H}_a(G_1, x) - \hat{H}_a(G_2, x) + w(a)^2 = \\ = \hat{H}(G_1, x) + \hat{H}(G_2, x) + \frac{1}{w(a)^2}\Big(\hat{H}_a(G_1, x) - w(a)^2\Big)\Big(\hat{H}_a(G_2, x) - w(a)^2\Big).$$

We used the fact that, because G_1 and G_2 share the vertex a,

$$\sum_{\substack{u \in V(G_1) \\ v \in V(G_2)}} = \sum_{\substack{u \in V(G_1) \\ v \in V(G_2) \\ u \neq v}} + \sum_{u=v=a} .$$

The second equation holds because

$$\hat{H}_a(G, x) = \sum_{v \in V(G)} w(a)w(v)x^{d(a,v)} = \\ = \sum_{v \in V(G_1)} w(a)w(v)x^{d(a,v)} + \sum_{v \in V(G_2)} w(a)w(v)x^{d(a,v)} - w(a)^2 = \\ = \hat{H}_a(G_1, x) + \hat{H}_a(G_2, x) - w(a)^2.$$

□

Remark 4.5. The statement of Lemma 4.4 is a generalization of Theorem 2.1 from [4] for the case when two double weighted connected graphs G_1 and G_2 are point-attached to obtain G. However, it is easy to see that Lemma 4.4 can be generalized to the general case with any finite number of graphs. As the proof is short, we write and prove the following theorem for completeness of presentation.

Theorem 4.6. *Let G_i be graphs with one common vertex a and let $G_i - a$ be disjoint. If $G = \bigcup_{i=1}^{n} G_i$, the Hosoya polynomial of the graph G equals to*

$$\hat{H}(G, x) = \sum_{i=1}^{n} \hat{H}(G_i, x) + \\ + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{w(a)^2}\Big(\hat{H}_a(G_i, x) - w(a)^2\Big)\Big(\hat{H}_a(G_j, x) - w(a)^2\Big) \quad (4.1)$$

and

$$\hat{H}_a(G, x) = \sum_{i=1}^{n} \hat{H}_a(G_i, x) - (n-1)w(a)^2.$$

Proof. For $n = 2$ the result follows from Lemma 4.4, and for arbitrary n the result follows by induction. Suppose the equation (4.1) is valid for $n - 1$, let $G_0 = \bigcup_{i=1}^{n-1} G_i$ and $G =$

$G_0 \cup G_n$. Then, using Lemma 4.4,

$$\begin{aligned}
\hat{H}(G,x) &= \hat{H}(G_0,x) + \hat{H}(G_n,x) + \\
&\quad + \frac{1}{w(a)^2}\Big(\hat{H}_a(G_0,x) - w(a)^2\Big)\Big(\hat{H}_a(G_n,x) - w(a)^2\Big) = \\
&= \sum_{i=1}^{n-1} \hat{H}(G_i,x) + \hat{H}(G_n,x) + \\
&\quad + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} \frac{1}{w(a)^2}\Big(\hat{H}_a(G_i,x) - w(a)^2\Big)\Big(\hat{H}_a(G_j,x) - w(a)^2\Big) + \\
&\quad + \frac{1}{w(a)^2}\Big(\Big(\sum_{i=1}^{n-1} \hat{H}_a(G_i,x)\Big) - (n-2)w(a)^2 - w(a)^2\Big)\Big(\hat{H}_a(G_n,x) - w(a)^2\Big) = \\
&= \sum_{i=1}^{n} \hat{H}(G_i,x) + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} \frac{1}{w(a)^2}\Big(\hat{H}_a(G_i,x) - w(a)^2\Big)\Big(\hat{H}_a(G_j,x) - w(a)^2\Big) + \\
&\quad + \sum_{i=1}^{n-1} \frac{1}{w(a)^2}\Big(\hat{H}_a(G_i,x) - w(a)^2\Big)\Big(\hat{H}_a(G_n,x) - w(a)^2\Big) = \\
&= \sum_{i=1}^{n} \hat{H}(G_i,x) + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{1}{w(a)^2}\Big(\hat{H}_a(G_i,x) - w(a)^2\Big)\Big(\hat{H}_a(G_j,x) - w(a)^2\Big)
\end{aligned}$$

and

$$\begin{aligned}
\hat{H}_a(G,x) &= \hat{H}_a(G_0,x) + \hat{H}_a(G_n,x) - w(a)^2 = \\
&= \sum_{i=1}^{n-1} \hat{H}_a(G_i,x) - (n-2)w(a)^2 + \hat{H}_a(G_n,x) - w(a)^2 = \\
&= \sum_{i=1}^{n} \hat{H}_a(G_i,x) - (n-1)w(a)^2 . \qquad \square
\end{aligned}$$

5 Cycle-like and unicyclic graphs

We now consider the case when the specific vertices $a_1, a_2, \ldots, a_n$ in G are vertices of a cycle.

Lemma 5.1. *Let $G_{a_1}, G_{a_2}, \ldots, G_{a_n}$ be disjoint rooted graphs and denote by G^C the union of $G_{a_1}, G_{a_2}, \ldots, G_{a_n}$, joined by the edges $a_1a_2, a_2a_3, \ldots, a_{n-1}a_n$ and a_na_1, see Figure 4. In this case the Hosoya polynomial of G^C equals to*

$$\begin{aligned}
\hat{H}(G^C,x) &= \sum_{i=1}^{n} \hat{H}(G_{a_i},x) + \\
&\quad + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{1}{w(a_i)w(a_j)} \hat{H}_{a_i}(G_{a_i},x) \cdot \hat{H}_{a_j}(G_{a_j},x) \cdot x^{d(a_i,a_j)} \qquad (5.1)
\end{aligned}$$

and

$$\hat{H}_{a_i}(G^C, x) = \hat{H}_{a_i}(G_{a_i}, x) + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{w(a_i)}{w(a_j)} \hat{H}_{a_j}(G_{a_j}, x) \cdot x^{d(a_i, a_j)}$$

for every $i = 1, 2, \dots, n$.

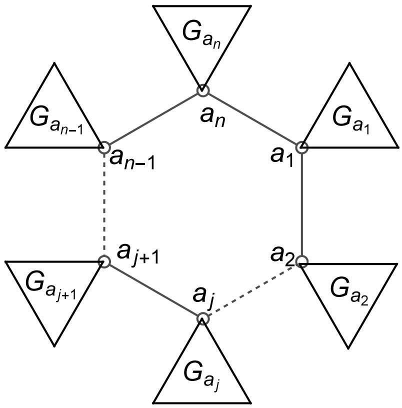

Figure 4: A cycle-like graph G^C.

Remark 5.2. In Lemma 5.1, the graph G^C can be any connected graph with a cycle such that all vertices $a_1, a_2, \dots, a_n$ of the cycle C are cut-vertices, and the G_{a_i} can be any subgraphs. Clearly, this includes as a special case the unicyclic graphs.

Proof of Lemma 5.1. Following the idea of the proof of Lemma 4.1, we write

$$\hat{H}(G^C, x) = \sum_{\{u,v\}\subseteq V(G)} w(u)w(v)x^{d(u,v)} =$$

$$= \sum_{i=1}^{n} \sum_{\{u,v\}\subseteq V(G_{a_i})} w(u)w(v)x^{d(u,v)} + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{\substack{u\in V(G_{a_i}) \\ v\in V(G_{a_j})}} w(u)w(v)x^{d(u,v)} =$$

$$= \sum_{i=1}^{n} \hat{H}(G_{a_i}, x) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{w(a_i)} \hat{H}_{a_i}(G_{a_i}, x) \cdot x^{d(a_i, a_j)} \cdot \frac{1}{w(a_j)} \hat{H}_{a_j}(G_{a_j}, x)$$

and

$$\hat{H}_{a_i}(G^C, x) = \sum_{v\in V(G)} w(a_i)w(v)x^{d(a_i,v)} =$$

$$= \sum_{v\in V(G_{a_i})} w(a_i)w(v)x^{d(a_i,v)} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \sum_{v\in V(G_{a_j})} w(a_i)w(v)x^{d(a_i,a_j)+d(a_j,v)} =$$

$$= \hat{H}_{a_i}(G_{a_i}, x) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{w(a_i)}{w(a_j)} \hat{H}_{a_j}(G_{a_j}, x) \cdot x^{d(a_i, a_j)},$$

as claimed in Lemma 4.1. □

Example 5.3. Let C be a cycle on three vertices a, b and c, with edges ab, bc and ca (see Figure 5). From Lemma 5.1 it follows

$$\begin{aligned}\hat{H}(C, x) &= \hat{H}(G_a, x) + \hat{H}(G_b, x) + \hat{H}(G_c, x) + \\ &\quad + \frac{1}{w(a)w(b)} \hat{H}_a(G_a, x) \hat{H}_b(G_b, x) x^{\lambda(ab)} + \\ &\quad + \frac{1}{w(b)w(c)} \hat{H}_b(G_b, x) \hat{H}_c(G_c, x) x^{\lambda(bc)} + \frac{1}{w(c)w(a)} \hat{H}_c(G_c, x) \hat{H}_a(G_a, x) x^{\lambda(ca)} = \\ &= w(a)w(b)x^{\lambda(ab)} + w(b)w(c)x^{\lambda(bc)} + w(c)w(a)x^{\lambda(ca)}.\end{aligned}$$

The result is the same as expected, from the definition (3.3). A similar reasoning applies to larger cycles.

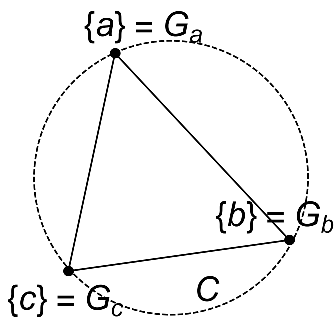

Figure 5: A cycle graph C.

Recall that our original motivation was to design a linear algorithm for calculating the Hosoya polynomial of a cactus graph. Observe that from equation (5.1) it appears that a double sum needs to be calculated which yields quadratic complexity. Therefore, we are going to consider this case more carefully and provide an alternative expression that will later be used to show the existence of a linear algorithm.

First, we will consider path-like graphs, and introduce, for technical reasons, polynomials of two variables that will in turn allow a natural generalization to handle cycle-like graphs.

Let G^P be a path-like graph, i.e. the union of disjoint graphs $G_{a_1}, G_{a_2}, \ldots, G_{a_n}$, rooted at $a_1, a_2, \ldots, a_n$ respectively, and joined by the edges $a_1a_2, a_2a_3, \ldots, a_{n-1}a_n$. The Hosoya polynomial $\hat{H}(G^P, x)$ and the polynomial $\hat{H}_{a_n}(G^P, x)$ can be calculated recursively using Lemma 4.1 $(n-1)$ times. For such a path-like graph G^P, we use the notation

$$H_1 = G_{a_1},$$
$$H_j = \bigcup_{i=1}^{j} G_{a_i} \cup \{a_1a_2, a_2a_3, \dots, a_{j-1}a_j\},$$

where $G^P = H_n$. By Lemma 4.1, the Hosoya polynomial and the corresponding polynomials $\hat{H}_{a_j}$ are, for $j = 1$:

$$\hat{H}(H_1, x) = \hat{H}(G_{a_1}, x),$$
$$\hat{H}_{a_1}(H_1, x) = \hat{H}_{a_1}(G_{a_1}, x),$$

and, for $j > 1$:

$$\hat{H}(H_j, x) = \hat{H}(G_{a_j}, x) + \hat{H}(H_{j-1}, x) + \\ + \frac{1}{w(a_j)w(a_{j-1})}\hat{H}_{a_j}(G_{a_j}, x)\hat{H}_{a_{j-1}}(H_{j-1}, x)x^{\lambda(a_{j-1}a_j)},$$
$$\hat{H}_{a_j}(H_j, x) = \hat{H}_{a_j}(G_{a_j}, x) + \frac{w(a_j)}{w(a_{j-1})}\hat{H}_{a_{j-1}}(H_{j-1}, x)x^{\lambda(a_{j-1}a_j)}.$$

The recursion above implies that, given polynomials $\hat{H}(G_{a_i}, x)$ and $\hat{H}_{a_i}(G_{a_i}, x)$, $i = 1, 2, \dots, n$, we need $3(n-1)$ additions and $2(n-1)$ multiplications (of polynomials) to obtain all $\hat{H}(H_i, x)$ and $\hat{H}_{a_i}(H_i, x)$.

From the definition of the graphs H_i and the recursions written above, we also have

Lemma 5.4. *For the graphs H_j, $j = 2, \dots, n$, the following is true*

$$\hat{H}(H_j, x) = \sum_{i=1}^{j} \hat{H}(G_{a_i}, x) + \\ + \sum_{i=1}^{j-1} \sum_{\ell=i+1}^{j} \frac{1}{w(a_i)w(a_\ell)}\hat{H}_{a_i}(G_{a_i}, x)\hat{H}_{a_\ell}(G_{a_\ell}, x)x^{\sum_{k=i}^{\ell-1}\lambda(a_ka_{k+1})},$$
$$\hat{H}_{a_j}(H_j, x) = \hat{H}_{a_j}(G_{a_j}, x) + \sum_{i=1}^{j-1} \frac{w(a_j)}{w(a_i)}\hat{H}_{a_i}(G_{a_i}, x)x^{\sum_{k=i}^{j-1}\lambda(a_ka_{k+1})}.$$

Proof. Lemma follows directly by the induction on j, using Lemma 4.1 and the recursive formulae above. □

Before generalizing from path-like to cycle-like graphs, we introduce auxiliary polynomials of two variables. For technical reasons, to distinguish the exponents based on the distance on the path and off the path, i.e. the exponents based on the distance within the graphs G_{a_i}, we introduce a second variable y. For example, assume that a shortest path from $u \in V(G_{a_i})$ to $v \in V(G_{a_j})$ has distance $d(u, v) = d(u, a_i) + d(a_i, a_j) + d(a_j, v)$. Then the contribution to the auxiliary polynomial is $w(u)w(v)x^{d(u,a_i)}y^{d(a_i,a_j)}x^{d(a_j,v)}$.

More formally,

$$\hat{H}(H_j,x,y) \coloneqq \sum_{i=1}^{j} \hat{H}(G_{a_i},x) + \tag{5.2}$$

$$+ \sum_{i=1}^{j-1}\sum_{\ell=i+1}^{j} \frac{1}{w(a_i)w(a_\ell)} \hat{H}_{a_i}(G_{a_i},x)\hat{H}_{a_\ell}(G_{a_\ell},x) y^{\sum_{k=i}^{\ell-1}\lambda(a_k a_{k+1})},$$

$$\hat{H}_{a_j}(H_j,x,y) \coloneqq \hat{H}_{a_j}(G_{a_j},x) + \sum_{i=1}^{j-1} \frac{w(a_j)}{w(a_i)} \hat{H}_{a_i}(G_{a_i},x) y^{\sum_{k=i}^{j-1}\lambda(a_k a_{k+1})}.$$

After the introduction of the new variable y, the recursion formulae become

$$\hat{H}(H_1,x,y) = \hat{H}(G_{a_1},x), \tag{5.3}$$

$$\hat{H}_{a_1}(H_1,x,y) = \hat{H}_{a_1}(G_{a_1},x) \tag{5.4}$$

for $j = 1$ and at every step of the recursion we have

$$\hat{H}(H_j,x,y) = \hat{H}(G_{a_j},x) + \hat{H}(H_{j-1},x) + \tag{5.5}$$

$$+ \frac{1}{w(a_j)w(a_{j-1})} \hat{H}_{a_j}(G_{a_j},x)\hat{H}_{a_{j-1}}(H_{j-1},x) y^{\lambda(a_{j-1}a_j)},$$

$$\hat{H}_{a_j}(H_j,x,y) = \hat{H}_{a_j}(G_{a_j},x) + \frac{w(a_j)}{w(a_{j-1})} \hat{H}_{a_{j-1}}(H_{j-1},x) y^{\lambda(a_{j-1}a_j)}. \tag{5.6}$$

It is obvious that

$$\hat{H}(H_j,x,x) = \hat{H}(H_j,x) \quad \text{and} \quad \hat{H}_{a_j}(H_j,x,x) = \hat{H}_{a_j}(H_j,x).$$

Let G^C be a cycle-like graph, i.e. the union of disjoint graphs $G_{a_1}, G_{a_2}, \ldots, G_{a_n}$, rooted at $a_1, a_2, \ldots, a_n$ respectively, and joined by the edges $a_1a_2, a_2a_3, \ldots, a_{n-1}a_n$ and a_na_1. Denote by L the girth of the cycle C on vertices $a_1, a_2, \ldots, a_n$, specifically

$$L = \lambda(a_na_1) + \sum_{i=1}^{n-1} \lambda(a_ia_{i+1}).$$

Define new modified polynomials of two variables of the path-like graph G^P as follows

$$\hat{H}^m(G^P,x,y) \coloneqq \sum_{i=1}^{n} \hat{H}(G_{a_i},x) +$$

$$+ \sum_{i=1}^{n-1}\sum_{\ell=i+1}^{n} \frac{1}{w(a_i)w(a_\ell)} \hat{H}(G_{a_i},x)\hat{H}(G_{a_\ell},x) y^{\min\{\sum_{k=i}^{\ell-1}\lambda(a_ka_{k+1}),\, L-\sum_{k=i}^{\ell-1}\lambda(a_ka_{k+1})\}},$$

$$\hat{H}^m_{a_n}(G^P,x,y) \coloneqq \hat{H}_{a_n}(G_{a_n},x) +$$

$$+ \sum_{i=1}^{n-1} \frac{w(a_n)}{w(a_i)} \hat{H}_{a_i}(G_{a_i},x) y^{\min\{\sum_{k=i}^{n-1}\lambda(a_ka_{k+1}),\, L-\sum_{k=i}^{n-1}\lambda(a_ka_{k+1})\}}.$$

According to Lemma 5.1, the next statement is obvious.

Proposition 5.5.

$$\hat{H}^m(G^P, x, x) = \hat{H}(G^C, x) \quad \textit{and} \quad \hat{H}^m_{a_n}(G^P, x, x) = \hat{H}_{a_n}(G^C, x)\,.$$

Example 5.6. Let G^C be a cycle-like graph which is the union of disjoint graphs G_{a_1}, G_{a_2}, G_{a_3} and G_{a_4} and edges a_1a_2, a_2a_3, a_3a_4 and a_4a_1. We assume that the polynomials $\hat{H}_{a_i}(G_{a_i}, x)$ and $\hat{H}(G_{a_i}, x)$ are given and that $\lambda(a_1a_2) = 2$, $\lambda(a_2a_3) = 5$, $\lambda(a_3a_4) = 3$ and $\lambda(a_4a_1) = 1$ as we see in Figure 6. In this case $L = 11$. The computations below

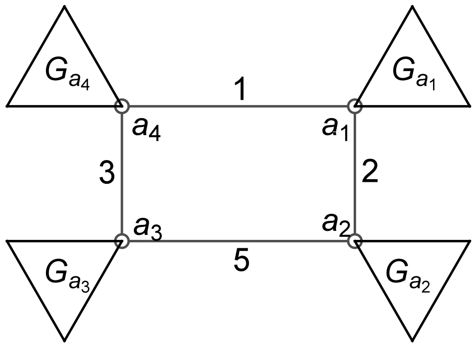

Figure 6: A cycle-like graph with given lengths of cycle's edges.

are following the recursion for the path-like graph and the idea of the separation of the exponents, and, in addition, we observe that the distances on the path change when the path is closed to a cycle with the edge a_4a_1. The base of the recursion is

$$\begin{aligned}\hat{H}_{a_1}(H_1, x, y) &= \hat{H}_{a_1}(G_{a_1}, x),\\ \hat{H}(H_1, x, y) &= \hat{H}(G_{a_1}, x)\,.\end{aligned}$$

Other steps are clearly

$$\begin{aligned}\hat{H}_{a_2}(H_2, x, y) &= \hat{H}_{a_2}(G_{a_2}, x) + \frac{w(a_2)}{w(a_1)}\hat{H}_{a_1}(H_1, x, y)y^2 = \\ &= \hat{H}_{a_2}(G_{a_2}, x) + \frac{w(a_2)}{w(a_1)}\hat{H}_{a_1}(G_{a_1}, x)y^2,\\ \hat{H}(H_2, x, y) &= \hat{H}(G_{a_2}, x) + \hat{H}(H_1, x, y) + \\ &\qquad + \frac{1}{w(a_2)w(a_1)}\hat{H}_{a_2}(G_{a_2}, x)\hat{H}_{a_1}(H_1, x, y)y^2 = \\ &= \hat{H}(G_{a_2}, x) + \hat{H}(G_{a_1}, x) + \frac{1}{w(a_2)w(a_1)}\hat{H}_{a_2}(G_{a_2}, x)\hat{H}_{a_1}(G_{a_1}, x)y^2,\\ \hat{H}_{a_3}(H_3, x, y) &= \hat{H}_{a_3}(G_{a_3}, x) + \frac{w(a_3)}{w(a_2)}\hat{H}_{a_2}(h_2, x, y)y^5 = \\ &= \hat{H}_{a_3}(G_{a_3}, x) + \frac{w(a_3)}{w(a_2)}\hat{H}_{a_2}(G_{a_2}, x)y^5 + \frac{w(a_3)}{w(a_1)}\hat{H}_{a_1}(G_{a_1}, x)y^7,\end{aligned}$$

$$\begin{aligned}\hat{H}(H_3,x,y) &= \hat{H}(G_{a_3},x)+\hat{H}(H_2,x,y)+\\ &\quad+\frac{1}{w(a_3)w(a_2)}\hat{H}_{a_3}(G_{a_3},x)\hat{H}_{a_2}(H_2,x,y)y^5=\\ &= \hat{H}(G_{a_3},x)+\hat{H}(G_{a_2},x)+\hat{H}(G_{a_1},x)+\\ &\quad+\frac{1}{w(a_2)w(a_1)}\hat{H}_{a_1}(G_{a_1},x)\hat{H}_{a_2}(G_{a_2},x)y^2+\\ &\quad+\frac{1}{w(a_3)w(a_2)}\hat{H}_{a_3}(G_{a_3},x)\hat{H}_{a_2}(G_{a_2},x)y^5+\\ &\quad+\frac{1}{w(a_3)w(a_1)}\hat{H}_{a_3}(G_{a_3},x)\hat{H}_{a_1}(G_{a_1},x)y^7\,.\end{aligned}$$

Similarly,

$$\begin{aligned}\hat{H}_{a_4}(H_4,x,y) &= \hat{H}_{a_4}(G_{a_4},x)+\frac{w(a_4)}{w(a_3)}\hat{H}_{a_3}(H_3,x,y)y^3=\\ &= \hat{H}_{a_4}(G_{a_4},x)+\frac{w(a_4)}{w(a_3)}\hat{H}_{a_3}(G_{a_3},x)y^3+\\ &\quad+\frac{w(a_4)}{w(a_2)}\hat{H}_{a_2}(G_{a_2},x)y^8+\frac{w(a_4)}{w(a_1)}\hat{H}_{a_1}(G_{a_1},x)y^{10}\end{aligned}$$

and

$$\begin{aligned}\hat{H}(H_4,x,y) &= \hat{H}(G_{a_4},x)+\hat{H}(H_3,x,y)+\\ &\quad+\frac{1}{w(a_4)w(a_3)}\hat{H}_{a_4}(G_{a_4},x)\hat{H}_{a_3}(H_3,x,y)y^3=\\ &= \hat{H}(G_{a_4},x)+\hat{H}(G_{a_3},x)+\hat{H}(G_{a_2},x)+\hat{H}(G_{a_1},x)+\\ &\quad+\frac{1}{w(a_2)w(a_1)}\hat{H}_{a_1}(G_{a_1},x)\hat{H}_{a_2}(G_{a_2},x)y^2+\\ &\quad+\frac{1}{w(a_3)w(a_2)}\hat{H}_{a_3}(G_{a_3},x)\hat{H}_{a_2}(G_{a_2},x)y^5+\\ &\quad+\frac{1}{w(a_3)w(a_1)}\hat{H}_{a_3}(G_{a_3},x)\hat{H}_{a_1}(G_{a_1},x)y^7+\\ &\quad+\frac{1}{w(a_4)w(a_3)}\hat{H}_{a_4}(G_{a_4},x)\hat{H}_{a_3}(G_{a_3},x)y^3+\\ &\quad+\frac{1}{w(a_4)w(a_2)}\hat{H}_{a_4}(G_{a_4},x)\hat{H}_{a_2}(G_{a_2},x)y^8+\\ &\quad+\frac{1}{w(a_4)w(a_1)}\hat{H}_{a_4}(G_{a_4},x)\hat{H}_{a_1}(G_{a_1},x)y^{10}\,.\end{aligned}$$

Than the modified polynomials of two variables are

$$\begin{aligned}\hat{H}^m_{a_4}(H_4,x,y) &= \hat{H}_{a_4}(G_{a_4},x)+\frac{w(a_4)}{w(a_3)}\hat{H}_{a_3}(G_{a_3},x)y^3+\\ &\quad+\frac{w(a_4)}{w(a_2)}\hat{H}_{a_2}(G_{a_2},x)y^3+\frac{w(a_4)}{w(a_1)}\hat{H}_{a_1}(G_{a_1},x)y=\end{aligned}$$

$$= \hat{H}_{a_4}(G_{a_4}, x) + \frac{w(a_4)}{w(a_1)}\hat{H}_{a_1}(G_{a_1}, x)y + \\ + \Big(\frac{w(a_4)}{w(a_3)}\hat{H}_{a_3}(G_{a_3}, x) + \frac{w(a_4)}{w(a_2)}\hat{H}_{a_2}(G_{a_2}, x)\Big)y^3$$

and

$$\begin{aligned}\hat{H}^m(H_4, x, y) = \hat{H}(G_{a_4}, x) + \hat{H}(G_{a_3}, x) + \hat{H}(G_{a_2}, x) + \hat{H}(G_{a_1}, x) + \\ + \frac{1}{w(a_2)w(a_1)}\hat{H}_{a_1}(G_{a_1}, x)\hat{H}_{a_2}(G_{a_2}, x)y^2 + \\ + \frac{1}{w(a_3)w(a_2)}\hat{H}_{a_3}(G_{a_3}, x)\hat{H}_{a_2}(G_{a_2}, x)y^5 + \\ + \frac{1}{w(a_3)w(a_1)}\hat{H}_{a_3}(G_{a_3}, x)\hat{H}_{a_1}(G_{a_1}, x)y^4 + \\ + \frac{1}{w(a_4)w(a_1)}\hat{H}_{a_4}(G_{a_4}, x)\hat{H}_{a_1}(G_{a_1}, x)y^1 + \\ + \Bigg(\frac{1}{w(a_4)w(a_3)}\hat{H}_{a_4}(G_{a_4}, x)\hat{H}_{a_3}(G_{a_3}, x) + \\ + \frac{1}{w(a_4)w(a_2)}\hat{H}_{a_4}(G_{a_4}, x)\hat{H}_{a_2}(G_{a_2}, x)\Bigg)y^3 .\end{aligned}$$

Hence, the Hosoya polynomial and the polynomial $\hat{H}_{a_4}(G^C, x)$ are

$$\hat{H}(G^C, x) = \hat{H}^m(H_4, x, x) \qquad \text{and} \qquad \hat{H}_{a_4}(G^C, x) = \hat{H}^m_{a_4}(H_4, x, x) .$$

Observe that the time complexity of the transformation of a polynomial of two variables x and y to the polynomial of one variable x (where $y \to x$) is comparable to time complexity of multiplication of polynomials. Thus, we can conclude:

Theorem 5.7. *The Hosoya polynomial of a cycle-like graph can be computed using the recursion (5.3), (5.4), (5.5), (5.6) in linear time, in the model where addition and multiplication of polynomials are atomic operations.*

6 The Hosoya polynomial in terms of edge contributions

It is well-known that the Wiener number can be expressed as a sum of edge contributions, see for example [19]. Recall, for example, the version for weighted graphs.

Lemma 6.1 ([22]). *For a weighted graph G,*

$$W(G) = \sum_{e=uv} \lambda(e) \cdot \sum_{P^*_{a,b} \ni e} \frac{n^*(a,b,e)}{n^*(a,b)}\, w(a)w(b),$$

where $P^*_{a,b} \ni e$ denotes a shortest path between a and b, $n^*(a,b)$ is the number of shortest paths with endpoints a and b and $n^*(a,b,e)$ is the number of all shortest paths with endpoints a and b traversing edge e.

Hence, the quotient $\frac{n^*(a,b,e)}{n^*(a,b)}$ represents the proportion of all shortest paths between a in b including e, among all shortest paths between a and b. On a tree, there is a unique shortest path between any pair of vertices, thus $n^*(a,b,e) = n^*(a,b) = 1$ for all a, b. If G is a cactus graph, $\frac{n^*(a,b,e)}{n^*(a,b)}$ can only have value 1 or $\frac{1}{2}$. Clearly, $\frac{n^*(a,b,e)}{n^*(a,b)} = \frac{1}{2}$ exactly when a and b are opposite vertices of a cycle and edge e is on this cycle. (More precisely, with *opposite vertices* of a cycle we mean that $d(a,b) = L/2$ where L is the girth of the cycle.)

It may be interesting to note that this formulation has an interesting meaning when considering the weighted graphs as communication networks [6, 18, 24]. In this case the Wiener number is interpreted as the total communication traffic in the network, where naturally the communication between nodes u and v contributes $d(u,v)w(u)w(v)$ (distance times population sizes of the two nodes). Assuming that all the communication is routed on the shortest paths and that it is evenly distributed among shortest paths if there are many of them, the edge contribution corresponds to the communication load on the edge.

Example 6.2. Let G be a communication network represented in Figure 7, where all edges have lengths 1. There are exactly three shortest paths between vertices u and v. Ratios indicating the part of the communication load are attached to the edges on the shortest paths.

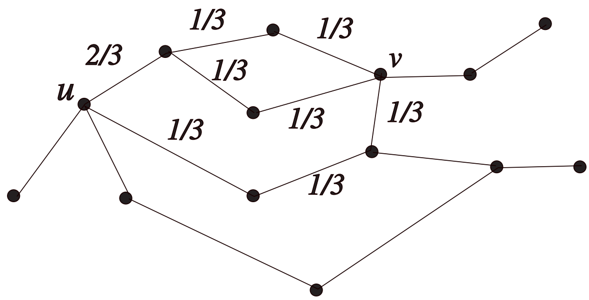

Figure 7: The quotients $\frac{n^*(u,v,e)}{n^*(u,v)}$ at edges on all shortest paths between vertices u and v.

As shown in [26], the Hosoya polynomial can be represented as a sum of the contributions of all shortest paths:

Lemma 6.3 ([26]). *For a weighted graph G,*

$$\hat{H}(G,x) = \sum_{\{a,b\} \subseteq V(G)} \sum_{P^*_{a,b}} \frac{1}{n^*(a,b)} w(a)w(b) \prod_{e \in P^*_{a,b}} x^{\lambda(e)},$$

where $P^*_{a,b}$ denotes a shortest path between a and b and $n^*(a,b)$ is the number of all shortest paths with endpoints a and b.

Representing the Hosoya polynomial in terms of edge contributions is hence somewhat more complicated: For each path crossing the edge, one needs to know the amount of traffic (the intensity of the traffic corresponds to $\frac{1}{n^*(a,b)} w(a)w(b)$) but also the length of the paths.

In case when G is a weighted tree, the Hosoya polynomial was expressed as a sum of edge contributions and a recursive formula for computing the Hosoya polynomial was given in [26].

In this section we show that the Hosoya polynomial can be similarly expressed as a sum of edge contributions on general graphs, and then provide a somewhat more elaborated expression that holds for cactus graphs.

Lemma 6.4. *The modified Hosoya polynomial, defined by (3.5), is a sum of edge contributions*

$$M(G,x) = \sum_{e\in E(G)} \lambda(e)\cdot M_e(G,x), \tag{6.1}$$

where $M_e(G,x)$ *is given by*

$$M_e(G,x) = \sum_{P^*_{u,v}\ni e} \frac{n^*(u,v,e)}{n^*(u,v)}\, w(u)w(v)x^{d(u,v)}. \tag{6.2}$$

Here $P^*_{u,v}\ni e$ *denotes a shortest path between* u *and* v *including* e*,* $n^*(u,v)$ *is the number of all shortest paths with endpoints* u *and* v *and* $n^*(u,v,e)$ *is the number of all shortest paths with endpoints* a *and* b *including edge* e*.*

Proof. To see this, it is enough to sum up the contribution of each edge to $M(G,x)$ in two different ways. Each pair of vertices u,v contributes $d(u,v)w(u)w(v)x^{d(u,v)}$ to the modified Hosoya polynomial. This can be regarded as a contribution of the pair u,v or it can be divided into $n^*(u,v,e)/n^*(u,v)$ path contributions, which can be further regarded as a sum of edge contributions along the path. An edge contributes as many times as it appears on various shortest paths. Hence, one can sum up the lengths of all shortest paths, or, equivalently, sum up the contributions of all edges. □

Let G be a cactus graph. Recall that each edge e of a cactus graph is on at most one cycle, in other words, either e is not on a cycle or there is a unique cycle C with $e\in C$. On the other hand, a vertex in a cactus graph can lie on more than one cycle.

In case when the edge $e=ab$ does not lie on a cycle, we can write our graph G as disjoint union of two graphs, denote them G_a and G_b, connected with edge $e=ab$ (defined in Lemma 4.1), see Figure 1.

On the other side, when edge e with endpoints a and b lies on a cycle C, we can use notations from Lemma 5.1, see Figure 4: e is one of the edges named a_ia_{i+1} with $a_i=a$ and $a_{i+1}=b$, $G_{a_i}=G_a$, $G_{a_{i+1}}=G_b$ for some $i\in\{1,2,\dots,n\}$. We can also say that G_a is the connected component of $G-E(C)$ with $a\in G-E(C)$, where $G-E(C)$ denotes the graph G without edges of the cycle C.

According to Lemma 4.1 and Lemma 5.1, we can derive the Hosoya polynomials $\hat{H}(G,x)$ and $M(G,x)$ for a cactus graph G as sums of edge contributions.

Theorem 6.5. *The modified Hosoya polynomial* $M(G,x)$ *on a weighted cactus graph* G *is a sum of edge contributions*

$$M(G,x) = \sum_{\substack{e=ab\in E(G)\\ e \text{ not on a cycle}}} \lambda(e)\frac{x^{\lambda(e)}}{w(a)w(b)}\left(\int_0^x \frac{M_a(G_a,t)}{t}\,dt\right)\cdot\left(\int_0^x \frac{M_b(G_b,t)}{t}\,dt\right) +$$

$$+\sum_{\substack{e=ab\in E(G)\\ e\text{ on a cycle}}}\lambda(e)\sum_{i=0}^{K}\sum_{j=0}^{M}\left(\frac{1}{2}\right)^{N_{ij}}\frac{x^{d(a_i,b_j)}}{w(a_i)w(b_j)}\left(\int_0^x\frac{M_{a_i}(G_{a_i},t)}{t}\,dt\right)\cdot\left(\int_0^x\frac{M_{b_j}(G_{b_j},t)}{t}\,dt\right).$$

In case when $e = ab$ is on a cycle C with girth L and vertices

$$b = b_0, b_1, b_2, \ldots, b_M, a_K, a_{K-1}, \ldots, a_1, a_0 = a,$$

we define $N_{ij} = \begin{cases} 1, & d(a_i, b_j) = L/2 \\ 0, & \textit{otherwise.} \end{cases}$

Proof. As every edge e of a cactus graph G does not lie on a cycle or there is unique cycle including e, we can discuss separately the two cases.

Case 1: The edge e with endpoints a and b does not lie on a cycle. Then $G = G_a \cup \{e\} \cup G_b$ (see Figure 1) and, obviously, $\frac{n^*(u,v,e)}{n^*(u,v)} = 1$ for all $u \in G_a$ and all $v \in G_b$.

Using the definition (3.6)

$$\hat{H}_a(G_a, x) \cdot \hat{H}_b(G_b, x) = w(a)w(b) \sum_{\substack{u\in V(G_a)\\ v\in V(G_b)}} w(u)w(v)x^{d(u,a)+d(b,v)}$$

and

$$\frac{\hat{H}_a(G_a, x) \cdot \hat{H}_b(G_b, x) \cdot x^{\lambda(e)}}{w(a)w(b)} = \sum_{\substack{u\in V(G_a)\\ v\in V(G_b)}} w(u)w(v)x^{d(u,v)},$$

since $d(u, a) + \lambda(e) + d(b, v) = d(u, v)$. So, the contribution (6.2) of the edge e in Case 1 is equal to

$$M_e(G, x) = \frac{\hat{H}_a(G_a, x) \cdot \hat{H}_b(G_b, x) \cdot x^{\lambda(e)}}{w(a)w(b)} = \\ = \frac{x^{\lambda(e)}}{w(a)w(b)}\left(\int_0^x\frac{M_a(G_a,t)}{t}\,dt\right)\cdot\left(\int_0^x\frac{M_b(G_b,t)}{t}\,dt\right).$$

Case 2: The edge e with endpoints a and b lies on a cycle C with girth L. Let $A = \{a = a_0, a_1, a_2, \ldots, a_K\}$ be the set of vertices of C that are closer to a than to b, i.e. $d(a, a_i) \leq d(b, a_i)$. $B = \{b = b_0, b_1, b_2, \ldots, b_M\}$ the set of vertices of C that are closer to b than to a, i.e. $d(b, b_i) < d(a, b_i)$.

Clearly, for a pair of vertices $a_i \in A$, $b_j \in B$ the edge e lies on the unique shortest path between them exactly when $d(a, b) = d(a, a_i) + \lambda(e) + d(b, b_j) < L/2$. Furthermore, e is on one of the two shortest paths exactly when $d(a, b) = d(a, a_i) + \lambda(e) + d(b, b_j) = L/2$ and is not on a shortest path between a_i and b_j when $d(a, b) < d(a, a_i) + \lambda(e) + d(b, b_j)$.

Denote

$$\hat{H}_{a_i}(G_{a_i}, x) = \sum_{u\in G_{a_i}} w(u)w(a_i)x^{d(u,a_i)} = w(a_i)\sum_{u\in G_{a_i}} w(u)x^{d(u,a_i)}, \qquad i = 0, 1, \ldots, K$$

$$\hat{H}_{b_j}(G_{b_j}, x) = \sum_{v \in G_{b_j}} w(b_j) w(v) x^{d(b_j,v)} = w(b_j) \sum_{v \in G_{b_j}} w(v) x^{d(b_j,v)}, \quad j = 0, 1, \dots, M.$$

Since

$$\frac{\hat{H}_{a_i}(G_{a_i}, x) \cdot \hat{H}_{b_j}(G_{b_j}, x) \cdot x^{d(a_i,b_j)}}{w(a_i) w(b_j)} = \sum_{\substack{u \in V(G_{a_i}) \\ v \in V(G_{b_j})}} w(u) w(v) x^{d(u,v)},$$

the contribution (6.2) of the edge e in Case 2 is equal to

$$M_e(G, x) = \sum_{i=0}^{K} \sum_{j=0}^{M} \left(\frac{1}{2}\right)^{N_{ij}} \frac{\hat{H}_{a_i}(G_{a_i}, x) \cdot \hat{H}_{b_j}(G_{b_j}, x) \cdot x^{d(a_i,b_j)}}{w(a_i) w(b_j)} =$$
$$= \sum_{i=0}^{K} \sum_{j=0}^{M} \left(\frac{1}{2}\right)^{N_{ij}} \frac{x^{d(a_i,b_j)}}{w(a_i) w(b_j)} \left(\int_0^x \frac{M_{a_i}(G_{a_i}, t)}{t} \, dt \right) \cdot \left(\int_0^x \frac{M_{b_j}(G_{b_j}, t)}{t} \, dt \right),$$

where

$$N_{ij} = \begin{cases} 1, & d(a_i, b_j) = L/2 \\ 0, & \text{otherwise.} \end{cases}$$

As we mentioned earlier, the case $\frac{n^*(u,v,e)}{n^*(u,v)} = \frac{1}{2}$ appears only when $u \in G_{a_i}$, $v \in G_{b_j}$ and a_i and b_j are opposite vertices of a cycle C, such that $d(a_i, b_j) = L/2$. In all other cases $\frac{n^*(u,v,e)}{n^*(u,v)} = 1$. □

7 Linear algorithm

In this section we give some details of the algorithm for computing Hosoya polynomial of a weighted cactus graph that is based on results provided in previous sections. Before writing the algorithm outline we recall the skeleton structure of a cactus graph and the depth first search algorithm. The algorithm and analysis of its time complexity are given in Subsection 7.2. The section is concluded with an example.

7.1 The structure of cactus graph and DFS algorithm

In the skeleton structure (elaborated for example in [2]) that corresponds to every cactus graph $G = (V(G), E(G))$, the vertices are of three types:

- C-vertex is a vertex on a cycle of degree 2,
- G-vertex is a vertex not included in any cycle,
- H-vertex or a hinge is a vertex which is included in at least one cycle and is of degree ≥ 3.

The depth first search (DFS) algorithm is a well known method for exploring graphs. It can be used for recognizing cactus graphs providing the data structure (see [17, 21, 22, 23]). Let G_r be a rooted cactus graph with a root r. After running the DFS algorithm, the vertices of G_r are DFS ordered. The order is given by the order in which DFS visits the vertices.

(Note that the DFS order of a graph is not unique as we can use any vertex as the starting vertex (the root) and can visit the neighbors of a vertex in any order. However, here we can assume that the DFS order is given and is fixed.)

For any vertex $v \in V(G)$ we denote by $\mathrm{DFN}(v)$ the position of v in the DFS order and we set $\mathrm{DFN}(r) = 0$. DFN is called the depth first number. Following [22], it is useful to store the information recorded during the DFS run in four arrays, called the DFS (cactus) data structure:

- $\mathrm{FATHER}(v)$ is the unique predecessor (father) of the vertex v in the rooted tree, constructed with the DFS.

- $\mathrm{ROOT}(v)$ is the root vertex of the cycle containing v i.e. the first vertex of the cycle (containing v) in the DFS order. If v does not lie on a cycle, then $\mathrm{ROOT}(v) = v$ by definition. We set $\mathrm{ROOT}(r) = r$. (In any DFS order, if $\mathrm{DFN}(w) < \mathrm{DFN}(v)$ and w is the root of the cycle containing v and v is the root of another cycle (it is a hinge), then $\mathrm{ROOT}(v) = w$.)

- For vertices on a cycle (i.e. $\mathrm{ROOT}(v) \neq v$), orientation of the cycle is given by $\mathrm{ORIEN}(v) = z$, where z is the son of $\mathrm{ROOT}(v)$ that is visited on the cycle first. If $\mathrm{ROOT}(v) = v$, then $\mathrm{ORIEN}(v) = v$.

- $\mathrm{IND}(v) := |\{u \mid \mathrm{FATHER}(u) = v\}|$ is the number of sons of v in the DFS tree.

We omit detailed description of DFS algorithm here, as it is well known, see for example [21]. The pseudocode of the DFS algorithm is also written in [17]. Some properties of the DFS ordered vertices of cactus and the relationship between the definitions of C, G, H-vertices in a rooted cactus G_r and arrays in the DFS table is described in [17].

In the rest of the paper the following notations are used. For a given cactus graph G and a vertex $v \in V(G)$ let G_v be the rooted induced subgraph of G with the root v. The set of vertices of G_v is the set $V(G_v) = \{w \in V(G) \mid \mathrm{DFN}(w) \geq \mathrm{DFN}(v)\}$. Let $u = \mathrm{FATHER}(v)$ and let the edge uv not be an edge of a cycle of G (i.e. $\mathrm{ROOT}(v) = v$). The graph $\tilde{G}_u$ is the induced rooted subgraph of G with the root u. The set of vertices of $\tilde{G}_u$ is the set $V(\tilde{G}_u) = \{w \in V(G) \mid \mathrm{DFN}(u) \leq \mathrm{DFN}(w) < \mathrm{DFN}(v)\}$. The sketch of rooted graphs G_v, $\tilde{G}_u$ and G_u is shown in Figure 8.

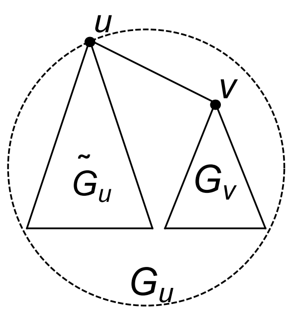

Figure 8: A rooted graph G_u.

7.2 The algorithm

The linear algorithm consists of three steps. First, the representation of a given weighted cactus is found, then, in Step 2 the initialization for the recursive algorithm is done and in the third step, the Hosoya polynomials of certain rooted subgraphs are computed recursively that finally give the Hosoya polynomial of the whole graph. More precisely, in Step 3 we traverse the DFS tree of the cactus in the reversed DFS order and for each vertex v compute $\hat{H}_v(G_v, x)$ and $\hat{H}(G_v, x)$. The algorithm continues until the last vertex in the back DFS order is considered, which is the root r of the cactus. The result follows from the fact that $\hat{H}(G, x) = \hat{H}(G_r, x)$.

We now give more details of each step.

Step 1: **Cactus recognition**

Using a DFS algorithm on the rooted cactus G (any vertex chosen for a root) the data structure of cactus graph can be derived, including arrays $\mathrm{DFN}(v)$, $\mathrm{FATHER}(v)$, $\mathrm{ROOT}(v)$, $\mathrm{ORIEN}(v)$ and $\mathrm{IND}(v)$.

Step 2: **Initialization**

For every vertex v we set

$$\hat{H}_v(G_v, x) = \hat{H}_v(\{v\}, x) = w(v)^2 \quad \text{and} \quad \hat{H}(G_v, x) = \hat{H}(\{v\}, x) = 0.$$

Step 3: **Computation of polynomials $\hat{H}$**

Start with v, the last vertex in the DFS order. Let $u = \mathrm{FATHER}(v)$.

While $v \neq u$ (i.e. $v = u$ is not the root of G) do (3a) or (3b):

(3a) If the edge $e = uv$ is not an edge of a cycle of G (i.e. $\mathrm{ROOT}(v) = v$):

- If $\mathrm{DFN}(u) \neq \mathrm{DFN}(v) - 1$ (i.e. $\mathrm{DFN}(u) < \mathrm{DFN}(v) - 1$), there exists rooted subgraph $\tilde{G}_u$ (see Figure 8). The algorithm calls itself recursively for the subgraph $\tilde{G}_u$, the rooted subcactus with root u and vertices in DFS table with $\mathrm{DFN}(u), \dots, \mathrm{DFN}(v) - 1$. We obtain $\hat{H}(G_u, x) = \hat{H}(\tilde{G}_u, x)$ and $\hat{H}_u(G_u, x) = \hat{H}_u(\tilde{G}_u, x)$.
- After the recursion or when u and v are the sequential vertices in the DFS order, polynomials $\hat{H}_u(G_u, x)$ and $\hat{H}(G_u, x)$ are calculated according to Lemma 4.1:

$$\hat{H}(G_u, x) = \hat{H}(G_u, x) + \hat{H}(G_v, x) + \tfrac{1}{w(u)w(v)} \hat{H}_u(G_u, x) \hat{H}_v(G_v, x) x^{\lambda(uv)}$$
$$\hat{H}_u(G_u, x) = \hat{H}_u(G_u, x) + \tfrac{w(u)}{w(v)} \hat{H}_v(G_v, x) x^{\lambda(uv)}.$$

- $v = u$ and $u = \mathrm{FATHER}(v)$.

(3b) If the edge $e = uv$ lies on a cycle C (i.e. $r = \mathrm{ROOT}(v) \neq v$):

- We have to read and remember all cycle's vertices. Denote them by $a_1, a_2, \dots,$ a_n where $a_1 = v$, $a_{n-1} = \mathrm{ORIEN}(v)$ and $a_n = r = \mathrm{ROOT}(v)$.
- If $\mathrm{DFN}(a_j) < \mathrm{DFN}(a_{j-1}) - 1$, denote by K_{a_j} the rooted subcacti on vertices with DFN: $\mathrm{DFN}(a_j) \leq \mathrm{DFN} < \mathrm{DFN}(a_{j-1})$ for $j = 2, 3, \dots, n-1$. Re-

cursively calculate polynomials $\hat{H}(K_{a_j}, x)$ and $\hat{H}_{a_j}(K_{a_j}, x)$ and repair polynomials $\hat{H}(G_{a_j}, x)$ and $\hat{H}_{a_j}(G_{a_j}, x)$ following Lemma 4.4:

$$\begin{aligned}\hat{H}(G_{a_j}, x) &= \hat{H}(G_{a_j}, x) + \hat{H}(K_{a_j}, x) + \\ &\quad + \tfrac{1}{w(a_j)^2}\big(\hat{H}_{a_j}(G_{a_j}, x) - w(a_j)^2\big)\big(\hat{H}_{a_j}(K_{a_j}, x) - w(a_j)^2\big) \\ \hat{H}_{a_j}(G_{a_j}, x) &= \hat{H}_{a_j}(G_{a_j}, x) + \hat{H}_{a_j}(K_{a_j}, x) - w(a_j)^2.\end{aligned}$$

- According to the discussion in Section 5 we calculate $\hat{H}_r(G_r, x)$ and $\hat{H}(G_r, x)$ using details $\hat{H}(G_{a_j}, x)$ and $\hat{H}_{a_j}(G_{a_j}, x)$, $j = 1, 2, \ldots, n$.
- u is the vertex with $\mathrm{DFN}(u) = \mathrm{DFN}(\mathrm{ORIEN}(v)) - 1$.
- $v = u$ and $u = \mathrm{FATHER}(v)$.

We conclude the subsection summarizing the time complexity. Step 1: It is well-known that traversing the graph with DFS algorithm and computing arrays $\mathrm{DFN}(v)$, $\mathrm{FATHER}(v)$, $\mathrm{ROOT}(v)$, $\mathrm{ORIEN}(v)$ and $\mathrm{IND}(v)$ can be done within $\mathcal{O}(m)$ time. Obviously, Step 2 can be computed in $\mathcal{O}(m)$ time. In Step 3, existence of implementation that uses $\mathcal{O}(m)$ additions and multiplications of polynomials follows from Lemmata 4.1, 4.4, 5.1, and Theorem 5.7. Hence we can conclude that the algorithm runs in linear time.

Theorem 7.1. *The algorithm for the Hosoya polynomial on a weighted cactus graph (given in Subsection 7.2) correctly calculates the polynomial, in linear time in the model where the addition and multiplication of polynomials are atomic operations.*

7.3 Example

Example 7.2. Let G be a cactus graph in Figure 9 (high) with representing DFS tree (one of possibilities) in Figure 9 (low) and its DFS structure in Table 1.

Table 1: The DFS structure of graph G.

v	$\mathrm{DFN}(v)$	$\mathrm{FATHER}(v)$	$\mathrm{ROOT}(v)$	$\mathrm{ORIEN}(v)$	$\mathrm{IND}(v)$
v_1	0	v_1	v_1	v_2	2
v_2	1	v_1	v_1	v_2	3
v_3	2	v_2	v_3	v_3	0
v_4	3	v_2	v_4	v_4	0
v_5	4	v_2	v_1	v_2	2
v_6	5	v_5	v_6	v_6	1
v_7	6	v_6	v_7	v_7	0
v_8	7	v_5	v_1	v_2	1
v_9	8	v_8	v_9	v_9	0

Starting from the initialization (Step 2) and following the algorithm (Step 3), we obtain

- $v = v_9, u = v_8$: $G_{v_9} = \{v_9\}$, $G_{v_8} = \{v_8, v_9\} \cup v_8v_9$,
  - step 2: $\hat{H}_{v_9}(G_{v_9}, x) = 1$, $\hat{H}(G_{v_9}, x) = 0$,
  - step (3a): $\hat{H}_{v_8}(G_{v_8}, x) = x + 1$, $\hat{H}(G_{v_8}, x) = x$.

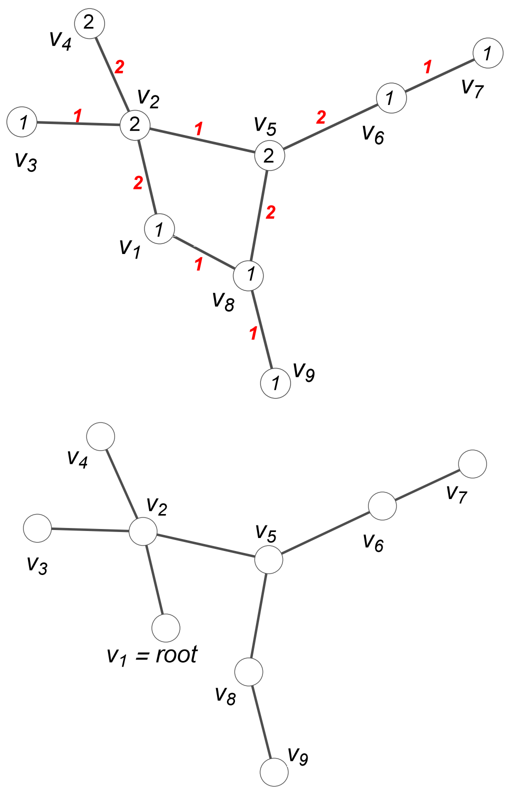

Figure 9: A weighted cactus graph G (high) and its DFS tree (low).

- $v = v_8, u = v_5$: v_5v_8 is on cycle with $a_1 = v_8$, $a_2 = v_5$, $a_3 = v_2$, $a_4 = v_1$,
  - step (3b): $K_{v_5} = \{v_5, v_6, v_7\} \cup v_5v_6 \cup v_6v_7$, $K_{v_2} = \{v_2, v_3, v_4\} \cup v_2v_3 \cup v_2v_4$,
  - recursively:

$$\hat{H}(K_{v_5}, x) = 2x^3 + 2x^2 + x, \qquad \hat{H}_{v_5}(K_{v_5}, x) = 2x^3 + 2x^2 + 4,$$
$$\hat{H}(K_{v_2}, x) = 2x^3 + 4x^2 + 2x, \qquad \hat{H}_{v_2}(K_{v_2}, x) = 4x^2 + 2x + 4,$$

  - from Lemma 4.4:

$$G_{v_5} = \{v_5, v_6, v_7, v_8, v_9\} \cup v_5v_6 \cup v_6v_7 \cup v_5v_8 \cup v_8v_9,$$
$$G_{v_2} = \{v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\} \cup v_2v_3 \cup v_2v_4 \cup v_2v_5 \cup v_5v_6 \cup \cup v_6v_7 \cup v_5v_8 \cup v_8v_9,$$
$$\hat{H}(G_{v_5}, x) = x^6 + 2x^5 + x^4 + 4x^3 + 4x^2 + 2x,$$
$$\hat{H}_{v_5}(G_{v_5}, x) = 4x^3 + 4x^2 + 4,$$
$$\hat{H}(G_{v_2}, x) = 5x^6 + 8x^5 + 7x^4 + 14x^3 + 14x^2 + 8x,$$
$$\hat{H}_{v_2}(G_{v_2}, x) = 4x^4 + 4x^3 + 4x^2 + 4x + 4,$$

– according to discussion in Section 5 we calculate ($G_{v_1} = G$)

$$\hat{H}(G_{v_1}, x) = 6x^6 + 9x^5 + 9x^4 + 17x^3 + 13x^2 + 9x,$$
$$\hat{H}_{v_1}(G_{v_1}, x) = x^6 + x^5 + 2x^4 + 3x^3 + 3x^2 + x + 1,$$

- $v = v_1, u = v_1$: end of the algorithm.

Finally, the Hosoya polynomial of graph G equals

$$\hat{H}(G, x) = \hat{H}(G_{v_1}, x) = 6x^6 + 9x^5 + 9x^4 + 17x^3 + 13x^2 + 9x.$$

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On Jacobian group and complexity of the I-graph $I(n, k, l)$ through Chebyshev polynomials

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Abstract

We consider a family of I-graphs $I(n, k, l)$, which is a generalization of the class of generalized Petersen graphs. In the present paper, we provide a new method for counting Jacobian group of the I-graph $I(n, k, l)$. We show that the minimum number of generators of $\mathrm{Jac}(I(n, k, l))$ is at least two and at most $2k + 2l - 1$. Also, we obtain a closed formula for the number of spanning trees of $I(n, k, l)$ in terms of Chebyshev polynomials. We investigate some arithmetical properties of this number and its asymptotic behaviour.

Keywords: Spanning tree, Jacobian group, I-graph, Petersen graph, Chebyshev polynomial.

Math. Subj. Class.: 05C30, 39A10

1 Introduction

The notion of the Jacobian group of a graph, which is also known as the Picard group, the critical group, and the dollar or sandpile group, was independently introduced by many authors ([1, 2, 4, 9]). This notion arises as a discrete version of the Jacobian in the classical theory of Riemann surfaces. It also admits a natural interpretation in various areas of physics, coding theory, and financial mathematics. The Jacobian group is an important algebraic invariant of a finite graph. In particular, its order coincides with the number of spanning trees of the graph, which is known for some simplest graphs, such as the wheel,

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fan, prism, ladder, and Möbius ladder [6], grids [23], lattices [25], prism and anti-prism [26]. At the same time, the structure of the Jacobian is known only in particular cases [4, 7, 9, 17, 20, 21] and [22]. We mention that the number of spanning trees for circulant graphs is expressed is terms of the Chebyshev polynomials; it was found in [8, 27], and [28]. We show that similar results are also true for the I-graph $I(n, k, l)$.

The generalized Petersen graph $GP(n, k)$ has vertex set and edge set given by

$$\begin{aligned} V(GP(n,k)) &= \{u_i, v_i \mid i = 1, 2, \ldots, n\} \\ E(GP(n,k)) &= \{u_iu_{i+1},\ u_iv_i,\ v_iv_{i+k} \mid i = 1, 2, \ldots, n\}, \end{aligned}$$

where the subscripts are expressed as integers modulo n. The classical Petersen graph is $GP(5, 2)$. The family of generalized Petersen graphs is a subset of so-called I-graphs ([3, 14]). The I-graph $I(n, k, l)$ is a graph of the following structure

$$\begin{aligned} V(I(n,k,l)) &= \{u_i, v_i \mid i = 1, 2, \ldots, n\} \\ E(I(n,k,l)) &= \{u_iu_{i+l},\ u_iv_i,\ v_iv_{i+k} \mid i = 1, 2, \ldots, n\}. \end{aligned}$$

where all subscripts are given modulo n.

Since $I(n, k, l) = I(n, l, k)$ we will usually assume that $k \leq l$. In this paper we will deal with 3-valent graphs only. This means that in the case of even n and $l = n/2$ the graph under consideration has multiple edges. The graph $I(n, l, k)$ is connected if and only if $\gcd(n, k, l) = 1$. If $\gcd(n, k, l) = m > 1$, then $I(n, k, l)$ is a union of m copies of the graph $I(n/m, k/m, l/m)$. If $m = 1$ and $\gcd(k, l) = d$, then the graphs $I(n, k, l)$ and $I(n, k/d, l/d)$ are isomorphic [5, 16, 24]. In the case of $l = 1$ it easy to see that the graph $I(n, k, 1)$ coincides with the generalized Petersen graph $GP(n, k)$. The number of spanning trees and the structure of Jacobian group for the generalized Petersen graph were investigated in [19]. The spectrum of the I-graph was found in [11]. Even though the number of spanning trees of a given graph can be computed through eigenvalues of its Laplacian matrix, it is not easy to find the number of spanning trees for $I(n, k, l)$ using them. In this paper, we obtained a closed formula for the number of spanning trees for $I(n, k, l)$, investigate some arithmetical properties of this number and provide its asymptotic behavior. Also, we suggest an effective way for calculating Jacobian of $I(n, k, l)$ and find sharp upper and lower bounds for the rank of $\mathrm{Jac}(I(n, k, l))$.

2 Basic definitions and preliminary facts

Consider a connected finite graph G, allowed to have multiple edges but without loops. We endow each edge of G with the two possible directions. Since G has no loops, this operation is well defined. Let $O = O(G)$ be the set of directed edges of G. Given $e \in O(G)$, we denote its initial and terminal vertices by $s(e)$ and $t(e)$, respectively. Recall that a closed directed path in G is a sequence of directed edges $e_i \in O(G)$, $i = 1, \ldots, n$ such that $t(e_i) = s(e_{i+1})$ for $i = 1, \ldots, n-1$ and $t(e_n) = s(e_1)$.

Following [1] and [2], *the Jacobian group*, or simply *Jacobian* $\mathrm{Jac}(G)$ of a graph G is defined as the (maximal) Abelian group generated by flows $\omega(e), e \in O(G)$, obeying the following two Kirchhoff laws:

K_1: the flow through each vertex of G vanishes, that is $\sum_{e\in O, t(e)=x} \omega(e) = 0$ for all $x \in V(G)$;

K_2: the flow along each closed directed path W in G vanishes, that is $\sum_{e\in W}\omega(e)=0$.

Equivalent definitions of the group $\mathrm{Jac}(G)$ can be found in papers [1, 2, 4, 9, 12, 18, 20].

We denote the vertex and edge set of G by $V(G)$ and $E(G)$, respectively. Given $u,v\in V(G)$, we set a_{uv} to be equal to the number of edges between vertices u and v. The matrix $A=A(G)=\{a_{uv}\}_{u,v\in V(G)}$, called *the adjacency matrix* of the graph G. The degree $d(v)$ of a vertex $v\in V(G)$ is defined by $d(v)=\sum_u a_{uv}$. Let $D=D(G)$ be the diagonal matrix indexed by the elements of $V(G)$ with $d_{vv}=d(v)$. Matrix $L=L(G)=D(G)-A(G)$ is called *the Laplacian matrix*, or simply *Laplacian*, of the graph G.

Recall [20] the following useful relation between the structure of the Laplacian matrix and the Jacobian of a graph G. Consider the Laplacian $L(G)$ as a homomorphism $\mathbb{Z}^{|V|}\to\mathbb{Z}^{|V|}$, where $|V|=|V(G)|$ is the number of vertices in G. The cokernel $\mathrm{coker}(L(G))=\mathbb{Z}^{|V|}/\,\mathrm{im}(L(G))$ — is an Abelian group. Let

$$\mathrm{coker}(L(G))\cong\mathbb{Z}_{d_1}\oplus\mathbb{Z}_{d_2}\oplus\cdots\oplus\mathbb{Z}_{d_{|V|}}$$

be its Smith normal form satisfying the conditions $d_i\big|d_{i+1}$, $(1\le i\le|V|)$. If the graph is connected, then the groups $\mathbb{Z}_{d_1},\mathbb{Z}_{d_2},\dots,\mathbb{Z}_{d_{|V|-1}}$ — are finite, and $\mathbb{Z}_{d_{|V|}}=\mathbb{Z}$. In this case,

$$\mathrm{Jac}(G)\cong\mathbb{Z}_{t_1}\oplus\mathbb{Z}_{t_2}\oplus\cdots\oplus\mathbb{Z}_{d_{|V|-1}}$$

is the Jacobian of the graph G. In other words, $\mathrm{Jac}(G)$ is isomorphic to the torsion subgroup of the cokernel $\mathrm{coker}(L(G))$.

Let M be an integer $n\times n$ matrix, then we can interpret M as a homomorphism from $\mathbb{Z}^n$ to $\mathbb{Z}^n$. In this interpretation M has a kernel $\ker M$, an image $\mathrm{im}\,M$, and a cokernel $\mathrm{coker}\,M=\mathbb{Z}^n/\,\mathrm{im}\,M$. We emphasize that $\mathrm{coker}\,M$ of the matrix M is completely determined by its Smith normal form.

In what follows, by I_n we denote the identity matrix of order n.

We call an $n\times n$ matrix *circulant,* and denote it by $\mathrm{circ}(a_0,a_1,\dots,a_{n-1})$ if it is of the form

$$\mathrm{circ}(a_0,a_1,\dots,a_{n-1})=\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1}\\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_1 & a_2 & a_3 & \dots & a_0\end{pmatrix}.$$

Recall [10] that the eigenvalues of matrix $C=\mathrm{circ}(a_0,a_1,\dots,a_{n-1})$ are given by the following simple formulas $\lambda_j=p(\varepsilon_n^j)$, $j=0,1,\dots,n-1$ where $p(x)=a_0+a_1x+\cdots+a_{n-1}x^{n-1}$ and ε_n is the order n primitive root of the unity. Moreover, the circulant matrix $C=p(T)$, where $T=\mathrm{circ}(0,1,0,\dots,0)$ is the matrix representation of the shift operator $T\colon(x_0,x_1,\dots,x_{n-2},x_{n-1})\to(x_1,x_2,\dots,x_{n-1},x_0)$.

By [15, Lemma 2.1] the $2n\times 2n$ adjacency matrix of the I-graph $I(n,k,l)$ has the following block form

$$A(I(n,k,l))=\begin{pmatrix} C_n^k & I_n\\ I_n & C_n^l\end{pmatrix},$$

where C_n^k is the $n\times n$ circulant matrix of the form

$$C_n^k=\mathrm{circ}(\underbrace{0,\dots,0}_{k\text{ times}},1,\underbrace{0,\dots,0}_{n-2k-1},1,\underbrace{0,\dots,0}_{k-1\text{ times}}).$$

Denote by $L = L(I(n,k,l))$ the Laplacian of $I(n,k,l)$. Since the graph $I(n,k,l)$ is three-valent, we have

$$L = 3I_{2n} - A(I(n,k,l)) = \begin{pmatrix} 3I_n - C_n^k & -I_n \\ -I_n & 3I_n - C_n^l \end{pmatrix}.$$

3 Cokernels of linear operators

Let $P(z)$ be a bimonic integer Laurent polynomial. That is $P(z) = z^p + a_1 z^{p+1} + \cdots + a_{s-1}z^{p+s-1} + z^{p+s}$ for some integers $p, a_1, a_2, \ldots, a_{s-1}$ and some positive integer s. Introduce the following companion matrix $\mathcal{A}$ for the polynomial $P(z)$:

$$\mathcal{A} = \left(\begin{array}{c|c} 0 & I_{s-1} \\ \hline \multicolumn{2}{c}{-1, -a_1, \ldots, -a_{s-1}} \end{array}\right),$$

where I_{s-1} is the identity $(s-1)\times(s-1)$ matrix. We will use the following properties of $\mathcal{A}$. Note that $\det \mathcal{A} = (-1)^s$. Hence $\mathcal{A}$ is invertible and inverse matrix $\mathcal{A}^{-1}$ is also integer matrix. The characteristic polynomial of $\mathcal{A}$ coincides with $z^{-p}P(z)$.

Let $\mathbb{A} = \langle \alpha_j,\ j \in \mathbb{Z}\rangle$ be a free Abelian group freely generated by elements $\alpha_j,\ j \in \mathbb{Z}$. Each element of $\mathbb{A}$ is a linear combination $\sum_j c_j\alpha_j$ with integer coefficients c_j.

Define the shift operator $T\colon \mathbb{A} \to \mathbb{A}$ as a $\mathbb{Z}$-linear operator acting on generators of $\mathbb{A}$ by the rule $T\colon \alpha_j \to \alpha_{j+1},\ j \in \mathbb{Z}$. Then T is an endomorphism of $\mathbb{A}$. Let $P(z)$ be an arbitrary Laurent polynomial with integer coefficients, then $A = P(T)$ is also an endomorphism of $\mathbb{A}$. Since A is a linear combination of powers of T, the action of A on generators α_j can be given by the infinite set of linear transformations $A\colon \alpha_j \to \sum_i a_{i,j}\alpha_i,\ j \in \mathbb{Z}$. Here all sums under consideration are finite. We set $\beta_j = \sum_i a_{i,j}\alpha_i$. Then $\operatorname{im} A$ is a subgroup of $\mathbb{A}$ generated by $\beta_j,\ j \in \mathbb{Z}$. Hence, $\operatorname{coker} A = \mathbb{A}/\operatorname{im} A$ is an abstract Abelian group $\langle x_i, i \in \mathbb{Z} \mid \sum_i a_{i,j}x_i = 0,\ j \in \mathbb{Z}\rangle$ generated by $x_i, i \in \mathbb{Z}$ with the set of defining relations $\sum_i a_{i,j}x_i = 0,\ j \in \mathbb{Z}$. Here x_j are images of α_j under the canonical homomorphism $\mathbb{A} \to \mathbb{A}/\operatorname{im} A$. Since T and $A = P(T)$ commute, subgroup $\operatorname{im} A$ is invariant under the action of T. Hence, the actions of T and A are well defined on the factor group $\mathbb{A}/\operatorname{im} A$ and are given by $T\colon x_j \to x_{j+1}$ and $A\colon x_j \to \sum_i a_{i,j}x_i$ respectively.

This allows to present the group $\mathbb{A}/\operatorname{im} A$ as follows $\langle x_i,\ i \in \mathbb{Z} \mid P(T)x_j = 0,\ j \in \mathbb{Z}\rangle$. In a similar way, given a set $P_1(z), P_2(z), \ldots, P_s(z)$ of Laurent polynomials with integer coefficients, one can define the group $\langle x_i,\ i \in \mathbb{Z} \mid P_1(T)x_j = 0, P_2(T)x_j = 0, \ldots, P_s(T)x_j = 0,\ j \in \mathbb{Z}\rangle$.

We will use the following lemma.

Lemma 3.1. *Let $T\colon \mathbb{A} \to \mathbb{A}$ be the shift operator. Consider endomorphisms A and B of the group $\mathbb{A}$ given by the formulas $A = P(T)$, $B = Q(T)$, where $P(z)$ and $Q(z)$ are Laurent polynomials with integer coefficients. Then $B\colon \mathbb{A} \to \mathbb{A}$ induces an endomorphism $B|_{\operatorname{coker} A}$ of the group* $\operatorname{coker} A = \mathbb{A}/\operatorname{im} A$ *defined by* $B|_{\operatorname{coker} A}(\alpha + \operatorname{im} A) = B(\alpha) + \operatorname{im} A$, $\alpha \in \mathbb{A}$. *Furthermore*

$$\langle x_i,\ i \in \mathbb{Z} \mid A(T)x_j = 0,\ B(T)x_j = 0,\ j \in \mathbb{Z}\rangle \cong \operatorname{coker} A/\operatorname{im}(B|_{\operatorname{coker} A}) \cong \operatorname{coker}(B|_{\operatorname{coker} A}).$$

Proof. The images $\operatorname{im} A$ and $\operatorname{im} B$ are subgroups in $\mathbb{A}$. Denote by $\langle \operatorname{im} A, \operatorname{im} B\rangle$ the subgroup generated by elements of $\operatorname{im} A$ and $\operatorname{im} B$. Since $P(z)$ and $Q(z)$ are Laurent polynomials, the operators $A = P(T)$ and $B = Q(T)$ do commute. Hence, subgroup $\operatorname{im} A$

is invariant under endomorphism B. Indeed for any $y = Ax \in \operatorname{im} A$, we have $By = B(Ax) = A(Bx) \in \operatorname{im} A$. This means that $B\colon \mathbb{A} \to \mathbb{A}$ induces an endomorphism of the group $\operatorname{coker} A = \mathbb{A}/\operatorname{im} A$. We denote this endomorphism by $B|_{\operatorname{coker} A}$. We note that the Abelian group $\langle x_i,\ i \in \mathbb{Z} \mid A(T)x_j = 0,\ B(T)x_j = 0,\ j \in \mathbb{Z}\rangle$ is naturally isomorphic to $\mathbb{A}/\langle \operatorname{im} A, \operatorname{im} B\rangle$. So we have

$$\mathbb{A}/\langle \operatorname{im} A, \operatorname{im} B\rangle \cong (\mathbb{A}/\operatorname{im} A)/\operatorname{im}(B|_{\operatorname{coker} A}) \cong \operatorname{coker} A/\operatorname{im}(B|_{\operatorname{coker} A}) \cong \operatorname{coker}(B|_{\operatorname{coker} A}).$$

The lemma is proved. □

4 Jacobian group for the I-graph $I(n, k, l)$

In this section we prove one of the main results of the paper. We start in the following theorem.

Theorem 4.1. *Let $L = L(I(n, k, l))$ be the Laplacian of a connected I-graph $I(n, k, l)$. Then*

$$\operatorname{coker} L \cong \operatorname{coker}(\mathcal{A}^n - I),$$

where $\mathcal{A}$ is $2(k + l) \times 2(k + l)$ companion matrix for the Laurent polynomial

$$(3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1.$$

Proof. Let L be the Laplacian matrix of the graph $I(n, k, l)$. Then, as it was mentioned above, L is a $2n \times 2n$ matrix of the form

$$L = \begin{pmatrix} 3I_n - C_n^k & -I_n \\ -I_n & 3I_n - C_n^l \end{pmatrix},$$

where $C_n^k = \operatorname{circ}(\underbrace{0, \ldots, 0}_{k \text{ times}}, 1, 0, \ldots, 0, 1, \underbrace{0, \ldots, 0}_{k-1 \text{ times}})$.

Consider L as a $\mathbb{Z}-$linear operator $L\colon \mathbb{Z}^{2n} \to \mathbb{Z}^{2n}$. In this case, $\operatorname{coker}(L)$ is an abstract Abelian group generated by elements $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ satisfying the system of linear equations $3x_j - x_{j-k} - x_{j+k} - y_j = 0, 3y_j - y_{j-l} - y_{j+l} - x_j = 0$ for any $j = 1, \ldots, n$. Here the indices are considered modulo n. By the property mentioned in Section 2, the Jacobian of the graph $I(n, k, l)$ is isomorphic to the finite part of cokernel of the operator L.

To study the structure of $\operatorname{coker}(L)$ we extend the list of generators to the two bi-infinite sequences of elements $(x_j)_{j\in\mathbb{Z}}$ and $(y_j)_{j\in\mathbb{Z}}$ setting $x_{j+mn} = x_j$ and $y_{j+mn} = y_j$ for any $m \in \mathbb{Z}$. Then we have the following representation for cokernel of L:

$$\operatorname{coker}(L) = \langle x_i, y_i, i \in \mathbb{Z} \mid 3x_j - x_{j+k} - x_{j-k} - y_j = 0, \\ 3y_j - y_{j+l} - y_{j-l} - x_j = 0, x_{j+n} = x_j, y_{j+n} = y_j, j \in \mathbb{Z}\rangle.$$

Let T be the shift operator defined by the rule $T\colon x_j \to x_{j+1},\ y_j \to y_{j+1},\ j \in \mathbb{Z}$. Consider the operator $P(T)$ defined by $P(T) = (3 - T^k - T^{-k})(3 - T^l - T^{-l}) - 1$. We use the operator notation from Section 3 to represent the cokernel of L. Then we have

$$\begin{aligned}\operatorname{coker}(L) &= \langle x_i,\, y_i, i \in \mathbb{Z} \mid (3 - T^k - T^{-k})x_j = y_j, (3 - T^l - T^{-l})y_j = x_j,\\ &\qquad T^n x_j = x_j, T^n y_j = y_j, j \in \mathbb{Z}\rangle\\ &= \langle x_i,\, i \in \mathbb{Z} \mid (3 - T^l - T^{-l})(3 - T^k - T^{-k})x_j = x_j, T^n x_j = x_j,\, j \in \mathbb{Z}\rangle\\ &= \langle x_i,\, i \in \mathbb{Z} \mid ((3 - T^k - T^{-k})(3 - T^l - T^{-l}) - 1)x_j = 0,\\ &\qquad (T^n - 1)x_j = 0,\, j \in \mathbb{Z}\rangle\\ &= \langle x_i,\, i \in \mathbb{Z} \mid P(T)x_j = 0, (T^n - 1)x_j = 0, j \in \mathbb{Z}\rangle.\end{aligned}$$

To finish the proof, we apply Lemma 3.1 to the operators $A = P(T)$ and $B = Q(T) = T^n - 1$.

Since the Laurent polynomial $P(z) = (3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$ is bimonic, it can be represented in the form $P(z) = z^{-k-l} + a_1 z^{-k-l+1} + \cdots + a_{2k+2l-1} z^{k+l-1} + z^{k+l}$, where $a_1, a_2, \ldots, a_{2k+2l-1}$ are integers. Then the corresponding companion matrix $\mathcal{A}$ is

$$\left(\begin{array}{c|c} 0 & I_{2k+2l-1} \\ \hline \multicolumn{2}{c}{-1, -a_1, \ldots, -a_{2k+2l-1}} \end{array}\right).$$

It is easy to see that $\det \mathcal{A} = 1$ and its inverse $\mathcal{A}^{-1}$ is also integer matrix.

For convenience we set $s = 2k + 2l$ to be the size of matrix $\mathcal{A}$.

Note that for any $j \in \mathbb{Z}$ the relations $P(T)x_j = 0$ can be rewritten as $x_{j+s} = -x_j - a_1 x_{j+1} - \cdots - a_{s-1} x_{j+s-1}$. Let $\mathbf{x}_j = (x_{j+1}, x_{j+2}, \ldots, x_{j+s})^t$ be s-tuple of generators $x_{j+1}, x_{j+2}, \ldots, x_{j+s}$. Then the relation $P(T)x_j = 0$ is equivalent to $\mathbf{x}_j = \mathcal{A}\,\mathbf{x}_{j-1}$. Hence, we have $\mathbf{x}_1 = \mathcal{A}\,\mathbf{x}_0$ and $\mathbf{x}_{-1} = \mathcal{A}^{-1}\,\mathbf{x}_0$, where $\mathbf{x}_0 = (x_1, x_2, \ldots, x_s)^t$. So, $\mathbf{x}_j = \mathcal{A}^j\,\mathbf{x}_0$ for any $j \in \mathbb{Z}$. Conversely, the latter implies $\mathbf{x}_j = \mathcal{A}\,\mathbf{x}_{j-1}$ and, as a consequence, $P(T)x_j = 0$ for all $j \in \mathbb{Z}$.

Consider $\operatorname{coker} A = \mathbb{A}/\operatorname{im} A$ as an abstract Abelian group with the following representation $\langle x_i, i \in \mathbb{Z} \mid P(T)x_j = 0, j \in \mathbb{Z}\rangle$.

Our present aim is to show that $\operatorname{coker} A \cong \mathbb{Z}^s$. We have

$$\begin{aligned}\operatorname{coker} A &= \langle x_i, i \in \mathbb{Z} \mid P(T)x_j = 0, j \in \mathbb{Z}\rangle\\ &= \langle x_j, j \in \mathbb{Z} \mid x_\ell + a_1 x_{\ell+1} + \cdots + a_{s-1} x_{\ell+s-1} + x_{\ell+s} = 0, \ell \in \mathbb{Z}\rangle\\ &= \langle x_j, j \in \mathbb{Z} \mid (x_{\ell+1}, x_{\ell+2}, \ldots, x_{\ell+s})^t = \mathcal{A}(x_\ell, x_{\ell+1}, \ldots, x_{\ell+s-1})^t, \ell \in \mathbb{Z}\rangle\\ &= \langle x_j, j \in \mathbb{Z} \mid (x_{\ell+1}, x_{\ell+2}, \ldots, x_{\ell+s})^t = \mathcal{A}^\ell(x_1, x_2, \ldots, x_s)^t, \ell \in \mathbb{Z}\rangle\\ &= \langle x_1, x_2, \ldots, x_s \mid \emptyset\rangle \cong \mathbb{Z}^s.\end{aligned}$$

Now we describe the action of the endomorphism $B|_{\operatorname{coker} A}$ on the $\operatorname{coker} A$. Since the operators $A = P(T)$ and T commute, the action $T|_{\operatorname{coker} A}\colon x_j \to x_{j+1}$, $j \in \mathbb{Z}$ on the $\operatorname{coker} A$ is well defined. First of all, we describe the action of $T|_{\operatorname{coker} A}$ on the set of generators $x_1, x_2, \ldots, x_s$. For any $i = 1, \ldots, s - 1$, we have $T|_{\operatorname{coker}}(x_i) = x_{i+1}$ and $T|_{\operatorname{coker} A}(x_s) = x_{s+1} = -x_1 - a_1 x_2 - \cdots - a_{s-2} x_{s-1} - a_{s-1} x_s$. Hence, the action of $T|_{\operatorname{coker} A}$ on the $\operatorname{coker} A$ is given by the matrix $\mathcal{A}$. Considering $\mathcal{A}$ as an endomorphism of the $\operatorname{coker} A$, we can write $T|_{\operatorname{coker} A} = \mathcal{A}$. Finally, $B|_{\operatorname{coker} A} = Q(T|_{\operatorname{coker} A}) = Q(\mathcal{A})$. Applying Lemma 3.1, we finish the proof of the theorem. □

Corollary 4.2. *The Jacobian group* $\operatorname{Jac}(I(n, k, l))$ *of a connected* I*-graph* $I(n, k, l)$ *is isomorphic to the torsion subgroup of* $\operatorname{coker}(\mathcal{A}^n - I)$*, where* $\mathcal{A}$ *is the companion matrix for the Laurent polynomial* $(3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$.

The Corollary 4.2 gives a simple way to find Jacobian group $\mathrm{Jac}(I(n,k,l))$ for small values of k, l and sufficiently large numbers n. The numerical results are given in the Tables 2 and 3.

5 Counting the number of spanning trees for the I-graph $I(n,k,l)$

In what follows, we always assume that the numbers k and l are relatively prime. To get the result for an arbitrary connected I-graph $I(n,k,l)$ with $\gcd(n,k,l)=1$ and $\gcd(k,l)=d>1$ we observe that $I(n,k,l)$ is isomorphic to $I(n,k',l')$, where the numbers $k'=k/d$ and $l'=l/d$ are relatively prime.

Theorem 5.1. *The number of spanning trees of the I-graph $I(n,k,l)$ is given by the formula*

$$\tau_{k,l}(n)=(-1)^{(n-1)(k+l)}n\prod_{s=1}^{k+l-1}\frac{T_n(w_s)-1}{w_s-1},$$

where w_s, $s=1,2,\dots,k+l-1$ are roots of the order $k+l-1$ algebraic equation

$$\frac{(3-2T_k(w))(3-2T_l(w))-1}{w-1}=0,$$

and $T_j(w)$ is the Chebyshev polynomial of the first kind.

Proof. By the celebrated Kirchhoff theorem, the number of spanning trees $\tau_{k,l}(n)$ is equal to the product of nonzero eigenvalues of the Laplacian of a graph $I(n,k,l)$ divided by the number of its vertices $2n$. To investigate the spectrum of Laplacian matrix we note that matrix $C_n^k=T^k+T^{-k}$, where $T=\mathrm{circ}(0,1,\dots,0)$ is the $n\times n$ shift operator. The latter equality easily follows from the identity $T^n=I_n$. Hence,

$$L=\begin{pmatrix}3I_n-T^k-T^{-k} & -I_n\\ -I_n & 3I_n-T^l-T^{-l}\end{pmatrix}.$$

The eigenvalues of circulant matrix T are ε_n^j, where $\varepsilon_n=e^{\frac{2\pi i}{n}}$. Since all eigenvalues of T are distinct, the matrix T is conjugate to the diagonal matrix $\mathbb{T}=\mathrm{diag}(1,\varepsilon_n,\dots,\varepsilon_n^{n-1})$, where diagonal entries of $\mathrm{diag}(1,\varepsilon_n,\dots,\varepsilon_n^{n-1})$ are $1,\varepsilon_n,\dots,\varepsilon_n^{n-1}$. To find spectrum of L, without loss of generality, one can assume that $T=\mathbb{T}$. Then the blocks of L are diagonal matrices. This essentially simplifies the problem of finding eigenvalues of L. Indeed, let λ be an eigenvalue of L and $(x,y)=(x_1,\dots,x_n,y_1,\dots,y_n)$ be the corresponding eigenvector. Then we have the following system of equations

$$\begin{cases}(3I_n-T^k-T^{-k})x-y=\lambda x\\ -x+(3I_n-T^l-T^{-l})y=\lambda y\end{cases}.$$

From here we conclude that $y=(3I_n-T^k-T^{-k})x-\lambda x=((3-\lambda)I_n-T^k-T^{-k})x$. Substituting y in the second equation, we have $(((3-\lambda)I_n-T^l-T^{-l})((3-\lambda)I_n-T^k-T^{-k})-1)x=0$.

Recall the matrices under consideration are diagonal and the $(j+1,j+1)$-th entry of T is equal to ε_n^j. Therefore, we have $((3-\lambda-\varepsilon_n^{jk}-\varepsilon_n^{-jk})(3-\lambda-\varepsilon_n^{jl}-\varepsilon_n^{-jl})-1)x_{j+1}=0$ and $y_{j+1}=(3-\lambda-\varepsilon_n^{jl}-\varepsilon_n^{-jl})x_{j+1}$.

So, for any $j = 0, \dots, n-1$ the matrix L has two eigenvalues, say $\lambda_{1,j}$ and $\lambda_{2,j}$ satisfying the quadratic equation $(3 - \lambda - \varepsilon_n^{jk} - \varepsilon_n^{-jk})(3 - \lambda - \varepsilon_n^{jl} - \varepsilon_n^{-jl}) - 1 = 0$. The corresponding eigenvectors are (x, y), where

$$x = \mathbf{e}_{j+1} = (0, \dots, \underbrace{1}_{(j+1)\text{-th}}, \dots, 0) \text{ and}$$
$$y = (3 - \lambda - T^k - T^{-k})\mathbf{e}_{j+1}.$$

In particular, if $j = 0$ for $\lambda_{1,0}, \lambda_{2,0}$ we have $(1-\lambda)(1-\lambda) - 1 = \lambda(\lambda - 2) = 0$. That is, $\lambda_{1,0} = 0$ and $\lambda_{2,0} = 2$. Since $\lambda_{1,j}$ and $\lambda_{2,j}$ are roots of the same quadratic equation, we obtain $\lambda_{1,j}\lambda_{2,j} = P(\varepsilon_n^j)$, where $P(z) = (3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$.

Now we have

$$\tau_{k,l}(n) = \frac{1}{2n}\lambda_{2,0}\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j} = \frac{1}{n}\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j} = \frac{1}{n}\prod_{j=1}^{n-1}P(\varepsilon_n^j).$$

To continue we need the following lemma.

Lemma 5.2. *The following identity holds*

$$(3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1 = (3 - 2T_k(w))(3 - 2T_l(w)) - 1,$$

where $T_k(w)$ is the Chebyshev polynomial of the first kind and $w = \frac{1}{2}(z + z^{-1})$. Moreover, if k and l are relatively prime then all roots of the Laurent polynomial

$$(3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$$

counted with multiplicities are $1, 1, z_1, 1/z_1, \dots, z_{k+l-1}, 1/z_{k+l-1}$, where we have $|z_s| \neq 1$, $s = 1, 2, \dots, k+l-1$. So, the right-hand polynomial has the roots $1, w_1, \dots, w_{k+l-1}$, where $w_s \neq 1$ for all $s = 1, 2, \dots, k+l-1$.

Proof. Let us substitute $z = e^{i\varphi}$. It is easy to see that $w = \frac{1}{2}(z + z^{-1}) = \cos\varphi$, so we have $T_k(w) = \cos(k \arccos w) = \cos(k\varphi)$. Then the first statement of the lemma is equivalent to the following trigonometric identity

$$(3 - 2\cos(k\varphi))(3 - 2\cos(l\varphi)) - 1 = (3 - 2T_k(w))(3 - 2T_l(w)) - 1.$$

To prove the second statement of the lemma we suppose that the Laurent polynomial $P(z) = (3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$ has a root z_0 such that $|z_0| = 1$. Then $z_0 = e^{i\varphi_0}$, $\varphi_0 \in \mathbb{R}$. Now we have $(3 - 2\cos(k\varphi_0))(3 - 2\cos(l\varphi_0)) - 1 = 0$. Since $3 - 2\cos(k\varphi_0) \geq 1$ and $3 - 2\cos(l\varphi_0) \geq 1$ the equations holds if and only if $\cos(k\varphi_0) = 1$ and $\cos(l\varphi_0) = 1$. So $k\varphi_0 = 2\pi s_0$ and $l\varphi_0 = 2\pi t_0$ for some integer s_0 and t_0. As k and l are relatively prime, so there exist two integers p and q such that $kp + ql = 1$. Hence $\varphi_0 = \varphi_0(kp + lq) = 2\pi(ps_0 + qt_0) \in 2\pi\mathbb{Z}$. As a result $z_0 = e^{i\varphi_0} = 1$. Now we have to show that the multiplicity of the root $z_0 = 1$ is 2. Indeed, $P(1) = P'(1) = 0$ and $P''(1) = -2(k^2 + l^2) \neq 0$. □

Let us set $H(z) = \prod_{s=1}^{m}(z - z_s)(z - z_s^{-1})$, where $m = k + l - 1$ and z_s are roots of $P(z)$ different from 1. Then by Lemma 5.2, we have $P(z) = \frac{(z-1)^2}{z^{k+l}}H(z)$.

Lemma 5.3. *Let $H(z)=\prod_{s=1}^{m}(z-z_s)(z-z_s^{-1})$ and $H(1)\neq 0$. Then*

$$\prod_{j=1}^{n-1} H(\varepsilon_n^j)=\prod_{s=1}^{m}\frac{T_n(w_s)-1}{w_s-1},$$

where $w_s=\frac{1}{2}(z_s+z_s^{-1})$, $s=1,\dots,m$ and $T_n(x)$ is the Chebyshev polynomial of the first kind.

Proof. It is easy to check that $\prod_{j=1}^{n-1}(z-\varepsilon_n^j)=\frac{z^n-1}{z-1}$ if $z\neq 1$. Also we note that $\frac{1}{2}(z^n+z^{-n})=T_n(\frac{1}{2}(z+z^{-1}))$. By the substitution $z=e^{i\,\varphi}$, the latter follows from the evident identity $\cos(n\varphi)=T_n(\cos\varphi)$. Then we have

$$\begin{aligned}\prod_{j=1}^{n-1} H(\varepsilon_n^j)&=\prod_{j=1}^{n-1}\prod_{s=1}^{m}(\varepsilon_n^j-z_s)(\varepsilon_n^j-z_s^{-1})\\&=\prod_{s=1}^{m}\prod_{j=1}^{n-1}(z_s-\varepsilon_n^j)(z_s^{-1}-\varepsilon_n^j)\\&=\prod_{s=1}^{m}\frac{z_s^n-1}{z_s-1}\,\frac{z_s^{-n}-1}{z_s^{-1}-1}=\prod_{s=1}^{m}\frac{T_n(w_s)-1}{w_s-1}.\end{aligned}$$ □

Note that $\prod_{j=1}^{n-1}(1-\varepsilon_n^j)=\lim\limits_{z\to 1}\prod_{j=1}^{n-1}(z-\varepsilon_n^j)=\lim\limits_{z\to 1}\frac{z^n-1}{z-1}=n$ and $\prod_{j=1}^{n-1}\varepsilon_n^j=(-1)^{n-1}$. As a result, taking into account Lemma 5.2 and Lemma 5.3, we obtain

$$\begin{aligned}\tau_{k,l}(n)&=\frac{1}{n}\prod_{j=1}^{n-1}P(\varepsilon_n^j)=\frac{1}{n}\prod_{j=1}^{n-1}\frac{(\varepsilon_n^j-1)^2}{(\varepsilon_n^j)^{k+l}}H(\varepsilon_n^j)\\&=\frac{(-1)^{(n-1)(k+l)}n^2}{n}\prod_{j=1}^{n-1}H(\varepsilon_n^j)\\&=(-1)^{(n-1)(k+l)}n\prod_{s=1}^{k+l-1}\frac{T_n(w_s)-1}{w_s-1}.\end{aligned}$$ □

Corollary 5.4. $\tau_{k,l}(n)=n\left|\prod_{s=1}^{k+l-1}U_{n-1}\left(\sqrt{\frac{1+w_s}{2}}\right)\right|^2$*, where w_s, $s=1,2,\dots,k$ are the same as in Theorem 5.1 and $U_{n-1}(w)$ is the Chebyshev polynomial of the second kind.*

Proof. Follows from the identity $\frac{T_n(w)-1}{w-1}=U_{n-1}^2\left(\sqrt{\frac{1+w}{2}}\right)$. □

The following theorem appeared after fruitful discussion with professor D. Lorenzini.

Theorem 5.5. *Let $\tau(n)=\tau_{k,l}(n)$ be the number of spanning trees of the graph $I(n,k,l)$. Then there exist an integer sequence $a(n)=a_{k,l}(n)$, $n\in\mathbb{N}$ such that*

1° $\tau(n)=n\,a^2(n)$ *when n is odd,*

2° $\tau(n)=6n\,a^2(n)$ *when n is even and $k+l$ is even,*

3° $\tau(n)=n\,a^2(n)$ *when n is even and $k+l$ is odd.*

Proof. Recall that all nonzero eigenvalues are given by the list $\{\lambda_{2,0}, \lambda_{1,j}, \lambda_{2,j}, j = 1, \ldots, n-1\}$. By the Kirchhoff theorem we have $2n\tau(n) = \lambda_{2,0}\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j}$.

Since $\lambda_{2,0} = 2$, we have $n\tau(n) = \prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j}$. We note that $\lambda_{1,j}\lambda_{2,j} = P(\varepsilon_n^j) = P(\varepsilon_n^{n-j}) = \lambda_{1,n-j}\lambda_{2,n-j}$. So, we get $n\tau(n) = (\prod_{j=1}^{(n-1)/2}\lambda_{1,j}\lambda_{2,j})^2$ if n is odd and $n\tau(n) = \lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}(\prod_{j=1}^{n/2-1}\lambda_{1,j}\lambda_{2,j})^2$, if n is even. The value $\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}} = P(-1) = (3-2(-1)^k)(3-2(-1)^l) - 1$ is equal to 4 if k and l are of different parity and 24 if both k and l are odd. The case when both k and l are even is impossible, since k and l are relatively prime.

The graph $I(n,k,l)$ admits a cyclic group of automorphisms isomorphic to $\mathbb{Z}_n$ which acts freely on the set of spanning trees. Therefore, the value $\tau(n)$ is a multiple of n. So $\frac{\tau(n)}{n}$ is an integer. Hence

1° $\frac{\tau(n)}{n} = \left(\frac{\prod_{j=1}^{(n-1)/2}\lambda_{1,j}\lambda_{2,j}}{n}\right)^2$ when n is odd,

2° $\frac{\tau(n)}{n} = 6\left(\frac{2\prod_{j=1}^{n/2-1}\lambda_{1,j}\lambda_{2,j}}{n}\right)^2$ when n is even and $k+l$ is even,

3° $\frac{\tau(n)}{n} = \left(\frac{2\prod_{j=1}^{n/2-1}\lambda_{1,j}\lambda_{2,j}}{n}\right)^2$ when n is even and $k+l$ is odd.

Each algebraic number $\lambda_{i,j}$ comes into both products $\prod_{j=1}^{(n-1)/2}\lambda_{1,j}\lambda_{2,j}$ and $\prod_{j=1}^{n/2-1}\lambda_{1,j}\lambda_{2,j}$ with all its Galois conjugate elements. Therefore, both products are integer numbers. From here we conclude that in equalities 1°, 2° and 3° the value that is squared is a rational number. Because $\frac{\tau(n)}{n}$ is integer and 6 is a squarefree, all these rational numbers are integer. Setting $a(n) = \frac{\prod_{j=1}^{(n-1)/2}\lambda_{1,j}\lambda_{2,j}}{n}$ if n is odd and $a(n) = \frac{2\prod_{j=1}^{n/2-1}\lambda_{1,j}\lambda_{2,j}}{n}$ if n is even, we finish the proof of the theorem. □

From now on, we aim to estimate the minimum number of generators for the Jacobian of I-graph $I(n,k,l)$.

Lemma 5.6. *For any given I-graph $I(n,k,l)$ the number of spanning trees $\tau(n)$ satisfies the inequality $\tau(n) \geq n^3$.*

Proof. Recall that for any $j = 0,\ldots,n-1$, the Laplacian matrix L of $I(n,k,l)$ has two eigenvalues, say $\lambda_{1,j}$ and $\lambda_{2,j}$, which are roots of the quadratic equation $Q_j(\lambda) = (3-\lambda-\varepsilon_n^{jk}-\varepsilon_n^{-jk})(3-\lambda-\varepsilon_n^{jl}-\varepsilon_n^{-jl})-1 = 0$. So, $\lambda_{1,j}\lambda_{2,j} = (3-\varepsilon_n^{jk}-\varepsilon_n^{-jk})(3-\varepsilon_n^{jl}-\varepsilon_n^{-jl})-1 = P(\varepsilon_n^j)$. Note that $\lambda_{1,0} = 0$ and $\lambda_{2,0} = 2$. Furthermore $\{\lambda_{1,j}, \lambda_{2,j} \mid j = 0,\ldots,n-1\}$ is the set of all eigenvalues of L. The Kirchhoff theorem states the following

$$2n\,\tau_{k,l}(n) = 2n\,\tau(n) = \lambda_{2,0}\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j} = 2\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j}.$$

Hence $n\tau(n) = \prod_{j=1}^{n-1}P(\varepsilon_n^j)$, where $P(\varepsilon_n^j) = (3-2\cos(\frac{2jk\pi}{n}))(3-2\cos(\frac{2jl\pi}{n}))-1$.

It is easy to prove the following trigonometric identity

$$\Big(3-2\cos\Big(\frac{2jk\pi}{n}\Big)\Big)\Big(3-2\cos\Big(\frac{2jl\pi}{n}\Big)\Big)-1=$$
$$4\sin^2\Big(\frac{jk\pi}{n}\Big)+4\sin^2\Big(\frac{jl\pi}{n}\Big)+16\sin^2\Big(\frac{jk\pi}{n}\Big)\sin^2\Big(\frac{jl\pi}{n}\Big).$$

Connectedness of I-graph implies $\gcd(n,k,l)=1$. It may happen that $\gcd(n,k)=m\neq 1$ and $\gcd(n,l)=m'\neq 1$. We will use the notation $n=m\,q=m'q'$, $k=p\,m$, $l=p'm'$. We introduce three sets, J, J_k and J_l in the following way

$$J=\{1,2,\ldots,n-1\},$$
$$J_k=\{j\mid j=d\,q,\ d=1,\ldots,m-1\}\text{ and}$$
$$J_l=\{j\mid j=d'\,q',\ d'=1,\ldots,m'-1\}.$$

If $j\in J_k$ then $\sin(\frac{j\,k\,\pi}{n})=0$ and if $j\in J_l$ then $\sin(\frac{j\,l\,\pi}{n})=0$. We note that J_k and J_l do not intersect. Otherwise, for $j\in J_k\cap J_l$ we have $\lambda_{1,j}\lambda_{2,j}=P(\varepsilon_n^j)=0$. Then at least one of the eigenvalues $\lambda_{1,j}$ and $\lambda_{2,j}$ is equal to zero. This leads to contradiction, as we have the unique zero eigenvalue $\lambda_{1,0}=0$. Now we are going to find a low bound for $\tau(n)$. As $n\,\tau(n)=\prod_{j=1}^{n-1}P(\varepsilon_n^j)$ we evaluate the product

$$\prod_{j=1}^{n-1}P(\varepsilon_n^j)=\prod_{j=1}^{n-1}\Big(4\sin^2\Big(\frac{jk\pi}{n}\Big)+4\sin^2\Big(\frac{jl\pi}{n}\Big)+16\sin^2\Big(\frac{jk\pi}{n}\Big)\sin^2\Big(\frac{jl\pi}{n}\Big)\Big)$$
$$\geq\prod_{j\in J_k}4\sin^2\Big(\frac{jl\pi}{n}\Big)\prod_{j\in J_l}4\sin^2\Big(\frac{jk\pi}{n}\Big)\prod_{j\in J\setminus(J_k\cup J_l)}16\sin^2\Big(\frac{jk\pi}{n}\Big)\sin^2\Big(\frac{jl\pi}{n}\Big)$$
$$=\prod_{j\in J\setminus J_k}4\sin^2\Big(\frac{jk\pi}{n}\Big)\prod_{j\in J\setminus J_l}4\sin^2\Big(\frac{jl\pi}{n}\Big).$$

Now we analyze individual component of the product. We make use of the following simple identity $\cos(\frac{2jp\pi}{q})=\cos(\frac{2(j+q)p\pi}{q})$.

$$\prod_{j\in J\setminus J_k}4\sin^2\Big(\frac{jk\pi}{n}\Big)=\prod_{j\in J\setminus J_k}\Big(2-2\cos\Big(\frac{2jk\pi}{n}\Big)\Big)=\prod_{j\in J\setminus J_k}\Big(2-2\cos\Big(\frac{2jmp\pi}{mq}\Big)\Big)$$
$$=\prod_{j\in J\setminus J_k}\Big(2-2\cos\Big(\frac{2jp\pi}{q}\Big)\Big)=\prod_{j=1}^{q-1}\Big(2-2\cos\Big(\frac{2jp\pi}{q}\Big)\Big)^m.$$

The Chebyshev polynomial $T_q(x)=\cos(q\arccos(x))$ has the following property. The roots of the equation $T_q(x)-1=0$ are $\cos(\frac{2j\pi}{q})$, $j=0,1,\ldots,q-1$. Since the leading coefficient of $T_q(x)$ is 2^{q-1}, for $x\neq 1$ we have the identity

$$\prod_{j=1}^{q-1}\Big(2x-2\cos\Big(\frac{2j\pi}{q}\Big)\Big)=\frac{T_q(x)-1}{x-1}.$$

As p and q are relatively prime we obtain

$$\prod_{j=1}^{q-1}\left(2-2\cos\left(\frac{2jp\pi}{q}\right)\right)^m=\prod_{j=1}^{q-1}\left(2-2\cos\left(\frac{2j\pi}{q}\right)\right)^m=$$

$$\left(\lim_{x\to 1}\frac{T_q(x)-1}{x-1}\right)^m=(q^2)^m=\left(\frac{n}{m}\right)^{2m}.$$

Hence

$$\prod_{j\in J\setminus J_k}4\sin^2\left(\frac{jk\pi}{n}\right)=\left(\frac{n}{m}\right)^{2m}.$$

In a similar way we obtain

$$\prod_{j\in J\setminus J_l}4\sin^2\left(\frac{jl\pi}{n}\right)=\left(\frac{n}{m'}\right)^{2m'}.$$

To get the final result we use the following trivial inequality. For any integers $a\geq 2$ and $b\geq 2$ we have $a^b\geq ab$. Since $q=n/m\geq 2$ and $q'=n/m'\geq 2$, we conclude

$$n\,\tau(n)=\prod_{j=1}^{n-1}P(\varepsilon_n^j)\geq\left(\frac{n}{m}\right)^{2m}\left(\frac{n}{m'}\right)^{2m'}\geq n^2n^2=n^4. \qquad \square$$

Using Lemma 5.6, one can show the following theorem.

Theorem 5.7. *For any given I-graph $I(n,k,l)$ the minimum number of generators for Jacobian $\mathrm{Jac}(I(n,k,l))$ is at least 2 and at most $2k+2l-1$.*

Proof. The upper bound for the number of generators follows from Theorem 4.1. Indeed, by this theorem the group $\mathrm{coker}(L(I(n,k,l))\cong\mathrm{Jac}(I(n,k,l))\oplus\mathbb{Z}$ is generated by $2k+2l$ elements. One of these generators is needed to generate the infinite cyclic group $\mathbb{Z}$. Hence $\mathrm{Jac}(I(n,k,l))$ is generated by $2k+2l-1$ elements.

To get the lower bound we use Lemma 5.6. Let us suppose that $\mathrm{Jac}(I(n,k,l))$ is generated by one element. Then it is the cyclic group of order $\tau(n)$. Denote by D be a product of all distinct nonzero eigenvalues of $I(n,k,l)$. By Proposition 2.6 from [20] the order of each element of $\mathrm{Jac}(I(n,k,l))$ is divisor of D. Hence, $\tau(n)$ is divisor of D and we have inequality $D\geq\tau(n)$. By the Kirchhoff theorem we have $2n\tau(n)=\lambda_{2,0}\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j}$. We note that all algebraic numbers $\lambda_{i,j}$ comes into product together with its Galois conjugate, so $2n\tau(n)$ is a multiple of D. In particular $2n\tau(n)\geq D$.

From the proof of Theorem 5.5 we have $n\tau(n)=(\prod_{j=1}^{(n-1)/2}\lambda_{1,j}\lambda_{2,j})^2$ if n is odd and $n\tau(n)=\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}(\prod_{j=1}^{n/2-1}\lambda_{1,j}\lambda_{2,j})^2$ if n is even. Moreover, the value $\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}$ is equal to 4 if k and l are of different parity and 24 if both k and l are odd. The case when both k and l are even is impossible as k and l are relatively prime.

Now, we have $4n\tau(n)=(2\prod_{j=1}^{(n-1)/2}\lambda_{1,j}\lambda_{2,j})^2$ if n is odd. Again, all algebraic numbers $\lambda_{i,j}$ comes into the product $\rho=2\prod_{j=1}^{(n-1)/2}\lambda_{1,j}\lambda_{2,j}$ together with its Galois conjugate. Therefore, the product ρ is an integer number and contains all distinct nonzero eigenvalues. Hence ρ is a multiple of D. So we obtain $4n\tau(n)=\rho^2\geq D^2\geq\tau(n)^2$.

Also we get $4n\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}\tau(n) = (2\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}\prod_{j=1}^{n/2-1}\lambda_{1,j}\lambda_{2,j})^2$ if n is even. By a similar argument, taking into account the inequality $24 \geq \lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}$ we obtain $96n\tau(n) \geq 4n\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}\tau(n) \geq D^2 \geq \tau(n)^2$.

As result, by Lemma 5.6 we have $4n \geq \tau(n) \geq n^3$ if n is odd and $96n \geq \tau(n) \geq n^3$ if n is even. For $n \geq 10$ this is impossible. So, the rank of $\mathrm{Jac}(I(n, k, l))$ is at least two for all $n \geq 10$. For n less than 10 this statement can be proved by direct calculation. □

For graphs $I(4, 2, 3)$ and $I(6, 3, 4)$, the Jacobian group $\mathrm{Jac}(I(n, k, l))$ is generated by 2 elements. The upper bound $2k + 2l - 1$ for the minimum number of generators of $\mathrm{Jac}(I(n, k, l))$ is attained for graph $I(34, 2, 3)$ and $I(170, 3, 4)$. See Tables 2 and 3 in Section 7.

So the lower bound 2 and the upper bound $2k + 2l - 1$ for the minimum number of generators of $\mathrm{Jac}(I(n, k, l))$ are sharp.

6 Asymptotic for the number of spanning trees

The asymptotic for the number of spanning trees of the graph $I(n, k, l)$ is given in the following theorem.

Theorem 6.1. *Let $P(z) = (3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$. Suppose that k and l are relatively prime and set $A_{k,l} = \prod_{P(z)=0,\,|z|>1}|z|$. Then the number $\tau_{k,l}(n)$ of spanning trees of the graph $I(n, k, l)$ has the asymptotic*

$$\tau_{k,l}(n) \sim \frac{n}{k^2 + l^2}A_{k,l}^n,\ n \to \infty.$$

Proof. By Theorem 5.1 we have

$$\tau_{k,l}(n) = (-1)^{(n-1)(k+l)}n\prod_{s=1}^{k+l-1}\frac{T_n(w_s) - 1}{w_s - 1},$$

where w_s, $s = 1, 2, \ldots, k + l - 1$ are roots of the polynomial

$$Q(w) = \frac{(3 - 2T_k(w))(3 - 2T_l(w)) - 1}{w - 1}.$$

So we obtain

$$\tau_{k,l}(n) = n\prod_{s=1}^{k+l-1}\left|\frac{T_n(w_s) - 1}{w_s - 1}\right| = n\prod_{s=1}^{k+l-1}|T_n(w_s) - 1|\Big/\prod_{s=1}^{k+l-1}|w_s - 1|.$$

By Lemma 5.2 we have $T_n(w_s) = \frac{1}{2}(z_s^n + z_s^{-n})$, where the z_s and $1/z_s$ are roots of the polynomial $P(z)$ with the property $|z_s| \neq 1$, $s = 1, 2, \ldots, k + l - 1$. Replacing z_s by $1/z_s$, if it is necessary, we can assume that all $|z_s| > 1$ for all $s = 1, 2, \ldots, k + l - 1$. Then $T_n(w_s) \sim \frac{1}{2}z_s^n$ as n tends to ∞. So $|T_n(w_s) - 1| \sim \frac{1}{2}|z_s|^n$ as $n \to \infty$. Hence

$$\prod_{s=1}^{k+l-1}|T_n(w_s) - 1| \sim \frac{1}{2^{k+l-1}}\prod_{s=1}^{k+l-1}|z_s|^n = \frac{1}{2^{k+l-1}}\prod_{P(z)=0,\,|z|>1}|z|^n = \frac{1}{2^{k+l-1}}A_{k,l}^n.$$

Now we directly evaluate the quantity $\prod_{s=1}^{k+l-1} |w_s - 1|$. We note that

$$Q(w) = a_0 w^{k+l-1} + a_1 w^{k+l-2} + \cdots + a_{k+l-2} w + a_{k+l-1}$$

is an integer polynomial with the leading coefficient $a_0 = 2^{k+l}$. From here we obtain

$$\prod_{s=1}^{k+l-1} |w_s - 1| = \prod_{s=1}^{k+l-1} |1 - w_s| = \left|\frac{1}{a_0} Q(1)\right| = \frac{2(k^2 + l^2)}{2^{k+l}} = \frac{k^2 + l^2}{2^{k+l-1}}.$$

Indeed,

$$\begin{aligned} Q(1) &= \lim_{w \to 1} \frac{(3 - 2T_k(w))(3 - 2T_l(w)) - 1}{w - 1} \\ &= -2T_k'(1)(3 - 2T_l(1)) - 2T_l'(1)(3 - 2T_k(1)) \\ &= -2kU_{k-1}(1)(3 - 2T_l(1)) - 2lU_{l-1}(1)(3 - 2T_k(1)) = -2(k^2 + l^2) \end{aligned}$$

and $a_0 = 2^{k+l}$.

In order to get the statement of the theorem we combine the above mentioned results. Then

$$\tau_{k,l}(n) \sim n \, \frac{A_{k,l}^n}{2^{k+l-1}} \Big/ \frac{k^2 + l^2}{2^{k+l-1}} = \frac{n}{k^2 + l^2} A_{k,l}^n \text{ as } n \to \infty. \qquad \square$$

Remark 6.2. It was noted by professor A. Yu. Vesnin that constant $A_{k,l}$ coincides with the Mahler measure of Laurent polynomial $P(z) = (3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$. It gives a simple way to evaluate $A_{k,l}$ using the following formula

$$A_{k,l} = \exp\left(\int_0^1 \log |P(e^{2\pi i t})| \mathrm{d}t\right).$$

See, for example, [13, p. 6] for the proof.

The numerical values for $A_{k,l}$, where k and l are relatively prime numbers $1 \leq k \leq l \leq 9$ will be given in Table 1 in the Section 7.

7 Examples and tables

7.1 Examples

1° The Prism graph $I(n, 1, 1)$. We have the following asymptotic

$$\tau_{1,1}(n) = n(T_n(2) - 1) \sim \frac{n}{2}(2 + \sqrt{3})^n, \; n \to \infty.$$

2° The generalized Petersen graph $GP(n, 2) = I(n, 1, 2)$. The the number of spanning trees (see [19]) behaves like $\tau_{1,2}(n) \sim \frac{n}{5} A_{1,2}^n$, $n \to \infty$, where

$$A_{1,2} = \frac{7 + \sqrt{5} + \sqrt{38 + 14\sqrt{5}}}{4} \cong 4.39026.$$

3° The smallest proper I-graph $I(n,2,3)$ has the following asymptotic for the number of spanning trees

$$\tau_{2,3}(n) \sim \frac{n}{13} A_{2,3}^n,\ n \to \infty.$$

Here $A_{2,3} \cong 4.84199$ is a suitable root of the algebraic equation

$$1 - 7x + 13x^2 - 35x^3 + 161x^4 - 287x^5 + 241x^6 - 371x^7 + 577x^8 \\ - 371x^9 + 241x^{10} - 287x^{11} + 161x^{12} - 35x^{13} + 13x^{14} - 7x^{15} + x^{16} = 0.$$

Here is the table for asymptotic constants $A_{k,l}$ for relatively prime numbers $1 \leq k \leq l \leq 9$.

Table 1: Asymptotic constants $A_{k,l}$.

$k \backslash l$	1	2	3	4	5	6	7	8	9
1	3.7320	4.3902	4.7201	4.8954	4.9953	5.0559	5.0945	5.1203	5.1382
2		-	4.8419	-	5.0249	-	5.1033	-	5.1414
3			-	5.0054	5.0541	-	5.1137	5.1320	-
4				-	5.0802	-	5.1244	-	5.1504
5					-	5.1201	5.1346	5.1461	5.1554
6						-	5.1438	-	-
7							-	5.1589	5.1649
8								-	5.1691

7.2 The tables of Jacobians of I-graphs

Theorem 4.1 is the first step to understand the structure of the Jacobian for $I(n,k,l)$. Also, it gives a simple way for numerical calculations of $\mathrm{Jac}(I(n,k,l))$ for small values of k and l. See Tables 2 and 3.

The first example of Jacobian $\mathrm{Jac}(I(n,3,4))$ with the maximum rank 13:

$$n = 170,$$
$$\mathrm{Jac}(I(170,3,4)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4^8 \oplus \mathbb{Z}_{6108} \oplus \mathbb{Z}_{30540} \oplus \mathbb{Z}_{2^2\cdot 3\cdot 5\cdot 103\cdot 509\cdot 1699\cdot 11593\cdot p\cdot q} \\ \oplus \mathbb{Z}_{2^2\cdot 3\cdot 5\cdot 17\cdot 103\cdot 509\cdot 1699\cdot 11593\cdot p\cdot q},$$

and

$$\tau_{3,4}(170) = 2^{25} \cdot 3^4 \cdot 5^3 \cdot 17 \cdot 103^2 \cdot 509^4 \cdot 1699^2 \cdot 11593^2 \cdot p^2 \cdot q^2,$$

where $p = 16901365279286026289$ and $q = 34652587005966540929$.

Table 2: Graph $I(n, 2, 3)$.

n	$\mathrm{Jac}(I(n,2,3))$	$\tau_{2,3}(n) = \lvert \mathrm{Jac}(I(n,2,3)) \rvert$
4	$\mathbb{Z}_7 \oplus \mathbb{Z}_{28}$	196
5	$\mathbb{Z}_{19} \oplus \mathbb{Z}_{95}$	1805
6	$\mathbb{Z}_{19} \oplus \mathbb{Z}_{114}$	2166
7	$\mathbb{Z}_{83} \oplus \mathbb{Z}_{581}$	48223
8	$\mathbb{Z}_{161} \oplus \mathbb{Z}_{1288}$	207368
9	$\mathbb{Z}_{289} \oplus \mathbb{Z}_{2601}$	751689
10	$\mathbb{Z}_{1558} \oplus \mathbb{Z}_{3895}$	6068410
11	$\mathbb{Z}_{1693} \oplus \mathbb{Z}_{18623}$	31528739
12	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{665} \oplus \mathbb{Z}_{7980}$	132667500
13	$\mathbb{Z}_{25} \oplus \mathbb{Z}_{325} \oplus \mathbb{Z}_{325} \oplus \mathbb{Z}_{325}$	858203125
14	$\mathbb{Z}_{17513} \oplus \mathbb{Z}_{245182}$	4293872366
15	$\mathbb{Z}_{37069} \oplus \mathbb{Z}_{556035}$	20611661415
16	$\mathbb{Z}_{84847} \oplus \mathbb{Z}_{1357552}$	115184214544
17	$\mathbb{Z}_2^6 \oplus \mathbb{Z}_{23186} \oplus \mathbb{Z}_{394162}$	584898568448
18	$\mathbb{Z}_{400843} \oplus \mathbb{Z}_{7215174}$	2892151991682
19	$\mathbb{Z}_{898243} \oplus \mathbb{Z}_{17066617}$	15329969253931
20	$\mathbb{Z}_{19}^4 \oplus \mathbb{Z}_{5453} \oplus \mathbb{Z}_{109060}$	77502443441780
21	$\mathbb{Z}_{4301807} \oplus \mathbb{Z}_{90337947}$	388616412770229
22	$\mathbb{Z}_{9536669} \oplus \mathbb{Z}_{209806718}$	2000857223542342
23	$\mathbb{Z}_{20949827} \oplus \mathbb{Z}_{481846021}$	10094590780588367
24	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{9192295} \oplus \mathbb{Z}_{220615080}$	50598972420215000
25	$\mathbb{Z}_{101468531} \oplus \mathbb{Z}_{2536713275}$	257396569582449025
26	$\mathbb{Z}_{25} \oplus \mathbb{Z}_{325} \oplus \mathbb{Z}_{8923525} \oplus \mathbb{Z}_{17847050}$	1293976099416406250
27	$\mathbb{Z}_{490309597} \oplus \mathbb{Z}_{13238359119}$	6490894524578165043
28	$\mathbb{Z}_{49} \oplus \mathbb{Z}_{154342069} \oplus \mathbb{Z}_{4321577932}$	32683062689111444092
29	$\mathbb{Z}_{2376466133} \oplus \mathbb{Z}_{68917517857}$	163780147157583236981
30	$\mathbb{Z}_{19} \oplus \mathbb{Z}_{19} \oplus \mathbb{Z}_{275089049} \oplus \mathbb{Z}_{8252671470}$	819549256247415262830
31	$\mathbb{Z}_{11507960491} \oplus \mathbb{Z}_{356746775221}$	4105427794534925793511
32	$\mathbb{Z}_{25318259953} \oplus \mathbb{Z}_{810184318496}$	20512457185525873990688
33	$\mathbb{Z}_{55700389051} \oplus \mathbb{Z}_{1838112838683}$	102383600234281102459833
34	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^6 \oplus \mathbb{Z}_{1915580948} \oplus \mathbb{Z}_{32564876116}$	511022336096582352633856
35	$\mathbb{Z}_{269747901677} \oplus \mathbb{Z}_{9441176558695}$	2546737566070056079431515

Table 3: Graph $I(n,3,4)$.

n	$\mathrm{Jac}(I(n,3,4))$	$\tau_{3,4}(n) = \lvert\mathrm{Jac}(I(n,3,4))\rvert$
5	$\mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$	2000
6	$\mathbb{Z}_{19} \oplus \mathbb{Z}_{114}$	2166
7	$\mathbb{Z}_{71} \oplus \mathbb{Z}_{497}$	35287
8	$\mathbb{Z}_{73} \oplus \mathbb{Z}_{584}$	42632
9	$\mathbb{Z}_{289} \oplus \mathbb{Z}_{2601}$	751689
10	$\mathbb{Z}_2 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{60} \oplus \mathbb{Z}_{60} \oplus \mathbb{Z}_{60}$	5184000
11	$\mathbb{Z}_{1541} \oplus \mathbb{Z}_{16951}$	26121491
12	$\mathbb{Z}_{11} \oplus \mathbb{Z}_{11} \oplus \mathbb{Z}_{209} \oplus \mathbb{Z}_{2508}$	63424812
13	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{1555} \oplus \mathbb{Z}_{20215}$	785858125
14	$\mathbb{Z}_{16969} \oplus \mathbb{Z}_{237566}$	4031257454
15	$\mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{17410} \oplus \mathbb{Z}_{52230}$	18186486000
16	$\mathbb{Z}_{71321} \oplus \mathbb{Z}_{1141136}$	81386960656
17	$\mathbb{Z}_2^6 \oplus \mathbb{Z}_{23186} \oplus \mathbb{Z}_{394162}$	584898568448
18	$\mathbb{Z}_{400843} \oplus \mathbb{Z}_{7215174}$	2892151991682
19	$\mathbb{Z}_{37} \oplus \mathbb{Z}_{37} \oplus \mathbb{Z}_{23939} \oplus \mathbb{Z}_{454841}$	14906272578931
20	$\mathbb{Z}_8 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{120} \oplus \mathbb{Z}_{79080} \oplus \mathbb{Z}_{79080}$	72042006528000
21	$\mathbb{Z}_{4487981} \oplus \mathbb{Z}_{94247601}$	422981442583581
22	$\mathbb{Z}_{10002631} \oplus \mathbb{Z}_{220057882}$	2201157792287542
23	$\mathbb{Z}_{22138559} \oplus \mathbb{Z}_{509186857}$	11272663275719063
24	$\mathbb{Z}_{187} \oplus \mathbb{Z}_{187} \oplus \mathbb{Z}_{259369} \oplus \mathbb{Z}_{6224856}$	56458663080288216
25	$\mathbb{Z}_{2114} \oplus \mathbb{Z}_{52850} \oplus \mathbb{Z}_{52850} \oplus \mathbb{Z}_{52850}$	312061332000250000

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The isolated-pentagon rule and nice substructures in fullerenes*

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Abstract

After fullerenes were discovered, Kroto in 1987 proposed first the isolated-pentagon rule (IPR): the most stable fullerenes are those in which no two pentagons share an edge, that is, each pentagon is completely surrounded by hexagons. To now the structures of the synthesized and isolated (neutral) fullerenes meet this rule. The IPR can be justified from local strain in geometry and π-electronic resonance energy of fullerenes. If two pentagons abut in a fullerene, a 8-circuit along the perimeter of the pentalene (a pair of abutting pentagons) occurs. This paper confirms that such a 8-circuit is always a conjugated cycle of the fullerene in a graph-theoretical approach. Since conjugated circuits of length 8 destabilize the molecule in conjugated circuit theory, this result gives a basis for the IPR in π-electronic resonance. We also prove that each 6-circuit (hexagon) and each 10-circuit along the perimeter of a pair of abutting hexagons are conjugated. Two such types of conjugated circuit satisfy the $(4n + 2)$-rule, and thus stabilise the molecule.

Keywords: Fullerene, patch, stability, isolated pentagon rule, Kekulé structure, conjugated cycle, cyclic edge-cut.

Math. Subj. Class.: 05C70, 05C10, 92E10

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1 Introduction

The *fullerenes* are closed carbon-cage molecules such that every carbon atom has bonds to three other atoms, and the length of each carbon ring is either 5 or 6. Ever since the first fullerene, Buckministerfullerene C_{60}, was discovered by Kroto et al. in 1985 [15], the stabilities of fullerenes have attracted many theorist's attentions. The simple Hückel molecular orbital model that predicts reliably the relative stabilities of planar aromatic hydrocarbons is not generally found to work so well for fullerenes. Kroto [14] in 1987 proposed first the isolated-pentagon rule (IPR): the most stable fullerenes are those in which no two pentagons share an edge, that is, each pentagon is completely surrounded by hexagons. Schmalz et al. [23] gave a more theoretical discussion of the rule in support of the fullerene hypothesis. Indeed the structures of the synthesized and isolated fullerenes meet this rule. The IPR can be justified from local strain and π-electronic resonance of fullerenes; for details, also see a book due to Fowler and Manolopoulos [7]. Pentagon adjacency leads to higher local curvature of the molecule surface and increases the strain energy. On the other hand, according to Hückel $(4n+2)$-rule, conjugated circuits of length $6, 10, 14, \ldots$ stabilize the molecule, whereas conjugated circuits of length $4, 8, 12, \ldots$ destabilize the molecule. Here a conjugated circuit is a cycle of alternating single and double bonds within a Kekulé structure. If two pentagons abut in a fullerene, the conjugated or resonant 8-circuit along the perimeter of the pentalene may occur, and this leads to resonance destabilization [22]. This is an interpretation of IPR in π-electronic resonation stabilization. However, a problem occurs: In a fullerene, is every 8-length circuit conjugated? To now we have not seen any definite answer to this problem in mathematics. In this article we investigate nice patches of a fullerene by applying some small cyclic edge-cuts of graphs and present a positive answer to the above problem (a patch of a fullerene is nice if its Kekulé structure can be extended to a Kekulé structure of the entire fullerene). As immediate consequences of our main theorems, we have that every 8-length circuit of a fullerene surrounds a pentalene (a pair of abutting pentagons) and is conjugated or alternating with respect to a Kekulé structure (see Corollary 3.4). This confirms the destabilization of any pentalene as a nice substructure to the entire fullerene and thus gives a mathematical support for the IPR of fullerenes. Furthermore we also show that in a fullerene every hexagon is a conjugated 6-circuit (see Corollary 3.3) and the boundary along a naphthalene (i.e. a pair of abutting hexagons) is a conjugated 10-circuit (see Corollary 4.2). The former has already been proved (see [26]). In conjugated circuit theory [10, 19, 20], conjugated 6-circuits and 10-circuits contribute stabilizations of fullerenes and the small conjugated circuits have the greatest effects (positive and negative) on stability. For recent discussions on the IPR of fullerenes about steric strain factor and π-electronic resonance factor, see [1, 2, 8, 13, 21]. For mathematical aspects of fullerenes, see a recent survey [3].

2 Preliminary

To obtain the above end we now start our arguments in a graph-theoretical approach. As a molecular graph of a fullerene, a *fullerene graph* is a 3-connected planar cubic graph with only pentagonal and hexagonal faces. It is well known that a fullerene graph on n vertices exists for every even $n \geq 20$ except $n = 22$ [9]. By Euler's polyhedron formula, every fullerene graph with n vertices has exactly 12 pentagonal faces and $(n/2 - 10)$ hexagonal faces.

Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. An edge set M of a graph

G is called a *matching* if no two edges in M have a common endvertex. A matching M of G is *perfect* if every vertex of G is incident with one edge in M. In organic molecular graphs, perfect matchings correspond to Kekulé structures, playing an important role in analysis of the resonance energy and stability of polycyclic aromatic hydrocarbons.

The following classical theorem is Tutte's 1-factor theorem on the existence of perfect matching of a graph [24]. For detailed monograph on matching theory, see Lovász and Plummer [17].

Theorem 2.1. *A graph G has a perfect matching if and only if* $\mathrm{odd}(G-S) \le |S|$ *for each* $S \subseteq V(G)$*, where* $\mathrm{odd}(G-S)$ *denotes the number of odd components in subgraph* $G-S$*.*

Subgraph G' of a graph G is called *nice* if $G - V(G')$ has a perfect matching. In particular, an even cycle C of a graph G is nice if G has a perfect matching M such that C is an M-alternating cycle, i.e. the edges of C alternate in M and $E(G) \setminus M$. A nice even cycle is also called *resonant* or *conjugated* cycle (or circuit) in chemical literature. For convenience, a cycle of length k is said to be a k-cycle or k-circuit.

For nonempty subsets X, Y of $V(G)$, let $[X, Y]$ denote the set of edges of G that each has one end-vertex in X and the other in Y. If $\overline{X} = V(G) \setminus X \neq \emptyset$, then $\nabla(X) := [X, \overline{X}]$ is called an *edge-cut* of G, and k*-edge-cut* whenever $|[X, \overline{X}]| = k$. The edges incident with a single vertex form a *trivial* edge-cut. For a subgraph H of G, let $\overline{H} := G - V(H)$. We simply write $\nabla(H)$ for $\nabla(V(H))$.

Lemma 2.2 ([25])**.** *Every* 3*-edge-cut of a fullerene graph is trivial.*

Lemma 2.3 ([25])**.** *Every* 4*-edge-cut of a fullerene graph isolates an edge.*

An edge-cut $S = \nabla(X)$ of G is *cyclic* if at least two components of $G-S$ each contains a cycle. The minimum size of cyclic edge-cuts of G is called *cyclic edge-connectivity* of G, denoted by $c\lambda(G)$.

Theorem 2.4 ([6, 12, 18])**.** *Let F be any fullerene graph. Then* $c\lambda(F) = 5$.

From the definition with the above properties we know that each fullerene graph has the girth 5 (the minimum length of cycles), and each of its 5-cycles and 6-cycles bounds a face. A cyclic k-edge-cut of a graph isolating just a k-cycle will be called *trivial*.

Theorem 2.5 ([12, 16])**.** *A fullerene graph with a non-trivial cyclic* 5*-edge-cut is a nanotube with two disjoint pentacaps (see Figure 1), and each non-trivial cyclic* 5*-edge-cut must be an edge set between two consecutive concentric cycles of length* 10.

A fullerene patch is a 2-connected plane graph with all faces pentagonal or hexagonal except one external face, all internal vertices (not incident with the external face) of degree 3 and those incident with the external face having degree 2 or 3. The cycle bounding the external face is the *boundary* of the patch. We can count the pentagons of a fullerene patch as internal faces as follows.

Lemma 2.6 ([4])**.** *For fullerene patch G, let p_5 denote the number of pentagonal faces other than the external face. Then*

$$p_5 = 6 + k_3 - k_2 = 6 + 2k_3 - l, \tag{2.1}$$

where k_2 and k_3 denote the number of vertices of degree 2 *and* 3 *on the boundary of G, respectively, and l is the boundary length.*

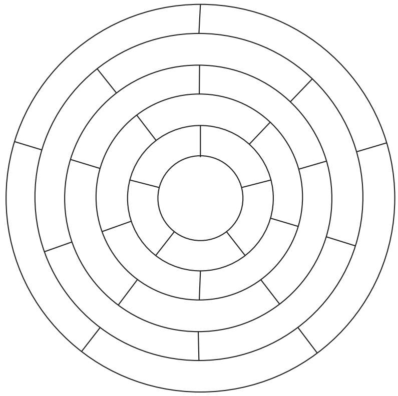

Figure 1: A fullerene with a non-trivial cyclic 5-edge-cut.

For $T \subseteq V(G)$, the induced subgraph of G by T consists of T and all edges whose endvertices are contained in T, denoted by $G[T]$.

In the next two sections we will investigate nice patches of fullerene graphs in cyclic 6-edge-cut and 8-edge-cut cases, respectively.

3 Cyclic 6-edge-cut

We first consider a more general case than fullerene patches.

Theorem 3.1. *Let F_0 be a connected induced subgraph of a fullerene graph F such that interior faces of F_0 exist and each one is a pentagon or hexagon. If F has exactly six edges from F_0 to the outside $\overline{F}_0 = F - V(F_0)$, then F_0 has a perfect matching.*

Proof. Let n_0 and ϵ_0 denote the numbers of vertices and edges of F_0 respectively. Then $3n_0 = 2\epsilon_0 + 6$, which implies that n_0 is even, i.e. F_0 has an even number of vertices.

We will prove that F_0 has a perfect matching by Tutte's theorem. To the contrary suppose that F_0 has no perfect matchings. By Theorem 2.1, there exists a subset $X_0 \subset V(F_0)$ such that

$$\mathrm{odd}(F_0 - X_0) > |X_0|. \tag{3.1}$$

For the sake of convenience, let $\alpha := \mathrm{odd}(F_0 - X_0)$. Since α and $|X_0|$ have the same parity, we have

$$\alpha \geq |X_0| + 2. \tag{3.2}$$

Let $G_1, \dots, G_\alpha$ and $G_{\alpha+1}, \dots, G_{\alpha+\beta}$ denote respectively the odd components and the even components of $F_0 - X_0$, where β denotes the number of even components of $F_0 - X_0$. For $i = 1, 2, \dots, \alpha+\beta$, let m_i denote the number of edges of F_0 which are sent to X_0 from G_i, and γ_i (resp. γ_0) the number of edges of F from G_i (resp. X_0) to $\overline{F}_0$. Since $\nabla(F_0)$ is a 6-edge-cut of F, we have

$$|\nabla(F_0)| = \sum_{i=0}^{\alpha+\beta} \gamma_i = 6. \tag{3.3}$$

Since F is 3-connected, for $i = 1, \dots, \alpha, \dots, \alpha+\beta$ we have

$$|\nabla(G_i)| = m_i + \gamma_i \geq 3. \tag{3.4}$$

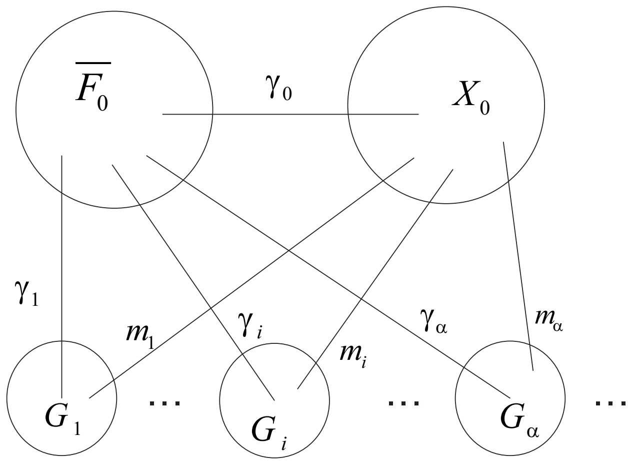

Figure 2: Illustration for the proof of Theorem 3.1.

By taking the number of edges of F from the components G_i to $\overline{F}_0$ and X_0 into account and by using Equation (3.3) and Inequalities (3.2) and (3.4) we have

$$\begin{aligned} 3(\alpha+\beta) &\leq \sum_{i=1}^{\alpha+\beta}(m_i+\gamma_i) \\ &\leq 3|X_0| - \gamma_0 + \sum_{i=1}^{\alpha+\beta}\gamma_i \\ &= 3|X_0| + 6 - 2\gamma_0 \\ &\leq 3\alpha - 2\gamma_0, \end{aligned} \tag{3.5}$$

which implies that $\beta = 0$, $\gamma_0 = 0$ and equalities always hold. Hence $\sum_{i=1}^{\alpha}\gamma_i = 6$, and $\alpha = |X_0| + 2$. Further, the second equality in (3.5) implies that X_0 is an independent set of F_0. The first equality in (3.5) implies that $m_i + \gamma_i = 3$ for each $1 \leq i \leq \alpha$, that is, $\nabla(G_i)$ is a 3-edge-cut of F. So by Lemma 2.2 it is a trivial edge-cut and each G_i is a singleton. Let Y_0 denote the set of all singletons G_i. Then F_0 is a bipartite graph with partite sets X_0 and Y_0.

If F_0 has no vertices of degree one, then F_0 is 2-connected. Otherwise, F_0 has a bridge, the deletion of which results in two components each containing a cycle. So the bridge together with at most three edges in $\nabla(F_0)$ form a cyclic edge-cut, contradicting that $c\lambda(F) = 5$ (Theorem 2.4). Hence F_0 is a fullerene patch. Since $k_2 = |\nabla(F_0)| = 6$, by Lemma 2.6 we have that the number p_5 of pentagons contained in F_0 is equal to the number k_3 of vertices of degree three lying on the boundary of F_0. Since F_0 is bipartite, $k_3 = p_5 = 0$, which implies that F_0 is just a hexagon, contradicting that $\alpha = |Y_0| = |X_0| + 2$.

If F_0 has a vertex x of degree one, let xy be the edge of F_0, and xy_1 and xy_2 be the other two edges in F incident with x. Then $\nabla(F_0 - x) = (\nabla(F_0) \setminus \{xy_1, xy_2\}) \cup \{xy\}$ forms a cyclic 5-edge-cut of F since $F_0 - x$ contains all cycles of F_0 and $\overline{F_0 - x}$ can be obtained from $F - F_0$ by adding a 2-length path y_1xy_2 and contains at least seven pentagons. Since $F_0 - x$ is bipartite, cyclic 5-edge-cut $\nabla(F_0 - x)$ is not trivial, and $F_0 - x$

is always 2-connected from Theorem 2.5. By Lemma 2.6 we have $p_5 = k_3 + 1$ for the fullerene patch $F_0 - x$, which implies that F_0 has at least one pentagon, contradicting that F_0 is bipartite. □

Corollary 3.2. *For each cyclic* 6*-edge cut* E_0 *of a fullerene graph* F*, both components of* $F - E_0$ *have a perfect matching.*

Proof. It follows that $F - E_0$ has exactly two components from Lemma 2.2 and 3-edge-connectedness of F. Such two components fulfil the conditions of Theorem 3.1 and thus each has a perfect matching. □

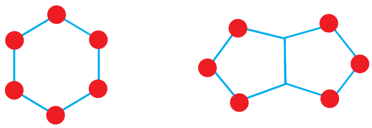

Figure 3: Some nice substructures of fullerene graphs.

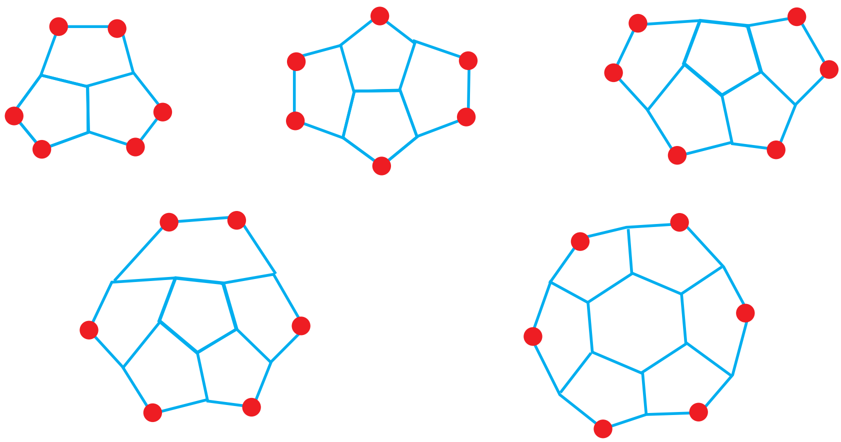

Figure 4: Some nice patches of fullerene graphs with six 2-degree vertices.

From Corollary 3.2 we can find many nice substructures of fullerene graphs, examples of which are shown in Figures 3 and 4. It should be mentioned that the third nice substructure fulvene in Figure 3 has been discovered by Došlić applying 2-extendability of fullerenes [5, 27], and the first one has been proved in investigating k-resonance [26, 11]; see the following.

Corollary 3.3 ([26])**.** *Each hexagon of a fullerene graph is resonant.*

Corollary 3.4. *Each* 8*-length cycle (if exists) of a fullerene graph bounds a pentalene (a pair of abutting pentagons) and is thus resonant.*

Proof. Let C be a 8-length cycle of a fullerene graph F. If F has an edge e whose endvertices both lie in C but $e \notin E(C)$, then e is called a *chord* of C. If C has no chords, then the eight edges issuing from C can be classified into two edge-cuts of size from 3 to 5, which lie in the interior and the exterior of C respectively. If one is a 3-edge-cut, then Lemma 2.2 implies that it is trivial, and thus a triangle or quadrilateral appear, a contradiction. If both are 4-edge-cuts, then Lemma 2.3 implies that F has only 12 vertices, also a contradiction. So C must have a chord. Further, this chord and C form a pair of 5-length cycles sharing this chord, which must bound pentagonal faces of F by Theorem 2.4. That is, C bounds a pentalene and is resonant from Corollary 3.2. □

4 Cyclic 8-edge-cut

Theorem 4.1. *If E_0 is a cyclic 8-edge-cut of a fullerene graph F and E_0 is a matching, then $F - E_0$ has a perfect matching.*

Proof. There exists a nonempty and proper subset X of vertex set $V(F)$ such that $E_0 = \nabla(X) = [X, \overline{X}]$. Let $F_0 := F[X]$ and $\overline{F}_0 := F[\overline{X}]$. We claim that both F_0 and $\overline{F}_0$ are connected and E_0 is a minimal edge-cut. If not, then one of F_0 and $\overline{F}_0$, say $\overline{F}_0$, is disconnected. Then $\overline{F}_0$ has exactly two components since F is 3-connected. Since E_0 is a matching, F_0 and each component of $\overline{F}_0$ have the minimum degree 2 and contain a cycle. So a cyclic edge-cut of at most four edges occurs in F, a contradiction. So the claim is verified. Hence each of F_0 and $\overline{F}_0$ has exactly one face of size more than six, which has exactly 8 two-degree vertices on its boundary.

We only show that F_0 has a perfect matching (the same for $\overline{F}_0$). If F_0 has a bridge, then it follows that F_0 can be obtained from two pentagons by adding one edge between them by Theorems 2.4 and 2.5. In this case F_0 has a perfect matching. So in the following we always suppose that F_0 is a patch of F. We adopt similar arguments and notations as in the proof of Theorem 3.1 (see Figure 2). It is known that F_0 has an even number of vertices. Suppose to the contrary that F_0 has no perfect matchings. By Tutte's theorem we can choose a *minimal* subset $X_0 \subset V(F_0)$ satisfying $\alpha := \text{odd}(F_0 - X_0) \geq |X_0| + 2$.

Let $G_1, \ldots, G_\alpha$ and $G_{\alpha+1}, \ldots, G_{\alpha+\beta}$ denote respectively the odd components and the even components of $F_0 - X_0$. For $i = 1, 2, \ldots, \alpha + \beta$, let m_i denote the number of edges of F_0 which are sent to X_0 from G_i, and γ_i (resp. γ_0) the number of edges of F from G_i (resp. X_0) to the patch $\overline{F}_0$. By $|\nabla(F_0)| = \sum_{i=0}^{\alpha+\beta} \gamma_i = 8$ and Inequality (3.4), we have

$$\begin{aligned} 3(\alpha+\beta) &\leq \sum_{i=1}^{\alpha+\beta}(m_i + \gamma_i) \\ &\leq 3|X_0| - \gamma_0 + \sum_{i=1}^{\alpha+\beta}\gamma_i \\ &= 3|X_0| + 8 - 2\gamma_0 \\ &\leq 3\alpha + 2 - 2\gamma_0, \end{aligned} \tag{4.1}$$

which implies that $\beta = 0$, $0 \leq \gamma_0 \leq 1$, and $|X_0| + 2 = \alpha$. So the forth equality in Inequality (4.1) holds.

If $\gamma_0 = 1$, then $|[X - X_0, \overline{X}]| = \sum_{i=1}^{\alpha+\beta} \gamma_i = 7$ and all equalities in Inequality (4.1) hold. Like the proof of Theorem 3.1 we have that X_0 is an independent set, $m_i + \gamma_i = 3$ for each $1 \leq i \leq \alpha$ and each G_i is a singleton. Hence F_0 is a bipartite graph. By Lemma 2.6 we have that F_0 has two three-degree vertices on the boundary of F_0. That implies that

F_0 is just the graph obtained by gluing two hexagons along an edge. So F_0 has the same cardinalities of two partite sets, which contradicts that $|X_0| + 2 = \alpha$.

From now on we suppose that $\gamma_0 = 0$. That is, each vertex of X_0 has degree 3 in F_0. We claim that second equality in Inequality (4.1) must hold. Otherwise, $F_0[X_0]$ has exactly one edge, say uv, and the first equality holds, so each G_i is a singleton. Without loss of generality, suppose that y_1 and y_2 are two neighbors of u other than v, and $V(G_1) = \{y_1\}$ and $V(G_2) = \{y_2\}$. Let $X_0' := X_0 \setminus \{u\}$, and $X_1 := \{u, y_1, y_2\}$. Then $G_1' := F_0[X_1]$ is a 3-vertex path obtained by combining G_1 and G_2 with vertex u. Hence $F_0 - X_0'$ has the odd components $G_1', G_3, \dots, G_\alpha$, and $\text{odd}(F_0 - X_0') = \alpha - 1 = |X_0'| + 2$, contradicting the minimality of X_0.

Hence X_0 is an independent set of F_0, and the first inequality is strict. Since for each $1 \le i \le \alpha$, $m_i + \gamma_i$ is always odd, there exists an i_0 such that $m_{i_0} + \gamma_{i_0} = 5$ and $m_i + \gamma_i = 3$ for all $i \ne i_0$. For convenience, we may suppose that $i_0 = 1$. So G_1 is an odd component with at least three vertices and $G_2, \dots, G_\alpha$ are all singletons. Let Y_0 denote the set of all singletons G_i ($2 \le i \le \alpha$). Then $H := (X_0, Y_0)$ is a bipartite graph as the induced subgraph of fullerene graph F.

If G_1 is a tree, then it is a 2-length path, say xyz, since $\nabla(G_1)$ has exactly five edges. For F_0, by Lemma 2.6 we have $p_5 = k_3 - 2$. Since E_0 is a matching, x and z both have neighbors in X_0, so $\gamma_1 \le 3$. The latter implies $\sum_{i=2}^{\alpha} \gamma_i \ge 5$. That is, the boundary of F_0 contains at least 5 two-degree vertices belonging to Y_0.

We assert that $p_5 \le 2$. Since H is bipartite, any pentagon P of F_0 must intersect G_1. If P only intersects a vertex of G_1, say z, then $P - z$ is a path of length 3 in H which connects two vertices of X_0, contradicting that any path between two vertices in the same partite set of a bipartite graph has an even length. Similarly we have that P cannot contain both edges of G_1. If F_0 has two distinct pentagons sharing the same edge of G_1, then one pentagon must have two edges G_1, a contradiction. So the assertion holds.

By the assertion and $p_5 = k_3 - 2$ we have $k_3 \le 4$. This implies that the boundary of F_0 has at most 4 vertices in X_0. Let C be the boundary of F_0. Then $C - V(C) \cap X_0$ has at most $|V(C) \cap X_0|$ components. On the other hand, $C - V(C) \cap X_0$ has all singletons in $V(C) \cap Y_0$ as components. But $|V(C) \cap Y_0| \ge 5$, contradicting $|V(C) \cap X_0| \le 4$.

From now on suppose that G_1 contains a cycle. Then $\nabla(G_1)$ is a cyclic 5-edge-cut of F. By Theorem 2.5 $\nabla(G_1)$ is a matching and G_1 is also a patch (precisely, G_1 is a pentagon or contains a pentacap according as the cyclic 5-edge-cut $\nabla(G_1)$ is trivial or not), so each vertex of H has degree at least two, and each component of H contains a cycle. If H is disconnected, then H has exactly two components H_1 and H_2 since F_0 is 2-edge-connected and $\nabla(G_1)$ has exactly five edges. Further, between G_1 and each H_i has at least two edges. So $\nabla(G_1)$ has two consecutive edges along the boundary of G_1 separately from G_1 to H_1 and H_2. These two edges must be contained in a cycle of length at least 8 bounding a face of F, a contradiction. Hence H is connected.

Since G_1 and $\overline{F}_0$ are two connected subgraphs of F with exactly one face of size more than six, there are two possible cases to be considered.

Case 1. G_1 and $\overline{F}_0$ lie in different faces of H. Suppose that G_1 lies in a bounded face f of H and $\overline{F}_0$ does in the exterior face of H. Then the boundary ∂f of f is a 10-length cycle since 5 neighbors of G_1 in H belong to X_0 and are separated by 5 vertices in Y_0. Hence F is a nanotube with two pentacaps and F_0 has exactly 6 pentagons. By Lemma 2.6 the boundary of F_0 has exactly 8 vertices of degree 3 in F_0. Hence the boundary of F_0 is an alternating cycle of three-degree and two-degree vertices. But in this nanotube there is only

10-length cycle as such boundary of a patch, a contradiction.

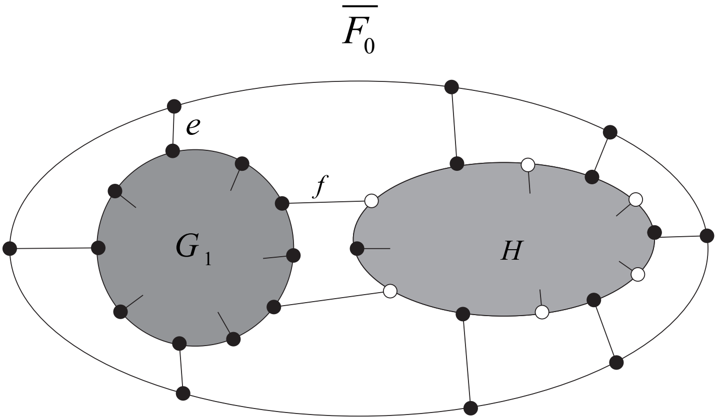

Figure 5: Illustration for Case 2 in the proof of Theorem 4.1 (the vertices in X_0 are colored white and other vertices black).

Case 2. G_1 and $\overline{F}_0$ lie in the exterior face of H. Then the boundary of F_0 is formed by a path P of H and a path P_1 of G_1 and two edges between them. So $0 \leq \gamma_1 \leq 3$, and there are $8 - \gamma_1$ two-degree vertices lying on P, which belong to Y_0 and are thus non-adjacent mutually. So there are at least $7 - \gamma_1$ three-degree vertices in X_0 on P that can separate them. Since the four end-vertices of P and P_1 are all of degree three in F_0, there are at least $11 - \gamma_1$ vertices of degree three of F_0 on the boundary. That is, for F_0, $k_3 \geq 11 - \gamma_1$. On the other hand, if G_1 is a pentagon, then F_0 has at most $5 - \gamma_1$ pentagons, so $k_3 \leq 7 - \gamma_1$ by Lemma 2.6, a contradiction. Otherwise, $\nabla(G_1)$ is a non-trivial cyclic 5-edge-cut and F_0 has exactly 6 pentagons. Hence, by Lemma 2.6 we have that for F_0, $k_3 = 8$. So $\gamma_1 = 3$. Take two consecutive edges e and f of $\nabla(G_1)$ along the boundary of G_1 separately from G_1 to $\overline{F}_0$ and H. Since $\nabla(G_1)$ is a non-trivial cyclic 5-edge-cut, by Theorem 2.5 we have that e and f have non-adjacent end-vertices in G_1. So these two edges belong to a cycle of length at least 7 bounding a face of F (see Figure 5). But this is impossible. □

From Theorem 4.1 we further find many nice substructures of fullerene graphs, which are listed in Figure 6. In particular, the first one is the naphthalene (a pair of abutting hexagons), whose boundary is a resonant cycle of length 10.

Corollary 4.2. *Any adjacent hexagons of a fullerene graph form a nice substructure, and the boundary (*10*-length cycle) is thus resonant.*

However, not all 10-length cycles of fullerene graphs are resonant. For example, see Figure 1. The following corollary gives a criterion for a 10-length cycle of a fullerene graph to be resonant.

Corollary 4.3. *A* 10*-length cycle C of a fullerene graph F is resonant if and only if it bounds either the naphthalene or the second patch in Figure 4.*

Proof. The sufficiency is immediate from Corollaries 3.2 and 4.2. So we only consider the necessity. Suppose that 10-length cycle C of a fullerene graph F is resonant. Let F_0 be

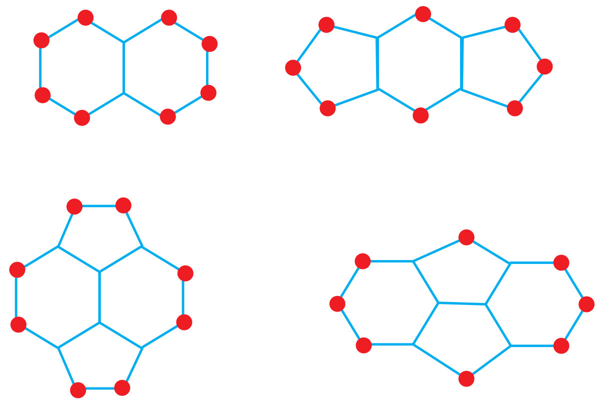

Figure 6: Some nice patches of fullerene graphs with eight 2-degree vertices.

the patch of F bounded by 10-length cycle C with $p_5 \leq 6$. So F_0 has an even number of vertices, and we can have that k_3 and k_2 both are even. By Lemma 2.6 we have $p_5 = 2k_3 - 4$ and $2 \leq k_3 \leq 5$. The possible values of k_3 are 2 and 4. If $k_3 = 2$, then C bounds a pair of adjacent hexagons. If $k_3 = 4$, then F_0 has exactly two vertices in the interior of C which are adjacent by Lemma 2.3. In fact, F_0 is the second patch in Figure 4. □

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Arc-transitive cyclic and dihedral covers of pentavalent symmetric graphs of order twice a prime*

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Abstract

A regular cover of a connected graph is called *cyclic* or *dihedral* if its transformation group is cyclic or dihedral respectively, and *arc-transitive* (or *symmetric*) if the fibre-preserving automorphism subgroup acts arc-transitively on the regular cover. In this paper, we give a classification of arc-transitive cyclic and dihedral covers of a connected pentavalent symmetric graph of order twice a prime. All those covers are explicitly constructed as Cayley graphs on some groups, and their full automorphism groups are determined.

Keywords: Symmetric graph, Cayley graph, bi-Cayley graph, regular cover.

Math. Subj. Class.: 05C25, 20B25

1 Introduction

All groups and graphs considered in this paper are finite, and all graphs are simple, connected and undirected, unless otherwise stated. Let G be a permutation group on a set Ω and let $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$, and *regular* if G is transitive and semiregular. Denote by $\mathbb{Z}_n$, $\mathbb{Z}_n^*$, D_n, A_n and S_n the cyclic group of order n, the multiplicative group of units of $\mathbb{Z}_n$, the dihedral group of order $2n$, the alternating and symmetric group of degree n, respectively. For two groups M and N, we use MN, $M.N$, $M \rtimes N$ and $M \times N$ to denote the product of M and N, an extension of M by N, a split extension of M by N and the direct product of M and N, respectively. For a subgroup H

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of a group G, $C_G(H)$ means the centralizer of H in G and $N_G(H)$ means the normalizer of H in G.

For a graph Γ, we denote its vertex set, edge set and full automorphism group by $V(\Gamma)$, $E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$, respectively. An *s-arc* in Γ is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of Γ such that $\{v_{i-1}, v_i\} \in E(\Gamma)$ for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. A 1-arc is just an *arc*. A graph Γ is (G,s)*-arc-transitive* for a subgroup G of $\mathrm{Aut}(\Gamma)$ if G acts transitively on the set of s-arcs of Γ, and (G,s)*-transitive* if Γ is (G,s)-arc-transitive but not $(G, s+1)$-arc-transitive. A graph Γ is said to be *s-arc-transitive* or *s-transitive* if it is $(\mathrm{Aut}(\Gamma), s)$-arc-transitive or $(\mathrm{Aut}(\Gamma), s)$-transitive, respectively. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. A graph Γ is *edge-transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on the edge set $E(\Gamma)$.

Let Γ be a graph and $N \leq \mathrm{Aut}(\Gamma)$. The *quotient graph* Γ_N of Γ relative to the orbits of N is defined as the graph with vertices the orbits of N on $V(\Gamma)$ and with two orbits adjacent if there is an edge in Γ between those two orbits. In particular, for a normal subgroup N of $\mathrm{Aut}(\Gamma)$, if Γ and Γ_N have the same valency, then Γ_N is a *normal quotient* of Γ, and if Γ has no proper normal quotient, then Γ is *basic*. To study a symmetric graph Γ, there is an extensive used strategy consisting of two steps: the first one is to investigate normal quotient graph Γ_N for some normal subgroup N of $\mathrm{Aut}(\Gamma)$ and the second one is to reconstruct the original graph Γ from the normal quotient Γ_N by using covering techniques. This strategy was first laid out by Praeger (see [31]), and it is usually done by taking the normal subgroup N as large as possible and then the graph Γ is reduced a basic graph. In the literature, there are many works about basic graphs (see [1, 14, 16] for example), while the works about the second step, that is, covers of graphs, are fewer.

An epimorphism $\pi\colon \widetilde{\Gamma} \to \Gamma$ of graphs is called a *regular covering projection* if $\mathrm{Aut}(\widetilde{\Gamma})$ has a semiregular subgroup K whose orbits in $V(\widetilde{\Gamma})$ coincide with the *vertex fibres* $\pi^{-1}(v)$, $v \in V(\Gamma)$, and whose arc and edge orbits coincide with the *arc fibres* $\pi^{-1}((u,v))$ and the *edge fibres* $\pi^{-1}(\{u,v\})$, $\{u,v\} \in E(\Gamma)$, respectively. In particular, we call the graph $\widetilde{\Gamma}$ a *regular cover* or a *K-cover* of the graph Γ, and the group K the *covering transformation group*. If K is dihedral, cyclic or elementary abelian, then $\widetilde{\Gamma}$ is called a *dihedral*, *cyclic* or *elementary abelian cover* of Γ, respectively. An automorphism of $\widetilde{\Gamma}$ is said to be *fibre-preserving* if it maps a vertex fibre to a vertex fibre, and all such fibre-preserving automorphisms form a group called the *fibre-preserving group*, denoted by F. It is easy to see that $F = N_{\mathrm{Aut}(\widetilde{\Gamma})}(K)$. If $\widetilde{\Gamma}$ is F-arc-transitive, we say that $\widetilde{\Gamma}$ is an *arc-transitive cover* or a *symmetric cover* of Γ. For an extensive treatment of regular cover, one can see [3, 26, 27].

Covering techniques have long been known as a powerful tool in algebraic and topological graph theory. Application of these techniques has resulted in many constructions and classifications of certain families of graphs with particular symmetry properties. For example, by using covering techniques, Djoković [10] constructed the first infinite family of 5-arc-transitive cubic graphs as covers of Tutte's 8-cage, and Biggs [4] constructed some 5-arc-transitive cubic graphs as covers of cubic graphs that are 4-arc-transitive but not 5-arc-transitive. Gross and Tucker [18] proved that every regular cover of a base graph can be reconstructed as a voltage graph on the base graph. Later, Malnič et al. [26] and Du et al. [12] developed these ideas further in a systematic study of regular covering projections of a graph along which a group of automorphisms lifts.

Based on the approaches studied in [12, 26], many arc-transitive covers of symmetric graphs of small orders and small valencies have been classified. For example, Pan et al. [29] studied arc-transitive cyclic covers of some complete graphs of small orders. One may see [2, 27] for other works. Moreover, a new approach was proposed by Conder and Ma [7, 8] by considering a presentation (quotient group) of a universal group, which can be obtained from Reidemeister-Schreier theory, and representation theory and other methods are applied when determining suitable quotients. As an application, arc-transitive abelian covers of the complete graph K_4, the complete bipartite graph $K_{3,3}$, the 3-dimensional hypercube Q_3, the Petersen graph and the Heawood graph, were classified. Later, arc-transitive dihedral covers of these graphs were determined by Ma [25].

For arc-transitive covers of infinite families of graphs, Du et al. studied 2-arc-transitive elementary abelian and cyclic covers of complete graphs K_n in [11, 13] and $K_{n,n} - nK_2$ in [32, 33]. Recently, Pan et al. [28] determined arc-transitive cyclic covers of the complete bipartite graph $K_{p,p}$ of order $2p$ for a prime p. Compared with symmetric covers of graphs of small orders and valencies, there are only a few contributions on symmetric covers of infinite families of graphs.

Arc-transitive covers of non-simple graphs were also considered in literature. For example, regular covers of the dipole Dip_k (a graph with two vertices and k parallel edges) were extensively studied in [2, 16, 26, 27, 34]. Such covers are called *Haar graphs*, and in particular, cyclic regular covers of dipoles are called *cyclic Haar graphs*, which can be regarded as a generalization of bipartite circulants and were studied in [21] (also see [15]). Construction of Haar graphs have aroused wide concern. Marušič et al. [26] studied elementary abelian covers of the dipole Dip_p for a prime p. In particular, symmetric elementary abelian covers and $\mathbb{Z}_p^2 \times \mathbb{Z}_p$-covers for a prime p of the dipole Dip_5 were classified completely in [16] and [34], respectively.

Let p be a prime. Pentavalent symmetric graphs of order $2p$ were classified by Cheng and Oxley in [6], which are the complete graph K_6 of order 6 and a family of Cayley graphs $\mathcal{CD}_p$ with $p = 5$ or $5 \mid (p-1)$ on dihedral groups (see Proposition 3.4). It has been shown that many pentavalent symmetric graphs are regular covers of them, see [16, 34]. In this paper, we consider arc-transitive cyclic and dihedral covers of these graphs. For K_6, the cyclic covers have been classified in [29], which should be the complete bipartite graph $K_{6,6}$ and the Icosahedron graph $\mathbf{I}_{12}$ (note that $\mathbf{I}_{12}$ is missed in [29]). For $\mathcal{CD}_p$, the cyclic covers consist of six infinite families of graphs, which are Cayley graphs on generalized dihedral groups. In particular, one family of graphs are cyclic Haar graphs and the other five families are non-cyclic Haar graphs. What is more, the full automorphism groups of them are determined. Arc-transitive dihedral covers of K_6 and $\mathcal{CD}_p$ are also classified, and there are only four sporadic graphs of order 24, 48, 60 and 120, respectively. A similar work about cubic graphs was done by Zhou and Feng [37].

Different from regular covers of graphs mentioned above, the method to classify arc-transitive cyclic covers used in this paper is related to the so called bi-Cayley graph. A graph Γ is a *bi-Cayley graph* over some group H if $\mathrm{Aut}(\Gamma)$ has a semiregular subgroup isomorphic to H having exactly two orbits on $V(\Gamma)$. Clearly, a Haar graph is a bipartite bi-Cayley graph. Recently, Zhou and Feng [38] gave a depiction of the automorphisms of bi-Cayley graphs (see Section 4), and based on this work, we classify the cyclic covers. In particular, all these covers are bi-Cayley graphs over some abelian groups. Note that vertex-transitive bi-Cayley graphs of valency 3 over abelian groups were determined in [36], while the case for valency 5 is still elusive. Indeed, even for arc-transitive pentavalent bi-Cayley graphs over abelian groups, it seems to be very difficult to give a classification, and one may see [2, 16, 34] for partial works.

The paper is organized as follows. After this introductory section, in Section 2 we give some notation and preliminary results. In Section 3, several infinite families of connected pentavalent symmetric graphs are constructed as Cayley graphs on generalized dihedral groups $\mathrm{Dih}(\mathbb{Z}_{mp^e} \times \mathbb{Z}_p)$, where e, m are two positive integers and p is a prime such that $(m, p) = 1$. In Section 4, it is proved that these Cayley graphs include all arc-transitive normal bipartite bi-Cayley graphs over $\mathbb{Z}_{mp^e} \times \mathbb{Z}_p$, and using this result, arc-transitive cyclic and dihedral covers of connected pentavalent symmetric graphs of order $2p$ are classified in Sections 5 and 6, respectively. In Section 7, the full automorphism groups of these covers are determined.

2 Preliminaries

In this section, we describe some preliminary results which will be used later. The following result is important to investigate symmetric pentavalent graphs.

Proposition 2.1 ([19, Theorem 1.1]). *Let Γ be a connected pentavalent (G, s)-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$ and $s \geq 1$, and let $v \in V(\Gamma)$. Then one of the following holds:*

(1) $s = 1$ *and* $G_v \cong \mathbb{Z}_5$, D_5 *or* D_{10}*;*

(2) $s = 2$ *and* $G_v \cong F_{20}$, $F_{20} \times \mathbb{Z}_2$, A_5 *or* S_5*, where* F_{20} *is the Frobenius group of order* 20*;*

(3) $s = 3$ *and* $G_v \cong F_{20} \times \mathbb{Z}_4$, $A_4 \times A_5$, $S_4 \times S_5$ *or* $(A_4 \times A_5) \rtimes \mathbb{Z}_2$ *with* $A_4 \rtimes \mathbb{Z}_2 = S_4$ *and* $A_5 \rtimes \mathbb{Z}_2 = S_5$*;*

(4) $s = 4$ *and* $G_v \cong \mathrm{ASL}(2, 4)$, $\mathrm{AGL}(2, 4)$, $\mathrm{A\Sigma L}(2, 4)$ *or* $\mathrm{A\Gamma L}(2, 4)$*;*

(5) $s = 5$ *and* $G_v \cong \mathbb{Z}_2^6 \rtimes \Gamma\mathrm{L}(2, 4)$.

From [24, Theorem 9], we have the following proposition.

Proposition 2.2. *Let Γ be a connected G-arc-transitive graph of prime valency, and let N be a normal subgroup of G. If N has at least three orbits, then it is semiregular on $V(\Gamma)$ and the kernel of G on the quotient graph Γ_N. Furthermore, Γ_N is G/N-arc-transitive, and Γ is a regular cover of Γ_N with N as the covering transformation group.*

Let G and E be two groups. We call an extension E of G by N a *central extension* of G if E has a central subgroup N such that $E/N \cong G$, and if further E is perfect, that is, if it equals its derived group E', we call E a *covering group* of G. Schur proved that for every non-abelian simple group G there is a unique maximal covering group M such that every covering group of G is a factor group of M (see [22, V, § 23]). This group M is called the *full covering group* of G, and the center of M is the *Schur multiplier* of G, denoted by $\mathrm{Mult}(G)$.

Lemma 2.3. *Let G be a group, and let N be an abelian normal subgroup of G such that G/N is a non-abelian simple group. If N is a proper subgroup of $C_G(N)$, then $G = G'N$ and $G' \cap N \lesssim \mathrm{Mult}(G/N)$.*

Proof. Since N is a proper subgroup of $C_G(N)$, we have $1 \neq C_G(N)/N \trianglelefteq G/N$, forcing $C_G(N)/N = G/N$ because G/N is simple. Thus $G = C_G(N)$ and it is a central extension of G/N by N. Since $G/N = (G/N)' = G'N/N \cong G'/(G' \cap N)$, we have $G = G'N$, and since $G' = (G'N)' = (G')'$, G' is a covering group of G/N. Hence $G' \cap N \lesssim \mathrm{Mult}(G/N)$. □

Denote by $\mathrm{soc}(G)$ the *socle* of G, that is, the product of all minimal normal subgroups of G. A list of all proper primitive permutation groups of degree less than 1000 was given by Dixon and Mortimer [9, Appendix B], and based on the list, we have:

Lemma 2.4. *Let G be a primitive permutation group on a set Ω and let $\alpha \in \Omega$, where $|\Omega| \in \{2,4,6,8,12,16,24,72,144,288,576\}$. If G_α is solvable, then either $G \lesssim \mathrm{AGL}(n,2)$ and $|\Omega| = 2^n$ with $1 \leq n \leq 4$, or $\mathrm{soc}(G) \cong \mathrm{PSL}(2,p)$, $\mathrm{PSL}(3,3)$ or $\mathrm{PSL}(2,q) \times \mathrm{PSL}(2,q)$ with $|\Omega| = p+1$, 144 or $(q+1)^2$ respectively, where $p \in \{5,7,11,23,71\}$ and $q \in \{11,23\}$.*

Proof. If $|\Omega| = 2$ or 4, then $G \leq S_2 \cong \mathrm{AGL}(1,2)$ or $G \leq S_4 \cong \mathrm{AGL}(2,2)$, respectively. Let $|\Omega| \geq 6$ and write $N := \mathrm{soc}(G)$. Then $N \trianglelefteq G$ and $N_\alpha \trianglelefteq G_\alpha$. Since G_α is solvable, N_α is solvable. By [9, Appendix B, Tables B.2 and B.3], G is an affine group, $N \cong A_{|\Omega|}$, or G is isomorphic to one group listed in [9, Tables B.2 and B.3]. If G is affine, then $|\Omega|$ is a prime power and thus $|\Omega| = 2^n$ with $n = 3$ or 4. By [9, Theorem 4.1A (a)], we have $G \lesssim \mathrm{AGL}(n,2)$. If $N \cong A_{|\Omega|}$ then $N_\alpha \cong A_{|\Omega|-1}$, which is insolvable because $|\Omega|-1 \geq 5$, a contradiction. In what follows we assume that G is isomorphic to one group listed in [9, Tables B.2 and B.3]. Note that all groups in the tables are collected into cohorts and all groups in a cohort have the same socle.

Assume that $|\Omega| = 144$. By [9, Table B.4, pp. 324], there are one cohort of type C, two cohorts of type H and four cohorts of type I (see [9, Table B.1, pp. 306] for types of cohorts of primitive groups) of primitive groups of degree 144. For the cohort of type C, by [9, Table B.2, pp. 314], $N \cong \mathrm{PSL}(3,3)$ and $N_\alpha \cong \mathbb{Z}_{13} \rtimes \mathbb{Z}_3$. For the two cohorts of type H, by [9, Table B.2, pp. 321], they have the same socle $N \cong \mathrm{M}_{12}$ and $N_\alpha \cong \mathrm{PSL}(2,11)$. For the four cohorts of type I, by [9, Table B.3, pp. 323], $N \cong A_{12} \times A_{12}$, $\mathrm{PSL}(2,11) \times \mathrm{PSL}(2,11)$, $\mathrm{M}_{11} \times \mathrm{M}_{11}$ or $\mathrm{M}_{12} \times \mathrm{M}_{12}$ and $N_\alpha \cong A_{11} \times A_{11}$, $(\mathbb{Z}_{11} \rtimes \mathbb{Z}_5) \times (\mathbb{Z}_{11} \rtimes \mathbb{Z}_5)$, $\mathrm{M}_{10} \times \mathrm{M}_{10}$ or $\mathrm{M}_{11} \times \mathrm{M}_{11}$, respectively. Since N_α is solvable, we have $N \cong \mathrm{PSL}(3,3)$ or $\mathrm{PSL}(2,11) \times \mathrm{PSL}(2,11)$.

For $|\Omega| \in \{6,8,12,16,24,72,288,576\}$, by [9, Tables B.2, B.3 and B.4], a similar argument to the above paragraph implies that either $N \cong \mathrm{PSL}(2,23) \times \mathrm{PSL}(2,23)$ with degree $23^2 = 576$ and $N_\alpha \cong (\mathbb{Z}_{23} \rtimes \mathbb{Z}_{11}) \times (\mathbb{Z}_{23} \rtimes \mathbb{Z}_{11})$, or $N \cong \mathrm{PSL}(2,p)$ with degree $p+1$ and $N_\alpha \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{\frac{p-1}{2}}$ where $p \in \{5,7,11,23,71\}$. □

3 Graph constructions as Cayley graphs

Let G be a finite group and S a subset of G with $1 \notin S$ and $S^{-1} = S$. The *Cayley graph* $\Gamma = \mathrm{Cay}(G,S)$ on G with respect to S is defined to have vertex set $V(\Gamma) = G$ and edge set $E(\Gamma) = \{\{g, sg\} \mid g \in G, s \in S\}$. It is well-known that $\mathrm{Aut}(\Gamma)$ contains the right regular representation $R(G)$ of G, the acting group of G by right multiplication, and Γ is connected if and only if $G = \langle S \rangle$, that is, S generates G. By Godsil [17], $N_{\mathrm{Aut}(\Gamma)}(R(G)) = R(G) \rtimes \mathrm{Aut}(G,S)$, where $\mathrm{Aut}(G,S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$. A Cayley graph $\Gamma = \mathrm{Cay}(G,S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\Gamma)$, and in this case, $\mathrm{Aut}(\Gamma) = R(G) \rtimes \mathrm{Aut}(G,S)$.

For an abelian group H, the *generalized dihedral group* $\mathrm{Dih}(H)$ is the semidirect product $H \rtimes \mathbb{Z}_2$, where the unique involution in $\mathbb{Z}_2$ maps each element of H to its inverse. In particular, if H is cyclic, then $\mathrm{Dih}(H)$ is a dihedral group. In this section, we introduce several infinite families of connected pentavalent symmetric graphs which are constructed as Cayley graphs on generalized dihedral groups.

Example 3.1. Let $\mathrm{Dih}(\mathbb{Z}_5^3) = \langle a, b, c, h \mid a^5 = b^5 = c^5 = h^2 = [a, b] = [a, c] = [b, c] = 1, a^h = a^{-1}, b^h = b^{-1}, c^h = c^{-1}\rangle$, and define

$$\mathcal{CGD}_{5^3} = \mathrm{Cay}(\mathrm{Dih}(\mathbb{Z}_5^3), \{h, ah, bh, ch, a^{-1}b^{-1}c^{-1}h\}).$$

By [34, Theorem 1.1], $\mathrm{Aut}(\mathcal{CGD}_{5^3}) \cong \mathrm{Dih}(\mathbb{Z}_5^3) \rtimes S_5$ and $\mathcal{CGD}_{5^3}$ is the unique connected pentavalent symmetric graph of order 250 up to isomorphism.

Let m be a positive integer. Consider the following equation in $\mathbb{Z}_m$

$$x^4 + x^3 + x^2 + x + 1 = 0. \tag{3.1}$$

In view of [15, Lemma 3.3], we have the following proposition.

Proposition 3.2. *Equation (3.1) has a solution r in $\mathbb{Z}_m$ if and only if $(r, m) \in \{(0, 1), (1, 5)\}$ or $m = 5^t p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$ and r is an element in $\mathbb{Z}_m^*$ of order 5, where $t \leq 1$, $s \geq 1$, $e_i \geq 1$ and p_i's are distinct primes such that $5 \mid (p_i - 1)$.*

The following infinite family of Cayley graphs was first constructed in [23].

Example 3.3. Let $m > 1$ be an integer such that Equation (3.1) has a solution r in $\mathbb{Z}_m$. Then $m = 5$, 11 or $m \geq 31$. Let

$$\mathcal{CD}_m = \mathrm{Cay}(D_m, \{b, ab, a^{r+1}b, a^{r^2+r+1}b, a^{r^3+r^2+r+1}b\})$$

be a Cayley graph on the dihedral group $D_m = \langle a, b \mid a^n = b^2 = 1, a^b = a^{-1}\rangle$. For $m = 5$ or 11, by [6], $\mathrm{Aut}(\mathcal{CD}_m) \cong (S_5 \times S_5) \rtimes \mathbb{Z}_2$ or $\mathrm{PGL}(2, 11)$, respectively. In particular, $\mathcal{CD}_5 \cong K_{5,5}$. For $m \geq 31$, by [23, Theorem B and Proposition 4.1], $\mathrm{Aut}(\mathcal{CD}_m) \cong D_m \rtimes \mathbb{Z}_5$, and obviously, if m has a prime divisor p with $p < m$, then $\mathrm{Aut}(\mathcal{CD}_m)$ has a normal subgroup $\mathbb{Z}_{m/p}$, and by Proposition 2.2, $\mathcal{CD}_m$ is a symmetric $\mathbb{Z}_{m/p}$-cover of a connected pentavalent symmetric graph of order $2p$.

By [6], we have the following proposition.

Proposition 3.4. *Let Γ be a connected pentavalent symmetric graph of order $2p$ for a prime p. Then $\Gamma \cong K_6$ or $\mathcal{CD}_p$ with $p = 5$ or $5 \mid (p - 1)$.*

In the remaining part of this section, we construct five infinite families of Cayley graphs on some generalized dihedral groups, and for convenience, we always assume that $G = \mathrm{Dih}(\mathbb{Z}_m \times \mathbb{Z}_{p^e} \times \mathbb{Z}_p) = \langle a, b, c, h \mid a^m = b^{p^e} = c^p = h^2 = [a, b] = [a, c] = [b, c] = 1, a^h = a^{-1}, b^h = b^{-1}, c^h = c^{-1}\rangle$ and r is a solution of Equation (3.1) in $\mathbb{Z}_m$, that is, $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod{m}$. By Proposition 3.2, m is odd and $5^2 \nmid m$.

Example 3.5. Assume that $e \geq 2$ and p is a prime such that $(m, p) = 1$ and $5 \mid (p - 1)$. Let λ be an element of order 5 in $\mathbb{Z}_{p^e}^*$. Then λ is a solution of Equation (3.1) in $\mathbb{Z}_{p^e}$. Set

$$\begin{aligned}
&T_1(r, \lambda) = \\
&\quad \{h, hab, ha^{r+1}b^{\lambda+1}c, ha^{r^2+r+1}b^{\lambda^2+\lambda+1}c^{\lambda^4+\lambda+1}, ha^{r^3+r^2+r+1}b^{\lambda^3+\lambda^2+\lambda+1}c\}, \\
&T_2(r, \lambda) = \\
&\quad \{h, hab, ha^{r+1}b^{\lambda+1}c, ha^{r^2+r+1}b^{\lambda^2+\lambda+1}c^{\lambda^3+\lambda+1}, ha^{r^3+r^2+r+1}b^{\lambda^3+\lambda^2+\lambda+1}c^{\lambda}\}, \\
&T_3(r, \lambda) = \\
&\quad \{h, hab, ha^{r+1}b^{\lambda+1}c, ha^{r^2+r+1}b^{\lambda^2+\lambda+1}c^{\lambda^2+\lambda+1}, ha^{r^3+r^2+r+1}b^{\lambda^3+\lambda^2+\lambda+1}c^{\lambda^2}\}.
\end{aligned}$$

It is easy to see that each of $T_1(r,\lambda)$, $T_2(r,\lambda)$ and $T_3(r,\lambda)$ generates G. Define

$$\mathcal{CGD}^i_{mp^e\times p} = \mathrm{Cay}(G, T_i(r,\lambda)),\ i = 1,2,3.$$

The maps

$$a\mapsto a^r,\ b\mapsto b^\lambda c,\ c\mapsto c^{\lambda^4},\ h\mapsto hab;$$
$$a\mapsto a^r,\ b\mapsto b^\lambda c,\ c\mapsto c^{\lambda^3},\ h\mapsto hab;$$
$$a\mapsto a^r,\ b\mapsto b^\lambda c,\ c\mapsto c^{\lambda^2},\ h\mapsto hab$$

induce three automorphisms of order 5 of G, denoted by α_1, α_2 and α_3 respectively, and α_i fixes the set $T_i(r,\lambda)$ and permutes its five elements cyclicly. It follows that for each $i = 1,2,3$, $\alpha_i \in \mathrm{Aut}(G, T_i(r,\lambda))$ and $\langle R(G), \alpha_i\rangle \cong G \rtimes \mathbb{Z}_5$, which is arc-transitive on $\mathcal{CGD}^i_{mp^e\times p}$.

The graphs $\mathcal{CGD}^1_{mp^e\times p}$, $\mathcal{CGD}^2_{mp^e\times p}$ and $\mathcal{CGD}^3_{mp^e\times p}$ for $(m,e) = (1,2)$ have been introduced in [34, Example 4.4], and they are not isomorphic to each other by [34, Lemma 4.5]. Indeed, we can also prove that the graphs $\mathcal{CGD}^1_{mp^e\times p}$, $\mathcal{CGD}^2_{mp^e\times p}$ and $\mathcal{CGD}^3_{mp^e\times p}$ for each integers $m \geq 1$ and $e \geq 2$ are not isomorphic to each other. Since the proof is similar to [34, Lemma 4.5], we omit it, and one can see [35] for a detailed proof.

Example 3.6. Let p be a prime such that $p = 5$ or $5 \mid (p \pm 1)$. Assume that $e = 1$ and $(m,p) = 1$. Then $G = \mathrm{Dih}(\mathbb{Z}_m \times \mathbb{Z}_p \times \mathbb{Z}_p)$. For $p = 5$, let $\lambda = 0$, and for $5 \mid (p\pm 1)$, let $\lambda \in \mathbb{Z}_p$ satisfying the equation $\lambda^2 = 5$ in $\mathbb{Z}_p$. Set

$$S(r,\lambda) = \{h, hab, ha^{r+1}c, ha^{r^2+r+1}b^{-2^{-1}(1+\lambda)}c^{2^{-1}(1+\lambda)}, ha^{r^3+r^2+r+1}b^{-2^{-1}(1+\lambda)}c\}.$$

It is easy to see that $S(r,\lambda)$ generates G. Define

$$\mathcal{CGD}^4_{mp\times p} = \mathrm{Cay}(G, S(r,\lambda)).$$

The map $a\mapsto a^r$, $b\mapsto b^{-1}c$, $c\mapsto b^{-2^{-1}(3+\lambda)}c^{2^{-1}(1+\lambda)}$ and $h\mapsto hab$ induces an automorphism of the group G, denoted by α_4, which permutes the elements in $S(r,\lambda)$ cyclicly. Then $\alpha_4 \in \mathrm{Aut}(G, S(r,\lambda))$ and $\langle R(G), \alpha_4\rangle \cong G \rtimes \mathbb{Z}_5$ acts arc-transitive on $\mathcal{CGD}^4_{mp\times p}$. Moreover, for $m = 1$ or 5, we have $r = 0$ or 1 respectively, and the map $a \mapsto a^{-1}$, $b\mapsto b^{-2^{-1}(1+\lambda)}c$, $c\mapsto b^{-2^{-1}(1+\lambda)}c^{2^{-1}(1+\lambda)}$, $h\mapsto h$ induces an automorphism β of G. It is easy to check that $\beta \in \mathrm{Aut}(G, S(r,\lambda))$ and $\langle\alpha_4, \beta\rangle \cong D_5$. In particular, by [16, Theorem 6.1], $\mathcal{CGD}^4_{5\times 5}$ is the unique connected pentavalent symmetric graph of order 50 up to isomorphism.

Example 3.7. Assume that $e = 1$ and p is a prime such that $(m,p) = 1$ and $5 \mid (p-1)$. By [34, Case 2, page 14], $x^4 + 10x^2 + 5 = 0$ has a root λ in $\mathbb{Z}_p$. Set

$$S(r,\lambda) = \{h,\ hab,\ ha^{r+1}c,\ ha^{r^2+r+1}b^{8^{-1}(\lambda^3-\lambda^2+7\lambda+1)}c^{2^{-1}(\lambda+1)},$$
$$ha^{r^3+r^2+r+1}b^{-8^{-1}(\lambda^3+\lambda^2+7\lambda-1)}c^{8^{-1}(\lambda^3+\lambda^2+11\lambda+3)}\}.$$

It is easy to check that $S(r,\lambda)$ generates G. Define

$$\mathcal{CGD}^5_{mp\times p} = \mathrm{Cay}(G, S(r,\lambda)).$$

The map $a \mapsto a^r$, $b \mapsto b^{-1}c$, $c \mapsto b^{8^{-1}(\lambda^3-\lambda^2+7\lambda-7)}c^{2^{-1}(\lambda+1)}$ and $h \mapsto hab$ induces an automorphism of the group G, denoted by α_5, which permutes the elements in $S(r,\lambda)$ cyclicly. Then $\alpha_5 \in \mathrm{Aut}(G, S(r,\lambda))$ and $\langle R(G), \alpha_5\rangle \cong G \rtimes \mathbb{Z}_5$ acts arc-transitive on $\mathcal{CGD}^5_{mp\times p}$.

Let $\Gamma_i = \mathcal{CGD}^i_{mp^e\times p}$ with $p = 5$ or $5 \mid (p-1)$, where $1 \le i \le 5$. By Examples 3.5 – 3.7, $\mathrm{Aut}(\Gamma_i)$ contains an arc-transitive subgroup $R(G) \rtimes \langle\alpha_i\rangle$ for each $1 \le i \le 5$. Let N_i be a subgroup of $R(G) \rtimes \langle\alpha_i\rangle$ as listed in Table 1. In particular, for Γ_4 with $5 \mid (p-1)$, since $\lambda^2 = 5$ in $\mathbb{Z}_p$, the equation $x^4 + 10x^2 + 5 = 0$ has a root t such that $t^2 = 2\lambda - 5$ (see Example 3.7). It is easy to compute that $N_i \cong \mathbb{Z}_{mp^e}$ and $N_i \trianglelefteq R(G) \rtimes \langle\alpha_i\rangle$ for each $1 \le i \le 5$ (see [35] for a detailed computation). By Proposition 2.2, we have the following lemma.

Table 1: Subgroups of $\mathrm{Aut}(\mathcal{CGD}^i_{mp^e\times p})$ for $1 \le i \le 5$.

Γ_i	p	N_i
$\mathcal{CGD}^1_{mp^e\times p}$	$5 \mid (p-1)$	$\langle R(a), R(b^5c^{3\lambda^4+2\lambda^2-\lambda+1})\rangle$
$\mathcal{CGD}^2_{mp^e\times p}$	$5 \mid (p-1)$	$\langle R(a), R(b^{-5}c^{2\lambda^3+4\lambda^2+\lambda+3})\rangle$
$\mathcal{CGD}^3_{mp^e\times p}$	$5 \mid (p-1)$	$\langle R(a), R(b^{-5}c^{4\lambda^3+3\lambda^2+2\lambda+1})\rangle$
$\mathcal{CGD}^4_{mp\times p}$	$p = 5$	$\langle R(a), R(b^2c^4)\rangle$
	$5 \mid (p-1)$	$\langle R(a), R(b^{t+1}c^{\lambda-3})\rangle$ ($t^2 = 2\lambda - 5$)
$\mathcal{CGD}^5_{mp\times p}$	$5 \mid (p-1)$	$\langle R(a), R(b^{2(\lambda^2+5)^{-1}(\lambda^3+10\lambda+5)-(\lambda+3)}c^4)\rangle$

Lemma 3.8. *Let p be a prime such that $p = 5$ or $5 \mid (p-1)$. Then for each $1 \le i \le 5$, $\mathcal{CGD}^i_{mp^e\times p}$ is a connected symmetric cyclic cover of a connected pentavalent symmetric graph of order $2p$.*

4 Pentavalent symmetric bi-Cayley graphs over abelian groups

Given a group H, let R, L and S be three subsets of H such that $R^{-1} = R$, $L^{-1} = L$, and $1 \notin R \cup L$. The *bi-Cayley graph* over H relative to the triple (R, L, S), denoted by $\mathrm{BiCay}(H, R, L, S)$, is the graph having vertex set $\{h_0 \mid h \in H\} \cup \{h_1 \mid h \in H\}$ and edge set $\{\{h_0, g_0\} \mid gh^{-1} \in R\} \cup \{\{h_1, g_1\} \mid gh^{-1} \in L\} \cup \{\{h_0, g_1\} \mid gh^{-1} \in S\}$. For a bi-Cayley graph $\Gamma = \mathrm{BiCay}(H, R, L, S)$, it is easy to see that $R(H)$ can be regarded as a semiregular subgroup of $\mathrm{Aut}(\Gamma)$ with two orbits, which acts on $V(\Gamma)$ by the rule $h_i^{R(g)} = (hg)_i,\ i = 0, 1,\ h, g \in H$. If $R(H)$ is normal in $\mathrm{Aut}(\Gamma)$, then Γ is a *normal bi-Cayley graph* over H.

Let $\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S)$ be a connected bi-Cayley graph over an abelian group H. Then Γ is bipartite. By [38, Lemma 3.1], we may always assume that $1 \in S$. Moreover, $\Gamma \cong \mathrm{BiCay}(H, \emptyset, \emptyset, S^\alpha)$ for $\alpha \in \mathrm{Aut}(H)$, and $H = \langle S\rangle$. Since H is abelian, there is an automorphism of H of order 2, denoted by γ, induced by $g \mapsto g^{-1}$, $\forall g \in H$. For

$\alpha \in \mathrm{Aut}(H)$ and $x \in H$, define

$$\delta_{\gamma,1,1}\colon h_0 \mapsto (h^{-1})_1,\ h_1 \mapsto (h^{-1})_0,\ \forall h \in H;$$
$$\sigma_{\alpha,x}\colon h_0 \mapsto (h^{\alpha})_0,\ h_1 \mapsto (xh^{\alpha})_1,\ \forall h \in H.$$

Set

$$\mathrm{F} = \{\sigma_{\alpha,x} \mid \alpha \in \mathrm{Aut}(H),\ S^{\alpha} = x^{-1}S\}.$$

Then $\delta_{\gamma,1,1} \in \mathrm{Aut}(\Gamma)$ and $\mathrm{F} \leq \mathrm{Aut}(\Gamma)_{1_0}$ (see [38, Lemma 3.3]). Since Γ is connected, F acts on $N_\Gamma(1_0)$ faithfully. By [38, Theorem 1.1 and Lemma 3.2], we have the following proposition.

Proposition 4.1. *Let $\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S)$ be a connected bi-Cayley graph over an abelian group H, and let $A = \mathrm{Aut}(\Gamma)$. Then $N_A(R(H)) = R(H)\langle \mathrm{F}, \delta_{\gamma,1,1}\rangle$ with vertex stabilizer $(N_A(R(H)))_{1_0} = \mathrm{F}$, and Γ is isomorphic to the Cayley graph $\mathrm{Cay}(\mathrm{Dih}(H), \gamma S)$, where $\mathrm{Dih}(H) = H \rtimes \langle\gamma\rangle$.*

The following lemma is from [2, Theorem 1.1].

Lemma 4.2. *Let n be a positive integer and p a prime such that $p \geq 5$. Let Γ be a connected pentavalent symmetric bi-Cayley graph over $\mathbb{Z}_{np}$. Then $\Gamma \cong \mathcal{CD}_{np}$, as defined in Example 3.3.*

Let $H = \langle x\rangle \times \langle y\rangle \times \langle z\rangle = \mathbb{Z}_m \times \mathbb{Z}_{p^e} \times \mathbb{Z}_p$, where m and e are two positive integers and p is a prime such that $p \geq 5$ and $(m,p) = 1$. In the remaining of this section, we always let $\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S)$ be a connected pentavalent bi-Cayley graph over H such that $N_{\mathrm{Aut}(\Gamma)}(R(H))$ is arc-transitive on Γ. Assume that $S = \{1, a, b, c, d\}$. Then $H = \langle a, b, c, d\rangle$. By Proposition 4.1, there exists a $\sigma_{\alpha,g} \in \mathrm{F}$ of order 5 permuting the neighborhood $\{1_1, a_1, b_1, c_1, d_1\}$ of 1_0 in Γ cyclicly. One may assume that $1_1^{\sigma_{\alpha,g}} = a_1$, which implies that $g = a$ because $1_1^{\sigma_{\alpha,g}} = g_1$, and that $b_1 = a_1^{\sigma_{\alpha,a}}$, $c_1 = b_1^{\sigma_{\alpha,a}}$, $d_1 = c_1^{\sigma_{\alpha,a}}$ and $1_1 = d_1^{\sigma_{\alpha,a}}$. It follows that

$$a^{\alpha} = ba^{-1},\ b^{\alpha} = ca^{-1},\ c^{\alpha} = da^{-1},\ d^{\alpha} = a^{-1}. \tag{4.1}$$

For $h \in H$, denote by $o(h)$ the order of h in H. Since $a^{\alpha} = ba^{-1}$ by Equation (4.1), $o(ba^{-1}) = o(a^{\alpha}) = o(a)$, forcing that $o(b) \mid o(a)$. Similarly, since $d^{\alpha} = a^{-1}$ and $c^{\alpha} = da^{-1}$, we have $o(d) = o(a)$ and $o(c) \mid o(a)$. Since $H = \langle a, b, c, d\rangle$, we have $o(x) \mid o(a)$ for any $x \in H$, and since $H = \mathbb{Z}_m \times \mathbb{Z}_{p^e} \times \mathbb{Z}_p$, we have $o(a) = mp^e$ and $|H : \langle a\rangle| = p$.

Suppose that $b \in \langle a\rangle$, say $b = a^i$ for some integer i. By Equation (4.1), $a^{\alpha} = ba^{-1} = a^{i-1} \in \langle a\rangle$ and $ca^{-1} = b^{\alpha} = (a^i)^{\alpha} = a^{i(i-1)} \in \langle a\rangle$, implying that $c \in \langle a\rangle$. Similarly, $d \in \langle a\rangle$ because $d = a \cdot c^{\alpha}$. Since $H = \langle a, b, c, d\rangle$, we have $H = \langle a\rangle \cong \mathbb{Z}_{mp^e}$, a contradiction. Hence $b \notin \langle a\rangle$, and since $|H : \langle a\rangle| = p$, we have $H = \langle a, b\rangle$ and $p \mid o(b)$.

Let $A = \mathrm{Aut}(\Gamma)$. Since Γ is $N_A(R(H))$-arc-transitive, $\mathrm{F} = N_A(R(H))_{1_0}$ acts transitively on $N_\Gamma(1_0)$. Let $\sigma_{\beta,g} \in \mathrm{F}$ for some $\beta \in \mathrm{Aut}(H)$ and $g \in H$ such that $1_1^{\sigma_{\beta,g}} = 1_1$. Then $1_1 = (1^{\beta}g)_1 = g_1$, forcing that $g = 1$. Hence $\mathrm{F}_{1_1} = \{\sigma_{\beta,1} \mid \beta \in \mathrm{Aut}(H), S^{\beta} = S\}$, that is, $\mathrm{F}_{1_1} \cong \mathrm{Aut}(H,S)$. By Proposition 4.1,

$$|N_A(R(H))| = 2|H||\mathrm{F}| = 2|H| \cdot |N_\Gamma(1_0)||\mathrm{F}_{1_1}| = 10|H||\mathrm{Aut}(H,S)|.$$

Observation 4.3. *$o(a) = mp^e$, $p \mid o(b)$, $H = \langle a, b\rangle$ and*

$$|N_A(R(H))| = 10|H||\operatorname{Aut}(H, S)|.$$

In the following two lemmas we consider the two cases: $e \geq 2$ and $e = 1$, respectively.

Lemma 4.4. *If $e \geq 2$, then $5 \mid (p-1)$, $\Gamma \cong \mathcal{CGD}^i_{mp^e \times p}$ for some $1 \leq i \leq 3$ and $|N_A(R(H))| = 10|H|$.*

Proof. By Observation 4.3, $o(a) = mp^e$, $p \mid o(b)$ and $H = \langle a, b\rangle = \langle x, y, z\rangle = \mathbb{Z}_m \times \mathbb{Z}_{p^e} \times \mathbb{Z}_p$, where $(m, p) = 1$. Then H has an automorphism mapping xy to a, and thus we may assume $a = xy$, which implies that $b = x^{r+1}y^{\lambda+1}z^{\iota}$ for some $r + 1 \in \mathbb{Z}_m$, $\lambda + 1 \in \mathbb{Z}_{p^e}$ and $0 \neq \iota \in \mathbb{Z}_p$ because $H = \langle a, b\rangle$. Furthermore, H has an automorphism fixing x, y and mapping z to z^{ι}, and so we may assume $b = x^{r+1}y^{\lambda+1}z$. Let $c = x^iy^jz^s$ and $d = x^ky^{\ell}z^t$, where $i, k \in \mathbb{Z}_m$, $j, \ell \in \mathbb{Z}_{p^e}$ and $s, t \in \mathbb{Z}_p$.

Note that both $\langle x\rangle = \mathbb{Z}_m$ and $\langle y, z\rangle = \mathbb{Z}_{p^e} \times \mathbb{Z}_p$ are characteristic in H. Since $a^{\alpha} = ba^{-1}$ by Equation (4.1), that is, $(xy)^{\alpha} = x^ry^{\lambda}z$, we have $x^{\alpha} = x^r$ and $y^{\alpha} = y^{\lambda}z$. Since $(x^{r+1}y^{\lambda+1}z)^{\alpha} = b^{\alpha} = ca^{-1} = x^{i-1}y^{j-1}z^s$, we have $z^{\alpha} = (x^{-r-1})^{\alpha} \cdot (y^{-\lambda-1})^{\alpha} \cdot (x^{i-1}y^{j-1}z^s) = x^{-r^2-r-1+i}y^{-\lambda^2-\lambda-1+j}z^{s-\lambda-1}$, implying that

$$z^{\alpha} = y^{-\lambda^2-\lambda-1+j}z^{s-\lambda-1}$$

and

$$-r^2 - r - 1 + i \equiv 0 \pmod{m}, \tag{4.2}$$
$$\lambda^2 - \lambda - 1 + j \equiv 0 \pmod{p^{e-1}}. \tag{4.3}$$

Similarly, since $c^{\alpha} = da^{-1}$ and $d^{\alpha} = a^{-1}$ by Equation (4.1), we have $x^{k-1}y^{\ell-1}z^t = c^{\alpha} = (x^iy^jz^s)^{\alpha} = x^{ir}y^{\lambda j+s(-\lambda^2-\lambda-1+j)}z^{j+s(s-\lambda-1)}$ and $x^{-1}y^{-1} = d^{\alpha} = (x^ky^{\ell}z^t)^{\alpha} = x^{kr}y^{\lambda\ell+t(-\lambda^2-\lambda-1+j)}z^{\ell+t(s-\lambda-1)}$, and by considering the powers of x, y and z, we have the following Equations (4.4) – (4.9).

$$ir \equiv k - 1 \pmod{m}; \tag{4.4}$$
$$kr \equiv -1 \pmod{m}; \tag{4.5}$$
$$\lambda j + s(-\lambda^2 - \lambda - 1 + j) \equiv \ell - 1 \pmod{p^e}; \tag{4.6}$$
$$\lambda\ell + t(-\lambda^2 - \lambda - 1 + j) \equiv -1 \pmod{p^e}; \tag{4.7}$$
$$j + s(s - \lambda - 1) \equiv t \pmod{p}; \tag{4.8}$$
$$\ell + t(s - \lambda - 1) \equiv 0 \pmod{p}. \tag{4.9}$$

By Equation (4.2), $i \equiv r^2 + r + 1 \pmod{m}$, and by Equations (4.4) and (4.5), $k \equiv r^3 + r^2 + r + 1 \pmod{m}$ and $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod{m}$. It follows from Proposition 3.2 that either $(r, m) \in \{(0, 1), (1, 5)\}$, or r is an element in $\mathbb{Z}^*_m$ of order 5 and $m = 5^tp_1^{e_1} \cdots p_f^{e_f}$ with $t \leq 1$, $f \geq 1$, $e_{\iota} \geq 1$ and p_{ι}'s are distinct primes such that $5 \mid (p_{\iota} - 1)$ for $1 \leq \iota \leq f$.

Note that $e \geq 2$. By Equation (4.3), $j \equiv \lambda^2+\lambda+1 \pmod{p^{e-1}}$ and by Equations (4.6) and (4.7), $\ell \equiv \lambda^3 + \lambda^2 + \lambda + 1 \pmod{p^{e-1}}$ and $\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1 \equiv 0 \pmod{p^{e-1}}$, implying $\lambda^5 \equiv 1 \pmod{p^{e-1}}$. It follows from Proposition 3.2 that either $(\lambda, p^{e-1}) =$

$(1,5)$, or $5 \mid (p-1)$ and λ is an element in $\mathbb{Z}^*_{p^{e-1}}$ of order 5, forcing that $\lambda \neq 0$ and $\ell^{-1} = (-\lambda^4)^{-1} = -\lambda$. Furthermore, one may assume that $j \equiv \lambda^2+\lambda+1+s_1p^{e-1} \pmod{p^e}$, $\ell \equiv \lambda^3+\lambda^2+\lambda+1+s_2p^{e-1} \pmod{p^e}$ and $\lambda^4+\lambda^3+\lambda^2+\lambda+1 \equiv \iota p^{e-1} \pmod{p^e}$ for some $s_1, s_2, \iota \in \mathbb{Z}_p$.

In what follows all equations are considered in $\mathbb{Z}_p$, unless otherwise stated. As $p \mid p^{e-1}$, the following equations are also true in $\mathbb{Z}_p$:

$$\begin{aligned} j &= \lambda^2+\lambda+1, \\ \ell &= \lambda^3+\lambda^2+\lambda+1, \\ \lambda^4+\lambda^3+\lambda^2+\lambda+1 &= 0, \\ \ell^{-1} &= -\lambda. \end{aligned}$$

By $s \times (4.9) - t \times (4.8)$, $s = \ell^{-1}(jt-t^2) = -\lambda(jt-t^2)$, and by Equation (4.9), we have $\lambda t^3 - (\lambda^3+\lambda^2+\lambda)t^2 - (\lambda+1)t + (\lambda^3+\lambda^2+\lambda+1) = 0$. Combined with $\lambda^4+\lambda^3+\lambda^2+\lambda+1 = 0$ and $\lambda \neq 0$, we have $(t-1)(t-\lambda)(t-\lambda^2) = 0$, which implies that $t = 1$, λ or λ^2. Recall that $j = \lambda^2+\lambda+1$ and $s = -\lambda(jt-t^2)$. Thus $(t,s) = (1, \lambda^4+\lambda+1)$, $(\lambda, \lambda^3+\lambda+1)$ or $(\lambda^2, \lambda^2+\lambda+1)$.

Since $j \equiv \lambda^2+\lambda+1+s_1p^{e-1} \pmod{p^e}$ and $\ell \equiv \lambda^3+\lambda^2+\lambda+1+s_2p^{e-1} \pmod{p^e}$, by Equations (4.6) and (4.7) we have:

$$\begin{cases} (\lambda+s)s_1p^{e-1} \equiv s_2p^{e-1} \pmod{p^e} \\ ts_1p^{e-1} + \lambda s_2p^{e-1} \equiv -(\lambda^4+\lambda^3+\lambda^2+\lambda+1) \pmod{p^e} \end{cases} \tag{4.10}$$

Recall that either $(\lambda, p^{e-1}) = (1,5)$ or $5 \mid (p-1)$. Suppose that $p^{e-1} = 5$. Then $p = 5$, $e = 2$ and $(\lambda, s, t) = (1,3,1)$. By Equation (4.10), we have $5s_2 = 20s_1$ and $5^2s_1 + 5 = 0$ in $\mathbb{Z}_{5^2}$, a contradiction. Hence $5 \mid (p-1)$. Again by Equation (4.10), we have $-(t+\lambda^2+\lambda s)s_1p^{e-1} \equiv \iota p^{e-1} \pmod{p^e}$, where $\iota p^{e-1} = \lambda^4+\lambda^3+\lambda^2+\lambda+1$. Furthermore,

$$\begin{cases} (t+\lambda^2+\lambda s)s_1 = -\iota \\ (t+\lambda^2+\lambda s)s_2 = -\iota(\lambda+s) \end{cases} \tag{4.11}$$

Since $(t,s) = (1, \lambda^4+\lambda+1)$, $(\lambda, \lambda^3+\lambda+1)$ or $(\lambda^2, \lambda^2+\lambda+1)$, we have $t+\lambda^2+\lambda s = 2\lambda^2+\lambda+2$, $\lambda^4+2\lambda^2+2\lambda$ or $\lambda^3+3\lambda^2+\lambda$, respectively, and since $(2\lambda^2+\lambda+2)(\lambda^4+2\lambda^2+2\lambda) = 6(\lambda^4+\lambda^3+\lambda^2+\lambda)+1 = -5$ and $(\lambda^3+3\lambda^2+\lambda)(\lambda^4-2\lambda^3+\lambda^2) = \lambda^4+\lambda^3+\lambda^2+\lambda-4 = -5$, we have $(t+\lambda^2+\lambda s)^{-1} = -5^{-1}(\lambda^4+2\lambda^2+2\lambda)$, $-5^{-1}(2\lambda^2+\lambda+2)$ or $-5^{-1}(\lambda^4-2\lambda^3+\lambda^2)$, respectively. By Equation (4.11), $(s_1,s_2) = (5^{-1}\iota(\lambda^4+2\lambda^2+2\lambda), 5^{-1}\iota(-3\lambda^4+\lambda^3+2\lambda^2))$, $(5^{-1}\iota(2\lambda^2+\lambda+2), 5^{-1}\iota(-3\lambda^4+2\lambda^3+\lambda))$ or $(5^{-1}\iota(\lambda^4-2\lambda^3+\lambda^2), 5^{-1}\iota(-2\lambda^4+\lambda^2+\lambda))$. Note that $a = xy$, $b = x^{r+1}y^{\lambda+1}z$, and

$$\begin{aligned} (c,d) = {} & (x^{r^2+r+1}y^{\lambda^2+\lambda+1+s_1p^{e-1}}z^{\lambda^4+\lambda+1}, x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1+s_2p^{e-1}}z), \\ & (x^{r^2+r+1}y^{\lambda^2+\lambda+1+s_1p^{e-1}}z^{\lambda^3+\lambda+1}, x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1+s_2p^{e-1}}z^{\lambda}) \text{ or} \\ & (x^{r^2+r+1}y^{\lambda^2+\lambda+1+s_1p^{e-1}}z^{\lambda^2+\lambda+1}, x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1+s_2p^{e-1}}z^{\lambda^2}), \end{aligned}$$

we have $S = S_1$, S_2 or S_3, where

$$S_1 = \{1, xy, x^{r+1}y^{\lambda+1}z, x^{r^2+r+1}y^{\lambda^2+\lambda+1+5^{-1}(\lambda^4+2\lambda^2+2\lambda)\iota p^{e-1}}z^{\lambda^4+\lambda+1}, \\ x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1+5^{-1}(-3\lambda^4+\lambda^3+2\lambda^2)\iota p^{e-1}}z\},$$

$$S_2 = \{1, xy, x^{r+1}y^{\lambda+1}z, x^{r^2+r+1}y^{\lambda^2+\lambda+1+5^{-1}(2\lambda^2+\lambda+2)\iota p^{e-1}}z^{\lambda^3+\lambda+1}, \\ x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1+5^{-1}(-3\lambda^4+2\lambda^3+\lambda)\iota p^{e-1}}z^{\lambda}\},$$

$$S_3 = \{1, xy, x^{r+1}y^{\lambda+1}z, x^{r^2+r+1}y^{\lambda^2+\lambda+1+5^{-1}(\lambda^4-2\lambda^3+\lambda^2)\iota p^{e-1}}z^{\lambda^2+\lambda+1}, \\ x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1+5^{-1}(-2\lambda^4+\lambda^2+\lambda)\iota p^{e-1}}z^{\lambda^2}\}.$$

Since $x^5 \equiv 1 \pmod{p^e}$ implies that $x^5 \equiv 1 \pmod{p^{e-1}}$, there exists $f \in \mathbb{Z}_p$ such that $\lambda_1 = \lambda + fp^{e-1}$ is an element of order 5 in $\mathbb{Z}_{p^e}^*$. Then $\lambda = \lambda_1 - fp^{e-1}$, $\lambda_1^5 = 1$ and $\lambda_1^4+\lambda_1^3+\lambda_1^2+\lambda_1+1 = 0$ in $\mathbb{Z}_{p^e}$. Hence $\iota p^{e-1} = \lambda^4+\lambda^3+\lambda^2+\lambda+1 = (\lambda_1 - fp^{e-1})^4 + (\lambda_1 - fp^{e-1})^3 + (\lambda_1 - fp^{e-1})^2 + (\lambda_1 - fp^{e-1}) + 1 = -(4\lambda_1^3 + 3\lambda_1^2 + 2\lambda_1 + 1)fp^{e-1}$ in $\mathbb{Z}_{p^e}$, and thus

$$\begin{aligned} S_1 &= \{1, xy, x^{r+1}y^{\lambda+1}z, x^{r^2+r+1}y^{\lambda^2+\lambda+1+5^{-1}(\lambda^4+2\lambda^2+2\lambda)\iota p^{e-1}}z^{\lambda^4+\lambda+1}, \\ &\qquad x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1+5^{-1}(-3\lambda^4+\lambda^3+2\lambda^2)\iota p^{e-1}}z\} \\ &= \{1, xy, x^{r+1}y^{\lambda_1+1}y^{-fp^{e-1}}z, x^{r^2+r+1}y^{\lambda_1^2+\lambda_1+1}y^{-(\lambda_1^4+\lambda_1+1)fp^{e-1}}z^{\lambda_1^4+\lambda_1+1}, \\ &\qquad x^{r^3+r^2+r+1}y^{\lambda_1^3+\lambda_1^2+\lambda_1+1}y^{-fp^{e-1}}z\}. \end{aligned}$$

Let φ be the automorphism of H induced by $x \mapsto x$, $y \mapsto y$ and $z \mapsto y^{fp^{e-1}}z$. Then $(S_1)^{\varphi} = \{1,\ xy,\ x^{r+1}y^{\lambda_1+1}z,\ x^{r^2+r+1}y^{\lambda_1^2+\lambda_1+1}z^{\lambda_1^4+\lambda_1+1},\ x^{r^3+r^2+r+1}y^{\lambda_1^3+\lambda_1^2+\lambda_1+1}z\}$. Since $\mathrm{BiCay}(H, \emptyset, \emptyset, S_1) \cong \mathrm{BiCay}(H, \emptyset, \emptyset, S_1^{\varphi})$, we may assume that $\lambda = \lambda_1$ is an element of order 5 in $\mathbb{Z}_{p^e}^*$, and

$$S_1 = \{1, xy, x^{r+1}y^{\lambda+1}z, x^{r^2+r+1}y^{\lambda^2+\lambda+1}z^{\lambda^4+\lambda+1}, x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1}z\}.$$

Similarly, we can also assume that

$$S_2 = \{1, xy, x^{r+1}y^{\lambda+1}z, x^{r^2+r+1}y^{\lambda^2+\lambda+1}z^{\lambda^3+\lambda+1}, x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1}z^{\lambda}\},$$

$$S_3 = \{1, xy, x^{r+1}y^{\lambda+1}z, x^{r^2+r+1}y^{\lambda^2+\lambda+1}z^{\lambda^2+\lambda+1}, x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1}z^{\lambda^2}\}.$$

By Proposition 4.1 and Example 3.5, $\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S) \cong \mathcal{CGD}^i_{mp^e \times p}$ with $1 \le i \le 3$.

Note that $|N_A(R(H))| = 10|H||\mathrm{Aut}(H, S)|$ (see Observation 4.3). For S_1, let $\beta \in \mathrm{Aut}(H, S_1)$. Then $S_1^{\beta} = S_1$. Since $\langle y, z\rangle$ is characteristic in $H = \langle x\rangle \times \langle y\rangle \times \langle z\rangle = \mathbb{Z}_m \times \mathbb{Z}_{p^e} \times \mathbb{Z}_p$, we have

$$\begin{aligned} &\{y,\ y^{\lambda+1}z,\ y^{\lambda^2+\lambda+1}z^{\lambda^4+\lambda+1},\ y^{\lambda^3+\lambda^2+\lambda+1}z\}^{\beta} \\ &\qquad = \{y,\ y^{\lambda+1}z,\ y^{\lambda^2+\lambda+1}z^{\lambda^4+\lambda+1},\ y^{\lambda^3+\lambda^2+\lambda+1}z\}. \end{aligned}$$

It follows that $y^{\beta} = y^s z^t$ with $(s,t) = (1,0)$, $(\lambda+1, 1)$, $(\lambda^2+\lambda+1, \lambda^4+\lambda+1)$, or $(\lambda^3+\lambda^2+\lambda+1, 1)$. Furthermore, we have

$$\begin{aligned} &(y \cdot y^{\lambda+1}z \cdot y^{\lambda^2+\lambda+1}z^{\lambda^4+\lambda+1} \cdot y^{\lambda^3+\lambda^2+\lambda+1}z)^{\beta} \\ &\qquad = y \cdot y^{\lambda+1}z \cdot y^{\lambda^2+\lambda+1}z^{\lambda^4+\lambda+1} \cdot y^{\lambda^3+\lambda^2+\lambda+1}z \end{aligned}$$

and

$$(y^\beta y^{-1})^{\lambda^3+2\lambda^2+3\lambda+4} = (z^{-\lambda^4-\lambda-3})^\beta z^{\lambda^4+\lambda+3}.$$

In particular, $(y^\beta y^{-1})^{(\lambda^3+2\lambda^2+3\lambda+4)p} = 1$. Note that $\lambda^4+\lambda^3+\lambda^2+\lambda+1 = 0$ in $\mathbb{Z}_{p^e}$ implies that $\lambda^4+\lambda^3+\lambda^2+\lambda+1 = 0$ in $\mathbb{Z}_p$. If $\lambda^3+2\lambda^2+3\lambda+4 = 0$ in $\mathbb{Z}_p$, then $\lambda^3 = -2\lambda^2-3\lambda-4$, $\lambda^4 = \lambda\cdot\lambda^3 = \lambda^2+2\lambda+8$, and thus $0 = \lambda^4+\lambda^3+\lambda^2+\lambda+1 = 5$, contrary to $5 \bigm| (p-1)$. Hence $\lambda^3+2\lambda^2+3\lambda+4 \neq 0$ in $\mathbb{Z}_p$ and $(y^\beta y^{-1})^p = 1$.

Suppose that $(s,t) \neq (1,0)$. Then $y^\beta y^{-1} = y^{s-1}z^t$ with $s-1 = \lambda, \lambda^2+\lambda$ or $\lambda^3+\lambda^2+\lambda$. Since $\lambda^4+\lambda^3+\lambda^2+\lambda+1 = 0$ in $\mathbb{Z}_p$, we have $\lambda \neq 0, -1$ and thus $(s-1,p) = 1$. This implies that $y^\beta y^{-1} = y^{s-1}z^t$ has order p^e, and since $e \geq 2$, we have $(y^\beta y^{-1})^p \neq 1$, a contradiction. Hence $(s,t) = (1,0)$, that is, $y^\beta = y$. It follows that $(y^{\lambda+1}z)^\beta = y^{\lambda+1}z^\beta \in \{y^{\lambda+1}z, y^{\lambda^2+\lambda+1}z^{\lambda^4+\lambda+1}, y^{\lambda^3+\lambda^2+\lambda+1}z\}$, and thus $z^\beta \in \{z, y^{\lambda^2}z^{\lambda^4+\lambda+1}, y^{\lambda^3+\lambda^2}z\}$. If $z^\beta = y^{\lambda^2}z^{\lambda^4+\lambda+1}$ or $y^{\lambda^3+\lambda^2}z$, then $(y^{\lambda^2})^p = 1$ or $(y^{\lambda^3+\lambda^2})^p = 1$. It forces that $\lambda^2 = 0$ or $\lambda^3+\lambda^2 = 0$ in $\mathbb{Z}_{p^{e-1}}$, and $\lambda = 0, -1$, a contradiction. Hence $z^\beta = z$. Noting that $\langle x\rangle$ is characteristic in H, we have $(xy)^\beta = x^\beta y \in S_1^\beta = S_1$. Then it is easy to check that $(xy)^\beta = xy$ and thus $x^\beta = x$. It implies that β is the identity automorphism. Hence $|\operatorname{Aut}(H,S_1)| = 1$ and $|N_A(R(H))| = 10|H|$. By a similar argument as above, for S_2 and S_3, we also have $|\operatorname{Aut}(H,S_2)| = |\operatorname{Aut}(H,S_3)| = 1$ and $|N_A(R(H))| = 10|H|$. □

Lemma 4.5. *If $e = 1$, that is, $H \cong \mathbb{Z}_m \times \mathbb{Z}_p \times \mathbb{Z}_p$, then one of the following holds:*

(1) $p = 5$ or $5 \bigm| (p\pm 1)$ and $\Gamma \cong \mathcal{CGD}^4_{mp\times p}$ as defined in Example 3.6. Furthermore,

(i) $|N_A(R(H))| = 10|H|$ if $m \neq 1, 5$;

(ii) $|N_A(R(H))| = 20|H|$ if $m = 5$;

(iii) $|N_A(R(H))| = 20|H|$ if $m = 1$ and $p \neq 5$; and

(iv) $|N_A(R(H))| = 40|H|$ if $m = 1$ and $p = 5$;

(2) $5 \bigm| (p-1)$, $\Gamma \cong \mathcal{CGD}^5_{mp\times p}$ as defined in Example 3.7 and $|N_A(R(H))| = 10|H|$.

Proof. Note that $(m,p) = 1$. By Observation 4.3, we have $o(a) = mp$, $p \bigm| o(b)$ and $H = \langle a,b\rangle = \langle x,y,z\rangle = \mathbb{Z}_m \times \mathbb{Z}_p \times \mathbb{Z}_p$. Then H has an automorphism mapping xy to a, and we may assume $a = xy$, implying that $b = x^{r+1}y^\lambda z^\iota$ for some $r+1 \in \mathbb{Z}_m$, $\iota, \lambda \in \mathbb{Z}_p$ and $\iota \neq 0$ because $H = \langle a,b\rangle$. The group H also has an automorphism fixing x, y and mapping z to $y^\lambda z^\iota$, and we may further assume $b = x^{r+1}z$. Let $c = x^iy^jz^s$ and $d = x^ky^\ell z^t$, where $i,k \in \mathbb{Z}_m$, $j,\ell,s,t \in \mathbb{Z}_p$.

By Equation (4.1), $a^\alpha = ba^{-1}$, that is, $(xy)^\alpha = x^ry^{-1}z$. Since both $\langle x\rangle$ and $\langle y,z\rangle$ are characteristic in H, we have $x^\alpha = x^r$ and $y^\alpha = y^{-1}z$. Again by Equation (4.1), since $(x^{r+1}z)^\alpha = b^\alpha = ca^{-1} = x^{i-1}y^{j-1}z^s$, we have $z^\alpha = (x^{-r-1})^\alpha \cdot b^\alpha = x^{-r^2-r-1+i}\cdot y^{j-1}z^s$, implying that $z^\alpha = y^{j-1}z^s$ and

$$-r^2-r-1+i \equiv 0 \pmod{m}. \tag{4.12}$$

Moreover, we have

$$\begin{aligned} x^{k-1}y^{\ell-1}z^t = da^{-1} = c^\alpha &= (x^iy^jz^s)^\alpha \\ &= (x^r)^i(y^{-1}z)^j(y^{j-1}z^s)^s = x^{ri}y^{-j+s(j-1)}z^{j+s^2}, \end{aligned}$$

and

$$x^{-1}y^{-1} = a^{-1} = d^{\alpha} = (x^k y^{\ell} z^t)^{\alpha} = (x^r)^k (y^{-1}z)^{\ell} (y^{j-1}z^s)^t = x^{rk} y^{-\ell+(j-1)t} z^{st+\ell}.$$

Considering the powers of x, y and z, we have Equations (4.13) – (4.18). As shown in these equations, in what follows all equations are considered in $\mathbb{Z}_p$, unless otherwise stated:

$$k - 1 \equiv ri \pmod m; \tag{4.13}$$

$$\ell - 1 = -j + s(j-1); \tag{4.14}$$

$$t = j + s^2; \tag{4.15}$$

$$-1 \equiv rk \pmod m; \tag{4.16}$$

$$-1 = -\ell + (j-1)t; \tag{4.17}$$

$$0 = st + \ell. \tag{4.18}$$

By Equation (4.12), we have $i \equiv r^2 + r + 1 \pmod m$ and by Equations (4.13) and (4.16), $k \equiv r^3 + r^2 + r + 1 \pmod m$ and $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod m$. It follows from Proposition 3.2 that either $(r, m) \in \{(0, 1), (1, 5)\}$ or r is an element of order 5 in $\mathbb{Z}_m^*$ and the prime decomposition of m is $5^t p_1^{e_1} \cdots p_f^{e_f}$ with $t \leq 1$, $f \geq 1$, $e_\iota \geq 1$ and $5 \mid (p_\iota - 1)$ for $1 \leq \iota \leq f$.

By Equation (4.15), $t = j + s^2$, and by Equations (4.14), (4.17) and (4.18), $\ell = 1 - j + s(j-1)$, $\ell = 1 + (j-1)t = 1 + (j-1)(j+s^2)$ and $\ell = -st = -sj - s^3$. It follows

$$j^2 + (s^2 - s)j - (s^2 - s) = 0; \tag{4.19}$$

$$(2s - 1)j + s^3 - s + 1 = 0. \tag{4.20}$$

By Equation (4.19), $(2s-1)^2 j^2 + (2s-1)^2(s^2-s)j - (2s-1)^2(s^2-s) = 0$, and since $(2s-1)j = -(s^3 - s + 1)$, we have $s^6 - 3s^5 + 5s^4 - 5s^3 + 2s - 1 = 0$, that is, $(s^2 - s - 1)(s^4 - 2s^3 + 4s^2 - 3s + 1) = 0$. Hence, either $s^2 - s - 1 = 0$ or $s^4 - 2s^3 + 4s^2 - 3s + 1 = 0$.

Case 1: $s^2 - s - 1 = 0$. Let $\lambda = 2s - 1$. Then $s = 2^{-1}(1 + \lambda)$ and $\lambda^2 = 5$, and thus $(\lambda, p) = (0, 5)$ or $5 \mid (p \pm 1)$ by [34, Example 4.6]. By Equations (4.19) and (4.20), $j^2 + j - 1 = 0$ and $(2s - 1)j + (s + 2) = 0$.

For $(\lambda, p) = (0, 5)$, $j^2 + j - 1 = 0$ implies that $j = 2 = -2^{-1}(1 + \lambda)$. For $5 \mid (p \pm 1)$, we have $\lambda \neq 0$, and since $2s - 1 = \lambda$ and $(2s-1)j + (s+2) = 0$, we have $j = -(2s-1)^{-1}(s+2) = -\lambda^{-1} \cdot 2^{-1}(\lambda + 5) = -2^{-1}(1 + \lambda)$ (note that $5 = \lambda^2$). It follows from Equations (4.15) and (4.18) that $t = j + s^2 = 1$ and $\ell = -st = -2^{-1}(1 + \lambda)$. Recall that $i \equiv r^2 + r + 1 \pmod m$ and $k \equiv i^3 + i^2 + i + 1 \pmod m$. Hence $c = x^{r^2+r+1} y^{-2^{-1}(1+\lambda)} z^{2^{-1}(1+\lambda)}$ and $d = x^{r^3+r^2+r+1} y^{-2^{-1}(1+\lambda)} z$. Now,

$$S = \{1, xy, x^{r+1}z, x^{r^2+r+1} y^{-2^{-1}(1+\lambda)} z^{2^{-1}(1+\lambda)}, x^{r^3+r^2+r+1} y^{-2^{-1}(1+\lambda)} z\}.$$

By Proposition 4.1 and Example 3.6, $\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S) \cong \mathcal{CGD}^4_{mp \times p}$.

For $(m, p) = (1, 5)$, we have $\lambda = 0$ and $S = \{1, y, z, y^{-3}z^3, y^{-3}z\}$. By MAGMA [5], $|N_A(R(H))| = 40|H|$. Assume that $(m, p) \neq (1, 5)$, and let $\beta \in \mathrm{Aut}(H, S)$. Then $S^\beta = S$, and since both $\langle x \rangle$ and $\langle y, z \rangle$ are characteristic subgroups of H, we have

$$\{x, x^{r+1}, x^{r^2+r+1}, x^{r^3+r^2+r+1}\}^\beta = \{x, x^{r+1}, x^{r^2+r+1}, x^{r^3+r^2+r+1}\},$$

$$\{y, z, y^{-2^{-1}(1+\lambda)}z^{2^{-1}(1+\lambda)}, y^{-2^{-1}(1+\lambda)}z\}^{\beta} = \{y, z, y^{-2^{-1}(1+\lambda)}z^{2^{-1}(1+\lambda)}, y^{-2^{-1}(1+\lambda)}z\}.$$

Similarly to Lemma 4.4, the two equations imply that for $m \neq 1, 5$ ($r \neq 0, \pm 1$), β is the identity automorphism of H, and for $m = 1$ or 5 ($r = 0$ or 1), β has order 2 that are induced by $x \mapsto x^{r^3+r^2+r+1}$, $y \mapsto y^{-2^{-1}(1+\lambda)}z$, and $z \mapsto y^{-2^{-1}(\lambda+1)}z^{2^{-1}(\lambda+1)}$ (one may also see [35] for a detailed computation). It implies that $|\operatorname{Aut}(H, S)| = 1$ for $m \neq 1, 5$ and $|\operatorname{Aut}(H, S)| = 2$ for $m = 1$ or 5. By Observation 4.3, we have $|N_A(R(H))| = 10|H|$ or $20|H|$, respectively.

Case 2: $s^4 - 2s^3 + 4s^2 - 3s + 1 = 0$. By Case 1, we may assume that $s^2 - s - 1 \neq 0$. If $p = 5$, then $s^4 - 2s^3 + 4s^2 - 3s + 1 = 0$ implies that $s = 3$ and thus $s^2 - s - 1 = 0$, a contradiction. Hence $p \neq 5$. By [34, Lemma 5.4, Case 2], we have $5 \mid (p - 1)$ and $s = 2^{-1}(1 + \lambda)$, where $\lambda^4 + 10\lambda^2 + 5 = 0$ and $\lambda \neq 0, \pm 1$.

Since $s^4 - 2s^3 + 4s^2 - 3s + 1 = 0$, we have $(2s - 1)(8s^3 - 12s^2 + 26s - 11) = -5$, and since $p \neq 5$, we have $(2s - 1)^{-1} = -5^{-1}(8s^3 - 12s^2 + 26s - 11)$. Noting that $s^4 = 2s^3 - 4s^2 + 3s - 1$, we have $s^5 = -5s^2 + 5s - 2$ and $s^6 = -5s^3 + 5s^2 - 2s$. By Equation (4.20), $j = -(2s - 1)^{-1}(s^3 - s + 1) = 5^{-1}(8s^3 - 12s^2 + 26s - 11)(s^3 - s + 1) = s^3 - 2s^2 + 3s - 1 = 8^{-1}(\lambda^3 - \lambda^2 + 7\lambda + 1)$ and by Equations (4.15) and (4.18), $t = j + s^2 = s^3 - s^2 + 3s - 1 = 8^{-1}(\lambda^3 + \lambda^2 + 11\lambda + 3)$ and $\ell = -st = -s^3 + s^2 - 2s + 1 = -8^{-1}(\lambda^3 + \lambda^2 + 7\lambda - 1)$. It follows that

$$S = \{1,\ xy,\ x^{r+1}z, x^{r^2+r+1}y^{8^{-1}(\lambda^3-\lambda^2+7\lambda+1)}z^{2^{-1}(1+\lambda)}, x^{r^3+r^2+r+1}y^{-8^{-1}(\lambda^3+\lambda^2+7\lambda-1)}z^{8^{-1}(\lambda^3+\lambda^2+11\lambda+3)}\}.$$

By Proposition 4.1 and Example 3.7, $\Gamma = \operatorname{BiCay}(H, \emptyset, \emptyset, S) \cong \mathcal{CGD}^5_{mp\times p}$.

Let $\beta \in \operatorname{Aut}(H, S)$. Then $S^\beta = S$. Since $\langle x\rangle$ and $\langle y, z\rangle$ are characteristic in H, we have

$$\{y, z, y^{8^{-1}(\lambda^3-\lambda^2+7\lambda+1)}z^{2^{-1}(1+\lambda)}, y^{-8^{-1}(\lambda^3+\lambda^2+7\lambda-1)}z^{8^{-1}(\lambda^3+\lambda^2+11\lambda+3)}\}^{\beta} = \{y, z, y^{8^{-1}(\lambda^3-\lambda^2+7\lambda+1)}z^{2^{-1}(1+\lambda)}, y^{-8^{-1}(\lambda^3+\lambda^2+7\lambda-1)}z^{8^{-1}(\lambda^3+\lambda^2+11\lambda+3)}\},$$

and since $\lambda \neq 0, \pm 1$, we have $y^\beta = y$ and $z^\beta = z$ (also see [35] for a detailed computation). Since $(xy)^\beta = x^\beta y \in S$, it is easy to check that $(xy)^\beta = xy$ and thus $x^\beta = x$. Hence β is the identity automorphism of H and $|\operatorname{Aut}(H, S)| = 1$. By Observation 4.3, $|N_A(R(H))| = 10|H|$. □

5 Cyclic covers

In this section, we classify connected symmetric cyclic covers of connected pentavalent symmetric graphs of order twice a prime. Denote by $K_{6,6} - 6K_2$ the complete bipartite graph of order 12 minus a one-factor and by $\mathbf{I}_{12}$ the Icosahedron graph. Edge-transitive cyclic covers of K_6 were classified in [29, Theorem 1.1], and by [29, Line 20, pp. 40], such graphs have order 12 and thus isomorphic to $K_{6,6} - 6K_2$ or $\mathbf{I}_{12}$ by [20, Proposition 2.7] (note that the graph $\mathbf{I}_{12}$ is missed in [29, Theorem 1.1]).

Theorem 5.1. *Let Γ be a connected pentavalent symmetric graph of order $2p$ for a prime p, and let $\widetilde{\Gamma}$ be a connected symmetric $\mathbb{Z}_n$-cover of Γ with $n \geq 2$. Then $\widetilde{\Gamma} \cong K_{6,6} - 6K_2$,*

$\mathbf{I}_{12}$, $\mathcal{CD}_{np}$, or $\mathcal{CGD}^i_{mp^e\times p}$ for $1 \le i \le 5$ with $n = mp^e$, $(m,p) = 1$, $5 \mid (p-1)$ and $e \ge 1$, which are defined in Examples 3.3, 3.5, 3.6 and 3.7.

Proof. By Proposition 3.4, $\Gamma \cong K_6$ for $p = 3$, $K_{5,5}$ for $p = 5$, or $\mathcal{CD}_p$ for $5 \mid (p-1)$. If $\Gamma \cong K_6$ then $\widetilde{\Gamma} \cong K_{6,6} - 6K_2$ or $\mathbf{I}_{12}$ by [29, Theorem 1.1] (also see the proof in [2, Theorem 3.6]). In the following, we assume that $p \ge 5$. Let $A = \mathrm{Aut}(\widetilde{\Gamma})$.

Let $K = \mathbb{Z}_n$ and $F = N_A(K)$. Since $\widetilde{\Gamma}$ is a symmetric K-cover of Γ, F is arc-transitive on $\widetilde{\Gamma}$ and F/K is arc-transitive on $\widetilde{\Gamma}_K = \Gamma$. Let B/K be a minimal arc-transitive subgroup of F/K. By Proposition 3.4, $B/K \cong D_p \rtimes \mathbb{Z}_5$ for $p > 11$; by MAGMA [5], $B/K \cong D_{11} \rtimes \mathbb{Z}_5$ for $p = 11$, and $B/K \cong \mathbb{Z}_5^2 \rtimes \mathbb{Z}_2$, $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_4$ or $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_8$ for $p = 5$. Each minimal normal subgroup of B/K is isomorphic to $\mathbb{Z}_p$ or $\mathbb{Z}_5^2$ with $p = 5$ and $B/K \cong \mathbb{Z}_5^2 \rtimes \mathbb{Z}_8$. Clearly, B is arc-transitive on $\widetilde{\Gamma}$ and B/K is non-abelian.

Set $C = C_B(K)$. Since K is abelian, $K \le Z(C) \le C$, where $Z(C)$ is the center of C. Suppose $K = C$. Then $B/K = B/C \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_n^*$, which forces that B/K is abelian, a contradiction. Hence $K < C$ and $1 \ne C/K \trianglelefteq B/K$. It follows that C/K contains a minimal normal subgroup of B/K, say L/K. Then $L \trianglelefteq B$ and $L \le C \trianglelefteq B$. Furthermore, $L/K \cong \mathbb{Z}_p$, or $L/K \cong \mathbb{Z}_5^2$ with $p = 5$ and $B/K \cong \mathbb{Z}_5^2 \rtimes \mathbb{Z}_8$.

Clearly, L and L/K have two orbits on $V(\widetilde{\Gamma})$ and $V(\widetilde{\Gamma}_K)$, and $\widetilde{\Gamma}$ and $\widetilde{\Gamma}_K$ are bipartite graphs with the two orbits of L and L/K as their bipartite sets, respectively. Since $K \le Z(C)$ and $L \le C$, $K \le Z(L)$.

First, assume $L/K \cong \mathbb{Z}_p$. Since $K \le Z(L)$, L is abelian, and so $L \cong \mathbb{Z}_{np}$ or $\mathbb{Z}_n \times \mathbb{Z}_p$ with $p \mid n$. For the latter, $L \cong \mathbb{Z}_m \times \mathbb{Z}_{p^e} \times \mathbb{Z}_p$ with $n = mp^e$, $(m,p) = 1$ and $e \ge 1$. Since L/K is semiregular on $V(\widetilde{\Gamma}_K)$, L is semiregular on $V(\widetilde{\Gamma})$ and thus $\widetilde{\Gamma}$ is a bi-Cayley graph over L. Noting that $L \lhd B$, we have that $N_A(L)$ is arc-transitive on $\widetilde{\Gamma}$, forcing that $\widetilde{\Gamma} \cong \mathrm{BiCay}(L, \emptyset, \emptyset, S)$ for some subset $S \subseteq L$. Recall that $p \ge 5$. By Lemmas 4.2 – 4.5, $\widetilde{\Gamma} \cong \mathcal{CD}_{np}$ or $\mathcal{CGD}^i_{mp^e\times p}$ $(1 \le i \le 5)$, as required.

Now, assume $L/K \cong \mathbb{Z}_5^2$. Then $p = 5$ and $B/K \cong \mathbb{Z}_5^2 \rtimes \mathbb{Z}_8$. Since $K \le Z(L)$ and $K = \mathbb{Z}_n$, $L = P \times H$, where P and H are the Sylow 5-subgroup and the Hall $5'$-subgroup of L, respectively. Note that $H \le K$ is abelian, but P may not. Since $(L/K)_{v^K} \cong \mathbb{Z}_5$, we have $L_v = P_v \cong \mathbb{Z}_5$, where $v \in V(\widetilde{\Gamma})$ and v^K is an orbit of K on $V(\widetilde{\Gamma})$ containing v. Note that $P \trianglelefteq B$ as P is characteristic in L and $L \trianglelefteq B$. By Proposition 2.2, P has at most two orbits on $V(\widetilde{\Gamma})$ because $P_v \ne 1$, and since L has exactly two orbits on $V(\widetilde{\Gamma})$, P and L have the same orbits. It follows that $L = PL_v = PP_v = P$, forcing that $H = 1$ and K is a 5-group.

Suppose $|K| = 5^t$ with $t \ge 2$. Since K is cyclic, K has a characteristic subgroup N such that $|K/N| = 25$, and since $K \trianglelefteq B$, $N \trianglelefteq B$. By Proposition 2.2, $\widetilde{\Gamma}_N$ is a connected pentavalent B/N-arc-transitive graph of order $10|K|/|N| = 250$, and by Example 3.1, $\Gamma_N \cong \mathcal{CGD}_{5^3}$. Since $B/K \cong \mathbb{Z}_5^2 \rtimes \mathbb{Z}_8$ and $|K/N| = 5^2$, all Sylow 2-subgroups of B/N are isomorphic to $\mathbb{Z}_8$ and $|B/N| = 8 \cdot 5^4$. However, by MAGMA [5], $\mathrm{Aut}(\mathcal{CGD}_{5^3})$ has no arc-transitive subgroup of order $8 \cdot 5^4$ that has a Sylow 2-subgroup isomorphic to $\mathbb{Z}_8$, a contradiction.

Since $K \ne 1$, we have $|K| = 5$ and $|V(\widetilde{\Gamma})| = 10|K| = 50$. By Example 3.6, $\widetilde{\Gamma} \cong \mathcal{CGD}^4_{5\times 5}$, as required. □

6 Dihedral covers

In this section, we aim to classify symmetric dihedral covers of connected pentavalent symmetric graphs of order twice a prime. First, we introduce four graphs which are from [30].

Example 6.1. Let

$$\mathbf{I}_{12}^{(2)} = \mathrm{Cay}(D_{12}, \{b, ba, ba^2, ba^4, ba^9\}),$$
$$\mathcal{G}_{48} = \mathrm{Cay}(D_{24}, \{b, ba, ba^3, ba^{11}, ba^{20}\}).$$

be two Cayley graphs on the dihedral groups $D_{12} = \langle a, b \mid a^{12} = b^2 = 1, a^b = a^{-1}\rangle$ and $D_{24} = \langle a, b \mid a^{24} = b^2 = 1, a^b = a^{-1}\rangle$, respectively. By MAGMA [5], $\mathrm{Aut}(\mathbf{I}_{12}^{(2)}) \cong A_5 \rtimes D_4$ and $\mathrm{Aut}(\mathcal{G}_{48}) \cong \mathrm{SL}(2,5) \rtimes D_4$, and their vertex stabilizers are isomorphic to F_{20}.

Example 6.2. Let

$$\mathcal{G}_{60} = \mathrm{Cay}(A_5,\ \{(1\ 4)(2\ 5),\ (1\ 3)(2\ 5),\ (1\ 3)(2\ 4),\ (2\ 4)(3\ 5),\ (1\ 4)(3\ 5)\})$$

be a Cayley graph on A_5. By MAGMA [5], it is a connected pentavalent symmetric graph of order 60 and $\mathrm{Aut}(\mathcal{G}_{60}) \cong A_5 \times D_5$ with vertex stabilizer isomorphic to D_5.

Example 6.3. Let G be a subgroup of S_7 generated by the elements $a = (1\ 4)(2\ 5)(6\ 7)$, $b = (1\ 3)(2\ 5)(6\ 7)$, $c = (1\ 3)(2\ 4)(6\ 7)$, $d = (2\ 4)(3\ 5)(6\ 7)$ and $e = (1\ 4)(3\ 5)(6\ 7)$, and define $\mathcal{G}_{120} = \mathrm{Cay}(G, \{a, b, c, d, e\})$. By MAGMA [5], $G \cong A_5 \times \mathbb{Z}_2$ and $\mathcal{G}_{120}$ is a connected pentavalent symmetric graph of order 120. Moreover, $\mathrm{Aut}(\mathcal{G}_{120}) \cong A_5 \times D_{10}$ with vertex stabilizer isomorphic to D_5.

A list of all pentavalent G-arc-transitive graphs on up to 500 vertices with the vertex stabilizer $G_v \cong \mathbb{Z}_5$, D_5 or F_{20} was given in MAGMA code by Potočnik [30]. Based on this list, we have the following lemma.

Lemma 6.4. *Let Γ be a G-arc-transitive graph of order 24, 48, 60, 120 or 240 with vertex stabilizer $G_v \cong \mathbb{Z}_5$, D_5 or F_{20} for some $G \leq \mathrm{Aut}(\Gamma)$ and $v \in V(\Gamma)$. Then Γ is a connected symmetric dihedral cover of K_6 if and only if $\Gamma \cong \mathbf{I}_{12}^{(2)}$, $\mathcal{G}_{48}$, $\mathcal{G}_{60}$ or $\mathcal{G}_{120}$.*

Proof. To show the necessity, let Γ be a connected symmetric dihedral cover of K_6. Then $\mathrm{Aut}(\Gamma)$ has an arc-transitive subgroup having a normal dihedral subgroup of order $|V(\Gamma)|/6$. Since Γ is G-arc-transitive with $G_v \cong \mathbb{Z}_5$, D_5 or F_{20}, by [30] Γ is isomorphic to one of the seven graphs: three graphs of order 24, 48 and 60 respectively, two graphs of order 120 and two graphs of order 240. For the orders 24, 48 and 60, $\Gamma \cong \mathbf{I}_{12}^{(2)}$, $\mathcal{G}_{48}$ or $\mathcal{G}_{60}$ by Examples 6.1 and 6.2. For the order 120, by MAGMA [5] one graph is isomorphic to $\mathcal{G}_{120}$ and the other has no arc-transitive group of automorphisms having a normal dihedral subgroup of order 20; in this case $\Gamma \cong \mathcal{G}_{120}$. For the order 240, again by MAGMA [5] none of the two graphs has an arc-transitive group of automorphisms having a normal dihedral subgroup of order 40.

Now, we show the sufficiency. By MAGMA [5], $\mathrm{Aut}(\mathbf{I}_{12}^{(2)})$ has a normal subgroup $N \cong D_2$. Clearly, N has more than two orbits on $V(\mathbf{I}_{12}^{(2)})$, and by Proposition 2.2, the quotient graph $(\mathbf{I}_{12}^{(2)})_N$ is a connected pentavalent symmetric graph of order 6, that is, the complete graph K_6. Thus $\mathbf{I}_{12}^{(2)}$ is a D_2-cover of K_6. Similarly, one may show that $\mathcal{G}_{48}$, $\mathcal{G}_{60}$ or $\mathcal{G}_{120}$ is a symmetric D_3-, D_5- or D_{10}-cover of K_6, respectively. □

Now, we are ready to classify symmetric dihedral covers of connected pentavalent symmetric graphs of order $2p$ for any prime p. Clearly, we have $p \geq 3$.

Theorem 6.5. *Let Γ be a connected pentavalent symmetric graph of order $2p$ with p a prime, and let $\widetilde{\Gamma}$ be a connected symmetric D_n-cover of Γ with $n \geq 2$. Then $\widetilde{\Gamma} \cong \mathbf{I}_{12}^{(2)}$, $\mathcal{G}_{48}$, $\mathcal{G}_{60}$ or $\mathcal{G}_{120}$.*

Proof. Let $K = D_n$ and let F be the fibre-preserving group. Since $\widetilde{\Gamma}$ is a symmetric K-cover of Γ, F is arc-transitive on $\widetilde{\Gamma}$ and F/K is arc-transitive on $\widetilde{\Gamma}_K = \Gamma$.

Assume $n = 2$. Then $|V(\widetilde{\Gamma})| = 2n \cdot |V(\Gamma)| = 8p$. Recall that $p \geq 3$. By [20, Proposition 2.9], $\widetilde{\Gamma} \cong \mathbf{I}_{12}^{(2)}$ or a graph $\mathcal{G}_{248}$ of order 248 with $\mathrm{Aut}(\mathcal{G}_{248}) = \mathrm{PSL}(2, 31)$. Since $\mathrm{PSL}(2, 31)$ has no proper subgroup of order divisible by 248 by MAGMA [5], $\mathrm{Aut}(\widetilde{\Gamma})$ is the unique arc-transitive group of automorphisms of $\widetilde{\Gamma}$, that is, $F \cong \mathrm{PSL}(2, 31)$. It implies that $\widetilde{\Gamma} \ncong \mathcal{G}_{248}$ because F has no normal subgroup isomorphic to D_n. Hence $\widetilde{\Gamma} \cong \mathbf{I}_{12}^{(2)}$.

Assume $n > 2$. Let $\mathbb{Z}_n$ be the cyclic subgroup of $K = D_n$ of order n. Then $\mathbb{Z}_n$ is characteristic in K and so $\mathbb{Z}_n \trianglelefteq F$ as $K \trianglelefteq F$. By Proposition 2.2, $\widetilde{\Gamma}_{\mathbb{Z}_n}$ is a connected pentavalent $F/\mathbb{Z}_n$-arc-transitive graph of order $4p$, and by [20, Proposition 2.7], $\widetilde{\Gamma}_{\mathbb{Z}_n} \cong \mathbf{I}_{12}$ or $K_{6,6} - 6K_2$. Thus $\widetilde{\Gamma}$ is a symmetric $\mathbb{Z}_n$-cover of $K_{6,6} - 6K_2$ or $\mathbf{I}_{12}$. Note that $|V(\widetilde{\Gamma})| = 12n$.

Let $\widetilde{\Gamma}_{\mathbb{Z}_n} \cong K_{6,6} - 6K_2$. Since each minimal arc-transitive subgroup of $\mathrm{Aut}(K_{6,6} - 6K_2)$ is isomorphic to $A_5 \times \mathbb{Z}_2$ or S_5 by MAGMA [5], $F/\mathbb{Z}_n$ has an arc-transitive subgroup $B/\mathbb{Z}_n = A_5 \times \mathbb{Z}_2$ or S_5. It follows that $|B_v| = 10$ for $v \in V(\widetilde{\Gamma})$, and form Proposition 2.1 that $B_v \cong D_5$. In particular, B is arc-transitive on $\widetilde{\Gamma}$ and $B/\mathbb{Z}_n$ has a normal subgroup $M/\mathbb{Z}_n = A_5$, which is edge-transitive on $\widetilde{\Gamma}_{\mathbb{Z}_n}$ and has exactly two orbits on $V(\widetilde{\Gamma}_{\mathbb{Z}_n})$. Thus $M \trianglelefteq B$ is edge-transitive and has two orbits on $V(\widetilde{\Gamma})$. Since $|B : M| = 2$, we have $M_v \cong D_5$.

Clearly, $\mathbb{Z}_n \leq C_M(\mathbb{Z}_n)$. If $\mathbb{Z}_n = C_M(\mathbb{Z}_n)$, then $A_5 = M/\mathbb{Z}_n = M/C_M(\mathbb{Z}_n) \leq \mathrm{Aut}(\mathbb{Z}_n) = \mathbb{Z}_n^*$, which is impossible. Hence $\mathbb{Z}_n$ is a proper subgroup of $C_M(\mathbb{Z}_n)$, and since $\mathrm{Mult}(A_5) = \mathbb{Z}_2$, Lemma 2.3 implies that either $M = M' \times \mathbb{Z}_n = A_5 \times \mathbb{Z}_n$ or $M = M'\mathbb{Z}_n = \mathrm{SL}(2, 5)\mathbb{Z}_n$ with $M' \cap \mathbb{Z}_n \cong \mathbb{Z}_2$. In particular, M/M' is cyclic. Since M' is characteristic in M and $M \trianglelefteq B$, we have $M' \trianglelefteq B$. If M' has at least three orbits on $V(\widetilde{\Gamma})$, by Proposition 2.2, M' is semiregular on $V(\widetilde{\Gamma})$ and $\widetilde{\Gamma}_{M'}$ is a connected pentavalent B/M'-arc-transitive graph. The stabilizer of $\alpha \in V(\widetilde{\Gamma}_{M'})$ in M/M' is isomorphic to $M_v \cong D_5$, but this is impossible because M/M' is cyclic. Thus M' has at most two orbits on $V(\widetilde{\Gamma})$ and so $|V(\widetilde{\Gamma})| \mid 2|M'|$, that is, $6n \mid |M'|$. If $M = A_5 \times \mathbb{Z}_n$, then $M' = A_5$ and $6n \mid |M'|$ implies that $n = 5$ or 10 as $n > 2$. It follows that $|V(\widetilde{\Gamma})| = 60$ or 120. Since $B_v \cong D_5$, we have $\widetilde{\Gamma} \cong \mathcal{G}_{60}$ or $\mathcal{G}_{120}$ by Lemma 6.4. If $M = \mathrm{SL}(2, 5)\mathbb{Z}_n$ with $M' = \mathrm{SL}(2, 5)$ and $\mathrm{SL}(2, 5) \cap \mathbb{Z}_n \cong \mathbb{Z}_2$, then n is even and $6n \mid |M'|$ implies that $n = 4$, 10 or 20. It follows that $|V(\widetilde{\Gamma})| = 48$, 120 or 240, and from Lemma 6.4 that $\widetilde{\Gamma} \cong \mathcal{G}_{48}$ or $\mathcal{G}_{120}$.

Let $\widetilde{\Gamma}_{\mathbb{Z}_n} \cong \mathbf{I}_{12}$. By MAGMA [5], under conjugation $\mathrm{Aut}(\mathbf{I}_{12})$ has only one minimal arc-transitive subgroup isomorphic to A_5, and so $F/\mathbb{Z}_n$ has an arc-transitive subgroup $B/\mathbb{Z}_n \cong A_5$. By a similar argument as the previous paragraph, $B = B'\mathbb{Z}_n$ and $B' \cap \mathbb{Z}_n \lesssim \mathrm{Mult}(A_5)$ by Lemma 2.3, forcing that either $B = B' \times \mathbb{Z}_n = A_5 \times \mathbb{Z}_n$ or $B = B'\mathbb{Z}_n = \mathrm{SL}(2, 5)\mathbb{Z}_n$ with $\mathrm{SL}(2, 5) \cap \mathbb{Z}_n \cong \mathbb{Z}_2$. Furthermore, B is arc-transitive on $\widetilde{\Gamma}$ with $B_v \cong \mathbb{Z}_5$ for

$v \in V(\widetilde{\Gamma})$, and B/B' is cyclic. If B' has more than two orbits on $V(\Gamma)$, then $\widetilde{\Gamma}_{B'}$ is a connected pentavalent B/B'-arc-transitive graph by Proposition 2.2, which is impossible because B/B' is abelian. Thus B' has at most two orbits on $V(\widetilde{\Gamma})$ and so $12n \mid 2|B'|$. If $B = A_5 \times \mathbb{Z}_n$, then $B' \cong A_5$, and $12n \mid 2|B'|$ implies that $n = 5$ or 10. It follows that $|V(\widetilde{\Gamma})| = 60$ or 120. Since $B_v \cong \mathbb{Z}_5$, we have $\widetilde{\Gamma} \cong \mathcal{G}_{60}$ or $\mathcal{G}_{120}$ by Lemma 6.4. If $B = \mathrm{SL}(2,5)\mathbb{Z}_n$ with $\mathrm{SL}(2,5) \cap \mathbb{Z}_n \cong \mathbb{Z}_2$, then $B' \cong \mathrm{SL}(2,5)$ and n is even. Since $12n \mid 2|B'|$, we have $n = 4$, 10 or 20, and so $|V(\widetilde{\Gamma})| = 48$, 120 or 240. It follows from Lemma 6.4 that $\widetilde{\Gamma} \cong \mathcal{G}_{48}$ or $\mathcal{G}_{120}$. □

7 Full automorphism groups of covers

Let Γ be a symmetric D_n- or $\mathbb{Z}_n$-cover of a connected symmetric pentavalent graph of order $2p$, where $n \geq 2$ is an integer and p is a prime. In this section, we aim to determine the full automorphism group of Γ. For D_n, by Theorem 6.5, $\Gamma \cong \mathbf{I}_{12}^{(2)}$, $\mathcal{G}_{48}$, $\mathcal{G}_{60}$ or $\mathcal{G}_{120}$ and by Examples 6.1 – 6.3, $\mathrm{Aut}(\Gamma)$ is known. For $\mathbb{Z}_n$, by Theorem 5.1, $\Gamma \cong K_{6,6} - 6K_2$, $\mathbf{I}_{12}$, $\mathcal{CD}_{np}$ (see Example 3.3), or $\mathcal{CGD}^i_{mp^e \times p}$ with $1 \leq i \leq 5$ (see Examples 3.5, 3.6 and 3.7). In particular, for the graph $\mathcal{CGD}^i_{mp^e \times p}$, we have $mp^e = n$ and m is given by

$$m = 5^t p_1^{e_1} \cdots p_s^{e_s} \quad \text{s.t.} \quad t \leq 1,\ s \geq 0,\ e_j \geq 1,\ 5 \mid (p_j - 1) \text{ for } 0 \leq j \leq s, \tag{7.1}$$

where m, p, e satisfy the conditions as listed in the second column in Table 2. Note that m is odd by Equation (7.1). By MAGMA [5], $\mathrm{Aut}(K_{6,6} - 6K_2) = S_6 \times \mathbb{Z}_2$ and $\mathrm{Aut}(\mathbf{I}_{12}) = A_5 \times \mathbb{Z}_2$, and by Example 3.3, $\mathrm{Aut}(\mathcal{CD}_{np}) = D_{np} \rtimes \mathbb{Z}_5$. Hence we only need to determine the full automorphism groups of $\mathcal{CGD}^i_{mp^e \times p}$ for $1 \leq i \leq 5$. All theses graphs are connected symmetric cyclic covers of some pentavalent symmetric graph of order $2p$ except $\mathcal{CGD}^4_{mp \times p}$ with $5 \mid (p+1)$, which are connected symmetric bi-Cayley graphs over $\mathbb{Z}_{mp} \times \mathbb{Z}_p$.

Theorem 7.1. $\mathrm{Aut}(\mathcal{CGD}^i_{mp^e \times p})$ *for* $1 \leq i \leq 5$ *is isomorphic to one group listed in Table 2.*

Table 2: Full automorphism groups of $\mathcal{CGD}^i_{mp^e \times p}$ for $1 \leq i \leq 5$.

Γ	Conditions: $(m,p) = 1$, m: Eq. (7.1)	$\mathrm{Aut}(\Gamma)$
$\mathcal{CGD}^i_{mp^e \times p}, i = 1,2$	$5 \mid (p-1)$ and $e \geq 2$	$\mathrm{Dih}(\mathbb{Z}_{mp^e} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_5$
$\mathcal{CGD}^4_{mp \times p}$	$m \neq 1, 5$, and $p = 5$ or $5 \mid (p \pm 1)$	$\mathrm{Dih}(\mathbb{Z}_{mp} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_5$
	$m = 1$ or 5, and $5 \mid (p \pm 1)$	$\mathrm{Dih}(\mathbb{Z}_{mp} \times \mathbb{Z}_p) \rtimes D_5$
	$m = 1$ and $p = 5$	$(\mathrm{Dih}(\mathbb{Z}_5^2) \times F_{20}).\mathbb{Z}_4$
$\mathcal{CGD}^5_{mp \times p}$	$5 \mid (p-1)$	$\mathrm{Dih}(\mathbb{Z}_{mp} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_5$

Proof. Let $\Gamma = \mathcal{CGD}^i_{mp^e \times p}$ for $1 \leq i \leq 5$ and $A = \mathrm{Aut}(\Gamma)$. For $(m,p) = (1,5)$, we have $\Gamma = \mathcal{CGD}^4_{5 \times 5}$ and by [16, Theorem 4.3 (1)], $\mathrm{Aut}(\Gamma) \cong (\mathrm{Dih}(\mathbb{Z}_5^2) \rtimes F_{20}).\mathbb{Z}_4$. In what follows we assume that $(m,p) \neq (1,5)$. By Examples 3.5, 3.6 and 3.7, A has an arc-transitive subgroup F isomorphic to $\mathrm{Dih}(\mathbb{Z}_{mp^e} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_5$ for $\mathcal{CGD}^i_{mp^e \times p}$ $(i = 1,2,3)$,

$\mathrm{Dih}(\mathbb{Z}_{mp} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_5$ for $\mathcal{CGD}^4_{mp\times p}$ with $m \neq 1, 5$ and $p = 5$ or $5 \mid (p \pm 1)$, $\mathrm{Dih}(\mathbb{Z}_{mp} \times \mathbb{Z}_p) \rtimes D_5$ for $\mathcal{CGD}^4_{mp\times p}$ with $m = 1$ or 5 and $5 \mid (p \pm 1)$, and $\mathrm{Dih}(\mathbb{Z}_{mp} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_5$ for $\mathcal{CGD}^5_{mp\times p}$ with $5 \mid (p - 1)$. Note that $F_v = \mathbb{Z}_5$ or D_5 for $v \in V(\Gamma)$. Furthermore, F has a normal semiregular subgroup $K = \mathbb{Z}_{mp^e} \times \mathbb{Z}_p$ having two orbits on $V(\Gamma)$, and hence Γ is an F-arc-transitive bi-Cayley graph over K. By Lemmas 4.4 and 4.5, $|N_A(K)| = |F|$, implying that $N_A(K) = F$. Note that $|F| = 10|K|$ or $20|K|$, that is, $|F| = 10mp^{e+1}$ or $20mp^{e+1}$ with $p = 5$ or $5 \mid (p \pm 1)$, and by Equation (7.1), both m and $|K|$ are odd. In particular, $|V(\Gamma)| = 2|K| = 2mp^{e+1}$ is twice an odd integer.

Clearly, $K = \mathbb{Z}_{mp^e} \times \mathbb{Z}_p$ has a characteristic Hall $5'$-subgroup, say H. Then $H \trianglelefteq F$ as $K \trianglelefteq F$. If $H \neq K$, then $5 \mid mp^{e+1}$ and H has at least three orbits. For $p \neq 5$, we have $5 \mid m$, and since $5^2 \nmid m$ by Equation (7.1), we have $|K : H| = 5$. For $p = 5$, by Table 2, $\Gamma = \mathcal{CGD}^4_{mp\times p}$ with $(m, 5) = 1$ and $K = \mathbb{Z}_m \times \mathbb{Z}_p \times \mathbb{Z}_p$, implying that $|K : H| = 5^2$. By Proposition 2.2, Γ_H is a connected pentavalent F/H-arc-transitive graph of order $2 \cdot 5$ or $2 \cdot 5^2$. By Proposition 3.4 and Example 3.6, $\Gamma_H \cong K_{5,5}$ or $\Gamma_H \cong \mathcal{CGD}^4_{5\times 5}$. Since $|F| = 10|K|$ or $20|K|$ and $|K|$ is odd, H is the characteristic Hall $\{2, 5\}'$-subgroup of F. Thus we have the following claim.

Claim 7.2. *H is the characteristic Hall $\{2, 5\}'$-subgroup of F, and we have $H = K$, or $|K : H| = 5$ and $\Gamma_H \cong K_{5,5}$, or $|K : H| = 25$ and $\Gamma_H \cong \mathcal{CGD}^4_{5\times 5}$.*

To finish the proof, we only need to show that $A = F$. Suppose to the contrary that $A \neq F$. Then A has a subgroup M such that F is a maximal subgroup of M. Since F is arc-transitive on Γ, M is arc-transitive, and since $N_A(K) = F$, we have $K \ntrianglelefteq M$.

By the definitions of the graphs $\mathcal{CGD}^i_{mp^e\times p}$ $(1 \leq i \leq 5)$ in Examples 3.5, 3.6 and 3.7, Γ has the 6-cycle $(1, h, a^{-r-1}b^{-\lambda-1}c^{-1}, ha^{-r}b^{-\lambda}c^{-1}, a^{-r}b^{-\lambda}c^{-1}, hab, 1)$ for $1 \leq i \leq 3$, and the 6-cycle $(1, h, a^{-r-1}c^{-1}, ha^{-r}bc^{-1}, a^{-r}bc^{-1}, hab, 1)$ for $4 \leq i \leq 5$. Suppose that Γ is $(M, 4)$-arc-transitive. Then each 4-arc lies in a 6-cycle in Γ and so Γ has diameter at most three. It follows that $|V(\Gamma)| = 2mp^{e+1} \leq 1 + 5 + 5 \cdot 4 + 5 \cdot 4 \cdot 4 = 106$, that is, $mp^{e+1} \leq 53$. Since $p = 5$ or $5 \mid (p \pm 1)$ and $e + 1 \geq 2$ (see the second column of Table 2), we have $p = 5$ and $m \leq 2$. Since m is odd, $(m, p) = (1, 5)$, contrary to assumption. Thus Γ is at most 3-arc-transitive, and by Proposition 2.1, we have $|M_v| \in \{5, 10, 20, 40, 60, 80, 120, 720, 1440, 2880\}$.

Note that $|M : F| = |M_v : F_v| \in \{2, 4, 6, 8, 12, 16, 24, 72, 144, 288, 576\}$ because $M \neq F$ and $|F_v| = 5$ or 10. Let $[M : F]$ be the set of right cosets of F in M. Consider the action of M on $[M : F]$ by right multiplication, and let F_M be the kernel of this action, that is, the largest normal subgroup of M contained in F. Then M/F_M is a primitive permutation group on $[M : F]$ because F/F_M is maximal in M/F_M, and $(M/F_M)_F = F/F_M$, the stabilizer of $F \in [M : F]$ in M/F_M. It follows that $|M/F_M| = |M : F||F/F_M|$ and so $|F/F_M| = |M/F_M|/|M : F|$. Since $|M : F| \in \{2, 4, 6, 8, 12, 16, 24, 72, 144, 288, 576\}$, by Lemma 2.4 we have $M/F_M \leq \mathrm{AGL}(t, 2)$ with $|M : F| = 2^t$ and $1 \leq t \leq 4$, or $\mathrm{soc}(M/F_M) \cong \mathrm{PSL}(2, q)$, $\mathrm{PSL}(3, 3)$ or $\mathrm{PSL}(2, r) \times \mathrm{PSL}(2, r)$ with $|M : F| = q + 1$, 144 or $(r + 1)^2$ respectively, where $q \in \{5, 7, 11, 23, 71\}$ and $r \in \{11, 23\}$.

Suppose $M/F_M \leq \mathrm{AGL}(2, 2)$ and $|M : F| = 4$. Since a 2-group cannot be primitive on $[M : F]$, we have $3 \mid |M/F_M|$ and so $3 \mid |M/F_M|/|M : F| = |F/F_M|$. Since $|F| = 10mp^{e+1}$ or $20mp^{e+1}$ with $p = 5$ or $5 \mid (p \pm 1)$, we have $3 \mid m$, which is impossible by Equation (7.1). Thus $M/F_M \nleq \mathrm{AGL}(2, 2)$. Similarly, since $7 \nmid m$, we have $M/F_M \nleq \mathrm{AGL}(3, 2)$, and if $M/F_M \leq \mathrm{AGL}(4, 2)$, then M/F_M is a $\{2, 5\}$-group. Furthermore, $\mathrm{soc}(M/F_M) \ncong \mathrm{PSL}(2, q)$, $\mathrm{PSL}(3, 3)$ or $\mathrm{PSL}(2, r) \times \mathrm{PSL}(2, r)$ for $q \in$

$\{7, 23, 71\}$ and $r = 23$ because otherwise one of $7, 23, 13, 23$ is a divisor of m. It follows that $M/F_M \cong \mathbb{Z}_2$ with $|M : F| = 2$, $M/F_M \leq \mathrm{AGL}(4, 2)$ with $|M : F| = 2^4$ and M/F_M a $\{2, 5\}$-group, $\mathrm{soc}(M/F_M) \cong \mathrm{PSL}(2, q)$ with $|M : F| = q + 1$ and $q \in \{5, 11\}$, or $\mathrm{soc}(M/F_M) \cong \mathrm{PSL}(2, 11) \times \mathrm{PSL}(2, 11)$ with $|M : F| = 144$.

First assume that $M/F_M \cong \mathbb{Z}_2$ with $|M : F| = 2$. Then $F \trianglelefteq M$ and $H \trianglelefteq M$ as H is characteristic in F by Claim 7.2. Let $C = C_M(H)$. Since K is abelian, $H \leq K \leq C$. Let P be a Sylow 5-subgroup of C containing the unique Sylow 5-subgroup of K. Since H is the Hall $5'$-group of K, $K \leq HP = H \times P$. Clearly, HP/H is a Sylow 5-subgroup of C/H. Recall that $|F/K| \bigm| 20$ and $|K/H| \bigm| 25$ (see Claim 7.2). Since $|M| = 2|F|$, we have $|M/H| \bigm| 2^3 \cdot 5^3$, and by Sylow theorem, M/H has a normal Sylow 5-subgroup. In particular, C/H has a normal Sylow 5-subgroup, that is, $HP/H \trianglelefteq C/H$. This implies $H \times P \trianglelefteq C$, and since $C \trianglelefteq M$ and P is characteristic in C, we have $P \trianglelefteq M$. Since $(m, p) \neq (1, 5)$ and $|V(\Gamma)| = 2mp^{e+1}$, P has at least three orbits on $V(\Gamma)$. By Proposition 2.2, P is semiregular on $V(\Gamma)$. Thus $|P| \bigm| |V(\Gamma)|$ and $|P| \bigm| |K|$. It follows that $|HP| = |H||P| \bigm| |K|$, and since $K \leq HP$, we have $K = HP \trianglelefteq M$, a contradiction.

Assume that $M/F_M \leq \mathrm{AGL}(4, 2)$ with $|M : F| = 2^4$ and M/F_M a $\{2, 5\}$-group. Then M/F_M has a regular normal subgroup of order 2^4, say L/F_M, and hence $L \trianglelefteq M$, $2^4 \bigm| |L|$ and $5 \bigm| |M : L|$. If L is semiregular then $2^4 \bigm| |V(\Gamma)| = 2mp^{e+1}$, which is impossible. Thus L is not semiregular, and so $5 \bigm| |L_v|$. By Proposition 2.2, L has one or two orbits, yielding that $|L| = |V(\Gamma)||L_v|$ or $|L| = |V(\Gamma)||L_v|/2$. Since $|M| = |V(\Gamma)||M_v|$, we have $|M : L| = |M_v : L_v|$ or $2|M_v : L_v|$, and since $5^2 \nmid |M_v|$, we have $5 \nmid |M : L|$, a contradiction.

Assume that $\mathrm{soc}(M/F_M) \cong \mathrm{PSL}(2, 5)$ with $|M : F| = 6$. Then $M/F_M = \mathrm{PSL}(2, 5)$ or $\mathrm{PGL}(2, 5)$, and $|F/F_M| = |M/F_M|/|M : F| = 10$ or 20. Since H is the unique normal Hall $\{2, 5\}'$-subgroup of F, we have $H \leq F_M$ and so H is characteristic in F_M. This implies $H \trianglelefteq M$ because $F_M \trianglelefteq M$. Since $M/F_M \cong (M/H)/(F_M/H)$, M/H is insolvable, and since $K \ntrianglelefteq M$, we have $H \neq K$. By Claim 7.2, $\Gamma_H \cong K_{5,5}$ or $\mathcal{CGD}^4_{5\times 5}$. If $\Gamma_H \cong \mathcal{CGD}^4_{5\times 5}$ then $\mathrm{Aut}(\Gamma_H) \cong (\mathrm{Dih}(\mathbb{Z}_5^2) \times F_{20}).\mathbb{Z}_4$ is solvable and so M/H is solvable, a contradiction. If $\Gamma_H \cong K_{5,5}$ then as $\mathrm{Aut}(K_{5,5}) = (S_5 \times S_5) \rtimes \mathbb{Z}_2$, it is easy to show that each insolvable arc-transitive group of $\mathrm{Aut}(K_{5,5})$ contains $A_5 \times A_5$ (this is also easily checked by MAGMA [5]), and so $|M/H| \geq 2 \cdot 60^2$. Noting that F_M is semiregular on $V(\Gamma)$, we have $|F_M| \bigm| |K|$. By Claim 7.2, $|K : H| \bigm| 5^2$, and hence $|F_M : H| \bigm| 5^2$. It follows that $|M/F_M| = |M/H|/|F_M/H| \geq 2 \cdot 60^2/5^2 > |\mathrm{PGL}(2, 5)|$, a contradiction.

Assume that $L/F_M := \mathrm{soc}(M/F_M) \cong \mathrm{PSL}(2, 11)$ with $|M : F| = 12$. Then $M/F_M = \mathrm{PSL}(2, 11)$ or $\mathrm{PGL}(2, 11)$, and $|F/F_M| = |M/F_M|/|M : F| = 55$ or 110. Moreover, $L \trianglelefteq M$ and $K \leq L$ as $|K|$ is odd and $|M : L| \leq 2$. Since $11 \bigm| |L/F_M|$, F_M has at least three orbits on $V(\Gamma)$, and by Proposition 2.2 F_M is semiregular and Γ_{F_M} is a pentavalent F/F_M-arc-transitive graph. Thus $|F_M| \bigm| |V(\Gamma)|$ and $|V(\Gamma_{F_M})|$ is even. Since $|V(\Gamma_{F_M})| = |V(\Gamma)|/|F_M| = 2|K|/|F_M|$, $|F_M|$ is odd and $|F_M| \bigm| |K|$.

Recall that H is the characteristic Hall $\{2, 5\}'$-subgroup of F by Claim 7.2. Set $N = H \cap F_M$. Since F_M has odd order, N is the characteristic Hall $5'$-subgroup of F_M, and since $F_M \trianglelefteq M$, we have $N \trianglelefteq M$. Hence F_M/N is a 5-subgroup. By Claim 7.2, $5^3 \nmid |K|$, and since $|F_M| \bigm| |K|$, we have $5^3 \nmid |F_M|$, that is, $|F_M/N| \bigm| 25$. Thus F_M/N is abelian, and $\mathrm{Aut}(F_M/N)$ is cyclic or $\mathrm{Aut}(F_M/N) \cong \mathrm{GL}(2, 5)$. If $F_M/N = C_{L/N}(F_M/N)$, then $\mathrm{PSL}(2, 11) \cong L/F_M \cong (L/N)/(F_M/N) \lesssim \mathrm{Aut}(F_M/N)$, which is impossible. Thus F_M/N is a proper subgroup of $C_{L/N}(F_M/N)$, and since $\mathrm{Mult}(\mathrm{PSL}(2, 11)) \cong \mathbb{Z}_2$, Lemma 2.3 implies that $L/N = (L/N)' \times F_M/N$ with $(L/N)' \cong \mathrm{PSL}(2, 11)$. Since

$|V(\Gamma_N)| = |V(\Gamma)|/|N| = 2|K|/|N|$ with $|K|$ odd, $(L/N)' \cong \mathrm{PSL}(2,11)$ cannot be semiregular on $V(\Gamma_N)$, implying that $5 \bigm| |(L/N)'_\alpha|$ for $\alpha \in V(\Gamma_N)$. It follows from Proposition 2.2 that $(L/N)'$ has at most two orbits on $V(\Gamma_N)$, and so $|(L/N)|/|(L/N)'| = |V(\Gamma_N)||(L/N)_\alpha|/(|V(\Gamma_N)||(L/N)'_\alpha|) = |(L/N)_\alpha|/|(L/N)'_\alpha|$ or $2|(L/N)_\alpha|/|(L/N)'_\alpha|$, implying that $5 \nmid (|L/N|/|(L/N)'|)$. Since $|(L/N)/(L/N)'| = |F_M/N|$ and F_M/N is a 5-group, we have $|F_M/N| = 1$, that is, $L/N = (L/N)' \cong \mathrm{PSL}(2,11)$.

Since $K \ntrianglelefteq M$ and $N \trianglelefteq M$, we have $N \neq K$, and since $K \leq C_L(N)$ and $|N|$ is odd, Lemma 2.3 implies $L = L' \times N$ with $L' \cong \mathrm{PSL}(2,11)$. Note that $L' \trianglelefteq M$. Since Γ has order twice an odd integer, L' cannot be semiregular on Γ, yielding $5 \bigm| |L'_v|$. By Proposition 2.2, L' has at most two orbits, and so $|\mathrm{PSL}(2,11)| = |L'| = |V(\Gamma)||L'_v|$ or $|V(\Gamma)||L'_v|/2$. It implies that $|V(\Gamma)| \bigm| 2|\mathrm{PSL}(2,11)|$, that is, $|V(\Gamma)| \bigm| 2^3 \cdot 3 \cdot 11$. Since $|V(\Gamma)| = 2mp^{e+1}$ and it is not divided by 3 or 2^2 by Equation (7.1), we have $|V(\Gamma)| = 22$, contrary to the fact that $e + 1 \geq 2$.

Assume that $L/F_M := \mathrm{soc}(M/F_M) \cong \mathrm{PSL}(2,11) \times \mathrm{PSL}(2,11)$ with $|M : F| = 144$. Then there exists $L_1/F_M \trianglelefteq L/F_M$ such that $L_1/F_M \cong \mathrm{PSL}(2,11)$ and $11 \bigm| |L : L_1|$. Since $11 \bigm| |L : F_M|$, F_M has at least three orbits and so Γ_{F_M} has order twice an odd integer. This implies that L/F_M cannot be semiregular, and by Proposition 2.2, L/F_M has one or two orbits. If L/F_M has one orbit then L_1/F_M is semiregular on Γ_{F_M} as $11 \bigm| |L : L_1|$ implies that L_1/F_M has at least three orbits, and so $4 \bigm| |V(\Gamma_{F_M})|$, a contradiction. If L/F_M has two orbits then Γ_{F_M} is bipartite and L/F_M is edge-transitive on Γ_{F_M}. Furthermore, L_1/F_M fixes the bipartite sets setwise. Since $11 \bigm| |L : L_1|$, L_1/F_M has at least two orbits on each bipartite set, and by [20, Proposition 2.4], L_1/F_M is semiregular on Γ_{F_M}. Since $L_1/F_M \cong \mathrm{PSL}(2,11)$, again we have the contradiction that $4 \bigm| |V(\Gamma_{F_M})|$. □

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Wonderful symmetric varieties and Schubert polynomials*

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Abstract

Extending results of Wyser, we determine formulas for the equivariant cohomology classes of closed orbits of certain families of spherical subgroups of the general linear group on the flag variety. Combining this with a slight extension of results of Can, Joyce and Wyser, we arrive at a family of polynomial identities which show that certain explicit sums of Schubert polynomials factor as products of linear forms.

Keywords: Symmetric varieties, Schubert polynomials, wonderful compactification, equivariant cohomology, weak order, parabolic induction.

Math. Subj. Class.: 14M27, 05E05, 14M15

1 Introduction

Suppose that G is a connected reductive algebraic group over $\mathbb{C}$. Suppose that $B \supseteq T$ are a Borel subgroup and a maximal torus of G, respectively, W is the Weyl group, and let $\mathfrak{t}$ denote the Lie algebra of T. By a classical theorem of Borel [1], the cohomology ring of G/B with rational coefficients is isomorphic to the coinvariant algebra $\mathbb{Q}[\mathfrak{t}^*]/I^W$, where

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I^W denotes the ideal generated by homogeneous W-invariant polynomials of positive degree. Any subvariety Y of G/B defines a cohomology class $[Y]$ in $H^*(G/B)$. It is then natural to ask for a polynomial in $\mathbb{Q}[\mathfrak{t}^*]$ which represents $[Y]$. In this paper, for certain families of subvarieties of certain G/B, we approach and answer this question in two different ways. Relating the two answers leads in the end to our main result, Theorem 4.1, which, roughly stated, says that certain non-negative linear combinations of Schubert polynomials factor completely into linear forms.

Our group of primary interest is $\mathbf{G} = \mathbf{GL}_n$, with $\mathbf{B}$ its Borel subgroup of lower-triangular matrices, and $\mathbf{T}$ its maximal torus of diagonal matrices. In this case, there is a canonical basis $x_1, \ldots, x_n$ of $\mathbf{t}^*$ that correspond to the Chern classes of the tautological quotient line bundles on the variety of complete flags $\mathbf{G}/\mathbf{B}$. Let $\mathbf{Z}_n$ denote the center of $\mathbf{GL}_n$, consisting of diagonal scalar matrices. Let $\mathbf{O}_n$ denote the orthogonal subgroup of $\mathbf{GL}_n$, and let $\mathbf{Sp}_{2n}$ denote the symplectic subgroup of $\mathbf{GL}_{2n}$. Denote by $\mathbf{GO}_n$ (resp. $\mathbf{GSp}_{2n}$) the central extension $\mathbf{Z}_n\mathbf{O}_n$ (resp. $\mathbf{Z}_{2n}\mathbf{Sp}_{2n}$). For any ordered sequence of positive integers $\mu = (\mu_1, \ldots, \mu_s)$ that sum to n, $\mathbf{GL}_n$ has a Levi subgroup $\mathbf{L}_\mu := \mathbf{GL}_{\mu_1} \times \cdots \times \mathbf{GL}_{\mu_s}$, as well as a parabolic subgroup $\mathbf{P}_\mu = \mathbf{L}_\mu \ltimes \mathbf{U}_\mu$ containing $\mathbf{B}$, where $\mathbf{U}_\mu$ denotes the unipotent radical of $\mathbf{P}_\mu$.

The subgroup

$$\mathbf{H}_\mu := (\mathbf{GO}_{\mu_1} \times \cdots \times \mathbf{GO}_{\mu_s}) \ltimes \mathbf{U}_\mu$$

of $\mathbf{GL}_n$ is *spherical*, meaning that it acts on $\mathbf{GL}_n/\mathbf{B}$ with finitely many orbits. Moreover, there is a unique *closed* $\mathbf{H}_\mu$-orbit $\mathbf{Y}_\mu$ on $\mathbf{GL}_n/\mathbf{B}$, which is our object of primary interest.

The reason for our interest in this family of orbits is that they correspond to the closed $\mathbf{B}$-orbits on the various $\mathbf{G}$-orbits of the **wonderful compactification** of the homogeneous space $\mathbf{GL}_n/\mathbf{GO}_n$. This homogeneous space is affine and symmetric, and it is classically known as the space of smooth quadrics in $\mathbb{P}^{n-1}$. Its wonderful compactification, classically known as the variety of complete quadrics [9, 13], is a G-equivariant projective embedding $\mathbf{X}$ which contains it as an open, dense $\mathbf{G}$-orbit, and whose boundary has particularly nice properties. (We recall the definition of the wonderful compactification in Section 2.1.)

It turns out that, with minor modifications, our techniques apply also to the wonderful compactification $\mathbf{X}'$ of the space $\mathbf{GL}_{2n}/\mathbf{GSp}_{2n}$, which parameterizes non-degenerate skew-symmetric bilinear forms on $\mathbb{C}^{2n}$, up to scalar. Letting $\mathbf{G} = \mathbf{GL}_{2n}$ in this case, the $\mathbf{G}$-orbits on $\mathbf{X}'$ are again parametrized by compositions

$$\mu = (\mu_1, \ldots, \mu_s)$$

of n; note that this is of course equivalent to parametrizing them by compositions of $2n$ with each part being even. Each $\mathbf{G}$-orbit has the form $\mathbf{G}/\mathbf{H}'_\mu$, with

$$\mathbf{H}'_\mu := (\mathbf{GSp}_{2\mu_1} \times \cdots \times \mathbf{GSp}_{2\mu_s}) \ltimes \mathbf{U}_\mu,$$

a spherical subgroup which again acts on $\mathbf{GL}_{2n}/\mathbf{B}$ with a unique closed orbit $\mathbf{Y}'_\mu$.

Let us consider two ways in which one might try to compute a polynomial representative of $[\mathbf{Y}_\mu]$ (or $[\mathbf{Y}'_\mu]$). For the first, note that $\mathbf{Y}_\mu$, being an orbit of $\mathbf{H}_\mu$, also admits an action of a maximal torus $\mathbf{S}_\mu$ of $\mathbf{H}_\mu$. Thus $\mathbf{Y}_\mu$ admits a class $[\mathbf{Y}_\mu]_{\mathbf{S}_\mu}$ in the $\mathbf{S}_\mu$-equivariant cohomology of $\mathbf{GL}_n/\mathbf{B}$, denoted by $H^*_{\mathbf{S}_\mu}(\mathbf{GL}_n/\mathbf{B})$. In brief, this is a cohomology theory which is sensitive to the geometry of the $\mathbf{S}_\mu$-action on $\mathbf{GL}_n/\mathbf{B}$. It admits a similar Borel-type presentation, this time as a polynomial ring in *two* sets of variables (the usual set of $\mathbf{x}$-variables referred to in the second paragraph, along with a second set which consists of

$\mathbf{y}$ and $\mathbf{z}$-variables) modulo an ideal. Moreover, the map $H^*_{\mathbf{S}_\mu}(\mathbf{GL}_n/\mathbf{B}) \to H^*(\mathbf{GL}_n/\mathbf{B})$ which sets all of the $\mathbf{y}$ and $\mathbf{z}$-variables to 0 sends the equivariant class of any $\mathbf{S}_\mu$-invariant subvariety of $\mathbf{GL}_n/\mathbf{B}$ to its ordinary (non-equivariant) class. Thus if a polynomial representative of $[\mathbf{Y}_\mu]_{\mathbf{S}_\mu}$ can be computed, one obtains a polynomial representative of $[\mathbf{Y}_\mu]$ by specializing $\mathbf{y}, \mathbf{z} \mapsto 0$.

In [15], this problem is solved for the case in which μ has only one part, in which case $\mathbf{H}_\mu = \mathbf{GO}_n$. Here, we extend the results of [15] to give a formula for the equivariant class $[\mathbf{Y}_\mu]_{\mathbf{S}_\mu}$ (and $[\mathbf{Y}'_\mu]_{\mathbf{S}'_\mu}$) for an arbitrary composition μ. The main general result is Proposition 3.4; it, together with Proposition 3.5, imply the case-specific equivariant formulas given in Corollaries 3.6 and 3.8.

The formulas for $[\mathbf{Y}_\mu]$ and $[\mathbf{Y}'_\mu]$ obtained from these corollaries (by specializing $\mathbf{y}$ and $\mathbf{z}$-variables to 0) are as follows:

Corollary 1.1. *The ordinary cohomology class of $[\mathbf{Y}_\mu]$ is represented in $H^*(\mathbf{G}/\mathbf{B})$ by the formula*

$$2^{d(\mu)} \left(\prod_{i=1}^{n} x_i^{R(\mu,i)+\delta(\mu,i)} \right) \prod_{i=1}^{s} \prod_{\nu_i+1 \le j \le k \le \nu_{i+1}-j} (x_j + x_k).$$

Corollary 1.2. *The ordinary cohomology class of $[\mathbf{Y}'_\mu]$ is represented in $H^*(\mathbf{G}/\mathbf{B})$ by the formula*

$$\left(\prod_{i=1}^{2n} x_i^{R(\mu,i)} \right) \prod_{i=1}^{s} \prod_{\nu_i+1 \le j < k \le \nu_{i+1}-j} (\mathbf{x}_j + \mathbf{x}_k).$$

The notations ν_i, $d(\mu)$, $R(\mu,i)$, $\delta(\mu,i)$, etc. will be defined in Sections 2 and 3. For now, note that the representatives we obtain are factored completely into linear forms. In fact, the formulas reflect the semi-direct decomposition of $\mathbf{H}_\mu$ (resp., $\mathbf{H}'_\mu$) as we will detail in Section 3.

A second possible way to approach the problem of computing $[\mathbf{Y}_\mu]$ is to write it as a non-negative integral linear combination of Schubert classes. For each Weyl group element w in the symmetric group $\mathbf{W} = S_n$, there is a **Schubert class** $[\mathbf{X}_w]$, the class of the **Schubert variety** $\mathbf{X}_w = \overline{\mathbf{B}^+ w\mathbf{B}/\mathbf{B}}$ in $\mathbf{GL_n}/\mathbf{B}$, where $\mathbf{B}^+$ denotes the Borel subgroup of upper-triangular elements of $\mathbf{GL}_n$. The Schubert classes form a $\mathbb{Z}$-basis for $H^*(\mathbf{GL}_n/\mathbf{B})$.

Assuming that one is able to compute the coefficients in

$$[\mathbf{Y}_\mu] = \sum_{w \in \mathbf{W}} c_w [\mathbf{X}_w], \tag{1.1}$$

then one may replace the Schubert classes in the above sum with the corresponding **Schubert polynomials** to obtain a polynomial in the $\mathbf{x}$-variables representing $[\mathbf{Y}_\mu]$. The Schubert polynomials $\mathfrak{S}_w$ are defined recursively by first explicitly setting

$$\mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-2}^2 x_{n-1},$$

and then declaring that $\mathfrak{S}_w = \partial_i \mathfrak{S}_{ws_i}$ if $ws_i < w$ in Bruhat order. Here, $s_i = (i, i+1)$ represents the ith simple reflection, and ∂_i represents the **divided difference operator** defined by

$$\partial_i(f)(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)}{x_i - x_{i+1}}.$$

It is well-known that $\mathfrak{S}_w$ represents $[\mathbf{X}_w]$ [10], and so if (1.1) can be computed, $[\mathbf{Y}_\mu]$ is represented by the polynomial

$$\sum_{w \in \mathbf{W}} c_w \mathfrak{S}_w.$$

In fact, a theorem due to M. Brion [2] tells us in principle how to compute the sum (1.1) in terms of certain combinatorial objects. More precisely, to the variety $\mathbf{Y}_\mu$ there is an associated subset of $\mathbf{W}$, which we call the W**-set** of $\mathbf{Y}_\mu$, and denote by $W(\mathbf{Y}_\mu)$. For each $w \in W(\mathbf{Y}_\mu)$, there is also an associated weight. In fact, this weight is always a power of 2, and the aforementioned theorem of Brion says that the sum (1.1) can be computed as

$$[\mathbf{Y}_\mu] = \sum_{w \in W(\mathbf{Y}_\mu)} 2^{d(\mathbf{Y}_\mu, w)} [\mathbf{X}_w] \tag{1.2}$$

for non-negative integers $d(\mathbf{Y}_\mu, w)$. This can be turned into an explicit polynomial representative via the aforementioned Schubert polynomial recipe, assuming that one can compute the sets $W(\mathbf{Y}_\mu)$, and the corresponding exponents $d(\mathbf{Y}_\mu, w)$ explicitly. In fact, in Section 2.4, we recall explicit descriptions of the W-sets $W(\mathbf{Y}_\mu)$ which have already been given in [5, 6], slightly extending those results to also give an explicit description of $W(\mathbf{Y}'_\mu)$ for arbitrary μ. (Previous results of [6] only described $W(\mathbf{Y}'_\mu)$ when μ consisted of a single part.) And as we will note, the exponents $d(\mathbf{Y}_\mu, w)$ are straightforward to compute.

This gives a second answer to our question, but note that it comes in a different form. Indeed, the formulas of Corollaries 1.1 and 1.2 are products of linear forms in the $\mathbf{x}$-variables which are not obviously equal to the corresponding weighted sums of Schubert polynomials. Of course, it is *a priori* possible that the two polynomial representatives are actually *not* equal, but simply differ by an element of $I^{\mathbf{W}}$, the ideal defining the Borel model of $H^*(\mathbf{G}/\mathbf{B})$. However, our main result, Theorem 4.1, states that in fact the apparent identity in $H^*(\mathbf{G}/\mathbf{B})$ is an equality *of polynomials*.

The paper is organized as follows. Section 2 is devoted mostly to recalling various background and preliminaries: We start by recalling necessary background on the wonderful compactification in Section 2.1. We then give the explicit details of the examples which we are concerned with in Section 2.2; this includes our conventions and notations regarding compositions, as well as our particular realizations of all groups, including the groups $\mathbf{H}_\mu$ and $\mathbf{H}'_\mu$. In Section 2.4, we review the notion of weak order and W-sets. We recall results of [5, 6] which are relevant to the current work, giving a slight extension of those results to the case of $W(\mathbf{Y}'_\mu)$ for arbitrary μ.

In Section 3, we briefly review the necessary details of equivariant cohomology and the localization theorem. We then use those facts to extend the formulas of [15] to the more general cases of this paper, obtaining Proposition 3.4 in a general setting, and its case-specific Corollaries 3.6 and 3.8. Corollaries 1.1 and 1.2 are immediate consequences of these.

Finally, in Section 4, we compare the representatives of $[\mathbf{Y}_\mu]$ and $[\mathbf{Y}'_\mu]$ obtained via our two different approaches, obtaining Theorem 4.1.

2 Background, notation, and conventions

Throughout the text, we use italicized notation when we give arguments that apply to general reductive groups. In that case, G is an arbitrary connected, reductive algebraic group defined over $\mathbb{C}$. We fix a Borel subgroup B of G and a maximal torus T of G contained in B. We let W denote the Weyl group of G and for every simple root α of T, we let $s_\alpha \in W$ denote the associated simple reflection and $P_\alpha = B \cup Bs_\alpha B$ denote the minimal parabolic subgroup containing B associated to α.

By contrast, we use bolded notation to denote our two main examples

$$\mathbf{G}/\mathbf{H} = \mathbf{GL}_n/\mathbf{O}_n \quad \text{and} \quad \mathbf{G}/\mathbf{H} = \mathbf{GL}_{2n}/\mathbf{Sp}_{2n}.$$

In these examples, we use the Borel subgroup $\mathbf{B}$ consisting of lower-triangular matrices and the maximal torus $\mathbf{T}$ consisting of diagonal matrices. Furthermore, the Weyl group $\mathbf{W}$ is isomorphic to the symmetric group S_n (respectively, S_{2n}). We hope this helps the reader distinguish our case-specific results from the general results that we need along the way.

2.1 The wonderful compactification

We review the notion of the wonderful compactification of a general spherical homogeneous space. Using our convention outlined above, let G be a connected, reductive algebraic group defined over $\mathbb{C}$. An algebraic subgroup H of G, as well as the homogeneous space G/H, is called **spherical** if a Borel subgroup B has finitely many orbits on G/H (or equivalently, if H has finitely many orbits on G/B). Some such homogeneous spaces, namely partial flag varieties, are complete, while others (for example, symmetric homogeneous spaces) are not. In the event that G/H is not complete, a *completion* of it is a complete G-variety X which contains an open dense subset X^0 G-equivariantly isomorphic to G/H. X is a *wonderful compactification* of G/H if it is a completion of G/H which is a "wonderful" spherical G-variety; this means that X is a smooth spherical G-variety whose boundary (the complement of X^0) is a union of smooth, irreducible G-stable divisors $D_1, \dots, D_r$ (the **boundary divisors**) with normal crossings and non-empty transverse intersections, such that the G-orbit closures on X are precisely the partial intersections of the D_i's. The number r is called the *rank* of the homogeneous space G/H.

The number of G-orbits on X is then 2^r, and they are parametrized by subsets of $\{1, \dots, r\}$, with a given subset determining the orbit by specifying the set of boundary divisors containing its closure. It is well-known that the subsets of $\{1, 2, \dots, r\}$ are in bijection with the compositions of $r + 1$. A **composition** of n is simply a tuple $\mu = (\mu_1, \dots, \mu_s)$ with $\sum_i \mu_i = n$. For a given composition $\mu = (\mu_1, \dots, \mu_s)$, we define the integers $\nu_2, \dots, \nu_s$ by the formula $\nu_i = \sum_{j=1}^{i-1} \mu_j$ for $i = 2, \dots, s$. By convention, we set $\nu_1 = 0$. In words, ν_i is the sum of the first $i-1$ parts of the composition μ. We parametrize the $\mathbf{G}$-orbits ($\mathbf{G} = \mathbf{GL}_n$ or $\mathbf{G} = \mathbf{GL}_{2n}$) in our examples by compositions of n, with $n-1$ being the rank of both of the symmetric spaces $\mathbf{GL}_n/\mathbf{GO}_n$ and $\mathbf{GL}_{2n}/\mathbf{GSp}_{2n}$.

2.2 Our examples

We now describe the two primary examples to which we will directly apply the general results of this paper. The first is the wonderful compactification $\mathbf{X}$ of the space of all smooth quadric hypersurfaces in $\mathbb{P}^{n-1}$, i.e. G/H, where $(\mathbf{G}, \mathbf{H}) = (\mathbf{GL}_n, \mathbf{GO}_n)$, classically known as the variety of complete quadrics.

Choose $\mathbf{B}$ to be the lower-triangular subgroup of $\mathbf{GL}_n$, and $\mathbf{T}$ to be the maximal torus consisting of diagonal matrices. We realize $\mathbf{H}_0 = \mathbf{O}_n$ as the fixed points of the involution given by $\theta(g) = J(g^t)^{-1}J$, where J is the $n \times n$ matrix with 1's on the antidiagonal, and 0's elsewhere. When $\mathbf{H}_0$ is realized in this way, $\mathbf{H}_0 \cap \mathbf{B}$, the lower-triangular subgroup of

$\mathbf{H}_0$, is a Borel subgroup, and $\mathbf{S}_0 := \mathbf{H}_0 \cap \mathbf{T}$ is a maximal torus of $\mathbf{H}_0$, consisting of all elements of the form

$$\mathrm{diag}(a_1, \dots, a_m, a_m^{-1}, \dots, a_1^{-1}), \tag{2.1}$$

where $a_i \in \mathbb{C}^*$ for $i = 1, \dots, m$ when $n = 2m$ is even, and of the form

$$\mathrm{diag}(a_1, \dots, a_m, 1, a_m^{-1}, \dots, a_1^{-1}),$$

where $a_i \in \mathbb{C}^*$ for $i = 1, \dots, m$ when $n = 2m + 1$ is odd. The Lie algebra $\mathfrak{s}_0$ of $\mathbf{S}_0$ then takes the form

$$\mathrm{diag}(a_1, \dots, a_m, -a_m, \dots, -a_1), \tag{2.2}$$

where $a_i \in \mathbb{C}$ for $i = 1, \dots, m$ in the even case, and

$$\mathrm{diag}(a_1, \dots, a_m, 0, -a_m, \dots, -a_1),$$

where $a_i \in \mathbb{C}$ for $i = 1, \dots, m$ in the odd case.

Note that the diagonal elements of H form a maximal torus $\mathbf{S}$ of dimension one greater than $\dim \mathbf{S_0}$. The general element of $\mathbf{S}$ is of the form

$$\mathrm{diag}(\lambda a_1, \dots, \lambda a_m, \lambda a_m^{-1}, \dots, \lambda a_1^{-1}) \tag{2.3}$$

in the even case, and of the form

$$\mathrm{diag}(\lambda a_1, \dots, \lambda a_m, \lambda, \lambda a_m^{-1}, \dots, \lambda a_1^{-1}) \tag{2.4}$$

in the odd case. Here, λ is an element of $\mathbb{C}^*$ and the a_i's are as before.

The Lie algebra $\mathfrak{s}$ of $\mathbf{S}$ then consists of diagonal matrices of the form

$$\mathrm{diag}(\lambda + a_1, \dots, \lambda + a_m, \lambda - a_m, \dots, \lambda - a_1) \tag{2.5}$$

in the even case, and of the form

$$\mathrm{diag}(\lambda + a_1, \dots, \lambda + a_m, \lambda, \lambda - a_m, \dots, \lambda - a_1). \tag{2.6}$$

in the odd case. Here, λ is an element of $\mathbb{C}$ and the a_i's are as before.

Thus we have described the homogeneous space $\mathbf{G}/\mathbf{H}$, where $\mathbf{G} = \mathbf{GL}_n$ and $\mathbf{H} = \mathbf{GO}_n$, which is the dense $\mathbf{G}$-orbit on $\mathbf{X}$. We now describe the other $\mathbf{G}$-orbits. As mentioned in Section 2.1, they are in bijection with compositions μ of n.

Corresponding to μ, we have a standard parabolic subgroup $\mathbf{P}_\mu = \mathbf{L}_\mu \ltimes \mathbf{U}_\mu$ containing $\mathbf{B}$ whose Levi factor $\mathbf{L}_\mu$ is $\mathbf{GL}_{\mu_1} \times \cdots \times \mathbf{GL}_{\mu_s}$, embedded in $\mathbf{GL}_n$ in the usual way, as block diagonal matrices. The $\mathbf{G}$-orbit $\mathcal{O}_\mu$ corresponding to μ is then isomorphic to $\mathbf{G}/\mathbf{H}_\mu$, where $\mathbf{H}_\mu$ is the group

$$(\mathbf{GO}_{\mu_1} \times \cdots \times \mathbf{GO}_{\mu_s}) \ltimes \mathbf{U}_\mu,$$

where $\mathbf{GO}_{\mu_i} = \mathbf{Z}_{\mu_i}\mathbf{O}_{\mu_i}$ is realized in $\mathbf{GL}_{\mu_i}$ as described above. Then $\mathbf{B} \cap \mathbf{H}_\mu$ is a Borel subgroup of $\mathbf{H}_\mu$, and $\mathbf{S}_\mu := \mathbf{T} \cap \mathbf{H}_\mu$ is a maximal torus of $\mathbf{H}_\mu$.

Note that $\mathbf{S}_\mu$ is diagonal, and consists of s "blocks", the ith block consisting of those diagonal entries in the range $\nu_i + 1, \dots, \nu_i + \mu_i$. If $\mu_i = 2m$ is even, then the ith block is of the form

$$\mathrm{diag}(\lambda_i a_{i,1}, \dots, \lambda_i a_{i,m}, \lambda_i a_{i,m}^{-1}, \dots, \lambda_i a_{i,1}^{-1}). \tag{2.7}$$

The corresponding ith block of an element of $\mathfrak{s}_\mu$ is then of the form

$$\mathrm{diag}(\lambda_i + a_{i,1}, \ldots, \lambda_i + a_{i,m}, \lambda_i - a_{i,m}, \ldots, \lambda_i - a_{i,1}). \tag{2.8}$$

If $\mu_i = 2m+1$ is odd, then the ith block is of the form

$$\mathrm{diag}(\lambda_i a_{i,1}, \ldots, \lambda_i a_{i,m}, \lambda_i, \lambda_i a_{i,m}^{-1}, \ldots, \lambda_i a_{i,1}^{-1}). \tag{2.9}$$

The ith block of an element of $\mathfrak{s}_\mu$ is correspondingly of the form

$$\mathrm{diag}(\lambda_i + a_{i,1}, \ldots, \lambda_i + a_{i,m}, \lambda_i, \lambda_i - a_{i,m}, \ldots, \lambda_i - a_{i,1}). \tag{2.10}$$

Our second primary example is the wonderful compactification of $(\mathbf{G}, \mathbf{H}')$, $(\mathbf{G}, \mathbf{H}') = (\mathbf{GL}_{2n}, \mathbf{GSp}_{2n})$. $\mathbf{H}'$ is a central extension of $\mathbf{H}'_0 = \mathbf{Sp}_{2n}$, the latter group being realized as the fixed points of the involutory automorphism of $\mathbf{GL}_{2n}$ given by $g \mapsto \widetilde{J}(g^t)^{-1}\widetilde{J}$, where $\widetilde{J}$ is the $2n \times 2n$ antidiagonal matrix whose antidiagonal consists of n 1's followed by n -1's, reading from the northeast corner to the southwest. Let $\mathbf{S}'_0 := \mathbf{Sp}_{2n} \cap \mathbf{T}$.

Once again taking $\mathbf{B}$ to be the lower-triangular Borel of $\mathbf{GL}_{2n}$, and $\mathbf{T}$ to be the diagonal maximal torus of $\mathbf{GL}_{2n}$, one checks that $\mathbf{H}'_0 \cap \mathbf{B}$ is a Borel subgroup of $\mathbf{H}'_0$, and that $\mathbf{S}'_0 := \mathbf{H}'_0 \cap \mathbf{T}$ is a maximal torus of $\mathbf{H}'_0$. The corresponding torus $\mathbf{S}'$ of $\mathbf{H}'$ is then of exactly the same format as indicated in (2.3), while its Lie algebra $\mathfrak{s}'$ is as indicated by (2.5).

The additional $\mathbf{G}$-orbits on the wonderful compactification $\mathbf{X}'$ of $\mathbf{GL}_{2n}/\mathbf{GSp}_{2n}$ again correspond to compositions $\mu = (\mu_1, \ldots, \mu_s)$ of n. For such a composition, we let $\mathbf{P}_\mu = \mathbf{L}_\mu \ltimes \mathbf{U}_\mu$ be the standard parabolic subgroup whose Levi factor is $\mathbf{GL}_{2\mu_1} \times \cdots \times \mathbf{GL}_{2\mu_s}$, embedded in $\mathbf{GL}_{2n}$ as block diagonal matrices. Then the $\mathbf{G}$-orbit corresponding to μ is isomorphic to $\mathbf{G}/\mathbf{H}'_\mu$, where

$$\mathbf{H}'_\mu = (\mathbf{GSp}_{2\mu_1} \times \cdots \times \mathbf{GSp}_{2\mu_s}) \ltimes \mathbf{U}_\mu,$$

with each $\mathbf{GSp}_{2\mu_i} = \mathbf{Z}_{2\mu_i}\mathbf{Sp}_{2\mu_i}$ embedded in the corresponding $\mathbf{GL}_{2\mu_i}$ just as described above.

The torus $\mathbf{S}'_\mu$ then consists of s "blocks", just as in the orthogonal case. This time, each block is of even dimension, so each is of the form described by (2.7). The corresponding block of the Lie algebra $\mathfrak{s}'_\mu$ is then of the form indicated in (2.8).

2.3 Some general results

We now introduce several general observations that are applicable to our chosen examples. In this subsection, returning to the conventions set forth in the beginning of Section 2, let G be an arbitrary connected, reductive algebraic group over $\mathbb{C}$ and let P be a parabolic subgroup of G containing the Borel subgroup B with Levi decomposition $P = L \ltimes U$, where L is a Levi subgroup of G containing T and U is the unipotent radical of P. Let H_L be a subgroup of L and consider the G-variety $V = G \times^P L/H_L$, the quotient of $G \times L/H_L$ by the action of P, where P acts on G by right multiplication and on L/H_L via its projection to L. Note that V is a G-variety via left multiplication of G on the first factor. V is a homogeneous G-variety, and the stabilizer subgroup of the point $[1, 1H_L/H_L]$ is $H := H_L U \subseteq P$. The construction of $V \cong G/H$ from L/H_L (or equivalently of $H \subseteq G$ from $H_L \subseteq L$) is called *parabolic induction*.

Note that in our examples, $\mathbf{H}_\mu$ is obtained via parabolic induction from $\mathbf{GO}_{\mu_1} \times \cdots \times \mathbf{GO}_{\mu_k}$ (playing the role of H_L) and $\mathbf{H}'_\mu$ is obtained via parabolic induction from $\mathbf{GSp}_{\mu_1} \times \cdots \times \mathbf{GSp}_{\mu_k}$ (likewise playing the role of H_L). We now summarize some results from the literature that describe the role of parabolic induction for wonderful varieties.

Definition 2.1. Let G be a reductive algebraic group. A subgroup H of G is said to be **symmetric** if there exists an algebraic involution $\theta\colon G \to G$ such that, letting $K = G^\theta = \{g \in G : \theta(g) = g\}$ and letting Z denote the center of G, we have $ZK^0 \subseteq H \subseteq ZK$.

Proposition 2.2.

1. *Let H be a symmetric subgroup of G such that G/H has a wonderful compactification X. Then every G-orbit of X is obtained via parabolic induction from some symmetric homogeneous space L/H_L associated to some Levi subgroup L of G.*
2. *If H_L is a spherical subgroup of L such that L/H_L contains a single closed B_L-orbit (with B_L a Borel subgroup of L) and H is the subgroup of G obtained by parabolic induction, i.e. $G/H \cong G \times^P L/H_L$, then H is a spherical subgroup of G and G/H contains a single closed B-orbit.*

Proof. The first result is a reformulation of a result of de Concini and Procesi [7, Theorem 5.2]. They show that a G-orbit V has a G-equivariant map $V \to G/P$ with fiber L/H_L. (In fact, they show that the closure of V maps G-equivariantly to G/P with fiber the wonderful compactifcation of L/H_L, from which our statement follows by restricting the map to V.) It follows that there is a bijective morphism $\phi\colon G \times^P L/H_L \to V$, which is an isomorphism if ϕ is separable [14, discussion after Theorem 2.2].

The second result follows from [4, Lemma 6], but we give a direct proof for completeness. We again consider the G-equivariant fibre bundle $\pi\colon G/H \to G/P$, with fiber L/H_L. Since H_L is a spherical subgroup of L and P is a spherical subgroup of G (by the classical Bruhat decomposition), it follows that H is a spherical subgroup of G. Moreover, since there is a unique closed B_L-orbit in L/H_L and a unique closed B-orbit in G/P, there is a unique closed B-orbit in G/H, namely the B-orbit which maps via π to the closed B-orbit of G/P and whose fiber over the base point $1P/P$ is identified with the closed B_L-orbit in L/H_L. □

2.4 Weak order and *W*-sets

Let H be a spherical subgroup of the connected, reductive algebraic group G and assume that there exists a wonderful compactification X of G/H. We continue with the notation of the previous subsection. In this subsection we review the notion of the weak order on the set of B-orbit closures of X. Note that X is a spherical variety, meaning that B has finitely many orbits on X, so the set of B-orbits equipped with the weak order is a finite poset.

The **weak order** on the set of B-orbit closures of X is the one whose covering relations are given by $Y \prec Y'$ if and only if $Y' = P_\alpha Y \neq Y$ for some simple root α of T relative to B. In general, $Y \leq Y'$ if and only if $Y' = P_{\alpha_s} \cdots P_{\alpha_1} Y$ for some sequence of simple roots $\alpha_1, \ldots, \alpha_s$.

When considering the weak order on X, it suffices to consider it on the individual G-orbits separately. Indeed, if Y and Y' are the closures of B-orbits Q and Q', respectively, and if $Y \leq Y'$ in weak order, then Q and Q' lie in the same G-orbit. Therefore, we focus on the weak order on B-orbit closures on a homogeneous space G/H. The Hasse diagram of the weak order poset can be drawn as a graph with labeled edges, each edge with a weight of either 1 or 2. This is done as follows: For each cover $Y \prec Y'$ with $Y' = P_\alpha Y$, we draw an edge from Y to Y', and label it by the simple reflection s_α. If the natural map $P_\alpha \times^B Y \to Y'$ is birational, then the edge has weight 1; if the map is generically 2-to-1, then the edge has weight 2. (These are the only two possibilities.) The edges of weight 2 are frequently depicted as double edges [4].

In the graph described above, there is a unique maximal element, since G/H is the closure of its dense B-orbit. Given a B-orbit closure Y, its ***W*-set**, denoted $W(Y)$, is defined as the set of all elements of W obtained by taking the product of edge labels of paths which start at Y and end at G/H. The weight $d(Y, w)$ alluded to before (1.2) is defined as the number of double edges in any such path whose edge labels multiply to w. (Note that there is one such path for each reduced expression of w, but all such paths have the same number of double edges, so that $d(Y, w)$ *is* well-defined [4].) We have now recalled all explanation necessary to understand (1.2).

Next, we briefly recall results of [5, 6] which give explicit descriptions of these W-sets in the cases described in Section 2.2.

We begin by addressing the case of the extended orthogonal group $\mathbf{H} = \mathbf{GO}_n$ and the variants $\mathbf{H}_\mu$. For a set $A \subseteq [n] := \{1, 2, \dots, n\}$, say that $a < b$ are *adjacent in* A if there does not exist $c \in A$ such that $a < c < b$. Let $\mathcal{W}_n$ denote the set of permutations $w \in S_n$ that have the following recursive property. Initialize $A_1 = [n]$. For $1 \le i \le \lfloor n/2 \rfloor$, assume that $w(1), \dots, w(i-1)$ and $w(n+2-i), \dots, w(n)$ have already been defined. (This condition is vacuous in the case $i = 1$.) Then $w(i)$ and $w(n+1-i)$ must be adjacent in A_i and $w(i)$ must be greater than $w(n+1-i)$. Define $A_{i+1} := A_i \setminus \{w(i), w(n+1-i)\}$. This completely defines w when n is even, and if $n = 2k+1$ is odd, then A_{k+1} will consist of a single element m, so define $w(k+1) = m$. For example, $\mathcal{W}_5$ consists of the eight elements of S_5 given in one-line notation by 24531, 25341, 34512, 35142, 42513, 45123, 52314, 53124.

Proposition 2.3 ([5]). *Let* $\mathbf{Y}$ *denote the closed* $\mathbf{B}$*-orbit in* $\mathbf{G}/\mathbf{H}$ *where* $\mathbf{G} = \mathbf{GL}_n$ *and* $\mathbf{H} = \mathbf{GO}_n$*. Then* $W(\mathbf{Y}) = \mathcal{W}_n$*.*

Remark 2.4. The "$\mathcal{W}$-set" of [5], which is denoted by D_n there, differs slightly from ours. More precisely, the relationship between D_n and our $\mathcal{W}$-set is

$$\mathcal{W}_n = \{w_0 w^{-1} w_0 : w \in D_n\}.$$

Let us explain the reason for the discrepancy. First, the partial order considered in [5] is the opposite of the weak order on Borel orbits considered here, which necessitates inverting the elements of D_n. Second, we consider here $\mathbf{B}$ to be the Borel subgroup of lower triangular matrices in GL_n, while [5] uses the Borel subgroup of upper triangular matrices. This necessitates conjugating the elements by w_0.

Similarly, we define a set $\mathcal{W}_\mu \subseteq S_n$ associated with a composition $\mu = (\mu_1, \dots, \mu_s)$ of n. We begin by recalling the notion of a μ-string [5]. Recall that we have defined $\nu_k = \sum_{j=0}^{k-1} \mu_j$ for $2 \le k \le s$; by convention $\nu_1 = 0$. The ith μ-string of a permutation $w \in S_n$, denoted by $\mathrm{str}_i(w)$ is the word $w(\nu_i + 1)\, w(\nu_i + 2) \dots w(\nu_{i+1})$. For example, if $w = 3715462$ is a permutation from S_7 (written in one-line notation) and $\mu = (2, 4, 1)$, then the second μ-string of w is the word 1546. Let $A \subseteq [n]$ have cardinality k and assume a word ω of length k is given that uses each letter of A exactly once. Define a bijection between $[k]$ and A by associating to $i \in [k]$ the ith largest element of A. Under this bijection, the word ω corresponds to the one-line notation of a permutation w in S_k. Call w the permutation associated to the word ω. Continuing the example, the permutation associated to the word $\omega = 1546$ is $1324 \in S_4$ (in one-line notation).

The set $\mathcal{W}_\mu$ consists of all $w \in S_n$ such that the letters of $\mathrm{str}_i(w)$ are precisely those j such that $n - \nu_{i+1} < j \leq n - \nu_i$ and the permutation associated to $\mathrm{str}_i(w)$ is an element of $\mathcal{W}_{\mu_i}$. For example, $\mathcal{W}_{(4,2)}$ consists of the three elements of S_6 given in one-line notation by $465321, 563421, 643521$.

Proposition 2.5 ([5]). *Let* $\mathbf{Y}_\mu$ *denote the closed* $\mathbf{B}$*-orbit in* $\mathbf{G}/\mathbf{H}_\mu$ *where* $\mathbf{G} = \mathbf{GL}_n$ *and* $\mathbf{H}_\mu = (\mathbf{GO}_{\mu_1} \times \cdots \times \mathbf{GO}_{\mu_s}) \ltimes \mathbf{U}_\mu$, *i.e.* $\mathbf{H}_\mu$ *is obtained by parabolic induction from* $\mathbf{H}_L = \mathbf{GO}_{\mu_1} \times \cdots \times \mathbf{GO}_{\mu_s} \subseteq \mathbf{L} = \mathbf{GL}_{\mu_1} \times \cdots \times \mathbf{GL}_{\mu_s}$ *to* $\mathbf{G} = \mathbf{GL}_n$*. Then* $W(\mathbf{Y}_\mu) = \mathcal{W}_\mu$.

Just as in Remark 2.4, the relation between $\mathcal{W}_\mu$ and the set D_μ defined in [5] is $\mathcal{W}_\mu = \{w_0 w^{-1} w_0 : w \in D_\mu\}$.

We now turn to the extended symplectic case $\mathbf{H}' = \mathbf{GSp}_{2n}$ and its variants $\mathbf{H}'_\mu$. Consider the inclusion of S_n into S_{2n} via the map $u \mapsto \phi(u) = v = v_1 v_2 \cdots v_{2n}$, where

$$[v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_{2n-1}, v_{2n}] = \\ [2u(1)-1, 2u(2)-1, \ldots, 2u(n)-1, 2u(n), \ldots, 2u(2), 2u(1)].$$

Let $\mathcal{W}'_{2n} = \{\phi(u) \in S_{2n} : u \in S_n\}$. For example, $\mathcal{W}'_6$ consists of the six elements of S_6 given in one-line notation by $135642, 153462, 315624, 351264, 513426, 531246$.

Proposition 2.6 ([6, 12]). *Let* $\mathbf{Y}'$ *denote the closed* $\mathbf{B}$*-orbit in* $\mathbf{G}/\mathbf{H}'$ *where* $\mathbf{G} = \mathbf{GL}_{2n}$ *and* $\mathbf{H}' = \mathbf{GSp}_{2n}$*. Then* $W(\mathbf{Y}') = \mathcal{W}'_n$.

We now proceed to define a set $\mathcal{W}'_\mu \subseteq S_{2n}$ for any composition $\mu = (\mu_1, \ldots, \mu_s)$ of n. The set $\mathcal{W}'_\mu$ consists of all $w \in S_n$ such that the letters of $\mathrm{str}_i(w)$ are precisely those j such that $n - \nu_{i+1} < j \leq n - \nu_i$ and the permutation associated to $\mathrm{str}_i(w)$ is an element of $\mathcal{W}'_{2\mu_i}$. For example, $\mathcal{W}'_{(2,4)}$ consists of the two elements of S_6 given in one-line notation by $123564, 125346$.

Proposition 2.7. *Let* $\mathbf{Y}'_\mu$ *denote the closed* $\mathbf{B}$*-orbit in* $\mathbf{G}/\mathbf{H}'_\mu$ *where* $\mathbf{G} = \mathbf{GL}_{2n}$ *and* $\mathbf{H}'_\mu = (\mathbf{GSp}_{2\mu_1} \times \cdots \times \mathbf{GSp}_{2\mu_s}) \ltimes \mathbf{U}_\mu$, *i.e.* $\mathbf{H}'_\mu$ *is obtained by parabolic induction from* $\mathbf{H}_L = \mathbf{GSp}_{2\mu_1} \times \cdots \times \mathbf{GSp}_{2\mu_s} \subseteq \mathbf{L} = \mathbf{GL}_{2\mu_1} \times \cdots \times \mathbf{GL}_{2\mu_s}$ *to* $\mathbf{G} = \mathbf{GL}_{2n}$*. Then* $W(\mathbf{Y}'_\mu) = \mathcal{W}'_\mu$.

Proposition 2.7 is proved in exactly the same manner as Proposition 2.5 is proven in [5, Theorem 4.11], so we omit its proof. Alternatively, it can be obtained as a corollary of Proposition 2.6 by applying a general result of Brion on W-sets for homogeneous spaces obtained by parabolic induction [2, Lemma 1.2].

3 Equivariant cohomology computations

3.1 Background

We start by reviewing the basic facts of equivariant cohomology that we will need to support our method of computation. All cohomology rings use $\mathbb{Q}$-coefficients. Results of this section are generally stated without proof, as they are fairly standard. To the reader seeking a reference we recommend [15] for an expository treatment, as well as references therein.

We will apply our results to equivariant cohomology with respect to the action of $\mathbf{S}_\mu$ (respectively, $\mathbf{S}'_\mu$) on $\mathbf{G}/\mathbf{B}$, these tori having been defined in Section 2.2. Given a variety X with an action of an algebraic torus S with Lie algebra $\mathfrak{s}$, the equivariant cohomology is, by definition,

$$H^*_S(X) := H^*((ES \times X)/S),$$

where ES denotes a contractible space with a free S-action. $H^*_S(X)$ is an algebra for the ring $\Lambda_S := H^*_S(\{\text{pt.}\})$, the Λ_S-action being given by pullback through the obvious map $X \to \{\text{pt.}\}$. The ring Λ_S is naturally isomorphic to the symmetric algebra $\mathrm{Sym}(\mathfrak{s}^*)$ on $\mathfrak{s}^*$. Thus, if $y_1, \ldots, y_n$ are a basis for $\mathfrak{s}^*$, then $\Lambda_S \simeq \mathrm{Sym}(\mathfrak{s}^*)$ is isomorphic to the polynomial ring $\mathbb{Q}[\mathbf{y}] = \mathbb{Q}[y_1, \ldots, y_n]$. When $X = G/B$ with G a reductive algebraic group and B a Borel subgroup, and if $S \subseteq T \subseteq B$ with T a maximal torus in G, then we have the following concrete description of $H^*_S(X)$:

Proposition 3.1. *Let $R = \mathrm{Sym}(\mathfrak{t}^*)$, $R' = \mathrm{Sym}(\mathfrak{s}^*)$. Then $H^*_S(X) = R' \otimes_{R^W} R$. If $X_1, \ldots, X_n$ are a basis for $\mathfrak{t}^*$, and $Y_1, \ldots, Y_m$ are a basis of $\mathfrak{s}^*$, elements of $H^*_S(X)$ are thus represented by polynomials in variables $x_i := 1 \otimes X_i$ and $y_i := Y_i \otimes 1$.*

To make this clear in the setting of our examples (cf. Section 2.2), if $\mathbf{S}$ is taken to be the full maximal torus $\mathbf{T}$ of $\mathbf{GL}_n$, we let $\mathbf{X}_i$ $(i = 1, \ldots, n)$ be the function on $\mathfrak{t}$ which evaluates to a_i on the element

$$t = \mathrm{diag}(a_1, \ldots, a_n).$$

We denote by $\mathbf{Y}_i$ $(i = 1, \ldots, n)$ a second copy of the same set of functions. We then have two sets of variables as in Proposition 3.1, typically denoted $\mathbf{x} = x_1, \ldots, x_n$ and $\mathbf{y} = y_1, \ldots, y_n$, and $\mathbf{T}$-equivariant classes are represented by polynomials in these variables.

If $\mu = (\mu_1, \ldots, \mu_s)$ is a composition of n, let $\mathbf{T}$ be the full torus of $\mathbf{GL}_n$, and let $\mathbf{S}_\mu$ be the torus of $\mathbf{H}_\mu$, as in Section 2.2. We denote by $\mathbf{X}_i$ the same function on $\mathfrak{t}$ as described above. We denote by $\mathbf{Y}_{i,j}$ the function on $\mathfrak{s}_\mu$ which evaluates to $a_{i,j}$ on an element of the form in (2.8) (if μ_i is even) or (2.10) (if μ_i is odd). We denote by $\mathbf{Z}_i$ the function which evaluates to λ_i on an element of the form (2.8) or (2.10). Then letting lower-case $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$-variables correspond to these coordinates (with matching indices), $H^*_{\mathbf{S}_\mu}(\mathbf{G}/\mathbf{B})$ is generated by these variables, and when we seek formulas for certain $\mathbf{S}_\mu$-equivariant classes, we are looking for polynomials in these particular variables.

We next recall the standard localization theorem for torus actions. For more on this fundamental result, the reader may consult, for example, [3].

Theorem 3.2. *Let X be an S-variety, and let $i\colon X^S \hookrightarrow X$ be the inclusion of the S-fixed locus of X. Then the pullback map of Λ_S-modules*

$$i^*\colon H^*_S(X) \to H^*_S(X^S)$$

is an isomorphism after a localization which inverts finitely many characters of S. In particular, if $H^*_S(X)$ is free over Λ_S, then i^* is injective.

When $X = G/B$, $H^*_S(X) = R' \otimes_{R^W} R$ is free over R' (as R is free over R^W), so any equivariant class is entirely determined by its image under i^*. We will only apply this result in the event that $S = T$ is the full maximal torus of G, so the S-fixed locus is finite, being parametrized by W. Then for us,

$$H_S^*(X^S) \cong \bigoplus_{w \in W} \Lambda_S,$$

so that in fact a class in $H_S^*(X)$ is determined by its image under i_w^* for each $w \in W$, where here i_w denotes the inclusion of the S-fixed point wB/B in G/B. Given a class $\beta \in H_S^*(X)$ and an S-fixed point wB/B, we may denote the restriction $i_w^*(\beta)$ at wB/B by $\beta|_w$.

We end the section by recalling how the restriction maps are computed.

Proposition 3.3 ([15]). *With the notation of the preceding paragraph, suppose that $\beta \in H_S^*(X)$ is represented by the polynomial $f = f(\mathbf{x}, \mathbf{y})$ in variables $\mathbf{x}$, $\mathbf{y}$. Then $\beta|_w \in \Lambda_S$ is the polynomial $f(wY|_{\mathfrak{s}}, Y)$.*

3.2 A general result

We continue with the general setup given at the beginning of Section 2. Let $P = LU$ be a parabolic subgroup of G containing B, with L a Levi factor containing T and U the unipotent radical of P contained in B. Let H_L denote a spherical subgroup of L and let $S := T \cap H_L$ be a maximal torus of H_L. Let X denote the generalized flag variety G/B.

Let H be the subgroup obtained by parabolic induction from $H_L \subseteq L$ to G, i.e. $H = H_L U$. As proven in Proposition 2.2, H is a spherical subgroup of G with a unique closed B-orbit Q. The torus S is a maximal torus of H, and we seek to describe $[Q] \in H_S^*(X)$.

Denote by B_L the Borel subgroup $B \cap L$ of L, and let Y denote the generalized flag variety L/B_L. Note that there is an S-equivariant embedding $j \colon Y \hookrightarrow X$, which induces a pushforward map in cohomology $j_* \colon H_S^*(Y) \to H_S^*(X)$. Given our setup, the orbit $Q' = H \cdot 1B_L/B_L$ is closed in Y. In fact, $j(Q') = Q$. To see this we just observe that $1B/B \cong 1B_L/B_L$ under the embedding j. Nonetheless, to avoid possible confusion, we refer to the closed orbit as Q' when thinking of it as a subvariety of Y, and as Q when thinking of it as a subvariety of X.

Let $W_L \subseteq W$ denote the Weyl group of L. In the root system Φ for (G, T), choose Φ^+ to be the positive system such that the roots of B are negative. Similarly, let Φ_L be the root system for (L, T) and let $\Phi_L^+ = \Phi_L \cap \Phi^+$ be the positive roots of L such that the roots of B_L are negative. Given a root $r \in \Phi$ (resp., $r \in \Phi_L$), let $\mathfrak{g}_r$ denote the associated one-dimensional root space in $\mathfrak{g}$ (resp., $\mathfrak{l}$).

The next result relates the class of Q' in $H_S^*(Y)$ to the class of Q in $H_S^*(X)$.

Proposition 3.4. *With notation as above, the classes $j^*[Q]$ and $[Q']$ are related via multiplication by the top S-equivariant Chern class of the normal bundle $N_Y X$, i.e. $j^*[Q] = c_d^S(N_Y X) \cap [Q']$. This Chern class is the restriction of a T-equivariant Chern class for the same normal bundle, and the latter class, which we denote by α, is uniquely determined by the following properties:*

$$\alpha|_w = \begin{cases} \prod_{r \in \Phi^+ \setminus \Phi_L^+} wr & \text{if } w \in W_L, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first statement follows easily from the equivariant self-intersection formula [8, p. 621, (4)], since $[Q] = j_*[Q']$. Since both X and Y have T-actions (not simply S-actions, as is the case for Q and Q'), there does exist a top T-equivariant Chern class of $N_X Y$, and clearly the S-equivariant version is simply the restriction of the T-equivariant one.

The properties which define α follow from analysis of tangent spaces at various fixed points. Indeed, it is clear that the T-fixed points of Q' lying in Y are those which correspond to elements of $W_L \subseteq W$. Thus for $w \notin W_L$, we have $\alpha|_w = 0$. For $w \in W_L$, the restriction is $c_d^T(N_X Y)|_w = c_d^T(N_X Y|_w) = c_d^T(T_w X / T_w Y)$, where $d = \operatorname{codim}_X(Y)$. Since both X and Y are flag varieties, it is straightforward to compute these two tangent spaces, and their decompositions as representations of T. Indeed, $T_w X$ is simply $\bigoplus_{r \in \Phi^+} \mathfrak{g}_{wr}$, while $T_w Y$ is $\bigoplus_{r \in \Phi_L^+} \mathfrak{g}_{wr}$. The quotient of the two spaces is then $\bigoplus_{r \in \Phi^+ \setminus \Phi_L^+} \mathfrak{g}_{wr}$, which implies our claim on $\alpha|_w$.

Finally, that these restrictions determine α follows from the localization theorem, Theorem 3.2. □

We now determine an explicit formula for the T-equivariant Chern class α that is defined in Proposition 3.4 when G is of type A. Let $\mu = (\mu_1, \dots, \mu_s)$ be a composition of n, let $\mathbf{L} = \mathbf{GL}_{\mu_1} \times \cdots \times \mathbf{GL}_{\mu_s}$, and let $\mathbf{T}$ be the full diagonal torus of $\mathbf{GL}_n$.

For each $1 \le k \le n$, let $\epsilon_k \in \mathfrak{t}^*$ be given by $\epsilon_k(\operatorname{diag}(t_1, \dots, t_n)) = t_k$. For any $1 \le k < l \le n$, let $\alpha_{k,l} = \epsilon_k - \epsilon_\ell \in \Phi^+$. Finally, let

$$h_\mu(\mathbf{x}, \mathbf{y}) := \prod_{\alpha_{k,\ell} \in \Phi^+ \setminus \Phi_L^+} (x_k - y_\ell).$$

An equivalent definition of h_μ showing the explicit dependence on the composition μ is as follows. For each pair i, j with $1 \le i < j \le s$, we define a polynomial

$$h_{i,j}(\mathbf{x}, \mathbf{y}) := \prod_{k=\nu_j+1}^{\nu_j+\mu_j} \prod_{\ell=\nu_i+1}^{\nu_i+\mu_i} (x_\ell - y_k).$$

Then it follows immediately that

$$h_\mu(\mathbf{x}, \mathbf{y}) = \prod_{1 \le i < j \le s} h_{i,j}(\mathbf{x}, \mathbf{y}).$$

Proposition 3.5. *The T-equivariant class α is represented by the polynomial $h_\mu(\mathbf{x}, \mathbf{y})$.*

Proof. It is straightforward to verify that the polynomial representative we give satisfies the restriction requirements of Proposition 3.4. Indeed, the Weyl group $W_{\mathbf{L}}$ of $\mathbf{L}$ is a parabolic subgroup of the symmetric group on n-letters, embedded as those permutations preserving separately the sets $\{\nu_i + 1, \dots, \nu_i + \mu_i\}$ for $i = 0, \dots, s-1$. Applying such a permutation to the representative above (with the action by permutation of the indices on the $\mathbf{x}$-variables, as in Proposition 3.3) gives the appropriate product of weights. On the other hand, applying any $w \notin W_{\mathbf{L}}$ will clearly give 0, since such a permutation necessarily sends some $l \in \{\nu_i + 1, \dots, \nu_i + \mu_i\}$ (for some i) to some $k \in \{\nu_j + 1, \dots, \nu_j + \mu_j\}$ with $i < j$. This permutation forces the factor $x_\ell - y_k$ to vanish, which, in turn, forces $h_\mu(\mathbf{x}, \mathbf{y})$ to vanish. By Proposition 3.4, h_μ represents α. □

3.3 The orthogonal case

We now apply these computations specifically to the two type A cases described in Section 2.2, using all of the notational conventions defined there. We start with the case of $(\mathbf{G}, \mathbf{H}) = (\mathbf{GL}_n, \mathbf{GO}_n)$. Let $\mu = (\mu_1, \dots, \mu_s)$ be a composition of n, and let $\mathbf{B}$, $\mathbf{T}$, and $\mathbf{S}_\mu$ be as defined in Section 2.2. Let $\mathbf{P}_\mu = \mathbf{L}_\mu \mathbf{U}_\mu$ be the standard parabolic subgroup containing $\mathbf{B}$ whose Levi factor $\mathbf{L}_\mu$ corresponds to μ.

With these choices made, let the $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ variables be the generators for $H^*_{\mathbf{S}_\mu}(\mathbf{G}/\mathbf{B})$ explicitly described after the statement of Proposition 3.1. We seek a polynomial in the $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$-variables which represents the class of $\mathbf{H}_\mu \cdot 1\mathbf{B}/\mathbf{B} \in H^*_{\mathbf{S}_\mu}(\mathbf{GL}_n/\mathbf{B})$.

To find such a formula, we use a known formula for the closed $\mathbf{H}_0 = \mathbf{O}_n$-orbit on $\mathbf{GL}_n/\mathbf{B}$ to deduce a formula for the $\mathbf{S}_\mu$-equivariant class of $\mathbf{H}_\mu \cdot 1\mathbf{B}/\mathbf{B}$ in $H^*_{\mathbf{S}_\mu}(\mathbf{L}_\mu/\mathbf{B}_{\mathbf{L}_\mu})$, and then apply Proposition 3.4. Indeed, we know from [15, 16] that when $\mathbf{H}_0 = \mathbf{O}_n$, the class of $\mathbf{H}_0 \cdot 1\mathbf{B}/\mathbf{B}$ in $H^*_{\mathbf{S}_0}(\mathbf{GL}_n/\mathbf{B})$ (here, $\mathbf{S}_0$ is the maximal torus of $\mathbf{H}_0$ described in Section 2.2) is given by

$$[\mathbf{H}_0 \cdot 1\mathbf{B}/\mathbf{B}]_{\mathbf{S}_0} = \prod_{1 \le i \le j \le n-i} (x_i + x_j) = 2^{\lfloor n/2 \rfloor} \prod_{i \le n/2} x_i \prod_{1 \le i < j \le n-i} (x_i + x_j). \tag{3.1}$$

The formula of (3.1) is given in [15] in the case that n is even, while an alternative formula is given in the case that n is odd. It is observed in [16] that the above formula applies equally well when n is odd.

Recall that when $\mathbf{H} = \mathbf{GO}_n$, a maximal torus $\mathbf{S}$ of $\mathbf{H}$ has dimension one greater than the corresponding maximal torus $\mathbf{S}_0$ of $\mathbf{H}_0$. Thus the $\mathbf{S}$-equivariant cohomology of $\mathbf{GL}_n/\mathbf{B}$ has one additional "equivariant variable", which we call z. In this case, it is no harder to show that the class of $\mathbf{H} \cdot 1\mathbf{B}/\mathbf{B}$ in $H^*_{\mathbf{S}}(\mathbf{GL}_n/\mathbf{B})$ is given by

$$[\mathbf{H} \cdot 1\mathbf{B}/\mathbf{B}]_{\mathbf{S}} = P_n(\mathbf{x}, \mathbf{y}, z) := \prod_{1 \le i \le j \le n-i} (x_i + x_j - 2z). \tag{3.2}$$

Note that by restricting from $H^*_{\mathbf{S}}(\mathbf{G}/\mathbf{B})$ to $H^*_{\mathbf{S}_0}(\mathbf{G}/\mathbf{B})$ (which amounts to setting the additional equivariant variable z to 0), we recover the original formula (3.1).

Combining these formulae with Proposition 3.4, for each composition μ of n we are now ready to give case-specific formulae for the unique closed $\mathbf{H}_\mu$-orbit $\mathbf{H}_\mu \cdot 1\mathbf{B}/\mathbf{B}$ on $\mathbf{GL}_n/\mathbf{B}$. (Recall that each of these orbits corresponds to the unique closed $\mathbf{B}$-orbit on the corresponding $\mathbf{G}$-orbit on the wonderful compactification of $\mathbf{G}/\mathbf{H}$.)

We consider the variables $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$, and $\mathbf{z} = (z_1, \dots, z_s)$, where s is the number of parts in the composition μ. We divide the variables into smaller clusters dictated by the composition μ. Let $\mathbf{x}^{(i)} = (x_{\nu_i+1}, \dots, x_{\nu_{i+1}})$ and $\mathbf{y}^{(i)} = (y_{\nu_i+1}, \dots, y_{\nu_{i+1}})$. Then define

$$P_\mu(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \prod_{i=1}^{s} P_{\mu_i}(\mathbf{x}^{(i)}, \mathbf{y}^{(i)}, z_i),$$

where P_{μ_i} is given by (3.2).

We now introduce an equivalent, but more explicit, description of the class $[\mathbf{H}_\mu \cdot 1\mathbf{B}/\mathbf{B}]_{S_\mu}$ which reflects the block decomposition associated to μ. To this end, we introduce new notation for a fixed composition $\mu = (\mu_1, \dots, \mu_s)$ of n. First, for each of the s blocks of $\mathbf{S}_\mu$, define a polynomial $f_i(\mathbf{x},\mathbf{z})$ as follows:

$$f_i(\mathbf{x},\mathbf{z}) = \prod_{j=\nu_i+1}^{\nu_i+\lfloor \mu_i/2 \rfloor} (x_j - z_i).$$

In words, the x_j occurring in the terms of this product are those occurring in the first half of their block, and from each, we subtract the $\mathbf{z}$-variable corresponding to that block. So for example, if $n = 11$, $\mu = (6, 5)$, then

$$f_1(\mathbf{x},\mathbf{z}) = (x_1 - z_1)(x_2 - z_1)(x_3 - z_1),$$

while

$$f_2(\mathbf{x},\mathbf{z}) = (x_7 - z_2)(x_8 - z_2).$$

Next, for each block, define $g_i(\mathbf{x},\mathbf{z})$ as follows:

$$g_i(\mathbf{x},\mathbf{z}) = \prod_{\nu_i+1 \le j < k \le 2\nu_i+\mu_i-j} (x_j + x_k - 2z_i).$$

(Note that $g_i(\mathbf{x},\mathbf{z}) = 1$ unless $\mu_i \ge 3$.) So for $\mu = (6, 5)$ as above, we have that

$$g_1(\mathbf{x},\mathbf{z}) = (x_1 + x_2 - 2z_1)(x_1 + x_3 - 2z_1)(x_1 + x_4 - 2z_1)(x_1 + x_5 - 2z_1)$$
$$(x_2 + x_3 - 2z_1)(x_2 + x_4 - 2z_1),$$

and

$$g_2(\mathbf{x},\mathbf{z}) = (x_7 + x_8 - 2z_2)(x_7 + x_9 - 2z_2)(x_7 + x_{10} - 2z_2)(x_8 + x_9 - 2z_2).$$

Finally, we define a third polynomial $h_\mu(\mathbf{x},\mathbf{y},\mathbf{z})$ in the $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$-variables to simply be $h_\mu(\mathbf{x},\rho(\mathbf{y}))$, where ρ denotes restriction from the variables $y_1,\dots,y_n$ corresponding to coordinates on the full torus $\mathbf{T}$ to the variables $y_{i,j}, z_i$ on the smaller torus $\mathbf{S}_\mu$. To be more explicit, for each i, j with $1 \le i < j \le s$ define $h_{i,j}(\mathbf{x},\mathbf{y},\mathbf{z})$ to be

$$\begin{cases} \prod_{k=1}^{\mu_i} \prod_{l=1}^{\mu_j/2} (x_{\nu_i+k} - y_{j,l} - z_j)(x_{\nu_i+k} + y_{j,l} - z_j) & \text{if } \mu_j \text{ is even,} \\ \prod_{k=1}^{\mu_i} (x_{\nu_i+k} - z_j) \prod_{l=1}^{\lfloor \mu_j/2 \rfloor} (x_{\nu_i+k} - y_{j,l} - z_j)(x_{\nu_i+k} + y_{j,l} - z_j) & \text{if } \mu_j \text{ is odd.} \end{cases}$$

So for the case $n = 4$, $\mu = (2, 2)$, we have

$$h_{1,2}(\mathbf{x},\mathbf{y},\mathbf{z}) = (x_1 - y_{2,1} - z_2)(x_1 + y_{2,1} - z_2)(x_2 - y_{2,1} - z_2)(x_2 + y_{2,1} - z_2),$$

while for the case $n = 5$, $\mu = (2, 3)$, we have

$$h_{1,2}(\mathbf{x},\mathbf{y},\mathbf{z}) = (x_1 - z_2)(x_2 - z_2)(x_1 - y_{2,1} - z_2)(x_1 + y_{2,1} - z_2)$$
$$(x_2 - y_{2,1} - z_2)(x_2 + y_{2,1} - z_2).$$

Then we define

$$h_\mu(\mathbf{x},\mathbf{y},\mathbf{z}) = \prod_{1 \le i < j \le s} h_{i,j}(\mathbf{x},\mathbf{y},\mathbf{z}).$$

Propositions 3.4 and 3.5 then imply the following formula for the $\mathbf{S}_\mu$-equivariant class of $\mathbf{H}_\mu \cdot 1\mathbf{B}/\mathbf{B}$ in this case.

Corollary 3.6. *The* $\mathbf{S}_\mu$*-equivariant class of the unique closed* $\mathbf{H}_\mu$*-orbit* $\mathbf{H}_\mu \cdot 1\mathbf{B}/\mathbf{B}$ *on* $\mathbf{G}/\mathbf{B}$ *is represented by the polynomial*

$$2^{d(\mu)} h_\mu(\mathbf{x}, \mathbf{y}, \mathbf{z}) \prod_{i=1}^{s} f_i(\mathbf{x}, \mathbf{z}) g_i(\mathbf{x}, \mathbf{z}),$$

where $d(\mu) = \sum_{i=1}^{s} \lfloor \mu_i/2 \rfloor$.

Proof. The fact that the product $2^{d(\mu)} \prod_{i=1}^{s} f_i(\mathbf{x}, \mathbf{z}) g_i(\mathbf{x}, \mathbf{z})$ is equal to the formula for $j^*[Q]$ follows from the formula of (3.2). It follows from Proposition 3.4 and Proposition 3.5 that the representative of $[Q]$ is obtained from $j^*[Q]$ by multiplying with the top S_μ-equivariant Chern class of the normal bundle, which is represented by the polynomial $h_\mu(\mathbf{x}, \mathbf{y}, \mathbf{z})$. □

Corollary 3.7. *The* $\mathbf{S}_\mu$*-equivariant class* $[\mathbf{H}_\mu \cdot 1\mathbf{B}/\mathbf{B}]$ *is represented by the polynomial* $P_\mu(\mathbf{x}, \mathbf{y}, \mathbf{z}) h_\mu(\mathbf{x}, \mathbf{y}, \mathbf{z})$.

Proof. This follows immediately from the observation that

$$P_{\mu_i}(\mathbf{x}^{(i)}, \mathbf{y}^{(i)}, z_i) = 2^{\lfloor \mu_i/2 \rfloor} f_i(\mathbf{x}, \mathbf{z}) g_i(\mathbf{x}, \mathbf{z}).$$ □

3.4 The symplectic case

We give similar (but simpler) formulas for the case when $(\mathbf{G}, \mathbf{H}') = (\mathbf{GL}_{2n}, \mathbf{GSp}_{2n})$. Recall from [15] that for the case $(\mathbf{G}, \mathbf{H}'_0) = (\mathbf{GL}_{2n}, \mathbf{Sp}_{2n})$, the $\mathbf{S}'_0$-equivariant class ($\mathbf{S}'_0$ the maximal torus of $\mathbf{H}'_0$ described in Section 2.2) of the unique closed orbit $\mathbf{H}'_0 \cdot 1\mathbf{B}/\mathbf{B}$ is given by

$$[\mathbf{H}'_0 \cdot 1\mathbf{B}/\mathbf{B}]_{\mathbf{S}'_0} = \prod_{1 \le i < j \le 2n-i} (x_i + x_j). \tag{3.3}$$

As before, it is no harder to see that if $\mathbf{S}'$ is the maximal torus of diagonal elements of $\mathbf{H}'$, then the $\mathbf{S}'$-equivariant class of the closed orbit $\mathbf{H}' \cdot 1\mathbf{B}/\mathbf{B}$ is represented by

$$[\mathbf{H}' \cdot 1\mathbf{B}/\mathbf{B}]_{\mathbf{S}'} = P'_n(\mathbf{x}, \mathbf{y}, z) := \prod_{1 \le i < j \le 2n-i} (x_i + x_j - 2z). \tag{3.4}$$

Now let $\mu = (\mu_1, \dots, \mu_s)$ be a composition of $2n$ with all even parts. Let $\mathbf{H}'_\mu$ be the spherical group

$$(\mathbf{GSp}_{\mu_1} \times \cdots \times \mathbf{GSp}_{\mu_s}) \ltimes \mathbf{U}_\mu,$$

as defined in Section 2.2. Let ${\mathbf{S}'}_\mu$ be the maximal torus of $\mathbf{H}'_\mu$, with the $\mathbf{y}$ and $\mathbf{z}$-variables as defined above.

Corollary 3.8. *In* $H^*_{{\mathbf{S}'}_\mu}(\mathbf{G}/\mathbf{B})$*, the class of the closed* $\mathbf{H}'_\mu$*-orbit* $\mathbf{H}'_\mu \cdot 1\mathbf{B}/\mathbf{B}$ *is represented by*

$$h_\mu(\mathbf{x}, \mathbf{y}, \mathbf{z}) \prod_{i=1}^{s} g_i(\mathbf{x}, \mathbf{z}).$$

Proof. The proof is identical to that of Corollary 3.6, using Proposition 3.4 combined with (3.4) in the same way. □

Remark 3.9. Note that $g_i(\mathbf{x}, \mathbf{z}) = P'_{\mu_i}(\mathbf{x}^{(i)}, \mathbf{y}^{(i)}, z_i)$, where P'_{μ_i} is given by (3.4), so that again $[\mathbf{H}'_\mu \cdot 1\mathbf{B}/\mathbf{B}]$ is equal to $P'_\mu(\mathbf{x}, \mathbf{y}, \mathbf{z})h_\mu(\mathbf{x}, \mathbf{y}, \mathbf{z})$, where

$$P'_\mu(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \prod_{i=1}^{s} P'_{\mu_i}(\mathbf{x}^{(i)}, \mathbf{y}^{(i)}, z_i).$$

We now specialize these formulas to ordinary cohomology (by setting all $\mathbf{y}$ and $\mathbf{z}$-variables to 0), in order to prove Corollaries 1.1 and 1.2. First, we define the notations used in those formulas which have not yet been defined. For each $i = 1, \ldots, n$, let $B(\mu, i)$ denote the block that the variable x_i occurs in, i.e. $B(\mu, i)$ is the smallest integer j such that

$$\sum_{\ell=1}^{j} \mu_\ell \geq i.$$

Then for each $i = 1, \ldots, n$, define $R(\mu, i)$ to be

$$R(\mu, i) := \sum_{B(\mu,i)<j\leq s} \mu_j.$$

This is the combined size of all blocks occurring strictly to the right of the block in which x_i occurs.

Finally, again for each such i, define $\delta(\mu, i)$ to be 1 if and only if x_i occurs in the first half of its block, and 0 otherwise. Note that by the "first half" we mean those positions less than or equal to $\ell/2$ where ℓ is the size of the block; in particular, for a block of odd size, the middle position is not considered to be in the first half of the block.

Proof of Corollaries 1.1 and 1.2. The formula of Corollary 1.1 comes from that of Corollary 3.6; we simply set $\mathbf{y} = \mathbf{z} = 0$. The binomial terms $x_j + x_k$ come from the polynomials $g_i(\mathbf{x}, 0)$. The monomial terms come from the polynomials $f_i(\mathbf{x}, 0)$ and $h_{i,j}(\mathbf{x}, 0, 0)$. The $x_i^{\delta(\mu,i)}$ term comes from $f_i(\mathbf{x}, 0)$, the latter being x_i if this variable occurs in the first half of its block, and 1 otherwise. The remaining $x_i^{R(\mu,i)}$ comes from the $h_{i,j}(\mathbf{x}, 0, 0)$. Indeed, it is evident that for an $\mathbf{x}$-variable in block i, for each $j > i$ the given $\mathbf{x}$-variable appears in precisely μ_j linear forms involving $\mathbf{y}, \mathbf{z}$ terms associated with block j.

The proof of Corollary 1.2 is almost identical, except simpler. □

4 Factoring sums of Schubert polynomials

We end by establishing explicit polynomial identities involving sums of Schubert polynomials, using the cohomological formulae of the preceding section together with the results of [5, 6] which were recalled in Section 2.4.

Note that by Brion's formula (1.2) combined with the fact that the Schubert polynomial $\mathfrak{S}_w$ is a representative of the class of the Schubert variety $\mathbf{X}_w$ in $H^*(\mathbf{G}/\mathbf{B})$, we have the following two families of identities in $H^*(\mathbf{G}/\mathbf{B})$:

$$\sum_{w\in W(\mathbf{Y}_\mu)} 2^{d(\mathbf{Y}_\mu,w)}\mathfrak{S}_w = 2^{d(\mu)} \prod_{i=1}^{n} x_i^{R(\mu,i)+\delta(\mu,i)} \prod_{i=1}^{s} \left(\prod_{\nu_i+1\leq j\leq k\leq \nu_{i+1}-j} (x_j + x_k) \right); \quad (4.1)$$

$$\sum_{w\in W(\mathbf{Y}'_\mu)} \mathfrak{S}_w = \prod_{i=1}^{2n} x_i^{R(\mu,i)} \prod_{i=1}^{s}\left(\prod_{\nu_i+1\le j<k\le \nu_{i+1}-j} (x_j+x_k)\right). \tag{4.2}$$

Equation (4.1) above simply combines (1.2) with Corollary 1.1. Likewise, (4.2) combines (1.2) with Corollary 1.2, together with the fact that all $\mathbf{B}$-orbit closures in the symplectic case are known to be multiplicity-free, meaning $d(\mathbf{Y}'_\mu, w) = 0$ for all $w \in W(\mathbf{Y}'_\mu)$.

In fact, also in (4.1) above, the powers of 2 can be completely eliminated from both sides of the equation. This follows from a result of Brion [4, Proposition 5], which states that whenever G is a simply laced group (recall that for us, $\mathbf{G} = \mathbf{GL}_n$ is simply laced), all of the coefficients appearing in (1.2) are the *same* power of 2. It is explained in [5, Section 5] that the coefficients appearing on the left-hand side of (4.1) are in fact all equal to $2^{d(\mu)}$. Since $H^*(\mathbf{G}/\mathbf{B})$ has no torsion, we have the simplified equality

$$\sum_{w\in W(\mathbf{Y}_\mu)} \mathfrak{S}_w = \prod_{i=1}^{n} x_i^{R(\mu,i)+\delta(\mu,i)} \prod_{i=1}^{s}\left(\prod_{\nu_i+1\le j\le k\le \nu_{i+1}-j} (x_j+x_k)\right). \tag{4.3}$$

Now, note that *a priori*, the identities (4.2) and (4.3) hold only in $H^*(\mathbf{G}/\mathbf{B})$. That is, we know only that the left and right-hand sides of the identities are congruent modulo the ideal $I^{\mathbf{W}}$. We end with a stronger result.

Theorem 4.1. *Both* (4.2) *and* (4.3) *are valid as polynomial identities.*

Proof. We use the fact that the Schubert polynomials $\{\mathfrak{S}_w \mid w \in S_n\}$ are a $\mathbb{Z}$-basis for the $\mathbb{Z}$-submodule Γ of $\mathbb{Z}[\mathbf{x}]$ spanned by monomials $\prod x_i^{c_i}$ with $c_i \le n-i$ for each i [11, Proposition 2.5.4]. We claim that on the right-hand side of (4.3) (resp. (4.2)), each x_i does in fact occur with exponent at most $n-i$ (resp. $2n-i$). To see this, note that since the right-hand side of each identity is a product of linear forms, it suffices to count, for each i, the number of these linear factors in which x_i appears.

In (4.3), we claim that x_i appears in precisely $\left(\sum_{j=B(\mu,i)}^{s} \mu_j\right) - i$ of the linear factors. Since $\sum_{j=B(\mu,i)}^{s} \mu_j \le \sum_{j=1}^{s} \mu_j = n$, this establishes our claim that the right-hand side lies in Γ. Indeed, clearly x_i appears $R(\mu,i)+\delta(\mu,i)$ times as a monomial factor, so we need only count the number of binomial factors of the form $x_j + x_k$ that it appears in. One checks easily that it appears in $\mu_i - i - 1$ such factors if x_i occurs in the first half of its block, and in $\mu_i - i$ such factors otherwise. In other words, if N_i is the number of binomial factors involving x_i, then we have $\delta(\mu,i) + N_i = \mu_i - i$. Thus

$$\begin{aligned} R(\mu,i)+\delta(\mu,i)+N_i &= R(\mu,i)+\mu_i - i \\ &= \left(\sum_{j=B(\mu,i)+1}^{s} \mu_j\right) + \mu_i - i \\ &= \left(\sum_{j=B(\mu,i)}^{s} \mu_j\right) - i, \end{aligned}$$

as claimed.

Clearly, the right-hand side of (4.2) also lies in Γ, applying the same argument with n replaced by $2n$. Indeed, the only difference is in the lack of the additional monomial factor

$x_i^{\delta(\mu,i)}$; thus x_i occurs in either $\left(\sum_{j=B(\mu,i)}^{s}\mu_j\right)-i$ or $\left(\sum_{j=B(\mu,i)}^{s}\mu_j\right)-i-1$ of the linear factors on the right-hand side of (4.2). In either event, this is at most $2n-i$, as required.

Now, since the right-hand side of each of (4.2) and (4.3) are in Γ, they are expressible as a sum of Schubert polynomials whose indexing permutations lie in S_{2n} (for (4.2)) or S_n (for (4.3)) in exactly one way. Furthermore, since the Schubert classes $\{[\mathbf{X}_w]\}$ are a $\mathbb{Z}$-basis for $H^*(\mathbf{G}/\mathbf{B})$, the cohomology class represented by the right-hand side of (4.2) and (4.3) is a $\mathbb{Z}$-linear combination of Schubert classes in precisely one way. Clearly, the same indexing permutations must arise with the same multiplicities in both the polynomial expansion and the cohomology expansion. Then since (4.2) and (4.3) are correct cohomologically, they must also be polynomial identities. □

Example 4.2. In the orthogonal case when $\mu=(3,4)$, the identity (4.3) becomes

$$\mathfrak{S}_{6752431}+\mathfrak{S}_{6753412}+\mathfrak{S}_{6754213}+\mathfrak{S}_{7562431}+\mathfrak{S}_{7563412}+\mathfrak{S}_{7564213}= x_1^5x_2^4x_3^4x_4x_5(x_1+x_2)(x_4+x_5)(x_4+x_6).$$

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