Some Considerations on the Series Solution of Differential Equations and its Engineering Applications

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Abstract: The ordinary linear differential equations with constant coefficients can be solved by the algebraic methods and the solutions are obtained by elementary functions. In practice, the class of this kind of differential equations is rather narrow. The most of the differential equations met in mathematics, physics and engineering sciences remain out of this class. In such cases, it is searched for solutions in form of infinite series. A new theory of functions named as higher transcendant functions or special functions were set up. The Legendre and Bessel equations are of this type. They appear in problems on vibrations, electric field, heat conduction, fluid flow, etc.

In this paper the authors intend to show by means of four interesting applications from the field of structural and mechanical engineering how still a powerful tool is the method of series solutions, beside the algebraic and numerical methods; how easy can the problems be handled by this method using the computer softwares prepared especially for mathematics. The results are compared with the numerical solutions.

Key words: Differential Equations, series solution, material and strength, engineering applications

INTRODUCTION

The ordinary differential equations whose coefficients are constant can be solved by the algebraic methods and the solutions are obtained by elementary functions. Except that the ordinary linear differential equations and the differential equations which reducible to this type, the other ordinary linear differential equations can not be solved in terms of algebraic and transcendent functions known as elementary functions, like trigonometric functions, trigonometric invert functions, transcendent and logarithmic functions. In practice, the class of the differential equations which can be reduced to the ordinary linear differential equations with constant

coefficients is rather narrow. The most of the differential equations met in mathematics, physics and engineering science remain out of this class. In such cases, it is searched for the solutions in form of infinite series. In this point, beyond the elementary functions, a new theory of functions named as higher transcendent functions or special functions come out. This theory was developed by the mathematicians in 18th and 19th centuries in perfect form. (WYLIE and BARRET (1985), KREYSZIG (1993), AYRES (1952), RICHTER (1968), KOÇAK (1983).

The method of separation of variable for the solution of the partial differential equations often leads to ordinary differential equations

with variable coefficients whose solutions are obtained either in the form of infinite series in term of special functions. The Legendre's and Bessel's equations are of this type. These and other equations and their solutions play an important role in applied mathematics; they appear in problems of vibrations, electric field, heat conduction, fluid flow etc. The Bessel equations are always to be expected when partial differential equations are used in the study of configurations possessing cylindrical symmetry. On the other hand, they arise in numerous applications when neither cylindrical symmetry nor partial differential equations are involved. The Legendre equations appear in problems showing spherical symmetry.

In this paper, the authors, who are self engineers, will try to show by four applications from the fields of structural and mechanical engineering, stability, dynamic, theory of plates and fluid mechanics how still a powerful tool is the method of the series solutions beside the algebraic and the numerical methods and how easy can the problems be handled by this method using the softwares (Mathematica(6), PC-Matlab (7)) prepared particularly for mathematics in the computer medium in our century and the results will be compared with the numerical solutions.

METHOD OF SOLUTION

Lets consider the differential equation given by equation (1).

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{1}$$

If the functions p(x) and q(x) of equation (1) is analytic at $x = x_0$, the every solution is analytic at $x = x_0$ and can be represented by a power series with radius of convergence R>0.

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m \tag{2}$$

The Legendre differential equation is

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \tag{3}$$

whose coefficients are analytic at $x = 0$. Thus the solution by the power series method gives Legendre polynomials of degree n

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n - 2m)!}{2^n\, m!\, (n-m)!\, (n-2m)!} x^{n-2m} \qquad M = \begin{cases} n/2 & ;n \;\; even \\ (n-1)/2 & ;n \;\; odd \end{cases} \tag{4}$$

which satisfy the orthogonality relations with respect to the weight function p(x)=1

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \begin{cases} 0 & ; m \neq n \\ 2/(2n+1) & ; m = n \end{cases} \tag{5}$$

If at least one of the functions p(x) and q(x) is not analytic at x=x_0, but the functions defined by the products (x-x_0)p(x) and (x-x_0)^2 q(x) are analytic at x=x_0, then there is at least one solution which can be represented by a extended power series (Frobenius method) with radius of convergence $0 \langle |x - x_0| \langle R$

$$y(x) = |x - x_0|^r \sum_{m=0}^{\infty} a_m (x - x_0)^m \tag{6}$$

Depending on the roots of the indicial equation

$$r(r-1) + p_0 r + q_0 = 0 \tag{7}$$

there can be three cases in the solution.
The Bessel differential equation of order ν with a parameter λ

$$y'' + \frac{1}{x} y' + (\lambda^2 - \frac{\nu^2}{x^2}) y = 0 \tag{8}$$

whose coefficients are not analytic at x = 0. If the independent variable of this equation is changed by the substitution of the new variable t = λ x, it is obtained

$$y'' + \frac{1}{t} y' + (1 - \frac{\nu^2}{t^2}) y = 0 \tag{9}$$

The solution of this equation by the extended power series method gives the Bessel functions of the first kind of order

$$J_\nu(t) = \sum_{m=0}^{\infty} \frac{(-1)^m t^{\nu+2m}}{2^{\nu+2m}\, m!\, \Gamma(\nu + m + 1)} \tag{10}$$

The Bessel and modified Bessel functions of the other kinds of order are derived by manipulating of these equations.
For each fixed nonnegative integer for ν = n, the Bessel functions satisfy the orthogonality relations with respect to the weight function p(x) = x

$$\int_{0}^{R} x\, J_n(\lambda_i x)\, .\, J_n(\lambda_j x)\, dx = 0 \qquad (i \neq j) \tag{11}$$

Eigenvalue problems can be written in form of the Sturm-Liouville equation with boundary conditions that includes the Legendre and Bessel differential equations. The solution of this equation gives the eigen values and the corresponding eigen functions which have the orthogonality properties. This leads to eigen function expansions.

APPLICATIONS

I - As a first application the buckling load of a simply supported steel column with variable cross-section (Figure1) will be determined.
The moment of inertia of the cross section in column varies according to the rule given as

$$J(x) = J_A (1 + ax)^2 \tag{12}$$

where $\delta = \delta_B / \delta_A$, $a = (\delta - 1)/l$.
The differential equation of this stability problem is

$$EJ_A (1 + ax)^2 \, y'' + P \, y = 0 \tag{13}$$

By changing of a variable by $e^z = 1+ax$, this equation is transformed into

$$y'' - y' + \beta^2 y = 0 \ , \qquad \beta^2 = \frac{P}{a^2 EJ_A} \tag{14}$$

The boundary conditions are:

$$y = 0 \quad \text{at} \quad z = 0 \tag{15}$$

$$y = 0 \quad \text{at} \quad z = \ln (1+al) \tag{16}$$

This homogenous linear differential equation of second order with constant coefficient can also be solved by the series method. If the power series

$$y = \sum_{m=0}^{\infty} a_m z^m \tag{17}$$

If the power series is put into the differential equation, it is derived for the coefficients of the independent variable z the recurrence formula

$$a_{m+2} = \frac{(m+1)a_{m+1} - a_m \beta^2}{(m+1)(m+2)} \ ; \quad m \geq 0 \tag{18}$$

and the solution is

$$y = a_0[1 - \frac{1}{2}\beta^2 z^2 - \frac{1}{6}\beta^2 z^3 + \frac{1}{24}(-\beta^2 + \beta^4)z^4 + \frac{1}{120}(-\beta^2 + 2\beta^4)z^5 + ...] +$$

$$a_1[z + \frac{1}{2}z^2 + \frac{1}{6}(1 - \beta^2)z^3 + \frac{1}{24}(1 - 2\beta^2)z^4 + \frac{1}{120}(1 - 3\beta^2 + \beta^4)z^5 + ...] \tag{19}$$

The condition for the non-trivial solution of the homogeneous equation system which is obtained by applying the boundary conditions to the above equation gives $\beta^2 = 64.46$. With this value, the critical load is calculated as $P_{cr} = 406.1$ kN. The value is % 7 higher than the value obtained by the analytical solution as 379.5 kN.

This problem is solved by the finite element method with a computer program coded by the second author. In this program the stiffness matrix of the column element is derived by accounting the second order effect of the axial forces.

The column is divided 10 finite elements. Beginning with a selected axial force the numerical value of the determinant of system stiffness matrix is calculated. This axial force value is increased giving an appropriate increment and this determinant is calculated at each step. The resulted axial force is the buckling force when the sign of the value of this determinant changed. All the operations are performed by the program. By this way the buckling force of this column is obtained as 382.6 kN. This value is %0.8 higher than the value obtained by the analytical solution and %6 less than the obtained value by the series solution.
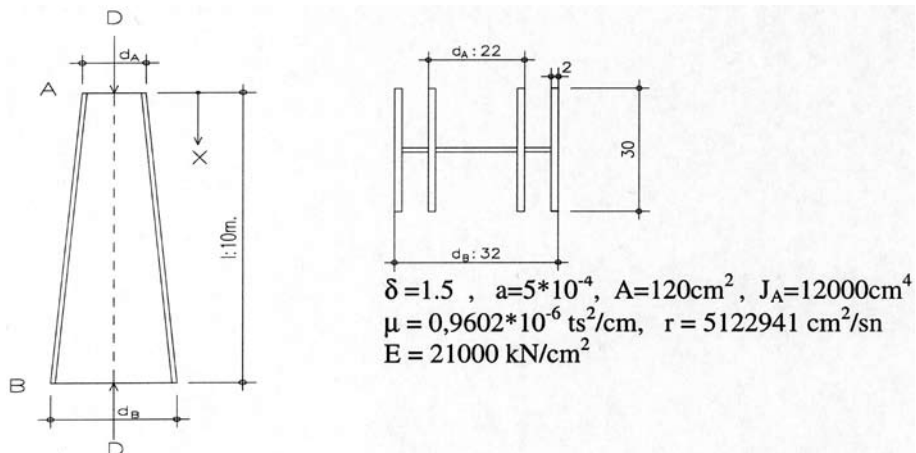

Figure 1. The properties of the material and strength of the first example of the steel structure

II – The eigen frequencies of the simply supported steel column given in (Figure 1.) will be determined for the transverse vibration.

The differential equation of motion [Nowacki (1974)] given as

$$E \frac{\partial^2}{\partial x^2}(J(x)\frac{\partial^2 y}{\partial x^2}) + \mu \frac{\partial^2 y}{\partial x^2} = 0$$

(20)

where μ is mass per unit length ($= A\gamma / g$ =constant). By the method of separation of variables, $y(x,t) = Y(x)\, e^{i\omega t}$, the partial differential equation is transformed into an ordinary differential equation as

$$\frac{d^2}{dx^2}(EJ(x)\frac{d^2 Y}{dx^2}) - \mu\omega^2 Y = 0$$

(21)

which can be written in another form

$$[\frac{1}{\mu}\frac{d}{dx}(\sqrt{\mu EJ(x)}\ \frac{d}{dx}) + \omega\,]\,[\frac{1}{\mu}\frac{d}{dx}(\sqrt{\mu EJ(x)}\ \frac{d}{dx}) - \omega\,]\,y = 0$$

(22)

The solution of this equation is

$$Y(x) = C_1 J_0(z) + C_2 Y_0(z) + C_3 I_0(z) + C_4 K_0(z)$$

(23)

where J_0, Y_0, I_0, K_0: The Bessel and modified Bessel functions of the first and second kinds of order zero, and

$$z = 2\lambda\sqrt{1 + ax} \quad , \quad \lambda^2 = \omega /(ra^2) \quad , \quad r = \sqrt{EJ_A / \mu} = \text{constant}$$

The four boundary conditions are

$$Y(0) = 0 \quad , \quad EJ(x)Y''(0) = 0 \qquad \text{at } x=0$$

(24)

$$Y(l) = 0 \quad , \quad EJ(x)Y''(l) = 0 \qquad \text{at } x=l$$

(25)

If these boundary conditions are applied to the solution, it comes out a homogeneous equation with the unknowns C_i. The eigen frequencies are calculated by using a computer software prepared for mathematics from the condition that the determinant of the coefficients matrix must be zero: $\omega_1 = 61.86\ rad / s, \omega_2 = 249.23\ rad / s,$ etc.

This problem is solved by the finite element method by using the well known structural analysis program package (SAP90)[13]. Dividing the column to ten elements the angular frequencies are obtained as $\omega_1 = 52.64\ rad/s, \omega_2 = 105.90\ rad/s,$ etc.

III– As the third application the deflections and the internal forces of a circular plate embedded on elastic foundation subject to vertical central load will be determined.
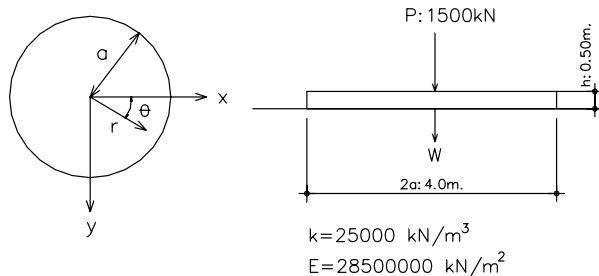

Figure 2. The plate, the coordinate system and the loading

The plate, the coordinate system and the loading are shown in Figure 2. The differential equation of the plate in polar coordinate system (Timoshenko (1964)) is

$$D(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})^2 w(r,\theta) = q(r,\theta) - k\ w(r,\theta)$$

(26)

where D is the rigidity of the plate and expressed with $D = Eh^3/[12(1-\nu^2)]$.
The differential equation will be independent on θ, if the loads act axisymmetric. Thus the equation will be

$$D(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr})(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}) = q - kw$$

(27)

For the case that the plate is loaded in the center by a concentrated load P, q will be zero, except at the origin, and by using the new variables $\lambda^4 = D/k,\ \ z = w/\lambda,\ \ t = r/\lambda$ the differential equation takes the form as

$$(\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt})\,(\frac{d^2z}{dt^2} + \frac{1}{t}\frac{dz}{dt}) + z = 0$$

(28)

or with Nabla or delta operators

$$\Delta\Delta\ z + z = 0 \qquad \text{or} \qquad \nabla^4 z + z = 0$$

(29)

The solution of this equation is

$$z(t) = \overline{C}_1 J_0(\sqrt{i}\ t) + \overline{C}_2 Y_0(\sqrt{i}\ t) + \overline{C}_3 I_0(\sqrt{i}\ t) + \overline{C}_4 K_0(\sqrt{i}\ t) \tag{30}$$

The boundary conditions are

i)
$$M_r = -D(\frac{d^2 w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr})_{r=a} = 0 \tag{31}$$

ii)
$$Q_r = -D\frac{d}{dr}(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr})_{r=a} = 0 \tag{32}$$

iii)   The deflection at the center of the plate remains finite.

iv)
$$\int_0^{2\pi} (Q_r r d\theta)_{r=\varepsilon} + P = 0 \quad or \quad -k\lambda^4 \frac{d}{dr}(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr})_{r=\varepsilon} 2\pi\varepsilon + P = 0 \tag{33}$$

where ε is the radius of an infinite small cylinder in the middle of the plate.
Considering the third boundary condition, the solution is written in Thomson's and Kelvin's

functions which are obtained by inserting the independent imaginer variable $\sqrt{i}\ t$ in the Bessel functions and separating the real and imaginer terms:

$$z(t) = C_1 ber_0 t + C_2 bei_0 t + C_3 kei_0 t \tag{34}$$

where

$$ber_0 t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{4k}}{2^{4k}[(2k)!]^2} \tag{35}$$

$$bei_0 t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{4k+2}}{2^{4k+2}[(2k+1)!]^2} \tag{36}$$

$$kei_0 t = Int.bei_0 t + \sum_{k=1}^{\infty} \frac{(-1)^k t^{4k-2}}{2^{4k-2}[(4k-2)!]^2} \sum_{n=1}^{2k-1} \frac{1}{n} \tag{37}$$

by using the remaining boundary conditions, the unknown coefficients C_1, C_2, C_3 are determined.

The entire operations have been programmed by using a computer software prepared for mathematics[6].

Comparison of the deflections which are calculated by the method given in this paper and the finite element method is given in Figure 3. In the finite element method, due to axial symmetry the plate is considered as one dimensional element in the radial direction. A finite element program written for this purpose is given in[14]. In this way 20 finite elements embedded on elastic foundation is considered.
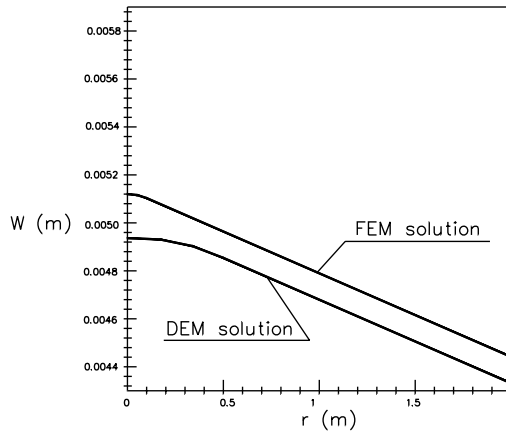
Figure 3. Comparison of the deflections which are obtained by the FEM and the dif.eq. method

IV – A fluid (wind) with a velocity 50 km/h flows uniformly over a spherical dome having a radius 5 m. The upstream pressure and temperature are equal to those inside the dome. The upward force on the dome will be estimated.
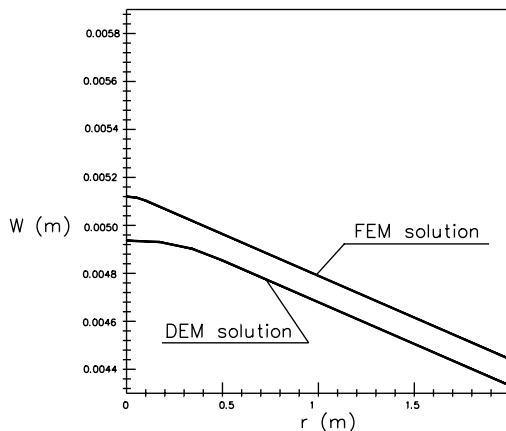
Figure 4. An uniform fluid acting on a spherical dome $(\rho = 1.225 \, kg / cm^3)$

The velocity potential function, Φ, satisfies the Laplace equation. This equation is written in spherical coordinates as[12]

$$\Delta\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial^2\Phi}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Phi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

(38)

The velocity components are:

$$V_r = \frac{\partial\Phi}{\partial r}$$

(39)

$$V_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial\theta}$$

(40)

$$V_\varphi = \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}$$

(41)

The velocity field satisfies the boundary conditions:

$$V_r = 0 \qquad\qquad at \quad r = R$$

(42)

$$V_r = V_\infty\sin\theta\sin\varphi \qquad at \quad r \to R$$

(43)

$$V_\varphi = V_\infty\cos\varphi \qquad\qquad at \quad r \to R$$

(44)

The solution of the equation by the method of separation of variables, $\Phi(r,\theta,\varphi) = P(r)\Psi(\theta)\Omega(\varphi)$, gives three ordinary differential equations:

$$\Omega''(\varphi) + m^2\Omega(\varphi) = 0$$

(45)

$$r^2 P''(r) + 2rP'(r) - n(n+1)P(r) = 0 \text{ (Euler differential equation)}$$

(46)

$$\sin^2\theta\Psi''(\theta) + \sin\theta\cos\theta\Psi'(\theta) + [n(n+1)\sin^2\theta - m^2]\Psi(\theta) = 0$$
(Associated Legendre eq.)

(47)

The solutions of this equation give

$$\Phi(r,\theta,\varphi) = \sum_{n=0}^{\infty}(a_n r^n + \frac{b_n}{r^{n+1}})\sum_{m=0}^{n}P_n^m(\cos\theta)[A_{nm}\cos m\varphi + B_{nm}\sin m\varphi]$$

(48)

where $P_n^m(\cos\theta)$ is Legendre polynomial. By using the boundary conditions, and for m=n=1 the velocity potential is obtained as

$$\Phi = V_\infty (r + \frac{R^3}{2r^2})\sin\theta \, \sin\varphi$$

(49)

If the Bernoulli equation is applied adjacent to the surface of the sphere where $V_r = 0$, it is written

$$\frac{p_\infty}{\rho} + \frac{V_\infty^2}{2} = \frac{p_s}{\rho} + \frac{V_\theta^2}{2}$$

(50)

from which it is obtained

$$p_s - p_\infty = \frac{\rho V_\infty}{8}[4 - 9\cos^2\theta \, \sin^2\theta \,]$$

(51)

The upward force is calculated by integration of the pressure over the surface as

$$L = \int_s (p_s - p_\infty)\vec{n}.\vec{k}ds = \rho\,\frac{V_\infty^2}{8}\int_0^{2\pi}\int_0^{\pi/2}(4 - 9\cos^2\theta\,\sin^2\varphi)\cos\theta R^2\,\sin\theta d\theta d\varphi = \frac{7\pi\rho R^2 V_\infty^2}{32}$$

(52)

where ds is surface element, $\vec{k}$ and $\vec{n}$ are unit vectors in the z axis and perpendicular to the surface. In this calculation it is assumed that the shear force $\tau_w = 0$. For the given numerical values, this force is calculated as 40.6 kN.

RESULTS AND CONCLUSIONS

In this paper four applications from the fields of structural and mechanical engineering, stability, dynamic, theory of plates and fluid mechanics are solved. In all applications, firstly, the differential equations and boundary conditions are given. From this point of view, these equations are considered as Sturm-Liouville equations.

In the first application, the buckling load of a simply supported steel column with vari-

able cross-section are calculated. The Euler differential equation is transformed into the linear differential equation with constant coefficients by changing of variable, and this equation is solved by applying the power series to demonstrate the usage and practicality of the method.

In the second application, the eigen frequencies for the transverse vibration of the column given in the first application are calculated. The partial differential equation of motion is transformed into an ordinary linear dif-

ferential equation of the fourth order whose solution is given by the Bessel functions. The eigen frequencies are calculated from the determinant of the coefficients matrix by using a computer software prepared for mathematics which has generally higher trancendant functions in its files.

In the third application, the deflections and the internal forces of a circular plate embedded on an elastic foundation and subject to a single load in its centre. Because of the circular symmetry of the plate, the partial differential equation of the plate reduces to the ordinary linear differential equation of the fourth order with variable coefficients. The solution of this equation is given under consideration of the boundary conditions in Thomson's and Kelvin's functions. The

whole operations have been performed by using a computer software. The deflections of the plate obtained by the series solution method are compared with the deflections obtained by the finite element method.

The last application belongs to a problem of fluid mechanics. The solution of the Laplace equation of velocity potential function written in spherical coordinates by the method of separation of variables gives three ordinary linear differential equation, Euler's differential equation and Legendre equation; the solutions of these equations give the potential function. The upward force is obtained by applying the Bernoulli equation and then integrating the pressure over the surface of the spherical dome.

APPENDIX

Notations

a_m	Coefficients in the series expansions
$C_i, \overline{C_i}$	Indefinite coefficients in the solution of the differential equations
ds	Surface element
D	Constant or longitudinal load
E	Modulus of elasticity
g	Acceleration of gravity
i	Imaginary unit
J_0, Y_0, I_0, K_0	Bessel, modified Bessel functions of first and second kinds of order zero
J(x)	Variation of the moment of inertia as a function of x
J_A	Moment of inertia at cross-section A
k	Modulus of subgrade reaction, real number, unit vector
m,M,n	Real numbers, unit vector perpendicular to the surface
M_r	Moment
p_s	Pressure of the fluid
p_0, q_0	Coefficients in the series expansions
$q(r, \theta)$	Lateral load
Q_r	Shear force

r   Exponent in the series expansions
R   Radius of convergence
t   Time, independent variable
T   Temperature of the fluid

V_r, V_θ, V_φ Velocities in the radial, longitudinal and latitudinal directions
w   Deflection
Y(x)   Amplitude
x,y,z   Dependent and independent variables
γ   Density of the steel per unit volume
Γ   Gamma function
Δ   Delta operator
∇   Nabla operator
μ   Mass per unit length
ν   Poisson's ratio
ρ   Density of the fluid
ω   Angular frequency

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