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Contents

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Perfect matchings, Hamiltonian cycles and edge-colourings in a class of cubic graphs

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Abstract

A graph G has the Perfect-Matching-Hamiltonian property (PMH-property) if for each one of its perfect matchings, there is another perfect matching of G such that the union of the two perfect matchings yields a Hamiltonian cycle of G. The study of graphs that have the PMH-property, initiated in the 1970s by Las Vergnas and Häggkvist, combines three well-studied properties of graphs, namely matchings, Hamiltonicity and edge-colourings. In this work, we study these concepts for cubic graphs in an attempt to characterise those cubic graphs for which every perfect matching corresponds to one of the colours of a proper 3-edge-colouring of the graph. We discuss that this is equivalent to saying that such graphs are even-2-factorable (E2F), that is, all 2-factors of the graph contain only even cycles. The case for bipartite cubic graphs is trivial, since if G is bipartite then it is E2F. Thus, we restrict our attention to non-bipartite cubic graphs. A sufficient, but not necessary, condition for a cubic graph to be E2F is that it has the PMH-property. The aim of this work

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is to introduce an infinite family of E2F non-bipartite cubic graphs on two parameters, which we coin *papillon graphs*, and determine the values of the respective parameters for which these graphs have the PMH-property or are just E2F. We also show that no two papillon graphs with different parameters are isomorphic.

Keywords: Cubic graph, perfect matching, Hamiltonian cycle, 3-edge-colouring.

Math. Subj. Class. (2020): 05C15, 05C45, 05C70

1 Introduction

Let G be a simple connected graph of even order with vertex set $V(G)$ and edge set $E(G)$. A *k-factor* of G is a k-regular spanning subgraph of G (not necessarily connected). Two very well-studied concepts in graph theory are *perfect matchings* and *Hamiltonian cycles*, where the former is the edge set of a 1-factor and the latter is a connected 2-factor of a graph. For $t \geq 3$, a *cycle* of length t (or a t-cycle), denoted by $C_t = (v_1, \dots, v_t)$, is a sequence of mutually distinct vertices $v_1, v_2, \dots, v_t$ with corresponding edge set $\{v_1v_2, \dots, v_{t-1}v_t, v_tv_1\}$. For definitions not explicitly stated here we refer the reader to [4]. A graph G admitting a perfect matching is said to have the *Perfect-Matching-Hamiltonian property* (for short the *PMH-property*) if for every perfect matching M of G there exists another perfect matching N of G such that the edges of $M \cup N$ induce a Hamiltonian cycle of G. For simplicity, a graph admitting this property is said to be PMH. This property was first studied in the 1970s by Las Vergnas [15] and Häggkvist [9], and for more recent results about the PMH-property we suggest the reader to [2, 1, 3, 7, 8]. In [3], a property stronger than the PMH-property is studied: the Pairing-Hamiltonian property, for short the PH-property. Before proceeding to the definition of this property, we first define what a pairing is. For any graph G, K_G denotes the complete graph on the same vertex set $V(G)$ of G. A perfect matching of K_G is said to be a *pairing* of G, and a graph G is said to have the *Pairing-Hamiltonian property* if every pairing M of G can be extended to a Hamiltonian cycle H of K_G such that $E(H) - M \subseteq E(G)$. Clearly, a graph having the PH-property is also PMH, although the converse is not necessarily true. Amongst other results, the authors of [3] show that the only cubic graphs admitting the PH-property are the complete graph K_4, the complete bipartite graph $K_{3,3}$, and the cube $\mathcal{Q}_3$. However, this does not mean that these are the only three cubic graphs admitting the PMH-property. For instance, all cubic 2-factor Hamiltonian graphs (all 2-factors of such a graph form a Hamiltonian cycle) are PMH (see for example [5, 6, 11, 12, 13]).

If a cubic graph G is PMH, then every perfect matching of G corresponds to one of the colours of a (proper) 3-edge-colouring of the graph, and we say that every perfect matching can be extended to a 3-edge-colouring. This is achieved by alternately colouring the edges of the Hamiltonian cycle containing a predetermined perfect matching using two colours, and then colouring the edges not belonging to the Hamiltonian cycle using a third colour. However, there are cubic graphs which are not PMH but have every one of their perfect matchings that can be extended to a 3-edge-colouring (see for example Figure 1). The following proposition characterises all cubic graphs for which every one of their perfect matchings can be extended to a 3-edge-colouring of the graph.

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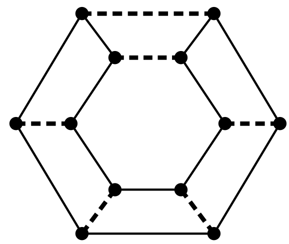

Figure 1: The bold dashed edges can be extended to a proper 3-edge-colouring but not to a Hamiltonian cycle.

Proposition 1.1. *Let G be a cubic graph admitting a perfect matching. Every perfect matching of G can be extended to a* 3*-edge-colouring of G if and only if all* 2*-factors of G contain only even cycles.*

Proof. Let F be a 2-factor of G, and let M be the perfect matching $E(G) - E(F)$. Since M can be extended to a 3-edge-colouring of G, F can be 2-edge-coloured, and hence F does not contain any odd cycles. Conversely, let M' be a perfect matching of G, and let F' be its complementary 2-factor, that is, $E(F') = E(G) - M'$. Since F' contains only even cycles, M' can be extended to a 3-edge-colouring, by assigning a first colour to all of its edges and then alternately colouring the edges of the 2-factor F' using another two colours. □

We shall call graphs in which all 2-factors consist only of even cycles as *even*-2-*factorable* graphs, denoted by E2F for short. In particular, from Proposition 1.1, if a cubic graph G has the PMH-property, then it is also E2F. As in the proof of Proposition 1.1, in the sequel, given a perfect matching M of a cubic graph G, the 2-factor obtained after deleting the edges of M from G is referred to as the *complementary* 2*-factor* of M.

If a cubic graph is bipartite, then trivially, each of its perfect matchings can be extended to a 3-edge-colouring, since it is E2F. But what about non-bipartite cubic graphs? In Table 1, we give the number of non-isomorphic non-bipartite 3-connected cubic graphs (having girth at least 4) such that each one of their perfect matchings can be extended to a 3-edge-colouring. As is the case of *snarks* (bridgeless cubic graphs which are not 3-edge-colourable), these seem to be difficult to find, as one can notice after comparing these numbers to the total number of non-isomorphic 3-edge-colourable (Class I) non-bipartite 3-connected cubic graphs having girth at least 4, also given in Table 1. The numbers shown in this table were obtained thanks to a computer check done by Jan Goedgebeur, and the data is sorted according to the cyclic connectivity of the graphs considered. We remark that E2F cubic graphs having girth 3 can be obtained by applying a star product between an E2F cubic graph of smaller order and the complete graph K_4—this has been investigated further by the last two authors in [14]. This is the reason why only graphs having girth at least 4 are considered in this work. More results on star products (also known in the literature as 3-cut connections) in cubic graphs can be found in [5, 6, 10, 11, 12, 13].

A complete characterisation of which cubic graphs are PMH is still elusive, so considering the Class I non-bipartite cubic graphs having the property that each one of their perfect matchings can be extended to a 3-edge-colouring may look presumptuous. As far as we

Number of vertices	Cyclic connectivity 3	4	5	6	Total no. of graphs E2F	Class I	ratio E2F : Class I
8	/	1	/	/	1	1	100%
10	/	/	/	/	0	3	0%
12	2	5	2	/	9	17	52.94%
14	2	2	2	/	6	92	6.52%
16	35	56	4	/	95	716	13.27%
18	84	21	9	/	114	7343	1.55%
20	926	655	15	2	1598	93946	1.70%
22	2978	331	17	6	3332	1400203	0.24%

Table 1: The number of non-isomorphic non-bipartite 3-connected cubic graphs with girth at least 4 which are E2F and Class I.

know this property and the corresponding characterisation problem were never considered before and tackling the following problem seems a reasonable step to take.

Problem 1.2. Characterise the Class I non-bipartite cubic graphs for which each one of their perfect matchings can be extended to a 3-edge-colouring, that is, are E2F.

We remark that although the PMH-property is an appealing property in its own right, Problem 1.2 continues to justify its study in relation to cubic graphs. Observe that in the family of cubic graphs, whilst snarks are not 3-edge-colourable, even-2-factorable graphs are quite the opposite being "very much 3-edge-colourable", since the latter can be 3-edge-coloured by assigning a colour to one of its perfect matchings, and then alternately colour the edges of the complementary 2-factor.

1.1 Cycle permutation graphs

Consider two disjoint cycles each of length t, referred to as the first and second t-cycles and denoted by $(x_1, \dots, x_t)$ and $(y_1, \dots, y_t)$, respectively. Let σ be a permutation of the symmetric group $\mathcal{S}_t$ on the t symbols $\{1, \dots, t\}$. The *cycle permutation graph corresponding to* σ is the cubic graph obtained by considering the first and second t-cycles in which x_i is adjacent to $y_{\sigma(i)}$, where $\sigma(i)$ is the image of i under the permutation σ.

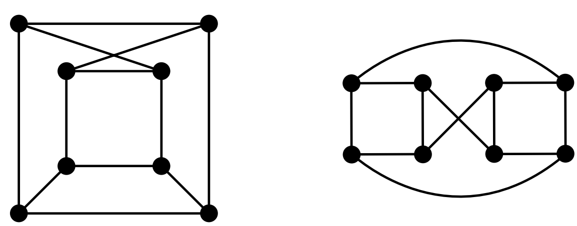

Figure 2: Two different drawings of the smallest non-bipartite E2F cubic graph.

The smallest non-bipartite cubic graph (from Table 1) which is E2F is in fact a cycle permutation graph corresponding to $\sigma = (1\ \ 2) \in \mathcal{S}_4$, where $\sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 3$, and $\sigma(4) = 4$ (see Figure 2). This shows that the edges between the vertices of the first and second 4-cycles of the cycle permutation graph are $x_1y_2, x_2y_1, x_3y_3, x_4y_4$. In what follows we shall denote permutations in cycle notation and, for simplicity, fixed points shall be suppressed. With the help of Wolfram Mathematica, in Table 2 we provide the number of non-isomorphic non-bipartite cycle permutation graphs up to 20 vertices which are PMH or just E2F. Recall that PMH cubic graphs are also E2F, and so, PMH cycle permutation graphs should be searched for from amongst the cycle permutation graphs which are E2F. We also remark that, in the sequel, cycle permutation graphs with total number of vertices equal to twice an odd number are not considered because, in this case, the first and second cycles form a 2-factor consisting of two odd cycles, and so they are trivially not E2F.

No. of vertices	E2F	PMH
8	1	0
12	5	1
16	28	2
20	175	0

Table 2: The number of non-isomorphic non-bipartite cycle permutation graphs with girth at least 4 which are E2F and PMH.

This work is a first structured attempt at tackling Problem 1.2. We give an infinite family of non-bipartite cycle permutation graphs which admit the PMH-property or are just E2F. In Section 2, we generalise the smallest cubic graph which is E2F into a family of non-bipartite cycle permutation graphs, namely papillon graphs $\mathcal{P}_{r,\ell}$ (for $r, \ell \in \mathbb{N}$), whose smallest member $\mathcal{P}_{1,1}$ is, in fact, the graph in Figure 2. We show that papillon graphs are E2F for all values of r and ℓ (Theorem 2.3) and PMH if and only if both r and ℓ are even (Theorem 3.8 and Theorem 3.9).

2 Papillon graphs

Let $[n] = \{1, \ldots, n\}$, for some positive integer n.

Definition 2.1. Let r and ℓ be two positive integers. The *papillon graph* $\mathcal{P}_{r,\ell}$ is the graph on $4r + 4\ell$ vertices such that $V(\mathcal{P}_{r,\ell}) = \{u_i, v_i : i \in [2r + 2\ell]\}$, where:

(i) $(u_1, u_2, \ldots, u_{2r+2\ell})$ is a cycle of length $2r + 2\ell$;

(ii) u_i is adjacent to v_i, for each $i \in [2r + 2\ell]$; and

(iii) the adjacencies between the vertices v_i, for $i \in [2r + 2\ell]$, form a cycle of length $2r + 2\ell$ given by the edge set

$$\{v_{2i-1}v_{2i} : i \in [r+\ell]\} \cup \{v_{2i-1}v_{2i+2} : i \in [r+\ell-1] \setminus \{s\}\} \\ \cup \{v_2v_{2s+2}, v_{2s-1}v_{2r+2\ell-1}\},$$

where $s = \min\{r, \ell\}$.

Clearly, the two papillon graphs $\mathcal{P}_{r,\ell}$ and $\mathcal{P}_{\ell,r}$ are isomorphic, and henceforth, without loss of generality, we shall tacitly assume that $r \leq \ell$. The papillon graph $\mathcal{P}_{r,\ell}$ for $r \geq 2$ is depicted in Figure 3. When r and ℓ are equal, say $r = \ell = n$, the papillon graph $\mathcal{P}_{r,\ell}$ is said to be *balanced*, and simply denoted by $\mathcal{P}_n$ (see, for example, Figure 4). Otherwise, $\mathcal{P}_{r,\ell}$ is said to be *unbalanced* (see, for example, Figure 11). The $(2r+2\ell)$-cycle induced by the sets of vertices $\{u_i : i \in [2r+2\ell]\}$ is referred to as the *outer-cycle*, whilst the $(2r+2\ell)$-cycle induced by the vertices $\{v_i : i \in [2r+2\ell]\}$ is referred to as the *inner-cycle*. The edges on these two $(2r+2\ell)$-cycles are said to be the *outer-edges* and *inner-edges* accordingly, whilst the edges u_iv_i are referred to as *spokes*. The edges $u_1u_{2r+2\ell}$, $v_{2r-1}v_{2r+2\ell-1}$, v_2v_{2r+2}, $u_{2r}u_{2r+1}$ are denoted by a, b, c, d, respectively, and we shall also denote the set $\{a, b, c, d\}$ by $\mathcal{X}$. The set $\mathcal{X}$ is referred to as the *principal* 4*-edge-cut* of $\mathcal{P}_{r,\ell}$.

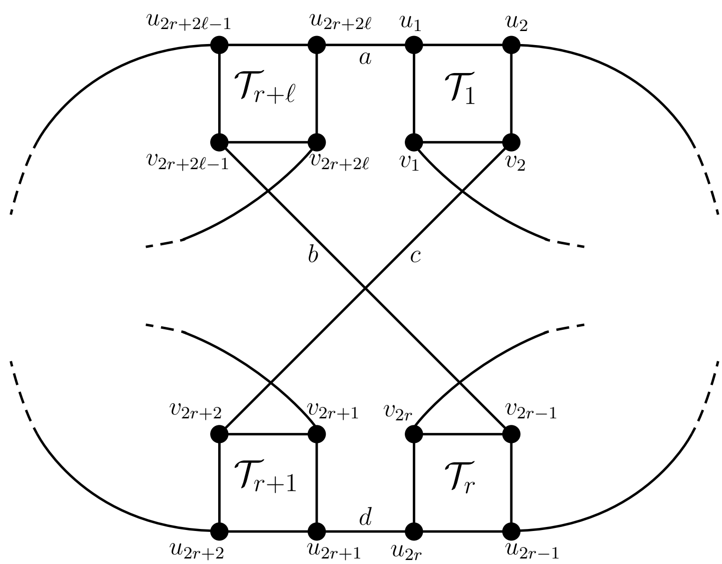

Figure 3: The papillon graph $\mathcal{P}_{r,\ell}$, for $\ell \geq r \geq 2$, and the 4-pole $\mathcal{T}_j$, for $j \in [r+\ell]$.

The graph in Figure 2 is actually the smallest (balanced) papillon graph $\mathcal{P}_1$. In general, since $\{u_i : i \in [2r+2\ell]\}$ and $\{v_i : i \in [2r+2\ell]\}$ induce two disjoint $(2r+2\ell)$-cycles in $\mathcal{P}_{r,\ell}$, and since every vertex belonging to the outer-cycle is adjacent to exactly one vertex on the inner-cycle, there exists an isomorphism π between the papillon graph $\mathcal{P}_{r,\ell}$ and a cycle permutation graph corresponding to some $\sigma \in \mathcal{S}_{2r+2\ell}$ satisfying $\pi(x_i) = u_i$ and $\pi(y_i) = v_{\sigma^{-1}(i)}$, for each $i \in [2r+2\ell]$. In fact, the papillon graph $\mathcal{P}_{r,\ell}$ is the cycle permutation graph, with $(u_1, \ldots, u_{2r+2\ell})$ as the first cycle, corresponding to the permutation:

- $\sigma_{1,\ell} := (3\ \ 4)\ldots(2\ell+1\ \ 2\ell+2)$, with fixed points 1 and 2, when $\ell \geq 1$;
- $\sigma_{2,2} := (1\ \ 2)(3\ \ 4)(5\ \ 7)(6\ \ 8)$;
- $\sigma_{r,3} := (1\ \ 2)\ldots(2r-1\ \ 2r)(2r+1\ \ 2r+5)(2r+2\ \ 2r+6)$, with fixed points $2r+3$ and $2r+4$, when $r \in \{2, 3\}$; and

- $\sigma_{r,\ell} := (1\ 2)\ldots(2r-1\ 2r)(2r+1\ 2r+2\ell-1)(2r+2\ 2r+2\ell)(2r+3\ 2r+2\ell-3)(2r+4\ 2r+2\ell-2)\ldots(\alpha\ \beta)$, when $\ell \geq r \geq 4$, where $(\alpha\ \beta) = (2r+\ell\ 2r+\ell+2)$ if ℓ is even, and $(\alpha\ \beta) = (2r+\ell-1\ 2r+\ell+3)$ if ℓ is odd.

We remark that when $r > 1$, the above permutations has no fixed points when ℓ is even, but, when ℓ is odd, $2r+\ell$ and $2r+\ell+1$ are fixed points, and thus, in this case, $x_{2r+\ell}$ is adjacent to $y_{2r+\ell}$, and $x_{2r+\ell+1}$ is adjacent to $y_{2r+\ell+1}$ in $\mathcal{P}_{r,\ell}$. Note that since $\sigma_{r,\ell}$ is an involution for all positive integers r and ℓ, the isomorphism π mentioned above can be rewritten as follows: $\pi(x_i) = u_i$ and $\pi(y_i) = v_{\sigma(i)}$, for each $i \in [2r+2\ell]$. The papillon graph $\mathcal{P}_{r,\ell}$ admits a natural automorphism ψ which exchanges the two cycles, given by $\psi(u_i) = v_{\sigma_{r,\ell}(i)}$ and $\psi(v_i) = u_{\sigma_{r,\ell}(i)}$, for each $i \in [2r+2\ell]$. In fact, the function ψ is clearly bijective. Moreover, it maps edges of the outer-cycle to edges of the inner-cycle (and vice-versa), and maps spokes to spokes, since the edges u_iv_i are mapped to $u_{\sigma_{r,\ell}(i)}v_{\sigma_{r,\ell}(i)}$.

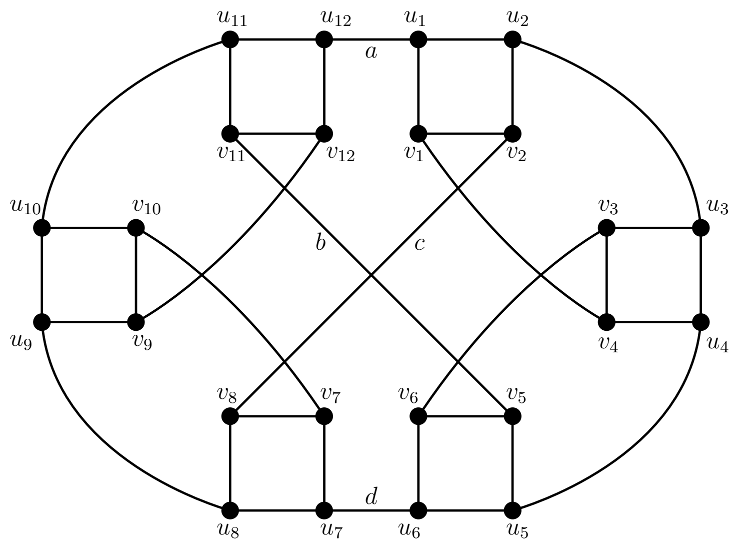

Figure 4: The balanced papillon graph $\mathcal{P}_3$ on 24 vertices.

Before proceeding, we introduce multipoles which generalise the notion of graphs. This will become useful when describing papillon graphs. A *multipole* $\mathcal{Z}$ consists of a set of vertices $V(\mathcal{Z})$ and a set of generalised edges such that each generalised edge is either an edge in the usual sense (that is, it has two endvertices) or a semiedge. A *semiedge* is a generalised edge having exactly one endvertex. The set of semiedges of $\mathcal{Z}$ is denoted by $\partial\mathcal{Z}$ whilst the set of edges of $\mathcal{Z}$ having two endvertices is denoted by $E(\mathcal{Z})$. Two semiedges are *joined* if they are both deleted and their endvertices are made adjacent. A *k-pole* is a multipole with k semiedges. A perfect matching M of a k-pole $\mathcal{Z}$ is a subset of generalised edges of $\mathcal{Z}$ such that every vertex of $\mathcal{Z}$ is incident with exactly one generalised edge of M. In what follows, we shall construct papillon graphs by joining together semiedges of a number of multipoles. In this sense, given a perfect matching M of a graph G, and a multipole $\mathcal{Z}$ used as a building block to construct G, we shall say that M contains a semiedge e of the multipole $\mathcal{Z}$, if M contains the edge in G obtained by joining e to another semiedge in the process of constructing G.

The 4-pole $\mathcal{Z}$ with vertex set $\{z_1, z_2, z_3, z_4\}$, such that $E(\mathcal{Z})$ induces the 4-cycle (z_1, z_2, z_3, z_4) and with exactly one semiedge incident to each of its vertices is referred to as a *C_4-pole* (see Figure 5). For each $i \in [4]$, let the semiedge incident to z_i be denoted by f_i. The semiedges f_1 and f_2 are referred to as the *upper left semiedge* and the *upper right semiedge* of $\mathcal{Z}$, respectively. On the other hand, the semiedges f_3 and f_4 are referred to as the *lower left semiedge* and the *lower right semiedge* of $\mathcal{Z}$, respectively (see Figure 5).

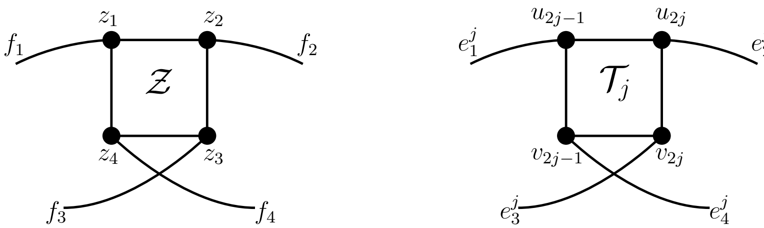

Figure 5: A C_4-pole $\mathcal{Z}$ and the 4-pole $\mathcal{T}_j$ in $\mathcal{P}_{r,\ell}$.

For some integer $n \geq 1$, let $\mathcal{Z}_1, \ldots, \mathcal{Z}_n$ be n copies of the above C_4-pole $\mathcal{Z}$. For each $j \in [n]$, let $V(\mathcal{Z}_j) = \{z_1^j, z_2^j, z_3^j, z_4^j\}$, and let $f_1^j, f_2^j, f_3^j, f_4^j$ be the semiedges of $\mathcal{Z}_j$ respectively incident to $z_1^j, z_2^j, z_3^j, z_4^j$ such that f_1^j and f_2^j are the upper left and upper right semiedges of $\mathcal{Z}_j$, whilst f_3^j and f_4^j are the lower left and lower right semiedges of $\mathcal{Z}_j$. A *chain of C_4-poles* of length $n \geq 2$, is the 4-pole obtained by respectively joining f_2^j and f_4^j (upper and lower right semiedges of $\mathcal{Z}_j$) to f_1^{j+1} and f_3^{j+1} (upper and lower left semiedges of $\mathcal{Z}_{j+1}$), for every $j \in [n-1]$. When $n = 1$, a chain of C_4-poles of length 1 is just a C_4-pole. For simplicity, we shall refer to a chain of C_4-poles of length n, as a *n-chain of C_4-poles*, or simply a *n-chain*. The semiedges f_1^1 and f_3^1 (similarly, f_2^n and f_4^n) are referred to as the upper left and lower left (respectively, upper right and lower right) semiedges of the n-chain. A chain of C_4-poles of any length has exactly four semiedges. For simplicity, when we say that e_1, e_2, e_3, e_4 are the four semiedges of a chain $\mathcal{Z}'$ of C_4-poles (possibly of length 1), we mean that e_1 and e_2 are respectively the upper left and upper right semiedges of $\mathcal{Z}'$, whilst e_3 and e_4 are respectively the lower left and lower right semiedges of the same chain $\mathcal{Z}'$ (see Figure 6). The semiedges e_1 and e_2 (similarly, e_3 and e_4) are referred to collectively as the *upper semiedges* (respectively, *lower semiedges*) of $\mathcal{Z}'$. In a similar way, the semiedges e_1 and e_3 (similarly, e_2 and e_4) are referred to collectively as the *left semiedges* (respectively, *right semiedges*) of $\mathcal{Z}'$.

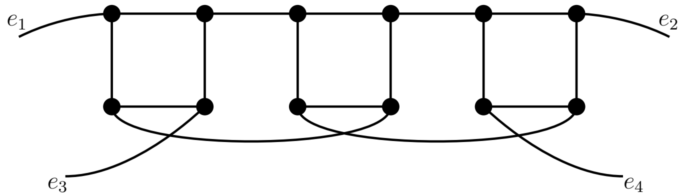

Figure 6: A chain of C_4-poles of length 3 having semiedges e_1, e_2, e_3, e_4.

In order to construct the papillon graph $\mathcal{P}_{r,\ell}$ using C_4-poles as building blocks, for each $j \in [r+\ell]$, we consider the 4-pole $\mathcal{T}_j$ arising from the cycle $(u_{2j-1}, u_{2j}, v_{2j}, v_{2j-1})$ of $\mathcal{P}_{r,\ell}$, whose semiedges are $e_1^j, e_2^j, e_3^j, e_4^j$ as in Figure 5. The r-chain and ℓ-chain giving rise to $\mathcal{P}_{r,\ell}$ consist of $\mathcal{T}_1, \ldots, \mathcal{T}_r$ (referred to as the *right r-chain* of $\mathcal{P}_{r,\ell}$), and $\mathcal{T}_{r+1}, \ldots, \mathcal{T}_{r+\ell}$ (referred to as the *left ℓ-chain* of $\mathcal{P}_{r,\ell}$), which have semiedges $e_1^1, e_2^r, e_3^1, e_4^r$, and $e_1^{r+1}, e_2^{r+\ell}$, $e_3^{r+1}, e_4^{r+\ell}$, respectively. The papillon graph $\mathcal{P}_{r,\ell}$ is then obtained by joining the semiedges in pairs as follows: e_1^1 to $e_2^{r+\ell}$, e_2^r to e_1^{r+1}, e_3^1 to e_3^{r+1}, and e_4^r to $e_4^{r+\ell}$.

2.1 Odd cycles and isomorphisms in the class of papillon graphs

In this section we shall discuss the presence and behaviour of odd cycles in papillon graphs. Consider the balanced papillon graph $\mathcal{P}_n$ and let C be an odd cycle in $\mathcal{P}_n$. Since cycles intersect C_4-poles in 2, 3 or 4 vertices, there must exist some $t_1 \in [2n]$, such that $|V(C) \cap V(\mathcal{T}_{t_1})| = 3$. Without loss of generality, assume that $t_1 \in [n]$, that is, $\mathcal{T}_{t_1}$ belongs to the right n-chain of $\mathcal{P}_n$. If $t_1 \notin \{1, n\}$, we must have exactly one of the following:

- $|V(C) \cap V(\mathcal{T}_i)| = 4$, for all $i \in \{1, \ldots, t_1 - 1\}$; or
- $|V(C) \cap V(\mathcal{T}_i)| = 4$, for all $i \in \{t_1 + 1, \ldots, n\}$.

Without loss of generality, assume that we either have $t_1 = 1$, or $|V(C) \cap V(\mathcal{T}_i)| = 4$, for all $i \in \{1, \ldots, t_1 - 1\}$. This implies that the number of vertices in C belonging to $\cup_{i=1}^{t_1} V(\mathcal{T}_i)$ is odd and at least 3. Moreover, the edges a and c must belong to C. We claim that $b \notin E(C)$. For, suppose that $b \in E(C)$. Since $\mathcal{X}$ is a 4-edge-cut, $d \in E(C)$ as well. This implies that $n > 1$ and there exist:

- $t_2 \in \{t_1 + 1, \ldots, n\}$, such that $|V(C) \cap V(\mathcal{T}_{t_2})| = 3$;
- $s_1 \in \{n + 1, \ldots, 2n - 1\}$, such that $|V(C) \cap V(\mathcal{T}_{s_1})| = 3$; and
- $s_2 \in \{s_1 + 1, \ldots, 2n\}$, such that $|V(C) \cap V(\mathcal{T}_{s_2})| = 3$.

Let $\Omega = \{1, \ldots, t_1\} \cup \{t_2, \ldots, n, n+1, \ldots, s_1\} \cup \{s_2, \ldots, 2n\}$. If $\Omega \setminus \{t_1, t_2, s_1, s_2\} \neq \emptyset$, then for any $j \in \Omega \setminus \{t_1, t_2, s_1, s_2\}$, $|V(C) \cap V(\mathcal{T}_j)| = 4$. Additionally, for any $k \in [2n] \setminus \Omega$, $|V(C) \cap V(\mathcal{T}_k)| = 0$. However, this means that C has even length, a contradiction. Thus, $\{b, d\} \cap E(C) = \emptyset$. As a result, C intersects none of the C_4-poles $\mathcal{T}_{t_1+1}, \ldots, \mathcal{T}_n$, but intersects each of the C_4-poles $\mathcal{T}_{n+1}, \ldots, \mathcal{T}_n$ in exactly 2 or 4 vertices. Hence, the length of C is at least $2n + 3$. When $n = 1$, $(u_1, u_2, v_2, v_4, u_4)$ is a 5-cycle, and when $n > 1$, $(u_1, u_2, v_2, v_{2n+2}, u_{2n+2}, u_{2n+3}, u_{2n+4}, \ldots, u_{4n})$ is an odd cycle of length exactly $2n+3$. Therefore, a shortest odd cycle in $\mathcal{P}_n$ has length $2n + 3$. By using similar arguments, a shortest odd cycle in $\mathcal{P}_{r,\ell}$ has length $2r + 3$.

Remark 2.2. The papillon graph $\mathcal{P}_{r,\ell}$ is not bipartite and has a shortest odd cycle of length $2r + 3$.

Consequently, we can show that any two distinct papillon graphs $\mathcal{P}_{r_1,\ell_1}$ and $\mathcal{P}_{r_2,\ell_2}$ are not isomorphic, where by distinct we mean that $(r_1, \ell_1) \neq (r_2, \ell_2)$. Suppose not, for contradiction. Since $\mathcal{P}_{r_1,\ell_1} \simeq \mathcal{P}_{r_2,\ell_2}$, we must have $r_1 + \ell_1 = r_2 + \ell_2$, and so if $r_1 = r_2$, then this implies that $\ell_1 = \ell_2$, and conversely. Hence, $r_1 \neq r_2$ and $\ell_1 \neq \ell_2$. Thus, without loss of generality, we can assume that $r_1 < r_2$. However, this means that a shortest odd cycle in $\mathcal{P}_{r_1,\ell_1}$ (of length $2r_1 + 3$), is shorter than a shortest odd cycle in $\mathcal{P}_{r_2,\ell_2}$ (of length $2r_2 + 3$), a contradiction.

We are now in a position to give our first result.

Theorem 2.3. *Every papillon graph $\mathcal{P}_{r,\ell}$ is E2F.*

Proof. Let $\mathcal{P}_{r,\ell}$ be a counterexample to the above statement, and let M be a perfect matching of $\mathcal{P}_{r,\ell}$ whose complementary 2-factor contains an odd cycle C. As previously discussed, C must intersect some $\mathcal{T}_j$, for some $j \in [r+\ell]$, in exactly 3 (consecutive) vertices. Without loss of generality, assume that these 3 vertices are u_{2j-1}, u_{2j}, v_{2j}. This means that both the left semiedges (e_1^j and e_3^j) of $\mathcal{T}_j$ belong to this odd cycle. However, since C is in the complementary 2-factor of M, the two edges $u_{2j-1}v_{2j-1}$ and $v_{2j-1}v_{2j}$ (which do not belong to $E(C)$) must both belong to M, a contradiction. □

3 The PMH-property in papillon graphs

3.1 The balanced case $r = \ell$

Let M be a perfect matching of the balanced papillon graph $\mathcal{P}_n$. Since $\mathcal{X} = \{a, b, c, d\}$ is a 4-edge-cut of $\mathcal{P}_n$, $|M \cap \mathcal{X}| \equiv 0 \pmod 2$, that is, $|M \cap \mathcal{X}|$ is $0, 2$ or 4. The following is a useful lemma which shall be used frequently in the results that follow.

Lemma 3.1. *Let M be a perfect matching of the balanced papillon graph $\mathcal{P}_n$ and let $\mathcal{X}$ be its principal 4-edge-cut. If $|M \cap \mathcal{X}| = k$, then $|M \cap \partial\mathcal{T}_j| = k$, for each $j \in [2n]$.*

Proof. Let M be a perfect matching of $\mathcal{P}_n$. We first note that the left semiedges of a C_4-pole are contained in a perfect matching if and only if the right semiedges of the C_4-pole are contained in the same perfect matching. The lemma is proved by considering three cases depending on the possible values of k, that is, $0, 2$ or 4. When $n = 1$, the result clearly follows since $\mathcal{X}$ is made up by joining $\partial\mathcal{T}_1$ and $\partial\mathcal{T}_2$ accordingly. So assume $n \geq 2$.

Case I: $k = 0$.
Since a and c do not belong to M, the left semiedges of $\mathcal{T}_1$ are not contained in M, and so M cannot contain its right semiedges. Therefore, $|M \cap \partial\mathcal{T}_1| = 0$. Consequently, the left semiedges of $\mathcal{T}_2$ are not contained in M implying again that $|M \cap \partial\mathcal{T}_2| = 0$. By repeating the same argument up till the n^{th} C_4-pole, we have that $|M \cap \partial\mathcal{T}_j| = 0$, for every $j \in [n]$. By noting that c and d do not belong to M and repeating a similar argument to the 4-poles in the left n-chain, we can deduce that $|M \cap \partial\mathcal{T}_j| = 0$ for every $j \in [2n]$.

Case II: $k = 4$.
Since a and c belong to M, the left semiedges of $\mathcal{T}_1$ are contained in M, and so M contains its right semiedges as well. Therefore, $|M \cap \partial\mathcal{T}_1| = 4$. Consequently, the left semiedges of $\mathcal{T}_2$ are contained in M implying again that $|M \cap \partial\mathcal{T}_2| = 4$. As in Case I, by noting that both c and d belong to M and repeating a similar argument to the 4-poles in the left n-chain, we can deduce that $|M \cap \partial\mathcal{T}_j| = 4$ for every $j \in [2n]$.

Case III: $k = 2$.
We first claim that when $k = 2$, $M \cap \mathcal{X}$ must be equal to $\{a, d\}$ or $\{b, c\}$. For, suppose that $M \cap \mathcal{X} = \{a, c\}$, without loss of generality. This means that the right semiedges of $\mathcal{T}_1$ are also contained in M, implying that $|M \cap \partial\mathcal{T}_1| = 4$. This implies that the left semiedges of $\mathcal{T}_2$ are contained in M, which forces $|M \cap \partial\mathcal{T}_j|$ to be equal to 4, for every $j \in [2n]$. In particular, $|M \cap \partial\mathcal{T}_n| = 4$, implying that the edges b and d belong to M, a contradiction since $M \cap \mathcal{X} = \{a, c\}$. This proves our claim. Since the natural automorphism ψ of $\mathcal{P}_n$,

which exchanges the outer- and inner-cycles, exchanges also $\{a, d\}$ with $\{b, c\}$, without loss of generality, we may assume that $M \cap \mathcal{X} = \{a, d\}$. Since $c \notin M$, $1 \leq |M \cap \partial \mathcal{T}_1| < 4$. But, $\partial \mathcal{T}_1$ corresponds to a 4-edge-cut in $\mathcal{P}_n$, and so, by using a parity argument, $|M \cap \partial \mathcal{T}_1|$ must be equal to 2, implying that exactly one of the right semiedges of $\mathcal{T}_1$ is contained in M. This means that exactly one left semiedge of $\mathcal{T}_2$ is contained in M, and consequently, by a similar argument now applied to $\mathcal{T}_2$, we obtain $|M \cap \partial \mathcal{T}_2| = 2$. By repeating the same argument and noting that $\mathcal{T}_{n+1}$ has exactly one left semiedge (corresponding to the edge d) contained in M, one can deduce that $|M \cap \partial \mathcal{T}_j| = 2$ for every $j \in [2n]$. □

The following two results are two consequences of the above lemma and they both follow directly from the proof of Case III. In a few words, if a perfect matching M of $\mathcal{P}_n$ intersects its principal 4-edge-cut in exactly two of its edges, then these two edges are either the pair $\{a, d\}$ or the pair $\{b, c\}$, and, for every $j \in [2n]$, M contains only one pair of semiedges of $\mathcal{T}_j$ which does not consist of the pair of left semiedges of $\mathcal{T}_j$ nor the pair of right semiedges of $\mathcal{T}_j$.

Corollary 3.2. *Let M be a perfect matching of $\mathcal{P}_n$ and let $\mathcal{X}$ be its principal 4-edge-cut. If $|M \cap \mathcal{X}| = 2$, then $M \cap \mathcal{X}$ is equal to $\{a, d\}$ or $\{b, c\}$.*

Corollary 3.3. *Let M be a perfect matching of $\mathcal{P}_n$ and let $\mathcal{X}$ be its principal 4-edge-cut such that $|M \cap \mathcal{X}| = 2$. For each $j \in [2n]$, M contains exactly one of the following sets of semiedges: $\{e_1^j, e_2^j\}, \{e_3^j, e_4^j\}, \{e_1^j, e_4^j\}, \{e_2^j, e_3^j\}$, that is, of all possible pairs of semiedges of $\mathcal{T}_j$, $\{e_1^j, e_3^j\}$ and $\{e_2^j, e_4^j\}$ cannot be contained in M.*

In the sequel, the process of traversing one path after another shall be called *concatenation of paths*. If two paths P and Q have endvertices x, y and y, z, respectively, we write PQ to denote the path starting at x and ending at z obtained by traversing P and then Q. Without loss of generality, if x is adjacent to y, that is, P is a path on two vertices, we may write xyQ instead of PQ.

Lemma 3.4. *Let M_1 be a perfect matching of $\mathcal{P}_n$ such that $|M_1 \cap \mathcal{X}| = 2$.*

(i) *There exists a perfect matching M_2 of $\mathcal{P}_n$ such that $|M_2 \cap \mathcal{X}| = 2$ and $M_1 \cap M_2 = \emptyset$.*

(ii) *The complementary 2-factors of M_1 and M_2 are both Hamiltonian cycles.*

Proof. (i) Since $|M_1 \cap \mathcal{X}| = 2$, by Lemma 3.1 we get that $|M_1 \cap \partial \mathcal{T}_j| = 2$ for every $j \in [2n]$. For each j, let $P^{(j)}$ be the subgraph of $\mathcal{P}_n$ which is induced by $E(\mathcal{T}_j) - M_1$. Note that $\cup_{j=1}^{2n} V(P^{(j)}) = V(\mathcal{P}_n)$. By Corollary 3.3, each $P^{(j)}$ is a path of length 3. Letting N be the unique perfect matching of $\mathcal{P}_n$ which intersects each $E(P^{(j)})$ in exactly two edges, we note that $M_1 \cap N = \emptyset$. Let $M_2 = E(\mathcal{P}_n) - (M_1 \cup N)$. Since M_1 and N are two disjoint perfect matchings, M_2 is also a perfect matching of $\mathcal{P}_n$ and, in particular, M_2 contains $\mathcal{X} - (M_1 \cap \mathcal{X})$. Thus, $|M_2 \cap \mathcal{X}| = 2$ and $M_1 \cap M_2 = \emptyset$, proving part (i).

(ii) Let M_2 be as in part (i), that is, $|M_2 \cap \mathcal{X}| = 2$ and $M_1 \cap M_2 = \emptyset$. When $n = 1$, the result clearly follows. So assume $n \geq 2$. For distinct i and j in $[2n]$, let $Q^{(i,j)}$ be the subgraph of $\mathcal{P}_n$ which is induced by $M_2 \cap \{xy \in E(\mathcal{P}_n) : x \in V(\mathcal{T}_i), y \in V(\mathcal{T}_j)\}$, that is, $E(Q^{(i,j)})$ is either empty or consists of exactly one edge, that is, $Q^{(i,j)}$ is a path of length 1. When $M_1 \cap \mathcal{X} = \{a, d\}$, we can form a Hamiltonian cycle of $\mathcal{P}_n$ (not containing M_1) by considering the following concatenation of paths:

$$P^{(1)}Q^{(1,2)} \dots Q^{(n-1,n)}P^{(n)}Q^{(n,2n)}P^{(2n)}Q^{(2n,2n-1)} \dots P^{(n+1)}Q^{(n+1,1)},$$

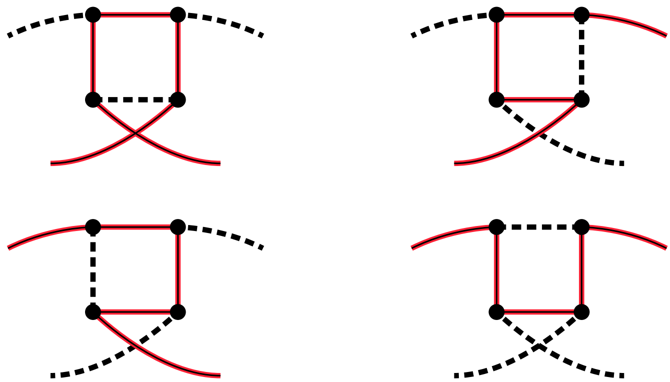

Figure 7: Perfect matching M_1 (bold dashed edges) with $|M_1 \cap \mathcal{X}| = 2$ and its complementary 2-factor (highlighted edges).

where $Q^{(1,2)}$ and $Q^{(2n,2n-1)}$ are respectively followed by $P^{(2)}$ and $P^{(2n-1)}$, and, $Q^{(n,2n)}$ and $Q^{(n+1,1)}$ consist of the edges b and c, respectively. On the other hand, when $M_1 \cap \mathcal{X} = \{b, c\}$, we can form a Hamiltonian cycle of $\mathcal{P}_n$ (not containing M_1) by considering the following concatenation of paths:

$$P^{(1)}Q^{(1,2)} \dots Q^{(n-1,n)}P^{(n)}Q^{(n,n+1)}P^{(n+1)}Q^{(n+1,n+2)} \dots P^{(2n)}Q^{(2n,1)},$$

where $Q^{(1,2)}$ and $Q^{(n+1,n+2)}$ are respectively followed by $P^{(2)}$ and $P^{(n+2)}$, and, $Q^{(n,n+1)}$ and $Q^{(2n,1)}$ consist of the edges d and a, respectively. Thus, the complementary 2-factor of M_1 is a Hamiltonian cycle. This is depicted in Figure 7. The proof that the complementary 2-factor of M_2 is a Hamiltonian cycle follows analogously. □

Proposition 3.5. *Let n be a positive odd integer. Then, the balanced papillon graph $\mathcal{P}_n$ is not PMH.*

Proof. Consider the following perfect matching of the balanced papillon graph $\mathcal{P}_n$:

$$M = \cup_{i=1}^{2n} \{u_{2i-1}u_{2i}, v_{2i-1}v_{2i}\}.$$

It is clear that when $n = 1$, the perfect matching M cannot be extended to a Hamiltonian cycle of the balanced papillon graph $\mathcal{P}_1$. So assume that $n \geq 3$. We claim that M cannot be extended to a Hamiltonian cycle of $\mathcal{P}_n$. For, let F be a 2-factor of $\mathcal{P}_n$ containing M. Since $u_1u_2 \in M$ and $\mathcal{P}_n$ is cubic, F contains exactly one of the following two edges: u_1u_{4n} or u_1v_1. In the former case, if $u_1u_{4n} \in E(F)$, then, $u_{2n}u_{2n+1}$ and all the edges of the outer- and inner-cycle will belong to F (at the same time, the choice of u_1u_{4n} forbids all the spokes of $\mathcal{P}_n$ to belong to F), yielding two disjoint cycles each of length $4n$. In the latter case, if $u_1v_1 \in E(F)$, then F must also contain all spokes u_iv_i, for $1 < i \leq 4n$. In fact, the subgraph induced by the set of spokes is exactly the complement of the 2-factor obtained in the former case. Consequently, F will consist of $2n$ disjoint 4-cycles. □

Consider $\mathcal{P}_n$, with $n \geq 2$, and let M be a perfect matching of $\mathcal{P}_n$ with $M \cap \mathcal{X} = 0$, which by Lemma 3.1 implies that $|M \cap \partial\mathcal{T}_j| = 0$ for all $j \in [2n]$. Now consider $j \in [2n] \setminus \{n, 2n\}$ and let $\mathcal{T}_{(j,j+1)}$ denote a 2-chain composed of $\mathcal{T}_j$ and $\mathcal{T}_{j+1}$. We say that $\mathcal{T}_{(j,j+1)}$ is *symmetric with respect to M* if exactly one of the following occurs:

(i) $\{u_{2j-1}v_{2j-1}, u_{2j}v_{2j}, u_{2j+1}v_{2j+1}, u_{2j+2}v_{2j+2}\} \subset M$; or

(ii) $\{u_{2j-1}u_{2j}, v_{2j-1}v_{2j}, u_{2j+1}u_{2j+2}, v_{2j+1}v_{2j+2}\} \subset M$.

If neither (i) nor (ii) occur, $\mathcal{T}_{(j,j+1)}$ is said to be *asymmetric with respect to* M. This is shown in Figure 8.

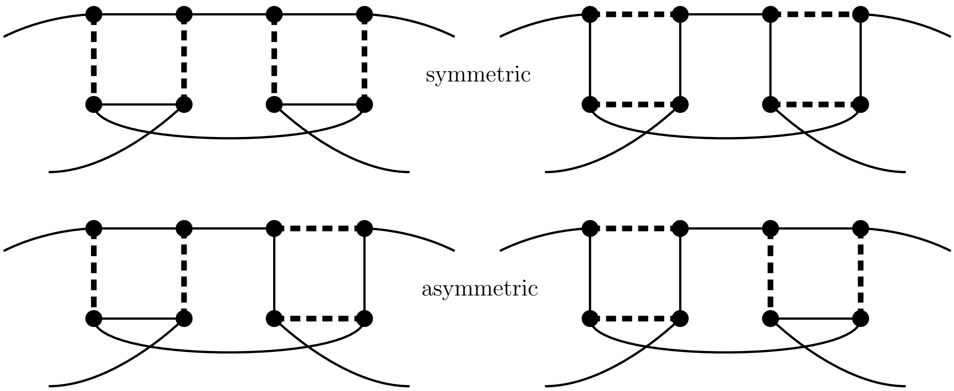

Figure 8: Symmetric and asymmetric 2-chains with the bold dashed edges belonging to M.

Remark 3.6. Let $n \geq 2$. Consider a perfect matching M_1 of $\mathcal{P}_n$ such that M_1 does not intersect the principal 4-edge-cut $\mathcal{X}$ of $\mathcal{P}_n$, that is, $M_1 \cap \mathcal{X} = \emptyset$, and consider a 2-chain of $\mathcal{P}_n$, say $\mathcal{T}_{(j,j+1)}$ with $j \in [2n] \setminus \{n, 2n\}$, having semiedges e_1, e_2, e_3, e_4, where $e_1 = e_1^j, e_2 = e_2^{j+1}, e_3 = e_3^j$ and $e_4 = e_4^{j+1}$. Assume there exists a perfect matching M_2 of $\mathcal{P}_n$ such that $|M_2 \cap \mathcal{X}| = 2$ and $M_1 \cap M_2 = \emptyset$ (see Figure 9). If $\mathcal{T}_{(j,j+1)}$ is symmetric with respect to M_1, then we have exactly one of the following instances:

$$M_2 \cap \partial\mathcal{T}_{(j,j+1)} = \{e_1, e_2\} \text{ (upper);} \quad \text{or} \quad M_2 \cap \partial\mathcal{T}_{(j,j+1)} = \{e_3, e_4\} \text{ (lower).}$$

Otherwise, if $\mathcal{T}_{(j,j+1)}$ is asymmetric with respect to M_1, then exactly one of the following must occur:

$$M_2 \cap \partial\mathcal{T}_{(j,j+1)} = \{e_1, e_4\} \text{ (upper left, lower right); or}$$
$$M_2 \cap \partial\mathcal{T}_{(j,j+1)} = \{e_2, e_3\} \text{ (upper right, lower left).}$$

Notwithstanding whether $\mathcal{T}_{(j,j+1)}$ is symmetric or asymmetric with respect to M_1, $(M_1 \cup M_2) \cap E(\mathcal{T}_{(j,j+1)})$ induces a path (see Figure 9) which contains all the vertices of $V(\mathcal{T}_{(j,j+1)})$, and whose endvertices are the endvertices of the semiedges in $M_2 \cap \partial\mathcal{T}_{(j,j+1)}$.

Remark 3.7. Let $n \geq 2$. Consider a perfect matching M_1 of $\mathcal{P}_n$ such that M_1 does not intersect the principal 4-edge-cut $\mathcal{X}$ of $\mathcal{P}_n$, that is, $M_1 \cap \mathcal{X} = \emptyset$, and consider a 2-chain of $\mathcal{P}_n$, say $\mathcal{T}_{(j,j+1)}$ with $j \in [2n] \setminus \{n, 2n\}$. Let M_2 be the perfect matching of $\mathcal{P}_n$ such that $|M_2 \cap \mathcal{X}| = 4$. Clearly $M_1 \cap M_2 = \emptyset$. Notwithstanding whether $\mathcal{T}_{(j,j+1)}$ is symmetric or asymmetric with respect to M_1, we have that $(M_1 \cup M_2) \cap E(\mathcal{T}_{(j,j+1)})$ induces two disjoint paths of equal length (see Figure 10) whose union contains all the vertices of $\mathcal{T}_j$

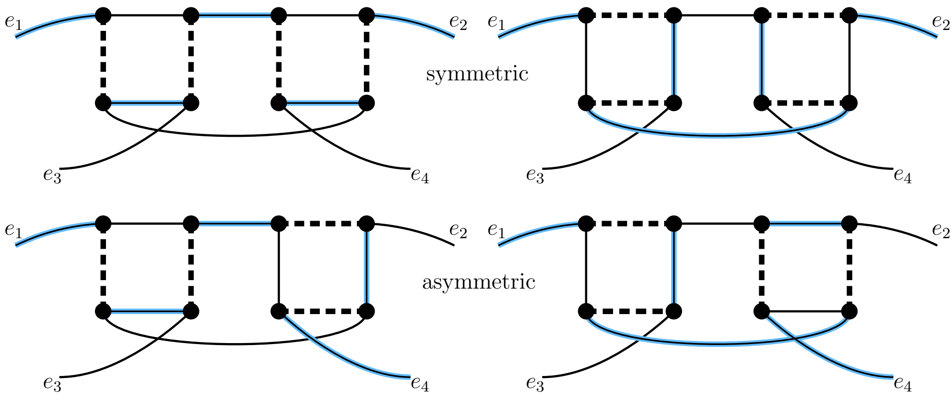

Figure 9: 2-chains when $M_1 \cap \mathcal{X} = \emptyset$ and $|M_2 \cap \mathcal{X}| = 2$ (bold dashed edges belong to M_1 and highlighted edges to M_2).

and $\mathcal{T}_{j+1}$. Let Q be one of these paths. We first note that Q contains exactly one vertex from $\{u_j, v_{j+1}\}$ and exactly one vertex from $\{u_{j+3}, v_{j+2}\}$. If $\mathcal{T}_{(j,j+1)}$ is symmetric with respect to M_1, then Q contains u_j if and only if Q contains u_{j+3}. Otherwise, if $\mathcal{T}_{(j,j+1)}$ is asymmetric with respect to M_1, then Q contains u_j if and only if Q contains v_{j+2}.

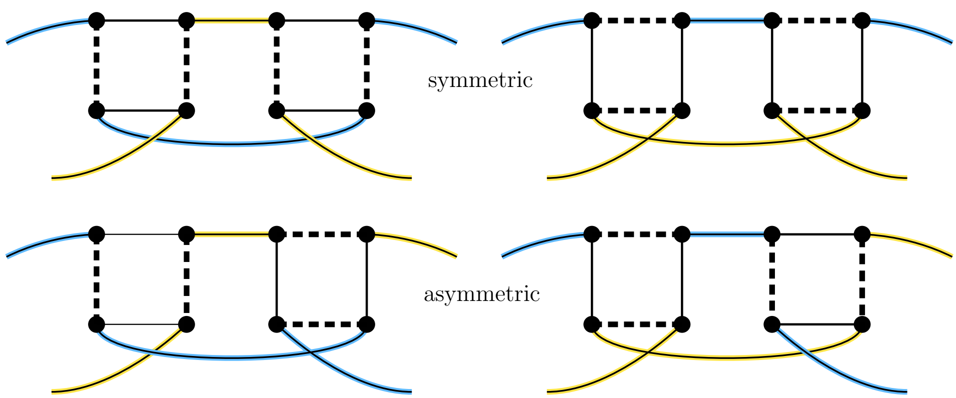

Figure 10: 2-chains when $M_1 \cap \mathcal{X} = \emptyset$ and $|M_2 \cap \mathcal{X}| = 4$ (bold dashed edges belong to M_1 and highlighted edges to M_2).

Theorem 3.8. *Let n be a positive even integer. Then, the balanced papillon graph $\mathcal{P}_n$ is PMH.*

Proof. Let M_1 be a perfect matching of $\mathcal{P}_n$. We need to show that there exists a perfect matching M_2 of $\mathcal{P}_n$ such that $M_1 \cup M_2$ induces a Hamiltonian cycle of $\mathcal{P}_n$. Three cases, depending on the intersection of M_1 with the principal 4-edge-cut $\mathcal{X}$ of $\mathcal{P}_n$, are considered. If $|M_1 \cap \mathcal{X}| = 2$, then, by Lemma 3.4, there exists a perfect matching N of $\mathcal{P}_n$ such that $|N \cap \mathcal{X}| = 2$ and $M_1 \cap N = \emptyset$. Moreover, the complementary 2-factor of N is a Hamiltonian cycle. Since M_1 is contained in the mentioned 2-factor, the result follows. When $|M_1 \cap \mathcal{X}| = 4$, we can define M_2 to be the following perfect matching:

$$M_2 = \{u_1v_1, u_2v_2\} \bigcup \cup_{j=2}^{2n}\{u_{2j-1}u_{2j}, v_{2j-1}v_{2j}\}.$$

In fact, $M_1 \cup M_2$ induces the following Hamiltonian cycle: $(u_1, v_1, v_4, \ldots, v_{2n}, v_{2n-1}, v_{4n-1}, v_{4n}, v_{4n-3}, \ldots, v_{2n+1}, v_{2n+2}, v_2, u_2, u_3, u_4, \ldots, u_{4n})$, where v_4 and v_{4n-3} are respectively followed by v_3 and v_{4n-2}.

What remains to be considered is the case when $|M_1 \cap \mathcal{X}| = 0$. Clearly, $|M_2 \cap \mathcal{X}|$ cannot be zero, because, if so, choosing M_2 to be disjoint from M_1, $M_1 \cup M_2$ induces $2n$ disjoint 4-cycles. Therefore, $|M_2 \cap \mathcal{X}|$ must be equal to 2 or 4. Let $\mathcal{R} = \{\mathcal{T}_{(1,2)}, \ldots, \mathcal{T}_{(n-1,n)}\}$ and $\mathcal{L} = \{\mathcal{T}_{(n+1,n+2)}, \ldots, \mathcal{T}_{(2n-1,2n)}\}$ be the sets of 2-chains within the left and right n-chains of $\mathcal{P}_n$—namely the right and left n-chains each split into $\frac{n}{2}$ 2-chains. We consider two cases depending on the parity of the number of 2-chains in $\mathcal{L}$ and $\mathcal{R}$ which are asymmetric with respect to M_1. Let the function $\Phi\colon \mathcal{R} \cup \mathcal{L} \to \{-1, +1\}$ be defined on the 2-chains $\mathcal{T} \in \mathcal{R} \cup \mathcal{L}$ such that:

$$\Phi(\mathcal{T}) = \begin{cases} +1 & \text{if } \mathcal{T} \text{ is symmetric with respect to } M_1, \\ -1 & \text{otherwise.} \end{cases}$$

Case 1: $\mathcal{L}$ and $\mathcal{R}$ each have an even number (possibly zero) of asymmetric 2-chains with respect to M_1.

We claim that there exists a perfect matching such that its union with M_1 gives a Hamiltonian cycle of $\mathcal{P}_n$. Since the number of asymmetric 2-chains in $\mathcal{R}$ is even, $\prod_{\mathcal{T} \in \mathcal{R}} \Phi(\mathcal{T}) = +1$, and consequently, by appropriately concatenating paths as in Remark 3.6, there exists a path R with endvertices u_1 and u_{2n} whose vertex set is $\cup_{i=1}^{2n}\{u_i, v_i\}$ such that it contains all the edges in $M_1 \cap (\cup_{i=1}^{n} E(\mathcal{T}_i))$. We remark that this path intersects exactly one edge of $\{xy \in E(\mathcal{P}_n) : x \in V(\mathcal{T}_j), y \in V(\mathcal{T}_{j+1})\}$, for each $j \in [n-1]$. By a similar reasoning, since $\prod_{\mathcal{T} \in \mathcal{L}} \Phi(\mathcal{T}) = +1$, there exists a path L with endvertices u_{2n+1} and u_{4n} whose vertex set is $\cup_{i=2n+1}^{4n}\{u_i, v_i\}$, such that it contains all the edges in $M_1 \cap (\cup_{i=n+1}^{2n} E(\mathcal{T}_i))$. Once again, this path intersects exactly one edge of $\{xy \in E(\mathcal{P}_n) : x \in V(\mathcal{T}_j), y \in V(\mathcal{T}_{j+1})\}$, for each $j \in \{n+1, \ldots, 2n-1\}$. These two paths, together with the edges a and d form the required Hamiltonian cycle of $\mathcal{P}_n$ containing M_1, proving our claim. We remark that this shows that there exists a perfect matching M_2 of $\mathcal{P}_n$ such that $M_2 \cap \mathcal{X} = \{a, d\}$, $M_1 \cap M_2 = \emptyset$ and with $M_1 \cup M_2$ inducing a Hamiltonian cycle of $\mathcal{P}_n$. One can similarly show that there exists a perfect matching M_2' of $\mathcal{P}_n$ such that $M_2' \cap \mathcal{X} = \{b, c\}$, $M_1 \cap M_2' = \emptyset$ and with $M_1 \cup M_2'$ inducing a Hamiltonian cycle of $\mathcal{P}_n$.

Case 2: One of $\mathcal{L}$ and $\mathcal{R}$ has an odd number of asymmetric 2-chains with respect to M_1.

Without loss of generality, assume that $\mathcal{R}$ has an odd number of asymmetric 2-chains with respect to M_1, that is, $\prod_{\mathcal{T} \in \mathcal{R}} \Phi(\mathcal{T}) = -1$. Let M_2 be the perfect matching of $\mathcal{P}_n$ such that $|M_2 \cap \mathcal{X}| = 4$. We claim that $M_1 \cup M_2$ induces a Hamiltonian cycle of $\mathcal{P}_n$. Since $\prod_{\mathcal{T} \in \mathcal{R}} \Phi(\mathcal{T}) = -1$, by appropriately concatenating paths as in Remark 3.7 we can deduce that $M_1 \cup M_2$ contains the edge set of two disjoint paths R_1 and R_2, such that:

(i) $|V(R_1)| = |V(R_2)| = 2n$;

(ii) $V(R_1) \cup V(R_2) = \cup_{i=1}^{2n}\{u_i, v_i\}$;

(iii) the endvertices of R_1 are u_1 and v_{2n-1}; and

(iv) the endvertices of R_2 are v_2 and u_{2n}.

Next, we consider two subcases depending on the value of $\prod_{\mathcal{T}\in\mathcal{L}} \Phi(\mathcal{T})$. We shall be using the fact that $\{u_1u_{4n}, v_{2n-1}v_{4n-1}, v_2v_{2n+2}, u_{2n}u_{2n+1}\} = \{a, b, c, d\} = \mathcal{X} \subset M_2$.

Case 2a: $\prod_{\mathcal{T}\in\mathcal{L}} \Phi(\mathcal{T}) = -1$

As above, by Remark 3.7, we can deduce that $M_1 \cup M_2$ contains the edge set of two disjoint paths L_1 and L_2, such that:

(i) $|V(L_1)| = |V(L_2)| = 2n$;

(ii) $V(L_1) \cup V(L_2) = \cup_{i=2n+1}^{4n}\{u_i, v_i\}$;

(iii) the endvertices of L_1 are u_{2n+1} and v_{4n-1}; and

(iv) the endvertices of L_2 are v_{2n+2} and u_{4n}.

The concatenation of the following paths and edges gives a Hamiltonian cycle of $\mathcal{P}_n$ containing M_1:

$$R_1v_{2n-1}v_{4n-1}L_1u_{2n+1}u_{2n}R_2v_2v_{2n+2}L_2u_{4n}u_1.$$

Case 2b: $\prod_{\mathcal{T}\in\mathcal{L}} \Phi(\mathcal{T}) = +1$.

Once again, by Remark 3.7 we can deduce that $M_1 \cup M_2$ contains the edge set of two disjoint paths L_1 and L_2, such that:

(i) $|V(L_1)| = |V(L_2)| = 2n$;

(ii) $V(L_1) \cup V(L_2) = \cup_{i=2n+1}^{4n}\{u_i, v_i\}$;

(iii) the endvertices of L_1 are u_{2n+1} and u_{4n}; and

(iv) the endvertices of L_2 are v_{2n+2} and v_{4n-1}.

The concatenation of the following paths and edges gives a Hamiltonian cycle of $\mathcal{P}_n$ containing M_1:

$$R_1v_{2n-1}v_{4n-1}L_2v_{2n+2}v_2R_2u_{2n}u_{2n+1}L_1u_{4n}u_1.$$

This completes the proof. □

3.2 The unbalanced case $r < \ell$ and final remarks

By following the proofs in Section 2, the results obtained for balanced papillon graphs are now extended to unbalanced papillon graphs.

Theorem 3.9. *The unbalanced papillon graph $\mathcal{P}_{r,\ell}$ is PMH if and only if r and ℓ are both even.*

Proof. This is an immediate consequence of Proposition 3.5 and Theorem 3.8. In particular, when at least one of r and ℓ is odd, $\mathcal{P}_{r,\ell}$ is not PMH because the perfect matching $\cup_{i=1}^{r+\ell}\{u_{2i-1}u_{2i}, v_{2i-1}v_{2i}\}$ of $\mathcal{P}_{r,\ell}$ (illustrated in Figure 11) cannot be extended to a Hamiltonian cycle. □

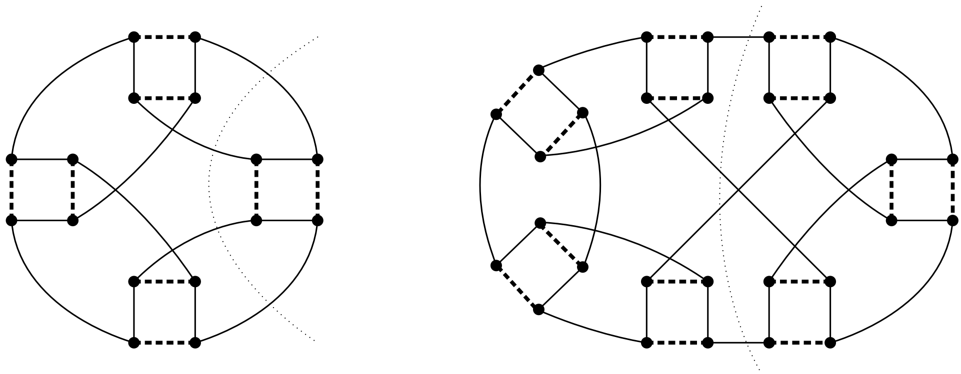

Figure 11: $\mathcal{P}_{1,3}$ and $\mathcal{P}_{3,4}$: unbalanced papillon graphs are not always PMH. The above perfect matchings do not extend to a Hamiltonian cycle.

Corollary 3.10. *The papillon graph $\mathcal{P}_{r,\ell}$ is PMH if and only if r and ℓ are both even.*

Finally, we remark that since $\mathcal{P}_n$ is PMH for every even $n \in \mathbb{N}$, balanced papillon graphs provide us with examples of non-bipartite PMH cubic graphs which are cyclically 4-edge-connected and have girth 4 such that their order is a multiple of 16. Additionally, by considering unbalanced papillon graphs, say $\mathcal{P}_{2,\ell}$, for some even $\ell > 2$, we can obtain non-bipartite PMH cubic graphs having the above characteristics (that is, cyclically 4-edge-connected and having girth 4) such that their order is 8ν, for odd $\nu \geq 3$.

It would also be very compelling to see whether there exist other 4-poles instead of the C_4-poles that can be used as building blocks when constructing papillon graphs and which yield non-bipartite PMH or just E2F cubic graphs.

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The search for small association schemes with noncyclotomic eigenvalues*

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Abstract

In this article we determine feasible parameter sets for (what could potentially be) commutative association schemes with noncyclotomic eigenvalues that are of smallest possible rank and order. A feasible parameter set for a commutative association scheme corresponds to a standard integral table algebra with integral multiplicities that satisfies all of the parameter restrictions known to hold for association schemes. For each rank and involution type, we generate an algebraic set for which any suitable integral solution corresponds to a standard integral table algebra with integral multiplicities, and then try to find the smallest suitable solution. The main results of this paper show the eigenvalues of association schemes of rank 4 and nonsymmetric association schemes of rank 5 will always be cyclotomic. In the rank 5 cases, the results rely on calculations done by computer for Gröbner bases or for bases of rational vector spaces spanned by polynomials. We give several examples of feasible parameter sets for small symmetric association schemes of rank 5 that have noncyclotomic eigenvalues.

Keywords: Association schemes, table algebras, character tables.

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1 Introduction

This paper investigates the Cyclotomic Eigenvalue Question for commutative association schemes that was posed by Simon Norton at Oberwolfach in 1980 [3]. This question asks if

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the eigenvalues of all the adjacency matrices of relations in the scheme lie in a cyclotomic number field, or equivalently if every entry of the character table (i.e., first eigenmatrix) of a commutative association scheme is cyclotomic. Showing this is a straightforward exercise for association schemes of rank 2 and 3. For commutative Schurian association schemes, this property is a consequence of the character theory of Hecke algebras and the fact that Morita equivalent algebras have isomorphic centers (see [9]). For commutative association schemes that are both P- and Q-polynomial, it follows from the fact that the splitting field of the scheme is quadratic extension of the rationals, a key ingredient of Bang, Dubickas, Koolen, and Moulten's proof of the Bannai-Ito conjecture ([2], see also [14]). Herman and Rahnamai Barghi proved it for commutative quasi-thin schemes [12], which were later shown by Muzychuk and Ponomarenko to always be Schurian [16]. Herman and Rahnamai Barghi also showed the cyclotomic eigenvalue property holds for commutative association schemes whose elements have valency ≤ 2 except for possibly one element of valency 3 and/or one element of valency > 4 [12, Theorem 3.3].

For association schemes in general we do not know if the character values have to be cyclotomic, but we do have noncommutative examples for which the eigenvalues are not cyclotomic – the smallest examples are two noncommutative Schurian association schemes of order 26, and three noncommutative Schur rings of order 32 (in the latter case the corresponding graphs are Cayley graphs on a nonabelian group of order 32).

In this article we investigate the cyclotomic eigenvalue question from a smallest counterexample perspective. For a given rank and involution type, our approach will be to generate an algebraic set in a multivariate polynomial ring in variables corresponding to the intersection numbers and character table parameters of such an association scheme. Each suitable integer point in this algebraic set corresponds to a standard integral table algebra with integral multiplicities (SITAwIM) that has the corresponding intersection matrices and character table via its regular representation. We use the algebraic set to search for small SITAwIMs of the given type that have some noncyclotomic eigenvalues.

The algebraic sets themselves are not easy to work with, as they are not monomial and the number of variables and polynomial generators is too large for available computer algebra systems to do efficient Gröbner basis calculations. After manually reducing the algebraic sets with all available linear substitutions, we search for solutions by specifying values for sufficiently many remaining parameters that the resulting algebraic set can be resolved with a Gröbner basis calculation. Using this approach, we are able to show the answer to the cyclotomic eigenvalue question is yes for all association schemes of rank 4 and for both involution types of nonsymmetric association schemes of rank 5. For commutative association schemes, the noncyclotomic eigenvalue property implies the Galois group of the splitting field will be non-Abelian, and so there must be an orbit of size at least 3 in its action on irreducible characters. We will say the Galois group acts *k-point transitively* if the size of its largest orbit on irreducible characters is k. So for symmetric association schemes of rank 5, noncylotomic eigenvalues can only occur when the Galois group of the splitting field is 3- or 4-point transitive. When this action is 4-point transitive, the association scheme will be pseudocyclic. This greatly reduces the number of cases we need to consider, and our searches have been able to produce six feasible examples of orders less than 1000, the smallest having order 249. When the action is 3-point transitive, the scheme is not pseudocyclic, so the search space is much larger. We have been able to generate all examples of order less than 100 and a few more with order less than 250, ten of which satisfy all available feasibility criteria. The smallest of these feasible examples have order

35, 45, 76, and 93. From the partial classification of association schemes of order 35, we know the order 35 example cannot be realized. The status of the larger feasible examples is open.

2 Preliminaries

In this section, we review some background results that are needed in this work. Recall that an involution ϕ of a finite-dimensional algebra A is a map $\phi\colon A \to A$ such that $\phi \circ \phi = id_A$.

2.1 SITA parameters

An *integral table algebra* $(A, \mathbf{B})$ is a finite-dimensional complex algebra A with distinguished basis $\mathbf{B} = \{b_i \mid i \in I = \{0, 1, \ldots, r-1\}\}$ such that

(i) $1 \in \mathbf{B}$,

(ii) A has an involution $*\colon A \to A$ that is additive, reverses multiplication, and acts as complex conjugation on scalars,

(iii) $\mathbf{B}$ is $*-$invariant,

(iv) $\mathbf{B}$ produces non-negative integer structure constants (see 2.2),

(v) $\mathbf{B}$ satisfies the pseudo-inverse condition: for all $b_i, b_j \in \mathbf{B}$, the coefficient of 1 in $b_i b_j^*$ is positive if and only if $b_j = b_i^*$.

Note that since $\mathbf{B}^* = \mathbf{B}$, the involution $*$ is a permutation of $\{0, 1, \ldots, r-1\}$. Therefore, the action of the involution $*$ can be defined by $(b_i)^* = b_{i^*}$ for all $b_i \in \mathbf{B}$.

In order to consider A as an algebra of square matrices over $\mathbb{C}$, we identify the elements of $\mathbf{B}$ with their left regular matrices in the basis $\mathbf{B}$. The basis $\mathbf{B}$ is called *standard* when, for all $b_i \in \mathbf{B}$, the coefficient of 1 in $b_i b_i^*$ is equal to the maximal eigenvalue of the regular matrix b_i. We refer to $r = |\mathbf{B}|$ as the *rank* of the table algebra, when the basis $\mathbf{B}$ is standard we say that $(A, \mathbf{B})$ is a standard integral table algebra, or SITA. The action of the involution $*$ on the basis $\mathbf{B}$ determines the *involution type* of the table algebra of a given rank.

The adjacency algebra of an association scheme is the prototypical example of a SITA, as the defining basis of adjacency matrices is a standard basis. Conversely, the structure constants determined by the basis of adjacency matrices of an association scheme determine a standard integral table algebra that is *realizable* as an association scheme. Many open problems concerning missing combinatorial objects correspond to standard integral table algebras that satisfy all the known conditions on their parameters for being realized by an association scheme, but are yet to be actually constructed. We call such standard integral table algebras (or their parameter sets) *feasible*.

Let $P = (\chi_i(b_j))_{i,j}$ be the character table of A with respect to the distinguished basis $\mathbf{B}$, whose rows are indexed by the irreducible characters of A and columns are indexed by the basis $\mathbf{B}$. As we can restrict ourselves to the commutative table algebras in this paper, P will be an $r \times r$ matrix. We order the irreducible characters so that the entries $P_{0,j} = \chi_0(b_j) = \delta_j$, $j = 0, 1, \ldots, r-1$ are equal to the Perron-Frobenius eigenvalues of the basis matrices (i.e., the *degrees* of standard basis elements, or in the association scheme case, the *valencies* of the scheme relations). The *order* of a standard integral table algebra is the sum of its degrees; that is, $n = \sum_{j=0}^{r-1} \delta_j$. The *multiplicity* m_i of each irreducible character χ_i can be computed by the following formula [4]

$$\sum_{j=0}^{r-1} \frac{|P_{ij}|^2}{\delta_j} = \frac{n}{m_i}, \text{ for } i = 0, 1, \dots, r-1.$$

For table algebras, the multiplicity m_i corresponds to the coefficient of χ_i when the standard feasible trace map $\rho(\sum_{j=0}^{r-1} \alpha_j b_j) = n\alpha_0$ is expressed as a (positive) linear combination of the irreducible characters of A. We always have $m_0 = 1$, but the other multiplicities m_i for $i = 1, \dots, r-1$ are only required to be positive real numbers. When the SITA is realized by an association scheme, the standard feasible trace is the character corresponding to the standard representation of the SITA, so the m_i's will be positive integers. This is just one of the feasibility conditions for the parameters of an association scheme. In this way, each feasible parameter set for association schemes determines a standard integral table algebra with integral multiplicities, i.e., a SITAwIM.

A SITA is called *pseudocyclic* if its multiplicities m_i for $i > 0$ are all equal to the same positive constant m. By a result of Blau and Xu [18], pseudocyclic SITAs are also *homogeneous*, that is, all degrees δ_i for $i > 0$ are equal to the same positive constant.

2.2 General conditions on SITA parameters

Let $\mathbf{B} = \{b_0, b_1, ..., b_{r-1}\}$ be the standard basis of a SITA $(A, \mathbf{B})$. Denote the *structure constants* relative to the basis $\mathbf{B}$ by $(\lambda_{ijk})_{i,j,k=0}^{r-1}$, so

$$b_i b_j = \sum_{k=0}^{r-1} \lambda_{ijk} b_k, \text{ for all } i, j \in \{0, 1, \dots, r-1\}.$$

Let $\chi_0(b_i) = \delta_i$ be the degree (or valency) of the basis element $b_i \in \mathbf{B}$, for all $i \in \{0, 1, \dots, r-1\}$. When the algebra has a standard basis, we have $\delta_i = \delta_{i^*} = \lambda_{i^* i 0}$ (see [5, Definition 1.3]).

Associativity of A and the pseudo-inverse condition on the standard basis can be used to prove two general properties of the structure constants relative to $\mathbf{B}$.

Lemma 2.1. *For all* $i, j, k \in \{0, 1, \dots, r-1\}$,

(i) $\lambda_{jki^*}\delta_i = \lambda_{kij^*}\delta_j = \lambda_{ijk^*}\delta_k$, *and*

(ii) $\sum_{k=0}^{r-1} \lambda_{jki} = \delta_j$.

Proof. (i) By the associativity of multiplication we have the following condition on the structure constants for all $i, j, k, \ell, m \in \{0, 1, \dots, r-1\}$,

$$\sum_{\ell} \lambda_{ij\ell}\lambda_{\ell km} = \sum_{\ell} \lambda_{i\ell m}\lambda_{jk\ell}$$

Now, fix k and let $m = 0$. Using the pseudo-inverse condition on $\mathbf{B}$ we have

$$\lambda_{jki^*}\delta_i = \lambda_{kij^*}\delta_j = \lambda_{ijk^*}\delta_k.$$

For (ii), we have that for all $i,j \in \{0,1,\dots,r-1\}$, $b_{j^*}b_i = \sum_{k=0}^{r-1} \lambda_{j^*ik}b_k = \sum_{k=0}^{r-1} \lambda_{i^*jk^*}b_k$. Since $\chi_0(b_j^*) = \chi_0(b_j)$ and the degree map is an algebra homomorphism from A to $\mathbb{C}$, we have

$$\delta_j\delta_i = \chi_0(b_{j^*}b_i) = \sum_{k=0}^{r-1} \lambda_{i^*jk^*}\delta_k,$$

which is equal by (i) to $\sum_{k=0}^{r-1} \lambda_{jki}\delta_i$. So, (ii) follows. □

Note that Lemma 2.1(ii) tells us that every row sum of the left regular matrix of $b_j \in \mathbf{B}$ is equal to the constant δ_j.

Next, we consider restrictions on the parameters of a SITA imposed by its fusions. If $I \subseteq \{1,\dots,r-1\}$, we let $b_I = \sum_{i\in I} b_i$. When $\Lambda = \{\{0\}, I_1,\dots,I_{s-1}\}$ is a partition of $\{0,1,\dots,r-1\}$ for which $\mathbf{B}_\Lambda = \{b_0, b_{I_1},\dots,b_{I_{s-1}}\}$ is the basis of a table algebra (which will automatically be the standard basis of a SITA in this case), then we say that $\mathbf{B}_\Lambda$ is a *fusion* of $\mathbf{B}$, and conversely say that $\mathbf{B}$ is a *fission* of $\mathbf{B}_\Lambda$. The next lemma shows that every SITA admits a rank 2 fusion.

Lemma 2.2. *Every table algebra $(A,\mathbf{B})$ with standard basis $\mathbf{B} = \{b_0,b_1,\dots,b_{r-1}\}$ of rank $r \geq 3$ has the trivial rank 2 fusion $\mathbf{B}_{\{\{0\},\{1,\dots,r-1\}\}} = \{b_0, b_1+\cdots+b_{r-1}\}$.*

Proof. Let $\mathbf{B}^+ = \sum_{j=0}^{r-1} b_j$. By [1] we have $(\mathbf{B}^+)^2 = \chi_0(\mathbf{B}^+)\mathbf{B}^+ = n\mathbf{B}^+$. It follows that $((\mathbf{B}-\{b_0\})^+)^2 = n\mathbf{B}^+ - 2\mathbf{B}^+ + b_0 = (n-2)\mathbf{B}^+ + b_0 = (n-2)(\mathbf{B}^+ - \{b_0\}) + (n-1)b_0$. This implies $\{b_0, \mathbf{B}^+ - b_0\}$ is a $*$-invariant subset of $\mathbf{B}$ that generates a 2-dimensional subalgebra of A. The lemma follows. □

The conditions imposed by fusion on the parameters of a commutative association scheme were studied by Bannai and Song in [4]. For structure constants the conditions are straightforward, for character table parameters the existence of a fusion imposes certain identities on partial row and column sums of P. Let $\Lambda = \{\{0\}, J_1,\dots,J_{d-1}\}$ be the partition inducing the fusion $\mathbf{B}_\Lambda = \{\tilde{b}_0, \tilde{b}_{J_1},\dots,\tilde{b}_{J_{d-1}}\}$ of our standard integral table algebra basis $\mathbf{B}$. If $E = \{e_0,e_1,\dots,e_{r-1}\}$ is the basis of primitive idempotents of A, then there is a (dual) partition $\Lambda^* = \{\{0\}, K_1,\dots,K_{d-1}\}$ of $\{0,1,\dots,r-1\}$, unique to the fusion, such that, if $\tilde{e}_0 = e_0$ and $\tilde{e}_{K_i} = \sum_{k\in K_i} e_k$ for $i = 1,\dots,d-1$, then $\tilde{E} = \{\tilde{e}_0, \tilde{e}_{K_1},\dots,\tilde{e}_{K_{d-1}}\}$ is the basis of primitive idempotents of the algebra $\mathbb{C}\mathbf{B}_\Lambda$.

Let $\tilde{P}$ be the character table of the fusion $\mathbf{B}_\Lambda$, so the rows of $\tilde{P}$ are indexed by the irreducible characters $\tilde{\chi}_I$ for $I \in \Lambda^*$, and the columns of $\tilde{P}$ are indexed by the basis elements b_J for $J \in \Lambda$. Let $\tilde{\delta} = \tilde{\chi}_0$ and $\delta = \chi_0$ be the respective degree maps. Let $\tilde{\delta}(b_J) = \tilde{k}_J$ for all $J \in \Lambda$, and $\delta(b_j) = k_j$ for all $j \in \{1,\dots,r-1\}$. Let $\tilde{m}_I$ and m_i denote the multiplicities of $\tilde{\chi}_I$ and χ_i, respectively. Then we have the following identities on partial row and column sums.

Theorem 2.3 (Theorem 1.4, [4])**.** *Let $J \in \Lambda$ and $I \in \Lambda^*$.*

(i) *For all $j \in J$, $\sum_{i\in I} m_i P_{i,j} = \frac{k_j \tilde{m}_I}{\tilde{k}_J}\tilde{P}_{I,J}$.*

(ii) *For all $i \in I$, then $\tilde{P}_{I,J} = \sum_{j\in J} P_{i,j}$.*

Proof. (i) We are assuming $\tilde{e}_I = \sum_{i \in I} e_i$. Using the formula for primitive idempotents in a standard table algebra [1],

$$\tilde{e}_I = \frac{\tilde{m}_I}{n} \sum_J \frac{\tilde{P}_{I,J}}{\tilde{k}_J} \tilde{b}_J^* = \sum_J \sum_{j \in J} \frac{\tilde{m}_I \tilde{P}_{I,J}}{n \tilde{k}_J} b_j^*.$$

On the other hand,

$$\begin{aligned} \tilde{e}_I &= \sum_{i \in I} e_i = \sum_{i \in I} \frac{m_i}{n} \sum_j \frac{P_{i,j}}{k_j} b_j^* \\ &= \sum_J \sum_{j \in J} \sum_{i \in I} \frac{m_i P_{i,j}}{n k_j} b_j^*. \end{aligned}$$

Therefore, for all $j \in J$, $\sum_{i \in I} m_i P_{i,j} = \frac{k_j \tilde{m}_I}{\tilde{k}_J} \tilde{P}_{I,J}$, as required.

(ii) When $\chi_i(b_0) = 1$, we have $b_j e_i = P_{i,j} e_i$ for all $b_j \in \mathbf{B}$. On the one hand,

$$\tilde{b}_J \tilde{e}_I = \tilde{P}_{I,J} \tilde{e}_I = \sum_{i \in I} \tilde{P}_{I,J} e_i,$$

and on the other hand, assuming $\chi_i(b_0) = 1$ for all $i \in I$,

$$\tilde{b}_J \tilde{e}_I = \sum_{j \in J} \sum_{i \in I} b_j e_i = \sum_{i \in I} \sum_{j \in J} P_{i,j} e_i.$$

Therefore, $\tilde{P}_{I,J} = \sum_{j \in J} P_{i,j}$ for all $i \in I$. □

We remark that the fusion condition (i) on partial column sums holds without change for noncommutative table algebras. Note that standard character considerations tell us $\sum_{j \in J} m_j \chi_j(b_0) = \tilde{m}_J$. Condition (ii) on partial row sums holds for the rows of $\tilde{P}$ indexed by the $\tilde{\chi}_I$ for which $\chi_i(b_0) = 1$ for all $i \in I$.

2.3 The splitting field and its Galois group

If $(A, \mathbf{B})$ is a commutative integral table algebra with standard basis $\mathbf{B} = \{b_0, b_1, \dots, b_{r-1}\}$ then the *splitting field* of $(A, \mathbf{B})$ is the field K obtained by adjoining all the eigenvalues of the regular matrices of elements of $\mathbf{B}$ to the rational field $\mathbb{Q}$, or equivalently, the smallest field K for which the character table P lies in $M_r(K)$, the algebra of $r \times r$ matrices over the field K. As each b_j in $\mathbf{B}$ is a nonnegative integer matrix, K is also the unique minimal Galois extension of $\mathbb{Q}$ that splits the characteristic polynomials of every $b_j \in \mathbf{B}$. Let $G = \mathrm{Gal}(K/\mathbb{Q})$ be the Galois group of this splitting field. Since the irreducible characters of A are also irreducible representations of A in the commutative case, G will act faithfully on the set of irreducible characters of A via $\chi_i^\sigma(b_j) = (\chi_i(b_j))^\sigma$, for all $\chi_i \in \mathrm{Irr}(A)$, $b_j \in \mathbf{B}$, and $\sigma \in G$. In this way G permutes the rows of the character table P, as well as the corresponding multiplicities. For SITAwIMs this means G can only permute sets of irreducible characters with the same multiplicity.

By the Kronecker-Weber theorem, a necessary and sufficient condition for $(A, \mathbf{B})$ to be a standard integral table algebra with noncyclotomic character values is for this Galois group G to be non-abelian. If G is non-abelian, the fact that the action of G on irreducible characters of A is faithful forces there to be at least one orbit of size 3 or more.

Theorem 2.4 ([15]). *Let $(A, \mathbf{B})$ be an integral table algebra (possibly noncommutative). Let H be the subset of $G = \mathrm{Gal}(K/\mathbb{Q})$ consisting of elements $\sigma \in G$ whose action on the character table $P = (\chi(b))_{\chi,b}$ can be realized by a permutation of the basis, that is, for all $b \in \mathbf{B}$ there exists $b^\sigma \in \mathbf{B}$ such that for all $\chi \in \mathrm{Irr}(A)$, $(P_{\chi,b})^\sigma = \chi(b^\sigma) = P_{\chi,b^\sigma}$. Then H is a central subgroup of G.*

Proof. To see that H is a subgroup of G, let $\sigma, \tau \in H$, $\chi \in \mathrm{Irr}(A)$, and $b \in \mathbf{B}$. Then,

$$(P_{\chi,b})^{\sigma\tau} = ((P_{\chi,b})^\sigma)^\tau = (P_{\chi,b^\sigma})^\tau = (P_{\chi,b^{\sigma\tau}}).$$

Therefore, $\sigma\tau \in H$. Since G is finite, H is a subgroup. To see that H is central, let $\tau \in H$, $\sigma \in G$, $\chi \in Irr(A)$, and $b \in \mathbf{B}$. Then,

$$(P_{\chi,b})^{\sigma\tau} = (P_{\chi^\sigma,b})^\tau = (P_{\chi^\sigma,b^\tau}) = (P_{\chi,b^\tau})^\sigma = ((P_{\chi b})^\tau)^\sigma = (P_{\chi,b})^{\tau\sigma}.$$

As the action of G on the rows of P is faithful, this implies $\sigma\tau = \tau\sigma$, so H is contained in $Z(G)$. □

Note that the above theorem always applies to commutative table algebras that are not symmetric.

Corollary 2.5. *Suppose $(A, \mathbf{B})$ is a commutative table algebra that is not symmetric. Then the restriction of complex conjugation to K is a nonidentity element of the center of G.*

Proof. Commutative table algebras that are not symmetric always have at least one irreducible character that is not real-valued. If otherwise, the identity $\chi_i(b_j^*) = \overline{\chi_i(b_j)}$, for all $\chi_i \in \mathrm{Irr}(A)$ and $b_j \in \mathbf{B}$, would imply the character table P would not be invertible. For the irreducible characters that are not real-valued, the restriction of complex conjugation to K will be a non-identity element of G that is realized by the permutation of $\mathbf{B}$ corresponding to the involution. By Theorem 2.4, this element lies in the center of G. □

2.4 Algebraic sets for SITAwIMs of a given rank and involution type.

As indicated in the introduction, we will obtain our results by searching for suitable nonnegative integer points in an algebraic set (i.e., the solution set to a system of polynomial equations) that is determined by the parameters of SITAwIMs of a given rank and involution type. To illustrate how the generating sets for the ideals corresponding to these algebraic sets are produced, we give the type $4A1$ case as an example. This is the algebraic set corresponding to rank 4 SITAwIMs whose basis $\mathbf{B}$ contains one asymmetric pair, i.e., $\mathbf{B} = \{b_0, b_1, b_2, b_2^*\}$. Using the properties of the involution, the row sum property, commutativity of the algebra, and the fact that $b_i b_j = \sum_{k=0}^{3} \lambda_{ijk} b_k$, the general form of the regular matrices for the nontrivial elements of this basis is

$$b_1 = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 1 & k_1 - 2x_1 - 1 & x_1 & x_1 \\ 0 & k_1 - x_2 - x_3 & x_2 & x_3 \\ 0 & k_1 - x_2 - x_3 & x_3 & x_2 \end{bmatrix},$$

$$b_2 = \begin{bmatrix} 0 & 0 & 0 & k_2 \\ 0 & x_1 & k_2 - x_1 - x_4 & x_4 \\ 1 & x_2 & k_2 - x_2 - x_5 - 1 & x_5 \\ 0 & x_3 & k_2 - x_3 - x_5 & x_5 \end{bmatrix},$$

$$b_2^* = \begin{bmatrix} 0 & 0 & k_2 & 0 \\ 0 & x_1 & x_4 & k_2 - x_1 - x_4 \\ 0 & x_3 & x_5 & k_2 - x_3 - x_5 \\ 1 & x_2 & x_5 & k_2 - x_2 - x_5 - 1 \end{bmatrix}.$$

Identifying entries in the matrix equations resulting from the identities that define the regular representation gives several linear and quadratic identities in the variables $x_1, \dots, x_5, k_1$, k_2, each of which corresponds to a multivariate polynomial equaling 0. For example, identifying entries on both sides of the matrix equation

$$b_1 b_2 = x_1 b_1 + x_2 b_2 + x_3 b_2^*$$

gives a list of 8 polynomials:

$$\begin{aligned}
&-x_2k_2 + x_4k_1,\\
&-x_1k_1 - x_3k_2 - x_4k_1 + k_1k_2,\\
&-x_1x_3 - x_2x_4 + x_3^2 - x_3x_4 + 2x_3x_5 - x_3k_2 + x_4k_1,\\
&x_1x_3 - x_1k_1 + x_2x_4 - x_2k_2 - x_3^2 + x_3x_4 - 2x_3x_5 - x_4k_1 + k_1k_2,\\
&-x_1x_2 + x_2x_3 - x_2x_4 - x_3x_4 + 2x_3x_5 - x_3k_2 + x_4k_1 + x_3,\\
&x_1x_2 - x_1k_1 - x_2x_3 + x_2x_4 - x_2k_2 + x_3x_4 - 2x_3x_5 - x_4k_1 + k_1k_2 - x_3,\\
&-x_1^2 + x_1x_3 - 2x_1x_4 + 2x_1x_5 - x_2x_4 + x_3x_4 - x_3k_2 + x_4k_1 - x_4 + k_2, \text{ and}\\
&x_1^2 - x_1x_3 + 2x_1x_4 - 2x_1x_5 - x_1k_1 + x_2x_4 - x_2k_2 - x_3x_4 - x_4k_1 + k_1k_2 + x_4 - k_2.
\end{aligned}$$

We get similar lists of polynomials from the defining identities for b_1^2, $b_1b_2^*$, b_2^2, $b_2b_2^*$, and $(b_2^*)^2$, and possibly still more from the commuting identities $b_1b_2 = b_2b_1$, $b_1b_2^* = b_2^*b_1$, and $b_2b_2^* = b_2^*b_2$. In the type $4A1$ case, up to sign, this process produces 16 distinct polynomials.

When we add the integral multiplicities condition, it leads to extra trace identities that can be added to our list. For each choice of multiplicities $m_i \in \mathbb{Z}^+$, $i = 1, \dots, r-1$, we have an identity satisfied by our character table P resulting from the column orthogonality relation:

$$k_j + \sum_{i=1}^{r-1} m_i P_{i,j} = 0.$$

In light of assumptions we can make regarding the Galois group, certain rows of P will be Galois conjugate, and the sums of $P_{i,j}$'s corresponding to these rows have to be rational algebraic integers, and thus integers. The multiplicities corresponding to Galois conjugate rows are the same. Summing these rows of P gives the rational character table, an integer matrix satisfying certain column and row orthogonality conditions. The entries in each column of this matrix are bounded in terms of the first entry k_j of the column, so we can search for the possible rational character tables for a given choice of multiplicities. For each possible rational character table, we can add linear trace identities

$$tr(b_j) = k_j + \sum_{i=1}^{r-1} P_{i,j}, \quad j = 1, \dots, r-1,$$

to our list of polynomials.

Let $\mathcal{S}$ be the set of polynomials produced by this process. Let $\mathcal{I}$ be the ideal generated by $\mathcal{S}$, and let $\mathcal{V}(\mathcal{I})$ be the corresponding algebraic set. The regular matrices of any SITAwIM of type $4A1$ with the given choice of multiplicities corresponds naturally to a point in $\mathcal{V}(\mathcal{I})$ with $x_1, \dots, x_5 \in \mathbb{N}$ and $k_1, k_2 \in \mathbb{Z}^+$. We will refer to this as a *suitable* integral point in the algebraic set. Conversely, any suitable integral point in $\mathcal{V}(\mathcal{I})$ corresponds to a SITAwIM of this rank, involution type, and choice of multiplicities.

For example, if we assume $m_1 = m_2 = m_3$ in the type $4A1$ case, it adds the trace identities $tr(b_j) = k_j - 1$ for $j = 1, 2, 3$, all of which reduce to $x_1 = x_2$. Since this pseudocyclic assumption implies the SITA is homogeneous, we also get $m_1 = k_1 = k_2$. Other linear identities, or ones that become linear after cancelling one of our nonzero degrees k_j, can also be used to reduce the number of variables we need to consider. For example, in the type $4A1$ case, one of the elements of $\mathcal{S}$ is $k_2(k_2 - 1 - x_2 - 2x_5)$, so we can substitute $x_2 = k_2 - 1 - 2x_5$ and reduce the number of variables by one. After we reduce by all available linear substitutions in the type $4A1$ case, only one polynomial remains:

$$f(x_5, k_1) = 36x_5^2 - 24x_5k_1 + 4k_1^2 + 32x_5 - 11k_1 + 7.$$

Putting this together with our linear substitutions, we can conclude that any pseudocyclic SITAwIM of type $4A1$ corresponds, via the above regular matrices, to an integer point $(x_1, x_2, x_3, x_4, x_5, k_1, k_2)$ for which $f(x_5, k_2) = 0$, $x_5 \geq 0$, $k_1 = k_2 > 0$, $x_1 = x_2 = x_4 = k_1 - 2x_5 - 1 \geq 0$, and $x_3 = 4x_5 - k_1 + 2 \geq 0$. This is an effective formula to generate pseudocyclic SITAwIMs of type $4A1$.

We refer the readers to [10] for the GAP implementation that produces the defining list of polynomials for rank 4 and 5 SITAwIMs of each involution type.

3 Rank 4 SITAwIMs have cyclotomic eigenvalues

In this section we show that rank 4 SITAwIMs have cyclotomic eigenvalues. In this case there are two involution types to consider: type $4A1$ and type $4S$.

Proposition 3.1. *Rank 4 SITAwIMs with one asymmetric pair of standard basis elements have cyclotomic eigenvalues. In fact, their eigenvalues lie in quadratic number fields.*

Proof. Suppose $(A, \mathbf{B})$ is a SITAwIM of rank 4 with $\mathbf{B} = \{b_0, b_1, b_2, b_2^*\}$. If there were nonidentity elements of $\mathbf{B}$ with noncyclotomic eigenvalues, the Galois group G of the splitting field K would have to be 3-point transitive; i.e., a transitive subgroup of the group $\mathrm{Sym}(\{\chi_1, \chi_2, \chi_3\})$. Since G would have to be non-abelian, it would have to be isomorphic to S_3. But $|Z(G)| > 1$ by Corollary 2.5, so this is a contradiction.

Since there are no 3-transitive groups with a central element of order 2, we can conclude that G is cyclic of order 2, and therefore K is a quadratic extension of $\mathbb{Q}$. □

Theorem 3.2. *Symmetric rank 4 SITAwIMs have cyclotomic eigenvalues.*

Proof. Suppose $(A, \mathbf{B})$ is a symmetric SITAwIM of rank 4 that has noncyclotomic eigenvalues. If G is the Galois group of its splitting field K, then as in the rank 4 one asymmetric pair case, G must act as the full symmetric group on the set $\{\chi_1, \chi_2, \chi_3\}$. In particular, this implies these three characters have the same multiplicity m. Therefore, $n = 1 + 3m$ and the character table P of $(A, \mathbf{B})$ has the form

	b_0	b_1	b_2	b_3	multiplicities
χ_0	1	δ_1	δ_2	δ_3	1
χ_1	1	α_1	β_1	γ_1	m
χ_2	1	α_2	β_2	γ_2	m
χ_3	1	α_3	β_3	γ_3	m

where $\{\delta_1, \alpha_1, \alpha_2, \alpha_3\}$, $\{\delta_2, \beta_1, \beta_2, \beta_3\}$, and $\{\delta_3, \gamma_1, \gamma_2, \gamma_3\}$ are the eigenvalues of b_1, b_2, and b_3, respectively. If we apply Theorem 2.3(i) to the column of P labeled by b_1, we get

$$m(\alpha_1 + \alpha_2 + \alpha_3) = \frac{\delta_1}{n-1}(n-1)(-1), \text{ so } \alpha_1 + \alpha_2 + \alpha_3 = \frac{-\delta_1}{m}.$$

Since $\alpha_1 + \alpha_2 + \alpha_3$ is an algebraic integer, we must have that m divides δ_1. Similarly m divides δ_2 and δ_3. Since $\delta_1 + \delta_2 + \delta_3 = n - 1 = 3m$ we must have $\delta_1 = \delta_2 = \delta_3 = m$.

Assume α_1 is a noncyclotomic eigenvalue of b_1. Since δ_1 is an integral eigenvalue of b_1, the minimal polynomial $\mu_{\alpha_1}(x)$ of α_1 in $\mathbb{Q}[x]$ will be a divisor of $(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$. If the degree of $\mu_{\alpha_1}(x)$ is 1 or 2, it would follow that α_1 is rational or lies in a quadratic extension of $\mathbb{Q}$, which runs contrary to our assumption that it is not cyclotomic. So $(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$ is the minimal polynomial of α_1 in $\mathbb{Q}[x]$. This implies $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ is the splitting field of α_1 over $\mathbb{Q}$. Since α_1 is not cyclotomic, this has to be an extension of $\mathbb{Q}$ with $[\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{Q}] = 6$. Since $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) \subseteq K$ and $[K : \mathbb{Q}] = |G| = 6$, we must have $K = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$.

Now consider the left regular matrices of b_1, b_2, b_3 in the basis $\mathbf{B}$. For convenience we write these in this form:

$$b_1 = \begin{bmatrix} 0 & m & 0 & 0 \\ 1 & u & x_1 & x_4 \\ 0 & v & x_2 & x_5 \\ 0 & w & x_3 & x_6 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 & 0 & m & 0 \\ 0 & x_1 & u' & x_7 \\ 1 & x_2 & v' & x_8 \\ 0 & x_3 & w' & x_9 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 0 & 0 & 0 & m \\ 0 & x_4 & x_7 & u'' \\ 0 & x_5 & x_8 & v'' \\ 1 & x_6 & x_9 & w'' \end{bmatrix},$$

where the u, v, and w entries are determined by the row sum criterion. Applying the structure constant identities which define the left regular matrices produces one polynomial identity in the variables $x_1, \dots, x_9, m$ for each entry of the product $b_i b_j$ for $i, j \in \{1, 2, 3\}$.

Since $\mathbf{B}$ is pseudocyclic, we have three more trace identities. On the one hand, we have $tr(b_1) = u + x_2 + x_6 = (m - 1 - x_1 - x_4) + x_2 + x_6$, and on the other, $tr(b_1) = \delta_1 + \alpha_1 + \alpha_2 + \alpha_3 = m + \alpha_1 + \alpha_2 + \alpha_3 = m - 1$, so we can restrict our algebraic set by adding the polynomial $x_2 + x_6 - x_1 - x_4$ to our list. Similar identities coming from $tr(b_2) = m - 1$ and $tr(b_3) = m - 1$ show we can add the polynomials $x_1 + x_9 - x_2 - x_8$ and $x_4 + x_8 - x_6 - x_9$ to our list.

Next, we reduce our list of polynomials using all available linear substitutions and obtain

$$\begin{aligned} x_1 &= v = m - x_2 - x_5 & x_6 &= u'' = m - x_4 - x_7 \\ x_2 &= u' = m - x_1 - x_7 & x_8 &= w' = m - x_3 - x_9 \\ x_3 &= x_5 = x_7 & x_9 &= v'' = m - x_5 - x_8. \\ x_4 &= w = m - x_3 - x_6 \end{aligned}$$

This implies the matrix of b_1 is

$$b_1 = \begin{bmatrix} 0 & m & 0 & 0 \\ 1 & u & x_1 & x_4 \\ 0 & x_1 & x_2 & x_3 \\ 0 & x_4 & x_3 & x_6 \end{bmatrix},$$

so by the row sum criterion $x_1 + x_2 + x_3 = x_4 + x_3 + x_6 = m$, which implies $x_1 + x_2 = x_4 + x_6$. Since the identity we obtained by considering $tr(b_1)$ was $x_1 + x_4 = x_2 + x_6$, we must conclude that $x_4 = x_2$ and hence $x_6 = x_1$. Similarly, we see that the matrix of b_2 is

$$b_2 = \begin{bmatrix} 0 & 0 & m & 0 \\ 0 & x_1 & x_2 & x_3 \\ 1 & x_2 & v' & x_8 \\ 0 & x_3 & x_8 & x_9 \end{bmatrix},$$

therefore, $x_3 + x_8 + x_9 = m$ and we must have $x_1 + x_2 = x_8 + x_9$. Comparing this to the identity $x_1 + x_9 = x_2 + x_8$ obtained by considering $tr(b_2)$, we see that $x_9 = x_2$ and it then follows that $x_8 = x_1$.

Therefore, we have

$$b_1 = \begin{bmatrix} 0 & m & 0 & 0 \\ 1 & x_3 - 1 & x_1 & x_2 \\ 0 & x_1 & x_2 & x_3 \\ 0 & x_2 & x_3 & x_1 \end{bmatrix}, b_2 = \begin{bmatrix} 0 & 0 & m & 0 \\ 0 & x_1 & x_2 & x_3 \\ 1 & x_2 & x_3 - 1 & x_1 \\ 0 & x_3 & x_1 & x_2 \end{bmatrix}, \text{ and}$$

$$b_3 = \begin{bmatrix} 0 & 0 & 0 & m \\ 0 & x_2 & x_3 & x_1 \\ 0 & x_3 & x_1 & x_2 \\ 1 & x_1 & x_2 & x_3 - 1 \end{bmatrix}.$$

If we take Q to be the permutation matrix

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

then we have $Q^{-1} b_1 Q = b_2$, $Q^{-1} b_2 Q = b_3$, and $Q^{-1} b_3 Q = b_1$. It follows that the regular matrices of b_1, b_2, and b_3 have the same characteristic polynomial, and that the Galois group G has a nontrivial central element of order 3 that permutes the corresponding columns in the character table. But this is contrary to G being isomorphic to S_3. We conclude that for symmetric SITAwIMs of rank 4, the eigenvalues of basis elements must be cyclotomic. □

Corollary 3.3. *All association schemes of rank* 4 *have cyclotomic eigenvalues.*

4 Rank 5 SITAwIMs

For rank 5 SITAwIMs we have three involution types to consider: type $5S$, type $5A1$, and type $5A2$.

4.1 Type $5A2$

Theorem 4.1. *Every rank* 5 *SITAwIM* $(A, \mathbf{B})$ *with* $\mathbf{B} = \{b_0, b_1, b_1^*, b_3, b_3^*\}$ *has cyclotomic eigenvalues.*

Proof. Let $\mathbf{B} = \{b_0, b_1, b_1^*, b_3, b_3^*\}$ be the standard basis of a SITAwIM of rank 5, with character table P, splitting field K, and Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$. As the table algebra is not symmetric, we know by Corollary 2.5 that G has a central element of order 2. If the character table P has a noncyclotomic entry, then G must also be a 3- or 4-point transitive non-Abelian subgroup of $\mathrm{Sym}(\{\chi_1, \chi_2, \chi_3, \chi_4\})$, so the only possibility is for $G \simeq D_4$, the dihedral group of order 8. This implies the action of G on the last 4 rows of P is 4-transitive, and so we must have that the multiplicities m_1, m_2, m_3, and m_4 are all equal to the same positive integer m. So as in the symmetric rank 4 case, this implies the table algebra is homogeneous: $\delta_1 = \delta_2 = \delta_3 = \delta_4 = m$.

This implies our regular matrices of $\mathbf{B}$ will have this pattern:

$$b_1 = \begin{bmatrix} 0 & 0 & m & 0 & 0 \\ 1 & x_1 & m-1-x_1-x_5-x_9 & x_5 & x_9 \\ 0 & x_2 & m-x_2-x_6-x_{10} & x_6 & x_{10} \\ 0 & x_3 & m-x_3-x_7-x_{11} & x_7 & x_{11} \\ 0 & x_4 & m-x_4-x_8-x_{12} & x_8 & x_{12} \end{bmatrix},$$

$$b_1^* = \begin{bmatrix} 0 & m & 0 & 0 & 0 \\ 0 & m-x_2-x_6-x_{10} & x_2 & x_{10} & x_6 \\ 1 & m-1-x_1-x_5-x_9 & x_1 & x_9 & x_5 \\ 0 & m-x_4-x_8-x_{12} & x_4 & x_{12} & x_8 \\ 0 & m-x_3-x_7-x_{11} & x_3 & x_{11} & x_7 \end{bmatrix},$$

$$b_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & m \\ 0 & x_5 & x_{10} & x_{13} & m-x_5-x_{10}-x_{13} \\ 0 & x_6 & x_9 & x_{14} & m-x_6-x_9-x_{14} \\ 1 & x_7 & x_{12} & x_{15} & m-1-x_7-x_{12}-x_{15} \\ 0 & x_8 & x_{11} & x_{16} & m-x_8-x_{11}-x_{16} \end{bmatrix},$$

$$b_3^* = \begin{bmatrix} 0 & 0 & 0 & m & 0 \\ 0 & x_9 & x_6 & m-x_6-x_9-x_{14} & x_{14} \\ 0 & x_{10} & x_5 & m-x_5-x_{10}-x_{13} & x_{13} \\ 0 & x_{11} & x_8 & m-x_8-x_{11}-x_{16} & x_{16} \\ 1 & x_{12} & x_7 & m-1-x_7-x_{12}-x_{15} & x_{15} \end{bmatrix}.$$

In addition to the set of polynomial identities in the variables $x_1, \dots, x_{16}, m$ we obtain by applying the structure constant identities to these regular matrices, we again have the additional trace identities coming from $tr(b_1) = tr(b_3) = m-1$, which adds the polynomial identities

$$x_1 + x_7 + x_{12} + 1 = x_2 + x_6 + x_{10} \text{ and } x_5 + x_9 + x_{15} + 1 = x_8 + x_{11} + x_{16}$$

to our list. The result is a list of 13 distinct polynomial generators, up to sign, for an ideal of $\mathbb{Q}[x_1,\ldots,x_{16},m]$. The available linear substitutions are:

$$\begin{cases} x_{16} = 2m - 6x_1 - x_2 - 2, \\ x_{15} = x_1, \\ x_{14} = x_8 \;=\; 3x_1 + x_2 + x_3 + 1 - m, \\ x_{13} = x_{11} \;=\; 2x_1 - x_3 + 1, \\ x_{12} = x_9 \;=\; x_7 \;=\; x_5 \;=\; \dfrac{m-1}{2} - x_1, \\ x_{10} = x_3, \\ x_6 = x_4 \;=\; m - x_1 - x_2 - x_3, \end{cases} \tag{4.1}$$

so the reduced ideal now lies in $\mathbb{Q}[x_1, x_2, x_3, x_4, m]$. With the above substitutions, the regular matrices have this pattern:

$$b_1 = \begin{bmatrix} 0 & 0 & m & 0 & 0 \\ 1 & x_1 & x_1 & x_5 & x_5 \\ 0 & x_2 & x_1 & x_4 & x_3 \\ 0 & x_3 & x_5 & x_5 & x_{11} \\ 0 & x_4 & x_5 & x_8 & x_5 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & m \\ 0 & x_5 & x_3 & x_{11} & x_5 \\ 0 & x_4 & x_5 & x_8 & x_5 \\ 1 & x_5 & x_5 & x_1 & x_1 \\ 0 & x_8 & x_{11} & x_{16} & x_1 \end{bmatrix}.$$

When we substitute $x_{16} = x_2 + y$ for an extra variable y, then reduce using the identities in (4.1) and calculate the Gröbner basis for the resulting ideal with respect to an ordering of variables with y maximal, we find that y^2 is one of the elements of the basis.

Therefore, x_{16} must be equal to x_2 for all points in our algebraic set. Substituting x_2 for x_{16} in the first equation of (4.1) gives $x_2 = m - 3x_1 - 1$, substituting this into the last equation makes $x_4 = 2x_1 - x_3 + 1 = x_{11}$, and substituting $x_2 = m - 3x_1 - 1$ into the third equation gives us $x_8 = x_3$. Hence b_1 and b_3 have the same characteristic polynomial, so they have the same eigenvalues. Consequently, b_1^* and b_3^* have the same four eigenvalues as b_1. This implies the Galois group of the splitting field will act transitively on the last four columns of the character table, hence the Galois group will be Abelian. It follows that any rank 5 SITAwIM whose standard basis has two distinct asymmetric pairs must have cyclotomic eigenvalues. □

4.2 Type $5A1$

Theorem 4.2. *Every SITAwIM* $(A, \mathbf{B})$ *of involution type* $5A1$ *has cyclotomic eigenvalues.*

Proof. Let $\mathbf{B} = \{b_0, b_1, b_2, b_3, b_3^*\}$ be the basis of a SITAwIM of type $5A1$. By Corollary 2.5, complex conjugation will be realized by a central element of the Galois group G of the splitting field K of $\mathbb{Q}\mathbf{B}$. If the character table P has an entry which is not cyclotomic, then as in the type $5A2$ case, we must have that $G \simeq D_4$ and acts 4-transitively on $\{\chi_1, \chi_2, \chi_3, \chi_4\}$. It follows that our SITAwIM $(A, \mathbf{B})$ is both pseudocyclic and homogeneous.

This implies that the pattern for our regular matrices in this case will be:

$$b_1 = \begin{bmatrix} 0 & m & 0 & 0 & 0 \\ 1 & m-1-x_1-2x_5 & x_1 & x_5 & x_5 \\ 0 & m-x_2-2x_6 & x_2 & x_6 & x_6 \\ 0 & m-x_3-x_7-x_8 & x_3 & x_7 & x_8 \\ 0 & m-x_4-x_8-x_7 & x_4 & x_8 & x_7 \end{bmatrix},$$

$$b_2 = \begin{bmatrix} 0 & 0 & m & 0 & 0 \\ 0 & x_1 & m-x_1-2x_9 & x_9 & x_9 \\ 1 & x_2 & m-1-x_2-2x_{10} & x_{10} & x_{10} \\ 0 & x_3 & m-x_3-x_{11}-x_{12} & x_{11} & x_{12} \\ 0 & x_4 & m-x_4-x_{11}-x_{12} & x_{12} & x_{11} \end{bmatrix},$$

$$b_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & m \\ 0 & x_5 & x_9 & x_{13} & m-x_5-x_9-x_{13} \\ 0 & x_6 & x_{10} & x_{14} & m-x_6-x_{10}-x_{14} \\ 1 & x_7 & x_{11} & x_{15} & m-1-x_7-x_{11}-x_{15} \\ 0 & x_8 & x_{12} & x_{16} & m-x_8-x_{12}-x_{16} \end{bmatrix},$$

$$b_3^* = \begin{bmatrix} 0 & 0 & 0 & m & 0 \\ 0 & x_5 & x_9 & m-x_5-x_9-x_{13} & x_{13} \\ 0 & x_6 & x_{10} & m-x_6-x_{10}-x_{14} & x_{14} \\ 0 & x_8 & x_{12} & m-x_8-x_{12}-x_{16} & x_{16} \\ 1 & x_7 & x_{11} & m-1-x_7-x_{11}-x_{15} & x_{15} \end{bmatrix}.$$

In addition to the polynomial identities obtained by applying the structure constant identities to these regular matrices, we again have three extra trace identities coming from $tr(b_1) = tr(b_2) = tr(b_3) = m-1$:

$x_1 + 2x_5 = x_2 + 2x_7$, $x_1 + 2x_{11} = x_2 + 2x_{10}$, and $x_5 + x_{10} + x_{15} + 1 = x_8 + x_{12} + x_{16}$.

In addition to these, the other available linear substitutions, including those that become linear after we cancel $m > 0$, are:

$$\begin{aligned} x_{16} &= m - x_8 - x_{12} - x_{15} \\ x_{14} &= x_{12} = m - x_6 - x_{10} - x_{11} \\ x_{13} &= x_8 = m - x_5 - x_6 - x_7 \\ x_9 &= x_6 = x_4 = x_3 = \frac{m}{2} - x_1 \\ x_2 &= x_1. \end{aligned}$$

Since we have the identity $x_3 = \frac{m}{2} - x_1$, integrality of x_3 and x_1 implies $m = 2k$ is even. Making as many substitutions as possible, we can leave ourselves with a set of 11 nonlinear polynomials in $\mathbb{Q}[x_1, x_5, x_{15}, m]$. Using a computer, we calculate the Gröbner basis of the ideal generated by these 11 polynomials, with m and x_{15} of highest weight. If we set $y = x_{15}$, the first polynomial in this Gröbner basis is the following element of $\mathbb{Q}[m, y]$:

$$\begin{aligned} W(y,m) = &\frac{1}{5184}(5184y^4 - 5184y^3m + 1944y^2m^2 - 324ym^3 + \frac{81}{4}m^4 + 7776y^3 \\ &- 6160y^2m + 1622ym^2 - 142m^3 + 4292y^2 - 2392ym + 330m^2 \\ &+ 1032y - 304m + 91). \end{aligned}$$

This means $5184 \cdot W(y,m)$ is an integer polynomial that must have a nonnegative solution with y an integer and m an even integer. But when we substitute $m = 2k$, $5184 \cdot W(y, 2k)$ has the form $2Q(y,k) + 1$ for some polynomial $Q(y,k) \in \mathbb{Z}[y,k]$, and it is impossible for $Q(y,k) = -\frac{1}{2}$ to have an integral solution. This implies there are no pseudocyclic SITAwIMs of involution type $5A1$. In particular this means we can conclude that all rank 5 SITAwIMs whose standard basis has exactly one asymmetric pair will have cyclotomic eigenvalues. □

Corollary 4.3. *The cyclotomic eigenvalue property holds for every nonsymmetric rank* 5 *association scheme.*

4.3 Type $5S$

If $(A, \mathbf{B})$ is a symmetric rank 5 SITAwIM with noncyclotomic eigenvalues, the action of the Galois group $G = Gal(K/\mathbb{Q})$ of the splitting field K on the irreducible characters of A will either be 3- or 4-point transitive. We begin with the 4-point transitive case.

4.3.1 Type $5S$ with 4-point transitive Galois group

Again in this case we deduce that $(A, \mathbf{B})$ is pseudocyclic and homogeneous from G being 4-point transitive. In addition to the polynomial identities obtained by applying the structure constant identities to our regular matrices, we also have four trace identities coming from $tr(b_1) = tr(b_2) = tr(b_3) = tr(b_4) = m - 1$. Altogether our initial list consists of 124 polynomials in 25 variables. By applying all available linear substitutions, we can reduce to a list of 21 polynomials in $\mathbb{Q}[x_1, x_2, x_3, x_5, x_7, x_{14}, x_{15}, m]$. Along the way our first trace identity reduces to

$$2(x_3 + x_5 + x_{14} - x_{23}) = m,$$

so we can conclude that m must be even. The Gröbner basis of this ideal generated by these 21 polynomials can be calculated in a few hours on our desktop implementation of GAP [6], but is too complicated for any easy interpretation. Instead, reducing to a basis of the rational span of these 21 polynomials leaves us with just 6 polynomials. Using these, we run a search for suitable nonnegative integer solutions, letting m run over increasing even integers and x_1, x_2, and x_3 over the sets of three nonnegative integers that sum to at most m. With these specifications, a Gröbner basis calculation solves for the possible values of the four remaining variables efficiently. When a suitable nonnegative integer solution is identified, we substitute its values back into our regular matrices and compute the factors of their characteristic polynomials. Noncyclotomic eigenvalues are detected by applying GAP's `GaloisType` command [6] to irreducible factors of degree 3 or 4. Our searches have found there is only one example with noncyclotomic eigenvalues with $m \leq 62$. We found more examples by carrying out a narrow search with the values of x_1, x_2, and x_3 set to within a 10% error of $\frac{m}{4}$ for $64 \leq m \leq 250$. Up to permutation equivalence, we have found *six* symmetric rank 5 SITAwIMs with 4-point transitive Galois group that have noncyclotomic eigenvalues. In all of these cases the Galois group is isomorphic to S_4. (Here we give the factorizations of the characteristic polynomials of their basis elements, from these it is possible to recover the character table P numerically, and from that their other parameters.)

Noncyclotomic SITAwIMs of type $5S$: 4-point transitive examples

$$\begin{aligned}
n = 249:\quad & (x-62)(x^4+x^3-93x^2-57x+12),\\
& (x-62)(x^4+x^3-93x^2-306x+261),\\
& (x-62)(x^4+x^3-93x^2-306x-237),\\
& (x-62)(x^4+x^3-93x^2-140x+925)
\end{aligned}$$

$$\begin{aligned}
n = 321:\quad & (x-80)(x^4+x^3-120x^2-341x-242),\\
& (x-80)(x^4+x^3-120x^2-20x+2968),\\
& (x-80)(x^4+x^3-120x^2-301x-400),\\
& (x-80)(x^4+x^3-120x^2+301x+1042)
\end{aligned}$$

$$\begin{aligned}
n = 473:\quad & (x-118)(x^4+x^3-177x^2-266x+279),\\
& (x-118)(x^4+x^3-177x^2-266x+3117),\\
& (x-118)(x^4+x^3-177x^2+680x-667),\\
& (x-118)(x^4+x^3-177x^2+207x+4536)
\end{aligned}$$

$$\begin{aligned}
n = 633:\quad & (x-158)(x^4+x^3-237x^2-356x+10897),\\
& (x-158)(x^4+x^3-237x^2-145x+11108),\\
& (x-158)(x^4+x^3-237x^2+1754x-3451),\\
& (x-158)(x^4+x^3-237x^2-778x+5411)
\end{aligned}$$

$$\begin{aligned}
n = 785:\quad & (x-196)(x^4+x^3-294x^2-1619x-1524),\\
& (x-196)(x^4+x^3-294x^2-49x+20456),\\
& (x-196)(x^4+x^3-294x^2+1521x+3186),\\
& (x-196)(x^4+x^3-294x^2+736x+7896)
\end{aligned}$$

$$\begin{aligned}
n = 993:\quad & (x-248)(x^4+x^3-372x^2+931x-128),\\
& (x-248)(x^4+x^3-372x^2+931x+9802),\\
& (x-248)(x^4+x^3-372x^2+2917x-6086),\\
& (x-248)(x^4+x^3-372x^2+1924x+7816).
\end{aligned}$$

For all of these examples, the noncyclotomic character table demands a certain algebraic structure of the Wedderburn decomposition of $\mathbb{Q}\mathbf{B}$. If the character table of $(A, \mathbf{B})$ is $P = (P_{i,j})_{i,j=0}^4 = (\chi_i(b_j))_{i,j=0}^4$, then

- for all $j \in \{1,2,3,4\}$, the four 4-dimensional primitive extension fields $\mathbb{Q}(P_{1,j})$, $\mathbb{Q}(P_{2,j})$, $\mathbb{Q}(P_{3,j})$, and $\mathbb{Q}(P_{4,j})$ are pairwise distinct and Galois conjugate over $\mathbb{Q}$;
- for all $i \in \{1,2,3,4\}$, the four primitive extension fields $\mathbb{Q}(P_{i,1})$, $\mathbb{Q}(P_{i,2})$, $\mathbb{Q}(P_{i,3})$, and $\mathbb{Q}(P_{i,4})$ are equal; and
- for all $i,j \in \{1,2,3,4\}$, $\mathbb{Q}\mathbf{B} \simeq \mathbb{Q} \oplus \mathbb{Q}(P_{i,j})$ as $\mathbb{Q}$-algebras.

Another interesting fact is that the field of Krein parameters will be equal to the splitting field K, this is the minimal field of realization for the dual intersection matrices.

In the last section we explain how to verify that these six SITAwIMs satisfy all the known feasibility conditions for being an association scheme. The first one is the smallest rank 5 example with 4-point transitive Galois group, we present its parameters in detail here.

Theorem 4.4. *The smallest symmetric rank* 5 *SITAwIM with noncyclotomic eigenvalues for which the Galois group of the splitting field is* 4*-point transitive has order* 249*. Up to permutation equivalence, its standard basis is given by:*

$$\mathbf{B} = \left\{ b_0, b_1 = \begin{bmatrix} 0 & 62 & 0 & 0 & 0 \\ 1 & 15 & 14 & 12 & 20 \\ 0 & 14 & 16 & 17 & 15 \\ 0 & 12 & 17 & 18 & 15 \\ 0 & 20 & 15 & 15 & 12 \end{bmatrix}, b_2 = \begin{bmatrix} 0 & 0 & 62 & 0 & 0 \\ 0 & 14 & 16 & 17 & 15 \\ 1 & 16 & 18 & 16 & 11 \\ 0 & 17 & 16 & 11 & 18 \\ 0 & 15 & 11 & 18 & 18 \end{bmatrix}, \right.$$

$$\left. b_3 = \begin{bmatrix} 0 & 0 & 0 & 62 & 0 \\ 0 & 12 & 17 & 18 & 15 \\ 0 & 17 & 16 & 11 & 18 \\ 1 & 18 & 11 & 18 & 14 \\ 0 & 15 & 18 & 14 & 15 \end{bmatrix}, b_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 62 \\ 0 & 20 & 15 & 15 & 12 \\ 0 & 15 & 11 & 18 & 18 \\ 0 & 15 & 18 & 14 & 15 \\ 1 & 12 & 18 & 15 & 16 \end{bmatrix} \right\}.$$

The character table of $(A, \mathbf{B})$ *is shown below. The roots of the degree* 4 *polynomials above have been approximated to six significant digits using* `Wolfram|Alpha` [17].

$$P = \begin{bmatrix} 1 & 62 & 62 & 62 & 62 \\ 1 & 9.45706 & -4.83450 & -8.21429 & 2.59173 \\ 1 & 0.165779 & -7.32957 & 10.6401 & -4.47634 \\ 1 & -0.777430 & 10.45989 & -2.18457 & -8.49789 \\ 1 & -9.84541 & 0.704180 & -1.24127 & 9.38250 \end{bmatrix}.$$

Since this algebra is self-dual, the second eigenmatrix is obtained by setting $Q_{i,j} = P_{j,i}$ *for* $i = 1, 2, 3, 4$ *and leaving the first row and column alone. The nontrivial dual intersection matrices are as follows, with irrational entries approximated to six significant digits:*

$$L_1^* = \begin{bmatrix} 0 & 62 & 0 & 0 & 0 \\ 1 & 16.2247 & 17.5718 & 15.3191 & 11.8843 \\ 0 & 17.5718 & 10.8695 & 18.0841 & 15.4745 \\ 0 & 15.3191 & 18.0841 & 13.9307 & 14.6661 \\ 0 & 11.8843 & 15.4745 & 14.6661 & 19.9751 \end{bmatrix},$$

$$L_2^* = \begin{bmatrix} 0 & 0 & 62 & 0 & 0 \\ 0 & 17.5718 & 10.8695 & 18.0841 & 15.4745 \\ 1 & 10.8695 & 18.3339 & 16.0173 & 15.7793 \\ 0 & 18.0841 & 16.0173 & 11.1233 & 16.7753 \\ 0 & 15.4745 & 15.7793 & 16.7753 & 13.9710 \end{bmatrix},$$

$$L_3^* = \begin{bmatrix} 0 & 0 & 0 & 62 & 0 \\ 0 & 15.3191 & 18.0841 & 13.9307 & 14.6661 \\ 0 & 18.0841 & 16.0173 & 11.1233 & 16.7753 \\ 1 & 13.9307 & 11.1233 & 17.5255 & 18.4206 \\ 0 & 14.6661 & 16.7753 & 18.4206 & 12.1381 \end{bmatrix},$$

$$L_4^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 62 \\ 0 & 11.8843 & 15.4745 & 14.6661 & 19.9751 \\ 0 & 15.4745 & 15.7793 & 16.7753 & 13.9710 \\ 0 & 14.6661 & 16.7753 & 18.4206 & 12.1381 \\ 1 & 19.9751 & 13.9710 & 12.1381 & 14.9159 \end{bmatrix}.$$

Remark 4.5. One might ask if there are metric association schemes of rank 5 with noncyclotomic splitting fields that have 4-point transitive Galois groups. With our method, this can be resolved by setting $x_4, x_5, x_9, x_{10}, x_{17} = 0$, calculating the Gröbner basis, and using known intersection array restrictions to bound tridiagonal entries of b_1. This approach allows one to make the same conclusion as Blau and Xu obtain for pseudocyclic metric association schemes in general, that the intersection array has to be $[2, 1, 1, 1; 1, 1, 1, 1]$ [18, Theorem 5.4]. But the splitting field of this association scheme has a 3-point transitive abelian Galois group, so the answer is no.

4.3.2 Type $5S$ with 3-point transitive Galois group

The other possibility for a symmetric SITAwIM of rank 5 with noncyclotomic eigenvalues is the case where the Galois group of the splitting field is non-abelian and acts 3-point transitively, so must be isomorphic to S_3. Let $(A, \mathbf{B})$ be such a SITAwIM, with splitting field K and Galois group G, and suppose the orbits of G on the irreducible characters of A are $\{\chi_0\}$, $\{\chi_1\}$, and $\{\chi_2, \chi_3, \chi_4\}$. In this situation the table algebra is not necessarily pseudocyclic, nor does it have to be homogeneous, so we do not have as many linear substitutions available to reduce our algebraic set initially. Instead, to find the SITAwIMs of a given order, we can first make a list of possible rationalized character tables for SITAwIMs of that order. The rationalized character table is an *integer* matrix with columns indexed by $\mathbf{B}$ and rows are indexed by the sums of irreducible characters of A up to Galois conjugacy over $\mathbb{Q}$. In our 3-point transitive case, it takes this form:

	b_0	b_1	b_2	b_3	b_4	multiplicities
χ_0	1	δ_1	δ_2	δ_3	δ_4	1
χ_1	1	a_1	a_2	a_3	a_4	m_1
$\chi_2 + \chi_3 + \chi_4$	3	t_1	t_2	t_3	t_4	$3m_2$

The rows and columns of the rationalized character table satisfy orthogonality relations induced by those of the usual character table. In our case the orthogonality relations give the following identities:

- $\delta_1 + \delta_2 + \delta_3 + \delta_4 = m_1 + 3m_2 = n - 1$;
- $a_1 + a_2 + a_3 + a_4 = -1$;
- $t_1 + t_2 + t_3 + t_4 = -3$;
- $1 + \frac{a_1^2}{\delta_1} + \frac{a_2^2}{\delta_2} + \frac{a_3^2}{\delta_3} + \frac{a_4^2}{\delta_4} = \frac{n}{m_1}$;
- $3 + \frac{a_1 t_1}{\delta_1} + \frac{a_2 t_2}{\delta_2} + \frac{a_3 t_3}{\delta_3} + \frac{a_4 t_4}{\delta_4} = 0$;
- $\delta_1 + m_1 a_1 + m_2 t_1 = 0$;
- $\delta_2 + m_1 a_2 + m_2 t_2 = 0$;
- $\delta_3 + m_1 a_3 + m_2 t_3 = 0$; and
- $\delta_4 + m_1 a_4 + m_2 t_4 = 0$.

These identities are subject to the restrictions $1 \leq m_1, m_2, \delta_1, \delta_2, \delta_3, \delta_4$, and $-\delta_i \leq a_i \leq \delta_i$ for $i = 1, 2, 3, 4$, and $-3\delta_i \leq t_i \leq 3\delta_i$ for $i = 1, 2, 3, 4$. So a straightforward search will produce all the rationalized character tables possible whose associated SITAwIM would have degree n.

Given a rationalized character table, we get four linear trace identities $tr(b_i) = \delta_i + a_i + t_i$, $i = 1, 2, 3, 4$ that can be added to our list of polynomial generators. This helps us to reduce our search space enough to allow the search and Gröbner basis calculations techniques to uncover suitable nonnegative solutions to the system and produce regular matrices for a SITAwIM with this rationalized character table. This approach has two computational barriers, which have limited our ability to guarantee a complete account only for orders up to 100. First, since we must consider every possibility for m_1 and m_2 with $1 + m_1 + 3m_2 = n$, the number of possible rational character tables of a given order can be very large and time-consuming to generate, and for almost all of these we find no SITAwIM. Secondly, the values of the x_i's are not as limited as they are in the homogeneous case, so when the minimum δ_i is large, the search space for all the values of x_1, x_2, and x_3 we need to check grows in size exponentially.

Our complete search for orders up to 100 found six examples. Their multiplicities and factorizations of the characteristic polynomials of their basis elements are as follows:

Noncyclotomic SITAwIMs of type $5S$: 3-point transitive examples

$n = 35:$ $m_1 = 4, m_2 = 10, \mu_{b_i}(x) = (x-4)(x+1)(x^3-6x+2), (x-6)^2(x+1)^3,$
$(x-12)(x+3)(x^3-12x-2), (x-12)(x+3)(x^3-12x+12);$

$n = 45:$ $m_1 = 8, m_2 = 12, \mu_{b_i}(x) = (x-4)^2(x+1)^3, (x-8)(x+1)(x^3-12x+14),$
$(x-8)(x+1)(x^3-12x+4), (x-24)(x+3)(x^3-18x+18);$

$n = 76:$ $m_1 = 18, m_2 = 19, \mu_{b_i}(x) = (x-3)^2(x+1)^3, (x-18)(x+1)(x^3-27x-18),$
$(x-18)(x+1)(x^3-27x-42), (x-36)(x+2)(x^3-36x-48);$

$n = 88^a:$ $m_1 = 66, m_2 = 7, \mu_{b_i}(x) = (x-3)^4(x+1), (x-14)(x)(x^3+2x^2-72x-16),$
$(x-35)(x)(x^3+5x^2-120x-360), (x-35)(x)(x^3+5x^2-120x+80);$

$n = 88^b:$ $m_1 = 66, m_2 = 7, \mu_{b_i}(x) = (x-3)^4(x+1), (x-21)(x)(x^3+3x^2-96x-384),$
$(x-21)(x)(x^3+3x^2-96x-472), (x-42)(x)(x^3+6x^2-120x-784);$ and

$n = 93:$ $m_1 = 2, m_2 = 30, \mu_{b_i}(x) = (x-12)(x+6)(x^3-15x+2), (x-20)(x+10)$
$(x^3-21x-16), (x-30)(x+15)(x^3-24x+8), (x-30)^2(x+1)^3.$

Narrow searches of orders 101 to 250, the first with $\delta_1 \leq 4$ and at least two of δ_2, δ_3, and δ_4 equal, and the second with $\delta_1 \leq 12$, $a_1 = k_1$, and at least two of δ_2, δ_3, and δ_4 equal produced a few more examples:

$n = 116:$ $m_1 = 58, m_2 = 19, \mu_{b_i}(x) = (x-1)^4(x+1), (x-19)(x)(x^3+x^2-48x+72)$
$(x-19)(x)(x^3+x^2-48x-44), (x-76)(x)(x^3+4x^2-72x-32);$

$n = 129:$ $m_1 = 86, m_2 = 14, \mu_{b_i}(x) = (x-2)^4(x+1), (x-28)(x)(x^3+2x^2-99x+150)$
$(x-28)(x)(x^3+2x^2-99x-108), (x-70)(x)(x^3+5x^2-135x-75);$

$n = 165:$ $m_1 = 32, m_2 = 44, \mu_{b_i}(x) = (x-4)^2(x+1)^3, (x-32)(x+1)(x^3-48x-32),$
$(x-32)(x+1)(x^3-48x-112), (x-96)(x+3)(x^3-72x-144);$

$n = 189:$ $m_1 = 20, m_2 = 56, \mu_{b_i}(x) = (x-8)^2(x+1)^3, (x-20)(x+1)(x^3-30x-20),$
$(x-80)(x+4)(x^3-75x+70), (x-80)(x+4)(x^3-75x-200);$

$n = 190:$ $m_1 = 18, m_2 = 57, \mu_{b_i}(x) = (x-9)^2(x+1)^3, (x-36)(x+2)(x^3-48x+32),$
$(x-36)(x+2)(x^3-48x+112), (x-108)(x+6)(x^3-72x+144);$

$n = 217:$ $m_1 = 30, m_2 = 62, \mu_{b_i}(x) = (x-6)^2(x+1)^3, (x-60)(x+2)(x^3-75x-100),$
$(x-60)(x+2)(x^3-75x-170), (x-90)(x+3)(x^3-90x-180);$

$n = 231^a:$ $m_1 = 32, m_2 = 66, \mu_{b_i}(x) = (x-6)^2(x+1)^3, (x-32)(x+1)(x^3-48x-96),$
$(x-96)(x+3)(x^3-96x-352), (x-96)(x+3)(x^3-96x-128);$

$n = 231^b:$ $m_1 = 32, m_2 = 66, \mu_{b_i}(x) = (x-6)^2(x+1)^3, (x-32)(x+1)(x^3-48x+16),$
$(x-96)(x+3)(x^3-96x-128), (x-96)(x+3)(x^3-96x+208).$

Example 4.6. The smallest noncyclotomic symmetric rank 5 SITAwIM with order $n = 35$ has regular matrices b_0,

$$b_1 = \begin{bmatrix} 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}, b_2 = \begin{bmatrix} 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 \\ 1 & 0 & 5 & 0 & 0 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 3 & 2 \end{bmatrix}, b_3 = \begin{bmatrix} 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 3 & 6 & 3 \\ 0 & 2 & 0 & 4 & 6 \\ 1 & 2 & 2 & 4 & 3 \\ 0 & 1 & 3 & 3 & 5 \end{bmatrix}, \text{ and}$$

$$b_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 12 \\ 0 & 3 & 3 & 3 & 3 \\ 0 & 2 & 0 & 6 & 4 \\ 0 & 1 & 3 & 3 & 5 \\ 1 & 1 & 2 & 5 & 3 \end{bmatrix}.$$

Its first and second eigenmatrices (with irrationals approximated to six significant digits) are as follows:

$$P = \begin{bmatrix} 1 & 4 & 6 & 12 & 12 \\ 1 & -1 & 6 & -3 & -3 \\ 1 & -2.60168 & -1 & -0.167055 & 2.768734 \\ 1 & 0.339877 & -1 & 3.54461 & -3.88448 \\ 1 & 2.26180 & -1 & -3.37755 & 1.11575 \end{bmatrix}, \text{ and}$$

$$Q = \begin{bmatrix} 1 & 4 & 10 & 10 & 10 \\ 1 & -1 & -6.50420 & 0.849692 & 5.65451 \\ 1 & 4 & -5/3 & -5/3 & -5/3 \\ 1 & -1 & -0.139212 & 2.95384 & -2.81463 \\ 1 & -1 & 2.30728 & -3.23707 & 0.929791 \end{bmatrix}.$$

Its dual intersection matrices, again with irrational entries approximated to six significant digits, are: $L_0^* = b_0$,

$$L_1^* = \begin{bmatrix} 0 & 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 5/3 & 5/3 \\ 0 & 0 & 5/3 & 2/3 & 5/3 \\ 0 & 0 & 5/3 & 5/3 & 2/3 \end{bmatrix},$$

$$L_2^* = \begin{bmatrix} 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 5/3 & 25/6 & 25/6 \\ 1 & 2/3 & 0.0541562 & 2.59972 & 5.67949 \\ 0 & 5/3 & 2.59972 & 3.51139 & 20/9 \\ 0 & 5/3 & 5.67946 & 20/9 & 0.431651 \end{bmatrix},$$

$$L_3^* = \begin{bmatrix} 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 25/6 & 5/3 & 25/6 \\ 0 & 5/3 & 2.59972 & 3.51139 & 20/9 \\ 1 & 2/3 & 3.51139 & 2.50545 & 2.31644 \\ 0 & 5/3 & 20/9 & 2.31169 & 3.79463 \end{bmatrix},$$

$$L_4^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 25/6 & 25/6 & 5/3 \\ 0 & 5/3 & 5.67946 & 20/9 & 0.431651 \\ 0 & 5/3 & 20/9 & 2.31649 & 3.79463 \\ 1 & 2/3 & 0.431651 & 3.79463 & 4.10706 \end{bmatrix}.$$

5 Checking feasibility

In this section we review the feasibility checks we have applied to the parameters of the noncyclotomic symmetric rank 5 SITAwIMs identified in the previous section. The parameters include the regular (a.k.a. intersection) matrices b_i ($i \in \{0, 1, \ldots, r-1\}$), the character table (first eigenmatrix) P, the dual character table (second eigenmatrix) Q, and the dual intersection matrices (Krein parameters) $L_i^* = (\kappa_{ijk})_{k,j=0}^{r-1}$ $(i \in \{0, 1, \ldots, r-1\})$. For commutative association schemes, we consider these to be equivalent since knowledge of any one of these determines the other.

We have tested our examples on the following feasibility conditions, which apply to general symmetric association schemes:

- the handshaking lemma: for $i, j \in \{1, \ldots, r-1\}$, if $i \neq j$, then $(b_i)_{i,j}k_j$ must be even (see [11, Lemma 7]);
- realizability of all closed subsets and quotients;
- the triangle count condition: for $j = 1, \ldots, r-1$, $\frac{1}{6}\sum_{i=0}^{r-1} m_i P_{i,j}^3 = t \in \mathbb{N}$;
- the absolute bound condition: for $i \in \{0, \ldots, r-1\}$

$$\sum_{k; q_{ijk} \neq 0} m_k \leq \begin{cases} m_i m_j & i \neq j \\ \binom{m_i+1}{2} & i = j; \end{cases}$$

- nonnegativity of Krein parameters: $(L_i^*)_{j,k} \geq 0$ for $i, j, k \in \{0, \ldots, r-1\}$; and
- Martin and Kodalen's Gegenbauer polynomial criterion (see [13, Theorem 3.7 and Corollary 3.8]).

We are aware of one more feasibility condition for symmetric association schemes, the *forbidden quadruple* condition described in [7, Corollary 4.2]. Our 4-point transitive examples do not have any nontrivial Krein parameters equal to zero, so they satisfy this condition vacuously. This is not the case for our 3-point transitive examples, to date these have not been tested for this condition.

We have ordered these feasibility conditions according to the ease we are able to check them. Since our algorithms require the multiplicities as part of the input and produce the intersection matrices, we have to compute P, then Q, then the dual intersection matrices in order from there. As our objective is only to report the examples that pass all conditions, once an example fails one of our conditions below it is removed and its status for subsequent conditions is not reported.

We will indicate our examples from the previous section by Galois group action and order: $3pt35$, $3pt45$, etc. Recall that $3pt35$ means the 3-point transitive example of order 35.

5.1 Handshaking lemma condition:

Only five of our examples have nontrivial basis elements of odd degree, of these five, three of them fail the handshaking lemma condition: $3pt88^a$, $3pt88^b$, and $3pt116$. $3pt76$ and $3pt190$ pass despite having a nontrivial basis element of odd degree.

5.2 Realizability of closed subsets and quotients:

All of our 4-point transitive examples are primitive, so there are no closed subsets or quotients to consider. On the other hand, all of the remaining 3-point transitive examples have a unique nontrivial closed subset of rank 2. For all but one of these, the quotient also has rank 2. The exception is $3pt129$, for which the quotient has rank 4. Since this quotient table algebra has an element of non-integral degree, it is not realizable as an association scheme.

5.3 Triangle count condition.

All of our examples pass.

5.4 Absolute bound condition.

All of our 4-point transitive examples pass. We can see from the multiplicities that $3pt45$, $3pt76$, and $3pt165$ will pass. $3pt35$ could potentially fail for $i = j = 1$ but passes because $\kappa_{1,1,k} = 0$ for $k = 2, 3, 4$. $3pt93$ and $3pt129$ also pass because enough nontrivial Krein parameters are 0.

5.5 Nonnegative Krein parameter condition.

For all of our 3- and 4-point transitive examples, we have calculated the dual intersection matrices and found them to be nonnegative.

5.6 Gegenbauer polynomial condition.

We check that $G_\ell^{m_i}(\frac{1}{m_i}L_i^*)$ is a nonnegative matrix for all $\ell \geq 1$ and $i = 1, \dots, 4$ using the approach of [13, §3.3].

We illustrate the process of checking this condition with $3pt35$. In the case $m_1 = 4$, it is not possible to find an ℓ^* satisfying the conditions of [13, Corollary 3.16]. However, L_1^* is a block matrix, and the upper left 2×2 block $\begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$ is the dual intersection matrix corresponding to the association scheme generated by the complete graph of order 5, in which it also occurs with nontrivial multiplicity 4. It follows that the first column of $G_\ell^{m_i}(\frac{1}{m_i}L_i^*)$ will always be nonnegative for all $\ell \geq 1$, so the result follows by [13, Corollary 3.8] and the remark following it.

In the cases $m_2 = m_3 = m_4 = 10$, we find that the minimum ℓ^* required for [13, Corollary 3.16] is $\ell^* = 6$, and we can check that $G_\ell^{10}(\frac{1}{10}L_i^*)$ has nonnegative entries for all $\ell \in \{1, \dots, 7\}$ and all $i = 2, 3, 4$. So, $3pt35$ passes all the feasibility conditions, with the possible exception of the forbidden quadruple condition.

In all of the remaining 3-transitive examples, $\mathbf{B}^*$ contains a rank 2 closed subset of order $m_i + 1$ for one i. So, a similar argument as in the $3pt35$ case applies for this m_i. For the other m_i a suitable ℓ^* can be found. After evaluating the appropriate Gegenbauer polynomials at $\frac{1}{m_i}L_i^*$, we found the result to be a nonnegative matrix.

All of our 4-point transitive examples pass the Gegenbauer polynomial test. In each case we have found a value of ℓ^* and shown all of the required evaluations result in nonnegative matrices.

In summary, we have verified that the six 4-point transitive examples pass all of the feasibility conditions, and ten of the 3-point transitive examples pass them: $3pt35$, $3pt45$, $3pt76$, $3pt93$, $3pt165$, $3pt189$, $3pt190$, $3pt217$, $3pt231^a$, and $3pt231^b$. Note that by the partial classification of association schemes of order 35 and rank 5 in [8], we know $3pt35$ cannot be realized.

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Comparing Wiener, Szeged and revised Szeged index on cactus graphs

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Abstract

We show that on cactus graphs the Szeged index is bounded above by twice the Wiener index. For the revised Szeged index the situation is reversed if the graph class is further restricted. Namely, if all blocks of a cactus graph are cycles, then its revised Szeged index is bounded below by twice its Wiener index. Additionally, we show that these bounds are sharp and examine the cases of equality. Along the way, we provide a formulation of the revised Szeged index as a sum over vertices, which proves very helpful, and may be interesting in other contexts.

Keywords: Wiener index, (Revised) Szeged index, cactus graphs.

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1 Introduction

Presumably the first topological graph index, the Wiener index, was invented in 1947 by the chemist Wiener [25], and is used to correlate physicochemical properties to the structure of chemical compounds [3, 11]. Since then it was and still is thoroughly studied, see e.g. [1, 4, 5, 6, 7, 17] for only some of the latest results. Over time many more topological graph indices were devised and investigated. One such topological graph index is the Szeged index that came up as an extension of a formula for the Wiener index of trees. It was first introduced in [10] without proper name. By its construction it has meaningful connections to the Wiener index. However, Randić found that the Szeged index is lacking something for chemical applications in comparison to the Wiener index, and thus introduced in [21] a slightly adapted variant of the Szeged index, the later so-called revised Szeged index. It produces better correlations in chemistry than the normal Szeged index [21] and both Szeged indices combined can be used to provide a measure of bipartivity of graphs [20].

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It is rather easy to see that the Wiener index and the (revised) Szeged index coincide on trees. Furthermore, in 1994 some conjectures about the relation of the Wiener and the Szeged index on connected graphs were made by Dobrynin and Gutman [8, 10]. A year later already Dobrynin and Gutman proved that the Wiener index and the Szeged index are equal if and only if every block of the graph is complete [9]. Another year later Klavžar et al. showed that the Szeged index is at least as big as the Wiener index [16]. Since then many more authors investigated the relation of the Wiener and the Szeged index, see [2, 13, 14, 19] and references therein. This research has been extended to the revised Szeged index [15, 29, 30] and to certain graph classes [12, 18, 24], with the most recent work on cactus graphs dating from this year. For further current research in the context of comparing graph indices with the Wiener index, we refer the interested reader to [26, 27, 28].

In this paper, we want to show new relations between Wiener, Szeged and revised Szeged index for the special case of cactus graphs. Namely, we prove that the Szeged index is bounded above by twice the Wiener index. In case of the revised Szeged index the situation is more complex. For bipartite cacti the revised Szeged is equal to the Szeged index, but if we limit the class of cactus graphs to those that have only cycles as blocks, we can reverse the above statement. That is, the revised Szeged index is bounded below by twice the Wiener index. Additionally, we show that these bounds are sharp and examine the cases of equality. Along the way, we provide a formulation of the revised Szeged index as a sum over vertices, which proves very helpful, and may be interesting in other contexts.

The paper is organized as follows. In Section 2, we first introduce the main definitions and directly afterwards show how the revised Szeged index can be written as a sum over vertices (Theorem 2.1). Then we introduce some auxiliary results needed in the following sections. The relation of the Szeged index and the Wiener index on cactus graphs is the main topic of Section 3. We show that the Szeged index is bounded above by twice the Wiener index (Theorem 3.1), and also look at equality cases. Section 4 starts with an example showing that arbitrary cactus graphs can have a revised Szeged index equal to twice its Wiener index. As a consequence, we look at a subclass of the cactus graphs to prove a reverse relation for the revised Szeged and the Wiener index (Theorem 4.2).

2 Preliminaries and the revised Szeged index as vertex sum

If not otherwise mentioned, we are working with a finite, simple and connected graph G, that has vertex set $V(G)$ and edge set $E(G)$. Let u, v be vertices of G. Then we denote with $d_G(u,v)$ the distance of u and v in G, that is, the length of the shortest path connecting u and v in G. For a path P, we use $|P|$ for its length. Furthermore, we write $n_G(u,v)$ for the number of vertices closer to u than to v, and $o_G(u,v) = o_G(v,u)$ for the number of vertices with equal distance to u and v. With this, the *Wiener index*, the *Szeged index*, and the *revised Szeged index* are defined respectively by

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v) = \frac{1}{2} \sum_{u,v\in V(G)} d_G(u,v),$$

$$\mathrm{Sz}(G) = \sum_{\{s,t\}\in E(G)} n_G(s,t)\, n_G(t,s),$$

$$\mathrm{Sz}^*(G) = \sum_{\{s,t\}\in E(G)} \Big(n_G(s,t) + \frac{1}{2}\, o_G(s,t)\Big)\Big(n_G(t,s) + \frac{1}{2}\, o_G(t,s)\Big).$$

Note, that the Wiener index is a sum over all unordered pairs of vertices, whereas the (revised) Szeged index is a sum over all edges.

In [22], Simić et al. introduced for vertices u, v and an edge $\{s,t\}$ the function

$$\mu_{u,v}(\{s,t\}) = \begin{cases} 1 & \text{if} \begin{cases} d_G(u,s) < d_G(u,t) \text{ and } d_G(v,s) > d_G(v,t), \\ \text{or} \\ d_G(u,s) > d_G(u,t) \text{ and } d_G(v,s) < d_G(v,t), \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

This can be considered an indicator function that is 1 if and only if the vertices u and v contribute to $n_G(s,t)\, n_G(t,s)$. Bonamy et al. [2] used $\mu_{u,v}$ to rewrite the Szeged index in the following way:

$$\mathrm{Sz}(G) = \sum_{\{u,v\}\subseteq V(G)} \sum_{\{s,t\}\in E(G)} \mu_{u,v}(\{s,t\}).$$

With this reformulation, the Szeged index is also a sum over all unordered pairs of vertices. Additionally, Bonamy et al. called all edges e satisfying $\mu_{u,v}(e) = 1$, 'good' for $\{u,v\}$, and referenced this again to Simić et al. [22]. However, Simić et al. used the term 'good edge' for a completely different concept. Because of this, and the fact that the term 'good' is not descriptive, we decided to use a different notation. We call edges e satisfying $\mu_{u,v}(e) = 1$, (u,v)-*distance-disparate*, and denote with $\mathrm{dis}_G(u,v)$ the number of (u,v)-distance-disparate edges in G. Hence, we can write for the Szeged index,

$$\mathrm{Sz}(G) = \sum_{\{u,v\}\subseteq V(G)} \mathrm{dis}_G(u,v) = \frac{1}{2} \sum_{u,v\in V(G)} \mathrm{dis}_G(u,v).$$

Since the revised Szeged index may not even be an integer, there cannot be a single indicator function as there is for the Szeged index. So it seems difficult to formulate the revised Szeged index as sum over vertices. Still a rather similar approach works. The first step is to consider an equivalent of $\mu_{u,v}$ for single vertices and edges having end points with the same distance to the vertex. Namely, we define for a vertex v and an edge $\{s,t\}$,

$$\nu_v(\{s,t\}) = \begin{cases} 1 & \text{if } d_G(v,s) = d_G(v,t), \\ 0 & \text{otherwise,} \end{cases}$$

an indicator function that is 1 if and only if the end points of the edge have the same distance to v. Now, similar to before, we call edges e satisfying $\nu_u(e) = 1$ and $\nu_v(e) = 1$ for vertices u and v, (u,v)-*distance-equal*, and denote with $\mathrm{deq}_G(u,v)$ the number of (u,v)-distance-equal edges in G. These are the ingredients necessary to write the revised Szeged index as sum over vertices.

Theorem 2.1. *The revised Szeged index of a graph G can be written as sum over vertices in the following form:*

$$\mathrm{Sz}^*(G) = \frac{1}{2} \sum_{u,v\in V(G)} \left(\mathrm{dis}_G(u,v) + \mathrm{deq}_G(u,u) - \frac{1}{2}\, \mathrm{deq}_G(u,v) \right).$$

Proof. Let n be the number of vertices in G. Use that $n = n_G(s,t) + n_G(t,s) + o_G(s,t)$ for all edges $\{s,t\}$ to rewrite the revised Szeged index:

$$\begin{aligned}\mathrm{Sz}^*(G) &= \sum_{\{s,t\}\in E(G)} \left(n_G(s,t) + \frac{1}{2}\, o_G(s,t)\right)\left(n_G(t,s) + \frac{1}{2}\, o_G(t,s)\right) \\ &= \sum_{\{s,t\}\in E(G)} \left(n_G(s,t)n_G(t,s) + \frac{1}{2}\, o_G(s,t)(n - o_G(s,t)) + \frac{1}{4}\, o_G(s,t)^2\right) \qquad (2.1)\\ &= \mathrm{Sz}(G) + \frac{1}{2}\, n \sum_{\{s,t\}\in E(G)} o_G(s,t) - \frac{1}{4} \sum_{\{s,t\}\in E(G)} o_G(s,t)^2.\end{aligned}$$

Since a vertex v is counted in $o_G(s,t)$ if and only if $d_G(v,s) = d_G(v,t)$, we can rewrite the second sum to

$$\sum_{\{s,t\}\in E(G)} o_G(s,t) = \sum_{u\in V(G)} \sum_{e\in E(G)} \nu_u(e) = \sum_{u\in V(G)} \mathrm{deq}_G(u,u).$$

For the third term notice that vertices u and v are involved in $o_G(s,t)\cdot o_G(s,t)$ if and only if $d_G(u,s) = d_G(u,t)$ and $d_G(v,s) = d_G(v,t)$, that is $\{s,t\}$ is counted in $\mathrm{deq}_G(u,v)$. Thus, we can reformulate this sum as well:

$$\sum_{\{s,t\}\in E(G)} o_G(s,t)^2 = \sum_{u,v\in V(G)} \mathrm{deq}_G(u,v).$$

Insert the reformulations and the Szeged index written as vertex sum in Equation (2.1) and write for n the sum over all vertices to get the desired result:

$$\begin{aligned}\mathrm{Sz}^*(G) &= \frac{1}{2} \sum_{u,v\in V(G)} \mathrm{dis}_G(u,v) + \frac{1}{2}\, n \sum_{u\in V(G)} \mathrm{deq}_G(u,u) - \frac{1}{4} \sum_{u,v\in V(G)} \mathrm{deq}_G(u,v) \\ &= \frac{1}{2} \sum_{u,v\in V(G)} \left(\mathrm{dis}_G(u,v) + \mathrm{deq}_G(u,u) - \frac{1}{2}\, \mathrm{deq}_G(u,v)\right).\end{aligned}$$ □

A noteworthy consequence of the above result is that the difference between the Szeged and the revised Szeged index can be nicely described.

Corollary 2.2. *The difference between the Szeged and the revised Szeged index of a graph G on n vertices satisfies*

$$\begin{aligned}\mathrm{Sz}^*(G) - \mathrm{Sz}(G) &= \frac{1}{2} \sum_{\{s,t\}\in E(G)} \left(n \cdot o_G(s,t) - \frac{1}{2}\, o_G(s,t)^2\right) \\ &= \frac{1}{2}\, n \sum_{u\in V(G)} \mathrm{deq}_G(u,u) - \frac{1}{4} \sum_{u,v\in V(G)} \mathrm{deq}_G(u,v).\end{aligned}$$

Before we come to the comparison of the Wiener index and the (revised) Szeged index on cactus graphs, we need some general results about graphs. The first is about the connection of dis_G and d_G on cycles.

Lemma 2.3. *Let u and v be two distinct vertices of a cycle C of length n. Then*

$$\mathrm{dis}_C(u,v) = \begin{cases} 2\,d_C(u,v) & \text{if } n \text{ is even,} \\ 2\,d_C(u,v) - 1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. To make things easier, we think of a suitable embedding of C in the plane and say right for counterclockwise, and left for clockwise. For some vertex w in C, let $P_r(w)$ be the path starting at w and going $\lfloor n/2 \rfloor$ edges to the right, and $P_l(w)$ the path starting at w, going $\lfloor n/2 \rfloor$ edges to the left. We denote the terminal vertices of $P_r(w)$ and $P_l(w)$ with w_r and w_l, respectively.

Let e be an edge in C. It is clear that if e is in $P_r(u)$, then the left vertex of e is closer to u, and vice versa, if e is in $P_l(u)$, then the right vertex of e is closer to u. For v the situation is the same. Thus, e is (u,v)-distance-disparate if and only if it is contained in the path $P_r(u) \cap P_l(v)$, or in the path $P_l(u) \cap P_r(v)$.

Without lost of generality, we can assume v is in $P_r(u)$, see Figure 1 for an exemplary illustration of the situation. In this case, $P_r(u) \cap P_l(v)$ is a shortest path from u to v, and $P_l(u) \cap P_r(v)$ is a shortest path from u_l to v_r. So we have

$$\mathrm{dis}_C(u,v) = d_C(u,v) + d_C(u_l, v_r). \tag{2.2}$$

By inclusion–exclusion principle, the distance from u_l to v_r can be determined by

$$\begin{aligned} d_C(u_l, v_r) &= |P_l(u) \cap P_r(v)| \\ &= |P_r(u) \cap P_l(v)| + |P_l(u)| + |P_r(v)| - |E(C)| \\ &= d_C(u,v) + 2\,\lfloor n/2 \rfloor - n. \end{aligned}$$

Now considering even and odd n respectively, and inserting $d_C(u_l, v_r)$ in (2.2) completes the proof. □

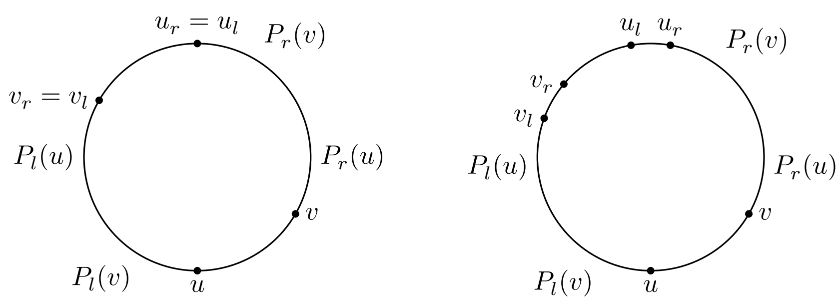

Figure 1: Cycle C of even length left and of odd length right, with vertices u, v, and the paths going right and left including their terminal vertices.

The next result is about splitting distances in a block-cut-vertex decomposition of the given graph. Recall, a block is a maximal 2-connected subgraph, and in a block-cut-vertex decomposition blocks only overlap at cut vertices. More information on blocks and the block-cut-vertex decomposition can be found in [23].

Proposition 2.4. *Let u and v be vertices of a graph G with set of blocks $\mathcal{B}$ obtained by the block-cut-vertex decomposition for G. For a block B in $\mathcal{B}$, denote by u_B and v_B the vertices in B closest to u and v, respectively. Then*

$$d_G(u,v) = \sum_{B\in\mathcal{B}} d_G(u_B\,, v_B). \tag{2.3}$$

Proof. Let P be a shortest path from u to v and $\mathcal{B}' \subseteq \mathcal{B}$ the set of blocks visited by P. Every block B in $\mathcal{B}'$ is entered by u_B and left by v_B , so P can be decomposed into subpaths P_B, where for a block B the subpath P_B starts at u_B and ends at v_B . Since every subpath of a shortest path is a shortest path itself, it follows that

$$d_G(u,v) = |P| = \sum_{B\in\mathcal{B}'} |P_B| = \sum_{B\in\mathcal{B}'} d_G(u_B\,, v_B). \tag{2.4}$$

Now consider a block B not visited by P. Since we have a block-cut-vertex decomposition, there is a unique vertex w in B minimizing the distance from the block B to the path P. This vertex is also a cut vertex and thus it minimises the distance from B to any vertex of the path P. Hence, it follows that $u_B = v_B = w$, and $d_G(u_B\,, v_B) = 0$. This finishes the proof. □

Note, in the proof of Proposition 2.4 we do not use that blocks are two-connected. That means, instead of blocks, we could split the graph into arbitrary subgraphs that only overlap at cut vertices.

In the remaining two sections, we apply the above tools to the so called cactus graphs. These are connected graphs where every two distinct cycles have at most one common vertex. Alternatively, the graph consists of a single vertex, or every block is either an edge or a cycle.

3 Comparing Wiener and Szeged index on cactus graphs

Since every edge on a shortest path from u to v is clearly (u,v)-distance-disparate, formulating the Szeged index as sum over vertices gives the first part of the following inequality:

$$W(G) \le \mathrm{Sz}(G) \le \mathrm{Sz}^*(G).$$

Already Simić et al. used the indicator function $\mu_{u,v}$ to show additionally that equality holds in the first part if and only if every block of G is complete, see [22, Theorem 2.1]. The inequality of the second part is clear by definition, whereas equality holds if and only if G is bipartite. This was shown by Pisanski and Randić, see [20, Theorem 1]. Besides, it follows from Corollary 2.2.

Here, we want to show a different inequality, true for the special class of cactus graphs.

Theorem 3.1. *Let G be a cactus graph, then*

$$\mathrm{Sz}(G) \le 2\,W(G),$$

with equality if and only if every block of G is a cycle of even length.

A special case of this result was already given in [18]. There, Li and Zhang showed that Theorem 3.1 holds for unicyclic graphs.

Proof. Let u, v be vertices in G and e be an edge in a block B. With u_B as in Proposition 2.4 every shortest path from e to u uses u_B. The same is true for v and v_B as in Proposition 2.4, respectively. Thus e is (u, v)-distance-disparate if and only if it is (u_B, v_B)-distance-disparate. Hence, with $\mathcal{B}$ as set of blocks, we can write

$$\mathrm{dis}_G(u, v) = \sum_{B \in \mathcal{B}} \mathrm{dis}_G(u_B, v_B). \tag{3.1}$$

Suppose that every block is a cycle of even length. Then by Lemma 2.3 and Proposition 2.4,

$$\begin{aligned} \mathrm{Sz}(G) &= \frac{1}{2} \sum_{u,v \in V(G)} \mathrm{dis}_G(u, v) = \frac{1}{2} \sum_{u,v \in V(G)} \sum_{B \in \mathcal{B}} \mathrm{dis}_G(u_B, v_B) \\ &= \frac{1}{2} \sum_{u,v \in V(G)} \sum_{B \in \mathcal{B}} 2 d_G(u_B, v_B) = \sum_{u,v \in V(G)} d_G(u, v) \\ &= 2\, W(G). \end{aligned} \tag{3.2}$$

Now if there is at least one odd cycle C, then again by Lemma 2.3, there is a strict inequality instead of the third equality in the above formula. Finally, if there is a block consisting of only a single edge $\{s, t\}$, then $\mathrm{dis}_G(s, t) = 1 = d_G(s, t)$, and thus also $\mathrm{Sz}(G) < 2\, W(G)$. □

Note, blocks consisting of two vertices connected with two edges considered as cycles of length two can be allowed in Theorem 3.1. Clearly, this is not a characterisation of graphs G satisfying $\mathrm{Sz}(G) \leq 2\, W(G)$, since every complete graph K_n satisfies $\mathrm{Sz}(K_n) = W(K_n)$. Unfortunately, it is also not a characterisation of graphs satisfying $\mathrm{Sz}(G) = 2\, W(G)$. Below, we give an example of a graph satisfying the equation that is not a cactus graph.

Example 3.2. Let G consist of three paths of length two joined at their end points. Attach on one side of the end points of the paths two edges by their end points and on the other side three edges. See Figure 2 for an exemplary drawing. It can be checked via a computer, or even easily by hand that

$$\mathrm{Sz}(G) = 192 = 2 \cdot 96 = 2\, W(G).$$

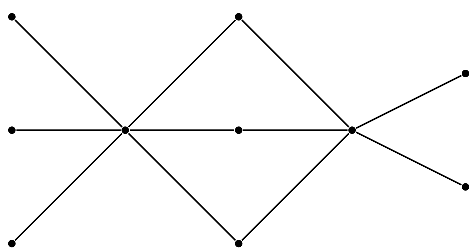

Figure 2: A bipartite non-cactus graph G satisfying $\mathrm{Sz}(G) = 2\, W(G)$.

By generalizing the graph in Example 3.2 to have k paths instead of only 3, more example graphs satisfying the equality can be found. Not for every k a suitable number of edges can be attached, but it seems there is no cap for k. The biggest example graph G we found has 783 paths of length $2, 28$ edges on one and 656 009 edges on the other side attached. It satisfies

$$\mathrm{Sz}(G) = 862\,902\,435\,600 = 2 \cdot 431\,451\,217\,800 = 2\,W(G).$$

This suggests that also if the cyclomatic number, which is just $|E(G)| - |V(G)| + 1$ for connected graphs, is large, $\mathrm{Sz}(G) = 2\,W(G)$ can still hold for non-cactus graphs.

4 Comparing Wiener and revised Szeged index on cactus graphs

From the last section, we can conclude that $\mathrm{Sz}^*(G) \leq 2\,W(G)$ holds for bipartite cactus graphs G. But in case of non-bipartite cactus graphs the situation becomes more complicated. There are even cactus graphs G satisfying $\mathrm{Sz}^*(G) = 2\,W(G)$, where not every block is a cycle of even length as the following example shows.

Example 4.1. Take a cycle of length 13, a cycle of length 11, six edges and join them at a single vertex to obtain a cactus graph G, as depicted in Figure 3. It can be checked that

$$\mathrm{Sz}^*(G) = 3636 = 2 \cdot 1818 = 2\,W(G).$$

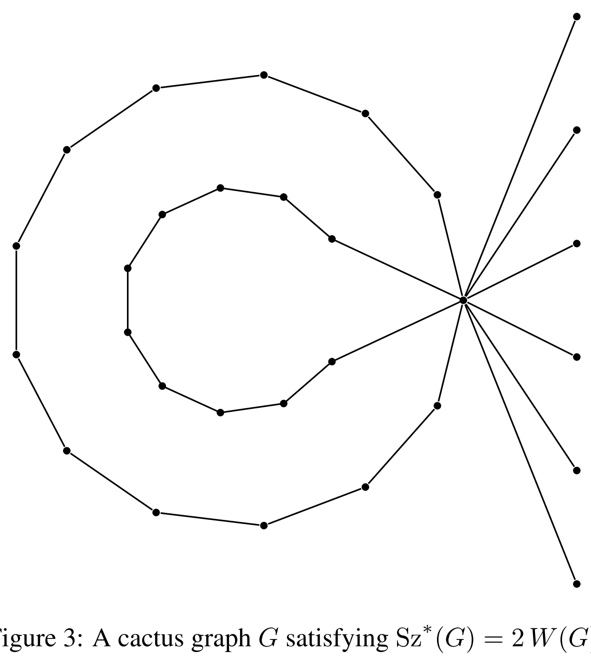

Figure 3: A cactus graph G satisfying $\mathrm{Sz}^*(G) = 2\,W(G)$.

With this in mind, it seems difficult to make any concrete statements about the connection of the revised Szeged and the Wiener index in the case of cactus graphs. Hence, we focused on a subclass of cactus graphs and found the following relation, which is in contrast to Theorem 3.1.

Theorem 4.2. *Suppose every block of a graph G is a cycle. Then*

$$2\,W(G) \leq \mathrm{Sz}^*(G),$$

with equality if and only if every cycle in G has even length.

Note, clearly a graph where every block is a cycle is a cactus graph.

Proof. Let u, v be vertices in G and e be an edge in a block B. Again, we use the notation of Proposition 2.4 with u_B and v_B for the vertices in B closest to u and v, respectively. Since u_B is on every shortest path from u to e, and the same is true for v_B and v, it is evident that $\mathrm{deq}_B(u,v) = \mathrm{deq}_B(u_B, v_B)$. Furthermore, the set of blocks $\mathcal{B}$ of G induces a partition of the edge set. Hence,

$$\mathrm{deq}_G(u,v) = \sum_{B \in \mathcal{B}} \mathrm{deq}_B(u,v) = \sum_{B \in \mathcal{B}} \mathrm{deq}_B(u_B, v_B).$$

Thus, with Theorem 2.1 we can formulate the revised Szeged index of G as

$$\begin{aligned} \mathrm{Sz}^*(G) &= \frac{1}{2} \sum_{u,v \in V(G)} \left(\mathrm{dis}_G(u,v) + \mathrm{deq}_G(u,u) - \frac{1}{2}\, \mathrm{deq}_G(u,v) \right) \\ &= \frac{1}{2} \sum_{u,v \in V(G)} \sum_{B \in \mathcal{B}} \left(\mathrm{dis}_G(u_B,v_B) + \mathrm{deq}_B(u_B,u_B) - \frac{1}{2}\, \mathrm{deq}_B(u_B,v_B) \right). \end{aligned} \tag{4.1}$$

Next we distinguish two cases, whereby the second case has two sub-cases, to show that for any vertices u_B and v_B in a block B,

$$2\, d_G(u_B,v_B) \leq \mathrm{dis}_G(u_B,v_B) + \mathrm{deq}_B(u_B,u_B) - \frac{1}{2}\, \mathrm{deq}_B(u_B,v_B). \tag{4.2}$$

Case 1: Suppose that B is a cycle of even length. Then,

$$\mathrm{deq}_B(u_B,u_B) = 0 = \mathrm{deq}_B(u_B,v_B),$$

and by Lemma 2.3,

$$\mathrm{dis}_G(u_B,v_B) = 2\, d_G(u_B,v_B).$$

Case 2: Suppose that B is a cycle of odd length.

Case 2.1: If $u_B \neq v_B$, then

$$\mathrm{deq}_B(u_B,u_B) = 1, \quad \mathrm{deq}_B(u_B,v_B) = 0,$$

and again by Lemma 2.3

$$\mathrm{dis}_G(u_B,v_B) = 2\, d_G(u_B,v_B) - 1.$$

Case 2.2: If $u_B = v_B$, then

$$\begin{aligned} \mathrm{deq}_B(u_B,u_B) &= 1 = \mathrm{deq}_B(u_B,v_B), \\ \mathrm{dis}_G(u_B,v_B) &= 0 = 2\, d_G(u_B,v_B). \end{aligned}$$

So in Case 1 and Case 2.1, we have equality in (4.2), and in Case 2.2, (4.2) is fulfilled with a strict inequality. Therefore by Proposition 2.4 and (4.1),

$$2\,W(G) = \frac{1}{2} \sum_{u,v \in V(G)} \sum_{B \in \mathcal{B}} 2\,d_G(u_B, v_B) \leq \mathrm{Sz}^*(G),$$

with equality if and only if every cycle in G has even length. □

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Component (edge) connectivity of pancake graphs*

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Abstract

The l-component (edge) connectivity of a graph G, denoted by $c\kappa_l(G)$ $(c\lambda_l(G))$, is the minimum number of vertices (edges) whose removal from G results in a disconnected graph with at least l components. The pancake graph P_n is a popular underlying topology for distributed systems. In the paper, we determine the $c\kappa_l(P_n)$ and $c\lambda_l(P_n)$ for $3 \leq l \leq 5$.

Keywords: Component connectivity, component edge connectivity, pancake graphs, fault tolerance.

Math. Subj. Class. (2020): 05C40, 05C75

1 Introduction

Multiprocessor systems are always built according to a graph which is called its interconnection network (network, for short). In a network, vertices correspond to processors, and edges correspond to communicating links between pairs of vertices. Since failures of processors and links are inevitable in multiprocessor systems, fault tolerance is an important issue in interconnection networks. Fault tolerance of interconnection networks becomes an essential problem and has been widely studied, such as, structure connectivity and substructure connectivity of hypercubes [20], extra connectivity of bubble sort star graphs [10], g-extra conditional diagnosability of hierarchical cubic networks [21], g-good-neighbor connectivity of graphs [25], conditional connectivity of Cayley graphs generated by unicyclic graphs [26].

Given a connected graph $G = (V, E)$, where V is the set of processors and E is the set of communication links between processors. The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices of G, if any, whose deletion disconnect G. The edge

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connectivity $\lambda(G)$ of a graph G is the minimum number of edges of G, if any, whose deletion disconnect G. The g-extra connectivity of G, denoted by $\kappa_g(G)$, is the minimum number of vertices whose removal separates G such that each component of the remaining graph has at least $g+1$ vertices.

The classic parameter is the connectivity $\kappa(G)$ and edge connectivity $\lambda(G)$. In general, the larger $\kappa(G)$ or $\lambda(G)$ is, the more stable the network is. The l-component connectivity of a graph was first introduced by Chartrand [8] and Sampathkumar [22], independently. Note that $c\kappa_2(G) = \kappa(G)$ and $c\lambda_2(G) = \lambda(G)$ for any graph if it is not a complete graph. Therefore, the l-component (edge) connectivity can be regarded as a generalization of the classical (edge) connectivity. The two parameters have been investigated in several interconnection networks. See for example [3, 7, 12, 16, 23, 27, 28, 29]. Recently, the relationship between extra connectivity and component connectivity of general networks has been investigated by Li et al. [14], while the relationship between extra edge connectivity and component edge connectivity of regular networks has been suggested by Hao et al. [15] and Guo et al. [13], independently.

The pancake graph, denoted by P_n, is one of alternative interconnection networks for multiprocessor systems, and it poses some attractive topological properties, such as $(n-1)$-regular, node-symmetric, bipartite and recursive [1]. The pancake graph has drawn considerable attention, such as, structure connectivity and substructure connectivity [6], super connectivity [19] and neighbor connectivity [9, 24] had been considered. For more examples, see [1, 5, 11, 17, 18, 30] and references therein.

The rest of the paper is organized as follows. Section 2 formally gives the definition of pancake graphs. In addition, we introduce some preliminary results. Section 3 determines the l-component connectivity of P_n for $l = 3, 4, 5$. Section 4 determines the l-component edge connectivity of P_n for $l = 3, 4, 5$. Concluding remarks are covered in Section 5.

2 Preliminaries

In this paper, graph-theoretical terminology and notation not defined here mostly follow [2].

For any two graphs G_1 and G_2, $G_1 \cap G_2 = (V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$. For any sets A and B, $A - B = \{x : x \in A \text{ but } x \notin B\}$ and we sometimes write $A - B$ as $A \setminus B$ if $B \subseteq A$. For $X, Y \subseteq V(G)$, $[X, Y]$ represents the edge set of G in which one end is in X and the other is in Y. The distance of two vertices u and v in a graph G, denote by $dis_G(u, v)$, is the length of a shortest path between u and v in G. Set $N_G(u) = \{v : dis_G(u, v) = 1\}$, and set $N_G(U) = \bigcup_{u \in U} N_G(u) - U$. For any vertex v, denote by $E(v)$ the edges incident to v. A *k-cycle*, denoted by C_k, is a cycle on k vertices, and a *k-path* $u_1u_2...u_k$, is a path on k vertices. Let $\langle n \rangle = \{1, 2, \cdots, n\}$.

Definition 2.1 ([1]). The n-dimensional pancake graph is denoted by P_n. The vertex set $V(P_n) = \{u = u_1u_2 \cdots u_n | u_i \in \langle n \rangle, u_i \neq u_j \text{ for } i \neq j\}$, the edge set $E(P_n) = \{uv | u = u_1u_2 \cdots u_k \cdots u_n, v = u^k = u_ku_{k-1} \cdots u_2u_1u_{k+1} \cdots u_{n-1}u_n \text{ and } 2 \leq k \leq n\}$, where u^k denotes the unique k-neighbour of u, the edge uu^k is called k-edge.

Clearly, P_n consists of $(n-1)$ kinds of edges. The pancake graphs P_2, P_3 and P_4 are shown in Figure 1. The pancake graphs are Cayley graphs with having the hierarchical (recursive) structure. The removal of all n-edges from P_n results in n connected components $P_n^1, P_n^2, \cdots, P_n^n$, where P_n^i is the subgraph of P_n induced by $\{u = u_1u_2 \cdots u_n \in V(P_n) :$

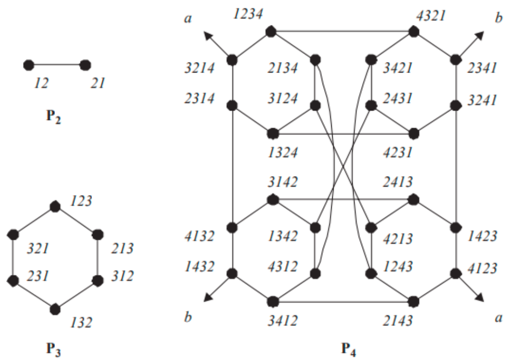

Figure 1: The pancake graphs P_2, P_3 and P_4.

$u_n = i\}$. Clearly, P_n^i is isomorphic to the $(n-1)$-dimensional pancake graph P_{n-1} [17]. We call P_n^i a sub-pancake of P_n. For any vertex $u \in V(P_n^i)$, just one vertex in $N(u)$ is not in $V(P_n^i)$, we define this vertex to be *out-neighbor* of u. For $i \neq j \in \langle n \rangle$, an edge is called a *cross-edge* if its two terminal vertices are in P_n^i and P_n^j, respectively.

Lemma 2.2 ([1, 5, 17, 18]). *An n-dimensional pancake graph P_n has the following combinatorial properties.*

(1) *P_n has $n!$ vertices, $(n-1)n!/2$ edges, $(n-1)$-regular.*

(2) *The girth of P_n is 6 for $n \geq 3$. Let the 6-cycle be presented as $u_1u_2u_3u_4u_5u_6$. Then u_1u_2, u_3u_4, u_5u_6 are 2-edges and u_2u_3, u_4u_5, u_1u_6 are 3-edges.*

(3) *For any $i \neq j$, the number of cross edges between P_n^i and P_n^j is $(n-2)!$.*

Remark 2.3. One and the same path of length 2 cannot be contained in two 6-cycles.

Lemma 2.4 ([30]). *Let F be a set of faulty vertices in P_n with $|F| \leq 2n-4$ for $n \geq 5$. If $P_n - F$ is disconnected, then it has exactly two components, one of which is a singleton or a single edge.*

In [4], Chen and Tan proposed the family of interconnection networks SP_n. It is obviously that P_n is one of the network of SP_n.

Lemma 2.5 ([11]). *Let F be a set of faulty vertices in P_n with $|F| \leq 2n-5$ for $n \geq 3$. If $P_n - F$ is disconnected, then it has exactly two components, one of which is a singleton.*

Lemma 2.6 ([11, 30]). *Let F be a set of faulty vertices in P_n with $|F| \leq 3n-8$ for $n \geq 5$. If $P_n - F$ is disconnected, then it either has two components, one of which is a singleton or a single edge, or has three components, two of which are singletons.*

Lemma 2.7 ([11]). *Let F be a set of faulty vertices in P_n with $|F| \leq 4n-11$ for $n \geq 6$. If $P_n - F$ is disconnected, then $P_n - F$ satisfies one of the following conditions:*

(1) *$P_n - F$ has two components, one of which is a singleton or a single edge or a 3-path;*

(2) *$P_n - F$ has three components, two of which are singletons;*

(3) *$P_n - F$ has three components, two of which are a singleton and an edge, respectively;*

(4) *$P_n - F$ has four components, three of which are singletons.*

Lemma 2.8 ([4, 11]). *$\kappa_1(P_n) = 2n - 4$ for $n \geq 3$, $\kappa_2(P_n) = 3n - 7$ for $n \geq 5$ and $\kappa_3(P_n) = 4n - 10$ for $n \geq 6$.*

Hereafter, we suppose that F is a vertex cut or an edge cut in P_n. For each $i \in \langle n \rangle$, let $F_i = F \cap V(P_n^i)$ or $F_i = F \cap E(P_n^i)$, and $f_i = |F_i|$. Let $I = \{i \in \langle n \rangle | f_i \geq n - 2\}$, $P_n^I = \bigcup_{i \in I} P_n^i$, $F_I = \bigcup_{i \in I} F_i$, and let $J = \langle n \rangle \setminus I$, $P_n^J = \bigcup_{j \in J} P_n^j$, $F_J = \bigcup_{j \in I} f_j$. Also, we Let H be the union of smaller components of $P_n - F$ and let $c(H)$ be the number of components of H.

3 The component connectivity of P_n

Lemma 3.1. *Let S be an independent set of $V(P_n)$ for $n \geq 4$. Then the following assertions hold.*

(1) *If $|S| = 2$, then $|N(S)| \geq 2n - 3$.*

(2) *If $|S| = 3$, then $|N(S)| \geq 3n - 6$.*

(3) *If $|S| = 4$, then $|N(S)| \geq 4n - 8$.*

Proof. For (1), let $S = \{v_1, v_2\}$. By Lemma 2.2, P_n contains no 4-cycle. Thus, v_1 and v_2 have at most one common neighbor, and $|N(S)| = |N(v_1)| + |N(v_2)| - |N(v_1) \cap N(v_2)| \geq 2n - 3$.

For (2), let $S = \{v_1, v_2, v_3\}$. By Lemma 2.2, P_n contains 6-cycle, there exists at most three common neighbors among these three singletons. Thus, we have $|N(S)| \geq \sum_{i=1}^{3} |N(v_i)| - 3 \geq 3n - 6$.

For (3), let $S = \{v_1, v_2, v_3, v_4\}$. Since P_n contains 8-cycle, and in order to make these four singletons contain as many common vertices as possible, we may assume that the 8-cycle is presented as $v_1 u_1 v_2 u_2 v_3 u_3 v_4 u_4$. Then there exists four common neighbors among these four singletons. If there exists five common neighbors among these four singletons, then it forms a cycle of length less than 6 or two 6-cycles with common 2-path, contradicting Lemma 2.2(2) and Remark 2.3, respectively. Thus, we have $|N(S)| \geq \sum_{i=1}^{4} |N(v_i)| - 4 \geq 4n - 8$. □

The following remark provides instances that attain the bounds for the assertions of Lemma 3.1.

Remark 3.2. Let $x = 123 \cdots n$, $y = (x^2)^3$, $z = (y^2)^3$, $w = (y^2)^n$, $o = (w^2)^3$. Clearly, $\{x, y, z\}$ is an independent set of P_n and $\{x, y, z\}$ lie on a 6-cycle in the subgraph of P_n, $\{x, y, w, o\}$ is an independent set of P_n and $\{x, y, w, o\}$ lie on a 8-cycle in the subgraph of P_n. Clearly, if $S = \{x, y\}$, then $|N(S)| = 2n - 3$. Since $P_n - F$ has three components, we have $c\kappa_3(P_n) \leq 2n - 3$. Similarly, if $S = \{x, y, z\}$, then $|N(S)| = 3n - 6$. Since $P_n - F$ has four components, we have $c\kappa_4(P_n) \leq 3n - 6$. Also, if $S = \{x, y, w, o\}$, then $|N(S)| = 4n - 8$. Since $P_n - F$ has five components, we have $c\kappa_5(P_n) \leq 4n - 8$.

Theorem 3.3. *For $n \geq 3$, $c\kappa_3(P_n) = 2n - 3$.*

Proof. It is true if $n = 3$. From Remark 3.2, we obtain the upper bound $c\kappa_3(P_n) \leq 2n - 3$ for $n \geq 4$. It suffices to show $c\kappa_3(P_n) \geq 2n - 3$. Suppose on the contrary that there is a vertex cut F with $|F| \leq 2n - 4$, and $P_n - F$ has at least three components.

We first consider that $n = 4$. Since $|F| \leq 4$, it is clear that $|I| \leq 2$. If $|I| = 1$, let $I = \{i\}$, then $f_i \in \{2, 3, 4\}$. If $|I| = 2$, let $I = \{i, j\}$, then $f_i = f_j = 2$. No matter which case, it's not hard to prove that $P_4 - F$ has at most two components, a contradiction. We now consider that $n \geq 5$. By Lemma 2.4, $P_n - F$ has exactly two components, a contradiction.

Thus, $c\kappa_3(P_n) \geq 2n - 3$. □

Next, we give a lemma which is used by Theorem 3.5 and 3.6.

Lemma 3.4. *For $n \geq 5$, if $|I| \leq 3$, then $P_n^J - F_J$ is connected.*

Proof. By the definition of J, we have $|J| = n - |I| \geq n - 3 \geq 2$ for $n \geq 5$ and $f_j \leq n - 3$ for $j \in J$. Since each subgraph P_n^j is isomorphic to P_{n-1}, by Lemma 2.2, we have $\kappa(P_n^j) = n - 2$. Thus, for each $j \in J$, $P_n^j - F_j$ is connected. For distinct $j, k \in J$, by Lemma 2.2, the number of cross edges between P_n^j and P_n^k is $(n - 2)!$, since $(n - 2)! > 2(n - 3)$ for $n \geq 5$, we have $P_n^j - F_j$ is connected to $P_n^k - F_k$. Therefore, $P_n^J - F_J$ is connected. □

Theorem 3.5. *For $n \geq 4$, $c\kappa_4(P_n) = 3n - 6$.*

Proof. Remark 3.2 acquires the upper bound $c\kappa_4(P_n) \leq 3n - 6$ for $n \geq 4$. It suffices to show $c\kappa_4(P_n) \geq 3n - 6$. Suppose that there is a vertex cut F with $|F| \leq 3n - 7$, and $P_n - F$ has at least four components.

We first consider that $n = 4$. By Theorem 3.3, $c\kappa_3(P_4) = 5$ and $|F| \leq 3n - 7 = 5$, we have know $P_n - F$ has at most three components, a contradiction.

Next, Let $n \geq 5$. Lemma 2.6 shows that the removal of a vertex cut with no more that $3n - 8$ vertices in P_n results in a disconnected graph with at most three components, a contradiction. To complete the proof, we need to show result holds when $|F| = 3n - 7$. Partition P_n into n disjoint copies $P_n^1, P_n^2, \cdots, P_n^n$ of P_{n-1} along dimension n. Recall that $I = \{i \in \langle n \rangle : f_i \geq n - 2\}$. Since $|F| = 3n - 7$, it is clear that $|I| \leq 2$. By Lemma 3.4, $P_n^J - F_J$ is connected. If $|I| = 0$, then $P_n - F = P_n^J - F_J$ is connected, a contradiction. Consider the following cases.

Case 1: $|I| = 1$. Let $I = \{i\}$.

Case 1.1: $n - 2 \leq f_i \leq 3(n - 1) - 8$.

Since each subgraph P_n^i is isomorphic to P_{n-1}, by Lemma 2.6, $P_n^i - F_i$ has at most three components, and all small components contain at most two vertices in total. Since $(n-1)! - 2 > 3n - 7$ for $n \geq 5$, the large component of $P_n^i - F_i$ is connected to $P_n^J - F_J$. This implies that $|V(H)| \leq 2$. It is clear that $c(H) \leq |V(H)| \leq 2$, a contradiction.

Case 1.2: $3n - 10 \leq f_i \leq 3n - 7$.

In this case, we have $F_J = |F| - f_i \leq (3n - 7) - (3n - 10) = 3$. Since every vertex of H has exactly one out-neighbor, we have $|V(H)| \leq 3$. If $|V(H)| = 3$, then $c(H) \leq 2$. Otherwise, $c(H) = 3$ and it implies that H is a set of three singletons. By Lemma 3.1, we have $|N_{P_n}(V(H))| \geq 3n - 6 > 3n - 7 = |F|$, a contradiction. If $|V(H)| \leq 2$, it is clear that $c(H) \leq |V(H)| \leq 2$, a contradiction.

Case 2: $|I| = 2$.

Let $I = \{i, j\}$. Without loss of generality, assume $f_i \leq f_j$. Since $|F| = 3n - 7$, we have $n - 2 \leq f_i \leq f_j \leq (3n - 7) - (n - 2) = 2n - 5$. If $f_i = 2n - 5$, then $f_i + f_j = 2(2n - 5) = 4n - 10 > 3n - 7$ for $n \geq 5$. Thus, it requires that $f_i \leq 2n - 6$.

Case 2.1: $n - 2 \leq f_i \leq f_j \leq 2n - 6$.

For $l \in \{i, j\}$, if $P_n^l - F_l$ is disconnected, by Lemma 2.4, $P_n^l - F_l$ has at most two components, one of which is a singleton or an edge. Since $(n-1)! - (n-2)! - 2 > 3n - 7$ for $n \geq 5$ and $l \in \{i, j\}$, then the large component of $P_n^l - F_l$ is connected to $P_n^J - F_J$. It implies that $c(H) \leq 2$, a contradiction.

Case 2.2: $f_j = 2n - 5$, and $f_i = n - 2$.

Since $|F| = 3n - 7$, we have $F_J = |F| - f_i - f_j = 0$. Thus, at most two vertices in $P_n^i \cup P_n^j - (F_i \cup F_j)$ cannot connect with $P_n^J - F_J$ in $P_n - F$, and the two vertices form an edge. Thus, $c(H) \leq 1$, a contradiction. □

Theorem 3.6. *For $n \geq 6$, $c\kappa_5(P_n) = 4n - 8$.*

Proof. Remark 3.2 acquires the upper bound $c\kappa_5(P_n) \leq 4n - 8$. It suffices to show $c\kappa_5(P_n) \geq 4n - 8$.

Suppose that there is a vertex cut F with $|F| \leq 4n - 9$, and $P_n - F$ has at least five components. Lemma 2.7 shows that the removal of a vertex cut with no more that $4n - 11$ vertices in P_n results in a disconnected graph with at most four components, a contradiction. To complete the proof, we need to show result holds when $4n - 10 \leq |F| \leq 4n - 9$. Partition P_n into n disjoint copies $P_n^1, P_n^2, \cdots, P_n^n$ of P_{n-1} along dimension n. Recall that $I = \{i \in \langle n \rangle : f_i \geq n - 2\}$. Since $|F| \leq 4n - 9$, it is clear that $|I| \leq 3$. By Lemma 3.4, $P_n^J - F_J$ is connected. If $|I| = 0$, then $P_n - F = P_n^J - F_J$ is connected, a contradiction.

Consider the following cases.

Case 1: $|I| = 1$. Let $I = \{i\}$.

Case 1.1: $n - 2 \leq f_i \leq 4(n - 1) - 11$.

Since each subgraph P_n^i is isomorphic to P_{n-1}, by Lemma 2.7, $P_n^i - F_i$ has at most four components, and all small components contain at most three vertices in total. Since $(n-1)! - 3 > 4n - 9$ for $n \geq 6$, the large component of $P_n^i - F_i$ is connected to $P_n^J - F_J$. This implies that $|V(H)| \leq 3$. It is clear that $c(H) \leq |V(H)| \leq 3$, a contradiction.

Case 1.2: $4n - 14 \leq f_i \leq 4n - 9$.

In this case, we have $F_J = |F| - f_i \leq (4n - 9) - (4n - 14) = 5$. Since every vertex of H has exactly one out-neighbor, we have $|V(H)| \leq 5$. If $|V(H)| = 5$, then $c(H) \leq 3$. Otherwise, $c(H) \geq 4$ and it implies that H contains five singletons or three singletons together with an edge. In the former case, let $H = H_0 \cup \{x\}$, where H_0 is a set of four singletons and x is a singleton. By Lemma 3.1(3), we have $|N_{P_n}(V(H_0))| \geq 4n - 8$. Clearly, $|N_{P_n}(V(H))| = |N_{P_n}(V(H_0))| + |N_{P_n}(x)| - |N_{P_n}(V(H)) \cap N_{P_n}(x)| \geq 4n - 8 + (n - 1) - 4 = 5n - 13 > 4n - 9 \geq |F|$ for $n \geq 6$, a contradiction. In the latter case, let $H = H_0 \cup \{u, v\}$, where H_0 is a set of three singletons and uv is an edge. Then, we have $|N_{P_n}(V(H_0))| \geq 3n - 6$ by Lemma 3.1(2) and $|N_{P_n}(\{u, v\})| = 2n - 4$ by Lemma 2.8. Also, the girth of P_n is 6 and it follows that $|N_{P_n}(V(H)) \cap N_{P_n}(\{u, v\})| \leq 3$. Thus $|N_{P_n}(V(H))| = |N_{P_n}(V(H_0))| + |N_{P_n}(\{u, v\})| - |N_{P_n}(V(H)) \cap N_{P_n}(\{u, v\})| \geq 3n - 6 + (2n - 4) - 3 = 5n - 13 > 4n - 9 \geq |F|$ for $n \geq 6$, a contradiction. If $|V(H)| = 4$, then $c(H) \leq 3$. Otherwise, H contains four singletons. By Lemma 3.1(3), we have $|N_{P_n}(V(H))| \geq 4n-8 > 4n-9 \geq |F|$, a contradiction. Also, if $|V(H)| \leq 3$, it is clear that $c(H) \leq |V(H)| \leq 3$, a contradiction.

Case 2: $|I| = 2$.

Let $I = \{i, j\}$. Without loss of generality, assume $f_i \leq f_j$. Since $|F| \leq 4n-9$, we have $n-2 \leq f_i \leq f_j \leq (4n-9)-(n-2) = 3n-7$. If $f_i \geq 3n-10$, then $f_i + f_j \geq 2(3n-10) = 6n-20 > 4n-9$ for $n \geq 6$. Thus, it requires that $f_i \leq 3n-11$.

Case 2.1: $n-2 \leq f_i \leq f_j \leq 3n-11$.

For each $l \in \{i, j\}$, if $P_n^l - F_l$ is disconnected, by Lemma 2.6, $P_n^l - F_l$ has at most three components and all smaller components contain at most two vertices in total. Since $(n-1)! - (n-2)! - 2 > 4n-9$ for $n \geq 6$, the large component of $P_n^l - F_l$ is connected to $P_n^J - F_J$. Thus, $|V(H)| \leq 2|I| = 4$. If $|V(H)| = 4$, then $c(H) \leq 3$. Otherwise, H contains four singletons. By Lemma 3.1(3), we have $|N_{P_n}(V(H))| \geq 4n-8 > 4n-9 \geq |F|$, a contradiction. Also, if $|V(H)| \leq 3$, it is clear that $c(H) \leq |V(H)| \leq 3$, a contradiction.

Case 2.2: $3n-10 \leq f_j \leq 3n-7$, and $n-2 \leq f_i \leq 4n-9-(3n-10) = n+1$.

$P_n^i - F_i$ has at most two components, one of which is a singleton. If $f_j = 3n-10$, by Theorem 3.5, $P_n^j - F_j$ has at most three components. Then $F_J = |F| - f_i - f_j \leq 4n-9-(n-2)-(3n-10) = 3$. If $3n-9 \leq f_j \leq 3n-7$, then $F_J = |F| - f_i - f_j \leq 4n-9-(n-2)-(3n-9) = 2$. No matter which case, $c(H) \leq 3$, a contradiction.

Case 3: $|I| = 3$.

Let $I = \{i, j, k\}$. Without loss of generality, assume $f_i \leq f_j \leq f_k$. Since $|F| \leq 4n-9$, we have $n-2 \leq f_i \leq f_j \leq f_k \leq (4n-9)-2(n-2) = 2n-5$. If $f_i \geq 2n-6$, then $f_i + f_j + f_k \geq 3(2n-6) = 6n-18 > 4n-9$ for $n \geq 6$. Thus, it requires that $f_i \leq 2n-7$. If $f_j \geq 2n-6$, then $f_i + f_j + f_k \geq n-2+2(2n-6) = 5n-14 > 4n-9$ for $n \geq 6$. Thus, it requires that $f_j \leq 2n-7$.

Case 3.1: $n-2 \leq f_i \leq f_j \leq f_k \leq 2n-7$.

For each $l \in \{i, j, k\}$, if $P_n^l - F_l$ is disconnected, by Lemma 2.5, $P_n^l - F_l$ has at most two components, one of which is a singleton. Since $(n-1)! - 2(n-2)! - 1 > 4n-9$ for $n \geq 6$, the large component of $P_n^l - F_l$ is connected to $P_n^J - F_J$. Thus, $|V(H)| \leq 3|I| = 3$. It is clear that $c(H) \leq |V(H)| \leq 3$, a contradiction.

Case 3.2: $n-2 \leq f_i \leq f_j \leq 2n-7 < f_k \leq 2n-5$.

For each $l \in \{i, j\}$, if $P_n^l - F_l$ is disconnected, by Lemma 2.5, $P_n^l - F_l$ has at most two components, one of which is a singleton. By a similar argument as Case 3.1, the large component of $P_n^l - F_l$ is connected to $P_n^J - F_J$. Since $|f_k| \leq 2n-5 \leq 3(n-1)-8$ for $n \geq 6$, by Lemma 2.6, either $P_n^k - F_k$ is connected or $P_n^k - F_k$ has at most three components and all smaller components contain at most two vertices in total. Since $(n-1)! - 2(n-2)! - 2 > 4n-9 \geq |F|$ for $n \geq 6$, the large component of $P_n^k - F_k$ is connected to $P_n^J - F_J$. Thus, $|V(H)| \leq 4$. Then, an argument similar to Case 2.1 shows that $c(H) \leq 3$, a contradiction. □

4 The edge component connectivity of P_n

Theorem 4.1. *For $n \geq 3$, $c\lambda_3(P_n) = 2n-3$.*

Proof. Take an edge $e = xy$ and $F = E(x) \cup E(y)$. Then $|F| = 2n-3$ and $P_n - F$ has at least three components. Hence $c\lambda_3(P_n) \leq 2n-3$. It suffices to show $c\lambda_3(P_n) \geq 2n-3$.

We consider an inductive proof as follows. The statement of theorem holds for $n = 3$. We assume that the result holds for P_{n-1}, and prove that it also holds for P_n, where $n \geq 4$. Suppose that there is an edge set F with $|F| \leq 2n - 4$, and $P_n - F$ has at least three components. Consider n disjoint copies $P_n^1, P_n^2, \cdots, P_n^n$. Since $I = \{i \in \langle n \rangle : f_i \geq n - 2\}$, and $|F| \leq 2n - 4$, it is clear that $|I| \leq 2$.

Consider the following cases.

Case 1: $|I| = 0$.

Each $P_n^i - F_i$ is connected for $i \in \langle n \rangle$. For distinct $i, j \in \langle n \rangle$, by Lemma 2.2, the number of cross edges between P_n^i and P_n^j is $(n-2)!$. Since $(2n - 4) < 3(n - 2)!$ for $n \geq 4$, there are at most two $[P_n^i, P_n^j]$'s which are contained in F for distinct $i, j \in \langle n \rangle$. Thus $P_n - F$ is connected, a contradiction.

Case 2: $|I| = 1$. Let $I = \{i\}$.

Case 2.1: $n - 2 \leq f_i \leq 2(n - 1) - 4$.

If each $P_n^i - F_i$ is connected for $i \in \langle n \rangle$, since $(2n - 4) - (n - 2) < 2(n - 2)!$ for $n \geq 4$, then there is at most one $[P_n^i, P_n^j]$ which is contained in F for distinct $i, j \in \langle n \rangle$. Thus, $P_n - F$ is connected, a contradiction. Hence, there exists i such that $P_n^i - F_i$ is not connected. By the inductive hypothesis, $P_n^i - F_i$ has at most two components.

Since $(2n - 4) - (n - 2) < 2(n - 2)!$ for $n \geq 4$, there is at most one $[P_n^j, P_n^k]$ which is contained in F for distinct $j, k \in \langle n \rangle \setminus \{i\}$. Thus $P_n^J - F_J$ is connected. Furthermore, $|[P_n^i, P_n^J - F_J]| = (n-1)! > 2n-4-(n-2)$ for $n \geq 4$. At least one component of $P_n^i - F_i$ is connected to $P_n^J - F_J$. Hence $P_n - F$ has at most two components, a contradiction.

Case 2.2: $2n - 5 \leq f_i \leq 2n - 4$.

In this case, we have $|F| - f_i \leq (2n - 4) - (2n - 5) = 1$. Then $P_n^J - F_J$ is connected. Note that at most one vertex of $P_n^i - F_i$ is disconnected to $P_n^J - F_J$. Hence $P_n - F$ has at most two components, a contradiction.

Case 3: $|I| = 2$.

Let $I = \{i, j\}$. Then $f_i = f_j = n - 2$ and $|F| - f_i - f_j = 0$. Thus $P_n^l - F_l$ has at most two components for any $l \in \{i, j\}$ and $P_n^J - F_J$ is connected. And so either any component of $P_n^l - F_l$ is connected to $P_n^J - F_J$ or two singletons are connected and the other component of $P_n^l - F_l$ is connected to $P_n^J - F_J$ if both $P_n^i - F_i$ and $P_n^j - F_j$ have a singleton, respectively. Thus $P_n - F$ has at most two components, a contradiction. □

Theorem 4.2. *For* $n \geq 3$, $c\lambda_4(P_n) = 3n - 5$.

Proof. Take a 3-path xyz and $F = E(x) \cup E(y) \cup E(z)$. Then $|F| = 3n-5$ and $P_n - F$ has at least four components. Hence $c\lambda_4(P_n) \leq 3n - 5$. It suffices to show $c\lambda_4(P_n) \geq 3n - 5$.

We consider an inductive proof as follows. The statement of theorem holds for $n = 3$. We assume that the result holds for P_{n-1}, and prove that it also holds for P_n, where $n \geq 4$. Suppose that there is an edge set F with $|F| \leq 3n - 6$, and $P_n - F$ has at least four components. Consider n disjoint copies $P_n^1, P_n^2, \cdots, P_n^n$. Since $I = \{i \in \langle n \rangle : f_i \geq n - 2\}$, and $|F| \leq 3n - 6$, it is clear that $|I| \leq 3$.

Consider the following cases.

Case 1: $|I| = 0$.

Similar to the proof of Case 1 of Theorem 4.1, we can show that $P_n - F$ is connected for $n \geq 5$, a contradiction. Consider that $n = 4$. Since $(4 - 2)! = 2$ and $|F| \leq 3n - 6 = 6$, there are at most three $[P_4^i, P_4^j]$'s which are contained in F. Hence $P_4 - F$ has at most two components, a contradiction.

Case 2: $|I| = 1$. Let $I = \{i\}$.

Case 2.1: $n - 2 \leq f_i \leq 3(n-1) - 6$.

Similar to the proof of Case 2.1 of Theorem 4.1, we can show that $P_n - F$ has at most three components for $n \geq 5$, a contradiction. Consider that $n = 4$. Then $2 \leq f_i \leq 3$. If $f_i = 2$, then $P_4^i - F_i$ has at most two components, and $|F| - f_i \leq (3n-6) - 2 = 4$. It is not hard to prove that $P_4 - F$ has at most two components, a contradiction. If $f_i = 3$, then $P_4^i - F_i$ has at most three components, and $|F| - f_i \leq (3n-6) - 3 = 3$. It is not hard to prove that $P_4 - F$ has at most three components, a contradiction.

Case 2.2: $3n - 8 \leq f_i \leq 3n - 6$.

In this case, we have $|F| - f_i \leq (3n-6) - (3n-8) = 2$. Furthermore, $P_n^J - F_J$ is connected. Note that at most two vertices of $P_n^i - F_i$ are disconnected to $P_n^J - F_J$. Hence $P_n - F$ has at most three components, a contradiction.

Case 3: $|I| = 2$.

Let $I = \{i, j\}$. Without loss of generality, assume $f_i \leq f_j$. Then $f_j \leq 3n - 6 - (n-2) = 2n - 4$.

Case 3.1: $n - 2 \leq f_j \leq 2(n-1) - 4$.

In this case, we have $n - 2 \leq f_i \leq f_j \leq 2(n-1) - 4$. By Theorem 4.1, both $P_n^i - F_i$ and $P_n^j - F_j$ have at most two components.

Consider that $n = 4$. Then $f_i = f_j = 2$, implying that $P_4^l - F_l$ has at most two components for $l \in \{i, j\}$, and $|F| - f_i - f_j \leq (3n-6) - 2 - 2 = 2$. It is not hard to prove that $P_4 - F$ has at most three components, a contradiction.

Consider that $n \geq 5$. Since $|[P_n^k, P_n^l]| = (n-2)! > 3n - 6 - 2(n-2)$ for $n \geq 5$ and $k, l \in \langle n \rangle \setminus \{i, j\}$, $P_n^J - F_J$ is connected. Furthermore, $|[P_n^i, P_n^J - F_J]| = (n-1)! - (n-2)! > 3n - 6 - 2(n-2)$ for $n \geq 5$. At least one component of $P_n^i - F_i$ is connected to $P_n^J - F_J$. Similarly, at least one component of $P_n^j - F_j$ is connected to $P_n^J - F_J$. Hence $P_n - F$ has at most three components, a contradiction.

Case 3.2: $f_j = 2n - 5$.

Then $n - 2 \leq f_i \leq 3n - 6 - (2n-5) = n - 1$.

If $f_i = n - 2$, $P_n^i - F_i$ has at most two components. Then $|F| - f_i - f_j \leq 1$, and so $P_n^J - F_J$ is connected. Thus $P_n - F$ has at most three components, a contradiction. If $f_i = n - 1$, then $|F| - f_i - f_j = 0$ and $P_n^J - F_J$ is connected. Thus $P_n - F$ has at most three components, a contradiction.

Case 3.3: $f_j = 2n - 4$.

Then $f_i = n - 2$ and $|F| - f_i - f_j = 0$. Thus $P_n^i - F_i$ has at most two components and $P_n^J - F_J$ is connected. Thus $P_n - F$ has at most two components, a contradiction.

Case 4: $|I| = 3$.

Let $I = \{i, j, k\}$. Then $f_i = f_j = f_k = n - 2$ and $|F| - f_i - f_j - f_k = 0$. Thus $P_n^l - F_l$ has at most two components for any $l \in \{i, j, k\}$. Thus $P_n - F$ has at most two components, a contradiction. □

Theorem 4.3. *For $n \geq 3$, $c\lambda_5(P_n) = 4n - 7$.*

Proof. Take a 4-path $xyzw$ and $F = E(x) \cup E(y) \cup E(z) \cup E(w)$. Then $|F| = 4n - 7$ and $P_n - F$ has at least five components. Hence $c\lambda_5(P_n) \leq 4n - 7$. It suffices to show $c\lambda_5(P_n) \geq 4n - 7$.

We consider an inductive proof as follows. The statement of theorem holds for $n = 3$. We assume that the result holds for P_{n-1}, and prove that it also holds for P_n, where $n \geq 4$. Suppose that there is an edge set F with $|F| \leq 4n - 8$, and $P_n - F$ has at least five components. Consider n disjoint copies $P_n^1, P_n^2, \cdots, P_n^n$. Since $I = \{i \in \langle n \rangle : f_i \geq n-2\}$, and $|F| \leq 4n - 8$, it is clear that $|I| \leq 4$.

Consider the following cases.

Case 1: $|I| = 0$.

Similar to the proof of Case 1 of Theorem 4.2, we can show that $P_n - F$ is connected for $n \geq 5$ and $P_4 - F$ has at most two components, a contradiction.

Case 2: $|I| = 1$. Let $I = \{i\}$.

Case 2.1: $n - 2 \leq f_i \leq 4(n-1) - 8$.

Similar to the proof of Case 2.1 of Theorem 4.1, we can show that $P_n - F$ has at most three components for $n \geq 5$, a contradiction. Consider that $n = 4$. Then $2 \leq f_i \leq 4$ and $(4-2)! = 2$. If $f_i = 2$, then $P_4^i - F_i$ has at most two components, and $|F| - f_i \leq (4n - 8) - 2 = 6$. It is not hard to prove that $P_4 - F$ has at most three components, a contradiction. If $f_i = 3$, then $P_4^i - F_i$ has at most three components, and $|F| - f_i \leq (4n - 8) - 3 = 5$. It is not hard to prove that $P_4 - F$ has at most three components, a contradiction. If $f_i = 4$, then $P_4^i - F_i$ has at most four components, three of which are singletons, and $|F| - f_i \leq (4n - 8) - 4 = 4$. It is not hard to prove that $P_4 - F$ has at most four components, a contradiction.

Case 2.2: $4n - 11 \leq f_i \leq 4n - 8$.

In this case, we have $|F| - f_i \leq (4n - 8) - (4n - 11) = 3$. Since $3 < 2(n-2)!$ for $n \geq 4$, there is at most one $[P_n^j, P_n^k]$ which is contained in F for $j, k \in \langle n \rangle \setminus \{i\}$, and so $P_n^J - F_J$ is connected. Note that at most three vertices of $P_n^i - F_i$ are disconnected to $P_n^J - F_J$. Hence $P_n - F$ has at most four components, a contradiction.

Case 3: $|I| = 2$.

Let $I = \{i, j\}$. Without loss of generality, assume $f_i \leq f_j$. Then $f_j \leq 4n - 8 - (n - 2) = 3n - 6$.

Case 3.1: $n - 2 \leq f_j \leq 2(n-1) - 4$.

Similar to the proof of Case 3.1 of Theorem 4.2, we can show that $P_n - F$ has at most three components, a contradiction.

Case 3.2: $2n - 5 \leq f_j \leq 3n - 9$.

Consider that $n = 4$. Then $f_j = 3$ and $2 \leq f_i \leq 3$. Then $P_4^j - F_j$ has at most three components. If $f_i = 2$, then $P_4^i - F_i$ has at most two components, and $|F| - f_i - f_j \leq (4n - 8) - 2 - 3 = 3$. It is not hard to prove that $P_4 - F$ has at most four components, a contradiction. If $f_i = 3$, then $P_4^i - F_i$ has at most three components, and $|F| - f_i - f_j \leq (4n-8) - 3 - 3 = 2$. Suppose either $P_4^i - F_i$ or $P_4^j - F_j$ contains no singleton, it is not hard to prove that $P_4 - F$ has at most three components, a contradiction. Suppose both $P_4^i - F_i$ and $P_4^j - F_j$ contain singletons, then $P_4 - F$ has at most four components, a contradiction. Otherwise, $P_4 - F$ have five components, four of which are singletons. If two singletons of P_4^l are not an edge of P_4^l for $l \in \{i, j\}$, then $f_l \geq 4$, a contradiction. Thus, two singletons form an edge of P_4^i and the other two singletons form an edge of P_4^j, implying that the four singletons form a 4-cycle, contradicting Lemma 2.2(2).

Consider that $n \geq 5$. If $f_i \geq 2n-3$, then $f_i + f_j \geq 2(2n-3) > 4n-8 \geq |F|$, a contradiction. Thus $n-2 \leq f_i \leq 2n-4$. Note that $2n-5 \leq f_j \leq 3n-9$. By Theorem 4.2, $P_n^j - F_j$ has at most three components. Since $|[P_n^k, P_n^l]| = (n-2)! > 4n-8-(n-2)-(2n-5)$ for $k, l \in \langle n \rangle \setminus \{i, j\}$, $P_n^J - F_J$ is connected. Furthermore, $|[P_n^i, P_n^J - F_J]| = (n-1)! - (n-2)! > 4n-8-(n-2)-(2n-5)$. At least one component of $P_n^i - F_i$ is connected to $P_n^J - F_J$. Similarly, at least one component of $P_n^j - F_j$ is connected to $P_n^J - F_J$.

If $n-2 \leq f_i \leq 2n-6$, by Theorem 4.1, $P_n^i - F_i$ has at most two components. Hence $P_n - F$ has at most four components, a contradiction. If $f_i = 2n-5$, and assume first that $f_j = 2n-5$. Then $|F| - f_i - f_j = 4n-8-(2n-5)-(2n-5) = 2$. By Theorem 4.1, we have $P_n^l - F_l$ has three components for $l \in \{i, j\}$. Similar to the case of $f_i = 3$ in the first paragraph of Case 3.2, $P_n - F$ has at most four components, a contradiction. Now assume that $2n-4 \leq f_j \leq 2n-3$, then $|F| - f_i - f_j \leq 4n-8-(2n-5)-(2n-4) = 1$. It is not hard to prove that $P_n - F$ has at most three components, a contradiction. If $f_i = 2n-4$, then $f_j = f_i = 2n-4$ and $|F| - f_i - f_j = 4n-8-2(2n-4) = 0$. By Theorem 4.2, $P_n^i - F_i$ has at most three components. Hence $P_n - F$ has at most three components, a contradiction.

Case 3.3: $f_j = 3n-8$.

It follows that $n-2 \leq f_i \leq n$. If $f_i = n-2$, by Theorem 4.1, $P_n^i - F_i$ has at most two components. Then $|F| - f_i - f_j \leq 4n-8-(3n-8)-(n-2) = 2$. It is not hard to prove that $P_n - F$ has at most four components, a contradiction. If $f_i = n-1$, by Theorem 4.2, $n-1 < 3(n-1)-5$ for $n \geq 4$, then $P_n^i - F_i$ has at most three components, and $|F| - f_i - f_j \leq 4n-8-(3n-8)-(n-1) = 1$. Then $P_n - F$ has at most four components, a contradiction. If $f_i = n$, then $|F| - f_i - f_j = 0$. By Theorem 4.2, both $P_n^i - F_i$ and $P_n^j - F_j$ have at most four components. Then $P_n - F$ has at most four components, a contradiction.

Case 3.4: $f_j = 3n-7$.

Similar to the proof of Case 3.3 of Theorem 4.3, we can show that $P_n - F$ has at most three components, a contradiction.

Case 3.5: $f_j = 3n-6$.

Then $f_i = n-2$ and $|F| - f_i - f_j = 0$. By Theorem 4.1, $P_n^i - F_i$ has at most two components. Thus $P_n - F$ has at most two components, a contradiction.

Case 4: $|I| = 3$.

Let $I = \{i, j, k\}$. Without loss of generality, assume $f_i \leq f_j \leq f_k$. Then $f_k \leq 4n-8-2(n-2) = 2n-4$. Consider $n = 4$. Then $f_i = 2, f_j = 2, f_k = 2$, or $f_i = 2, f_j = 2, f_k = 3$, or $f_i = 2, f_j = 2, f_k = 4$, or $f_i = 2, f_j = 3, f_k = 3$. No matter which case, it's not hard to prove that $P_4 - F$ has at most four components, a contradiction.

Next, we consider $n \geq 5$. If $f_j \geq 2n-5$, then $f_i + f_j + f_k \geq n-2+2(2n-5) = 5n-12 > 4n-8 \geq |F|$ for $n \geq 5$, a contradiction. Then $f_j \leq 2n-6$.

Case 4.1: $n-2 \leq f_i \leq f_j \leq f_k \leq 2n-6$.

By Theorem 4.1, $P_n^l - F_l$ has at most two components for any $l \in \{i, j, k\}$. Since $|[P_n^x, P_n^y]| = (n-2)! > 4n-8-3(n-2)$ for $n \geq 5$ and $x, y \in \langle n \rangle \setminus \{i, j, k\}$, $P_n^J - F_J$ is connected. Furthermore, $|[P_n^l, P_n^J - F_J]| = (n-1)! - 2(n-2)! > 4n-8-3(n-2)$ for $n \geq 5$. At least one component of $P_n^l - F_l$ is connected to $P_n^J - F_J$. Hence $P_n - F$ has at most four components, a contradiction.

Case 4.2: $n-2 \leq f_i \leq f_j \leq 2n-6 < f_k \leq 2n-4$.

By Theorem 4.1, $P_n^l - F_l$ has at most two components for any $l \in \{i, j\}$, and by Theorem 4.2, $P_n^k - F_k$ has at most three components. Since $|F| - f_i - f_j - f_k \leq 4n - 8 - 2(n-2) - (2n-5) = 1$, $P_n^J - F_J$ is connected, and at most four vertices of $P_n^i - F_i$, $P_n^j - F_j$ and $P_n^k - F_k$ are disconnected to $P_n^J - F_J$. If four vertices of $P_n^i - F_i$, $P_n^j - F_j$ and $P_n^k - F_k$ are disconnected to $P_n^J - F_J$, then two of which forms an edge, and then $P_n - F$ has at most four components, a contradiction. Otherwise, $P_n - F$ has at most four components, a contradiction.

Case 5: $|I| = 4$.

Let $I = \{i, j, k, p\}$. Then $f_i = f_j = f_k = f_p = n-2$ and $|F| - f_i - f_j - f_k - f_p = 0$. Thus $P_n^l - F_l$ has at most two components for any $l \in \{i, j, k, p\}$. Thus $P_n - F$ has at most three components, a contradiction. □

5 Concluding remarks

In this paper, we study the l-component (edge) connectivity of P_n for $3 \leq l \leq 5$. We have know that the l-component connectivity of P_n are $c\kappa_3(P_n) = 2n - 3$ for $n \geq 3$, $c\kappa_4(P_n) = 3n - 6$ for $n \geq 4$, $c\kappa_5(P_n) = 4n - 8$ for $n \geq 6$. Also, for $n \geq 3$, we have know the l-component edge connectivity of P_n are $c\lambda_3(P_n) = 2n - 3$, $c\lambda_4(P_n) = 3n - 5$, $c\lambda_5(P_n) = 4n - 7$. We study the larger component (edge) connectivity of P_n in the future work.

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Hamilton cycles in primitive graphs of order $2rs$*

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Abstract

After long term efforts, it was recently proved by Du, Kutnar and Marušič in 2021 that except for the Petersen graph, every connected vertex-transitive graph of order rs has a Hamilton cycle, where r and s are primes. A natural topic is to solve the hamiltonian problem for connected vertex-transitive graphs of $2rs$. This topic is quite nontrivial, as the problem is still unsolved even for that of $r = 3$ and 5. In this paper, it is shown that except for the Coxeter graph, every connected vertex-transitive graph of order $2rs$ contains a Hamilton cycle, provided the automorphism group acts primitively on vertices.

Keywords: Vertex-transitive graph, Hamilton cycle, primitive group, automorphism group, orbital graph.

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1 Introduction

Throughout this paper graphs are finite, simple and undirected, and groups are finite. Given a graph X, by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ we denote the vertex set, the edge set and the automorphism group of X, respectively. A graph X is *vertex-* or *arc-transitive* if $\mathrm{Aut}(X)$ acts transitively on vertices or arcs, respectively.

Given a transitive group G on Ω, a subset B of Ω is called a *block* of G if, for any $g \in G$, we have either $B = B^g$ or $B \cap B^g = \emptyset$. Clearly, G has blocks Ω and $\{\alpha\}$ for any $\alpha \in \Omega$, which are said to be *trivial*. Then G is said to be *primitive* if it has no nontrivial blocks. Moreover, a vertex-transitive graph X is said to be *primitive* if $\mathrm{Aut}(X)$ is primitive on vertices.

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A simple path (resp. cycle) containing all vertices of a graph is called a *Hamilton path* (resp. *cycle*) of this graph. For convenience, a Hamilton-cycle (resp. path) is usually abbreviated by a H-cycle (resp. H-path). A graph containing a Hamilton cycle will be sometimes referred as a *hamiltonian graph.*

In 1970, Lovász asked in [1] that

Does every finite connected vertex-transitive graph have a Hamilton path?

Up to now, this question remains unresolved and no connected vertex-transitive graph without a Hamilton path is known to exist. Moreover, only four (families) of connected vertex-transitive graphs on at least three vertices not having a Hamilton cycle are known, which are Petersen graph, Coxeter graph and triangle-replaced graphs from them. Since all of these graphs are not Cayley graph, we may ask if every connected Cayley graph has a Hamilton cycle.

It has been shown that connected vertex-transitive graphs of orders kp, $k \leq 6$, $10p$ ($p \geq 11$), p^j ($j \leq 5$) and $2p^2$, where p is a prime contain a Hamilton path, see [2, 5, 18, 19, 20, 25, 26, 27, 28, 31]. Furthermore, for all of these families, except for the graphs of order $6p$ and $10p$ and that four exceptions, they contain a Hamilton cycle. With the exception of the Petersen graph, Hamilton cycles are also known to exist in connected vertex-transitive graphs whose automorphism groups contain a transitive subgroup with a cyclic commutator subgroup of prime-power order (see [6] and also [9, 17, 24]).

So far we know that Cayley graphs of the following groups contain a Hamilton cycle: nilpotent groups of odd order, with cyclic commutator subgroups (see [6, 11, 12]); dihedral groups of order divisible by 4 (see [3]); and arbitrary p-groups (see [30]). A Hamilton path and in some cases even a Hamilton cycle was proved to exist in cubic Cayley graphs arising from $(2, s, 3)$-generated groups (see [13, 14, 15]).

Recently, Kutnar, Marusic and the first author proved that vertex transitive graphs of order rs have a Hamilton cycle, except for the Petersen graph (see [7, 8]). This work took many years, because of a difficult case, which is a primitive graph with automorphism group $\mathrm{PSL}(2, p)$ and a point-stabilizer $\mathbb{D}_{p-1}$. A natural question is to consider hamiltonian problem for vertex-transitive graphs of order $2rs$. As mentioned above, some special cases have been solved such as that of graphs of order $4p$, $6p$, $10p$ and $2p^2$, where p is a prime (Hamilton path or cycle). To solve the general case, a necessary step is to deal with all primitive graphs of such order. The main result of this paper is the following theorem.

Theorem 1.1. *Except for Coxeter graph, every connected vertex-transitive graph of order* $2rs$ *contains a Hamilton cycle provided the automorphism group acts primitively on its vertices, where* r *and* s *are primes.*

After this introductory section, some notations, basic definitions and useful facts will be given in Section 2 and Theorem 1.1 will be proved in Section 3.

2 Preliminaries

By $\lfloor a \rfloor$ and $\lceil a \rceil$, we denote the largest integer that is smaller than a and smallest integer that is larger than a, respectively. For a prime q, a finite field of order q will be denoted by $\mathbb{F}_q$. Set $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$, $S = \{t^2 \mid t \in \mathbb{F}_q\}$, $S^* = S \cap \mathbb{F}_q^*$ and $N = \mathbb{F}_q^* \setminus S^*$. Then the elements in S and N are called to be *squares* and *non-squares*, respectively. By $\mathbb{Z}_n$ and $\mathbb{D}_{2n}$ we denote a cycle group of order n and dihedral group of order $2n$, respectively. For a group G and $L \subset G$, by $C_G(L)$ and $N_G(L)$ we denote the centralizer and normalizer of L in G, respectively. A semi-product of K and H is denoted by $K \rtimes H$, where K is normal. Let G be a group with a normal subgroup N, we denote the image of $g \in G$ under the natural homomorphism of G to G/N by $\overline{g}$. For a group G and its subgroup H, $[G : H]$ denotes the set of right cosets of H in G; HgH denotes the orbit containing Hg under the action of H. Recall that the socle of G which is denoted by $\mathrm{soc}(G)$ is defined to be the product of all minimal normal subgroups of G.

Let G act on some set Ω. For some $\alpha \in \Omega$ and $g \in G$, set $\alpha^G = \{\alpha^g \mid g \in G\}$. For $\alpha \in \Omega$, set $H = G_\alpha$. Then the action of G on Ω is equivalent to its right multiplication action on right cosets $[G : H]$ relative to H. For a subset Δ of Ω, by $G_{(\Delta)}$ and $G_{\{\Delta\}}$, we denote the pointwise and setwise stabilizer of Δ in G, respectively.

In a graph X, let $a \in V(X)$ and $B \subset V(X)$, by $d(a, B)$ we denote the number of neighbors of a in B. Given $A, B \subset V(X)$, if $d(a, B) = d(a', B)$ for any $a, a' \in A$, then we denote $d(a, B)$ by $d(A, B)$. Moreover, set $d(B) = d(B, B)$. The neighborhood of any vertex a in the graph X is denoted by $X_1(a)$.

In what follows we recall some definitions related to orbital graphs and semiregular automorphisms.

Let G be a transitive permutation group on Ω. Then G induces a natural action on $\Omega \times \Omega$. We call the orbits of G on $\Omega \times \Omega$ the *orbitals* of G, and in particular the *trivial* orbital is referred to $\{(\alpha, \alpha) \mid \alpha \in \Omega\}$. The *orbital digraph* $X(G, \Gamma)$ relative to an orbital Γ is defined to be the directed graph with vertex set Ω and edge set Γ. Each orbital Γ has an associated *paired orbital* Γ' defined by $\Gamma' = \{(\beta, \alpha) \mid (\alpha, \beta) \in \Gamma\}$, and of course, Γ is said to be *self-paired* if $\Gamma = \Gamma'$ in which case $X(G, \Gamma)$ can be viewed as an undirected graph (*orbital graph*). The G-arc-transitive graphs with vertex-set Ω are precisely the orbital graphs $X(G, \Gamma)$ for the nontrivial self-paired orbitals Γ. In addition, take a point $\alpha \in \Omega$, the orbits of the stabilizer G_α on Ω are called *suborbits* of G relative to α. There is a one-to-one correspondence between the suborbits and the orbitals of G. Each orbital Γ_i corresponds to a suborbit $\Delta_i = \{\beta \in \Omega \mid (\alpha, \beta) \in \Gamma_i\}$. Conversely, each suborbit Δ_i corresponds to an orbital $\Gamma_i = \{(\alpha, \beta)^g \mid g \in G, \beta \in \Delta_i\}$. A suborbit of G is said to be *self-paired* if the corresponding orbital is self-paired. Thus we often use $X(G, \Delta_i)$ and $X(G, \Delta_i \cup \Delta_i')$ to denote graphs $X(G, \Gamma)$ and $X(G, \Gamma \cup \Gamma')$ respectively.

Let $m \geq 1$ and $n \geq 2$ be integers. An automorphism ρ of a graph X is called (m, n)-*semiregular* (in short, *semiregular*) if as a permutation on $V(X)$ it has a cycle decomposition consisting of m cycles of length n. If $m = 1$ then X is called a *circulant*; it is in fact a Cayley graph of a cyclic group of order n. Let $\mathcal{P}$ be the set of orbits of ρ, that is, the orbits of the cyclic subgroup $\langle\rho\rangle$ generated by ρ. We let the *quotient graph corresponding to* $\mathcal{P}$ be the graph $X_{\mathcal{P}}$ whose vertex set equals $\mathcal{P}$ with $A, B \in \mathcal{P}$ adjacent if there exist vertices $a \in A$ and $b \in B$, such that $a \sim b$ in X.

The following four results will be used later.

Proposition 2.1 ([29, page 167]). *Let F_q be the finite field of odd prime order q. Then*

$$|(S^* + 1) \cap (-S^*)| = \begin{cases} (q-5)/4 & q \equiv 1 \pmod 4, \\ (q+1)/4 & q \equiv 3 \pmod 4. \end{cases}$$

This implies that if $q \equiv 1 \pmod 4$ then

$$|S^* \cap (S^* + 1)| = (q-5)/4, \quad |N \cap (N+1)| = (q-1)/4, \quad |S^* \cap (N \pm 1)| = (q-1)/4.$$

No.	soc(G)	$2rs$	*Action*	*Comment*
1	PSL$(2,\, q)$	$q(q+1)/2$	$G_\alpha \cap \mathrm{soc}(G) = \mathbb{D}_{2(q-1)/d}$	$d = (2,\, q-1)$, $G = \mathrm{PGL}(2,\, 11)$ for $q = 11$
2	PSL$(2,\, q)$	$q(q-1)/2$	$G_\alpha \cap \mathrm{soc}(G) = \mathbb{D}_{2(q+1)/d}$	$d = (2,\, q-1)$
3	PSL$(2, 47)$	$2 \times 47 \times 23$	S_4	
4	PSL$(2, 17)$	$2 \times 17 \times 3$	S_4	
5	PSL$(2, 41)$	$2 \times 41 \times 7$	A_5	

Table 1: Primitive groups of degree $2rs$, where the socle PSL$(2, q)$.

Proposition 2.2 ([16, Theorem 6] (Jackson's Theorem)). *Every* 2*-connected regular graph of order* n *and valency at least* $n/3$ *contains a Hamilton cycle.*

Proposition 2.3 ([4, Corollary 3]). *If* X *is a connected Cayley graph of an abelian group of order at least* 3*, then every edge of* X *lies in a hamiltonian cycle.*

Lemma 2.4 ([27, Lemma 5]). *Let* X *be a graph admitting an* $(m,\, p)$*-semiregular automorphism* ρ*, where* p *is a prime. Let* C *be a cycle of length* m *in the quotient graph* $X_{\mathcal{P}}$*, where* $\mathcal{P}$ *is the set of orbits of* ρ*. Then, the lift of* C *either contains a cycle of length* mp *or it consists of* p *disjoint* m*-cycles. In the latter case we have* $d(S,\, S') = 1$ *for every edge* SS' *of* C*.*

3 Proof of Theorem 1.1

To prove Theorem 1.1, let X be a connected vertex-transitive graph of order $2rs$, where r and s are primes. Set $G = \mathrm{Aut}(X)$. It has been proved that X contains a Hamilton cycle if $2rs = 2p^2$ or $4p$ for a prime p, provided X is not the Coxeter graph which is of order 28. Therefore, in what follows we assume that $r < s$. If G acts 2-transitively on $V(X)$, then X is a complete graph, which contains a H-cycle. Now we need to consider all the primitive groups of degree $2rs$ of rank at least 3 from [10] (or [21]), where r and s are distinct odd primes. Let H be a point stabilizer in soc(G). Checking [10], all the possible groups are listed in Tables 1 and 2.

Table 1 gives the these groups with the socle PSL$(2,\, q)$. The first two cases $H = \mathbb{D}_{q-1}$ and $H = \mathbb{D}_{q+1}$ will be dealt with in Subsections 3.1 and 3.2, respectively. With the help of Magma, we can show that every vertex-transitive graph is hamiltonian, arising from other three groups in Table 1.

Table 2 gives these groups whose socle is a classical simple group which is not PSL$(2,\, q)$, an alternative group or a sporadic simple group. These groups will be dealt with in Subsection 3.3.

3.1 $\mathrm{soc}(G) = \mathrm{PSL}(2,\, q)$ and $H = \mathbb{D}_{q-1}$

Let $G = \mathrm{PSL}(2,\, q)$ and $H = \mathbb{D}_{q-1}$. Consider the action of G on the set $[G : H]$ of cosets of H in G, see row 1 of Table 1. Then the degree is $q(q+1)/2 = 2rs$, thus $q \equiv 3 (\mathrm{mod}\ 4)$ and in particular $-1 \in N$, the set of non-squares. Set $\mathbb{F}_q^* = \langle \theta \rangle$.

No.	soc(G)	$2rs$	*Action*	*Comment*
1	PSL(4, q)	$\frac{q^3-1}{q-1}(q^2+1)$	2-spaces	$q=3$; or $q=5$; or $q\equiv 11,\ 29 (\mathrm{mod}\ 30)$ and q prime and $q\geq 59$
2	PSL(5, q)	$\frac{q^5-1}{q-1}(q^2+1)$	2-spaces	$q\equiv -1(\mathrm{mod}\ 10)$, q prime and $q\geq 29$
3	$\mathrm{P}\Omega^-(2m,\ q)$	$\frac{(q^m+1)(q^{m-1}-1)}{q-1}$	on t.s. 1-spaces	m even
4	$\mathrm{P}\Omega^+(2m,\ q)$	$\frac{(q^m-1)(q^{m-1}+1)}{q-1}$	on t.s. 1-spaces	m odd
5	PSL(3, 5)	$2\times 31\times 3$	on (1, 2)-dim. flags	$G=$ PSL(3, 5).2
6	A_c	$\frac{c(c+1)}{2}$	on 2-sets	$c\geq 5$
7	M_{11}	66	S_5	
8	M_{12}	66	$M_{10}:2$	
9	M_{23}	506	A_8	
10	J_1	266	PSL(2, 11)	

Table 2: Primitive groups G of degree $2rs$, where $\mathrm{soc}(G)\neq \mathrm{PSL}(2,q)$.

For any $g\in \mathrm{SL}(2,\ q)$, set $\overline{g}=gZ(\mathrm{SL}(2,\ q))$. In SL(2, q), set

$$u=\begin{pmatrix}1&1\\0&1\end{pmatrix},\ u'=\begin{pmatrix}1&0\\1&1\end{pmatrix},\ l=\begin{pmatrix}\theta&0\\0&\theta^{-1}\end{pmatrix},\ t=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

Since PSL(2, q) has only one conjugacy class of subgroups isomorphic to $\mathbb{D}_{q-1}$, we may set $H=\langle \bar{l},\ \bar{t}\rangle$. Let V be the row vector space so that the action of $g\in \mathrm{GL}(2,\ q)$ on a vector $(x,\ y)$ is just defined as $(x,\ y)\cdot g$. Set $\frac{y}{x}=\langle (x,\ y)\rangle$. Then all the projective points are $\{\infty,\ 0,\ 1,\ 2,\ \cdots,\ q-1\}$. The action of G on $[G:H]$ is equivalent to its action on the set of unordered pairs of distinct projective points, where $H=G_{\{0,\ \infty\}}$. Thus we have

$$\begin{aligned}&\overline{u}':\{\infty,\ 0\}\to\{1,\ 0\},\quad \overline{l^i}:\{j,\ j+1\}\to\{j\theta^{-2i},\ (j+1)\theta^{-2i}\},\\&\overline{l^it}:\{j,\ j+1\}\to\{-j^{-1}\theta^{2i},\ -(j+1)^{-1}\theta^{2i}\}.\end{aligned}$$

Then the all $\langle\overline{u}\rangle$-orbits are

$$B_\infty=\{\{\infty,\ i\}|i\in\mathbb{F}_q\},\quad B_j=\{\{i,\ i+j\}|i\in\mathbb{F}_q\},\ j\in\{1,\ 2,\ 3,\ \cdots,\ \frac{q-1}{2}\}.$$

Set $\mathbf{B}=\{B_j\ \big|\ j\in\{1,\ 2,\ 3,\ \cdots,\ \frac{q-1}{2}\}\}$. Considering the action of $N_G(\langle\overline{u}\rangle)=\langle\overline{u}\rangle\rtimes\langle\bar{l}\rangle$ on the vertices, we know that $N_G(\langle\overline{u}\rangle)$ fixes the block B_∞ setwise and acts transitively on other vertices. In particular, $\langle\bar{l}\rangle$ fixes B_∞ and acts regularly on $\frac{q-1}{2}$ remaining blocks B_j in $\mathbf{B}$.

The suborbits of G have been determined in [22] and an alternative description is given below.

Lemma 3.1. *Suppose $q\equiv 3(\mathrm{mod}\ 4)$. Then every nontrivial suborbit of G relative to H can be written as $\{j,\ j+1\}^H$, where $j\in\mathbb{F}_q$, with length $\frac{q-1}{2}$ and $q-1$ if and only if $j^2+j\in N$ and $j^2+j\in S$, respectively. Moreover, $\{j,\ j+1\}^H$ is self-paired if and only if either $j+1\in N$ or $j\in S$, and if it is non self-paired, then its paired suborbit is $\{-j,\ -j-1\}^H$.*

Proof. For $i \in \mathbb{F}_q^*$, direct computations show that $\{\infty, i\}$ belongs to $\{0, 1\}^H$ or $\{0, -1\}^H$ depending on whether $i \in S^*$ or $i \in N$, respectively. Since $\langle \bar{l} \rangle \leq H$ acts regularly on $\mathbf{B}$, any other suborbits can also be written as $\{j, j+1\}^H$. The length of $\{j, j+1\}^H$ is $\frac{q-1}{2}$ and $q-1$ if and only if the order of the stabilizer for $\{j, j+1\}$ in H is 2 and 1, respectively. But the former holds if and only if there exists some $k \in \mathbb{Z}_q$ such that $\overline{l^k t}$ fixes $\{j, j+1\}$, i.e., $j+1 = -j^{-1}\theta^{2k}$. Therefore we deduce that the length of the suborbit is $\frac{q-1}{2}$ or $q-1$ depending on $j^2 + j \in N$ or $j^2 + j \in S$, respectively.

Let $\Delta = \{j, j+1\}^H$. If $j+1 = 0$, then $\Delta^* = \{0, 1\}^H$. If $j+1 \neq 0$, then $\Delta^* = \{\frac{-j}{j+1}, -1\}^H$. Now, Δ is self-paired if and only if there exists some element of H mapping $\{j, j+1\}$ to $\{\frac{-j}{j+1}, -1\}$. From

$$\{\overline{l^k}(j), \overline{l^k}(j+1)\} = \{j\theta^{-2k}, (j+1)\theta^{-2k}\} = \{\frac{-j}{j+1}, -1\} \quad \text{and}$$

$$\{\overline{l^k t}(j), \overline{l^k t}(j+1)\} = \{-j^{-1}\theta^{2k}, -(j+1)^{-1}\theta^{2k}\} = \{\frac{-j}{j+1}, -1\},$$

we know that such element of H exists if and only if $j+1 \in N$ or $j \in S$, as desired.

Suppose that Δ is not self-paired and $j+1 \neq 0$. Then $j+1 = \theta^{-2k} \in S$ and $\overline{l^k}$ maps $\{\frac{-j}{j+1}, -1\}$ to $\{-j, -j-1\}$, that is $\Delta^* = \{\frac{-j}{j+1}, -1\}^H = \{-j, -j-1\}^H$. □

Remark 3.2. By Lemma 3.1, it is easy to determine the number of nontrivial suborbits of length $\frac{q-1}{2}$ or $q-1$, and the number of nontrivial paired suborbits. But we do not need these numbers in here.

Before going to prove the main result, we first give a technical lemma on number theory.

Lemma 3.3. *Suppose that q is an odd prime. If $a, b \in \mathbb{F}_q^*$ and $a \neq b$. Then*

$$\begin{aligned}
|(S^* + a) \cap (S^* + b) \cap N| &\leq \lceil \tfrac{1}{8}(q + 11 + 2\sqrt{q}) \rceil, \\
|(S^* + a) \cap (N + b) \cap N| &\leq \lceil \tfrac{1}{8}(q + 11 + 2\sqrt{q}) \rceil, \\
|(S^* + a) \cap (N + b) \cap S^*| &\geq \lfloor \tfrac{1}{8}(q - 11 - 2\sqrt{q}) \rfloor, \\
|(N + a) \cap (N + b) \cap S^*| &\geq \lfloor \tfrac{1}{8}(q - 11 - 2\sqrt{q}) \rfloor.
\end{aligned}$$

Proof. Set $\eta\colon \mathbb{F}_q^* \to \{\pm 1\}$ by assigning the elements of S^* to 1 and that of N to -1 and moreover, set $\eta(0) = 0$. This η is exactly that in [23, Example 5.10]. Also we need to quote the following three results from [23, Theorems 5.4, 5.48, 5.41]:

(i) $\sum_{x \in \mathbb{F}_q} \eta(x) = 0$;

(ii) $\sum_{x \in \mathbb{F}_q} \eta(x^2 + Ax + B) = q - 1$ for $A^2 - 4B = 0$ or -1 for otherwise, where $A, B \in \mathbb{F}_q$;

(iii) $|m| \leq 2\sqrt{q}$, where $m := \sum_{x \in \mathbb{F}_q} \eta(x(x-1)(x-t))$ and $t \in \mathbb{F}_q$.

For four inequalities of the lemma, we have the same arguments and here we just prove the first one. Set $W = (S^* + a) \cap (S^* + b) \cap N$, that is

$$W = \{x \in \mathbb{F}_q \mid \eta(x - a) = \eta(x - b) = 1, \eta(x) = -1\}.$$

Now let $a, b \in S^*$. Then by the above three formulas (i) – (iii), we have

$$
\begin{aligned}
|W| &= \frac{1}{8} \sum_{x \in \mathbb{F}_q \setminus \{0,a,b\}} (1+\eta(x-a))(1+\eta(x-b))(1-\eta(x)) \\
&= \frac{1}{8} \sum_{x \in \mathbb{F}_q \setminus \{0,a,b\}} (1-\eta(x)+\eta(x-a)+\eta(x-b)-\eta(x(x-a))-\eta(x(x-b)) \\
&+ \eta((x-a)(x-b)) - \eta((x-a)(x-b)x) \\
&= \frac{1}{8}[(q-3)-(-\eta(b)-\eta(a))-(\eta(-a)+\eta(b-a))-(\eta(-b)+\eta(a-b)) \\
&- (-1-\eta b(b-a)) - (-1-\eta a(a-b)) + (-1-\eta(ab)) + m] \\
&\le \lceil \frac{1}{8}(q+11+2\sqrt{q}) \rceil.
\end{aligned}
$$

□

According to Lemma 3.1, we shall deal with the orbital graphs $X = X(G, \Delta)$ or $X = X(G, \Delta \cup \Delta^*)$, according to that Δ is self-paired and of length $\frac{q-1}{2}$, non self-paired and of length $\frac{q-1}{2}$, self-paired and of length $q-1$, and non self-paired and of length $q-1$, respectively, in the following four lemmas.

Lemma 3.4. *Suppose that Δ is a self-paired suborbit of length $\frac{q-1}{2}$. Then $X(G, \Delta)$ is hamiltonian.*

Proof. Let $X = X(G, \Delta)$, where Δ is self-paired and of length $\frac{q-1}{2}$. Let Y be the quotient graph induced by $\langle \overline{u} \rangle$, with vertices $\mathbf{B} \cup \{B_\infty\}$. Then by Lemma 3.1, we may set $\Delta = \{j, j+1\}^H$, where $j(j+1) \in N$, $j+1 \in N$ and $j \in \mathbb{F}_q$. Then the neighborhood of $\{0, \infty\}$ is:

$$X_1(\{0, \infty\}) = \Delta = \{\{j\theta^{-2k}, (j+1)\theta^{-2k}\} \mid k \in \mathbb{F}_q\}.$$

Since $|\Delta| = \frac{q-1}{2}$ and $\langle \bar{l} \rangle$ acts regularly on $\mathbf{B}$, $d(B_\infty, B_i) = 1$ for any $i = 1, 2, 3, \cdots, \frac{q-1}{2}$.

The lemma will be proved by the following three steps:

Step 1: Show $d(B_m, B_i) \le 2$ for any $i, m = 1, 2, 3, \cdots, \frac{q-1}{2}$.

Since $\langle \bar{l} \rangle$ is regular on $\mathbf{B}$ and $\{0, 1\} \in B_1$, we may just consider $d(B_1, B_i) = d(\{0, 1\}, B_i)$ for any $i = 1, 2, 3, \cdots, \frac{q-1}{2}$. Since $\overline{u}'$ maps $\{\infty, 0\}$ to $\{0, 1\}$, we know that

$$
\begin{aligned}
X_1(\{0, 1\}) = \Delta^{\overline{u}'} &= \{\{j\theta^{-2k}, (j+1)\theta^{-2k}\} \mid k \in \mathbb{F}_q\}^{\overline{u}'} \\
&= \{\{\frac{j\theta^{-2k}}{1+j\theta^{-2k}}, \frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}}\} \mid k \in \mathbb{F}_q\}.
\end{aligned}
$$

So a vertex in $X_1(\{0, 1\})$ is contained in B_i if and only if

$$\{\frac{j\theta^{-2k}}{1+j\theta^{-2k}}, \frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}}\} = \{t, t+i\} \text{ for some } t,$$

if and only if one of the following two systems of equations has solutions:

$$\frac{j\theta^{-2k}}{1+j\theta^{-2k}} = t, \quad \frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}} = t+i; \tag{3.1}$$

and

$$\frac{j\theta^{-2k}}{1+j\theta^{-2k}} = t+i, \quad \frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}} = t. \tag{3.2}$$

Solving Equation (3.1), we get

$$ij(j+1)u^2 + (2ij+i-1)u + i = 0,$$

where $u = \theta^{-2k}$. This equation has solutions for u if and only if

$$\delta_1 := (2ij+i-1)^2 - 4i^2j(j+1) = i^2 - (2+4j)i + 1 \in S^*.$$

Suppose that the above equation has solutions, say u_1 and u_2. Since $u_1u_2 = (j(j+1))^{-1}$, a non-square, we know that $u_1, u_2 \neq 0$, one of them is a non-square and the other one is a square. Therefore, there exists exactly one solution for $\theta^{-2k} = u$ if and only if $\delta_1 \in S^*$, noting that every θ^{-2k} gives a unique t, equivalently, a unique vertex in the block B_i.

Solving Equation (3.2), we get

$$ij(j+1)u^2 + (2ij+i+1)u + i = 0,$$

where $u = \theta^{-2k}$. This equation has solutions for u if and only if

$$\delta_2 := (2ij+i+1)^2 - 4i^2j(j+1) = i^2 + (2+4j)i + 1 \in S^*.$$

Similarly, there exists exactly one solution for θ^{-2k} if and only if $\delta_2 \in S^*$.

Summarizing Equation (3.1) and Equation (3.2), we get $d(\{0, 1\}, B_i) \leq 2$.

Step 2: Show that for a given j, there exists some i such that $d(B_j, B_i) = 2$.

It suffices to show $d(\{0, 1\}, B_i) = 2$ for some $i \neq 0$, equivalently, to show that the number of B_i ($i \neq 1$) such that $d(B_1, B_i) = 1$ is less than $\frac{q-1}{2} - 1 - 2 = \frac{q-7}{2}$.

Now, $d(B_1, B_i) = 1$ if and only if

$$\delta_1\delta_2 = (i^2 - (2+4j)i + 1)(i^2 + (2+4j)i + 1) = y \in N,$$

that is

$$u^2 + (2 - (2+4j)^2)u + 1 - y = 0, \tag{3.3}$$

where $u = i^2$. Note that for a given $u \in S^*$, i and $-i$ give the same block B_i. Thus a solution of u can provide at most one block B_i satisfying our conditions.

In what follows, we analyse the number of solutions for u.

Equation (3.3) has some solutions for u if and only if

$$\delta := (2 - (2+4j)^2)^2 - 4(1-y) \in S,$$

that is

$$y \in S + t, \text{ where } t = -4j(j+1) \in S.$$

Now $y \in (S+t) \cap N$. First suppose that $1 - y \in N$. Then $y \in (S+t) \cap N \cap (1+S)$. By Lemma 3.3, we have at most $\lceil \frac{1}{8}(q + 11 + 2\sqrt{q}) \rceil + 1$ choices for y, and then for u as well.

Secondly, suppose that $1-y \in S$. Then $y \in (S+t) \cap N \cap (1+N)$. By Lemma 3.3, we have at most $\lceil \frac{1}{8}(q+11+2\sqrt{q})\rceil + 1$ choices for y. Since every y may give two solutions for u, we have at most $2\lceil \frac{1}{8}(q+11+2\sqrt{q})\rceil + 2$ solutions for u.

In summary, we have at most

$$\lceil \frac{1}{8}(q+11+2\sqrt{q})\rceil + 2\lceil \frac{1}{8}(q+11+2\sqrt{q})\rceil + 3$$

blocks B_i such that $d(B_0,\, B_i) = 1$. Now

$$\lceil \frac{1}{8}(q+11+2\sqrt{q})\rceil + 2\lceil \frac{1}{8}(q+11+2\sqrt{q})\rceil + 3 \le \frac{q-7}{2},$$

provided $q > 169$. In other words, if $q > 169$ there exists some i such that $d(B_0,\, B_i) = 2$. For $7 \le q \le 169$, only the primes $19, 43, 67$ and 163 satisfy $\frac{q(q+1)}{2} = 2rs$. For these primes, we can get a Hamilton cycle by Magma.

Step 3: Show the existence of a H-cycle.

Let us come back to the proof of the lemma. Let $Y_1 = Y[\mathbf{B}]$, the subgraph of Y induced by $\mathbf{B}$. Then Y_1 is a Cayley graph on $\mathbb{Z}_{\frac{q-1}{2}}$. Since the valency of X is $\frac{q-1}{2}$, $d(B_1,\, B_\infty) = 1$, and $d(B_1,\, B_i) \le 2$, it follows from

$$\frac{1}{2}(\frac{q-1}{2} - 1 - 2) \ge \frac{1}{3} \cdot \frac{q-1}{2}$$

that Y_1 has at most two connected components. Since $\frac{q-1}{2}$ is odd, Y_1 must be connected. Now there are double edges between B_1 and B_i for some i. By Proposition 2.3, Y_1 contains a cycle passing the edge B_1B_i, say $\cdots B_jB_1B_i\cdots$. In Y, replacing the edge B_jB_1 by the path $B_jB_\infty B_1$, we get a H-cycle, say C for Y. By Proposition 2.4, C can be lifted to a H-cycle of X. □

Lemma 3.5. *Suppose that Δ is a non self-paired suborbit of length $\frac{q-1}{2}$. Then $X(G,\, \Delta)$ is hamiltonian.*

Proof. Let $X = X(G,\, \Delta \cup \Delta^*)$, where Δ is non self-paired and of length $\frac{q-1}{2}$. Let Y be the quotient graph induced by $\mathbf{B} \cup \{B_\infty\}$. Then by Lemma 3.1, we may set $\Delta = \{j,\, j+1\}^H$ and $\Delta^* = \{-j,\, -j-1\}^H$ where $j(j+1) \in N$, $j+1 \in S$, $j \in N$ and $j \in \mathbb{F}_q$. Then the neighborhood of $\{0,\, \infty\}$ is:

$$X_1(\{0,\, \infty\}) = \Delta \cup \Delta^* = \{\{j\theta^{-2k},\, (j+1)\theta^{-2k}\},\, \{(-j)\theta^{-2k},\, (-j-1)\theta^{-2k}\} \bigm| k \in \mathbb{F}_q\}.$$

Since $|\Delta \cup \Delta^*| = q-1$ and $\langle \bar{l} \rangle$ acts regularly on $\mathbf{B}$, $d(B_\infty,\, B_i) = 2$ for any $i = 1, 2, \cdots, \frac{q-1}{2}$.

The lemma will be proved by the following two steps:

Step 1: $d(B_k,\, B_i) \in \{0, 2, 4\}$ for any $i, k = 1, 2, \cdots, \frac{q-1}{2}$.

Since $\langle \bar{l} \rangle$ is regular on $\mathbf{B}$ and $\{0,\, 1\} \in B_1$, we may just consider $d(B_1,\, B_i) = d(\{0, 1\},\, B_i)$ for any $i = 1, 2, 3, \cdots, \frac{q-1}{2}$. Since $\overline{u}'$ maps $\{\infty,\, 0\}$ to $\{0,\, 1\}$, we know that

$$X_1(\{0,\,1\}) = \{\Delta,\,\Delta^*\}^{\overline{u}'}$$
$$= \{\{\frac{j\theta^{-2k}}{1+j\theta^{-2k}},\,\frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}}\},\,\{\frac{(-j)\theta^{-2k}}{1+(-j)\theta^{-2k}},\,\frac{(-j-1)\theta^{-2k}}{1+(-j-1)\theta^{-2k}}\}\mid k\in\mathbb{F}_q\}.$$

A vertex in $X_1(\{0,\,1\})$ is contained in B_i if and only if some of the following four systems of equations has solutions:

$$\frac{j\theta^{-2k}}{1+j\theta^{-2k}} = t, \quad \frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}} = t+i; \tag{3.4}$$

$$\frac{j\theta^{-2k}}{1+j\theta^{-2k}} = t+i, \quad \frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}} = t; \tag{3.5}$$

$$\frac{(-j)\theta^{-2k}}{1+(-j)\theta^{-2k}} = t, \quad \frac{(-j-1)\theta^{-2k}}{1+(-j-1)\theta^{-2k}} = t+i; \tag{3.6}$$

$$\frac{(-j)\theta^{-2k}}{1+(-j)\theta^{-2k}} = t+i, \quad \frac{(-j-1)\theta^{-2k}}{1+(-j-1)\theta^{-2k}} = t. \tag{3.7}$$

Solving Equation (3.4) and Equation (3.6), we get the respective equation

$$ij(j+1)u^2 \pm (2ij+i-1)u + i = 0,$$

where $u = \theta^{-2k}$. For each of these two equations, it has solutions for u if and only if

$$\delta_1 := (2ij+i-1)^2 - 4i^2j(j+1) = i^2 - (2+4j)i + 1 \in S^*.$$

Since the product of two solutions u_1 and u_2 is $(j(j+1))^{-1}$, a non-square, we know that either $u_1 \in S^*$ or $u_2 \in S^*$ if the above equation has solutions. Therefore, there exists exactly one solution for $\theta^{-2k} = u$ if and only if $\delta_1 \in S^*$, noting that every θ^{-2k} gives a unique t, equivalently, a unique vertex in the block B_i. Totally, two systems of equations give two vertices in the B_i.

Solving Equation (3.5) and Equation (3.7), we get respective equation

$$ij(j+1)u^2 \pm (2ij+i+1)u + i = 0,$$

where $u = \theta^{-2k}$. This equation has solutions for u if and only if

$$\delta_2 := (2ij+i+1)^2 - 4i^2j(j+1) = i^2 + (2+4j)i + 1 \in S^*.$$

Similarly, there exists exactly one solution for θ^{-2k} if and only if $\delta_2 \in S^*$. Totally, two systems of equations give two vertices in the B_i.

In summary, $d(B_1,\,B_i) = 2$ if and only if $\delta_1\delta_2 \in N$; and $d(B_1,\,B_i) = 0$ or 4 provided $\delta_1\delta_2 \in S$.

Step 2: Show the existence of a H*-cycle.*

Let $Y_1 = Y[\mathbf{B}]$, the subgraph of Y induced by $\mathbf{B}$. Then Y_1 is a Cayley graph on $\mathbb{Z}_{\frac{q-1}{2}}$. Since the valency of X is $q-1$, $d(B_1, B_\infty) = 2$, and $d(B_1, B_i) \le 4$, it follows from

$$\frac{1}{4}(q-1-4-2) \ge \frac{1}{3} \cdot \frac{q-1}{2}$$

that Y_1 has at most two connected components. Then, using the same arguments in Step 3 of Lemma 3.4, one may get a H-cycle of X. □

Lemma 3.6. *Suppose that* Δ *is a self-paired suborbit of length* $q-1$*. Then* $X(G, \Delta)$ *is hamiltonian.*

Proof. Let $X = X(G, \Delta)$, where Δ is self-paired and of length $q-1$. Let Y be the quotient graph induced by $\mathbf{B} \cup \{B_\infty\}$. Then by Lemma 3.1, we may set $\Delta = \{j, j+1\}^H$ where $j(j+1) \in S^*$ and either $j+1 \in N$ or $j \in S^*$. Then the neighborhood of $\{0, \infty\}$ is:

$$X_1(\{0, \infty\}) = \Delta = \{\{j\theta^{-2k}, (j+1)\theta^{-2k}\}, \{(-j)\theta^{-2k}, (-j-1)\theta^{-2k}\} \mid k \in \mathbb{F}_q\}.$$

Since $|\Delta| = q-1$ and $\langle \bar{l} \rangle$ acts regularly on $\mathbf{B}$, $d(B_\infty, B_i) = 2$ for any $i = 1, 2, \cdots, \frac{q-1}{2}$.

The lemma will be proved by the following two steps:

Step 1: $d(B_m, B_i) \le 4$ *for any* $i, m \in \mathbb{F}_q^*$.

Since $\langle \bar{l} \rangle$ is regular on $\mathbf{B}$ and $\{0, 1\} \in B_1$, we may just consider $d(B_1, B_i) = d(\{0, 1\}, B_i)$ for any $i \in \mathbb{F}_q^*$. Now,

$$X_1(\{0, 1\}) = \{\Delta\}^{\overline{u}'} = \{\{\frac{j\theta^{-2k}}{1+j\theta^{-2k}}, \frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}}\},$$

$$\{\frac{(-j)\theta^{-2k}}{1+(-j)\theta^{-2k}}, \frac{(-j-1)\theta^{-2k}}{1+(-j-1)\theta^{-2k}}\} \mid k \in \mathbb{F}_q\}.$$

A vertex in $X_1(\{0, 1\})$ is contained in B_i if and only if one of the following four systems of equations has solutions:

$$\frac{j\theta^{-2k}}{1+j\theta^{-2k}} = t, \quad \frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}} = t+i; \tag{3.8}$$

$$\frac{j\theta^{-2k}}{1+j\theta^{-2k}} = t+i, \quad \frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}} = t; \tag{3.9}$$

$$\frac{(-j)\theta^{-2k}}{1+(-j)\theta^{-2k}} = t, \quad \frac{(-j-1)\theta^{-2k}}{1+(-j-1)\theta^{-2k}} = t+i; \tag{3.10}$$

$$\frac{(-j)\theta^{-2k}}{1+(-j)\theta^{-2k}} = t+i, \quad \frac{(-j-1)\theta^{-2k}}{1+(-j-1)\theta^{-2k}} = t. \tag{3.11}$$

Solving Equation (3.8) and Equation (3.10) we get the respective equation

$$ij(j+1)u^2 \pm (2ij+i-1)u + i = 0,$$

where $u = \theta^{-2k}$. Each of these two equations has solutions for u only if

$$\delta_1 := (2ij+i-1)^2 - 4i^2j(j+1) = i^2 - (2+4j)i + 1 \in S.$$

(1) $\delta_1 \in S^*$: Since the product of two solutions u_1 and u_2 is $(j(j+1))^{-1}$, a square, we know that either $u_1,\, u_2 \in S^*$ or $u_1,\, u_2 \in N^*$. Therefore, there exist two solutions for $\theta^{-2k} = u$ only if $\delta_1 \in S^*$. Noting that every θ^{-2k} gives a unique t, equivalently, one vertex in the block B_i. Thus two systems of equations give two vertices in B_i.

(2) $\delta_1 = 0$: For these two equations, there is just one solution for u and it gives a unique t. Thus two systems of equations give one vertice in B_i.

Solving Equation (3.9) and Equation (3.11), we get respective equation

$$ij(j+1)u^2 \pm (2ij+i+1)u + i = 0,$$

where $u = \theta^{-2k}$. This equation has solutions for u if and only if

$$\delta_2 := (2ij+i+1)^2 - 4i^2j(j+1) = i^2 + (2+4j)i + 1 \in S.$$

Similarly, if $\delta_2 \in S^*$, there exist exactly two solutions for θ^{-2k}. Thus two equations give two vertices in B_i. If $\delta_2 = 0$, there exists one solution for θ^{-2k}. Thus we only get one vertex in B_i.

In summary, $d(B_1,\, B_i) = 2$ if and only if $\delta_1\delta_2 \in N$; $d(B_1,\, B_i) = 0$ or 4, provided $\delta_1\delta_2 \in S^*$; and $d(B_1,\, B_i) = 1$ or 3 if and only if $\delta_1\delta_2 = 0$.

Step 2: Show the existence of a H-cycle.

Let $Y_1 = Y[\mathbf{B}]$ be the subgraph of Y induced by $\mathbf{B}$. Then Y_1 is a Cayley graph on $\mathbb{Z}_{\frac{q-1}{2}}$. Since the valency of X is $q-1$, $d(B_1,\, B_\infty) = 2$, and $d(B_1,\, B_i) \le 4$, it follows from

$$\frac{1}{4}(q-1-4-2) \ge \frac{1}{3} \cdot \frac{q-1}{2}.$$

Then we get a H-cycle, with the same arguments as in Step 3 of Lemma 3.4. □

Lemma 3.7. *Suppose that Δ is a non self-paired suborbit of length $q-1$. Then $X(G,\, \Delta)$ is hamiltonian.*

Proof. In this case, $\Delta = \{1,\, 0\}^H$ and $\Delta^* = \{-1,\, 0\}^H$. Let $X = X(\Delta \cup \Delta^*)$ and Y the quotient graph induced by $\mathbf{B} \cup \{B_\infty\}$. Then the neighborhood of $\{0,\, \infty\}$ is:

$$X_1(\{0,\, \infty\}) = \Delta \cup \Delta^* = \{\{0,\, \theta^k\},\, \{\infty,\, \theta^k\} \mid k \in \mathbb{F}_q\}.$$

By observing the vertices of block B_∞, we get $d(B_\infty) = q-1$, and since $\langle \bar{l} \rangle$ is regular on $\mathbf{B}$, $d(B_\infty,\, B_i) = 2$ for any $i = 1,\, 2,\, \cdots,\, \frac{q-1}{2}$. Since $\overline{u}'$ maps $\{\infty,\, 0\}$ to $\{0,\, 1\}$, we know that

$$X_1(\{0,\, 1\}) = \{\Delta,\, \Delta^*\}^{\overline{u}'} = \{\{0,\, \frac{\theta^k}{1+\theta^k}\},\, \{1,\, \frac{\theta^k}{1+\theta^k}\} \mid k \in \mathbb{F}_q\}.$$

A direct computation shows $d(B_1) = 2$. Moreover, $d(B_1,\, B_i)$ is exactly the number of union of solutions of the following two equations:

$$\{0,\, \frac{\theta^k}{1+\theta^k}\} = \{t+i,\, t\} \quad \text{and} \quad \{1,\, \frac{\theta^k}{1+\theta^k}\} = \{t+i,\, t\}.$$

Solving them, we get four solutions:

$$\theta^k = \frac{-i}{1+i},\, t = -i; \quad \theta^k = \frac{i}{1-i},\, t = 0;$$

$$\theta^k = \frac{1-i}{i},\, t = 1-i; \quad \theta^k = \frac{-i-1}{i},\, t = 1.$$

Therefore, $d(B_1,\, B_i) = 4$.

Since $\langle \bar{l} \rangle$ is regular on $\mathbf{B}$, $d(B_j,\, B_i) = d(B_1,\, B_{i'})$ for some i' and $d(B_i) = d(B_1)$. Then we conclude that $d(B_i,\, B_j) = 4$ and $d(B_i) = 2$. Thus the graph $Y \setminus \{B_\infty\}$ is a complete graph. As before, X is hamiltonian. □

3.2 $\mathrm{soc}(G) = \mathrm{PSL}(2,\, q)$ and $H = \mathbb{D}_{q+1}$

Let $G = \mathrm{PSL}(2,\, q)$ and $H = \mathbb{D}_{q+1}$. Consider the action of G on the set $[G : H]$ of cosets of H in G, see row 2 of Table 1. Then $n = \frac{q(q-1)}{2} = 2rs$. This implies that $q \equiv 1 (\mathrm{mod}\ 4)$ and both q and $\frac{q-1}{4}$ are primes. So $r = \frac{q-1}{4}$ and $s = q$. Set $\mathbb{F}_q^* = \langle \theta \rangle$ and $\sqrt{-1} = \theta^{\frac{q-1}{4}}$. In $\mathrm{GL}(2,\, q)$, we set

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\, u' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\, l = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix},\, t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$t(x,\, y) = \begin{pmatrix} x & y\theta \\ y & x \end{pmatrix},\, t'(x,\, y) = \sqrt{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} t(x,\, y) = \sqrt{-1} \begin{pmatrix} x & -y\theta \\ y & -x \end{pmatrix},\, x \neq 0.$$

Then up to conjugacy, H may be chosen as

$$H = \{\overline{t(x,\, y)}, \overline{t'(x,\, y)} \mid x^2 - y^2\theta = 1\}.$$

Consider the action of $N_G(\langle \overline{u} \rangle) = \langle \overline{u} \rangle \rtimes \langle \bar{l} \rangle$ on the set of $\langle \overline{u} \rangle$-orbits (blocks) on $[G : H]$. Then $[G : H]$ can be divided into two parts, say $\mathbf{B}$ and $\mathbf{B}'$, where

$$\mathbf{B} = \{B_1,\, B_2,\, \cdots,\, B_{\frac{q-1}{4}}\}, \quad \mathbf{B}' = \{B_1',\, B_2',\, \cdots,\, B_{\frac{q-1}{4}}'\},$$

where $B_i = \{H\overline{u^j l^i} \mid j \in \mathbb{Z}_q\}$ and $B_i' = \{H\overline{tu^j l^i} \mid j \in \mathbb{Z}_q\}$, where $1 \leq i \leq \frac{q-1}{4}$.

Lemma 3.8. *Suppose $q \equiv 1 (\mathrm{mod}\ 4)$. Then for G acting on $[G : H]$,*

(1) *there are $\frac{q-3}{2}$ suborbits of length $\frac{q+1}{2}$, while $\frac{q-1}{4}$ of them $\{H\overline{l^i t}H \mid 1 \leq i \leq \frac{q-1}{4}\}$ are self-paired and $\frac{q-5}{4}$ of them $\{H\overline{l^i}H \mid 1 \leq i \leq \frac{q-1}{4}\}$ are non-self-paired suborbits;*

(2) *there are $\frac{q-1}{4}$ suborbits of length $q+1$, with the form $H\overline{u^i}H$, where $i^2 \in S^* \cap (4\theta + N)$. All of them are self-paired.*

Proof. Since $q+1 \equiv 3(\text{mod } 4)$, for any $g \in G$, $H \cap H^g$ is either $\mathbb{Z}_2$ or 1, so every suborbit is of length either $\frac{q+1}{2}$ or $q+1$.

(1) $|\Delta| = \frac{q+1}{2}$

Let $\Delta = HgH$ be a suborbit of length $\frac{q+1}{2}$. Then $H^g \cap H \cong Z_2$ and so α^g is an involution of H, where $\alpha = \overline{l^{\frac{q-1}{4}}} \in H$. Then $\alpha^g = \alpha^h$ for some $h \in H$, and so $gh^{-1} \in C_G(\alpha) = \langle \bar{l}, \bar{t} \rangle$. Since $HgH = Hgh^{-1}H$, we may choose $h = 1$ so that $g \in C_G(\alpha)$. Set $g = \overline{l^i}$ or $\overline{l^it}$ for some i. Moreover, direct computations show that for any two distinct elements $g_1, g_2 \in C_G(\alpha) = \langle \bar{l}, \bar{t} \rangle$, $Hg_1H = Hg_2H$ if and only if $g_1 = g_2\alpha$. Therefore, we have $\frac{q-1}{2}$ suborbits of length $\frac{q+1}{2}$. In particular, $HgH = Hg^{-1}H$ if and only if either $g^2 = 1$ or $g^{-1} = g\alpha$, where the second case gives $g \in H$. So we get $\frac{q-1}{4}$ self-paired suborbits HgH where g is non-central involution in $C_G(\alpha)$, noting $Hg\alpha H = HgH$. So the remaining $\frac{q-5}{4}$ suborbits of length $\frac{q+1}{2}$ are non self-paired.

(2) $|\Delta| = q+1$

Let first consider the suborbits $D = H\overline{u^i}H$ where $i \in \mathbb{Z}_q^*$. From the arguments in (1), we know that $|\Delta| = q+1$. Since $H\overline{u^i}H = H\alpha\overline{u^i}\alpha H = H\overline{u^{-i}}H$, Δ is self-paired. Set $g = \overline{u^i}$.

Suppose that $H^g \cap H = Z_2$, that is

$$\overline{u^{-i}t'(x_1, y_1)u^i} \in H,$$

which implies $2x_1 - iy_1 = 0$. Insetting it in $x_1^2 - y_1^2\theta = 1$, we get

$$i^2 = 4\theta + 4x_1^{-2} \in S^* \cap (4\theta + S^*).$$

Therefore, Δ is of length $q+1$ if and only if $i^2 \in S^* \cap (4\theta + N)$. By Proposition 2.1, $|S^* \cap (4\theta + N)| = \frac{q-1}{4}$. Check that $H\overline{u^i}H = H\overline{u^j}H$ if and only if $i = \pm j$. Therefore, we get $\frac{q-1}{4}$ suborbits of length $q+1$.

Since $1+\frac{q-3}{2}\frac{q+1}{2}+\frac{q-1}{4}(q+1) = \frac{q(q-1)}{2} = |[G:H]|$, we already find all suborbits. □

In what follows we deal with all cases of suborbits Δ in Lemma 3.8, separately.

Lemma 3.9. *Suppose that Δ is a self-paired suborbit of length $\frac{q+1}{2}$. Then $X(G, \Delta)$ is hamiltonian.*

Proof. Let $X = X(G, \Delta)$, where Δ is self-paired and of length $\frac{q+1}{2}$. From the last lemma, $\Delta = H\overline{l^kt}H$ for some k. Note $\frac{q-1}{4} = r$ is a prime, the two smallest values for q are 13 and 29. One may find a H-cycle by Magma for $q = 13$ and 29. So let $q \neq 13, 29$. First we give a remark.

Remark: Suppose we may get two facts: ① for any $B' \in \mathbf{B}'$, $d(H, B') = 0, 2$ or 4; ② $d(H, \cup_{B' \in \mathbf{B}'} B') \geq 5$. Then H is adjacent to at least two blocks B'_i, B'_j in $\mathbf{B}$ such that $d(H, B'_i) = 2$ or 4. Let Y be the block graph. Then Y is a bipartite graph of order $2r$, where $r = \frac{q-1}{4}$ is a prime. Note that $H \in B_r$. Since $\langle \bar{l} \rangle / \langle \bar{l}^r \rangle$ acts regularly on both $\mathbf{B}$ and $\mathbf{B}'$, we may set $B'^d_i = B'_j$ for some $d \in \langle \bar{l} \rangle / \langle \bar{l}^r \rangle$. Then we get a H-cycle of Y:

$$B'_i, B_r, B'^d_i, B^d_r, B'^{d^2}_i, \cdots, B^{d^{r-1}}_r, B'_i.$$

Then by Proposition 2.4, we may find a H-cycle for $X(G, \Delta)$.

Now we continue to prove the lemma. Clearly, the neighborhood of H is:

$$X_1(H) = \Delta = H\overline{l^k t}H = \{H\overline{l^k tt(x_1, y_1)} \mid x_1^2 - y_1^2\theta = 1\}.$$

The vertex $H\overline{l^k tt(x_1, y_1)}$ is contained in B_i' if and only if

$$H\overline{l^k tt(x_1, y_1)} = H\overline{tu^j l^i}, \text{ for some } j,$$

if and only if

$$\overline{l^k tt(x_1, y_1)}(\overline{tu^j l^i})^{-1} \in H,$$

if and only if one of the following two systems of equations with unknowns j, i, x_1 and y_1 has solutions corresponding to $(\varepsilon, \eta) = (1, -1)$ and $(-1, 1)$:

$$\begin{cases} y_1 j\theta^{2k} & = x_1(\theta^{2k+2i} - \varepsilon), \\ y_1(\theta^{2i+2} + \eta\theta^{2k}) & = x_1\theta j, \\ x_1^2 - y_1^2\theta & = 1. \end{cases} \tag{3.12}$$

Every such system has the same solutions with

$$\begin{cases} y_1^2 = \frac{\theta^{2k+1}}{\eta\theta^{4k}+\theta^2\varepsilon}\theta^{2i} - \frac{\varepsilon\theta}{\eta\theta^{4k}+\theta^2\varepsilon}, & (i) \\ y_1^2 = \theta^{-1}x_1^2 - \theta^{-1}, & (ii) \\ j \ = (\theta^{2i} - \varepsilon\theta^{-2k})\frac{x_1}{y_1}. & (iii) \end{cases} \tag{3.13}$$

From (iii), we know that given a solution for x_1^2, y_1^2 and i, we have two values of j, that is $\pm j$. Then the possible values for $d(H, B_i')$ is $0, 2$ or 4, noting we have two choices for (ε, η), showing fact ①.

Set $b = -\theta^{-1}$, $a_1 = \frac{\theta^{2k+1}}{\eta\theta^{4k}+\theta^2\varepsilon}$ and $a_2 = -\frac{\varepsilon\theta}{\eta\theta^{4k}+\theta^2\varepsilon}$. Then $a_1, a_2 \neq 0$ and $a_2 \neq b$. From (i) and (ii), we get that either

$$y_1^2 \in S^* \cap (S^* + a_2) \cap (N + b) \text{ if } a_1 \in S^* \quad \text{or} \quad y_1^2 \in S^* \cap (N + a_2) \cap (N + b) \text{ if } a_1 \in N.$$

By using Lemma 3.3, we get that the number of solutions for y_1^2 is at least $\lfloor \frac{1}{8}(q - 11 - 2\sqrt{q}) \rfloor$, which implies that the number of solutions for j, i, x_1, y_1 is at least $2\lfloor \frac{1}{8}(q - 11 - 2\sqrt{q}) \rfloor$, for given (ε, η). In other words, $d(H, \cup_{B' \in \mathbf{B}'} B')$ is at least $2\lfloor \frac{1}{8}(q - 11 - 2\sqrt{q}) \rfloor$. Moreover, $2\lfloor \frac{1}{8}(q - 11 - 2\sqrt{q}) \rfloor \geq 5$, showing fact ②. □

Lemma 3.10. *Suppose that Δ is a non self-paired suborbit of length $\frac{q+1}{2}$. Then $X(G, \Delta \cup \Delta^*)$ is hamiltonian.*

Proof. Let $X = X(G, \Delta \cup \Delta^*)$, where Δ is non self-paired and of length $\frac{q+1}{2}$. From Lemma 3.8, $\Delta = H\overline{l^k}H$ and $\Delta^* = H\overline{l^{-k}}H$ for some integer k. Note $\frac{q-1}{4} = r$ is a prime, the three smallest values for q are 13, 29 and 53. One may find a H-cycle by Magma for $q = 13, 29$ and 53. So let $q \neq 13, 29, 53$.

From the remark in last lemma, it suffices to show two facts: (i) for any $B' \in \mathbf{B}'$, $d(H, B') = 0$, 2, 4, 6 or 8; (ii) $d(H, \cup_{B' \in \mathbf{B}'} B') \geq 9$.

Check that the neighborhood of H is:

$$X_1(H) = \Delta \cup \Delta^* = \{H\overline{l^k t(x_1,\, y_1)},\, H\overline{l^{-k} t(x_1,\, y_1)} \mid x_1^2 - y_1^2\theta = 1\}.$$

The vertex $H\overline{l^k t(x_1,\, y_1)}$ and $H\overline{l^{-k} t(x_1,\, y_1)}$ are contained in B_i' if and only if either

$$H\overline{l^k t(x_1,\, y_1)} = H\overline{t u^j l^i},\ \text{or}\ H\overline{l^{-k} t(x_1,\, y_1)} = H\overline{t u^j l^i},\ \text{for some}\ j$$

if and only if either

$$\overline{l^k t(x_1,\, y_1)}(\overline{t u^j l^i})^{-1} \in H,\ \text{or}\ \overline{l^{-k} t(x_1,\, y_1)}(\overline{t u^j l^i})^{-1} \in H$$

if and only if one of the following four systems of equations with unknowns j, i, x_1 and y_1 has solutions corresponding to $(\varepsilon,\, \eta) = (1,\, -1),\ (1,\, 1),\ (-1,\, -1)$ or $(-1,\, 1)$:

$$\begin{cases} y_1(\theta^{i+\epsilon k+1} - \eta\theta^{-i-\varepsilon k}) & = x_1 j \theta^{\varepsilon k - i}, \\ y_1 j \theta^{-\varepsilon k - i + 1} & = x_1(\theta^{i-\epsilon k+1} - \eta\theta^{-i+\varepsilon k}), \\ x_1^2 - y_1^2\theta & = 1. \end{cases} \tag{3.14}$$

Every such system has the same solutions with

$$\begin{cases} y_1^2 = \frac{\theta^{2i}\theta}{\eta\theta\theta^{2\varepsilon k} - \eta\theta\theta^{-2\varepsilon k}} - \frac{\eta\theta^{2\varepsilon k}}{\eta\theta\theta^{2\varepsilon k} - \eta\theta\theta^{-2\varepsilon k}}, & (i) \\ y_1^2 = \theta^{-1}x_1^2 - \theta^{-1}, & (ii) \\ j\ = \frac{\theta^{i+\varepsilon k+1} - \eta\theta^{-i-\varepsilon k}}{\theta^{\varepsilon k - i}} \frac{y_1}{x_1}. & (iii) \end{cases} \tag{3.15}$$

From (iii), we know that given a solution for x_1^2, y_1^2 and i, we have two values of j, that is $\pm j$. Then the possible values for $d(H,\, B_i')$ is $0, 2, 4, 6$ or 8, noting we have four choices for $(\varepsilon,\, \eta)$, showing fact (i).

Set $b = -\theta^{-1}$, $a_1 = \eta\theta\theta^{2\varepsilon k} - \eta\theta\theta^{-2\varepsilon k}$ and $a_2 = -\frac{\eta\theta^{2\varepsilon k}}{\eta\theta\theta^{2\varepsilon k} - \eta\theta\theta^{-2\varepsilon k}}$. Then $a_1, a_2 \neq 0$ and $a_2 \neq b$. From (i) and (ii), we get that either

$$y_1^2 \in S^* \cap (N + a_2) \cap (N + b)\ \text{if}\ a_1 \in S^* \quad \text{or} \quad y_1^2 \in S^* \cap (S^* + a_2) \cap (N + b)\ \text{if}\ a_1 \in N.$$

By using Lemma 3.3, we get that the number of solutions for y_1^2 is at least $\lfloor \frac{1}{8}(q - 11 - 2\sqrt{q}) \rfloor$, which implies that the number of solutions for j, i, x_1, y_1 is at least $2\lfloor \frac{1}{8}(q - 11 - 2\sqrt{q}) \rfloor$, for given $(\varepsilon,\, \eta)$. In other words, $d(H,\, \cup_{B' \in \mathbf{B}'} B')$ is at least $2\lfloor \frac{1}{8}(q - 11 - 2\sqrt{q}) \rfloor$. Moreover, $2\lfloor \frac{1}{8}(q - 11 - 2\sqrt{q}) \rfloor \geq 9$, showing fact (ii). □

Lemma 3.11. *Suppose that Δ is a self-paired suborbit of length $q + 1$. Then $X(G,\, \Delta)$ is hamiltonian.*

Proof. Let $X = X(G,\, \Delta)$, where Δ is self-paired and of length $q + 1$. From Lemma 3.8, $\Delta = H\overline{u^k}H$ for some integer k. Note $\frac{q-1}{4} = r$ is a prime.

If we may get two facts: (i) for any $B' \in \mathbf{B}'$, $d(H,\, B') = 0, 2$ or 4; (ii) for any $B' \in \mathbf{B}'$, $d(H,\, B') = 0, 2$ or 4, then every vertex in block graph has the valency at least $\frac{(q+1)-2}{4} = \frac{q-1}{4} = \frac{1}{2}\frac{q-1}{2}$. So Y contains a H-cycle. Since $d(B_i,\, B_j')$ is even, this cycle can lift a H-cycle for $X(G,\, \Delta)$ by Proposition 2.4.

In fact, check that the neighborhood of H is:

$$X_1(H) = \Delta = H\overline{u^k}H = \{H\overline{u^k t(x_1,\, y_1)}, H\overline{u^k t'(x_1,\, y_1)} \mid x_1^2 - y_1^2\theta = 1\}.$$

By observing the neighbor, one can see these neighbors contained in $B_{\frac{q-1}{4}}$ are just $H\overline{u^k}$ and $H\overline{u^{-k}}$, which implies $d(B_{\frac{q-1}{4}}) = 2$. The vertex $H\overline{u^k t(x_1, y_1)}$ and $H\overline{u^k t'(x_1, y_1)}$ are contained in B_i if and only if either:

$$H\overline{u^k t(x_1, y_1)} = H\overline{u^j l^i}, \text{ or } H\overline{u^k t'(x_1, y_1)} = H\overline{u^j l^i}$$

if and only if either:

$$\overline{u^k t(x_1, y_1)}(\overline{u^j l^i})^{-1} \in H, \text{ or } \overline{u^k t'(x_1, y_1)}(\overline{u^j l^i})^{-1} \in H$$

if and only if one of the following systems of equations with unknowns j, i, x_1 and y_1 has solutions corresponding to $(\epsilon, \eta, \gamma, \delta) = (-1, 1, -1, 1)$, $(1, -1, 1, -1)$, $(-1, -1, -1, -1)$ or $(1, 1, 1, 1)$:

$$\begin{cases} \epsilon\theta^{-i} y_1 j & = (x_1 + ky_1)\theta^{-i} - \eta x_1 \theta^i, \\ \gamma(x_1 + ky_1)\theta^{-i} j & = y_1\theta^{-i}\theta - \delta\theta^i(y_1\theta + kx_1), \\ x_1^2 - y_1^2\theta & = 1. \end{cases} \tag{3.16}$$

This system has the same solutions with

$$j = \frac{(\theta^{-i} - \eta\theta^i)x_1}{\varepsilon\theta^{-i}y_1} + k\varepsilon^{-1} \text{ where } \delta\varepsilon\theta^{2i} = \gamma(k^2y_1^2 + 2kx_1y_1 + 1).$$

Calculating the equation $\delta\varepsilon\theta^{2i} = \gamma(k^2y_1^2 + 2kx_1y_1 + 1)$ we could get

$$(4k^2\theta - k^4)u^2 + (2k^2 + 2\delta\varepsilon\gamma\theta^{2i}k^2)u - (\delta\varepsilon\theta^{2i} - \gamma)^2 = 0,$$

where $u = y_1^2$. Since the product of the two solutions is $\frac{-(\delta\varepsilon\theta^{2i}-\gamma)^2}{4k^2\theta-k^4}$, a non-square (as $4\theta - k^2 \in N$), there exists at most one solution for $u = y_1^2$. It is easy to see that there are two solutions for j. Since there are just two different equations for $\delta\varepsilon\theta^{2i} = \gamma(k^2y_1^2 + 2kx_1y_1 + 1)$, there are at most 4 solutions for j, that is $d(H, B_i) = 0$, 2 or 4, showing fact (i).

The vertex $H\overline{u^k t(x_1, y_1)}$ and $H\overline{u^k t'(x_1, y_1)}$ are contained in B_i' if and only if either

$$H\overline{u^k t(x_1, y_1)} = H\overline{tu^j l^i} \text{ or } H\overline{u^k t'(x_1, y_1)} = H\overline{tu^j l^i}$$

if and only if either

$$\overline{u^k t(x_1, y_1)}(\overline{tu^j l^i})^{-1} \in H \text{ or } \overline{u^k t'(x_1, y_1)}(\overline{tu^j l^i})^{-1} \in H$$

if and only if one of the following systems of equations with unknowns j, i, x_1 and y_1 has solutions corresponding to $(\epsilon, \eta, \gamma, \delta) = (1, -1, 1, -1)$, $(1, 1, 1, 1)$, $(-1, -1, -1, -1)$ or $(-1, 1, -1, 1)$:

$$\begin{cases} -(x_1 + ky_1)\theta^{-i} j & = \eta y_1 \theta^{-i} - \epsilon(y_1\theta + kx_1)\theta^{i-1} \\ -y_1\theta^{-i}\theta j & = \delta(x_1 + ky_1)\theta^{-i} - \gamma x_1\theta^{i-1}\theta \\ x_1^2 - y_1^2\theta & = 1. \end{cases} \tag{3.17}$$

This system has the same solutions with

$$j = \frac{\delta\theta^{-i} - \gamma\theta^{i}\theta}{-\theta^{-i}\theta}\frac{x_1}{y_1} - \frac{k\delta}{\theta} \text{ where } \gamma\theta^{2i}\theta = \delta(k^2y_1^2 + 2kx_1y_1 + 1).$$

Calculating the equation $\gamma\theta^{2i}\theta = \delta(k^2y_1^2 + 2kx_1y_1 + 1)$ we could get

$$(4k^2\theta - k^4)u^2 + (2k^2 + 2\delta\gamma k^2\theta^{2i}\theta)u - (-\gamma\theta^{2i}\theta + \delta)^2 = 0,$$

where $u = y_1^2$. Since the product of the two solutions is $\frac{-(-\gamma\theta^{2i}\theta+\delta)^2}{4k^2\theta-k^4}$, a non-square (as $4\theta - k^2 \in N$), there exists at most one solution for $u = y_1^2$ and it is easy to see there are two solutions for j. Since there are just two different equations for $\gamma\theta^{2i} = \delta(k^2y_1^2 + 2kx_1y_1 + 1)$, there are at most 4 solutions for j, that is $d(B_{\frac{q-1}{4}}, B_i') = 0,\ 2$ or 4, showing fact (ii). □

3.3 Groups in Table 2

In this subsection, we shall deal with the groups in Table 2, separately.

Lemma 3.12. *Let G be one of groups in rows* 1 *and* 2 *of* Table 2. *Then every orbital graph of G contains a Hamilton cycle.*

Proof. Let $T = \mathrm{PSL}(m, q)$ where $m = 4$ or 5. It suffices to consider the group T. We shall deal with two cases: $m = 4$ and $m = 5$, separately.

Case 1: $m = 4$.

Let Ω be the set of 2-dim. subspaces of a space V of dimension 4. Then $n = \frac{(q^4-1)(q^3-1)}{(q-1)(q^2-1)} = (q^2 + q + 1)(q^2 + 1)$, where $s = q^2 + q + 1$ and $r = \frac{q^2+1}{2}$ are two primes. Consider a subspace W_0 of dimension $d(W_0) = 2$. Then T has two nontrivial suborbits relative to W_0:

$$\Delta_1 = \{W \in \Omega \mid d(W \cap W_0) = 1\} \quad \text{and} \quad \Delta_2 = \{W \in \Omega \mid d(W \cap W_0) = 0\},$$

where $r_1 := |\Delta_1| = \frac{q^4-q}{q^2-q} = \frac{q^3-1}{q-1}$ and $r_2 := |\Delta_2| = n - 1 - r_1$. Since $r_2 \geq \frac{n}{2}$, the corresponding orbital graph $\Gamma(T, \Delta_2)$ has a H-cycle.

Now we are considering $X(T, \Delta_1)$. Take a projective point $\langle\alpha\rangle$ and extend it into a base $\alpha, \alpha_1, \alpha_2, \alpha_3$ of V. Let $\Sigma(\alpha)$ be the set of all 2-dim. subspaces containing α. Then $|\Sigma(\alpha)| = q^2 + q + 1$. Since $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ contains exactly $q^2 + q + 1$ points and for any two distinct points β, β' in $\langle\alpha_1, \alpha_2, \alpha_3\rangle$, $\langle\alpha, \beta\rangle \neq \langle\alpha, \beta'\rangle$, one may see

$$\Sigma(\alpha) = \{\langle\alpha, \beta\rangle \mid \beta \in \langle\alpha_1, \alpha_2, \alpha_3\rangle\}.$$

Let $\langle h\rangle$ be the Singer subgroup of $\mathrm{PSL}(3, q)$ and $\beta \in \langle\alpha_1, \alpha_2, \alpha_3\rangle$. Since $s = q^2 + q + 1$ is a prime, $\langle\beta\rangle, \langle\beta^h\rangle, \langle\beta^{h^2}\rangle, \cdots, \langle\beta^{h^{s-1}}\rangle$ are all the projective points of $\langle\alpha_1, \alpha_2, \alpha_3\rangle$. Denote $\beta^{h^i} = \beta_i$. Since the subgraph induced by $\Sigma(\alpha)$ is a complete graph, we may consider a H-cycle of the subgraph, say

$$\langle\alpha, \beta_0\rangle, \langle\alpha, \beta_1\rangle, \langle\alpha, \beta_2\rangle, \cdots, \langle\alpha, \beta_{s-2}\rangle, \langle\alpha, \beta_{s-1}\rangle, \langle\alpha, \beta_0\rangle,$$

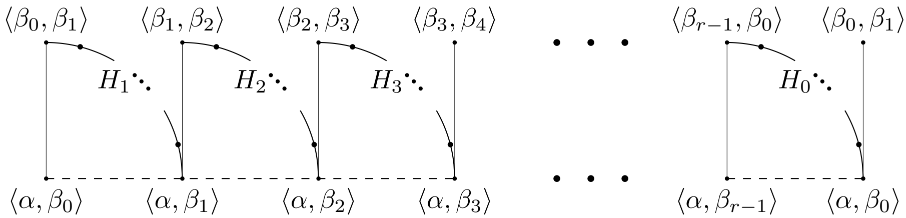

Figure 1:

where $\beta_i \in \langle\alpha_1,\, \alpha_2,\, \alpha_3\rangle$ and $s = q^2 + q + 1$.

Set

$$A = \{\langle\beta_i,\, \beta_{i+1}\rangle,\, \langle\beta_{s-1},\, \beta_0\rangle | i = 0,\, 1,\, \cdots,\, s-2\} = \{\langle\beta,\, \beta^h\rangle^{h^i} | i = 0,\, 1,\, \cdots,\, s-1\},$$

$$X_i = \Sigma(\beta_i) \setminus (\bigcup_{j=1}^{i-1} \Sigma(\beta_j) \bigcup A),\ i \in \{1,\, 2,\, \cdots,\, s-1\},$$

$$X_0 = \Sigma(\beta_0) \setminus (\bigcup_{j=1}^{s-1} \Sigma(\beta_j) \bigcup A).$$

Since every 2-subspace $\langle\eta, \gamma\rangle$ can be expressed as $\langle\eta,\, \gamma - \frac{b_0}{a_0}\eta\rangle$, where $\eta = a_0\alpha + a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$ and $\gamma = b_0\alpha + b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3$, every 2-subspace of V is contained in $(\bigcup_{i=0}^{s-1} X_i) \bigcup A$. Moreover, from the definition, we know that $X_0,\, X_1,\, \cdots,\, X_{s-1},\, A$ are mutually disjoint.

Now we are ready to find a H-cycle for $X(T,\, \Delta_1)$. For $i = 0,\, 1,\, \cdots,\, r-2$, consider a H-path H_{i+1} in the subgraph induced by $X_{i+1} \bigcup \{\langle\beta_i,\, \beta_{i+1}\rangle\}$ with the starting vertex $\langle\beta_i,\, \beta_{i+1}\rangle$ and the ending vertex $\langle\alpha,\, \beta_{i+1}\rangle$. Consider H_0 in the subgraph induced by $X_0 \bigcup \{\langle\beta_{s-1},\, \beta_0\rangle\}$ with the starting vertex $\langle\beta_{s-1},\, \beta_0\rangle$ and the ending vertex $\langle\alpha,\, \beta_0\rangle$. Then by replacing every arc $(\langle\alpha,\, \beta_i\rangle, \langle\alpha,\, \beta_{i+1}\rangle)$ by the path $(\langle\alpha,\, \beta_i\rangle,\, H_{i+1})$ and the arc $(\langle\alpha,\, \beta_{s-1}\rangle,\, \langle\alpha,\, \beta_0\rangle)$ by the path $(\langle\alpha,\, \beta_{s-1}\rangle,\, H_0)$, we get a cycle:

$$\langle\alpha,\, \beta_0\rangle,\, H_1,\, H_2,\, \cdots,\, H_{s-1},\, H_0,$$

which is clearly a H-cycle of $X(T,\, \Delta_1)$, as shown in Figure 1.

Case 2: $m = 5$.

Let Ω be the set of 2-dim. subspaces of V. Then

$$n = |\Omega| = \frac{(q^5-1)(q^4-1)}{(q-1)(q^2-1)} = (q^4 + \cdots + 1)(q^2+1) = 2rs.$$

Then $s = q^4 + \cdots + 1$ is a prime and $r = \frac{q^2+1}{2}$ are two prime. Let $S = \langle h\rangle$ be a Singer subgroup of PSL$(5, q)$, where $|S| = s$. Take a projective point α. Then $\alpha,\, \alpha^h,\, \cdots,\, \alpha^{h^{s-1}}$ are all the projective points. Set $W_i = \langle\alpha,\, \alpha^{h^i}\rangle$ where $i = 1,\, 2,\, \cdots,\, s-1$. Then G has two nontrivial suborbits relative to W_1:

$$\Delta_1 = \{W \in \Omega | d(W \cap W_1) = 1\} \quad \text{and} \quad \Delta_2 = \{W \in \Omega | d(W \cap W_1) = 0\},$$

where

$$r_1 := |\Delta_1| = (\frac{q^4}{q-1} - 1)(q+1) = q(q+1)(q^2+q+1),$$
$$r_2 := |\Delta_2| = \frac{(q^5-q^2)(q^5-q^3)}{(q^2-1)(q^2-q)} = q^4(q^2+q+1).$$

Since $r_2 \geq \frac{n}{2}$, the corresponding orbital graph $X(T, \Delta_2)$ has a H-cycle.

Now we are considering $X(T, \Delta_1)$. Let S_i be the path

$$W_i,\, W_i^{h^i},\, W_i^{h^{2i}},\, W_i^{h^{3i}},\, \cdots,\, W_i^{h^{(s-1)i}}.$$

Since $\langle h^i \rangle$ acts nontrivially on W_i and it is of order a prime s, $\langle h^i \rangle$ moves W_i. Since every 2-subspace must be contained in some clique and either $|S_i \cap S_j| = 0$ or $S_i = S_j$ for any two distinct cliques S_i and S_j, we could pick up $q^2 + 1$ distinct cliques which cover all 2-dim. subspaces, denoted by $W_{\mu_1}, W_{\mu_2}, \cdots, W_{\mu_{q^2+1}}$. Then we can get a H-cycle of $X(T, \Delta_1) : W_{\mu_1}, W_{\mu_1}^{h^{\mu_1}}, W_{\mu_1}^{h^{2\mu_1}}, \cdots, W_{\mu_1}^{h^{(s-1)\mu_1}}, W_{\mu_2}, W_{\mu_2}^{h^{\mu_2}}, W_{\mu_2}^{h^{2\mu_2}}, W_{\mu_2}^{h^{3\mu_2}}, \cdots,$ $W_{\mu_{q^2+1}}^{h^{(s-1)\mu_{q^2+1}}}, W_{\mu_1}$. □

Lemma 3.13. *Every orbital graph of* $G = \mathrm{P}\Omega^-(2m, q)$ *in row* 3 *of* Table 2 *is hamiltonian.*

Proof. Let $G = \mathrm{P}\Omega^-(2m, q)$ act on n totally singular (t.s.) 1-spaces, where $n = \frac{(q^m+1)(q^{m-1}-1)}{q-1} = 2rs$ and $m = 2^{2^l}$. Then $m-1$ is a prime. Since $m - 1 = (2^{2^{l-1}} - 1)(2^{2^{l-1}} + 1)$, we get $2^{2^{l-1}} - 1 = 1$, which implies $l = 1$ and then $m = 4$. Now $r =: \frac{q^3-1}{q-1}$ is a prime. Let Ω be the set of all t.s.1-spaces. Recall that $\mathrm{SO}^-(8, q) \leq \mathrm{GL}(8, q)$ and $|\mathrm{GL}(8, q)| = q^{28}\Pi_{i=1}^{8}(q^i - 1)$. To describe $\mathrm{SO}^-(8, q)$, take a symmetric bilinear form, given by the following matrix:

$$J = \begin{pmatrix} 0 & E_3 & 0 \\ E_3 & 0 & 0 \\ 0 & 0 & J_2 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 & 0 \\ 0 & -t \end{pmatrix}, \quad t \in N.$$

Let $\langle A \rangle$ be a Singer subgroup of $\mathrm{GL}(3, q)$, $C = A^{q-1}$ and $D = (C^{-1})'$, where C' denotes the transpose of C. Set $B = C \bigoplus (C')^{-1} \bigoplus E_2$, the block diagonal matrix. Then we have $BJB' = J$, which means $B \in \mathrm{SO}^-(8, q)$. Since $\overline{B}$ is of prime order, $\overline{B} \in (\mathrm{PSO}^-(8, q))' = \mathrm{P}\Omega^-(8, q)$. Set $S = \langle \overline{B} \rangle$ and $\alpha = (1, 0, \cdots, 0)$. Then there are two nontrivial suborbits for the action of $G_{\langle \alpha \rangle}$ relative to $\langle \alpha \rangle$, see [22]:

$$\Delta_1 = \{\langle \beta \rangle \in \Omega \setminus \{\langle a \rangle\} \mid (\alpha, \beta) = 0\} \quad \text{and} \quad \Delta_2 = \{\langle \beta \rangle \in \Omega \setminus \{\langle a \rangle\} \mid (\alpha, \beta) \neq 0\},$$

where $|\Delta_1| = q^5 + q^4 + q^2 + q$ and $|\Delta_2| = q^6$. Since $|\Delta_2| \geq \frac{1}{2}n$, we only need to consider $X(G, \Delta_1)$.

Noting that S acts semiregularly on Ω, we consider the block graph $\overline{X}$ induced by S-orbits, where $V(\overline{X}) = q^4 + 1$. For any $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \Omega$, where $\gamma_1 = (c_1, c_2, c_3)$, $\gamma_2 = (c_4, c_5, c_6)$ and $\gamma_3 = (c_7, c_8)$, we have $\gamma \overline{B}^i J \alpha' = 0$ if and only if $\gamma_2 D^i \alpha' = 0$, that is $\gamma_2 D^i = (0, c_5', c_6')$ for some c_5', c_6'. Since $\langle C \rangle$ (and so $\langle D \rangle$) is regular on nonzero 1-spaces, we know that α has $q + 1$ (resp. $q^2 + q$) neighbors in the block γ^S if $\gamma \notin \alpha^S$ (resp. $\gamma \in \alpha^S$). From $((q^5 + q^4 + q^2 + q) - (q^2 + q))/(q + 1) = q^4$ we know that $\overline{X}$ is a complete graph. By Propsosition 2.4, $X(G, \Delta_1)$ is hamiltonian. □

Lemma 3.14. *Every orbital graph of* $G = \mathrm{P}\Omega^+(2m, q)$ *in row* 4 *of* Table 2 *is hamiltonian.*

Proof. Let $G = \mathrm{P}\Omega^+(2m, q)$ act on n totally singular 1-spaces, where the degree $n = \frac{(q^m-1)(q^{m-1}+1)}{q-1} = 2rs$, $m = 2^{2^l}+1$, and $s = \frac{q^m-1}{q-1}$ and $r = \frac{q^{m-1}+1}{2}$ are primes. Let Ω be the set of all totally singular 1-spaces. Recall that $\mathrm{SO}^+(2m, q) \le \mathrm{GL}(2m, q)$. To describe $\mathrm{SO}^+(2m, q)$, take a symmetric bilinear form, given by the following matrix:

$$J = \begin{pmatrix} 0 & E_m \\ E_m & 0 \end{pmatrix}.$$

Let $\langle A\rangle$ be a Singer subgroup of $\mathrm{GL}(m, q)$, $C = A^{q-1}$ and $D = (C^{-1})'$, where C' denotes the transpose of C. Set $B = C\bigoplus(C')^{-1}$. Then we have $BJB' = J$, which means $B \in \mathrm{SO}^+(2m, q)$. Since B is of prime order, $\overline{B} \in (\mathrm{PSO}^+(m,q))' = \mathrm{P}\Omega^+(m, q)$. Set $S = \langle\overline{B}\rangle$ and $\alpha = (1, 0, \cdots, 0)$. Then there are two nontrivial suborbits for the action of $G_{\langle\alpha\rangle}$ relative to $\langle\alpha\rangle$, see By [22]:

$$\Delta_1 = \{\langle\beta\rangle \in \Omega \setminus \{\langle a\rangle\} \mid (\alpha, \beta) = 0\} \quad \text{and} \quad \Delta_2 = \{\langle\beta\rangle \in \Omega \setminus \{\langle a\rangle\} \mid (\alpha, \beta) \ne 0\},$$

where $|\Delta_1| = \frac{(q^{m-1}+q)(q^{m-1}-1)}{q-1}$ and $|\Delta_2| = q^{2m-2}$. Since $|\Delta_2| \ge \frac{1}{2}n$, we only need to consider $X(G, \Delta_1)$.

Noting that S acts semiregularly on Ω, we consider the block graph $\overline{X}$ induced by S-orbits, where $V(\overline{X}) = q^{m-1}+1$. For any $\gamma = (\gamma_1, \gamma_2) \in \Omega$, we have $\gamma\overline{s}^i J\alpha' = 0$ if and only if $\gamma_2 D^i\alpha' = 0$, which implies that the first coordinate of $\gamma_2 D^i$ is 0. Since $\langle C\rangle$ (and so $\langle D\rangle$) is regular on nonzero 1-spaces, we know that α has $\frac{q^{m-1}-1}{q-1}$ (resp. $\frac{q^m-1}{q-1} - 1$) neighbors in the block γ^S if $\gamma \notin \alpha^S$ (resp. $\gamma \in \alpha^S$). From $(\frac{(q^{m-1}+q)(q^{m-1}-1)}{q-1} - (\frac{q^m-1}{q-1} - 1))/(\frac{q^{m-1}-1}{q-1}) = q^{m-1}$ we know that $\overline{X}$ is a complete graph. By Proposition 2.4, $X(G, \Delta_1)$ is hamiltonian. □

Lemma 3.15. *Vertex-transitive graphs arising from the action of A_c on 2-subsets given in row 6 of* Table 2 *are hamiltonian.*

Proof. Let $\Omega = \{\alpha_1, \alpha_2, \cdots, \alpha_c\}$, where $c \ge 5$. Then we only have the following two orbital graphs:

(1) Two subsets are adjacent if and only if they intersect at a single point. In this case, the orbital graph is the Johnson graph $J(c, 2)$. Then we may get a H-cycle as the following way:

first pick up a cycle of c vertices, say $\{\alpha_1, \alpha_2\}, \{\alpha_2, \alpha_3\}, \{\alpha_3, \alpha_4\}, \cdots, \{\alpha_{c-1}, \alpha_c\}, \{\alpha_c, \alpha_1\}, \{\alpha_1, \alpha_2\}$; then

replace the edge $\{\alpha_1, \alpha_2\}, \{\alpha_2, \alpha_3\}$ by any H-path of all 2-subsets containing α_2, with the starting vertex $\{\alpha_1, \alpha_2\}$ and the ending vertex $\{\alpha_2, \alpha_3\}$; then

replace the edge $\{\alpha_2, \alpha_3\}, \{\alpha_3, \alpha_4\}$ by any H-path of all 2-subsets containing α_3, with the starting vertex $\{\alpha_2, \alpha_3\}$ and the ending vertex $\{\alpha_3, \alpha_4\}$; then for $5 \le i \le c$,

replace the edge $\{\alpha_{i-2}, \alpha_{i-1}\}, \{\alpha_{i-1}, \alpha_i\}$ by any H-path of all 2-subsets containing α_{i-1} but removing $\{\{\alpha_2, \alpha_{i-1}\}, \{\alpha_3, \alpha_{i-1}\}, \cdots, \{\alpha_{i-3}, \alpha_{i-1}\}\}$, with the starting vertex $\{\alpha_{i-2}, \alpha_{i-1}\}$ and the ending vertex $\{\alpha_{i-1}, \alpha_i\}$.

(2) Two subsets are adjacent if and only if they have no intersecting point. Then the orbital graph is the Kneser graph $K(c,2)$. If $c \geq 8$, then the degree of the graph is more than $\frac{n}{2}$ and so it is hamiltonian, where n is the order of the graph. For the cases when $c \leq 7$, we do it just by Magma. □

Lemma 3.16. *Let G be one of the groups listed in row $5, 7-10$ of* Table 2*. Then every orbital graph of G is hamiltonian.*

Proof. Using Magma, we compute the suborbits for these groups and show that every corresponding orbital graph is hamiltonian.

(1) The action of $\mathrm{PSL}(3,5).2$ on the flags has three nontrivial suborbits, with the respective length $10, 50$ and 125;

(2) The action of M_{11} on the cosets of a subgroup isomorphic to S_5 has three nontrivial suborbits, with the respective length $15, 20$ and 30;

(3) The action of M_{12} on the cosets of a subgroup isomorphic to $M_{10} : 2$ has two nontrivial suborbits, with the respective length 20 and 45;

(4) The action of M_{23} on the cosets of a subgroup isomorphic to A_8 has three nontrivial suborbits, with the respective length 15, 210 and 280;

(5) The action of J_1 on the cosets of a subgroup isomorphic to $\mathrm{PSL}(2,11)$ has four nontrivial suborbits, with the respective length 11, 12, 110 and 132. □

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Bootstrap percolation via automated conjecturing*

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Abstract

Bootstrap percolation is a simple monotone cellular automaton with a long history in physics, computer science, and discrete mathematics. In k-neighbor bootstrap percolation, a collection of vertices are initially infected. Vertices with at least k infected neighbors subsequently become infected; the process continues until no new vertices become infected. In this paper, we hunt for graphs which can become entirely infected from initial sets which are as small as possible. We use automated conjecture-generating software and a large group lab-based model as a fundamental part of our exploration.

Keywords: Bootstrap percolation, automated conjecturing, graph theory, percolation, cellular automata.

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1 Introduction

1.1 History

Bootstrap percolation can be thought of as a graph process in which an arbitrary initial configuration of infected vertices is selected from a graph; remaining uninfected vertices with many infected neighbors are successively added to the infected set until the system stabilizes. Bootstrap percolation serves as a model of nucleation and growth [12] and has been applied in the study of crack formation [1], magnetic alloys [13], hydrogen mixtures [2], and computer storage arrays [20]. More generally, it provides an important stepping stone towards understanding other cellular automaton models with applications in physics, biology, information technology, epidemiology, and more.

The k-bootstrap model has a long and interesting history. First introduced by Chalupa, Leath, and Reich [13] in 1979 as a way to model magnetic materials, it is perhaps the simplest example of a monotone cellular automata (which were introduced by von Neumann [26], based on a suggestion of Ulam [25]). Most of the work related to bootstrap percolation has focused on finding thresholds for growing families of graphs, in which the initially infected sets are chosen at random. For the interested reader, this fascinating direction can be explored through, e.g., [3, 4, 5, 10, 11, 19]). These bootstrap models have been generalized significantly in recent years, with the advent of graph bootstrap percolation [8].

Here, however, we go in a somewhat different direction. Rather than selecting the initially infected vertices at random, we allow them to be chosen very carefully. How small could such an initial set be, given that it eventually infects the entire graph?

It is clear that if the infection can only spread to vertices with at least k infected neighbors, then such an initial infected set can contain no fewer than k vertices (otherwise, not even a single uninfected vertex can become infected). In this note, we search for those graphs which can be infected from a set *exactly* k vertices – that is, those graphs which can be infected as easily as possible.

This work is inspired by earlier results of Dairyko, Ferrara, Lidický, Martin, Pfender and Uzzell [15], and Freund, Poloczek, and Reichman [17]. These groups gave degree conditions for graphs to be infectable from a small set, in the case that $k = 2$. Similar results were later proven by Gunderson [18] and Wesolek [27] for the $k \geq 3$ cases. We stay with the $k = 2$ case, but provide non-degree based conditions for percolation from small sets.

1.2 Conjecturing

The conjectures reported below are the product of the property-relations version of the CONJECTURING program of Larson and Van Cleemput [21, 22]. While the program is described in these papers it is worth mentioning that produced conjectures are produced if they are both true for all input (graph) examples and *significant*—here this means that the produced conjecture was an improvement of either temporarily stored potential conjectures or user-supplied theoretical knowledge (theorems). The CONJECTURING program is open-source, and written to work with Sage; the code, examples, and set-up instructions are available at: `nvcleemp.github.io/conjecturing/`. A substantial effort has also been made to code graph-theoretic knowledge; this is available at: `math1um.github.io/objects-invariants-properties/`.

CONJECTURING is used a *tool* in this research; while we don't mean to add anything to the papers that describe how the program works, we will add some context for interested readers. In this paper we investigate both sufficient and necessary conditions for a graph to be 2-*bootstrap-good*. Sufficient conditions are themselves properties, often themselves boolean functions of more basic properties. The CONJECTURING program allows the user to input any list of pre-coded properties to use as "ingredients" for these sufficient conditions. These input (or basic, or pre-coded) properties have minimum complexity—or "complexity-1". A unary boolean operator, such as negation, applied to a complexity-1 property yields a complexity-2 property. The CONJECTURING will systematically build every possible property-expression from these input properties and (built-in) boolean operators. The CONJECTURING program also allows the user to input a list of graphs. A property P will be considered to be a possible sufficient condition for a graph to be 2-*bootstrap-good* if every input graph G that has property P is also 2-*bootstrap-good*. A possible sufficient condition P will only be added as a (potential) *conjecture* if it is true for some input graph G which is false for every other currently stored sufficient condition conjecture.

The user of the CONJECTURING can improve the quality of the produced conjectures by adding more pre-coded properties, and by adding as input graphs any graphs that have been found to be counterexamples to previous conjectures. The CONJECTURING program simply systematizes and automates what a human mathematician already does: a human mathematician's sufficient condition condition conjectures for a graph to be 2-*bootstrap-good* are necessarily properties "built" from graph properties she already knows and should be true at least for the specific graphs she has tested them on. In a precise sense, a human cannot make a "better" conjecture for a graph to be 2-*bootstrap-good* than the conjectures the CONJECTURING program makes (from the same inputs). Maybe the most important feature of the program is its ability to systematically consider every property up to some complexity—no human can do this.

A last feature we will mention of the CONJECTURING program is its ability to use theorems or *theoretical knowledge*. Suppose it is known that property P is a sufficient condition for a graph to be 2-*bootstrap-good*. This can be added as an input to the program: any conjectured sufficient condition property Q must be true for some input graph G which does not have property P. This feature of the program can be useful to "grow" a theory. In fact, some simple theorems may later be superseded. There is utility still in simple theorems: Dirac's Theorem, for instance, is still of interest—even though it is now implied by less-simple, more comprehensive, theorems.

1.3 Definitions

Here, we define precisely the k-bootstrap percolation model. Wherever possible, we use standard graph theoretic notation (see, e.g., [7]).

Let k be a natural number, G a graph, and let $\mathcal{I} \subseteq V(G)$ be a set of vertices which we think of as being initially *infected*. We then grow the infected set as follows: if an uninfected vertex v has at least k neighbors which are infected, then we add it to $\mathcal{I}$. That is, whenever we have a vertex $v \in V(G) \setminus \mathcal{I}$ with $|N(v) \cap \mathcal{I}| \geq k$, then we move v to $\mathcal{I}$.[1] Eventually, this process stabilizes – either every remaining uninfected vertex has fewer

[1] There is some abiguity here – we have described this process as happening a single vertex at a time. That is, each vertex of the graph in sequence checks its number of infected neighbors, and becomes infected if this is large. It is more standard to think of this infection as occurring in 'rounds', where *every* vertex with lots of infected neighbors is infected simultaneously. Because the process is monotone (infected vertices never uninfect), both versions reach the same final percolating set. We won't be concerned with things like the *time to percolate*, which

than k neighbors in $\mathcal{I}$, or every vertex has joined $\mathcal{I}$. We denote this final infected set by $\langle \mathcal{I} \rangle$. When $\langle \mathcal{I} \rangle = V(G)$, we say that G *k-percolates from $\mathcal{I}$*. When G is clear from context, we will say that $\mathcal{I}$ *k-percolates*; when k is also clear from context, we simply say that $\mathcal{I}$ *percolates*.

For a graph with more than k vertices, any set $\mathcal{I}$ which k-percolates must have $|\mathcal{I}| \geq k$; otherwise, there are not enough vertices in total for *any* uninfected vertex to join $\mathcal{I}$. With this minimum size in mind, we call a graph *k-bootstrap-good* if there is a set of size exactly k which k-percolates[2]. A graph which is not k-bootstrap-good is k-bootstrap-bad.

We define $m(G, k)$ to be the minimum size of a set $\mathcal{I}$ such that G k-percolates from $\mathcal{I}$. As such, our k-bootstrap-good graphs are those which have $m(G, k) = k$. In the rest of this paper, we focus on finding conditions related to 2-bootstrap-good graphs[3].

2 Lemmata (useful lemmas)

In this section, we note a few very simple results which we shall use frequently in the remainder of the paper. To be explicit, since we're only interested in graphs which might be 2-bootstrap-good, all theorems and conjectures following should be assumed to have the following extra conditions:

1. We focus exclusively on graphs with at least 3 vertices, since it requires two neighbors to become infected.
2. All graphs are connected. (The only disconnected graph which is 2-bootstrap-good is the graph with two isolated vertices)
3. All graphs have at most two blocks (as discussed in the following paragraph.).

Recall that a *block* in a graph G is a maximal connected subgraph with no cut vertex. We enforce the third condition above due to the following lemma.

Lemma 2.1. *If a graph is 2-bootstrap-good, then it has at most two blocks.*

Proof. Assume G is a connected graph with three blocks B_1, B_2, B_3. Since G is connected, the blocks are nontrivial (that is, the blocks are either K_2 or 2-connected graph). If both infected vertices are in a single block (say B_1), then at most one vertex vertex of B_2 will be infected – the cut vertex separating B_1 and B_2, if such a vertex exists. Thus B_2 will not be infected, and so the set cannot percolate.

If, instead, the infected vertices are in different blocks, then either no infection will spread, or if the two vertices are adjacent to a common cut vertex, they will infect first only that common cut vertex. This can then spread to the two blocks, but as before it will not move to the remaining block (since it cannot spread beyond the cut vertex). □

As a consequence of this, we note that in particular if G contains a cut edge between two bad subgraphs, then G is itself 2-bootstrap-bad.

We shall frequently make use the following two lemmas, which help us to decompose graphs which are k-bootstrap-good.

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is itself a fascinating area of research, and so we shall use either the 'vertex-by-vertex' or 'rounds' perspective as we see fit.

[2]It is worth noting here that we only require the existence of a single small percolating set – not that *every* set of k vertices percolates.

[3]Thus wherever it is not stated, the reader should assume that we are discussing 2-bootstrap percolation and that a graph declared 'good' is in fact '2-bootstrap-good'.

Lemma 2.2. *If G is k-bootstrap-good and H is formed by adding a vertex v with at least k neighbors inside G, then H is also good.*

By infecting the initial percolating set $\mathcal{I}$ of size k in G, all of G will become infected, including (at least) k neighbors of v, and so v will also become infected. And so, our initial set $\mathcal{I}$ inside G actually percolates to all of H.

Lemma 2.3. *If G is an n vertex graph which is k-bootstrap-good, then it can be constructed from an $n-1$ vertex k-bootstrap-good graph G' and adding a new vertex adjacent to at least k vertices of G'.*

This is immediate – consider a minimum size infecting set, and let v be the very last vertex which becomes infected.

To be explicit, as we are only interested in graphs which might be 2-bootstrap-good, in order to avoid very long theorem statements, we really wish all of our theorems to have the following additional conditions:

1. As we stated in the introduction, we focus exclusively on graphs with at least 3 vertices.
2. All graphs are connected. (The only disconnected graph which is 2-bootstrap-good is the graph with two isolated vertices)
3. All graphs have at most two blocks (as discussed in the Lemmata).

We collect together this set of 'potentially bootstrap good' graphs in the definition below. This will allow us to simplify our theorems tremendously; rather than, e.g., "every connected chordal graph with at least three vertices and at most two blocks is 2-bootstrap-good", we can simply say "A graph in $\mathcal{G}$ which is chordal is 2-bootstrap-good".

Definition 2.4. We let $\mathcal{G}$ denote the set of all connected graphs of order at least three which have at most two blocks. We emphasize that the all large graphs which are 2-bootstrap-good are in $\mathcal{G}$.

2.1 Which graphs are bad?

In this section, we collect some easy properties of 2-bootstrap-bad graphs. While none of these results are new (and several seem to be folklore), we give their very short proofs here for completeness. The first two of these rely on the simple observation that pendant vertices must be initially infected in any percolating set.

Proposition 2.5. *A graph with at least two leaves with distinct parents is 2-bootstrap-bad.*

Again, this is straightforward – leaves can never become infected if they are not initially infected; thus any leaves must be initially infected. So, both leaves must be initially infected, and since these have distinct parents the infection does not spread.

Proposition 2.6. *Any graph with at least three leaves is 2-bootstrap-bad.*

As above, leaves must be initially infected, and there are simply too many to infect.

Proposition 2.7. *The path graph P_k of order $k \geq 4$ is 2-bootstrap-bad.*

Initially infected vertices x and y are either adjacent (and no spread happens), or not (in which case they infect exactly the vertex in between them if $d(x, y) = 2$ and no vertices otherwise).

Proposition 2.8. *The cycle C_k of order $k \geq 4$ is 2-bootstrap-bad.*

Consider a cycle $x_1x_2 \dots x_k$, with initially infected vertices x_i, x_j with $i < j$. If $j - i \in 1, 3, 4, \dots, k-1$, then no new vertices are infected; otherwise, x_{i+1} is infected and the spread stops.

For the next results, we denote by $\overline{d}(G)$ the average degree of G, we denote the maximum average degree by $mad(G) := \max_{G' \subseteq G} \overline{d}(G')$; this is a well known graph parameter arising in chromatic theory. We will prove the following using a simple counting technique due to Riedl (who also uses *wasted* and *used* edges similar to our *usable edges* above) [23].

Theorem 2.9. *Let $\varepsilon > 0$. Then there is some $N = N(\varepsilon)$ such that every graph with $mad(G) < 4 - \varepsilon$ and $|G| > N$ is 2-bootstrap-bad.*

It is worth noting that this theorem is sharp, as is seen by the square of the cycle C_n^2 for each n; such graphs have $mad(G) = 4$ and are 2-bootstrap-good for each n. In fact, we will prove the corresponding result for the more general k-bootstrap model; this is again shown to be sharp by the k^{th} power of the cycle.

Theorem 2.10. *Let $\varepsilon > 0$. Then there is some $N = N(\varepsilon)$ such that every graph with $mad(G) < 2k - \varepsilon$ and $|G| > N$ is k-bootstrap-bad.*

Proof. Assume G is k-bootstrap-good, with vertices infected one at a time; let H_t be the graph induced by those vertices which are infected within the first t steps. Then since we initially infected k vertices, followed by one vertex at each time step, we have $|H_t| = t + k$. Further, each vertex was infected because it had at least k edges to the preceding infected vertices and so $\|H_t\| \geq kt$. Thus $\overline{d}(G) \geq \frac{kt}{t+k}$, and for t sufficiently large this is larger than $2k - \varepsilon$; this contradicts the maximum average degree condition. □

2.2 What is required to be good?

As is common in such problems, we provide only a few necessary conditions for a graph to be 2-bootstrap-good. The first of these is immediate from Lemma 2.3.

Proposition 2.11. *If G is good, then $\|G\| \geq 2(|G| - 2)$.*

The next result will be of considerable use to us later. Recall that the girth of a graph is the minimum of the cycle lengths present.

Proposition 2.12. *If G is 2-bootstrap-good and not P_3, then it has girth less than five.*

Proof. Consider two initially infected vertices u and v which percolate. Since we're assuming our graphs have at least 3 vertices, there is some vertex w which is becomes infected next – it is adjacent to both u and v. If $uv \in E(G)$, then we already have a triangle. Otherwise, if G is not $K_{1,2}$, then there is a fourth vertex which becomes infected; say x. Then x must be adjacent to two of $\{u, v, w\}$ – if it is adjacent to both v, w, we form a triangle; if it is adjacent to u, w or u, v we form a C_4. □

Note that this result shows that the Petersen graph is not 2-bootstrap-good.

2.3 What will guarantee goodness?

In this section, we provide a number of theorems giving sufficient conditions for a graph to be 2-bootstrap-good. The first of these require little to prove; however, they were the first conjectures provided by the CONJECTURING program, so we record them here for completeness.

Proposition 2.13. *Complete graphs are* 2*-bootstrap-good.*

Proposition 2.14. *Complete bipartite graphs are* 2*-bootstrap-good.*

Proof. Since $|G| > 2$, one of the bipartition classes class has at least two vertices; assume that G has bipartition (X, Y) with $|X| > 1$. Initially infect two vertices of X. Since the graph is complete bipartite, every vertex of Y is infected immediately. Then, the remaining vertices of X become infected in the next step. □

Indeed, this remains true for the similar class of split graphs – those graphs whose vertex set can be partitioned into a clique and an independent set.

Theorem 2.15. *If G is a split graph with at most two blocks, then G is* 2*-bootstrap-good.*

Proof. First, notice that if the complete side has only one vertex, then the graph is a star (and thus either K_2 or $K_{1,2}$, since it has at most two blocks, and thus good.) The graph can have at most one pendant, v, which must lie in the independent set. Choosing v and any vertex of the complete graph which is not the parent of v will infect the entire graph, since the complete graph will become immediately infected and each non-pendant in the independent set must be adjacent to at least two vertices of the complete graph. If there is no pendant, then infecting any two vertices of the complete graph will suffice. □

The above classes of graphs percolate very quickly (in at most 3 steps). Next, we see a class of graphs which percolates, but not necessarily in a fixed number of steps. Recall that a graph is *locally connected* if the open neighborhood of every vertex is a connected graph.

Theorem 2.16. *If a graph $G \in \mathcal{G}$ is locally connected, then it is* 2*-bootstrap-good.*

Proof. First, note that a locally connected graph has no pendants – otherwise, the neighborhood of the pendant vertex's parent contains an isolated vertex. Hence let G be a locally-connected graph, v be any vertex and w be any neighbor of v. We initially infect $\{v, w\}$. Recall that $\langle\{v, w\}\rangle$ is then the set of vertices eventually infected from $\{v, w\}$.

As the (open) neighborhood $N(v)$ is connected there is a path $w = x_1 ... x_k = u$ from w to any other vertex u in the graph $H = G[N(v)]$ induced by $N(v)$. Note that each vertex x_i in this path is necessarily a neighbor of v. Since v and w are infected and x_2 is a neighbor of both, x_2 is also infected. Similarly $x_3, ..., x_k = u$ must all be infected. So $N(v)$ is a subset of $\langle\{v, w\}\rangle$. By a symmetric argument $N(w)$ is also a subset of $\langle\{v, w\}\rangle$.

Suppose $\langle\{v, w\}\rangle$ does not equal V. Let x be any vertex in $V \setminus \langle\{v, w\}\rangle$ that is adjacent to some vertex $y \in \langle\{v, w\}\rangle$. Since $y \in \langle\{v, w\}\rangle$ and our graph is connected, there must also be a neighbor z of y in $\langle\{v, w\}\rangle$. By the reasoning above it follows that $N(y)$ must be a subset of $\langle\{v, w\}\rangle$. But then x must be in $\langle\{v, w\}\rangle$. □

It is worth noting that the above proof in fact shows that if G is locally connected and pendant-free, then G 2-percolates from *any* set of two adjacent vertices.

The CONJECTURING program made several conjectures of the form that a known sufficient condition for graph Hamiltonicity is a sufficient condition for 2-bootstrap-goodness. It is a well-known result that Dirac graphs are Hamiltonian; indeed Freund, Poloczek, and Reichmann [16] proved that they are also 2-bootstrap-good. As we will use similar techniques later, we provide a short proof here.

Theorem 2.17. *If a graph in $\mathcal{G}$ is Dirac then it is* 2*-bootstrap-good.*

Proof. It is easy to check that graphs with order three with the Dirac property are 2-bootstrap-good. Assume that Dirac graphs with fewer than n vertices are 2-bootstrap-good. Let G be a Dirac graph with n vertices; so every vertex in G has degree at least $\frac{n}{2}$.

Let H be a 2-bootstrap-good subgraph of H with a maximum number of vertices. Note that no vertex in $V \setminus H$ has more than one neighbor in H, otherwise H would not be a maximum 2-bootstrap-good subgraph of G. So every vertex in $V \setminus H$ has at least $\frac{n}{2} - 1$ neighbors in $V \setminus H$. So $V \setminus H$ induces a Dirac subgraph of G. By our inductive assumption the graph $G[V \setminus H]$ is 2-bootstrap-good. Since every vertex in $G[V \setminus H]$ has degree at least $\frac{n}{2} - 1$ and the order of $G[V \setminus H]$ is no more than the order of h, it follows that both H and $V \setminus H$ have order $\frac{n}{2}$.

So G has the structure of two complete order $\frac{n}{2}$ complete sugraphs with a matching from H to $V \setminus H$. Let v be a vertex in H, v' be the vertex it is matched to in $V \setminus H$ and w be any other vertex in $V \setminus H$. It is easy to see that $\{v, w\}$ percolates G and thus G is 2-bootstrap-good. □

As a consequence, we obtain the following easy corollary.

Corollary 2.18. *If a graph in $\mathcal{G}$ is* 2*-bootstrap-good then it is either not cubic or it is Dirac.*

Proof. A graph which is both Dirac and cubic has order at most six (and no cubic graph has order seven). Hence it suffices to prove that if G is cubic with at least eight vertices, then it is not 2-bootstrap-good. We'll call an edge 'usable' at any particular step in the infection process if it has one infected endpoint and one uninfected endpoint. If an infected graph has less than two usable edges, then the infection cannot spread any further. Consider an initial set of two infected vertices in G. There are at most six usable edges leaving this set, since G is cubic. Any new infected vertex will make two edges unusable, and add at most one usable edges; thus the total number of usable edges drops by at least one with each newly infected vertex. Hence, the final number of infected vertices can be at most five, since at this point there will be at most one usable edge remaining. □

A graph with order n is *Ore* if every pair of non-adjacent vertices have degree sum at least n: being Ore is also a sufficient condition for being Hamiltonian. The CONJECTURING program also conjectured that Ore graphs are 2-bootstrap-good – this is strictly weaker than the result proven in [15], who prove that in fact degree sum at least $n - 2$ is enough.

Recall that a graph is *chordal* if it has no induced cycle of length longer than three. In order to prove that all chordal graphs are 2-bootstrap-good, we will need the following lemma.

Lemma 2.19. *Let $G \in \mathcal{G}$ be a* 2*-connected chordal graph and $S \subsetneq V(G)$ such that $|S| \geq 2$ and $G[S]$ is connected. Then there is some $x \in V(G) \setminus S$ such that x is adjacent to at least two vertices $v, w \in S$.*

Proof. Consider building an auxiliary graph G' by adding a new vertex y adjacent to everything in S. By its construction, G' is 2-connected. Pick $x \in V \setminus S$ to minimize the total length of two internally disjoint paths from x to p; call these paths $P_1 = xv_1 \dots v_n y$ and $P_2 = xw_1 \dots w_m y$. By minimality, both v_n and w_m are both in S. Since $G[S]$ is connected, there is a path from v_n to w_m in S; let $P = v_n p_1 \dots p_k w_m$ be a minimum length such path. By taking the union of $P_1 v_n$, $P_2 v_m$ and P, we find a cycle, which by minimality must be induced. But since G is chordal, this means the cycle is a triangle, and it means that x is the vertex we wanted. □

From this lemma, we easily deduce that all chordal graphs are 2-bootstrap-good.

Theorem 2.20. *If $G \in \mathcal{G}$ is chordal, then it is 2-bootstrap-good.*

Proof. If G contains a single block, infect any two adjacent vertices. Otherwise, infect one vertex of each block (both of which are adjacent to the cut vertex). Thus in each block we have two infected adjacent vertices, and so by repeatedly applying Lemma 2.19 we infect a new vertex as long as there is some uninfected vertex. □

In fact, something somewhat stronger is true – every chordal graph is a *strangulated graph*; this somewhat less common class of graphs consists of those graphs in which deleting the edges of any induced cycle of length greater than three would disconnect the remaining graph.

Theorem 2.21. *If $G \in \mathcal{G}$ is a strangulated graph, then it is 2-bootstrap-good.*

Proof. A strangulated graph can be constructed from chordal graphs and maximal planar graphs by gluing along cliques [24]. If such a graph is a block, then all the gluings occur along cliques of size at least two. We argue that any two adjacent vertices will infect the graph since both chordal graphs and maximal planar graphs can be infected from any initial adjacent pair (this is argued above for chordal graphs).

Indeed, let H be a maximal 2-bootstrap-good subgraph of a maximal planar graph G (such maximal planar graphs are well known to be triangulations). If $H \neq G$, then there must be some vertex $v \in G - H$ with a neighbor $w \in H \cap N(v)$. We orient the vertices around w clockwise as $x_1, x_2, \dots, x_k$. Note that $v = x_i$ for some $i \in [k]$, and there is at least one x_j from H (possibly with $i = j$). Hence at some point there must be a pair $x_\ell, x_{\ell+1}$ (we think cyclically, allowing x_k, x_1) with exactly of the pair in H and the other in $G - H$. But then $\{w, x_\ell, x_{\ell+1}\}$ lie on a common face, which must be a triangle. As such, any infection which percolates on H also spreads to all of $w, x_\ell, x_{\ell+1}$ contradicting maximality. Thus it must be that $H = G$, and so each triangulation can be percolated from any adjacent pair. Note that if a strangulated graph has two blocks, then there it has only one gluing that is along a single vertex; infecting a single neighbor from each adjacent block will infect the entire graph via the infection processes described above inside each block. □

A different superclass of chordal graphs is that of *dually chordal graphs* (so named because they are the clique graphs of chordal graphs, and thus dual in nature to chordal graphs). An alternate characterization is that a graph is dually chordal if and only if the hypergraph of its maximal cliques is the *dual* is a hypertree [9] (we give a more technical version of this somewhat non-standard term inside the proof). These graphs, like chordal graphs, are always 2-bootstrap-good.

Theorem 2.22. *If $G \in \mathcal{G}$ is dually chordal, then it is 2-bootstrap-good.*

Proof. We first make use of an alternate characterization. If a graph is dually chordal if the auxiliary hypergraph formed with $V(H) = V(G)$ and $E(H) = \{X : G[X]$ is a maximal clique$\}$ is a hypertree (that is, it is connected and has no cycles). If G contains a single block, then each pair of cliques intersects in at least two vertices; thus infecting any adjacent vertices will infect the entire block (as it percolates through the cliques of the instersection hypertree). For a graph with two blocks, we once again infect a single vertex of each block, with both adjacent to the cut vertex. □

Next, recall that a graph is called a *cograph* (short for complement reducible) when it contains no induced copy of the path P_4. The CONJECTURING program conjectured that such graphs are 2-bootstrap-good.

Theorem 2.23. *If $G \in \mathcal{G}$ is a cograph, then it is 2-bootstrap-good.*

Proof. Cographs can be constructed by taking disjoint unions and joins of cographs, starting from single vertices [14]. We proceed by strong induction on order of our cograph; the base cases are trivial. Consider next a cograph which is a single block. Since $G \in \mathcal{G}$, we know that G is connected and thus it arises from taking the join of two cographs G_1 and G_2. Consider infecting two vertices from G_1; this will infect all of G_2 in the next step, and these will infect the remainder of G_2 in the second step. Note that this shows something slightly stronger – we can infect any two vertices in either part of their block. Therefore, if G is constructed from two blocks G_1 and G_2 sharing a cut-vertex, and each G_i was constructed by taking the join of $H_{i,1}$ and $H_{i,2}$ with at least two vertices each, then we simply select a vertex in $H_{1,k}$ and $H_{2,j}$ which are adjacent to the cut vertex; this will infect the cut vertex, and then spread to the blocks by the preceding argument. □

3 Which Kneser graphs are good?

Finally, we make a somewhat different attack; rather than proving a general condition is sufficient, we explore a particular class of graphs and characterize those which are good. In particular, recall that the *Kneser graph* $KG_s^{(t)}$ is a graph whose vertices are the t element subsets of $[s]$, with two vertices adjacent when their corresponding subsets are disjoint. Trivially, this graph is an independent set whenever $s < 2t$. But when is it 2-bootstrap-good?

$KG_1^{(1)}$ and $KG_2^{(1)}$ are both trivially 2-bootstrap-good, and these are the only interesting Kneser graphs with $s \leq 2$. Further, we note that for $k \geq 2$, the graph $KG_{2t}^{(t)}$ is a collection of disjoint edges; this is clearly not 2-bootstrap-good, so we may assume that $s \geq 2t + 1$. All remaining possibilities are covered by the following theorem.

Theorem 3.1. *Assume $s \geq 3$. A Kneser graph $KG_s^{(t)} \in \mathcal{G}$ is 2-bootstrap-good if and only if $s \geq \min\{3t, 2t + 3\}$.*

Proof of Theorem 3.1. *Necessity*: Assume that $s < 3t$, that $s \leq 2t + 2$, and let v, w be vertices of $KG_s^{(t)}$ (that is, v and w are size t subsets of $[s]$) with which our infection begins. Note that since v and w are t element sets, we have $|v \cap w| \in [0, t-1]$. Let $A := v \cup w \subseteq [s]$, and let $B := [s] \setminus A$. Note that if a vertex x is adjacent to both v and w (that is, x can be infected by $\{v, w\}$), then x must be disjoint from A – and thus $x \subseteq B$.

Since $s \geq 3$, we have at least three vertices in $KG_s^{(t)}$, and so if $|B| < t$, then there are no vertices disjoint from $|A|$ and so v and w cannot infect any vertices. So, if our infection is to percolate we must choose v and w in order to guarantee $|B| \geq t$, and so $|A| = s - |B| \leq t + 2$. Now we need only consider two cases – either $|A| = t + 1$ (and so v and w share $t - 1$ common elements) or $|A| = t + 2$ (and so v, w share $t - 2$ common elements). Most of our work will lie in proving the first case; the second will fall shortly after.

Assume $|A| = t + 1$; then $|B| = t$ or $|B| = t + 1$. If $|B| = t$, then there is only a single vertex x, which will become infected by v and w. Since $s < 3t$, there are no vertices adjacent to both x and v or to both x and w. Thus the infection stops at precisely three vertices, and since $\binom{s}{t} \geq 4$ for all s, t satisfying our conditions, this is not the entire graph.

If $|B| = t + 1$, then similarly v and w can infect $t + 1$ new vertices. At the next stage, any two of these vertices will infect all vertices disjoint from B – these are precisely $X = \{y : |y| = t \text{ and } y \subseteq A\}$. In these last two steps, we've built a complete bipartite infected $K_{t+1,t+1}$. We will show that the infection can spread no further.

Let X be as above, and let Y be the corresponding vertices from B. Since $s < 3t$, there is again no vertex adjacent to both a vertex from X and a vertex from Y. Further, any two vertices $a, b \in X$ (or both in Y) contain $t - 1$ common elements, so between them they both contain all $t + 1$ elements of A (or of B). Then, the only vertices adjacent to both a, b are those vertices in the other half of our bipartite graph – which are already infected. Again, the infection process must stop. Since vertices with some elements from A and some elements from B are not yet infected, the initial set has not percolated.

Finally, assume $|A| = t + 2$. Then, $|B| = t$ and there is only one vertex infected by v and w. As above, the infection cannot spread any further, and since there are more than three vertices in $KG_s^{(t)}$ we cannot infect the whole graph.

Sufficiency: Suppose we have $s \geq 3$, along with $s \geq 3t$ or $s \geq 2t + 3$. We note that there are only two Kneser graphs for which $s \geq 3t$ but $s < 2t + 3$ – these are $KG_3^{(1)}$ and $KG_6^{(2)}$. Since $KG_3^{(1)} \cong K_3$, this is trivially 2-bootstrap-good. Further, one can easily check that $\{\{1, 2\}, \{2, 3\}\}$ percolates in $KG_6^{(2)}$ (as does any other pair of vertices sharing a common element). For all other values of our parameters, we may assume $s \geq 2t + 3$ (since $s \geq 3t$ will guarantee this).

As in $KG_6^{(2)}$ we choose two vertices v and w with $|v \cap w| = t - 1$. Then, letting $A := v \cup w \subseteq [s]$ and $B := [s] \setminus A$ as before, we have $|A| = t + 1$ and $|B| \geq t + 2$. Now, we partition the vertices x of $KG_s^{(t)}$ according to the size of $|A \cap x|$, noting that $|B \cap x| = t - |A \cap x|$. We denote these sets $A_0, A_1, \ldots, A_t$, where $|A_i| = i$.

Initially, we infect vertices v and w. In the second round, v and w infect all vertices disjoint from A; that is, all those vertices in A_0. These vertices in A_0 then infect all those vertices in A_t (which are disjoint from B). Since $|B| \geq t + 2$, we can choose two vertices b_1 and b_2 in A_0 which overlap which share $t - 1$ elements, so that $|b_1 \cup b_2| = t + 1$. Then, there will be at least one element of $y \in B \setminus (b_1 \cup b_2)$, so b_1 and b_2 can infect the vertices in A_{t-1}. Finally, we show that our infection percolates from this point.

Claim 3.2. *If all vertices in A_t, A_{t-1}, and A_0 are infected, the entire graph will become infected.*

Proof of Claim 3.2. For any choice of $k \in [1, t]$ we can choose two vertices of A_k such that we can from them infect a vertex in A_{t-k} and in A_{t-k+1}. Choose two vertices v and

w from A_k for which $|v \cap A| = |w \cap A|$ and $|(v \cap B) \cap (w \cap B)| = t - k - 1$. Now, since $|A| = t + 1$, we have $t + 1 - k$ elements of A at our disposal, and since $|B| \geq t + 2$ we have at least $t + 2 - (t - k + 1) = k + 1$ elements available from B. Since a vertex of A_{t-k+1} requires $t - k + 1$ elements of A and $t - (t - k + 1) = k - 1$ elements of B, and so such a vertex exists. Further, since we have every element of A_k already infected, we can choose $v \cap A$ and $w \cap A$ such that any $t + 1 - k$ elements are available, and since our choice of $v \cap B$ and $w \cap B$ is independent of our choice these, we can choose $v \cap B$ and $w \cap B$ so that any $k + 1$ entries are available from B; this allows us to infect our desired vertex in A_{t-k+1}.

To infect a vertex in A_{t-k}, choose v and w such that $|(v \cap A) \cap (w \cap A)| = k - 1$ and such that $v \cap B = w \cap B$. Then we have again $t + 1 - (k + 1) = t - k$ elements available from A, and at least $t + 2 - (t - k) = k + 2$ elements of B. Thus we can find a vertex in A_{t-k} adjacent to both v and w, and which will thus become infected. As before, we can infect any vertex of A_{t-k} this way. □

End of Proof of Theorem 3.1. □

4 Conclusion and further work

This is an introductory exploration to the area of very small percolating sets. Building on the work of Dairkyo et al. [15], and others, we used the automated conjecturing framework to begin a systematic search for classes of graphs which are 2-bootstrap-good (or 2-bootstrap-bad). From this starting point, we've given a number of not-so-hard-to-prove but quite-hard-to-discover conditions (both necessary and sufficient) for a graph to be 2-bootstrap-good. It remains an intriguing open question to find a full characterization of such graphs (however, at this early stage we do not even have a conjecture of what such a characterization might look like).

Further, there are many natural generalizations of these results to explore. In particular, what properties will guarantee that a graph has a k-element percolating set in k-bootstrap percolation? This paper explores the $k = 2$ case, but $k = 3$ and higher are as interesting. In addition, bootstrap percolation is just one of many monotone cellular automata which one can define on a graph (as a group, these are all examples of *graph bootstrap percolation* defined in the 1960s by Bollobás under the name *weak saturation* [6]). What graphs have the smallest possible percolating sets in these more general models?

Finally we report a conjecture that attracted our interest but which we did not resolve. The *diameter* of a (connected) graph is the maximum distance between any pair of its vertices. Notice that a graph with diameter no more than two has at most two blocks. A graph is *perfect* if the chromatic number and clique number of every subgraph is equal. This class of graphs includes, for instance, bipartite graphs and chordal graphs. As such, there is relation between this conjecture and Theorem 2.20.

Conjecture 4.1. *If a graph in $\mathcal{G}$ is perfect and its diameter is no more than two then the graph is* 2*-bootstrap-good.*

This conjecture was produced by the CONJECTURING program and, like the proved conjectures reported above, only guaranteed to be true for the input graphs used when the program was run. It is a surprising fact that many conjectures of the program are in fact true.

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On the existence of zero-sum perfect matchings of complete graphs

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Abstract

In this paper, we prove that given a 2-edge-coloured complete graph K_{4n} that has the same number of edges of each colour, we can always find a perfect matching with an equal number of edges of each colour. This solves a problem posed by Caro, Hansberg, Lauri, and Zarb. The problem is also independently solved by Ehard, Mohr, and Rautenbach.

Keywords: Graphs, zero-sum perfect matching.

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1 Introduction

Note that in this paper, we will use the word 'matching' when in fact we mean 'perfect matching'.

For an edge-colouring function $f\colon E(G) \to S$ of a graph G where $S \subseteq \mathbb{Z}$ and a subgraph H of G, if $\sum_{e\in E(H)} f(e) = 0$ then H is called a *zero-sum subgraph* of G.

The research in zero-sum problems can be traced back to the three theorems that give them the algebraic foundation. These are the Erdős-Ginzberg-Ziv Theorem [13], the Cauchy-Davenport Theorem [11], and Chevalley's Theorem [10]. Early zero-sum results concern with the sum taken in additive group $\mathbb{Z}_k$, the area is called Zero-sum Ramsey Theory. This theory studies the zero-sum Ramsey number $R(G, \mathbb{Z}_k)$ which is the smallest number n such that in every $\mathbb{Z}_k$-edge-colouring of r-uniform hypergraph on n vertices $K_n^{(r)}$ there exists a zero-sum modulo k copy of G. It also studies the zero-sum bipartite Ramsey number $B(G, \mathbb{Z}_k)$ which is the smallest number n such that in every $\mathbb{Z}_k$-edge-colouring of $K_{n,n}$ there exists a zero-sum modulo k copy of G. For more complete developments of the topic consult [1, 3].

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In [2], Caro gave the complete characterization of the zero-sum modulo 2 Ramsey number $R(G, \mathbb{Z}_2)$. In [7], Caro and Yuster gave the characterization of zero-sum modulo 2 bipartite Ramsey numbers. Along with [8] and [14], these four papers completely solved the zero-sum Ramsey theory over $\mathbb{Z}_2$. Caro and Yuster [9] were the first to consider zero-sum problems over $\mathbb{Z}$. Recently, several variants of the zero-sum problems have been studied (see, e.g. [5, 6]).

In [4], Caro, Hansberg, Lauri, and Zarb had studied zero-sum subgraphs where $S = \{-1, 1\}$ from various host graphs and various kinds of subgraphs. They then proved what they call the 'Master Theorem' which covers many results of this kind. However there is a remarkable variation of zero-sum subgraph problem that has not been decided by their work in that paper. So they posed it at the end of the paper and the problem is the following:

Problem 1.1. Suppose $f\colon E(K_{4n}) \to \{-1, 1\}$ is such that it is a zero-sum graph. Does a zero-sum matching always exist?

Observe that this problem essentially wants us to find a matching that has an equal number of edges that were assigned with -1 and 1 out of a complete graph of degree $4n$ that had been assigned an equal number of -1 and 1 to their edges. This allows us to discard the arithmetic meanings of -1 and 1 and replace them with general colour names. In this paper, we choose to use black and red.

Our main result is the following theorem whose proof is equivalent to the solution of Problem 1.1.

Theorem 1.2. *For any 2-edge-colouring of K_{4n} with an equal number of edges of each colour, there exists a matching with an equal number of edges of each colour.*

Recently, Problem 1.1 had been independently resolved by Ehard, Mohr, and Rautenbach [12].

2 Terminology

To facilitate the language of our proof, we introduce the following notations and terminologies.

For a graph G, we define $\mathcal{M}(G)$ to denote the set of all matchings in G.

In K_{4n}, $n \in \mathbb{N}$, we define the operation $S\colon \mathcal{M}(G) \times V(G)^4 \to \mathcal{M}(G)$ by

$$S(M, u, v, x, y) = \begin{cases} (V(M), E(M) \cup \{ux, vy\} - \{uv, xy\}) & \text{if } uv \in M \text{ and } xy \in M, \\ M & \text{otherwise.} \end{cases}$$

This operation will be called a *swapping* (see Figure 1).

If M is a matching extracted from 2-edge-coloured K_{4n}, $V_B(M)$ will denote the set of all vertices of M incident to a black edge in M, while $V_R(M)$ will denote the same thing for red.

For any disjoint subsets $S, T \subseteq V(G)$, $E(S, T)$ denotes the set of all edges with one endpoint in S and one endpoint in T.

Let $b(M)$ and $r(M)$ denote the number of black edges and red edges in M respectively.

Lastly, sometimes we will shorten the phrase 'the difference between the number of edges of each colour' to merely 'the difference'. As there will be only one kind of difference in this work, this should cause no ambiguity.

Now we have all the terminologies needed for our proof.

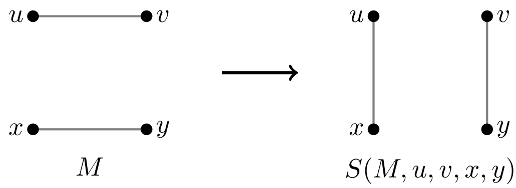


Figure 1: The result of a swapping. All other edges in the matching stays the same.

3 Proof of the Theorem

We first state an important observation as a lemma.

Lemma 3.1. *For $M \in \mathcal{M}(G)$, if there are more red edges than black edges in $E(V_R(M), V_B(M))$, then we can make a swapping that will increase the number of red edges and decrease the number of black edges in M by 1 each. In particular, if there are more black edges than red edges in M and there are more red edges than black edges in $E(V_R(M), V_B(M))$, then we can make a swapping that will reduce the difference by 2.*

Proof. Consider two edges of M, one black and one red, uv and xy respectively. The edges joining between the vertices of these two edges will be among the following six varieties

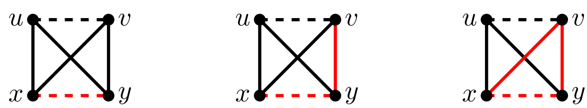


Figure 2: The first three varieties.

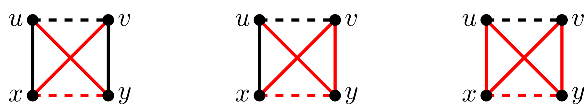
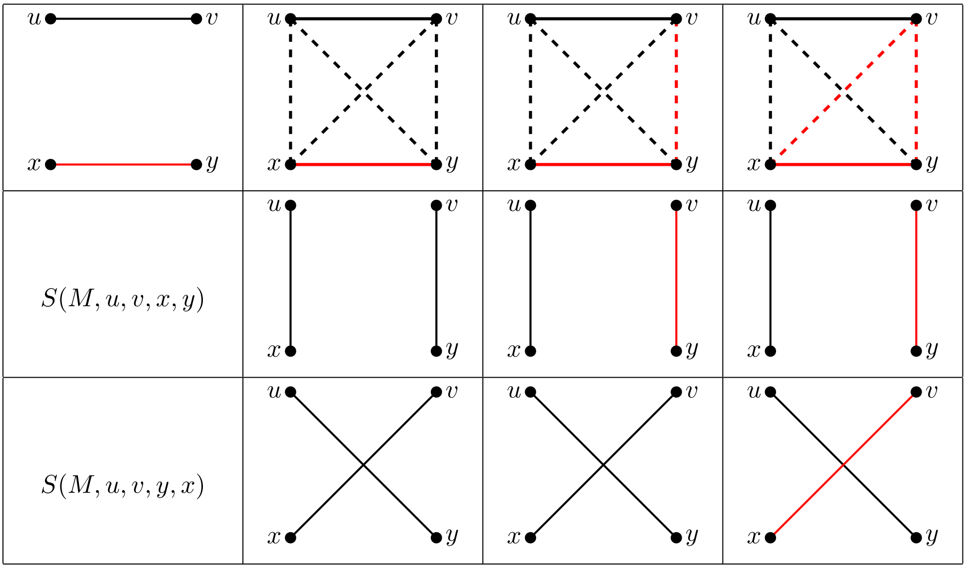
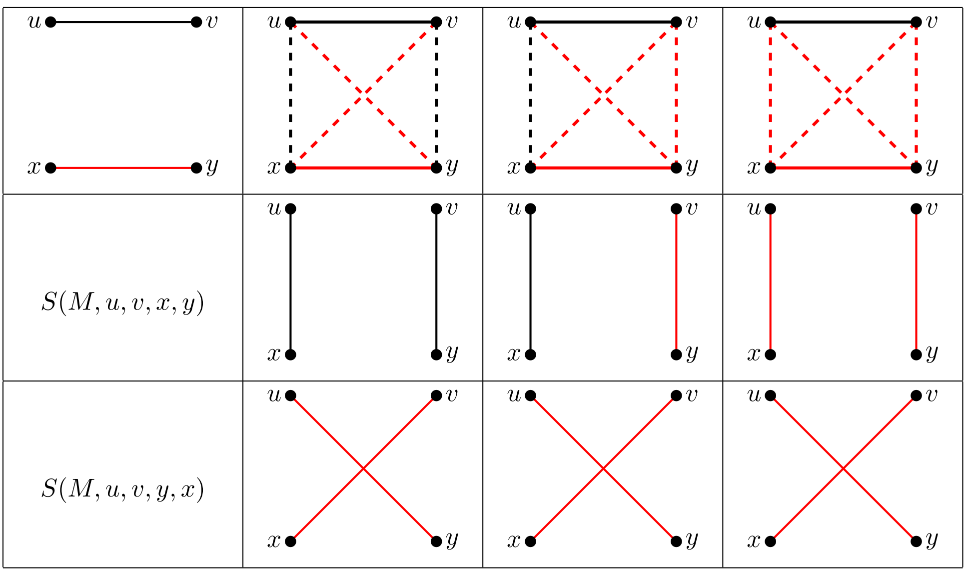


Figure 3: The first three varieties.

Observe that in the first three varieties the number of black edges are no fewer than the numbers of red edges and that the resulting matchings from $S(M, u, v, x, y)$ and $S(M, u, v, y, x)$ will never reduce the difference between the numbers of black edges and red edges that we had from M.



While in the last three varieties the numbers of red edges are no fewer than the numbers of black edges and that at least one of the resulting matchings from $S(M, u, v, x, y)$ or $S(M, u, v, y, x)$ reduces the difference between the numbers of black edges and red edges that we had from M.



So if there are more red edges than black edges joining between $V_B(M)$ and $V_R(M)$, we can guarantee the existence of a pair of edges, one black and one red, in M such that the edges joining between them form one of the latter three (in fact two) varieties.

Using this pair of edges and appropriate order of vertices, we can make a swapping that will increase the number of red edges and decrease the number of black edges that we had from M by 1 each.

Note that in any cases, we have replaced one black edge and one red edge with two red edges, so that the difference between the number of black edges and red edges that we had from M will change by 2 in the resulting matching. If initially there are more black edges than red edges in M, the difference will reduce by 2. But if initially there are more red edges than black edges in M, the difference will increase by 2. □

Remark 3.2. If we are to read the proof of this lemma with the colour red and black in place of each other, the same thing will happen to the colour black when there are more black edges than red edges joining between $V_B(M)$ and $V_R(M)$.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. In our attempt to prove this statement, we will first take an arbitrary matching of K_{4n}, then gradually reduce the difference between the numbers of edges of each colour.

To achieve that, we will construct a finite sequence of matchings in $\mathcal{M}(G)$ which as our sequence progress $|b(M) - r(M)|$, the difference between the numbers of edges of each colour, will gradually and strictly decrease until it reaches zero.

To start the proof, we first pick an arbitrary matching M of G.

We take this M as M_0, the zeroth term of our sequence.

Next, we proceed to obtain next terms of our sequence by the following method.

For nonnegative integer i, if M_i is a term in our sequence, it is without loss of generality to assume that $b(M_i) \geqslant r(M_i)$.

Now we have three cases to consider.

Case 1: $b(M_i) = r(M_i)$

In this case, we end our sequence and take M_i as the matching we have been looking for.

Case 2: $b(M_i) - r(M_i) > 2$

Case 2.1: $G[V_B(M_i)]$ is monochromatic.

We claim that there are more red edges than black edges between $V_B(M_i)$ and $V_R(M_i)$.

Since $b(M_i) > r(M_i)$, $|V_B(M_i)| > |V_R(M_i)|$ so there are more edges in $G[V_B(M_i)]$ than in $G[V_R(M_i)]$.

Recall that in our graph, the number of red edges and the number of black edges are equal.

Since $G[V_B(M_i)]$ is monochromatic(black) and $e(G[V_B(M_i)]) > e(G[V_R(M_i)])$, there must be more red edges between $V_B(M_i)$ and $V_R(M_i)$ than black edges.

By applying the lemma to M_i, we can make a swapping that will reduce the difference by 2.

We take the resulting matching of this swapping as M_{i+1} in our sequence.

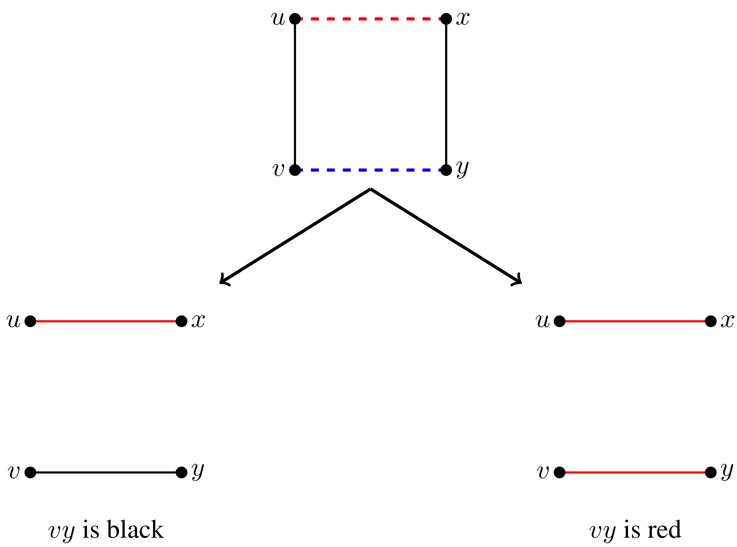
Case 2.2: $G[V_B(M_i)]$ is not monochromatic.

Since $G[V_B(M_i)]$ is not monochromatic, there is a red edge, ux, in $G[V_B(M_i)]$.

Since $u, x \in V_B(M_i)$ and ux is red, there must be $v, y \in V_B(M_i)$ such that $uv, xy \in M_i$.

We take $S(M_i, u, v, x, y)$ to be M_{i+1} in our sequence.

This resulting matching will reduce the difference between the number of black edges and red edges by 2 or 4 compared to that of M_i, depending on the colour of vy (see Figure 4).



If vy is black, the difference reduces by 2. If vy is red, the difference reduces by 4.

Figure 4: Two possibilities of swapping.

Case 3: $b(M_i) - r(M_i) = 2$

Case 3.1: $G[V_B(M_i)]$ is monochromatic.

The reasoning and execution of this case are the same as the Case 2.1.

Case 3.2: $G[V_B(M_i)]$ is not monochromatic.

Claim A. If there are more red edges than black edges between $V_B(M_i)$ and $V_R(M_i)$, then we are done.

Proof. By applying the lemma to M_i, we can make a swapping that will reduce the difference by 2.

We take the resulting matching of this swapping as M_{i+1} in our sequence. □

Thus we may assume that there are not more red edges than black edges between $V_B(M_i)$ and $V_R(M_i)$.

Claim B. If there are $u, v, x, y \in V_B(M_i)$ such that $uv, xy \in E(M_i)$, ux is red and vy is black in G, then we are done.

Proof. This claim is the same as Case 2.2 where vy is black. We take $S(M_i, u, v, x, y)$ to be M_{i+1}.

This reduce the difference by 2, so that it becomes 0. □

Thus we may assume that for all $u, v, x, y \in V_B(M_i)$ such that $uv, xy \in E(M_i)$, if ux is red then vy is also red in G

We observe that , as a consequence of this assumption, red edges always appear in pairs and each red edge only involve two edges of M_i, namely those that share a vertex with it. So if we count the number of red edges in $G[V_B(M_i)]$, it must be an even number.

As it will become important in the rest of the proof, let us make explicit that , as a consequence of this assumption, there must be $n+1$ black edges and $n-1$ red edges in M_i.

So $|V_B(M_i)| = 2n+2$, $|V_R(M_i)| = 2n-2$, $e(G[V_B(M_i)]) = \binom{2n+2}{2} = 2n^2+3n+1$, $e(G[V_R(M_i)]) = \binom{2n-2}{2} = 2n^2 - 5n + 3$, the number of edges between $V_B(M_i)$ and $V_R(M_i)$ is $(2n+2)(2n-2) = 4n^2 - 4$, and finally the total number of edges of each colour is $\frac{1}{2}\binom{4n}{2} = 4n^2 - n$.

Claim C. If there is an equal number of red edges and black edges between $V_B(M_i)$ and $V_R(M_i)$, then we are done.

Proof. We claim that there is an odd number of black edges in $G[V_R(M_i)]$. Note that there is an even number of red edges in $G[V_B(M_i)]$ and an even number of edges of each colour joining $V_B(M_i)$ and $V_R(M_i)$.

Since the argument is just a simple parity analysis, to avoid a verbose and confusing argument, we present the following table as our argument.

n	total black edges	$e(G[V_B(M_i)])$	black edges in $G[V_R(M_i)]$
even	even	odd	odd
odd	odd	even	odd

So there is an odd number of black edges in $G[V_R(M_i)]$.

Thus there are $p, q, r, s \in V_R(M_i)$ such that $pq, rs \in E(M_i)$, pr is red and qs is black in G.

Let $M_i' = S(M_i, p, q, r, s)$. Now M_i' has $n+2$ black edges and $n-2$ red edges.

Observe that after the latest swapping occurs, those vertices and edges originally in $G[V_B(M_i)]$ are all contained in $G[V_B(M_i')]$.

As a premise of this Case 3.2 states that there is a red edge in $G[V_B(M_i)]$, there must be $u, v, x, y \in V_B(M_i) \subset V_B(M_i')$ such that $uv, xy \in E(M_i)$ and ux, vy are red in G.

Since $uv, xy \in E(M_i')$, we take M_{i+1} to be $S(M_i', u, v, x, y)$.

This reduce the difference by 2, so that it becomes 0. □

Thus we may assume that there are more black edges than red edges between $V_B(M_i)$ and $V_R(M_i)$.

For the sake of clarity of how we divide our next cases, we will consider the question "What is the least number of red edges that have to be in $G[V_B(M_i)]$?".

So we will have to maximize the number of red edges outside of $G[V_B(M_i)]$.

Thus all the edges in $G[V_R(M_i)]$ and $\frac{4n^2-4}{2} - 1 = 2n^2 - 3$ edges between $V_B(M_i)$ and $V_R(M_i)$ have to be red. Since there are $4n^2 - n$ red edges in total, there are at least $(4n^2 - n) - (2n^2 - 5n + 3) - (2n^2 - 3) = 4n$ red edges in $G[V_B(M_i)]$

Claim D. If there are more than $4n$ red edges in $G[V_B(M_i)]$, then we are done.

Proof. Since there are more black edges than red edges joining $V_B(M_i)$ and $V_R(M_i)$, from our lemma, there must be a black edge uv and a red edge xy of M_i such that $S(M_i, u, v, x, y)$ will increase the number of black edges from that of M_i by one.

But before we make the swapping, we consider that there are $4n$ edges adjacent to uv in $G[V_B(M_i)]$. So that there is a red edge not adjacent to uv.

As a consequence of the assumption right after Claim B, a red edge implies an existence of another red edge, there are $p, q, r, s \in V_B(M_i) - \{u, v\}$ such that $pq, rs \in E(M_i)$ and pr, qs are red in G.

Let $M_i' = S(M_i, u, v, x, y)$ so that there are $n + 2$ black edges and $n - 2$ red edges.

We take $S(M_i', p, q, r, s)$ to be our M_{i+1}.

The first swap increase the difference by 2 to be 4, then the second swap reduce the difference by 4 to 0. □

Thus we may assume that there are exactly $4n$ red edges in $G[V_B(M_i)]$

As a consequence of this assumption, there are $\frac{4n^2-4}{2} - 1 = 2n^2 - 3$ red edges between $V_B(M_i)$ and $V_R(M_i)$ and $G[V_R(M_i)]$ is monochromatic (red).

Claim E. If there is a pair of edges in M_i of different colours such that the edges that lie between them are of the latter three varieties shown in the proof of our lemma, then we are done.

Proof. In this case we are guaranteed a swapping that will reduce the difference between the numbers of black edges and red edges by 2 to 0. We take the resulting matching of that swapping to be M_{i+1} □

Thus we may assume that each pair of edges of different colours in M_i have edges of the first three varieties between them.

Since there are $2n^2 - 1$ black edges and $2n^2 - 3$ red edges between $V_B(M_i)$ and $V_R(M_i)$ and the edges are from just the first three varieties, there is only one possibility.

That is there is a black uv and a red xy connected to each other by edges of type two, while other pairs of edges in M_i are connected by edges of type three in G.

Without loss of generality let ux be red in G.

Claim F. There is a red edge not adjacent to uv in $G[V_B(M_i)]$

Proof. As in Claim D, there must be $p, q, r, s \in V_B(M_i) - \{u, v\}$ such that $pq, rs \in E(M_i)$ and pq, rs are red in G.

As in Claim D, we take $S(S(M_i, u, v, y, x), p, q, r, s)$ as our M_{i+1}.

This effectively reduce the difference from 2 to 0. □

Thus we may assume that all $4n$ red edges are adjacent to uv in $G[V_B(M_i)]$

In $G[V_B(M_i)]$, u and v each connecting to $2n$ vertices apart from each other.

Thus they collectively involve $4n$ edges, so that all of those edges are red.

Claim G. There is a member of $V_B(M_i) - \{u, v\}$ that is joined with y by a black edge.

Proof. Let this member of $V_B(M_i) - \{u, v\}$ be called z.

Since $z \in V_B(M_i)$, there is a $w \in V_B(M_i)$ such that $zw \in E(M_i)$.

Now vw is red and vy, yz, zw are black in G.

We take $S(S(M_i, u, v, x, y), v, y, w, z)$ as our M_{i+1} (see Figure 5).

First swapping does not change the difference, while the second one reduce it by 2 to 0. □

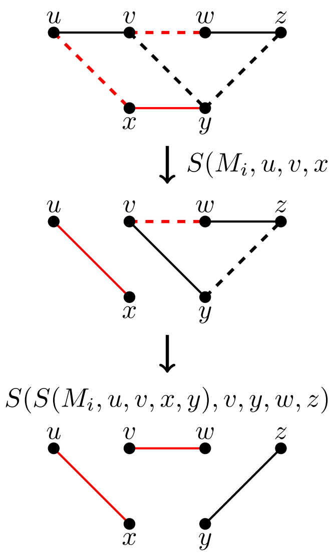


Figure 5

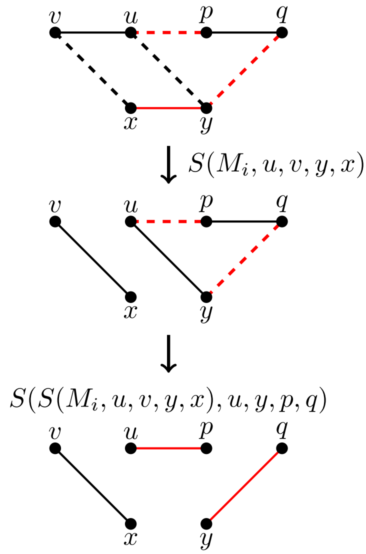


Figure 6

Thus we may assume that all members of $V_B(M_i) - \{u, v\}$ are joined with y by red edges.

Observe that from our premises every red edge in $G[V_B(M_i)]$ has to have either u or v as an endpoint and that all the edges joining u or v with any member in $V_B(M_i) - \{u, v\}$ are red.

Now choose any $p, q \in V_B(M_i) - \{u, v\}$. The edge pq is black while up and yq are red.

We take $S(S(M_i, u, v, y, x), u, y, p, q)$ as our M_{i+1} (see Figure 6).

First swapping increase the difference by 2 to be 4, then the second one reduce it by 4 to 0.

Now, every case, except for Case 1 which is the terminal case, strictly reduces the difference between the numbers of black edges and red edges as we are creating new terms for our sequence.

Note that in no case the difference was reduced by more than its value.

Thus as our sequence progresses the difference strictly decreases and when it terminates, the last term of the sequence has the difference of 0.

That is by following this method, we can always guarantee a matching which has the same number of black edges and red edges.

This proves our theorem. □

4 Concluding Remarks

We have proved Theorem 1.2 which settles the problem posed in [4]. Now we know that if we are given a 2-edge-coloured complete graph of order $4n$ with the same number of edges of each colour, we can extract a matching that has the same number of edges of each colour.

We would like to pose two problems that are related to the result that we have proven.

The first problem is a generalization of our result. In our theorem, we study only the case where the complete graph K_{4n} is 2-edge-coloured and those two colours colour an equal number of edges. An obvious generalization of this problem is to consider the similar situation when there are more than two colours involved. Now we pose the following question which is a generalization of our result.

Problem 4.1. For any k-edge-colouring of K_{2kn} such that there are an equal number of edges of each colour. Does there exist a matching such that there are an equal number of edges of each colour?

The second problem comes from the fact that when we take a matching with an equal number of edges of each colour out of our original complete graph, we are left with a graph that has the same number of edges of each colour. One question that comes up is 'can we take another such matching?'. If we can, can we continue until all edges are gone? If we cannot exhaust the edges with an arbitrary order of taking matchings out, is there any sequence of taking matchings out that would use all edges? This leads us to pose the following problem.

Problem 4.2. Given a 2-edge-coloured K_{4n} with an equal number of edges of each colour. Can the graph be decomposed into perfect matchings such that each matching has the same number of edges of each colour?

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The A-Möbius function of a finite group

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Abstract

The Möbius function of the subgroup lattice of a finite group G has been introduced by Hall and applied to investigate several different questions. We propose the following generalization. Let A be a subgroup of the automorphism group $\mathrm{Aut}(G)$ of a finite group G and denote by $\mathcal{C}_A(G)$ the set of A-conjugacy classes of subgroups of G. For $H \leq G$ let $[H]_A = \{ H^a \mid a \in A\}$ be the element of $\mathcal{C}_A(G)$ containing H. We may define an ordering in $\mathcal{C}_A(G)$ in the following way: $[H]_A \leq [K]_A$ if $H^a \leq K$ for some $a \in A$. We consider the Möbius function μ_A of the corresponding poset and analyse its properties and possible applications.

Keywords: Groups, subgroup lattice, Möbius function.

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1 Introduction

The Möbius function of a finite partially ordered set (poset) P is the map $\mu_P\colon\ P \times P \to \mathbb{Z}$ satisfying $\mu_P(x, y) = 0$ unless $x \leq y$, in which case it is defined inductively by the equations $\mu_P(x, x) = 1$ and $\sum_{x \leq z \leq y} \mu_P(x, z) = 0$ for $x < y$.

In a celebrated paper [5], P. Hall used for the first time the Möbius function μ of the subgroup lattice of a finite group G to investigate some properties of G, in particular to compute the number of generating t-tuples of G. A detailed investigation of the properties of the function μ associated to a finite group G is given by T. Hawkes, I. M. Isaacs and

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M. Özaydin in [6]. In that paper, the authors also consider the Möbius function λ of the poset of conjugacy classes of subgroups of G, where $[H] \leq [K]$ if $H \leq K^g$ for some $g \in G$ (see [6, Section 7]). In particular, they propose the interesting and intriguing question of comparing the values of μ and λ.

In this paper we aim to generalize the definitions and main properties of the functions μ and λ to a more general context. Let G and A be a finite group and a subgroup of the automorphism group $\mathrm{Aut}(G)$ of G, respectively. Denote by $\mathcal{C}_A(G)$ the set of A-conjugacy classes of subgroups of G. For $H \leq G$ let $[H]_A = \{ H^a \mid a \in A\}$ be the element of $\mathcal{C}_A(G)$ containing H. We may define an ordering in $\mathcal{C}_A(G)$ in the following way: $[H]_A \leq [K]_A$ if $H^a \leq K$ for some $a \in A$; we consider the Möbius function μ_A of the corresponding poset. We will write $\mu_A(H, K)$ in place of $\mu_A([H]_A, [K]_A)$. When $A = \mathrm{Inn}(G)$, we write $\mathcal{C}(G)$ and $[H]$, in place of $\mathcal{C}_{\mathrm{Inn}(G)}(G)$ and $[H]_{\mathrm{Inn}(G)}$. When $A = 1$, $\mu_A = \mu$ is the Möbius function in the subgroup lattice of G, introduced by P. Hall. In the case when $A = \mathrm{Inn}(G)$ is the group of the inner automorphism, $\mu_{\mathrm{Inn}(G)}$ coincides the Möbius function λ of the poset of conjugacy classes of subgroups of G, defined above. Note that for any subgroup A of $\mathrm{Aut}(G)$, we get $[G]_A = \{G\}$.

In Section 2, we prove some general properties of μ_A. In particular we prove the following result:

Proposition 1.1. *Let G be a finite solvable group. If $G' \leq K \leq G$ and A is the subgroup of $\mathrm{Inn}(G)$ obtained by considering the conjugation with the elements of K, then $\mu_A(H, G) = \lambda(H, G)$ for every $H \leq G$.*

To illustrate the meaning of the previous proposition, consider the following example. Let $G = A_4$ be the alternating group of degree 4 and A the subgroup of $\mathrm{Inn}(G)$ induced by conjugation with the elements of $G' \cong C_2 \times C_2$. The posets $\mathcal{C}(G)$ and $\mathcal{C}_A(G)$ are different. For example there are three subgroups of G of order 2, which are conjugated in G, but not A-conjugated. However $\lambda(H, G) = \mu_A(H, G)$ for any $H \leq G$.

In Section 3, we generalize some result given by Hall in [5], about the cardinality $\phi(G, t)$ of the set $\Phi(G, t)$ of t-tuples $(g_1, \dots, g_t)$ of group elements g_i such that $G = \langle g_1, \dots, g_t\rangle$. As observed by P. Hall, using the Möbius inversion formula, it can be proved that

$$\phi(G, t) = \sum_{H \leq G} \mu(H, G)|H|^t. \tag{1.1}$$

We generalize this formula, showing that $\phi(G, t)$ can be computed with a formula involving μ_A for any possible choice of A.

Theorem 1.2. *For any finite group G and any subgroup A of $\mathrm{Aut}(G)$,*

$$\phi(G, t) = \sum_{[H]_A \in \mathcal{C}_A(G)} \mu_A(H, G)| \cup_{a \in A} (H^a)^t|.$$

If G is not cyclic, then $\phi(G, 1) = 0$, so we obtain the following equality, involving the values of μ_A.

Corollary 1.3. *If G is not cyclic, then*

$$0 = \sum_{[H]_A \in \mathcal{C}_A(G)} \mu_A(H, G)| \cup_{a \in A} H^a|.$$

Further generalizations are given in Section 4, where we consider the function $\phi^*(G,t)$, which is an analogue of $\phi(G,t)$: actually, $\phi^*(G,t)$ denotes the cardinality of the set of of t-tuples $(H_1,\dots,H_t)$ of subgroups of G such that $G=\langle H_1,\dots,H_t\rangle$. As a corollary of our formula for computing $\phi^*(G,t)$, we obtain we following unexpected result.

Proposition 1.4. *Let $\sigma(X)$ denote the number of subgroups of a finite group X. For any finite group G, the following equality holds:*

$$1=\sum_{H\le G}\mu(H,G)\sigma(H).$$

Finally, in Section 5, we consider one question originated from a result given by Hawkes, Isaacs and Özaydin in [6]: they proved that the equality

$$\mu(1,G)=|G'|\lambda(1,G)$$

holds for any finite solvable group G; later Pahlings [7] generalized the result proving that

$$\mu(H,G)=|N_{G'}(H):G'\cap H|\cdot\lambda(H,G) \tag{1.2}$$

holds for any $H\le G$ whenever G is finite and solvable. Following [3], we say that G satisfies the (μ,λ)-property if (1.2) holds for any $H\le G$. Several classes of non-solvable groups satisfy the (μ,λ)-property, for example all the minimal non-solvable groups (see [3]). However it is known that the (μ,λ)-property does not hold for every finite group. For instance, it does not hold for the following finite almost simple groups: A_9, S_9, A_{10}, S_{10}, A_{11}, S_{11}, A_{12}, S_{12}, A_{13}, S_{13}, J_2, $PSU(3,3)$, $PSU(4,3)$, $PSU(5,2)$, M_{12}, M_{23}, M_{24}, $PSL(3,11)$, HS, $\mathrm{Aut}(HS)$, He $\mathrm{Aut}(H)$, McL, $PSL(5,2)$, $G_2(4)$, Co_3, $P\Omega^-(8,2)$, $P\Omega^+(8,2)$. It is somehow intriguing to notice that although the (μ,λ)-property fails for the sporadic groups M_{12}, J_2, McL, it holds for their automorphism groups.

We prove the following generalization of Pahlings's result.

Theorem 1.5. *Let N be a solvable normal subgroup of a finite group G. If G/N satisfies the (μ,λ)-property, then G also satisfies the (μ,λ)-property.*

An almost immediate consequence of the previous theorem is the following.

Corollary 1.6. *$PSU(3,3)$ is the smallest group which does not satisfy the (μ,λ) property.*

In the last part of Section 5, we use Theorem 1.2 to deduce some consequences of the (μ,λ)-property. In particular we prove the following theorem.

Theorem 1.7. *Suppose that a finite group G satisfies the (μ,λ)-property. Then, for every positive integer t, the following equality is satisfied:*

$$\sum_{[H]\in\mathcal{C}(G)}\lambda(H,G)\left(\frac{|H|^{t-1}|G||G'H|}{|G'N_G(H)|}-|\cup_{a\in A}(H^a)^t|\right)=0.$$

Some open questions are proposed along the paper.

2 Applying some general properties of the Möbius function

Given a poset P, a closure on P is a function $^-\colon P \to P$ satisfying the following three conditions:

(a) $x \le \bar{x}$ for all $x \in P$;

(b) if $x, y \in P$ with $x \le y$, then $\bar{x} \le \bar{y}$;

(c) $\bar{\bar{x}} = \bar{x}$ for all $x \in P$.

If $^-$ is a closure map on P, then $\overline{P} = \{x \in P \mid \bar{x} = x\}$ is a poset with order induced by the order on P. We have:

Theorem 2.1 (The closure theorem of Crapo [2])**.** *Let P be a finite poset and let $^-\colon P \to P$ be a closure map. Fix $x, y \in P$ such that $y \in \overline{P}$. Then*

$$\sum_{x \le z \le y, \bar{z} = y} \mu_P(x, z) = \begin{cases} \mu_{\bar{P}}(x, y) \text{ if } x = \bar{x} \\ 0 \text{ otherwise.} \end{cases}$$

In [5], P. Hall proved that if $H < G$, then $\mu(H, G) \ne 0$ only if H is an intersection of maximal subgroups of G. Using the previous theorem, the following more general statement can be obtained.

Proposition 2.2. *If $H < G$ and $\mu_A(H, G) \ne 0$, then H can be obtained as intersection of maximal subgroups of G.*

Proof. Let H be a proper subgroup of G and let $\overline{H}$ be the intersection of the maximal subgroups of G containing H. Moreover let $\overline{G} = G$. The map $[H]_A \mapsto [\overline{H}]_A$ is a well defined closure map on $\mathcal{C}_A(G)$. Apply Theorem 2.1, with $x = [H]_A$ and $y = [G]_A$. Since $\overline{K} = G$ if and only if $K = G$, we have that $\mu_A(H, G) = 0$ if $H \ne \overline{H}$. □

An element a of a poset $\mathcal{P}$ is called conjunctive if the pair $\{a, x\}$ has a least upper bound, written $a \vee x$, for each $x \in \mathcal{P}$.

Lemma 2.3 ([6, Lemma 2.7])**.** *Let $\mathcal{P}$ be a poset with a least element 0, and let $a > 0$ be a conjunctive element of $\mathcal{P}$. Then, for each $b > a$, we have*

$$\sum_{a \vee x = b} \mu_{\mathcal{P}}(0, x) = 0.$$

From the above 2.3, the following Lemma 2.4 follows easily. Together with Lemma 2.5 and Lemma 2.7, this allows us to prove Proposition 1.1.

Lemma 2.4. *Let N be an A-invariant normal subgroup of G and $H \le G$. If $H < HN < G$, then*

$$\mu_A(H, G) = -\sum_{[Y]_A \in \mathcal{S}_A(H, N)} \mu_A(H, Y),$$

with $\mathcal{S}_A(H, N) = \{[Y]_A \in \mathcal{C}_A(G) \mid [H]_A \le [Y]_A < [G]_A$ and $YN = G\}$.

Proof. Let $\mathcal{P}$ be the interval $\{[K]_A \in \mathcal{C}_G(A) \mid [H]_A \le [K]_A \le [G]_A]\}$. Notice that $[HN]_A$ is a conjunctive element of $\mathcal{P}$. Indeed $[HN]_A \vee [K]_A = [KN]_A$ for every $[K]_A \in \mathcal{P}$. So the conclusion follows immediately from Lemma 2.3. □

Lemma 2.5. *Let K and A be a subgroup of G and the subgroup of $\mathrm{Inn}(G)$ induced by the conjugation with the elements of K, respectively. Assume that N is an abelian minimal normal subgroup of G contained in K and $H < HN \leq G$. Then*

$$\mu_A(H,G) = -\mu_A(HN,G)\gamma_A(N,H),$$

where $\gamma_A(N,H)$ is the number of A-conjugacy classes of complements of N in G containing H.

Proof. If $HN = G$, then H is a maximal subgroup of G, hence $\mu_A(H,G) = -1$, while $\mu_A(HN,G) = \mu_A(G,G) = 1$ and $\gamma_A(N,H) = 1$, so the statement is true. So we may assume $HN < G$ and apply Lemma 2.4. Suppose $[Y]_A \in \mathcal{S}_A(H,N)$. Notice that, since $YN = G$ and N is abelian, $Y \cap N$ is normal in G. Moreover $N \not\leq G$, since $Y < G = YN$. By the minimality of N as normal subgroup, we conclude $Y \cap N = 1$. Let

$$\mathcal{C} = \{J \leq G \mid H \leq J \leq Y\}, \quad \mathcal{D} = \{L \leq G \mid HN \leq L\}$$
$$\mathcal{C}_A = \{[J]_A \in \mathcal{C}_A(G) \mid [H]_A \leq [J]_A \leq [Y]_A\}, \mathcal{D}_A = \{[L]_A \in \mathcal{C}_A(G) \mid [HN]_A \leq [L]_A\}.$$

The map $\eta\colon \mathcal{C} \to \mathcal{D}$ sending J to JN is an order preserving bijection. Clearly, if $J_2 = J_1^x$ for some $x \in K$, then $\eta(J_2) = NJ_2 = NJ_1^x = (NJ_1)^x = (\eta(J_1))^x$. Conversely assume $\eta(J_2) = (\eta(J_1))^x$ with $x \in K$. Since $YN = G$, $x = yn$ with $n \in N$ and $y \in Y \cap K$. Thus $J_2N = (J_1N)^x = (J_1N)^y$ and consequently, applying the Dedekind law, $J_2 = J_2(Y \cap N) = J_2N \cap Y = (J_1N)^y \cap Y = (J_1N \cap Y)^y = J_1^y$. It follows that η induces an order preserving bijection from $\mathcal{C}_A$ to $\mathcal{D}_A$, but then $\mu_A(H,Y) = \mu_A(HN,YN) = \mu_A(HN,G)$. □

The statement of the previous lemma leads to the following open question.

Question 2.6. Let G be a finite group, $A \leq \mathrm{Aut}(G)$ and N an A-invariant normal subgroup of G. Does $\mu_A(HN,G)$ divide $\mu_A(H,G)$ for every $H \leq G$?

The following lemma is straightforward.

Lemma 2.7. *Let A be a subgroup of $\mathrm{Aut}(G)$ and N an A-invariant normal subgroup of G. Every $a \in A$ induces an automorphism $\overline{a}$ of G/N. Let $\overline{A} = \{\overline{a} \mid a \in A\}$. Then, for any $H \leq G$, $\mu_A(HN,G) = \mu_{\overline{A}}(HN/N,G/N)$.*

Proof of Proposition 1.1. We work by induction on $|G| \cdot |G:H|$. The statement is true if G is abelian. Assume that G' contains a minimal normal subgroup, say N, of G. If $N \leq H$, then, by Lemma 2.7

$$\lambda(H,G) = \lambda(H/N,G/N) = \mu_{\overline{A}}(H/N,G/N) = \mu_A(H,G).$$

So we may assume $N \not\leq H$. If H is not an intersection of maximal subgroups of G, then $\lambda(H,G) = \mu_A(H,G) = 0$. Suppose $H = M_1 \cap \cdots \cap M_t$ where $M_1, \ldots, M_t$ are maximal subgroups of G. In particular N is not contained in M_i for some i, so M_i is a complement of N in G containing H and $N \cap H = 1$. By Lemma 2.5, we have

$$\lambda(H,G) = -\lambda(HN,G)\gamma(N,H), \quad \mu_A(H,G) = -\mu_A(HN,G)\gamma_A(N,H),$$

where $\gamma(N,H)$ is the number of conjugacy classes of complements of N in G containing H and $\gamma_A(N,H)$ is the number of A-conjugacy classes of these complements. Suppose that K_1, K_2 are two conjugated complements of N in G containing H. Then $K_2 = K_1^n$ for some $n \in N_N(H)$. Since $N \leq G' \leq K$, it follows $\gamma(N,H) = \gamma_A(N,H)$. Moreover, by induction, $\lambda(HN,G) = \mu_A(HN,G)$, hence we conclude $\lambda(H,G) = \mu_A(H,G)$. □

3 Generalizing a formula of Philip Hall

We begin with introducing the functions $\Psi_A(H,t)$ and $\psi_A(H,t)$, analogue of $\Phi(H,t)$ and $\phi(H,t)$ in the general case of any possible subgroup A of $\mathrm{Aut}(G)$.

For any $H \in \mathcal{C}_A(G)$ and any positive integer t, let

1. $\Omega_A(H,t) = \bigcup_{a\in A}(H^a)^t$;
2. $\omega_A(H,t) = |\Omega_A(H,t)|$;
3. $\Psi_A(H,t) = \{(g_1,\dots,g_t) \in G^t \mid \langle g_1,\dots,g_t\rangle = H^a \text{ for some } a \in A\}$;
4. $\psi_A(H,t) = |\Psi_A(H,t)|$.

If $(x_1,\dots,x_t) \in \Omega_A(H,t)$, then $\langle x_1,\dots,x_t\rangle \le H^a$ for some $a \in A$, hence $\langle x_1,\dots,x_t\rangle = K$ for some $K \le G$ with $[K]_A \le [H]_A$. Thus

$$\sum_{[K]\le_A[H]} \psi_A(K,t) = \omega_A(H,t)$$

and therefore, by the Möbius inversion formula,

$$\sum_{[H]\in\mathcal{C}_A(G)} \mu_A(H,G)\omega_A(H,t) = \psi_A(G,t).$$

On the other hand $\psi_A(G,t) = \phi(G,t)$ so we have proved the following formula.

Theorem 3.1. *For any finite group G and any subgroup A of* $\mathrm{Aut}(G)$,

$$\phi(G,t) = \sum_{[H]\in\mathcal{C}_A(G)} \mu_A(H,G)\omega_A(H,t).$$

Notice that if $A = 1$, then $\omega_A(H,t) = |H^t|$, so that the result by Hall given in (1.1) is a particular case of the previous theorem.

Corollary 3.2. *If G is not cyclic, then*

$$0 = \phi(G,1) = \sum_{[H]\in\mathcal{C}_A(G)} \mu_A(H,G)\omega_A(H,1).$$

Taking $A = \mathrm{Inn}(G)$, we deduce in particular that if G is not cyclic, then

$$\sum_{H\in\mathcal{C}(H)} \lambda(H,G)\omega_{\mathrm{Inn}(G)}(H,1) = \sum_{H\in\mathcal{C}(H)} \lambda(H,G)|\cup_g H^g| = 0.$$

For example, if $G = S_4$, then the values of $\lambda(H,G)$ and $|\cup_g H^g|$ are as in the following table and $24 - 12 - 16 - 15 + 4 + 9 + 7 - 1 = 0$.

	$\lambda(H,G)$	$\vert\cup_g H^g\vert$
S_4	1	24
A_4	-1	12
D_4	-1	16
S_3	-1	15
K	1	4
$\langle(1,2,3,4)\rangle$	0	10
$\langle(1,2,3)\rangle$	1	9
$\langle(1,2\rangle$	1	7
$\langle(1,2)(3,4)\rangle$	0	4
1	-1	1

If $G = A_5$, then the values of $\lambda(H,G)$, $\omega_{\mathrm{Inn}(G)}(H,1) = |\cup_g H^g|$, $\omega_{\mathrm{Inn}(G)}(H,2) = |\cup_g (H^g)^2|$ (taking only the subgroups H with $\lambda(H,G) \neq 0$) are as in the following table and $60 - 36 - 36 - 40 + 21 + 32 - 1 = 0$.

	$\lambda(H,G)$	$\vert\cup_g H^g\vert$	$\vert\cup_g (H^g)^2\vert$
A_5	1	60	3600
A_4	-1	36	636
S_3	-1	36	306
D_5	-1	40	550
$\langle(1,2,3)\rangle$	1	21	81
$\langle(1,2)(3,4)\rangle$	2	16	46
1	-1	1	1

Moreover

$$3600 - 636 - 306 - 550 + 81 + 2 \cdot 46 - 1 = 2280 = \frac{19}{30} \cdot 3600 = \phi(A_5, 2).$$

If $G = D_p = \langle a, b \mid a^p = 1, b^2 = 1, a^b = a^{-1}\rangle$ and p is an odd prime, then the behaviour of the subgroups in $\mathcal{C}(G)$ is described by the following table.

	$\lambda(H,G)$	$\vert\cup_g H^g\vert$
D_p	1	$2p$
$\langle a\rangle$	-1	p
$\langle b\rangle$	-1	$p+1$
1	-1	1

Another interesting example is given by considering $G = C_p^n$ and $A = \mathrm{Aut}(G)$. Let $H \cong C_p^{n-1}$ be a maximal subgroup of G. Then, for $K \leq G$, $\mu_A(K,G) \neq 0$ if and only if either $[K]_A = [G]_A$ or $[K]_A = [H]_A$. Clearly $\cup_{\alpha\in\mathrm{Aut}(G)} H^\alpha = G$ so $\mu_A(G,G)\omega_A(G,1) - \mu_A(H,G)\omega_A(H,1) = |G| - |G| = 0$. More generally, $\Omega_A(H,t)$ is the set of t-tuples $(x_1,\ldots,x_t)$ such that $(x_1,\ldots,x_t) \in K^t$ for some maximal subgroup K of G, so $\mu_A(G,G)\omega_A(G,t) - \mu_A(H,G)\omega_A(H,t) = |G|^t - \omega_A(H,t)$ is the number of generating t-tuples of G.

Another generalization of (1.1), essentially due to Gaschütz, has been described by Brown in [1, Section 2.2]. Let N be a normal subgroup of G and suppose that G/N admits t generators for some integer t. Let $y = (y_1, \dots, y_t)$ be a generating t-tuple of G/N and denote by $P(G, N, t)$ the probability that a random lift of y to a t-tuple of G generates G. Then $P(G, N, t) = \phi(G, N, t)/|N|^t$, where $\phi(G, N, t)$ is the number of generating t-tuples of G lying over y. As is showed in [1, Section 2.2], using again the Möbius inversion formula it can be proved:

$$\phi(G, N, t) = \sum_{H \le G, HN = G} \mu(H, G)|H \cap N|^t. \tag{3.1}$$

This formula can be generalized in our contest in the following way:

Theorem 3.3. *Let N be an A-invariant normal subgroup of G and fix $g_1, \dots, g_t \in G$ with the property that $G = \langle g_1, \dots, g_t \rangle N$. Define*

- $\Omega_A(H, N, t) = \{(n_1, \dots, n_t) \mid \langle g_1 n_1, \dots, g_t n_t \rangle \le H^a$ *for some* $a \in A\}$;
- $\omega_A(H, N) = |\Omega_A(H, N, t)|$

and let $\mathcal{C}_A(G, N) = \{[H]_A \in \mathcal{C}_A(G) \mid HN = G\}$. Then

$$\phi(G, N, t) = \sum_{[H]_A \in \mathcal{C}_A(G,N)} \mu_A(H, G)\omega_A(H, N, t).$$

Proof. Fix $g_1, \dots, g_t \in G$ with the property that $G = \langle g_1, \dots, g_t \rangle N$. Then $\phi(G, N, t)$ is the cardinality of the set

$$\Phi(G, N, g_1, \dots, g_t) = \{(n_1, \dots, n_t) \in N^t \mid \langle g_1 n_1, \dots, g_t n_t \rangle = G\}.$$

Set:

$$\Psi_A(H, N, g_1, \dots, g_t) = \{(n_1, \dots, n_t) \in N^t \mid \langle g_1 n_1, \dots, g_t n_t \rangle = H^a \text{ for some } a \in A\};$$
$$\psi_A(H, N, t) = |\Psi_A(H, N, g_1, \dots, g_t)|.$$

Notice that $\omega_A(H, N, t) \neq 0$ if and only if $[H]_A \in \mathcal{C}_A(G, N)$. If $(n_1, \dots, n_t) \in \Omega_A(H, N, t)$, then $\langle g_1 n_1, \dots, g_t n_t \rangle \le H^a$ for some $a \in A$, and $\langle g_1 n_1, \dots, g_t n_t \rangle = K$ for some $K \le G$ with $[K]_A \le [H]_A$. Thus

$$\sum_{[K]_A \le [H]_A} \psi_A(K, N, t) = \omega_A(H, N, t)$$

and therefore, by the Möbius inversion formula

$$\sum_{[H] \in \mathcal{C}_A(G,N)} \mu_A(H, G)\omega_A(H, N, t) = \psi_A(G, N, t) = \phi(G, N, t) \qquad \square$$

4 Another application of Möbius inversion formula

Denote by $\Phi^*(G,t)$ the set of t-tuples $(H_1,\dots,H_t)$ of subgroups of G such that $G = \langle H_1,\dots,H_t\rangle$ and by $\phi^*(G,t)$ the cardinality of this set. For any $H \in \mathcal{C}_A(G)$ and any positive integer t, let

1. $\Sigma_A(H,t) = \{(H_1,\dots,H_t) \mid \langle H_1,\dots,H_t\rangle \le H^a$ for some $a \in A\}$;
2. $\sigma_A(H,t) = |\Sigma_A(H,t)|$;
3. $\Gamma_A(H,t) = \{(H_1,\dots,H_t) \mid \langle H_1,\dots,H_t\rangle = H^a$ for some $a \in A\}$;
4. $\gamma_A(H,t) = |\Gamma_A(H,t)|$.

Theorem 4.1.

$$\phi^*(G,t) = \sum_{[H]\in\mathcal{C}_A(G)} \mu_A(H,G)\sigma_A(H,t).$$

Proof. If $(H_1,\dots,H_t) \in \Sigma_A(H,t)$, then $\langle H_1,\dots,H_t\rangle = K$ for some $K \le G$ with $[K]_A \le [H]_A$. Thus

$$\sum_{[K]\le_A[H]} \gamma_A(K,t) = \sigma_A(H,t)$$

and therefore, by the Möbius inversion formula,

$$\sum_{[H]\in\mathcal{C}_A(G)} \mu_A(H,G)\sigma_A(H,t) = \gamma_A(G,t) = \phi^*(G,t). \qquad \square$$

In the particular case when $A = 1$, $\sigma_A(H,t) = \sigma(H)^t$, denoting with $\sigma(H)$ the number of subgroups of H. So we obtain the following corollary:

Corollary 4.2.

$$\phi^*(G,t) = \sum_{H\le G} \mu(H,G)\sigma(H)^t.$$

Clearly $\Sigma^*(G,t) = \{G\}$, so $\phi^*(G,1) = 1$ and therefore it follows:

Corollary 4.3.

$$1 = \sum_{H\in H_A} \mu_A(H,G)\sigma_A(H,1).$$

In particular:

Corollary 4.4.

$$1 = \sum_{H\le G} \mu(H,G)\sigma(H).$$

For example, if $G = A_5$ then the subgroups of G with $\mu(H,G) \ne 0$ are listed in the following table (where $\kappa(H,G)$ denote the numbers of conjugate of H in G).

	$\mu(H,G)$	$\kappa(H,G)$	$\sigma(H)$
A_5	1	1	59
A_4	-1	5	10
S_3	-1	10	6
D_5	-1	6	8
$\langle(1,2,3)\rangle$	2	10	2
$\langle(1,2)(3,4)\rangle$	4	15	2
1	-60	1	1

According with Corollary 4.4, $1 = 59 - 5\cdot 10 - 10\cdot 6 - 6\cdot 8 + 2\cdot 10\cdot 2 + 4\cdot 15\cdot 2 - 60$.

For a finite group G, denote by $P(G,t)$ and $P^*(G,t)$ the probability of generating G with, respectively, t elements or t subgroups. It can be easily seen that $P(G,t) = P(G/\operatorname{Frat}(G),t)$, but in general $P^*(G,t) \neq P^*(G/\operatorname{Frat}(G),t)$. For example, if $G \cong C_{p^a}$, then G and $H \cong C_{p^{a-1}}$ are the unique subgroups of G with non trivial Möbius number and therefore

$$P(G,t) = \frac{|G|^t - |H|^t}{|G|^t} = 1 - \frac{1}{p^t},$$
$$P^*(G,t) = \frac{\sigma(G)^t - \sigma(H)^t}{\sigma(G)^t} = 1 - \frac{a^t}{(a+1)^t}.$$

So $P(G,t)$ is independent of a, while $P^*(G,t)$ tends to 0 when a tends to infinity.

5 The (μ, λ)-property

Proof of Thereom 1.5. Working by induction on the order of G, it suffices to prove the statement in the particular case when N is an abelian minimal normal subgroup of G. Let H be a subgroup of G. If $N \leq H$, then

$$\begin{aligned}\mu(H,G) &= \mu(H/N,G/N) = \lambda(H/N,G/N)|N_{G'N/N}(H/N) : H/N \cap G'N/N| \\ &= \lambda(H,G)|N_{G'N}(H) : H \cap G'N| = \lambda(H,G)|NN_{G'}(H) : N(H\cap G')| \\ &= \lambda(H,G)\frac{|N_{G'}(H) : H\cap G'|}{|N\cap N_{G'}(H) : N\cap H\cap G'|} = \lambda(H,G)\frac{|N_{G'}(H) : H\cap G'|}{|N\cap G' : N\cap G'|} \\ &= \lambda(H,G)|N_{G'}(H) : H\cap G'|.\end{aligned}$$

So we may assume $N \not\leq H$. If H is not an intersection of maximal subgroups of G, then $\mu(G,H) = \lambda(G,H) = 0$. So we may assume $H = M_1 \cap \cdots \cap M_t$ where $M_1,\dots,M_t$ are maximal subgroups of G. Since N is not contained in H, then N is not contained in M_i for some i, but then M_i is a complement of N in G containing H and $N\cap H = 1$. If $g \in N_G(HN)$, then $g = xn$ with $x \in M_i$ and $n \in N$. In particular $H^x \leq HN \cap M_i = H(N\cap M_i) = H$, so $N_G(HN) = N_G(H)N$. By Lemma 2.5, we have

$$\frac{\mu(H,G)}{\lambda(H,G)} = \frac{\mu(HN,G)}{\lambda(HN,G)}\frac{\kappa}{\delta} = |N_{G'N}(HN) : HN\cap G'N|\frac{\kappa}{\delta} = |NN_{G'}(H) : HN\cap G'N|\frac{\kappa}{\delta}$$

where k is the number of complements of N in G containing H and δ is the number of conjugacy classes of these complements. First assume that $N \leq Z(G)$. Then $\kappa = \delta$,

$G' = M_i' \leq M_i$, $N \cap G' = 1$ and

$$\begin{aligned}\frac{\mu(H,G)}{\lambda(H,G)} &= |NN_{G'}(H) : HN \cap G'N|\frac{\kappa}{\delta} = |NN_{G'}(H) : HN \cap G'N| \\ &= |NN_{G'}(H) : N(H \cap G')| = |N_{G'}(H) : H \cap G'|.\end{aligned}$$

Finally assume $N \not\leq Z(G)$. Then $N \leq G'$, $\kappa/\delta = |N_N(H)|$ and

$$\begin{aligned}\frac{\mu(H,G)}{\lambda(H,G)} &= |NN_{G'}(H) : HN \cap G'N|\frac{\kappa}{\delta} = |NN_{G'}(H) : N(H \cap G')||N_N(H)| \\ &= \frac{|N||N_{G'}(H)|}{|N_N(H)|}\frac{|N_N(H)|}{|N||H \cap G'|} = |N_{G'}(H) : H \cap G'|.\end{aligned}$$ □

Proof of Corollary 1.6. Suppose that G has minimal order with respect to the property that G does not satisfy the (μ, λ) property. By the previous proposition, G contains no abelian minimal normal subgroup and therefore $\mathrm{soc}(G) = S_1 \times \cdots \times S_t$ is a direct product of nonabelian finite simple groups. If $|G| \leq |PSU(3,3)| = 6048$, then either $t = 1$ or $G = \mathrm{soc}(G) = A_5 \times A_5$. So it suffices to check that $A_5 \times A_5$ and any almost simple group of order at most 6048 satisfies the (μ, λ) property. Recall that the table of marks of a finite group G is a matrix whose rows and columns are labelled by the conjugacy classes of subgroups of G and where for two subgroups A and B the (A, B)-entry is the number of fixed points of B in the transitive action of G on the cosets of A in G. Since, for every $H \leq G$, $\lambda(H, G)$ and $\mu(H, G)$ can be computed from the table of marks of G (see [7, Proposition 1]), our proof can be easily completed using the library of table of marks available in GAP [4]. □

We may use Theorem 3.1 to deduce some consequences of the (μ, λ)-property.

Theorem 5.1. *Suppose that a finite group G satisfies the (μ, λ)-property. Then*

$$\sum_{[H]\in\mathcal{C}(G)} \lambda(H,G)\left(\frac{|H|^{t-1}|G||G'H|}{|G'N_G(H)|} - \omega(H,t)\right) = 0. \tag{5.1}$$

Proof. By Theorem 3.1,

$$\begin{aligned}\sum_{H\in\mathcal{C}(G)} \lambda(H,G)\omega(H,t) = \phi(G,t) &= \sum_{H\leq G} \mu(H,G)|H|^t \\ &= \sum_{H\in\mathcal{C}(G)} \mu(H,G)|G : N_G(H)||H^t| \\ &= \sum_{H\in\mathcal{C}(G)} \lambda(H,G)|N_{G'}(H) : G' \cap H||G : N_G(H)||H^t| \\ &= \sum_{H\in\mathcal{C}(G)} \lambda(H,G)\frac{|H|^t|G||N_{G'}(H)|}{|G' \cap H||N_G(H)|} \\ &= \sum_{H\in\mathcal{C}(G)} \lambda(H,G)\frac{|H|^{t-1}|G||G'H|}{|G'N_G(H)|}.\end{aligned}$$ □

A natural question is whether (5.1) is also a sufficient condition for the (μ, λ)-property. For any $H \leq G$, set $\mu^*(H,G) = |N_{G'}(H) : G' \cap H|\lambda(H,G)$. The validity of (5.1) is equivalent to

$$\sum_{H \in \mathcal{C}(G)} \lambda(H,G)\omega(H,t) - \sum_{H \in \mathcal{C}(G)} \mu^*(H,G)|H|^t|G : N_G(H)| = 0.$$

In any case we must have

$$\sum_{H \in \mathcal{C}(G)} \lambda(H,G)\omega(H,t) - \sum_{H \in \mathcal{C}(G)} \mu(H,G)|H|^t|G : N_G(H)| = 0.$$

So (5.1) is equivalent to

$$\sum_{H \in \mathcal{C}(G)} \frac{(\mu(H,G) - \mu^*(H,G))|H|^t}{|N_G(H)|} = 0.$$

Let $\mathcal{T} = \{[H] \in \mathcal{C}(G) \mid \mu(H,G) \neq \mu^*(H,G)\}$. Then (5.1) is true if and only if

$$\sum_{[H] \in \mathcal{T}} \frac{(\mu(H,G) - \mu^*(H,G))|H|^t}{|N_G(H)|} = 0. \tag{5.2}$$

For example, if $G = PSU(3,3)$, then $\mathcal{T}$ consists of four conjugacy classes of subgroups and the corresponding values are given by the following table:

$\mu(H,G)$	$\mu^*(H,G)$	$\lvert H\rvert$	$\lvert N_G(H)\rvert$
-48	0	2	96
3	0	6	18
0	-4	8	32
1	2	24	24

In this case (5.2) is equivalent to

$$2^{t-1} - 6^{t-1} - 8^{t-1} + 24^{t-1} = 0$$

which is true only if $t = 1$.

For any positive integer n let

$$\tau(n) = \sum_{H \in \mathcal{T}, |H|=n} \frac{\mu(H,g) - \mu^*(H,G)}{|N_G(H)|}.$$

Proposition 5.2. *A finite group G satisfies* (5.1) *for every positive integer t if and only if $\tau(n) = 0$ for any $\in \mathbb{N}$.*

Question 5.3. Does $\tau(n) = 0$ for all $n \in \mathbb{N}$ imply $\mu^*(H,G) = \mu(H,G)$ for all $H \leq G$?

For any $H \leq G$, consider

$$\alpha(H,t) = \frac{|H|^{t-1}|G||G'H|}{|G'N_G(H)|}, \quad \beta(H,t) = \alpha(H,t) - \omega(H,t).$$

Let $\mathcal{C}^*(G) = \{[H] \in \mathcal{C}(H) \mid [H] < [G] \text{ and } \lambda(H,G) \neq 0\}$. If G satisfies the (λ,μ)-property, then for any $t \in \mathbb{N}$, the vector

$$\beta_t(G) = (\beta(H,t))_{[H]\in\mathcal{C}^*(G)}$$

is an integer solution of the linear equation

$$\sum_{[H]\in C^*(G)} \lambda(H,G)x_H = 0. \tag{5.3}$$

One could investigate about the dimension of the vector space generated by the vectors $\beta_t(G)$, $t \in \mathbb{N}$. For example, if $G = A_5$, then we may order the elements of $\mathcal{C}^*(G)$ so that $H_1 = A_4$, $H_2 = S_3$, $H_3 = D_5$, $H_4 = \langle(1,2,3)\rangle$, $H_5 = \langle(1,2)(3,4)\rangle$, $H_6 = 1$. Then (5.3) can be written in the form

$$\sum_{[H]\in\mathcal{C}^*(G)} \lambda(H,G)x_H = -x_{H_1} - x_{H_2} - x_{H_3} + x_{H_4} + 2x_{H_5} - x_{H_6}$$

and

$$\begin{aligned}
\beta_1(G) &= (24, 24, 20, 39, 44, 59),\\
\beta_2(G) &= (84, 54, 50, 99, 74, 59),\\
\beta_3(G) &= (264, 114, 110, 279, 134, 59),\\
\beta_4(G) &= (804, 234, 230, 819, 254, 59),\\
\beta_5(G) &= (2424, 474, 470, 2439, 494, 59),\\
\beta_6(G) &= (7284, 954, 950, 7299, 974, 59).
\end{aligned}$$

The first three vectors $\beta_1(G), \beta_2(G), \beta_3(G)$ are linearly independent, while $\beta_4(G)$, $\beta_5(G)$ and $\beta_6(G)$ can be obtained as linear combinations of $\beta_1(G), \beta_2(G), \beta_3(G)$.

The situation is completely different when $G = S_3$. We may order the elements of $C^*(G)$ so that $H_1 = \langle(1,2,3)\rangle$, $H_2 = \langle(1,2)\rangle$, $H_3 = 1$. The equation (5.3) has in this case the form $x_{H_1} + x_{H_2} - x_{H_3} = 0$ and $\beta_t(G) = (0,2,2)$ independently on the choice of t.

Some properties of the vectors $\beta_t(G)$ are described in the following propositions.

Proposition 5.4. *If $H \in \mathcal{C}^*(G)$, then $\beta(H,t) \geq 0$ with equality if and only if $G' \leq H$. In particular $\beta_t(G)$ is a non-negative vector and $\beta_t(G) = 0$ if and only if G is nilpotent.*

Proof. Notice that $\omega(H,t) \leq |G : N_G(H)|(|H|^t - 1) + 1$. So

$$\begin{aligned}
\beta(H,t) &\geq \frac{|H|^{t-1}|G||G'H|}{|G'N_G(H)|} - |G : N_G(H)|(|H|^t - 1) - 1\\
&= |H|^t|G : N_G(H)|\frac{|G' \cap N_G(H)|}{|G' \cap H|} - |G : N_G(H)|(|H|^t - 1) - 1 \geq 0
\end{aligned}$$

with equality if and only if $H \geq G'$. □

Proposition 5.5. *The vector $\beta_t(G)$ is independent on the choice of t if and only if G is a nilpotent group or a primitive Frobenius group, with cyclic Frobenius complement.*

Proof. By the previous proposition, if G is nilpotent then $\beta_t(G)$ is the zero vector for any $t \in \mathbb{N}$, so we may assume that G is not nilpotent. Assume that $\beta_t(G)$ is independent on the choice of t. Let H be a maximal non-normal subgroup of G. Then $\alpha(H,t) = |H|^t \cdot u$ with $u = |G : H|$. Let $H_1, \dots, H_u$ be the conjugates of H in G. For any $J \subseteq \{1, \dots, u\}$, let $\alpha_J = |\bigcap_{j \in J} H_j|$. Then

$$\beta(H,t) = \sum_{J \neq \{1,\dots,u\}} (-1)^{|J|+1} |\alpha_J|^t.$$

We must have $\alpha_J = 1$ for every choice of J, otherwise $\lim_{t\to\infty} \beta(H,t) = \infty$. Hence H is a Frobenius complement and, since H is a maximal subgroup, the Frobenius kernel V is an irreducible H-module. Since $\beta(V,t) = |V|^t(|H'| - 1)$ does not depends on t, H must be abelian, and consequently cyclic. So if $\beta_t(G)$ is independent of the choice of t, then G is a primitive Frobenius group with a cyclic Frobenius complement. Conversely assume $G = V \rtimes H$, where H is cyclic and V and irreducible H-module. If $X \in \mathcal{C}^*(G)$, then $\lambda(X,G) \neq 0$, so X is an intersection of maximal subgroups of G and therefore either $V = G' \leq X$, or X is conjugate to a subgroup of H. In the first case $\beta(H,t) = 0$. Assume $X = K^v$ for some $K \leq H$ and $v \in V$. Then $\beta(H,t) = |K|^t|V| - \omega(K,t) = |K|^t|V| - (|V|(|K|^t - 1) + 1) = |V| - 1$. □

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On adjacency and Laplacian cospectral switching non-isomorphic signed graphs*

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Abstract

Let $\Gamma = (G, \sigma)$ be a signed graph, where σ is the sign function on the edges of G. In this paper, we use the operation of partial transpose to obtain switching non-isomorphic Laplacian cospectral signed graphs. We will introduce a new operation on signed graphs. This operation will establish a relationship between the adjacency spectrum of one signed graph with the Laplacian spectrum of another signed graph. As an application, this new operation will be utilized to construct several pairs of switching non-isomorphic cospectral signed graphs. Finally, we construct integral signed graphs.

Keywords: Signed graph, partial transpose, cospectral signed graphs, Laplacian cospectral signed graphs, equienergetic signed graphs, integral signed graph.

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1 Introduction

Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. A signed graph is defined to be a pair $\Gamma = (G, \sigma)$, with $G = (V(G), E(G))$ as the underlying graph and $\sigma\colon E(G) \to \{-1, 1\}$ as the signing function. In this manuscript, bold lines denote positive edges, and dashed lines

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denote negative edges. Signed graphs are a generalization of graphs, since they are signed graphs with each edge positive. The sign of a cycle in a signed graph is defined to be the product of the signs of its edges. A signed cycle is said to be positive (resp. negative) if its sign is positive (resp. negative). A signed graph is said to be balanced if none of its cycles is negative, otherwise unbalanced.

In a signed graph $\Gamma = (G, \sigma)$, the degree of a vertex v is the same as its degree in the underlying graph G (denoted by $d_v(G)$). For a signed graph Γ with vertex set $V(G)$, let $X \subset V(G)$ be a nonempty set. Let Γ^X denote the signed graph obtained from Γ by reversing signs of edges between X and $V(G) - X$. Then, we say Γ^X is switching equivalent to Γ. Here, we note that the switching is an equivalence relation and preserves the eigenvalues of the adjacency and the Laplacian matrix including their multiplicities. A switching class is represented by a single signed graph.

The adjacency matrix of a signed graph Γ with vertex set $\{v_1, v_2, \ldots, v_n\}$, is the $n \times n$ matrix $A(\Gamma) = (a_{ij})$, where

$$a_{ij} = \begin{cases} \sigma(v_i, v_j), & \text{if there is an edge from } v_i \text{ to } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

For a graph G, the Laplacian matrix is $L(G) = D(G) - A(G)$ and signless Laplacian matrix is $Q(G) = D(G) + A(G)$, where $A(G)$ and $D(G)$ are respectively the adjacency matrix and the diagonal matrix of vertex degrees of G. The Laplacian matrix of Γ is $L(\Gamma) = L(G, \sigma) = D(G) - A(\Gamma)$. Note that $L(G, +) = L(G)$ and $L(G, -) = Q(G)$. The characteristic polynomial $|xI - A(\Gamma)|$ and eigenvalues of the adjacency matrix $A(\Gamma)$ of Γ are denoted by $\phi_\Gamma(x)$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. The characteristic polynomial $|xI - L(\Gamma)|$ and eigenvalues of the Laplacian matrix $L(\Gamma)$ of Γ are denoted by $\psi_\Gamma(x)$ and $\mu_1, \mu_2, \ldots, \mu_n$, respectively. For a graph G (resp. signed graph Γ), eigenvalues of its adjacency matrix and Laplacian matrix are called adjacency and Laplacian eigenvalues of G (resp. Γ). Clearly, $A(\Gamma)$ and $L(\Gamma)$ are real symmetric and so all their eigenvalues are real. Let the signed graph Γ of order n has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and let their respective multiplicities be $m_1, m_2, \ldots, m_k$. The adjacency spectrum of Γ is written as $Spec(\Gamma) = \{\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, \ldots, \lambda_k^{(m_k)}\}$. A signed graph is said to be an integral signed graph if its adjacency spectrum consists of integers only.

Given a graph G, its subdivision graph $S(G)$ is obtained from G by replacing each of its edge by a path of length 2, or, equivalently, by inserting an additional vertex into each edge of G. If two signed graphs have the same adjacency spectrum (resp. Laplacian spectrum), they are said to be cospectral (resp. Laplacian cospectral); otherwise, they are noncospectral (resp. Laplacian noncospectral). Any two switching isomorphic signed graphs are cospectral (resp. Laplacian cospectral). A signed graph is said to be determined by its adjacency spectrum if cospectral signed graphs are switching isomorphic. It is well-known that in general the adjacency spectrum does not determine the signed graph and this problem has attracted to identify, if any, switching non-isomorphic cospectral signed graphs for a given class of signed graphs. For open problems in signed graphs, we refer to [2].

The energy of a graph G is the sum of the absolute values of its adjacency eigenvalues. This concept was extended to signed graphs by Germina, Hameed and Zaslavsky [9]. The energy of a signed graph Γ with eigenvalues $x_1, x_2, \ldots, x_n$ is defined as $\mathcal{E}(\Gamma) = \sum_{j=1}^{n} |x_j|$. Two signed graphs of same order are said to be equienergetic if they have the same energy.

Harary [12] pioneered the use of signed graphs in connection with the study of social balance theory. Signed graphs have been intensively explored in a variety of fields such as group theory, topological graph theory and classical root system. The reader is referred to [17] for a complete bibliography on signed graphs.

The rest of the paper is organized as follows. In Section 2, we present some preliminary results which will be used in the sequel. In Section 3, we define the concept of partial transpose in signed graphs and use it to obtain switching non-isomorphic Laplacian cospectral signed graphs. In Section 4, we introduce a new operation on signed graphs and this will be utilized to construct switching non-isomorphic cospectral signed graphs, noncospectral equienergetic signed graphs and integral signed graphs.

2 Preliminaries

In this section, we recall some previously established results which will be required in the subsequent sections.

Definition 2.1 ([6]). Let $P = (p_{ij}) \in M_{m\times n}(\mathbb{R})$ and $Q \in M_{p\times q}(\mathbb{R})$. The Kronecker product of P and Q, denoted by $P \otimes Q$, is defined as

$$P \otimes Q = \begin{pmatrix} p_{11}Q & p_{12}Q & \dots & p_{1n}Q \\ p_{21}Q & p_{22}Q & \dots & p_{2n}Q \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1}Q & p_{m2}Q & \dots & p_{mn}Q \end{pmatrix}.$$

Lemma 2.2 ([6]). *Let P, $Q \in M_n(\mathbb{R})$ be two square matrices of order n. Let λ be an eigenvalue of matrix P with corresponding eigenvector x and μ be an eigenvalue of matrix Q with corresponding eigenvector y. Then $\lambda\mu$ is an eigenvalue of $P\otimes Q$ with corresponding eigenvector $x \otimes y$.*

The Cartesian product (or sum) of two signed graphs $\Gamma_1 = (V(G_1), E(G_1), \sigma_1)$ and $\Gamma_2 = (V(G_2), E(G_2), \sigma_2)$, denoted by $\Gamma_1 \times \Gamma_2$, is the signed graph $(V(G_1)\times V(G_2), E, \sigma)$, where the edge set is that of the Cartesian product of underlying unsigned graphs and the sign function is defined by

$$\sigma((u_i, v_j), (u_k, v_l)) = \begin{cases} \sigma_1(u_i, u_k), & \text{if } j = l, \\ \sigma_2(v_j, v_l), & \text{if } i = k. \end{cases}$$

The Kronecker product (or conjunction) of two signed graphs $\Gamma_1 = (V(G_1), E(G_1), \sigma_1)$ and $\Gamma_2 = (V(G_2), E(G_2), \sigma_2)$, denoted by $\Gamma_1 \otimes \Gamma_2$, is the signed graph $(V(G_1) \times V(G_2), E, \sigma)$, where the edge set is that of the Kronecker product of underlying unsigned graphs and the sign function is defined by $\sigma((u_i, v_j), (u_k, v_l)) = \sigma_1(u_i, u_k)\sigma_2(v_j, v_l)$.

Lemma 2.3 ([9]). *Let Γ_1 and Γ_2 be two signed graphs with respective eigenvalues $x_1, x_2, \dots, x_{n_1}$ and $y_1, y_2, \dots, y_{n_2}$. Then*

(i) *the eigenvalues of $\Gamma_1 \times \Gamma_2$ are $x_i + y_j$, for all $i = 1, 2, \dots, n_1$ and $j = 1, 2, \dots, n_2$,*

(ii) *the eigenvalues of $\Gamma_1 \otimes \Gamma_2$ are $x_i y_j$, for all $i = 1, 2, \dots, n_1$ and $j = 1, 2, \dots, n_2$.*

Lemma 2.4 ([4]). *Let Γ be an unbalanced signed graph with at least one edge, whose spectrum is symmetric about the origin, having eigenvalues $\xi_1, \xi_2, \ldots, \xi_n$. Then $\Gamma \times K_2$ and $\Gamma \otimes K_2$, where K_2 is a complete signed graph on 2 vertices, are unbalanced, noncospectral and equienergetic if and only if $|\xi_j| \geq 1$, for all $j = 1, 2, \ldots, n$.*

Lemma 2.5 ([14]). *Let $P(\bullet)$ be a given polynomial. If μ is an eigenvalue of $A \in M_n$, while y is an associated eigenvector, then $P(\mu)$ is an eigenvalue of the matrix $P(A)$ and y is an eigenvector associated with $P(\mu)$.*

Lemma 2.6 ([4]). *Let Γ be a signed graph of order n. Then the following statements are equivalent.*

(i) *The spectrum of Γ is symmetric about the origin,*

(ii) *$\phi_\Gamma(x) = x^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b_{2k} x^{n-2k}$, where b_{2k} are non negative integers for all $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$,*

(iii) *Γ and $-\Gamma$ are cospectral, where $-\Gamma$ is the signed graph obtained by negating sign of each edge of Γ.*

Lemma 2.7 ([16]). *For infinitely many n, there exists a family of 2^k pairwise nonisomorphic Laplacian integral, Laplacian cospectral graphs on n vertices, where $k > \frac{n}{(2log_2(n))}$.*

3 Constructing Laplacian cospectral non-isomorphic signed graphs

Dutta [8] constructed large families of non-isomorphic signless Laplacian cospectral graphs using partial transpose on graphs. In this section, we define the partial transpose in signed graphs. Let $\Gamma = (G, \sigma)$ be a signed graph on $2n$ vertices with vertex set $V(G) = V_1 \cup V_2$, such that $V_1 \cap V_2 = \emptyset$, and $V_1 = \{u_1, u_2, \ldots u_n\}$, $V_2 = \{v_1, v_2, \ldots v_n\}$. We denote by $\langle V_1 \rangle_\Gamma$ and $\langle V_2 \rangle_\Gamma$ as the induced signed subgraphs of Γ formed by V_1 and V_2 respectively. The spanning signed subgraph of Γ consisting of the signed edge set $\{(u_i, v_j) \in E(\Gamma) : u_i \in V_1, v_j \in V_2\}$ is denoted by $\langle V_1, V_2 \rangle_\Gamma$. Consider the set of signed edges $E(\widehat{\langle V_1, V_2 \rangle}_\Gamma) = \{(u_j, v_i) : (u_i, v_j) \in E(\langle V_1, V_2 \rangle_\Gamma)\}$. The definition of $E(\widehat{\langle V_1, V_2 \rangle}_\Gamma)$ suggests that given any signed edge $(u_i, v_j) \in E(\langle V_1, V_2 \rangle_\Gamma)$ there is a unique signed edge $(u_j, v_i) \in E(\widehat{\langle V_1, V_2 \rangle}_\Gamma)$ with the same sign as the sign of edge (u_i, v_j) in $\langle V_1, V_2 \rangle_\Gamma$.

The partial transpose of a signed graph Γ, denoted by Γ^τ, is defined as $\Gamma^\tau = \Gamma - E(\langle V_1, V_2 \rangle_\Gamma) + E(\widehat{\langle V_1, V_2 \rangle}_\Gamma)$. Note that, subtracting $E(\langle V_1, V_2 \rangle_\Gamma)$ indicates to remove all the existing signed edges in Γ of the form $(u_i, v_j) \in E(\langle V_1, V_2 \rangle_\Gamma)$. Then we include the signed edges $(u_j, v_i) \in E(\widehat{\langle V_1, V_2 \rangle}_\Gamma)$ to construct Γ^τ. If $i = j$, then the edge (u_i, v_i) will be removed and added again, that is the edge (u_i, v_i) is unaltered under partial transpose. Therefore, partial transpose of a signed graph Γ is an operation on the edge set which replaces the signed edge (u_i, v_j) with the sign $\sigma = \pm 1$, with the corresponding signed edge (u_j, v_i) with the same sign σ.

Example 3.1. Consider the signed graphs Γ_1 and Γ_1^τ as shown in Figure 1. Here, we have $V_1 = \{u_1, u_2, u_3\}$, $V_2 = \{v_1, v_2, v_3\}$ and $E\left(\langle V_1, V_2 \rangle_{\Gamma_1}\right) = \{(u_1, v_1), (u_1, v_3)\}$. Thus, $E\left(\widehat{\langle V_1, V_2 \rangle}_{\Gamma_1}\right) = \{(u_1, v_1), (u_3, v_1)\}$. Here, we replace the signed edge (u_1, v_3) with the signed edge (u_3, v_1).

Remark 3.2. The partial transpose of a signed graph is labelling dependent. Therefore, switching isomorphic signed graphs may have switching non-isomorphic partial transposes, depending on the labellings. The partial transpose keeps $\langle V_1\rangle$ and $\langle V_2\rangle$ unaltered. The total number of vertices and edges remains the same.

A cycle $C_l^\sigma(v_1, v_2, \ldots, v_l, v_1)$ in a signed graph $\Gamma = (G, \sigma)$ is a finite sequence of distinct vertices such that $(v_i, v_{i+1}) \in E(\Gamma)$ for all $i = 1, 2, \ldots, l-1$ and $(v_l, v_1) \in E(\Gamma)$. We denote the negative edges in the signed cycle $C_l^\sigma(v_1, v_2, \ldots, v_l, v_1)$ by putting the bar over the corresponding adjacent vertices. For example, the cycle $C_4^\sigma(v_1, v_2, v_3, v_4, v_1)$ on four vertices such that the only edge $(v_1, v_2) \in E(\Gamma)$ has negative sign will be denoted by $C_4^-(\overline{v_1, v_2}, v_3, v_4, v_1)$. Similarly if only two consecutive edges $(v_1, v_2), (v_2, v_3) \in E(\Gamma)$ have negative signs, then the cycle $C_4^\sigma(v_1, v_2, v_3, v_4, v_1)$ will be denoted by $C_4^+(\overline{v_1, v_2, v_3}, v_4, v_1)$. In a signed graph Γ, a signed TU-subgraph H is a signed subgraph whose components are trees or unbalanced unicyclic graphs, namely the unique cycle containing an odd number of negative edges. Thus, if H is a signed TU-subgraph, then $H = T_1 \cup T_2 \cup \cdots \cup T_p \cup U_1 \cup U_2 \cup \cdots \cup U_q$, where $T_i's$ are trees and $U_i's$ are unbalanced unicyclic graphs. The weight of the signed TU-subgraph H is defined as $w(H) = 4^q \prod_{i=1}^p |T_i|$, where $|T_i|$ is the number of vertices in the tree T_i. Note that we define $\prod_{i=1}^p |T_i| = 1$ when $p = 0$. The relation between the coefficients of the Laplacian characteristic polynomial with the TU-subgraphs of a signed graph can be seen in [[3],Theorem 3.9]. If Γ is a signed graph with Laplacian characteristic polynomial $\psi(\Gamma, x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, then its coefficients are given by

$$a_i = (-1)^i \sum_{H \in \mathcal{H}_i(\Gamma)} w(H) \quad (i = 1, 2, \ldots, n), \tag{3.1}$$

where $\mathcal{H}_i(\Gamma)$ denotes the set of signed TU-subgraphs of Γ containing i edges. Two sets of signed TU-subgraphs $\mathcal{H}_i(\Gamma)$ and $\mathcal{H}_i(\Gamma')$ are comparable if

$$\sum_{H \in \mathcal{H}_i(\Gamma)} w(H) = \sum_{H \in \mathcal{H}_i(\Gamma')} w(H).$$

Now, Equation (3.1) suggests that Γ and Γ' are Laplacian cospectral if and only if the sets of their signed TU-subgraphs are comparable for all $i = 1, 2, \ldots m$, where m is the number of edges in the signed graph Γ. We say two signed graphs Γ_1 and Γ_2 are comparable if $\mathcal{H}_i(\Gamma_1)$ and $\mathcal{H}_i(\Gamma_2)$ are comparable for all i. As an example, two signed paths with equal number of vertices are comparable.

Example 3.3. Consider the signed graphs Γ_1 and Γ_1^τ as shown in Figure 1. We observe that Γ_1 contains two cycles $C_3^+(u_1, u_2, u_3, u_1)$ and $C_4^-(\overline{u_1, v_1}, v_2, v_3, u_1)$. The partial transpose Γ_1^τ of Γ_1 is obtained by replacing the signed edge (u_1, v_3) with (u_3, v_1). Clearly, Γ_1^τ contains three cycles $C_3^+(u_1, u_2, u_3, u_1)$, $C_4^-(\overline{u_1, v_1}, u_3, u_2, u_1)$ and $C_3^-(\overline{u_1, v_1}, u_3, u_1)$. The cycle $C_3^+(u_1, u_2, u_3, u_1)$ remains invariant under partial transpose on Γ_1. If the cycle has an odd number of negative edges, then it contributes an unbalanced unicyclic graph in the formation of signed TU-subgraphs. Therefore, the balanced cycle $C_3^+(u_1, u_2, u_3, u_1)$ does not contribute in $\psi_{\Gamma_1}(x)$ and $\psi_{\Gamma_1^\tau}(x)$.

The unbalanced cycle $C_4^-(\overline{u_1, v_1}, v_2, v_3, u_1)$ in Γ_1 is replaced by $C_4^-(\overline{u_1, v_1}, u_3, u_2, u_1)$ in Γ_1^τ. Therefore, signed TU-subgraphs whose components contain cycles $C_4^-(\overline{u_1, v_1}, v_2, v_3, u_1)$ and $C_4^-(\overline{u_1, v_1}, u_3, u_2, u_1)$ in Γ_1 and Γ_1^τ, respectively, have equal contribution in $\psi_{\Gamma_1}(x)$ and $\psi_{\Gamma_1^\tau}(x)$.

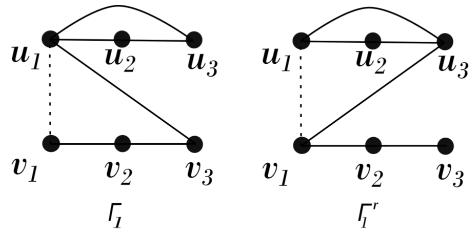

Figure 1: Signed graph Γ_1 and its partial transpose Γ_1^τ.

The signed edges $(u_1, v_1), (u_1, v_3)$ and (u_1, u_3) induce star signed graph $K_{1,3}$ in Γ_1. It is replaced by an unbalanced unicyclic TU-subgraph $C_3^-(\overline{u_1, v_1}, u_3, u_1)$ in Γ_1^τ. The signed graphs $K_{1,3}$ and $C_3^-(\overline{u_1, v_1}, u_3, u_1)$ have the same weights equal to 4 as a components to the signed TU-subgraphs. Therefore, the signed TU-subgraphs formed by $K_{1,3}$ and $C_3^-(\overline{u_1, v_1}, u_3, u_1)$ in Γ_1 and Γ_1^τ, respectively, have equal contribution in $\psi_{\Gamma_1}(x)$ and $\psi_{\Gamma_1^\tau}(x)$. Moreover, the role of signed TU-subgraphs which contain the signed edges $(u_1, v_1), (u_1, v_3)$ and (u_1, u_3) in Γ_1 are replaced by the signed TU-subgraphs which contain the signed cycle $C_3^-(\overline{u_1, v_1}, u_3, u_1)$ in Γ_1^τ. Thus, all the signed TU-subgraphs of Γ_1 and Γ_1^τ are comparable. Hence they have same Laplacian characteristic polynomials, which can be calculated by Equation (3.1) and is given as

$$\psi_{\Gamma_1}(x) = \psi_{\Gamma_1^\tau}(x) = x^6 - 14x^5 + 73x^4 - 176x^3 + 196x^2 - 88x + 12.$$

In a signed graph $\Gamma = (G, \sigma)$ with $w_i, w_j \in V(\Gamma)$, if the signed edge (w_i, w_j) is added, then the resultant signed graph is denoted by $\Gamma' = \Gamma + \{(w_i, w_j)\}$. Similarly, $\Gamma' = \Gamma - \{(w_i, w_j)\}$ denotes the signed graph obtained by removing an edge (w_i, w_j). Whether the added/removed edge (w_i, w_j) is positive or negative, we denote a negative edge by $\overline{(w_i, w_j)}$, and a positive edge without a bar over the edge (w_i, w_j).

Theorem 3.4. *Let the signed subgraphs $\langle V_1 \rangle_\Gamma$ and $\langle V_2 \rangle_\Gamma$ of the signed graph Γ be two paths on n vertices with each edge being positive. If $\langle V_1, V_2 \rangle_\Gamma$ is an empty signed graph, then*

(i) *the signed graph $\Gamma_1 = \Gamma + \left\{(u_1, u_n), \overline{(u_1, v_1)}, (u_1, v_n)\right\}$ is switching non-isomorphic and Laplacian cospectral to its partial transpose Γ_1^τ.*

(ii) *the signed graphs $\Gamma_2 = \Gamma_1 - \left\{(u_{n-1}, u_n), \overline{(u_1, v_1)}\right\} + \left\{\overline{(u_{n-1}, u_n)}, (u_1, v_1)\right\}$ and $\Gamma_3 = \Gamma_1^\tau - \{(u_{n-1}, u_n)\} + \left\{\overline{(u_{n-1}, u_n)}\right\}$ are switching non-isomorphic and Laplacian cospectral.*

Proof. (i) The cycles in Γ_1 generated by additional three edges and their incidence with existing edges of Γ are $C_n^+(u_1, u_2, u_3, \ldots, u_n, u_1)$ and $C_{n+1}^-(\overline{u_1, v_1}, v_2, \ldots, v_n, u_1)$. The signed spanning subgraph $\langle V_1, V_2 \rangle_\Gamma$ contains only two signed edges which are $\overline{(u_1, v_1)}$ and (u_1, v_n). Partial transpose replaces (u_1, v_n) with (u_n, v_1). Clearly, Γ_1^τ contains three cycles $C_n^+(u_1, u_2, u_3, \ldots, u_n, u_1)$, $C_{n+1}^-(\overline{u_1, v_1}, u_n, \ldots, u_2, u_1)$ and $C_3^-(\overline{u_1, v_1}, u_n, u_1)$.

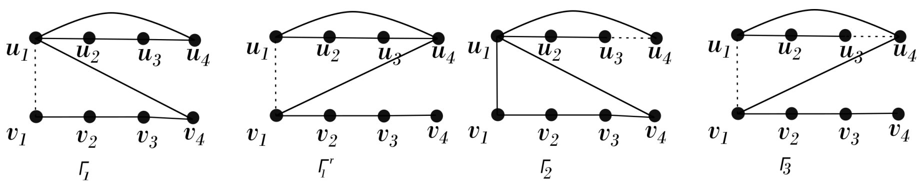

Figure 2: Signed graphs Γ_1, Γ_1^τ, Γ_2 and Γ_3.

The cycle $C_n^+\,(u_1, u_2, u_3, \ldots, u_n, u_1)$ remains invariant under partial transpose on Γ_1. Removing an edge from a cycle generates a tree. If the cycle has an odd number of negative edges, then it contributes an unbalanced unicyclic graph in the formation of signed TU-subgraphs. Therefore, the balanced cycle $C_n^+\,(u_1, u_2, u_3, \ldots, u_n, u_1)$ does not contribute in $\psi_{\Gamma_1}(x)$ and $\psi_{\Gamma_1^\tau}(x)$.

The unbalanced cycle $C_{n+1}^-\,(\overline{u_1, v_1}, v_2, \ldots, v_n, u_1)$ in Γ_1 is replaced by $C_{n+1}^-(\overline{u_1, v_1}, u_n, \ldots, u_2, u_1)$ in Γ_1^τ. Therefore, signed TU-subgraphs whose components contain cycles $C_{n+1}^-\,(\overline{u_1, v_1}, v_2, \ldots, v_n, u_1)$ and $C_{n+1}^-\,(\overline{u_1, v_1}, u_n, \ldots, u_2, u_1)$ in Γ_1 and Γ_1^τ, respectively, have equal contribution in $\psi_{\Gamma_1}(x)$ and $\psi_{\Gamma_1^\tau}(x)$.

The signed edges $\overline{(u_1, v_1)}, (u_1, v_n)$ and (u_1, u_n) induce star signed graph $K_{1,3}$ in Γ_1. It is replaced by an unbalanced unicyclic TU-subgraph $C_3^-\,(\overline{u_1, v_1}, u_n, u_1)$ in Γ_1^τ. They have equal contribution in $\psi_{\Gamma_1}(x)$ and $\psi_{\Gamma_1^\tau}(x)$ which is seen in Example 3.3. Moreover, the role of signed TU-subgraphs which contain the signed edges $\overline{(u_1, v_1)}, (u_1, v_n)$ and (u_1, u_n) in Γ_1 are replaced by the signed TU-subgraphs which contain the signed cycle $C_3^-\,(\overline{u_1, v_1}, u_n, u_1)$ in Γ_1^τ. Therefore, all the signed TU-subgraphs of Γ_1 and Γ_1^τ are comparable. Thus, they have the same Laplacian characteristic polynomials, which proves the result in this case.

(ii) If a signed graph is switching equivalent to a signed graph whose each edge is negative, then its Laplacian matrix coincides with the signless Laplacian matrix of its underlying graph. If n is odd, then the result follows by Corollary 2 of [8]. For even n, we observe that Γ_2 contains two cycles $C_n^-\,(u_1, u_2, \ldots, \overline{u_{n-1}, u_n}, u_1)$ and $C_{n+1}^+\,(u_1, v_1, \ldots, v_n, u_1)$. The underlying graph of Γ_3 is the partial transpose of the underlying graph of Γ_2. The signed graph Γ_3 contains three cycles $C_n^-\,(u_1, u_2, \ldots, \overline{u_{n-1}, u_n}, u_1)$, $C_{n+1}^+(\overline{u_1, v_1}, \overline{u_n, u_{n-1}}, u_{n-2}, \ldots, u_2, u_1)$ and $C_3^-(\overline{u_1, v_1}, u_n, u_1)$. The cycle $C_n^-\,(u_1, u_2, \ldots, \overline{u_{n-1}, u_n}, u_1)$ is common in Γ_2 and Γ_3. Therefore, the signed $TU-$subgraphs formed by the unbalanced cycle $C_n^-\,(u_1, u_2, \ldots, \overline{u_{n-1}, u_n}, u_1)$ in Γ_2 and Γ_3 contribute same in $\psi_{\Gamma_2}(x)$ and $\psi_{\Gamma_3}(x)$. The balanced cycle $C_{n+1}^+\,(u_1, v_1, \ldots, v_n, u_1)$ in Γ_2 is replaced by a balanced cycle $C_{n+1}^+(\overline{u_1, v_1}, \overline{u_n, u_{n-1}}, u_{n-2}, \ldots, u_2, u_1)$ in Γ_3. Therefore, they do not contribute in $\psi_{\Gamma_2}(x)$ and $\psi_{\Gamma_3}(x)$.

The signed edges $(u_1, v_1), (u_1, v_n)$ and (u_1, u_n) induce star signed graph $K_{1,3}$ in Γ_2. It is replaced by an unbalanced unicyclic TU-subgraph $C_3^-(\overline{u_1, v_1}, u_n, u_1)$ in Γ_3 and proceeding similarly as in (i), we get the result. □

Example 3.5. Consider the signed graphs Γ_1, Γ_1^τ, Γ_2 and Γ_3 as given in Figure 2. They are constructed by using Theorem 3.4. Their Laplacian characteristic polynomials are respectively given as.

$$\psi_{\Gamma_1}(x) = \psi_{\Gamma_1^\tau}(x) = x^8 - 18x^7 + 131x^6 - 498x^5 + 1061x^4 - 1256x^3 + 764x^2 - 200x + 16$$

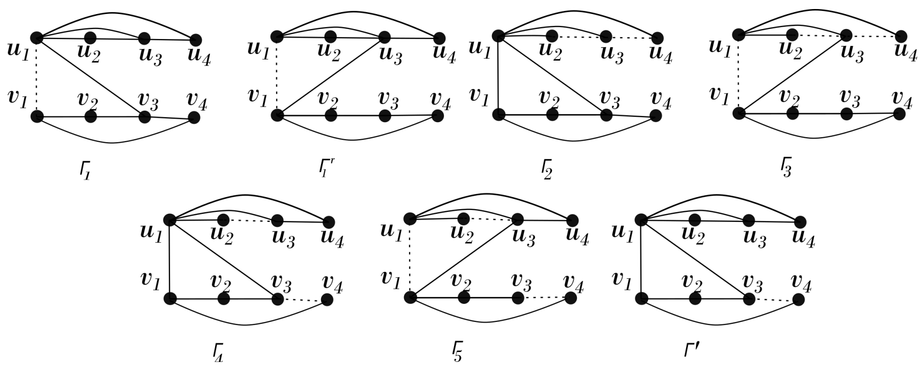

Figure 3: Signed graphs Γ_1 , Γ_1^τ, Γ_2, Γ_3, Γ_4, Γ_5 and Γ'.

and

$$\psi_{\Gamma_2}(x) = \psi_{\Gamma_3}(x) = x^8 - 18x^7 + 131x^6 - 498x^5 + 1065x^4 - 1284x^3 + 824x^2 - 240x + 20.$$

Clearly, the signed graphs Γ_1 and Γ_1^τ are switching non-isomorphic and Laplacian cospectral. Also, Γ_2 and Γ_3 are switching non-isomorphic and Laplacian cospectral signed graphs.

The proof of the following result is similar to Theorem 1 of [8] but for the self containment of the paper, we include it here.

Theorem 3.6. *Let the signed subgraphs $\langle V_1\rangle_\Gamma$ and $\langle V_2\rangle_\Gamma$ of Γ be two cycles on n vertices with each edge being positive. Let $\langle V_1, V_2\rangle_\Gamma$ be an empty signed graph. Given two non-adjacent vertices u_i and u_j with $i < j$, construct a new signed graph $\Gamma_1 = \Gamma + \left\{(u_i, u_j), \overline{(u_i, v_i)}, (u_i, v_j)\right\}$. Then*

(i) *the signed graph Γ_1 is switching non-isomorphic and Laplacian cospectral to its partial transpose Γ_1^τ,*

(ii) *the signed graphs $\Gamma_2 = \Gamma_1 - \left\{(u_{n-1}, u_n), \overline{(u_i, v_i)}, (u_{j-1}, u_j)\right\} + \left\{\overline{(u_{n-1}, u_n)}, (u_1, v_1), \overline{(u_{j-1}, u_j)}\right\}$ and $\Gamma_3 = \Gamma_1^\tau - \{(u_{n-1}, u_n), (u_{j-1}, u_j)\} + \left\{\overline{(u_{n-1}, u_n)}, \overline{(u_{j-1}, u_j)}\right\}$ are switching non-isomorphic and Laplacian cospectral,*

(iii) *the signed graphs $\Gamma_4 = \Gamma_1 - \left\{(v_{n-1}, v_n), \overline{(u_i, v_i)}, (u_{j-1}, u_j)\right\} + \left\{\overline{(v_{n-1}, v_n)}, (u_1, v_1), \overline{(u_{j-1}, u_j)}\right\}$ and $\Gamma_5 = \Gamma_1^\tau - \{(v_{n-1}, v_n), (u_{j-1}, u_j)\} + \left\{\overline{(v_{n-1}, v_n)}, \overline{(u_{j-1}, u_j)}\right\}$ are switching non-isomorphic and Laplacian cospectral.*

Proof. (i) The set of all cycles in Γ_1 includes two cycles of Γ. They are denoted by γ_1 and γ_2. The new cycles formed by additional three signed edges and their incidence with existing edges in Γ are $\gamma_3 = C^+_{j-i+1}(u_i, u_{i+1}, u_{i+2}, \dots, u_j, u_i)$, $\gamma_4 = C^+_{n-(j-i)+1}(u_1, u_2, \dots, u_i, u_j, u_{j+1}, \dots, u_n, u_1)$, $\gamma_5 = C^-_{j-i+2}(\overline{u_i, v_i}, v_{i+1}, v_{i+2}, \dots v_j, u_i)$, and $\gamma_6 =$

$C^-_{n-(j-i)+2}(v_1, v_2, \ldots, \overline{v_i, u_i}, v_j, v_{j+1}, \ldots, v_n, v_1)$. Note that, $\langle V_1, V_2\rangle_{\Gamma_1}$ contains only two edges which are (u_i, v_i) and (u_i, v_j). Partial transpose replaces (u_i, v_j) with (u_j, v_i). The cycles $\gamma_1, \gamma_2, \gamma_3$ and γ_4 remain invariant under partial transpose on Γ_1. If the cycle has an odd number of negative edges, then it contributes an unbalanced unicyclic graph in the formation of signed TU-subgraphs. Therefore, signed TU-subgraphs formed by the cycles $\gamma_1, \gamma_2, \gamma_3$ and γ_4 in Γ_1 and Γ_1^τ have equal contribution in $\psi_{\Gamma_1}(x)$ and $\psi_{\Gamma_1^\tau}(x)$.

Now, the signed cycle γ_5 in Γ_1 is replaced by $\gamma_5' = C^-_{j-i+2}(\overline{u_i, v_i}, u_j, u_{j-1}, \ldots, u_{i+1}, u_i)$ in Γ_1^τ. They have equal length and equal contribution in the characteristic coefficients. The cycle γ_6 in Γ_1 and its counterpart $\gamma_6' = C^-_{n-(j-i)+2}(u_1, u_2, \ldots, u_i, v_i, u_j, u_{j+1}, \ldots, u_q, u_1)$ in Γ_1^τ have equal lengths. If $(v_k, v_{k+1}) \in \gamma_6 \cap \langle V_2\rangle_{\Gamma_1}$ in Γ_1, then $(u_k, u_{k+1}) \in \gamma_6' \cap \langle V_1\rangle_{\Gamma_1^\tau}$ in Γ_1^τ. A signed TU-subgraphs containing more than $n-(j-i)+2$ edges contains edges from γ_1 in Γ_1. The role of γ_1 in Γ_1 is replaced by the edges of γ_2 in Γ_1^τ. We have assumed that γ_1 and γ_2 have equal length. Therefore, replacement of γ_6 in Γ_1^τ does not make any difference in the characteristic coefficients.

The new edges $(u_i, u_j), (u_i, v_i)$ and (u_i, v_j) form a star $K_{1,3}$ in Γ_1. It is replaced by a unicyclic signed TU-subgraph $C_3^-(u_i, u_j, \overline{v_i, u_i})$ in Γ_1^τ. They have equal contribution in $\psi_{\Gamma_1}(x)$ and $\psi_{\Gamma_1^\tau}(x)$ which is seen in Example 3.3. Therefore, all the signed $TU-$subgraphs of Γ_1 and Γ_1^τ are comparable as well as they form equal characteristic polynomials. This proves (i) The proof of (ii) and (iii) is similar to (i). Hence, the result follows. □

Example 3.7. Consider the signed graphs $\Gamma_1, \Gamma_1^\tau, \Gamma_2, \Gamma_3, \Gamma_4$ and Γ_5 as shown in Figure 3. Here $n = 4, i = 1$ and $j = 3$. The signed graph Γ_1, which is constructed by Theorem 3.6 is switching non-isomorphic and Laplacian cospectral to its partial transpose Γ_1^τ. We obtain the signed graph Γ_2 from Γ_1 by replacing the positive edges (u_2, u_3) and (u_3, u_4) with negative edges $\overline{(u_2, u_3)}$ and $\overline{(u_3, u_4)}$ and negative edge $\overline{(u_1, v_1)}$ with the positive edge (u_1, v_1). Also, the signed graph Γ_3 is obtained from Γ_1^τ by replacing the positive edges (u_2, u_3) and (u_3, u_4) with negative edges $\overline{(u_2, u_3)}$ and $\overline{(u_3, u_4)}$. The signed graphs Γ_2 and Γ_3 are switching non-isomorphic and Laplacian cospectral. Similarly the switching non-isomorphic and Laplacian cospectral signed graphs Γ_4 and Γ_5 are obtained from Γ_1 and Γ_1^τ, respectively, as in Theorem 3.6.

Remark 3.8. In Example 3.7, we have seen that Γ_1 and Γ_1^τ are Laplacian cospectral signed graphs. Also, we have mentioned that Γ_1^τ is the partial transpose of Γ_1. But, not all signed graphs are Laplacian cospectral to their partial transpose, for instance, consider the signed graphs Γ and Γ^τ as given in Figure 4. It is easy to calculate that the Laplacian characteristic polynomials of Γ and Γ^τ are $\psi_\Gamma(x) = x^6 - 12x^5 + 51x^4 - 96x^3 + 81x^2 - 30x + 4$ and $\psi_{\Gamma^\tau}(x) = x^6 - 12x^5 + 51x^4 - 94x^3 + 72x^2 - 18x$.

Let G be a graph and $\Gamma = (G, \sigma)$ be a signed graph on G. Hou et al. [15] raised the following two problems.

Problem 1. Let G be a graph, $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ be two signed graphs on G, and $\det(L(\Gamma_1)) = \det(L(\Gamma_2))$. Are $L(\Gamma_1)$ and $L(\Gamma_2)$ cospectral?

Problem 2. Do there exist pairs $\Gamma_1 = (G_1, \sigma_1)$ and $\Gamma_2 = (G_2, \sigma_2)$ of signed graphs that have either of the following properties (i) and (ii)?

(i) Γ_1 and Γ_2 are not balanced but Laplacian cospectral such that G_1 and G_2 are nonisomorphic.

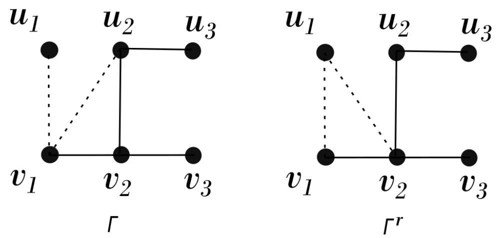

Figure 4: Signed graphs Γ and Γ^{τ}.

(ii) Γ_1 and Γ_2 are not balanced but Laplacian cospectral such that G_1 and G_2 are not cospectral.

The statement of Problem 1 is not always true. To see this, let Γ_1 be a signed graph as shown in Figure 3. Let Γ' (shown in Figure 3) be the signed graph obtained from Γ_1 by replacing the negative edge $\overline{(u_1, v_1)}$ with positive edge (u_1, v_1) and positive edge (v_3, v_4) with negative edge $\overline{(v_3, v_4)}$. The Laplacian characteristic polynomials of Γ_1 and Γ' are respectively given by

$$\psi_{\Gamma_1}(x) = x^8 - 22x^7 + 197x^6 - 928x^5 + 2476x^4 - 3736x^3 + 2976x^2 - 1056x + 128$$

and

$$\psi_{\Gamma'}(x) = x^8 - 22x^7 + 197x^6 - 928x^5 + 2476x^4 - 3748x^3 + 3048x^2 - 1152x + 128.$$

The underlying graphs of Γ_1 and Γ' are isomorphic and $\det(L(\Gamma_1)) = \det(L(\Gamma'))$. It is clear that the signed graphs Γ_1 and Γ' are not Laplacian cospectral and this answers Problem 1.

For Problem 2, consider the signed graph Γ_1 and its partial transpose Γ_1^{τ} as given in Figure 3. Clearly, the underlying graphs of Γ_1 and Γ_1^{τ} are non-isomorphic. The unbalanced signed graphs Γ_1 and Γ_1^{τ} are Laplacian cospectral. Also, it is easy to see that the underlying graph of Γ_1 and Γ_1^{τ} are not cospectral and this answers Problem 2.

4 Constructing switching non-isomorphic cospectral signed graphs, integral signed graphs and equienergetic signed graphs

The novel non-isomorphic cospectral graph constructions have implications for the complexity of the graph isomorphism problem. This necessitates the creation of methods for detecting and/or creating non-isomorphic cospectral graphs. Seidel switching, Godsil–McKay (GM) switching, and others are well-known approaches for constructing cospectral graphs. In 2019, Belardo et al. [1] used the Godsil-Mckay-type procedures developed for graphs to construct the pairs of switching non-isomorphic cospectral signed graphs. In this section, we will introduce a new operation on signed graphs. This operation establishes the relationship of the adjacency spectrum of one signed graph with the Laplacian spectrum of another signed graph. Furthermore, this operation will be utilized to construct the pairs of switching non-isomorphic cospectral signed graphs and integral signed graphs. Before that, we need the following motivation which can also be seen in [3].

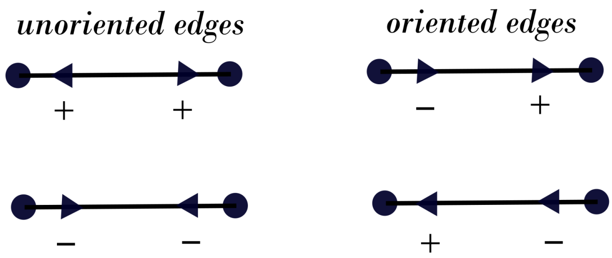

Figure 5: Bidirected edges in signed graphs.

The usual orientation of edges in digraphs differs from the orientation of signed graphs. In fact in signed graphs, instead of one arrow, we can use two arrows assigned to edges, which results in bidirected graphs. An orientated signed graph, more exactly, is an ordered pair $\Gamma_\vartheta = (\Gamma, \vartheta)$, where

$$\vartheta \colon V(G) \times E(G) \to \{0, 1, -1\} \tag{4.1}$$

satisfying the following three conditions.

(a) $\vartheta(u, vw) = 0$ whenever $u \neq v, w; u, v, w \in V(G)$ and $vw \in E(G)$,

(b) $\vartheta(v, vw) = 1$ (or -1) if an arrow at v is going into (rep. out of) v. For illustration, see Figure 5,

(c) $\vartheta(v, vw)\, \vartheta(w, vw) = -\sigma(vw)$.

As a result, positive edges are oriented edges, whereas negative edges are unoriented (see Figure 5). Therefore, every bidirected graph is also a signed graph. The converse is likewise true, however, one arrow (at any end) can be taken at random, whereas the other arrow (in light of (c) above) cannot. For an oriented signed graph Γ_ϑ, its incidence matrix $B_\vartheta = (b_{ij})$ is a matrix, whose rows correspond to vertices and columns to edges of G, with $b_{ij} = \vartheta(v_i, e_j)$ (here $v_i \in V(G)$, $e_j \in E(G)$). Usually, when only Γ is given, then we use an arbitrary orientation. So each row of the incidence matrix corresponding to vertex v_i contains d_{v_i} non-zero entries, each equal to $+1$ or -1. On the other hand, each column of the incidence matrix corresponding to edge e_j contains two non-zero entries, each equal to $+1$ or -1. Therefore, even in the case that multiple edges exist, we easily obtain

$$B_\vartheta B_\vartheta^T = D(G) - A(\Gamma_\vartheta) = L(\Gamma_\vartheta), \tag{4.2}$$

where $D(G)$ is the diagonal matrix of vertex degrees of G. It is easy to observe that $L(\Gamma_\vartheta)$ is positive-semidefinite.

The subdivision signed graph $S(\Gamma_\vartheta)$ is the signed graph whose underlying graph is $S(G)$ with vertex set $V(G) \cup E(G)$. It preserves the orientation ϑ and its adjacency matrix can be represented in the block form as follows

$$A(S(\Gamma_\vartheta)) = \begin{pmatrix} O_n & B_\vartheta \\ B_\vartheta^T & O_m \end{pmatrix},$$

where $O_r \in M_r(\mathbb{R})$. It is easy to see that the signature σ of the subdivision signed graph is defined by $\sigma(v_i e_j) = \vartheta_{ij}$. An example of a subdivision signed graph of a signed graph is shown in Figure 6.

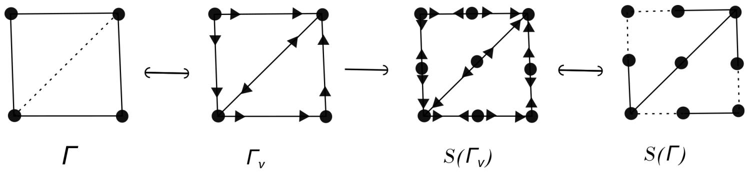

Figure 6: A signed graph and the corresponding subdivision signed graph.

Remark 4.1. Any orientation (random) ϑ to the edges of Γ gives rise to the same matrices $A(\Gamma_\vartheta) = A(\Gamma)$ and $L(\Gamma_\vartheta) = L(\Gamma)$, while the matrix $A(S(\Gamma_\vartheta))$ does depend on ϑ. Let S be a ± 1 diagonal matrix such that $B_{\vartheta'} = B_\vartheta S$. Clearly, $A(S(\Gamma_{\vartheta'})) = [I_n \dot{+} S]A(S(\Gamma_\vartheta))[I_n \dot{+} S]$, where $\dot{+}$ denotes the direct sum of two matrices. From now on, the subscript ϑ in B_ϑ will be not specified anymore.

Lemma 4.2 ([3]). *If B is the incidence matrix of a connected signed graph $\Gamma = (G, \sigma)$ having n vertices. Then*

$$rank(B) = \begin{cases} n-1 & \text{if } \Gamma \text{ is balanced,} \\ n & \text{if } \Gamma \text{ is unbalanced.} \end{cases}$$

Operation. Let Γ be a signed graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. Let $S(\Gamma)$ be a subdivision signed graph of a signed graph Γ with vertex set $V(G) \cup E(G)$. In $S(\Gamma)$, replace each vertex v_i, $i = 1, 2, \dots, n$, by k vertices and join every vertex to the neighbours of v_i with the same sign as that of the signed edge joining v_i with corresponding neighbours in $S(\Gamma)$. Then in the resulting signed graph, replace each vertex e_j, $j = 1, 2, \dots, m$, by p vertices and join every vertex to the neighbours of e_j with the same sign as that of the signed edge joining e_j with corresponding neighbours in $S(\Gamma)$. The resulting signed graph is denoted by $S_{k,p}(\Gamma)$. That is, for a given signed graph Γ with a compatible orientation ϑ, the signed graph $S_{k,p}(\Gamma)$ has vertex set $V(\Gamma) \times \{1, 2, \dots, k\} \cup E(\Gamma) \times \{1, 2, \dots, p\}$($k$ copies of $V(\Gamma)$ and p copies of $E(\Gamma)$) and the edges of $S_{k,p}(\Gamma)$ are all between pairs of vertices (v, i) and (e, j), where $e \in E(\Gamma)$ is incident to $v \in V(\Gamma)$ in Γ, and with sign given by $\vartheta(v, e)$.

If $k = p = 1$, then $S_{k,p}(\Gamma)$ coincides with the subdivision signed graph $S(\Gamma)$. For convenience, if $k = 1$, then $S_{k,p}(\Gamma)$ will be denoted by $S_p(\Gamma)$. To illustrate the above operation, $S_2(\Gamma)$ is shown in Figure 7 and $S_{2,2}(\Gamma)$ for $k = p = 2$ is shown in Figure 8.

Theorem 4.3. *Let Γ be a signed graph with n vertices and m edges. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n \geq 0$ be the Laplacian eigenvalues of the signed graph Γ. Then the adjacency spectrum of $S_p(\Gamma)$ is $Spec(S_p(\Gamma)) =$*

$$\begin{cases} \{0^{(pm-n+2)}, \pm\sqrt{p\mu_1}^{(1)}, \pm\sqrt{p\mu_2}^{(1)}, \dots, \pm\sqrt{p\mu_{n-1}}^{(1)}\} & \text{if } \Gamma \text{ is balanced,} \\ \{0^{(pm-n)}, \pm\sqrt{p\mu_1}^{(1)}, \pm\sqrt{p\mu_2}^{(1)}, \dots, \pm\sqrt{p\mu_{n-1}}^{(1)}, \pm\sqrt{p\mu_n}^{(1)}\} & \text{if } \Gamma \text{ is unbalanced.} \end{cases}$$

Proof. We first label the vertices of $S_p(\Gamma)$ as follows. Let $V(\Gamma) = \{v_1, v_2, \dots, v_n\}$ and let $\{u_1^j, u_2^j, \dots, u_p^j\}$, $j = 1, 2, \dots, m$, denote the vertex set replaced corresponding to the vertex e_j in $S(\Gamma)$. Denote by

$$V_i = \left\{u_i^1, u_i^2, \dots, u_i^m\right\}, \quad i = 1, 2, \dots, p.$$

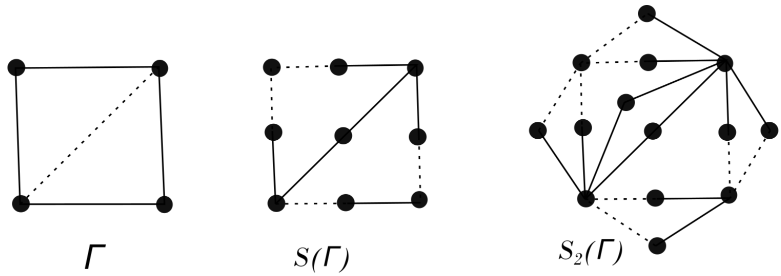

Figure 7: Signed graphs Γ, $S(\Gamma)$ and $S_2(\Gamma)$.

Then $V(\Gamma)\cup V_1\cup V_2\cup\cdots\cup V_p$ is a partition of $V(S_p(\Gamma))$. With this partition, the adjacency matrix of $S_p(\Gamma)$ can be written as

$$A(S_p(\Gamma)) = \begin{pmatrix} O & B & B & \dots & B \\ B^T & O & O & \dots & O \\ B^T & O & O & \dots & O \\ \vdots & \vdots & \ddots & \vdots & \\ B^T & O & O & \dots & O \end{pmatrix}.$$

Now, we have

$$A(S_p(\Gamma))^2 = \begin{pmatrix} O & B & B & \dots & B \\ B^T & O & O & \dots & O \\ B^T & O & O & \dots & O \\ \vdots & \vdots & \ddots & \vdots & \\ B^T & O & O & \dots & O \end{pmatrix} \begin{pmatrix} O & B & B & \dots & B \\ B^T & O & O & \dots & O \\ B^T & O & O & \dots & O \\ \vdots & \vdots & \ddots & \vdots & \\ B^T & O & O & \dots & O \end{pmatrix}$$

$$= \begin{pmatrix} pBB^T & O & O & \dots & O \\ O & B^TB & B^TB & \dots & B^TB \\ O & B^TB & B^TB & \dots & B^TB \\ \vdots & \vdots & \ddots & \vdots & \\ O & B^TB & B^TB & \dots & B^TB \end{pmatrix}$$

$$= \begin{pmatrix} pBB^T & O_{1\times p} \\ O_{p\times 1} & J_{p\times p}\otimes B^TB \end{pmatrix},$$

where $J_{p\times p}$ is a square matrix whose all entries are equal to 1. Therefore

$$Spec(A(S_p(\Gamma))^2) = Spec(pBB^T)\cup Spec(J_{p\times p}\otimes B^TB).$$

As B^TB is a real symmetric matrix of order m, so all its eigenvalues are real. Let $x_1\geq x_2\geq\cdots\geq x_m$ be the eigenvalues of the matrix B^TB. Note that $rank(BB^T) = rank(B^TB) = rank(B)$. Therefore, by Lemma 4.2, we have

$$Spec(B^TB) = \begin{cases} \{0^{(m-n+1)}, x_1, x_2, \dots, x_{n-1}\} & \text{if } \Gamma \text{ is balanced,} \\ \{0^{(m-n)}, x_1, x_2, \cdots, x_{n-1}, x_n\} & \text{if } \Gamma \text{ is unbalanced,} \end{cases}$$

and

$$Spec(BB^T) = Spec(L(\Gamma)) = \begin{cases} \{0, \mu_1, \mu_2, \ldots, \mu_{n-1}\} & \text{if } \Gamma \text{ is balanced,} \\ \{\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n\} & \text{if } \Gamma \text{ is unbalanced,} \end{cases}$$

where $x_n \neq 0$ and $\mu_n \neq 0$. As Spec($J_{p\times p}$) is $\{0^{p-1}, p\}$, then by Lemma 2.2, we have

$$Spec(J_{p\times p} \otimes B^T B) = \begin{cases} \{0^{(pm-n+1)}, px_1, px_2, \cdots, px_{n-1}\} & \text{if } \Gamma \text{ is balanced,} \\ \{0^{(pm-n)}, px_1, px_2, \ldots, px_{n-1}, px_n\} & \text{if } \Gamma \text{ is unbalanced.} \end{cases}$$

We know that the underlying graph of a subdivision signed graph is always bipartite. Similarly, the underlying graph of $S_p(\Gamma)$ is always bipartite. Note that the eigenvalues of $B^T B$ are given by the eigenvalues of BB^T, together with 0 of multiplicity $m - n$. Therefore, by Lemmas 2.5 and 2.6, we have $Spec(S_p(\Gamma)) =$

$$\begin{cases} \{0^{(pm-n+2)}, \pm\sqrt{p\mu_1}^{(1)}, \pm\sqrt{p\mu_2}^{(1)}, \ldots, \pm\sqrt{p\mu_{n-1}}^{(1)}\} & \text{if } \Gamma \text{ is balanced,} \\ \{0^{(pm-n)}, \pm\sqrt{p\mu_1}^{(1)}, \pm\sqrt{p\mu_2}^{(1)}, \ldots, \pm\sqrt{p\mu_{n-1}}^{(1)}, \pm\sqrt{p\mu_n}^{(1)}\} & \text{if } \Gamma \text{ is unbalanced.} \end{cases}$$

Hence, the result follows. □

The following result can also be seen in Theorem 2.2 of [3].

Corollary 4.4. *Let Γ be a signed graph with n vertices and m edges. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n \geq 0$ be the Laplacian eigenvalues of the signed graph Γ. Then the adjacency spectrum of $S(\Gamma)$ is Spec$(S(\Gamma)) =$*

$$\begin{cases} \{0^{(m-n+2)}, \pm\sqrt{\mu_1}^{(1)}, \pm\sqrt{\mu_2}^{(1)}, \ldots, \pm\sqrt{\mu_{n-1}}^{(1)}\} & \text{if } \Gamma \text{ is balanced,} \\ \{0^{(m-n)}, \pm\sqrt{\mu_1}^{(1)}, \pm\sqrt{\mu_2}^{(1)}, \ldots, \pm\sqrt{\mu_{n-1}}^{(1)}, \pm\sqrt{\mu_n}^{(1)}\} & \text{if } \Gamma \text{ is unbalanced.} \end{cases}$$

Theorem 4.5. *Let Γ be a signed graph with n vertices and m edges. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n \geq 0$ be the Laplacian eigenvalues of the signed graph Γ. Then the adjacency spectrum of $S_{k,p}(\Gamma)$, where $p \in \{k, k-1\}$, is Spec$(S_{k,p}(\Gamma)) =$*

$$\begin{cases} \{0^{((k-2)n+pm+2)}, \pm\sqrt{pk\mu_1}^{(1)}, \ldots, \pm\sqrt{pk\mu_{n-1}}^{(1)}\} & \text{if } \Gamma \text{ is balanced,} \\ \{0^{((k-2)n+pm)}, \pm\sqrt{pk\mu_1}^{(1)}, \ldots, \pm\sqrt{pk\mu_{n-1}}^{(1)}, \pm\sqrt{pk\mu_n}^{(1)}\} & \text{if } \Gamma \text{ is unbalanced.} \end{cases}$$

Proof. We first label the vertices of $S_{k,p}(\Gamma)$ as follows. Let $\left\{v_1^j, v_2^j, \ldots, v_k^j\right\}$, $j = 1, 2, \ldots, n$, denote the vertex set replaced corresponding to the vertex v_j and $\{u_1^j, u_2^j, \ldots, u_p^j\}$, $j = 1, 2, \ldots, m$, denote the vertex set replaced corresponding to the vertex e_j in $S(\Gamma)$. Denote by

$$V^i = \left\{v_i^1, v_i^2, \ldots, v_i^n\right\}, \quad i = 1, 2, \ldots, k,$$

and

$$V_i = \left\{u_i^1, u_i^2, \ldots, u_i^m\right\}, \quad i = 1, 2, \ldots, p.$$

Then $V^1 \cup V_1 \cup V^2 \cup V_2 \cup \cdots \cup V^k \cup V_p$ is a partition of $V(S_{k,p}(\Gamma))$ when $k = p$. With this partition, the adjacency matrix of $S_{k,p}(\Gamma)$ can be written as

$$A(S_{k,p}(\Gamma)) = \begin{pmatrix} O & B & O & \dots & O & B \\ B^T & O & B^T & \dots & B^T & O \\ O & B & O & \dots & O & B \\ \vdots & \vdots & \ddots & \vdots & & \\ O & B & O & \dots & O & B \\ B^T & O & B^T & \dots & B^T & O \end{pmatrix}.$$

If $p = k - 1$, then $V^1 \cup V_1 \cup V^2 \cup V_2 \cup \cdots \cup V_{k-1} \cup V^k$ is a partition of $V(S_{k,p}(\Gamma))$. With this partition, the adjacency matrix of $S_{k,p}(\Gamma)$ is given by

$$A(S_{k,p}(\Gamma)) = \begin{pmatrix} O & B & O & \dots & B & O \\ B^T & O & B^T & \dots & O & B^T \\ O & B & O & \dots & B & O \\ \vdots & \vdots & \ddots & \vdots & & \\ B^T & O & B^T & \dots & O & B^T \\ O & B & O & \dots & B & O \end{pmatrix}.$$

To prove the result, the following two cases arise.

Case 1. Let Γ be a balanced signed graph with n vertices and m edges. Let $Z = \begin{pmatrix} X \\ Y \end{pmatrix} \in M_{(n+m)\times 1}(\mathbb{R})$, where $X \in M_{n\times 1}(\mathbb{R})$ and $Y \in M_{m\times 1}(\mathbb{R})$, be an eigenvector corresponding to the non-zero eigenvalue λ_i, $1 \leq i \leq 2n-2$, of $S(\Gamma)$. Then $A(S(\Gamma))Z = \lambda_i Z$ implies that $BY = \lambda_i X$ and $B^T X = \lambda_i Y$. To find the eigenvalues of $S_{k,p}(\Gamma)$, consider the following two subcases.

Subcase 1.1. If $k = p$, then let $U = \begin{pmatrix} X \\ Y \\ \vdots \\ X \\ Y \end{pmatrix}$. Clearly, $U \in M_{(kn+pm)\times 1}(\mathbb{R})$ is a non-zero column vector. We have

$$\begin{aligned} A(S_{k,p}(\Gamma))U &= \begin{pmatrix} O & B & O & \dots & O & B \\ B^T & O & B^T & \dots & B^T & O \\ O & B & O & \dots & O & B \\ \vdots & \vdots & \ddots & \vdots & & \\ O & B & O & \dots & O & B \\ B^T & O & B^T & \dots & B^T & O \end{pmatrix} \begin{pmatrix} X \\ Y \\ \vdots \\ X \\ Y \end{pmatrix} = \begin{pmatrix} p\lambda_i X \\ p\lambda_i Y \\ \vdots \\ p\lambda_i X \\ p\lambda_i Y \end{pmatrix} \\ &= p\lambda_i U. \end{aligned}$$

Therefore $p\lambda_i$ is an eigenvalue of $S_{k,p}(\Gamma)$ corresponding to an eigenvector U. As $k = p$, thus $p\lambda_i$ can be written as $\sqrt{kp}\lambda_i$. Hence the result follows by Corollary 4.4.

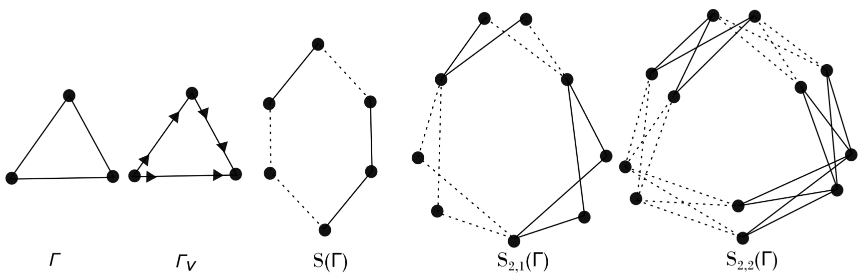

Figure 8: Signed graphs Γ, Γ_ϑ, $S(\Gamma)$, $S_{2,1}(\Gamma)$ and $S_{2,2}(\Gamma)$.

Subcase 1.2. If $p = k-1$, then let $U = \begin{pmatrix} \sqrt{p}X \\ \sqrt{k}Y \\ \vdots \\ \sqrt{p}X \\ \sqrt{k}Y \\ \sqrt{p}X \end{pmatrix}$. Clearly, $U \in M_{(kn+pm)\times 1}(\mathbb{R})$ is a non-zero column vector. We have

$$A(S_{k,p}(\Gamma))U = \begin{pmatrix} O & B & O & \dots & B & O \\ B^T & O & B^T & \dots & O & B^T \\ O & B & O & \dots & B & O \\ \vdots & \vdots & \ddots & \vdots & & \\ B^T & O & B^T & \dots & O & B^T \\ O & B & O & \dots & B & O \end{pmatrix} \begin{pmatrix} \sqrt{p}X \\ \sqrt{k}Y \\ \vdots \\ \sqrt{p}X \\ \sqrt{k}Y \\ \sqrt{p}X \end{pmatrix} = \begin{pmatrix} p\lambda_i\sqrt{k}X \\ k\lambda_i\sqrt{p}Y \\ \vdots \\ p\lambda_i\sqrt{k}X \\ k\lambda_i\sqrt{p}Y \\ p\lambda_i\sqrt{k}X \end{pmatrix}$$
$$= \sqrt{kp}\lambda_i U.$$

Therefore $\sqrt{kp}\lambda_i$ is an eigenvalue of $S_{k,p}(\Gamma)$ corresponding to an eigenvector U. Hence the result follows by Corollary 4.4.

Case 2. When Γ is an unbalanced signed graph with n vertices and m edges, the proof is similar to that of Case 1. □

Various constructions for non-isomorphic cospectral regular graphs, non-isomorphic Laplacian cospectral graphs and non-isomorphic signless Laplacian cospectral graphs can be seen in [5, 7, 8, 10, 11, 13]. The following results show that these constructions including the constructions obtained in the last section can be utilized to obtain infinite families of switching non-isomorphic cospectral signed graphs.

Corollary 4.6. *Let Γ_1 and Γ_2 be two switching non-isomorphic signed graphs which are Laplacian cospectral. Then*

(i) *the signed graphs $S_p(\Gamma_1)$ and $S_p(\Gamma_2)$ are switching non-isomorphic and cospectral,*

(ii) *for $p \in \{k, k-1\}$, the signed graphs $S_{k,p}(\Gamma_1)$ and $S_{k,p}(\Gamma_2)$ are switching non-isomorphic and cospectral.*

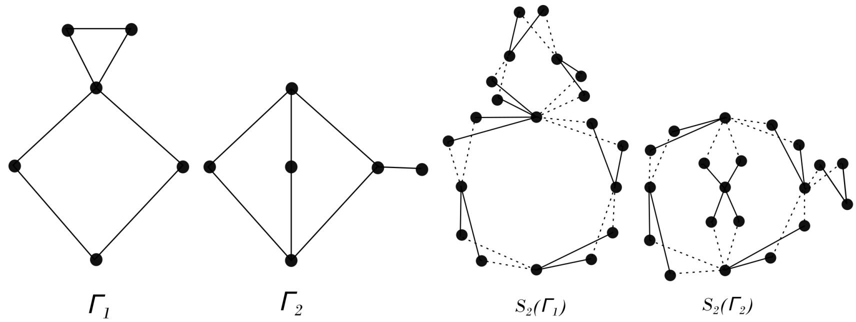

Figure 9: Cospectral signed graphs $S_2(\Gamma_1)$ and $S_2(\Gamma_2)$.

Proof. Let Γ_1 and Γ_2 be two switching non-isomorphic signed graphs. Then, clearly $S_p(\Gamma_1)$ and $S_p(\Gamma_2)$ are switching non-isomorphic signed graphs and $S_{k,p}(\Gamma_1)$ and $S_{k,p}(\Gamma_2)$ are switching non-isomorphic signed graphs. Hence the result follows by Theorems 4.3 and 4.5. □

Example 4.7. Consider the two non-isomorphic signed graphs Γ_1 and Γ_2 as shown in Figure 9. Their Laplacian spectrum is respectively given by $Spec_L(\Gamma_1) = \{0, 2, 3^{(2)}, 3 + \sqrt{5}, 3 - \sqrt{5}\}$ and $Spec_L(\Gamma_2) = \{0, 2, 3^{(2)}, 3 + \sqrt{5}, 3 - \sqrt{5}\}$. So Γ_1 and Γ_2 are Laplacian cospectral. It is easy to see that $S_2(\Gamma_1)$ and $S_2(\Gamma_2)$ are non-isomorphic signed graphs which are cospectral as their adjacency spectrum are, respectively, given by $Spec(S_2(\Gamma_1))$ $= \{0^{(10)}, \pm 2, \pm\sqrt{6}^{(2)}, \pm(\sqrt{6+\sqrt{20}}), \pm(\sqrt{6-\sqrt{20}})\}$ and $Spec(S_2(\Gamma_2)) = \{0^{(10)}, \pm 2,$ $\pm\sqrt{6}^{(2)}, \pm(\sqrt{6+\sqrt{20}}), \pm(\sqrt{6-\sqrt{20}})\}$.

Corollary 4.8. *Let Γ_1 and Γ_2 be two switching non-isomorphic cospectral $r-$regular signed graphs. Then*

(i) *the signed graphs $S_p(\Gamma_1)$ and $S_p(\Gamma_2)$ are switching non-isomorphic and cospectral,*

(ii) *for $p \in \{k, k-1\}$, the signed graphs $S_{k,p}(\Gamma_1)$ and $S_{k,p}(\Gamma_2)$ are switching non-isomorphic and cospectral.*

Proof. If Γ_1 and Γ_2 are two switching non-isomorphic cospectral regular signed graphs, then $L(\Gamma_1) = D(\Gamma_1) - A(\Gamma_1)$ and $L(\Gamma_2) = D(\Gamma_2) - A(\Gamma_2)$ are cospectral. Hence the result follows by Corollary 4.6. □

Corollary 4.9. *Let Γ be a signed graph whose all Laplacian eigenvalues are perfect squares. Then*

(i) *the signed graph $S_p(\Gamma)$ is integral, if p is a perfect square,*

(ii) *for $p \in \{k, k-1\}$, the signed graph $S_{k,p}(\Gamma)$ is integral, if kp is a perfect square.*

Example 4.10. Let K_n be a balanced complete signed graph on n vertices, where $n = t^2$, $t \geq 2$ is a positive integer. Then

(i) the signed graph $S_p(K_n)$ is integral, if p is a perfect square,

(ii) the signed graph $S_{k,p}(K_n)$ is integral, if kp is a perfect square.

The following result is the graceful implication of Lemma 2.7 and Corollaries 4.6 and 4.8.

Theorem 4.11. *For infinitely many n, there exists a family of 2^k pairwise switching nonisomorphic cospectral signed graphs on n vertices, where $k > \frac{n}{(2log_2(n))}$.*

The next result directly follows from Theorems 4.3 and 4.5.

Theorem 4.12. *Let Γ be a signed graph with n vertices and m edges. Then*

(i) $\mathcal{E}(S_p(\Gamma)) = \sqrt{p}\mathcal{E}(S(\Gamma))$,

(ii) $\mathcal{E}(S_{k,p}(\Gamma)) = \sqrt{pk}\mathcal{E}(S(\Gamma))$, *where* $p \in \{k, k-1\}$.

Theorem 4.13. *Let Γ be an unbalanced unicyclic signed graph with at least one edge and having Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n > 0$. Then $S(\Gamma) \times K_2$ and $S(\Gamma) \otimes K_2$, where K_2 is a complete signed graph on 2 vertices, are noncospectral and equienergetic if and only if $\mu_n \geq 1$.*

Proof. Let Γ be an unbalanced unicyclic signed graph. Then, by Theorem 4.3, we have

$$Spec(S(\Gamma)) = \{\pm\sqrt{\mu_1}^{(1)}, \pm\sqrt{\mu_2}^{(1)}, \ldots, \pm\sqrt{\mu_{n-1}}^{(1)}, \pm\sqrt{\mu_n}^{(1)}\}.$$

First, assume that $\mu_n \geq 1$. This implies that $|\sqrt{\mu_j}| \geq 1$, for all $j = 1, 2, \ldots, n$. Also,

$$\mathcal{E}(S(\Gamma) \times K_2) = 2\sum_{j=1}^{n}(|\sqrt{\mu_j} + 1| + |\sqrt{\mu_j} - 1|).$$

As $|\sqrt{\mu_j}| \geq 1$, for all $j = 1, 2, \ldots, n$, we have

$$\begin{aligned}\mathcal{E}(S(\Gamma) \times K_2) &= 2\sum_{j=1}^{n}(|\sqrt{\mu_j}| + 1 + |\sqrt{\mu_j}| - 1)\\ &= 2\mathcal{E}(S(\Gamma))\\ &= \mathcal{E}(S(\Gamma))\mathcal{E}(K_2) = \mathcal{E}(S(\Gamma) \otimes K_2).\end{aligned}$$

Note that $\sqrt{\mu_1} + 1 \in Spec(S(\Gamma) \times K_2)$ but $\sqrt{\mu_1} + 1 \notin Spec(S(\Gamma) \otimes K_2)$. Therefore $S(\Gamma) \times K_2$ and $S(\Gamma) \otimes K_2$ are noncospectral. The converse is similar to that of the converse in Lemma 2.4. □

Example 4.14. Let $C_3^- = (C_3, -)$ be an unbalanced unicyclic signed graph on 3 vertices. Its Laplacian spectrum is given by $Spec_L(C_3^-) = \{4, 1, 1\}$. Therefore C_3^- meets the requirement of Theorem 4.13. Hence $S(C_3^-) \times K_2$ and $S(C_3^-) \otimes K_2$ are noncospectral and equienergetic.

The following corollary directly follows from Theorem 4.12.

Corollary 4.15. *Let Γ_1 and Γ_2 be two signed graphs whose subdivision signed graphs are noncospectral and equienergetic. Then*

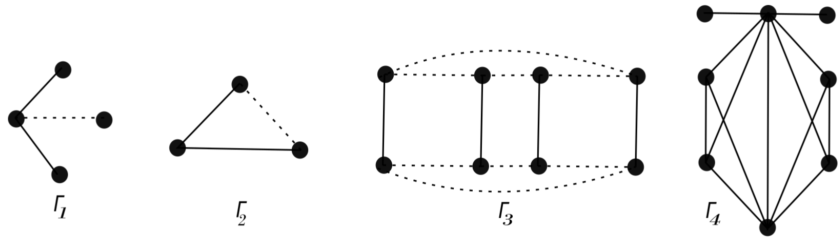

Figure 10: Signed graphs Γ_1, Γ_2, Γ_3 and Γ_4.

(i) *the signed graphs $S_p(\Gamma_1)$ and $S_p(\Gamma_2)$are noncospectral and equienergetic,*

(ii) *for $p \in \{k, k-1\}$, the signed graphs $S_{k,p}(\Gamma_1)$ and $S_{k,p}(\Gamma_2)$are noncospectral and equienergetic.*

Example 4.16. Consider the signed graphs Γ_1, Γ_2, Γ_3 and Γ_4 as shown in Figure 10. The adjacency spectrum of their subdivision signed graphs is respectively given by $Spec(S(\Gamma_1)) = \{\pm 2, \pm 1^{(2)}, 0\}$, $Spec(S(\Gamma_2)) = \{\pm 2, \pm 1^{(2)}\}$, $Spec(S(\Gamma_3)) = \{\pm 2^{(3)}, \pm\sqrt{2}^{(3)}, \pm\sqrt{6}, 0^{(6)}\}$ and $Spec(S(\Gamma_4)) = \{\pm 1^{(2)}, \pm 2^{(2)}, \pm\sqrt{2}, \pm 2\sqrt{2}, \pm\sqrt{6}, 0^{(7)}\}$. Clearly, the signed graphs $S(\Gamma_1)$ and $S(\Gamma_2)$ are noncospectral and equienergetic. Similarly, the signed graphs $S(\Gamma_3)$ and $S(\Gamma_4)$ are noncospectral and equienergetic. Thus, by Corollary 4.15, we have

(i) the signed graphs $S_p(\Gamma_1)$ and $S_p(\Gamma_2)$ are noncospectral and equienergetic,

(ii) the signed graphs $S_p(\Gamma_3)$ and $S_p(\Gamma_4)$ are noncospectral and equienergetic,

(iii) for $p \in \{k, k-1\}$, the signed graphs $S_{k,p}(\Gamma_3)$ and $S_{k,p}(\Gamma_4)$ are noncospectral and equienergetic.

Conclusion. In this paper, we generalized the construction of the subdivision graph $S(\Gamma)$ to $S_{k,p}(\Gamma)$ of a signed graph Γ. The adjacency spectrum of $S_{1,p}(\Gamma)(S_p(\Gamma))$, $S_{p,p-1}(\Gamma)$ and $S_{p,p}(\Gamma)$ is completely determined by the Laplacian spectrum of Γ. Now, it remains a problem to investigate the adjacency spectrum of $S_{k,p}(\Gamma)$ for other values of k and p.

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The role of the Axiom of Choice in proper and distinguishing colourings

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Abstract

Call a colouring of a graph *distinguishing* if the only automorphism which preserves it is the identity. We investigate the role of the Axiom of Choice in the existence of certain proper or distinguishing colourings in both vertex and edge variants with special emphasis on locally finite connected graphs. We show that every locally finite connected graph has a distinguishing colouring with at most countable number of colours or every locally finite connected graph has a proper colouring with at most countable number of colours if and only if Kőnig's Lemma holds. This statement holds for both vertex and edge colourings. Furthermore, we show that it is not provable in ZF that such colourings exist even for every connected graph with maximum degree 3. We also formulate a few conditions about distinguishing and proper colourings which are equivalent to the Axiom of Choice.

Keywords: Proper colourings, distinguishing colourings, asymmetric colourings, infinite graphs, graph automorphisms, Axiom of Choice.

Math. Subj. Class. (2020): 05C15, 03E25, 05C25, 05C63

1 Introduction

Let c be a vertex or an edge colouring of a graph G. We say that an automorphism φ of G *preserves* c if each vertex of G is mapped to a vertex of the same colour or each edge of G is mapped to an edge of the same colour. Call a colouring c *distinguishing* if the only automorphism which preserves c is the identity. If a colouring c is a mapping into ordinal numbers (or any well-ordered set) we can think about the number of colours in c. The *distinguishing number* $D(G)$ of a graph G is the least number of colours in a distinguishing vertex colouring of G. Similarly, the *distinguishing index* $D'(G)$ of a graph G is the least number of colours in a distinguishing edge colouring of G. Distinguishing vertex colourings were introduced by Babai [1] in 1977 under the name *asymmetric* colourings during

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his study of the complexity of the graph isomorphism problem [2]. Distinguishing edge colourings were introduced by Kalinowski and Pilśniak [9].

In this paper we study proper and distinguishing colourings in ZF, hence without assuming the Axiom of Choice. Proper vertex colourings in ZF were investigated by Galvin and Komjáth [5]. They proved that the existence of the chromatic number of each graph is equivalent to the Axiom of Choice. Distinguishing colourings and proper edge colourings in ZF were not previously investigated.

In most of the papers about infinite graphs some version of the Axiom of Choice is used though not always explicitly. The most popular methods often involve Zorn's Lemma or Kőnig's Lemma. In particular, proofs of general bounds by a function of $\Delta(G)$ for chromatic number [3], chromatic index [10], distinguishing number [11] and distinguishing index [12] of connected infinite graphs all use Kőnig's Lemma in the case of locally finite graphs and the Axiom of Choice in the form of Hessenberg's Theorem for general bounds. We show that in all these cases the use of Kőnig's Lemma or respectively the Axiom of Choice is necessary.

Similar problems for graphs without the assumption of connectivity were previously investigated for proper vertex colourings. The statement that every graph has a proper vertex colouring using at most two colours if and only if each of its finite subgraphs has such a colouring is equivalent to the Axiom of Choice for Pairs. If we replace two colours with three colours, then we obtain the statement equivalent to the Prime Ideal Theorem. See [6] p. 109–116 for details and further examples.

Arguably, most of the results related to proper or distinguishing colourings in graph theory concern only locally finite connected graphs. From the results mentioned in the previous paragraph, it follows that one cannot prove in ZF that every locally finite connected graph has a distinguishing or a proper colouring with at most countable number of colours. We show that one cannot prove the existence of such colourings in ZF even in the simplest case of connected graphs with maximum degree 3.

2 Preliminaries

By a *cardinal number* we mean an *initial ordinal* i.e. an ordinal which is not equinumerous with any smaller ordinal. For every set there exists a cardinal number equinumerous with it if and only if the Axiom of Choice holds.

Well-Ordering Theorem states that for every set X there exists a well-order on X. Well-Ordering Theorem is equivalent to the Axiom of Choice.

We now present some weak choice principles. The axiom $\mathbf{AC}^{\omega}_{\leqslant\kappa}$ states that every countable family of non-empty sets of cardinality at most κ has a choice function. The axiom $\mathbf{AC}^{\omega}_{fin}$ states that every countable family of non-empty finite sets has a choice function. The axiom $\mathbf{AC}^{\omega}_{2}$ is the same as $\mathbf{AC}^{\omega}_{\leqslant 2}$. The axiom $\mathbf{AC}^{\omega}_{fin}$ is equivalent to Kőnig's Lemma stating that every locally finite infinite connected graph has a ray. The axiom $\mathbf{AC}^{\omega}_{fin}$ is also equivalent to the statement that every countable union of finite sets is countable. More about the Axiom of Choice, weak choice principles and their equivalent forms may be found in the extensive monograph of Howard and Rubin [7].

Let G be a graph. Denote by $\Delta(G)$ the supremum over the degrees of all vertices of G. If there exists a vertex $v \in V(G)$ such that $d(v) = \Delta(G)$, then $\Delta(G)$ is called the *maximum degree* of G. Graphs with maximum degree 3 are called *subcubic*. We say that a graph is *locally finite* if each of its vertices has finite degree.

Let Γ be a group acting on a set Ω and let A be a subset of Ω. The *orbit* of A is the set $\{\varphi(a) : a \in A, \varphi \in \Gamma\}$. We say that A is *fixed* if every $\varphi \in \Gamma$ acts trivially on A i.e. if $\varphi(a) = a$ for every $a \in A$. We say that A is *stabilized* if for every $\varphi \in \Gamma$, we have $\varphi(A) \subseteq A$. In the definitions in this paragraph, if $A = \{a\}$ is a singleton, then we often refer to a instead of $\{a\}$. An *automorphism* of a graph G is a bijection $\varphi\colon V(G) \to V(G)$ such that uv is an edge in G if and only if $\varphi(u)\varphi(v)$ is an edge in G. They form a group with composition as the operation. If not written explicitly, the meaning of Γ and Ω shall follow from the context. In this paper Γ is usually a group of some automorphisms of a graph G and Ω is a set of some vertices of G or some edges of G.

Colourings in this paper are not necessarily proper unless stated otherwise. For notions which are not defined here, see [4] or [8].

3 The Axiom of Choice in proper and distinguishing colourings

Let κ be an arbitrary non-zero cardinal. Call a family $\mathcal{A} = \{A_i : i \in \omega\}$ *acceptable* if $\mathcal{A}$ is a countable family of pairwise disjoint non-empty sets. We say that a family $\mathcal{A}$ is *almost κ-acceptable* if $\mathcal{A}$ is acceptable and every set in $\mathcal{A}$ has cardinality less than κ. We say that $\mathcal{A}$ is *κ-acceptable* if it is almost κ^+-acceptable. In other words, $\mathcal{A}$ is *κ-acceptable* if $\mathcal{A}$ is acceptable and every set in $\mathcal{A}$ has cardinality at most κ.

Let $\mathcal{A} = \{A_i : i \in \omega\}$ be an acceptable family and let $Y = \bigcup \mathcal{A}$. Let $Z = \{z_i : i \in \omega\}$ and $Z' = \{z'_i : i \in \omega\}$ be disjoint sets which are also disjoint from Y. We now define graphs $G_\mathcal{A}$ and $H_\mathcal{A}$ by

$$\begin{aligned}
&V(G_\mathcal{A}) = V(H_\mathcal{A}) = Y \cup Z \cup Z',\\
&E(G_\mathcal{A}) = \{z_i z_{i+1} : i \in \omega\} \cup \{z_i z'_i : i \in \omega\} \cup \{a_i z'_i : i \in \omega, a_i \in A_i\},\\
&E(H_\mathcal{A}) = E(G_\mathcal{A}) \cup \{ab : a \neq b, a, b \in A_i, i \in \omega\}.
\end{aligned}$$

From the definitions of $G_\mathcal{A}$ and $H_\mathcal{A}$ it follows that every vertex in $Z \setminus \{z_0\}$ has degree 3 in both graphs, and vertex z_0 has degree 2. For the rest of the paragraph, assume that for every $i \in \omega$ the set A_i is well-orderable. The vertex z'_i has degree $|A_i| + 1$ for every $i \in \omega$ in both $G_\mathcal{A}$ and $H_\mathcal{A}$. Every vertex in Y has degree 1 in $G_\mathcal{A}$. However, every vertex $a \in A_i$ has degree $|A_i|$ in $H_\mathcal{A}$ since it has edges to every other vertex in A_i and to z'_i. Summarizing, we obtain $\Delta(G_\mathcal{A}) = \Delta(H_\mathcal{A}) = \max\{3, \sup\{|A_i| + 1 : i \in \omega\}\}$.

Claim 3.1. *Let $\mathcal{A} = \{A_i : i \in \omega\}$ be an acceptable family. Then for every natural number i, the vertices in A_i form an orbit with respect to the groups of automorphisms of $G_\mathcal{A}$ and $H_\mathcal{A}$. The rest of the vertices in both graphs are fixed with respect to these groups.*

Proof. First we show that z_0 is fixed in both graphs. Suppose that z_0 may be mapped into z_i for some $i \neq 0$. Let R_i be the maximal induced ray with endvertex z_i. Clearly R_0 is a tail of the ray R induced by Z. Let P be a maximal induced path with endvertex z_0 which is edge disjoint from R, and let P_i be a maximal induced path with endvertex z_i which is edge disjoint from R_i and contains z_0. If P_i contains z_0, then the length of P_i is larger than the length of P. Notice that in this case P_i is the longest induced path with endvertex z_i which is edge disjoint from R_i. This leads to contradiction because P_i cannot be mapped into P. Hence, z_0 is fixed. The ray R is the only induced ray with endvertex z_0. Therefore, R is fixed.

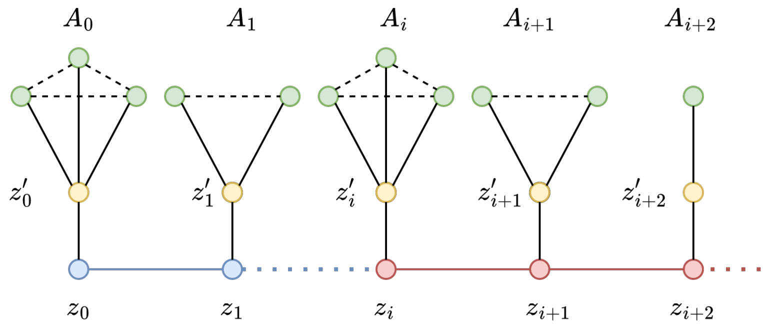

Figure 1: Graphs $G_{\mathcal{A}}$ and $H_{\mathcal{A}}$. Graph $H_{\mathcal{A}}$ is obtained by adding the dashed edges. The ray R_i is pink, and the remaining part of the ray R is blue.

Since every vertex of the fixed set Z has exactly one neighbour outside Z, all of these neighbours are fixed. Hence, Z' is fixed, and A_i is stabilized for every $i \in \omega$. The vertices in A_i form an independent set in $G_{\mathcal{A}}$ (or a clique in $H_{\mathcal{A}}$). Hence, they form an orbit with respect to the group of automorphisms of $\mathcal{G}_{\mathcal{A}}$, and also with respect to the group of automorphisms of $\mathcal{H}_{\mathcal{A}}$. □

Notice that from Claim 3.1 it follows that the group of automorphisms of $G_{\mathcal{A}}$ is the same as the group of automorphism of $H_{\mathcal{A}}$. Now, we prove a lemma which allow us to restrict part of the later considerations to the problem of the existence of the distinguishing number for graphs of the form $G_{\mathcal{A}}$. With the lemma below we are able to simultaneously obtain results about distinguishing colourings and proper colourings in both vertex and edge versions.

Lemma 3.2. *Let $\mathcal{A}$ be an acceptable family. Then the following conditions are equivalent.*

(a) *There exists the distinguishing number of $G_{\mathcal{A}}$.*

(b) *There exists the distinguishing index of $G_{\mathcal{A}}$.*

(c) *There exists the chromatic index of $G_{\mathcal{A}}$.*

(d) *There exists the chromatic number of $H_{\mathcal{A}}$.*

Proof. By Claim 3.1 if c is a distinguishing vertex colouring of $G_{\mathcal{A}}$, then for every $i \in \omega$ vertices in A_i have distinct colours. If for every $i \in \omega$ and each vertex $v \in A_i$ we colour the edge vz'_i with colour $c(v)$, and we colour the rest of the edges of $G_{\mathcal{A}}$ arbitrarily, then we obtain a distinguishing edge colouring of $G_{\mathcal{A}}$. Hence, condition (a) implies (b).

Now, let c be a distinguishing edge colouring of $G_{\mathcal{A}}$. Let c' be a colouring in which the edges incident to vertices in Y have the same colour as in c, the edges between Z and Z' are coloured with the same new colour, and the edges between the vertices in Z are coloured alternately with two new colours. The colouring c' is a proper edge colouring. Therefore, condition (b) implies condition (c).

Let c be a proper edge colouring of $G_{\mathcal{A}}$. We now define a proper vertex colouring c' of $H_{\mathcal{A}}$. First, for every $i \in \omega$, we colour each vertex $v \in A_i$ with colour $c'(v) = c(vz'_i)$.

Next, we colour the vertices in Z' with the same new colour, and we colour the vertices in Z alternately with two new colours. Colouring c' is a proper vertex colouring of $H_{\mathcal{A}}$. Hence, condition (c) implies condition (d).

Implication between (d) and (a) follows directly from Claim 3.1 since every proper vertex colouring of $H_{\mathcal{A}}$ is a distinguishing vertex colouring of $G_{\mathcal{A}}$. □

We can now proceed to the study of relations between the existence of certain colourings and the Axiom of Choice. The first step is Lemma 3.3, which shows that the existence of the distinguishing number of $G_{\mathcal{A}}$ implies the existence of a choice function for $\mathcal{A}$.

Lemma 3.3. *Let $\mathcal{A}$ be an acceptable family and assume that there exists the distinguishing number of $G_{\mathcal{A}}$. Then there exists a choice function for $\mathcal{A}$.*

Proof. Let c be a vertex colouring of $G_{\mathcal{A}}$ with elements of some cardinal number κ. Then $f(A) = \arg\min\{c(a) : a \in A\}$ is a choice function for $\mathcal{A}$. □

We now prove the next lemma which in the case of non-zero natural number k and k-acceptable family $\mathcal{A}$ allows us to construct a distinguishing colouring of $G_{\mathcal{A}}$ using a choice function for $\mathcal{A}$.

Lemma 3.4. *Let k be an arbitrary non-zero natural number and assume $\mathbf{AC}^{\omega}_{\leqslant k}$. Then for every k-acceptable family $\mathcal{A}$ graph $G_{\mathcal{A}}$ has distinguishing number at most k.*

Proof. The proof is by induction on k. Let $\mathcal{A}$ be a k-acceptable family. If $k = 1$, then G has no non-trivial automorphism. Hence, its distinguishing number is equal to 1. Assume that $k \geqslant 2$, and that the statement of the lemma holds for every $l < k$. Let f be a choice function for $\mathcal{A}$. From the inductive hypothesis $G_{\mathcal{A}} - f(\mathcal{A})$ has a distinguishing vertex colouring c' using at most $k-1$ colours. Colouring c which agrees with colouring c' on $V(G_{\mathcal{A}}) \setminus f(\mathcal{A})$ and which assigns the rest of vertices of $G_{\mathcal{A}}$ the same new colour is a distinguishing colouring using at most k colours. □

Lemmas 3.2 – 3.4 allows us to formulate the following corollary about the existence of certain parameters for k-acceptable families in the case of finite k.

Theorem 3.5. *Let $k \geqslant 2$ be an arbitrary natural number. Then the following conditions are equivalent.*

(a) $\mathbf{AC}^{\omega}_{\leqslant k}$.

(b) *For every k-acceptable family $\mathcal{A}$ the graph $G_{\mathcal{A}}$ has the distinguishing number.*

(c) *For every k-acceptable family $\mathcal{A}$ the graph $G_{\mathcal{A}}$ has the distinguishing index.*

(d) *For every k-acceptable family $\mathcal{A}$ the graph $G_{\mathcal{A}}$ has the chromatic index.*

(e) *For every k-acceptable family $\mathcal{A}$ the graph $H_{\mathcal{A}}$ has the chromatic number.*

In particular for $k = 2$ condition (a) is the axiom $\mathbf{AC}^{\omega}_{2}$ which is independent of ZF. It follows that in ZF one cannot prove the existence of the above parameters even for every connected subcubic graph.

Theorem 3.5 tells us about the existence of certain parameters for connected graphs with finite maximal degree. Now, we establish the relations between Kőnig's Lemma and the existence of proper colourings and distinguishing colourings using at most countable number of colours in the case of locally finite connected graphs.

Theorem 3.6. *The following conditions are equivalent.*

(KL) *Kőnig's Lemma.*

(KL1) *Every infinite locally finite connected graph has the distinguishing number.*

(KL2) *Every infinite locally finite connected graph has the distinguishing index.*

(KL3) *Every infinite locally finite connected graph has the chromatic index.*

(KL4) *Every infinite locally finite connected graph has the chromatic number.*

(KL5) *For every almost $\aleph_0$-acceptable family $\mathcal{A}$ the graph $G_{\mathcal{A}}$ has the distinguishing number.*

(KL6) *For every almost $\aleph_0$-acceptable family $\mathcal{A}$ the graph $G_{\mathcal{A}}$ has the distinguishing index.*

(KL7) *For every almost $\aleph_0$-acceptable family $\mathcal{A}$ the graph $G_{\mathcal{A}}$ has the distinguishing index.*

(KL8) *For every almost $\aleph_0$-acceptable family $\mathcal{A}$ the graph $H_{\mathcal{A}}$ has the chromatic number.*

Proof. First, we show that Kőnig's Lemma implies conditions (KL1) – (KL4). Let G be an infinite locally finite connected graph. Let v be some vertex of G. For a natural number d denote $B(v, d) = \{x \in V(G) : d(v, x) = d\}$. By local finiteness of G each $B(v, d)$ is finite. By connectivity of G the vertex set of G may be represented as the countable union of finite sets $V(G) = \bigcup\{B(v, d) : d < \omega\}$. Recall that Kőnig's Lemma is equivalent to the statement that the sum of every countable family of finite sets is countable. Hence, $V(G)$ is countable. As $E(G) \subseteq V(G) \times V(G)$, then $E(G)$ is also countable. Since both sets $V(G)$ and $E(G)$ are countable, we can obtain the desired colourings by assigning to each vertex (edge respectively) a unique natural number.

Implications (KL1) ⇒ (KL5), (KL2) ⇒ (KL6), (KL3) ⇒ (KL7) and (KL4) ⇒ (KL8) are trivial. The equivalence of the conditions (KL5) – (KL8) follows from Lemma 3.2.

It remains to show that the condition (KL5) implies Kőnig's Lemma. From (KL5) and Lemma 3.3, we have that for every countable family of finite sets there exists its choice function. This is the axiom $\mathbf{AC}^{\omega}_{fin}$, which is equivalent to Kőnig's Lemma. □

As we have shown, the existence of the distinguishing number of $G_{\mathcal{A}}$ for every almost $\aleph_0$-acceptable family $\mathcal{A}$ is equivalent to Kőnig's Lemma and therefore to the Axiom of Countable Choice for Finite Sets. One may think that the existence of the distinguishing number of every graph of the form $G_{\mathcal{A}}$ for some acceptable family $\mathcal{A}$ is equivalent to the Axiom of Countable Choice. It turns out that this condition is much stronger and it implies the full Axiom of Choice.

Theorem 3.7. *If for every acceptable family $\mathcal{A}$ the graph $G_{\mathcal{A}}$ has the distinguishing number, then the Axiom of Choice holds.*

Proof. Let X be a non-empty set and let $\mathcal{A}$ be an acceptable family such that $X \in \mathcal{A}$. By the assumption there exists a distinguishing vertex colouring c of the graph $G_{\mathcal{A}}$ using colours from some cardinal κ. As the colouring c is distinguishing, the elements of X have distinct colours in c. It follows that $c|_X$ is an injection from X to cardinal number κ. Hence, the Well-Ordering Theorem holds and so does the Axiom of Choice. □

Theorem 3.7 allows to formulate a list of conditions equivalent to the Axiom of Choice. The conditions (AC1) – (AC4) in the theorem below are equivalent to their restrictions to connected graphs. Recall that the equivalence of the Axiom of Choice, the existence of the chromatic number of every graph, and the existence of the chromatic number of every connected graph was proved by Galvin and Komjáth [5].

Theorem 3.8. *The following conditions are equivalent.*

(AC) *The Axiom of Choice.*

(AC1) *Every graph has the distinguishing number.*

(AC2) *Every graph without a component isomorphic to K_1 or K_2 has the distinguishing index.*

(AC3) *Every graph has the chromatic index.*

(AC4) *Every graph has the chromatic number.*

(AC5) *For every acceptable family $\mathcal{A}$ the graph $G_\mathcal{A}$ has the distinguishing number.*

(AC6) *For every acceptable family $\mathcal{A}$ the graph $G_\mathcal{A}$ has the distinguishing index.*

(AC7) *For every acceptable family $\mathcal{A}$ the graph $G_\mathcal{A}$ has the chromatic index.*

(AC8) *For every acceptable family $\mathcal{A}$ the graph $H_\mathcal{A}$ has the chromatic number.*

Proof. From the Well-Ordering Theorem we can well-order the set of vertices and the set of edges of a given graph and then colour each vertex (edge respectively) of the said graph with a unique colour. This means that the Axiom of Choice implies conditions (AC1) – (AC4). Each of the condition (AC1) – (AC4) implies its restriction to connected graphs and also the corresponding condition (AC5) – (AC8). By Lemma 3.2 conditions (AC5) – (AC8) are equivalent. The implication between condition (AC5) and the Axiom of Choice is Theorem 3.7. □

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