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The number of rooted forests in circulant graphs*

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Abstract

In this paper, we develop a new method to produce explicit formulas for the number $f_G(n)$ of rooted spanning forests in the circulant graphs $G = C_n(s_1, s_2, \ldots, s_k)$ and $G = C_{2n}(s_1, s_2, \ldots, s_k, n)$. These formulas are expressed through Chebyshev polynomials. We prove that in both cases the number of rooted spanning forests can be represented in the form $f_G(n) = p\, a(n)^2$, where $a(n)$ is an integer sequence and p is a certain natural number depending on the parity of n. Finally, we find an asymptotic formula for $f_G(n)$ through the Mahler measure of the associated Laurent polynomial $P(z) = 2k+1-\sum_{i=1}^{k}(z^{s_i}+z^{-s_i})$.

Keywords: Rooted tree, spanning forest, circulant graph, Laplacian matrix, Chebyshev polynomial, Mahler measure.

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1 Introduction

The famous Kirchhoff's Matrix Tree Theorem [15] states that the number of spanning trees in a graph can be expressed as the product of its non-zero Laplacian eigenvalues divided by the number of vertices. Since then, a lot of papers on enumeration of spanning trees for various classes of graphs were published. In particular, explicit formulae were derived for complete multipartite graphs [1, 5], almost complete graphs [33], wheels [3], fans [12], prisms [2], ladders [26], Moebius ladders [27], lattices [28], anti-prisms [31],

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complete prisms [25] and for many other families. For the circulant graphs some explicit and recursive formulae are given in [8, 23, 34, 35].

Along with the number of spanning trees in a given graph one can be interested in the number of rooted spanning forests in the graph. According to the classical result [14] (see also more recent paper [7, 16]) this value can be found with the use of determinant of the matrix $\det(I + L)$. Here L is the Laplacian matrix of the graph. At the same time, it is known very little about analytic formulas for the number of spanning forests. In particular, P. Chebotarev [6] enumerated the rooted spanning forests in path and cyclic graphs and O. Knill [16] proved that the number of rooted spanning forests in the complete graph K_n on n vertices is equal to $(n+1)^{n-1}$. The rooted spanning forests in complete bipartite graphs were enumerated in [13]. Explicit formulas for the number of rooted spanning forests for cyclic, star, line and some other graphs were given by [16]. As for the number of unrooted forests, it has much more complicated structure [4, 19, 32].

Starting with Boesch and Prodinger [3] the idea to apply Chebyshev polynomials for counting various invariants of graphs arose. This idea provided a way to find complexity, that is the number of spanning tress, of circulant graphs and their natural generalisations in [8, 17, 23, 24, 35].

Recently, asymptotical behavior of complexity for some families of graphs was investigated from the point of view of so called Malher measure [11, 29]. Mahler measure of a polynomial $P(z)$, with complex coefficients, is the absolute value of the product of all the roots of $P(z)$ whose modulus is greater than 1 multiplied by the leading coefficient. For general properties of the Mahler measure see survey [30] and monograph [10].

The purpose of this paper is to present new formulas for the number of rooted spanning forests in circulant graphs and investigate their arithmetical properties and asymptotics.

We arrange the paper in the following way. First, in Sections 3 and 4, we present new explicit formulas for the number of spanning forests in the undirected circulant graphs $C_n(s_1, s_2, \ldots, s_k)$ and $C_{2n}(s_1, s_2, \ldots, s_k, n)$ of even and odd valency respectively. They will be given in terms of Chebyshev polynomials. Next, in Section 5, some arithmetic properties of the number of spanning forests are investigated. More precisely, it is shown that the number of spanning forests of the circulant graph G can be represented in the form $f_G(n) = p\,a(n)^2$, where $a(n)$ is an integer sequence and p is a certain natural number. At last, in Section 6, we use explicit formulas for $f_G(n)$ in order to find its asymptotics in terms of Mahler measure of the associated polynomials. For circulant graphs of even valency the associated polynomial is $P(z) = 2k + 1 - \sum_{j=1}^{k}(z^{s_j} + z^{-s_j})$. In this case (Theorem 6.1) we have $f_G(n) \sim A^n$, $n \to \infty$, where A is the Mahler measure of $P(z)$. For circulant graphs of odd valency we use the polynomial $R(z) = P(z)(P(z) + 2)$. Then the respective asymptotics (Theorem 6.2) is $f_G(n) \sim K^n$, $n \to \infty$, where $K = M(R)$. In the last Section 7, we illustrate the obtained results by a series of examples.

2 Basic definitions and preliminary facts

Consider a finite and not necessary connected graph G without loops. A *rooted tree* is a tree with one marked vertex called *root*. A *rooted forest* is a graph whose connected components are rooted trees. A *rooted spanning forest* F in the graph G is a subgraph that is a rooted forest containing all the vertices of G. We denote the vertex and edge set of G by $V(G)$ and $E(G)$, respectively. Given $u, v \in V(G)$, we set a_{uv} to be equal to the number of edges between vertices u and v. The matrix $A = A(G) = (a_{uv})_{u,\,v\in V(G)}$ is called *the* *adjacency matrix* of the graph G. The degree $d(v)$ of a vertex $v \in V(G)$ is defined by $d(v) = \sum_{u \in V(G)} a_{uv}$. Let $D = D(G)$ be the diagonal matrix indexed by the elements of $V(G)$ with $d_{vv} = d(v)$. The matrix $L = L(G) = D(G) - A(G)$ is called *the Laplacian matrix*, or simply *Laplacian*, of the graph G.

By I_n we denote the identity matrix of order n.

Denote by $\chi_G(\lambda) = \det(\lambda I_n - L(G))$ the characteristic polynomial of the Laplacian matrix of a graph G on n vertices. Its extended form is

$$\chi_G(\lambda) = c_1\lambda + \ldots + c_{n-1}\lambda^{n-1} + \lambda^n.$$

The theorem by Kelmans and Chelnokov [14] states that the absolute value of coefficient c_k of $\chi_G(\lambda)$ coincides with the number of rooted spanning k-forests in the graph G. Since all the Laplacian eigenvalues of G are non-negative, the sequence c_k is alternating. So, the number of rooted spanning forests of the graph G can be found by the formula

$$\begin{aligned} f_G(n) &= f_1 + f_2 + \ldots + f_n = |c_1 - c_2 + c_3 - \ldots + (-1)^{n-1}| \\ &= (-1)^n \chi_G(-1) = \det(I_n + L(G)). \end{aligned} \tag{2.1}$$

This result was independently obtained by many authors (P. Chebotarev and E. Shamis [7] O. Knill [16] and others).

Let $s_1, s_2, \ldots, s_k$ be integers such that $1 \le s_1 < s_2 < \ldots < s_k \le \frac{n}{2}$. The graph $C_n(s_1, s_2, \ldots, s_k)$ with n vertices $0, 1, 2, \ldots, n-1$ is called *circulant graph* if the vertex i, $0 \le i \le n-1$ is adjacent to the vertices $i \pm s_1, i \pm s_2, \ldots, i \pm s_k \pmod n$. When $s_k < \frac{n}{2}$ all vertices of the graph have even degree $2k$. If n is even and $s_k = \frac{n}{2}$, then all vertices have odd degree $2k-1$. Two circulant graphs $C_n(s_1, s_2, \ldots, s_k)$ and $C_n(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_k)$ of the same order are said to be conjugate by multiplier if there exists an integer r coprime to n such that $\{\tilde{s}_1, \tilde{s}_2, ..., \tilde{s}_k\} = \{rs_1, rs_2, \ldots, rs_k\}$ as subsets of $\mathbb{Z}_n$. In this case, the graphs are isomorphic, with multiplication by the unit $r \pmod n$ giving an isomorphism.

We call an $n \times n$ matrix *circulant,* and denote it by $\mathrm{circ}(a_0, a_1, \ldots, a_{n-1})$ if it is of the form

$$\mathrm{circ}(a_0, a_1, \ldots, a_{n-1}) = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \ldots & a_{n-2} \\ & \vdots & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{pmatrix}$$

It is easy to see that adjacency and Laplacian matrices of the circulant graph are circulant matrices. The converse is also true. If the Laplacian matrix of a graph is circulant then the graph is also circulant.

Recall [9] that the eigenvalues of matrix $\mathrm{C} = \mathrm{circ}(a_0, a_1, \ldots, a_{n-1})$ are given by the following simple formulas $\lambda_j = P(\varepsilon_n^j)$, $j = 0, 1, \ldots, n-1$, where $P(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}$ and ε_n is an order n primitive root of unity. Moreover, the circulant matrix $\mathrm{T} = \mathrm{circ}(0, 1, 0, \ldots, 0)$ is the matrix representation of the shift operator $T\colon (x_0, x_1, \ldots, x_{n-2}, x_{n-1}) \to (x_1, x_2, \ldots, x_{n-1}, x_0)$.

Let $P(z) = a_0 z^s + \ldots + a_s = a_0 \prod_{i=1}^{s}(z - \alpha_i)$ be a nonconstant polynomial with complex coefficients. Then, following Mahler [21] its *Mahler measure* is defined to be

$$M(P) := \exp(\int_0^1 \log |P(e^{2\pi \mathrm{i} t})| dt), \tag{2.2}$$

the geometric mean of $|P(z)|$ for z on the unit circle. However, $M(P)$ had appeared earlier in a paper by Lehmer [18], in an alternative form

$$M(P) = |a_0| \prod_{|\alpha_i|>1} |\alpha_i|. \tag{2.3}$$

See, for example [10], for the proof of equivalence of these definitions.

The concept of Mahler measure can be naturally extended to the class of Laurent polynomials $P(z) = a_0 z^{p+s} + a_1 z^{p+s-1} + \ldots + a_{s-1} z^{s+1} + a_s z^s = a_0 z^p \prod_{i=1}^{s}(z - \alpha_i)$, where $a_0 \neq 0,$ s is a positive integer and p is an arbitrary and not necessarily positive integer.

Let $T_n(z) = \cos(n \arccos z)$ be the Chebyshev polynomial of the first kind. The following property of the Chebyshev polynomials is widely used in the paper $T_n(\frac{1}{2}(z + z^{-1})) = \frac{1}{2}(z^n + z^{-n})$. See [22] for more properties of Chebyshev polynomials.

3 The number of rooted spanning forests of circulant graphs of even valency

The aim of this section is to find new formulas for the numbers of rooted spanning forests of circulant graph $C_n(s_1, s_2, \ldots, s_k)$ in terms of Chebyshev polynomials. Here and below, we will use G to denote the circulant graph under consideration.

Theorem 3.1. *The number of rooted spanning forests $f_G(n)$ in the circulant graph $G = C_n(s_1, s_2, \ldots, s_k)$, $1 \leq s_1 < s_2 < \ldots < s_k < \frac{n}{2}$, is given by the formula*

$$f_G(n) = \prod_{p=1}^{s_k} |2T_n(w_p) - 2|,$$

where w_p, $p = 1, 2, \ldots, s_k$ are all the roots of the algebraic equation $\sum_{j=1}^{k}(2T_{s_j}(w) - 2) = 1$ and $T_s(w)$ is the Chebyshev polynomial of the first kind.

Proof. The number of rooted spanning forests of the graph G can be found by the formula $f_G(n) = \det(I_n + L(G))$. The latter value is equal to the product of all eigenvalues of the matrix $I_n + L(G)$. We denote by $T = \mathrm{circ}(0, 1, \ldots, 0)$ the $n \times n$ cyclic shift operator. Consider the Laurent polynomial $P(z) = 2k + 1 - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i})$. Then the matrix $I_n + L(G)$ has the following form

$$I_n + L(G) = P(T) = (2k+1)I_n - \sum_{i=1}^{k}(T^{s_i} + T^{-s_i}).$$

The eigenvalues of circulant matrix T are ε_n^j, $j = 0, 1, \ldots, n-1$, where $\varepsilon_n = e^{\frac{2\pi \mathrm{i}}{n}}$. Since all of them are distinct, the matrix T is similar to the matrix $\mathbb{T} = \mathrm{diag}(1, \varepsilon_n, \ldots, \varepsilon_n^{n-1})$ with diagonal entries $1, \varepsilon_n, \ldots, \varepsilon_n^{n-1}$. So the matrix $I_n + L(G)$ is similar to the diagonal matrix $P(\mathbb{T})$. This essentially simplifies the problem of finding eigenvalues of $I_n + L(G)$. Indeed, let λ be an eigenvalue of $P(\mathbb{T})$ and x be the corresponding eigenvector. Then we have the following system of linear equations

$$((2k+1-\lambda)I_n - \sum_{i=1}^{k}(\mathbb{T}^{s_i} + \mathbb{T}^{-s_i}))x = 0.$$

Recall that the matrices under consideration are diagonal and the $(j+1, j+1)$-th entry of $\mathbb{T}$ is equal to ε_n^j, where $\varepsilon_n = e^{\frac{2\pi i}{n}}$. Then, for any $j = 0, \dots, n-1$, matrix $P(\mathbb{T})$ has an eigenvalue $\lambda_j = P(\varepsilon_n^j) = 2k + 1 - \sum_{i=1}^{k}(\varepsilon_n^{js_i} + \varepsilon_n^{-js_i})$. Hence we have

$$f_G(n) = \prod_{j=0}^{n-1} P(\varepsilon_n^j). \tag{3.1}$$

To continue the proof of the theorem we need the following lemma.

Lemma 3.2.

$$\prod_{j=0}^{n-1} P(\varepsilon_n^j) = \prod_{p=1}^{s_k} |2T_n(w_p) - 2|,$$

where $w_p,\ p = 1, \dots, s_k$ *are all the roots of the algebraic equation* $\sum_{j=1}^{k}(2T_{s_j}(w) - 2) = 1$.

To prove the above formula we introduce integer polynomial $\widetilde{P}(z) = -z^{s_k}P(z)$. This is a monic polynomial with the same roots as $P(z)$ and its degree is $2s_k$. As $P(z) = P(\frac{1}{z})$, its roots look like $z_1, \frac{1}{z_1}, \dots, z_{s_k}, \frac{1}{z_{s_k}}$.

We have $\prod_{j=0}^{n-1} P(\varepsilon_n^j) = \prod_{j=0}^{n-1}(-\varepsilon_n^{-s_kj}\widetilde{P}(\varepsilon_n^j)) = (-1)^{(s_k+1)(n+1)-1}\prod_{j=0}^{n-1}\widetilde{P}(\varepsilon_n^j)$. Recall one of the basic properties of resultants

$$\text{Res}\,(\widetilde{P}(z), \widetilde{Q}(z)) = (-1)^{\deg(\widetilde{P})\deg(\widetilde{Q})}\text{Res}\,(\widetilde{Q}(z), \widetilde{P}(z)),$$

where $\widetilde{P}(z)$ and $\widetilde{Q}(z)$ are monic polynomials of degree $\deg(\widetilde{P})$ and $\deg(\widetilde{Q})$ respectively. We set $\widetilde{Q}(z) = z^n - 1$ and note that $\deg(\widetilde{P}) = 2s_k$ is even. Then we obtain

$$\begin{aligned}
\prod_{j=0}^{n-1}\widetilde{P}(\varepsilon_n^j) &= \text{Res}\,(\widetilde{P}(z), z^n - 1) = \text{Res}\,(z^n - 1, \widetilde{P}(z)) \\
&= \prod_{z:\widetilde{P}(z)=0}(z^n - 1) = \prod_{z:P(z)=0}(z^n - 1) \\
&= \prod_{p=1}^{s_k}(z_p^n - 1)(z_p^{-n} - 1) = (-1)^{s_k}\prod_{p=1}^{s_k}(2T_n(w_p) - 2).
\end{aligned}$$

We used the identity $T_n(\frac{1}{2}(z + z^{-1})) = \frac{1}{2}(z^n + z^{-n})$. Here $w_p = \frac{1}{2}(z_p + \frac{1}{z_p})$, $p = 1, \dots, s_k$. These numbers are the roots of algebraic equation $\sum_{j=1}^{k}(2T_{s_j}(w) - 2) = 1$.

To finish the proof of the theorem we use Lemma 3.2 and take absolute value of the righthand side of the Equation 3.1. □

4 The number of rooted spanning forests in circulant graphs of odd valency

This section is devoted to investigation of the numbers of rooted spanning forests in circulant graph $C_{2n}(s_1, s_2, \dots, s_k, n)$ in terms of Chebyshev polynomials.

Theorem 4.1. *Let $C_{2n}(s_1, s_2, \dots, s_k, n)$, $1 \le s_1 < s_2 < \dots < s_k < n$, be a circulant graph of odd degree. Then the number $f_G(n)$ of rooted spanning forests in the graph $G = C_{2n}(s_1, s_2, \dots, s_k, n)$ is given by the formula*

$$f_G(n) = \prod_{p=1}^{s_k} (2T_n(u_p) - 2)(2T_n(v_p) + 2),$$

where the numbers u_p and v_p, $p = 1, 2, \dots, s_k$ are respectively the roots of the algebraic equations $Q(u) - 1 = 0$ and $Q(v) + 1 = 0$, $Q(w) = 2k + 2 - 2\sum_{i=1}^{k} T_{s_i}(w)$ and $T_s(w)$ is the Chebyshev polynomial of the first kind.

Proof. In order to find the number of rooted spanning forests $f_G(n)$ in the graph G we need to find the determinant $\det(I_{2n} + L(G))$. The matrix $I_{2n} + L(G)$ can be represented in the form

$$I_{2n} + L(G) = (2k + 2)I_{2n} - \sum_{j=1}^{k} (T^{s_j} + T^{-s_j}) - T^n,$$

where T is $2n \times 2n$ circulant matrix circ$(0, 1, 0, \dots, 0)$. The eigenvalues of circulant matrix T are ε_{2n}^j, $j = 0, 1, \dots, 2n - 1$, where $\varepsilon_{2n} = e^{\frac{2\pi i}{2n}}$. Since all of them are distinct, the matrix T is similar to the matrix $\mathbb{T} = \text{diag}(1, \varepsilon_{2n}, \dots, \varepsilon_{2n}^{2n-1})$ with diagonal entries $1, \varepsilon_{2n}, \dots, \varepsilon_{2n}^{2n-1}$. To find the determinant $\det(I_{2n} + L(G))$ we use the product of all eigenvalues of matrix $I_{2n} + L(G)$. The matrix $I_{2n} + L(G)$ is similar to the diagonal matrix with eigenvalues

$$\lambda_j = 2k + 2 - \sum_{l=1}^{k} (\varepsilon_{2n}^{j\, s_l} + \varepsilon_{2n}^{-j\, s_l}) - \varepsilon_{2n}^{jn},\ j = 0, 1, \dots, 2n - 1.$$

All of them are non-zero.

Consider the following Laurent polynomial $P(z) = 2k + 2 - \sum_{i=1}^{k} (z^{s_i} + z^{-s_i})$. Since $\varepsilon_{2n}^n = -1$, we can write $\lambda_j = P(\varepsilon_{2n}^j) - 1$ if j is even and $\lambda_j = P(\varepsilon_{2n}^j) + 1$ if j is odd. By Formula 2.1 we have

$$\begin{aligned} f_G(n) &= \prod_{j=0}^{2n-1} \lambda_j = \prod_{s=0}^{n-1} (P(\varepsilon_{2n}^{2s}) - 1) \prod_{s=0}^{n-1} (P(\varepsilon_{2n}^{2s+1}) + 1) \\ &= \prod_{s=0}^{n-1} (P(\varepsilon_{2n}^{2s}) - 1) \frac{\prod_{p=0}^{2n-1} (P(\varepsilon_{2n}^p) + 1)}{\prod_{s=0}^{n-1} (P(\varepsilon_{2n}^{2s}) + 1)} \\ &= \prod_{s=0}^{n-1} (P(\varepsilon_n^s) - 1) \frac{\prod_{p=0}^{2n-1} (P(\varepsilon_{2n}^p) + 1)}{\prod_{s=0}^{n-1} (P(\varepsilon_n^s) + 1)}. \end{aligned}$$

By making use of Lemma 3.2 and arguments from the proof of Theorem 3.1 we obtain

(i) $\prod_{s=0}^{n-1} (P(\varepsilon_n^s) - 1) = (-1)^{n\,(s_k+1)} \prod_{p=1}^{s_k} (2T_n(u_p) - 2)$,

(ii) $\prod_{s=0}^{n-1} (P(\varepsilon_n^s) + 1) = (-1)^{n\,(s_k+1)} \prod_{p=1}^{s_k} (2T_n(v_p) - 2)$, and

(iii) $\prod_{p=0}^{2n-1} (P(\varepsilon_{2n}^p) + 1) = \prod_{p=1}^{s_k} (2T_{2n}(v_p) - 2)$, where u_p and v_p are the same as in the statement of the theorem. Hence,

$$f_G(n) = \prod_{p=1}^{s_k}(2T_n(u_p)-2)\prod_{p=1}^{s_k}\frac{T_{2n}(v_p)-1}{T_n(v_p)-1}.$$

Finally, taking into account the identity $T_{2n}(w)-1 = 2(T_n(w)-1)(T_n(w)+1)$ we obtain

$$f_G(n) = \prod_{p=1}^{s_k}(2T_n(u_p)-2)(2T_n(v_p)+2). \qquad \square$$

5 Arithmetic properties of the number of rooted spanning forests for circulant graphs

It has been proved in the paper [23] that the number of spanning trees $\tau(n)$ in circulant graph $C_n(s_1, s_2, \ldots, s_k)$ is given by the formula $\tau(n) = p\,n\,a(n)^2$, where $a(n)$ is an integer sequence and p is a natural number depending only on the parity of n. The aim of the next theorem is to find a similar property for the number of rooted spanning forests.

Recall that any positive integer p can be uniquely represented in the form $p = q\,r^2$, where p and q are positive integers and q is square-free. We will call q the *square-free part* of p.

Theorem 5.1. *Let $f_G(n)$ be the number of rooted spanning forests in the circulant graph*

$$C_n(s_1, s_2, \ldots, s_k),\ 1 \le s_1 < s_2 < \ldots < s_k < \frac{n}{2}.$$

Denote by p the number of odd elements in the sequence $s_1, s_2, \ldots, s_k$ and let q be the square-free part of $4p+1$. Then there exists an integer sequence $a(n)$ such that

(1) $f_G(n) = a(n)^2$, *if n is odd;*

(2) $f_G(n) = q\,a(n)^2$, *if n is even.*

Proof. The number of odd elements in the sequence $s_1, s_2, \ldots, s_k$ is counted by the formula $p = \sum_{i=1}^{k}\frac{1-(-1)^{s_i}}{2}$.

We already know that all eigenvalues of the $I_n + L(G)$ are given by the formulas $\lambda_j = P(\varepsilon_n^j),\ j = 0, \ldots, n-1$, where $P(z) = 2k+1-\sum_{i=1}^{k}(z^{s_i}+z^{-s_i})$ and $\varepsilon_n = e^{\frac{2\pi \mathrm{i}}{n}}$. We note that $\lambda_{n-j} = P(\varepsilon_n^{n-j}) = P(\varepsilon_n^j) = \lambda_j$.

Since $\lambda_0 = P(\varepsilon_n^0) = P(1) = 1$ by Formula 2.1 we have $f_G(n) = \prod_{j=1}^{n-1}\lambda_j$. Since $\lambda_{n-j} = \lambda_j$, we obtain $f_G(n) = (\prod_{j=1}^{\frac{n-1}{2}}\lambda_j)^2$ if n is odd and $f_G(n) = \lambda_{\frac{n}{2}}(\prod_{j=1}^{\frac{n}{2}-1}\lambda_j)^2$ if n is even. We note that each algebraic number λ_j comes with all its Galois conjugates [20]. So, the numbers $b(n) = \prod_{j=1}^{\frac{n-1}{2}}\lambda_j$ and $c(n) = \prod_{j=1}^{\frac{n}{2}-1}\lambda_j$ are integers. Also, for even n we have $\lambda_{\frac{n}{2}} = 2k+1-\sum_{i=1}^{k}((-1)^{s_i}+(-1)^{-s_i}) = 1+2\sum_{i=1}^{k}(1-(-1)^{s_i}) = 4p+1$. Hence, $f_G(n) = b(n)^2$ if n is odd and $f_G(n) = (4p+1)\,c(n)^2$ if n is even. Let q be the square-free part of $4p+1$ and $4p+1 = q\,r^2$. Setting $a(n) = b(n)$ in the first case and $a(n) = r\,c(n)$ in the second, we conclude that number $a(n)$ is always integer which completes the proof. $\square$

The following theorem clarifies some number-theoretical properties of the number of rooted spanning forests $f_G(n)$ for circulant graphs of odd valency.

Theorem 5.2. *Let $f_G(n)$ be the number of rooted spanning forests in the circulant graph*

$$G = C_{2n}(s_1, s_2, \ldots, s_k, n),\ 1 \le s_1 < s_2 < \ldots < s_k < n.$$

Denote by p the number of odd elements in the sequence $s_1, s_2, \ldots, s_k$. Let q be the square-free part of $4p+1$ and r be the square-free part of $4p+3$. Then there exists an integer sequence $a(n)$ such that

(1) $f_G(n) = q\,a(n)^2$, *if n is even;*

(2) $f_G(n) = r\,a(n)^2$, *if n is odd.*

Proof. The number p of odd elements in the sequence $s_1, s_2, \ldots, s_k$ is equal to $\sum_{i=1}^{k} \frac{1-(-1)^{s_i}}{2}$. The eigenvalues of the matrix $I_{2n} + L(G)$ are given by the formulas

$$\lambda_j = P(\varepsilon_{2n}^j) - (-1)^j,\ j = 0, 1, \ldots, 2n-1,$$

where $P(z) = 2k + 2 - \sum_{l=1}^{k}(z^{s_l} + z^{-s_l})$ and $\varepsilon_{2n} = e^{\frac{\pi \mathrm{i}}{n}}$.

Since $\lambda_0 = P(1) - 1 = 1$ by the Formula 2.1 we have $f_G(n) = \prod_{j=1}^{2n-1} \lambda_j$. Since $\lambda_{2n-j} = \lambda_j$, we obtain $f_G(n) = \lambda_n (\prod_{j=1}^{n-1} \lambda_j)^2$, where $\lambda_n = P(-1) - (-1)^n$. Now we have

$$\lambda_n = 2k + 2 - (-1)^n - 2\sum_{l=1}^{k}(-1)^{s_l} = 2 - (-1)^n + 4\sum_{l=1}^{k}\frac{1-(-1)^{s_l}}{2} = 4\,p + 2 - (-1)^n.$$

So, $\lambda_n = 4\,p + 1$, if n is even and $\lambda_n = 4\,p + 3$, if n is odd. We note that each algebraic number λ_j comes into the product $\prod_{j=1}^{n-1} \lambda_j$ together with all its Galois conjugates, so the number $c(n) = \prod_{j=1}^{n-1} \lambda_j$ is an integer [20].

Hence, $f_G(n) = (4\,p+1)c(n)^2$, if n is even and $f_G(n) = (4\,p+3)\,c(n)^2$, if n is odd. Let q and r be the square-free parts of $4\,p+1$ and of $4p+3$ respectively. Then for some integers x and y we have $4\,p+1 = q\,x^2$ and $4\,p+3 = r\,y^2$.

Now, the integer number $f_G(n)$ can be represented in the form

(1) $f_G(n) = q\,(x\,c(n))^2$ if n is even and

(2) $f_G(n) = r\,(y\,c(n))^2$ if n is odd.

Setting $a(n) = x\,c(n)$ in the first case and $a(n) = y\,c(n)$ in the second, we conclude that number $a(n)$ is always integer. The theorem is proved. □

6 Asymptotics for the number of spanning forests

In this section we give asymptotic formulas for the number of rooted spanning forests in circulant graphs.

Theorem 6.1. *The number of rooted spanning forests in the circulant graph*

$$G = C_n(s_1, s_2, \ldots, s_k),\ 1 \le s_1 < s_2 < \ldots < s_k < \frac{n}{2}$$

has the following asymptotics

$$f_G(n) \sim A^n, \text{ as } n \to \infty$$

where $A = \exp(\int_0^1 \log(P(e^{2\pi \mathrm{i} t}))dt)$ *is the Mahler measure of Laurent polynomial* $P(z) = 2k + 1 - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i})$.

Proof. By Theorem 3.1, the number of rooted spanning forests $f_G(n)$ is given by

$$f_G(n) = \prod_{s=1}^{s_k} |2\,T_n(w_s) - 2|,$$

where $w_s = (z_s + z_s^{-1})/2$. We have $T_n(w_s) = \frac{1}{2}(z_s^n + z_s^{-n})$, where z_s and $1/z_s$, $s = 1, \ldots, s_k$ are all the roots of the polynomial $P(z)$. If $\varphi \in \mathbb{R}$ then $P(e^{\mathrm{i}\varphi}) = 2k + 1 - \sum_{j=1}^{k}(e^{s_j \mathrm{i}\varphi} + e^{-s_j \mathrm{i}\varphi}) = 2k + 1 - 2\sum_{j=1}^{k}\cos(s_j\varphi) \geq 1$, so $|z_s| \neq 1$ for all s. Replacing z_s by $1/z_s$, if it is necessary, we can assume that $|z_s| > 1$ for all s. Then $T_n(w_s) \sim \frac{1}{2}z_s^n$, as n tends to ∞. So, $|2T_n(w_s) - 2| \sim |z_s|^n,\ n \to \infty$. Hence

$$\prod_{s=1}^{s_k} |2\,T_n(w_s) - 2| \sim \prod_{s=1}^{s_k} |z_s|^n = \prod_{P(z)=0,\,|z|>1} |z|^n = A^n,$$

where $A = \prod_{P(z)=0,\,|z|>1} |z|$ is the Mahler measure of $P(z)$. By the results mentioned in the preliminary part, it can be found by the formula $A = \exp(\int_0^1 \log(P(e^{2\pi \mathrm{i} t}))dt)$.

Finally,

$$f_G(n) = \prod_{s=1}^{s_k} |2\,T_n(w_s) - 2| \sim A^n,\ n \to \infty. \qquad \square$$

The next theorem is a direct consequence of Theorem 4.1 and can be proved by the same arguments as Theorem 6.1.

Theorem 6.2. *The number of rooted spanning forests* $f_G(n)$ *in the circulant graph* $G = C_{2n}(s_1, s_2, \ldots, s_k, n)$, $1 \leq s_1 < s_2 < \ldots < s_k < n$ *has the following asymptotics*

$$f_G(n) \sim K^n, \text{ as } n \to \infty.$$

Here $K = \exp(\int_0^1 \log|P(e^{2\pi \mathrm{i} t})^2 - 1|dt)$ *is the Mahler measure of the Laurent polynomial* $P(z)^2 - 1$, *where* $P(z) = 2k + 2 - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i})$.

7 Examples

(1) **Cycle graph** $G = C_n(1) = C_n$.

We need to solve the equation $1 + 2 - 2T_1(w) = 0$. We have $w = 3/2$. So, $f_G(n) = 2T_n(3/2) - 2$. Then $f_G(n) \underset{n\to\infty}{\sim} (\frac{3+\sqrt{5}}{2})^n$. Also, we have $f_G(n) = 5F_n^2$, if n is even, and $f_G(n) = L_n^2$, if n is odd, where F_n and L_n are the Fibonacci and Lucas numbers respectively. The latter result was independently obtained in [6].

(2) **Graph** $G = C_n(1, 2)$.

We need to solve the equation $1 + 4 - 2T_1(w) - 2T_2(w) = 0$. Its roots are $w_1 = \frac{1}{4}(-1 + \sqrt{29})$ and $w_2 = \frac{1}{4}(-1 - \sqrt{29})$.

By Theorem 5.1, there exists an integer sequence $a(n)$ such that $f_G(n) = 5a(n)^2$, if n is even, and $f_G(n) = a(n)^2$, if n is odd.

(3) **Graph** $G = C_n(1, 3)$.

Let w_1, w_2 and w_3 be the roots of the cubic equation $1+4-2T_1(w)-2T_3(w) = 0$. Then $f_G(n) = (2T_n(w_1) - 2)(2T_n(w_2) - 2)(2T_n(w_3) - 2)$. In this case, $f_G(n) \sim A_{1,3}^n$, $n \to \infty$, where $A_{1,3} \approx 4.48461\ldots$ is the Mahler measure of the Laurent polynomial $5 - z - z^{-1} - z^3 - z^{-3}$. One can check that $A_{1,3}$ is a root of the equation $1 - x - 2x^2 - 4x^3 + x^4 = 0$. By Theorem 5.1, we have $f_G(n) = a(n)^2$, where $a(n)$ is an integer sequence.

(4) **Graph Möbius ladder** $G = C_{2n}(1, n)$.

We have to solve the equations $3 - 2T_1(w) = 0$ and $5 - 2T_1(w) = 0$. Their roots are $u_1 = 3/2$ and $v_1 = 5/2$ respectively. Then $f_G(n) = (2T_n(3/2) - 2)(2T_n(5/2) + 2) \sim K^n$, where $K = \frac{1}{4}(3 + \sqrt{5})(5 + \sqrt{21}) \approx 12.5438\ldots$. By Theorem 5.2, $f_G(n) = 5a(n)^2$, if n is even, and $f_G(n) = 7a(n)^2$, if n is odd for some integer sequence $a(n)$.

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