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On local operations that preserve symmetries and on preserving polyhedrality of maps

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Abstract

We prove that local operations that preserve all symmetries, as e.g. dual, truncation, medial, or join, as well as local operations that are only guaranteed to preserve all orientation-preserving symmetries, as e.g. gyro or snub, preserve the polyhedrality of simple maps. This generalizes a result by Mohar proving this for the operation dual. We give the proof based on an abstract characterization of these operations, prove that the operations are well defined, and also demonstrate the close connection between these operations and Delaney-Dress symbols.

Keywords: Embedded graph, map, polyhedral embedding, operation, symmetry, tiling.

Math. Subj. Class. (2020): 05C76, 05C10, 05C40, 52B05

1 Introduction

Symmetry-preserving operations on polyhedra have been studied for a very long time. They were first applied in ancient Greece. Some of the Archimedean solids can be obtained from Platonic solids by applying the operation which was later called truncation by Kepler. Over the centuries, polyhedra and specific operations on them have been studied extensively [3, 11, 12, 18, 22]. However, a general definition of the concept *local symmetry-preserving operation* and a systematic way of describing such operations was only presented in 2017 [2]. This description covers a large class of operations on maps, including all well-known symmetry-preserving operations such as truncation, dual, or those operations known as achiral Goldberg-Coxeter operations [4, 5]. Goldberg-Coxeter operations were in fact introduced by Caspar and Klug [4] and can be used to construct all fullerenes or certain viruses with icosahedral symmetry.

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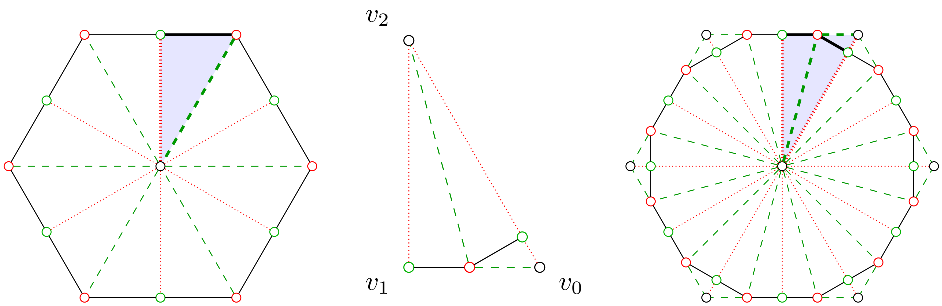


Figure 1: On the left, the barycentric subdivision of a hexagonal face is shown. In the middle, the lsp-operation truncation is given and on the right the barycentric subdivision of the result of applying the operation. The blue shaded area shows one chamber of the original hexagon.

In addition to these local symmetry-preserving operations (lsp-operations), which preserve all the symmetries of a map, there are also operations that are only guaranteed to preserve the orientation-preserving symmetries. Well-known examples of such operations are snub and gyro [23], or the chiral Goldberg-Coxeter operations. In [2], a general description of such *local orientation-preserving symmetry-preserving operations* (lopsp-operations) was also presented. The very general way of describing lsp- and lopsp-operations in [2] allows to tackle various problems from a more abstract perspective, and also allows to prove general theorems about the whole class of operations instead of considering each operation separately. In this paper we will use the new description to prove that all those operations (e.g. dual, medial, truncation, snub, ...) preserve polyhedrality of maps i.e., if an lsp- or lopsp-operation is applied to a simple 3-connected map of face-width at least 3, then the result is also simple and 3-connected and it has face-width at least 3.

As the description in [2] was aimed at a broader audience than just mathematicians, the approach was described in a more intuitive way. In that article an operation is defined as a triangle 'cut' out of a simple periodic 3-connected tiling, and it is applied by gluing copies of that triangle into the barycentric subdivision of a map. Another way of looking at it is that the faces of the barycentric subdivision, which are triangles, are further subdivided into smaller triangles. This is done in a way that the subdivisions of the faces of the barycentric subdivision are identical or mirror images of each other, or – in case only orientation preserving symmetries must be preserved – in a way that each pair of two triangular faces of the subdivision that share the same edge as well as the same face of the map is subdivided in the same way. In the remainder of this text we will give the conditions for these subdivisions that guarantee that the result is the barycentric subdivision of another map – the result of the operation. An example of an lsp-operation and its application is shown in Figure 1. In this article, we will give the more direct definition based on Delaney-Dress symbols that forms the base of this approach and show the connection to the original description. We will also show that for every lsp-operation there is an equivalent lopsp-operation, i.e. a lopsp-operation that has the same result as the lsp-operation when applied to a map.

In [2], it is proved that the result of applying an lsp-operation to a polyhedron – that is: a simple 3-connected map embedded in the plane [17] – is also a polyhedron. In [14] this result is also announced for all lopsp-operations. We will modify some concepts that are used in that paper, but due to some serious problems in that paper we will not use the results given there.

Originally, lsp- as well as lopsp-operations were only defined for simple plane maps because of their origin in the study of polyhedra. However, there is no mathematical reason why these definitions should not be applied to maps with multiple edges or loops and embeddings of higher genus. The question then arises in how far we can extend the theorem for 3-connected simple plane maps to 3-connected maps of higher genus.

In general, lopsp-operations do not necessarily preserve 3-connectivity for maps that are not plane. This is obvious for maps with faces of size 1 or 2, but it is also true for simple maps in general, even if we require the result to be simple. The most striking example of a local symmetry-preserving operation that can turn 3-connected maps into (even simple) maps with lower connectivity is *dual*. In [1] it is proven that for any $k \geq 1$, there exist embeddings of k-connected simple maps M so that the dual M^* is simple and has a 1-cut.

However, even dual always preserves 3-connectivity in simple maps of face-width at least three, as proven in [20]. In Definition 4.1 and Definition 4.8 we will define $\mathbf{c}k$-maps and $\mathbf{c}k$-operations. A map is $\mathbf{c}k$ if it is k-connected, it has face-width at least k, and all of its faces have size at least k. In this paper we will prove the general Theorem 4.9 from which the following key result is a corollary. The result in [20] for dual is a special case of this result. The map $O(M)$ is the result of applying the operation O to the map M:

Corollary 1.1. *Let $k \in \{1, 2, 3\}$. If M is a $\mathbf{c}k$-map, and O is a $\mathbf{c}k$-lsp- or $\mathbf{c}k$-lopsp-operation, then $O(M)$ is also $\mathbf{c}k$.*

This theorem is most interesting and relevant for $k = 3$. This has two reasons. Firstly, the set of $\mathbf{c}3$-operations contains all well-known and intensely studied operations. Lsp-operations that are not $\mathbf{c}3$-lsp-operations were not even included in the original definition of lsp-operations [2]. Secondly, $\mathbf{c}3$-maps, which are in fact simple embedded 3-connected maps of face-width at least three, have some very interesting properties. These maps are also known as *polyhedral maps* or polyhedral embeddings [20]. They can be defined equivalently as simple maps where every facial walk is a simple cycle and any two faces are either disjoint or their intersection consists of only one vertex or one edge. As the name suggests, polyhedral maps are a generalisation of polyhedra to surfaces of higher genus. It turns out that the key property that these operations preserve is not 3-connectivity but polyhedrality. This property is equivalent to being simple and 3-connected in the plane, but only in the plane. The main result of this article follows immediately from Corollary 1.1: If M is a polyhedral map and O is a $\mathbf{c}3$-lsp- or $\mathbf{c}3$-lopsp-operation, then $O(M)$ is also a polyhedral map (Theorem 4.10).

In Section 2 we give the definitions of the terminology we will use in this text. It starts with some basic concepts and then the definitions of lsp- and lopsp-operations are given. There is some freedom in the way that lopsp-operations are applied. However, in Section 3 we will prove that the result of applying a lopsp-operation is independent of the choices that are made in its application. Section 4 holds the main results of this paper: We prove a general result that implies that all lopsp-operations preserve polyhedrality of maps. To show that the definition of lopsp-operations we give is equivalent to the original definition in [2], we explore the strong connection between lsp- and lopsp-operations and tilings in Section 5.

2 Definitions

There are many different, often equivalent, definitions of a map. A short description is that a *map* is a cellular decomposition of a surface into vertices (0-cells), edges (1-cells), and faces (2-cells). Perhaps more intuitively, a map is an embedding of a topological representation of a graph G onto a surface S. In this text we will only consider 2-cell embeddings, which means that all the connected components of $S \setminus G$ are homeomorphic to 2-dimensional disks. We will only consider *oriented* surfaces. What we will refer to as map is often called an oriented map in texts where more general maps are also studied. Maps are often studied from a topological point of view. To make some technical details easier to describe rigorously, we will use the combinatorial approach that is given below. This definition is equivalent to the topological ones [16, 21]. We will define a map as a graph together with a rotation system, which for every vertex imposes a cyclic rotational order on the edges incident to that vertex.

A *graph* is a tuple (V, E) where V is the set of *vertices* and E is the set of *edges*. Every edge is incident to two vertices that are not necessarily different. If they are the same vertex, then that edge is a *loop*. Though we are mainly interested in graphs without loops or multiple edges, they will occur in a natural way — e.g. as tools or as the result of an operation — so that we will in general assume that the underlying graph of a map may have multiple edges and loops and explicitly restrict the class where necessary.

With every edge of a graph G, we associate two oriented edges, each starting in one vertex of the edge and ending in the other. In the literature these are also called directed edges or darts. If e is one oriented edge, then e^{-1} is the other oriented edge associated with the same edge of G. For every vertex v of G, a cyclic order is assigned to all oriented edges starting at v. This way, every oriented edge e has a 'successor' $\sigma(e)$. A *map* – also known as embedded graph or graph embedding – is a connected graph together with such a successor function σ. In a more general context, our maps could be referred to as oriented maps. As we will not consider unoriented maps we just use the term map. The vertices and edges of a map are the vertices and edges of the underlying graph. When drawing maps, the cyclic order around the vertices induced by σ corresponds to the clockwise order of edges around that vertex in the drawing.

A map is *simple* if it has no loops and no multiple edges that are incident with the same 2 vertices.

A map is *k-connected* if it has at least $k + 1$ vertices and it has no vertex-cut of fewer than k vertices.

Consider three oriented edges e_1, e_2, and e_3 incident with a vertex v. We say that e_2 is *between* e_1 and e_3 if e_1, e_2, e_3 occur in this order in the cyclic order around v, i.e. the cyclic order of edges around v is of the form $(\dots e_1 \dots e_2 \dots e_3 \dots)$ and not $(\dots e_3 \dots e_2 \dots e_1 \dots)$.

We say that e and $\sigma(e^{-1})$ form an *angle* in the map. A *face* of a map M is a cyclic sequence of oriented edges such that every two consecutive edges form an angle. We will use the term *facial walk* to refer to the closed walk in M corresponding to this cyclic sequence of oriented edges. This definition of face corresponds to the topological notion of a face.

For a map M, with V_M, E_M, and F_M denoting the sets of vertices, edges, and faces of M respectively, $\chi(M) = |V_M| - |E_M| + |F_M|$ is the *Euler characteristic* of M. The *genus* of M is defined as $gen(M) = \frac{2-\chi(M)}{2}$. If a map has genus 0, it is called *plane*. Note that a plane map is not the same as a planar graph. A planar graph is a graph (not embedded) that

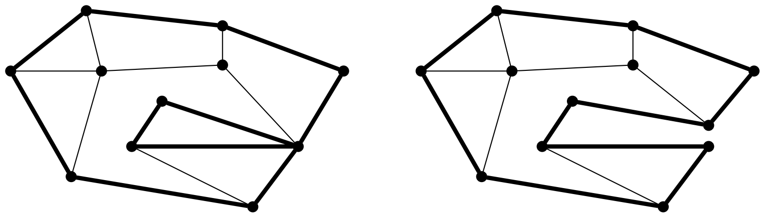


Figure 2: The left figure shows a map and one of its submaps with bold edges.The right figure shows the internal component of the only face of the submap that has bridges.

can be embedded such that it has genus 0. A plane map is one specific genus 0 embedding of a graph.

Let M be a map and G' a subgraph of the underlying graph of M. The map M' which is the graph G' with the embedding induced by that of M is called a *submap* of M.

To formalize when a vertex or edge is 'in' a face of one of its submaps we will now define what a *bridge* for a submap M' of M is. There are two kinds of bridges:

- If $e \in E_M \setminus E_{M'}$ is an edge with endpoints $v, w \in V_{M'}$, then the submap with vertex set $\{v, w\}$ and edge set $\{e\}$ is a bridge.
- Let C be a component of the submap of M induced by the vertices of M that are not in M', and define $E'_C = \{e \in E_M \mid e \cap V_C \neq \emptyset\}$ and $V'_C = \{v \in V_M \mid \exists e \in E'_C : v \in e\}$. Then the submap with vertex set V'_C and edge set E'_C is a bridge.

If a bridge has an edge that is between two edges e and e' so that e^{-1} and e' form an angle in a face of M', then the bridge is *in that face*. All the vertices and edges of the bridge are also said to be *in the face*. The *boundary* ∂f of a face f is the submap of M consisting of all the vertices and edges in the facial walk of f. A vertex or edge of M is in the *interior* of a face of M' if it is in that face and it is not in the boundary.

If a bridge is in more than one face, we say that those faces are *bridged*. A face that is not bridged is called *simple*.

Let f be a simple face of M'. We will define the *internal component* of f as follows. Start with the submap N of M that consists of the boundary of f together with all bridges in f. Intuitively, we cut along the boundary of f in N in such a way that the facial walk becomes a simple cycle. More formally, we replace every vertex v of N that appears $k > 1$ times in the facial walk of f by k pairwise different vertices $v_1, ..., v_k$. If both oriented edges associated with an edge of M' appear in the facial walk, this edge is also split into two different edges between different copies of its vertices. Let (x, v) and (v, y) be the oriented edges that form the angle in M' at the i-th occurrence of v. Then we define the rotational order (and also the neighbours) of v_i to be the same as the rotational order around v in M, but restricted to the edges between (v, x) and (v, y). Of course some vertices may be replaced by their copies. The result of this is the internal component $IC(f)$ of f. An example of an internal component is illustrated in Figure 2. If $IC(f)$ is plane, we call f *internally plane*.

An important concept in the definition of lsp- and lopsp-operations is the barycentric subdivision of a map. It is obtained by subdividing every face into triangular faces, which we will call chambers. We will also use the barycentric subdivision to define contractible cycles and face-width in a combinatorial way.

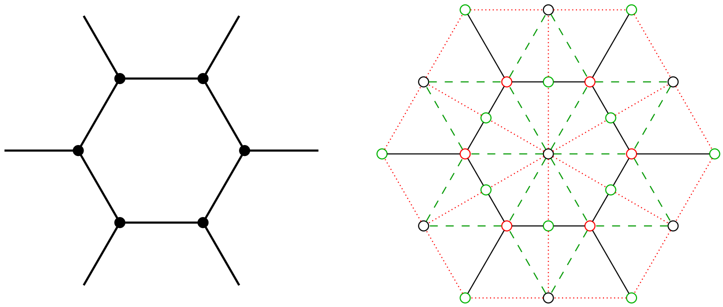


Figure 3: A face in a map M and the corresponding part of B_M. Edges of colour 1 are dashed and edges of colour 2 are dotted.

The *barycentric subdivision* B_M of a map M is a map that has a unique vertex for every vertex, for every edge and for every face of M. We always assume that B_M comes with the natural vertex-colouring that assigns colours 0, 1, and 2 to vertices that correspond to vertices, edges, and faces of M respectively. These colours correspond to their topological dimension. There are edges between vertices of colour 0 and 1 if the corresponding vertex and edge are incident. There are edges between vertices of colour 0 or 1 and colour 2 if the corresponding vertex or edge appears in the boundary of the corresponding face. There are no edges between vertices of the same colour. For $i \in \{0, 1, 2\}$, an edge is *of colour* i if it is not incident with a vertex of colour i. We will also refer to vertices and edges of colour i as i-vertices and i-edges. The rotational order of the edges adjacent to a vertex of colour 2 follows the order of the vertices and edges in the corresponding facial walk of M, and similarly for vertices of colour 0 and 1. This is illustrated in Figure 3. Every face of B_M is a triangle. Note that in every figure in this text, colours are represented by colours in the order rgb, that is: a red is colour 0, green is colour 1, and black is colour 2. The edges of colour 1 are dashed and the edges of colour 2 are dotted, so that when looking at the figures printed in black and white it should still be clear which edges have which colour. With this rotation system, a short calculation of the Euler characteristic shows that $gen(B_M) = gen(M)$. If x is a face, edge or vertex of M, then to keep notation simple we will also write x for the corresponding vertex of B_M.

Every face of B_M is a triangle, with exactly one vertex and one edge of each colour. We call such a triangle a *chamber*. Two chambers are *adjacent* if they share an edge. In the literature, chambers are also called *flags*. The *flag graph* of M is the dual of B_M, i.e. it is the 3-regular graph that has the chambers as its vertices, and there is an edge between two vertices if their corresponding chambers are adjacent. In some papers flags are defined as triples (v, e, f) where v, e, and f are respectively a vertex, an edge, and a face such that v is a vertex of e and v and e are in face f [7, 19]. We cannot use that approach here because with our general definition of a map there is no 1-to-1 correspondence between chambers and triples (v, e, f). For example, an edge can have the same face on both sides so that there are multiple chambers with the same vertices.

Lemma 2.1. *A map M, vertex-coloured with colours* 0,1, *and* 2 *is the barycentric subdivision of another map if and only if:*

(i) *Every face of M is a triangle.*

(ii) *There are no edges between vertices of the same colour.*

(iii) *Every vertex of colour* 1 *has degree* 4*.*

Proof. Let V_G and E_G be the sets of vertices of M with colours 0 and 1 respectively. Conditions (i) and (ii) imply that every face has exactly one vertex of each colour. It now follows from (iii) that a vertex $e \in E_G$ of colour 1 has two neighbours in V_G and two neighbours f and g of colour 2. This induces an incidence relation on the vertex set V_G and the edge set E_G that defines a graph G. The rotation system of M induces a rotation system on G. Let N be the map that consists of G with this rotation system. It is not difficult to check that $M = B_N$. □

The *double chamber map* D_M of a map M is the submap of B_M that only contains the edges of colours 1 and 2. A *double chamber* of a map M is a face in D_M. Every double chamber has length four: two (in case of no loops different) vertices of colour 0, one of colour 1, and one of colour 2. Two double chambers are *adjacent* if they share a 1-edge or two 2-edges.

In [2], lsp- and lopsp-operations are – following Goldberg [15] – defined in a geometric way as triangles 'cut' out of the barycentric subdivision of a 3-connected tiling of the plane, such that in case of lsp-operations the sides of the triangle are on symmetry axes of the tiling. In this article we give purely combinatorial definitions of lsp- and lopsp-operations, similar to [14] and [13]. The definitions given here are equivalent to those in [2] when restricted to what we will later call **c**3-operations. The equivalence can be seen by applying operations as defined here to some special periodic tiling, but readers who want to see the equivalence already before starting on the main results of this paper and who want to have a deeper insight into the relation of operations and periodic tilings encoded by Delaney-Dress symbols, can find a direct proof without applications of the operations in Section 5.

Definition 2.2. Let O be a 2-connected plane map with vertex set V, together with a colouring $c\colon V \to \{0,1,2\}$. One of the faces is called the outer face. This face contains three special vertices marked as v_0, v_1, and v_2. We say that a vertex v has *colour* i if $c(v) = i$. This 3-coloured map O is a *local symmetry preserving operation*, lsp-operation for short, if the following properties hold:

(1) Every inner face — i.e. every face that is not the outer face — is a triangle.

(2) There are no edges between vertices of the same colour, i.e. the colouring is proper.

(3) For each vertex that is not in the outer face:

$$c(v) = 1 \Rightarrow deg(v) = 4$$

For each vertex v in the outer face, different from v_0, v_1, and v_2:

$$c(v) = 1 \Rightarrow deg(v) = 3$$

and

$$c(v_0), c(v_2) \neq 1$$

$$c(v_1) = 1 \Rightarrow deg(v_1) = 2$$

An example of an lsp-operation is shown in the middle of Figure 1.

Just like for barycentric subdivisions we say that an edge is of *colour* i if it is not incident to a vertex of colour i. This is well-defined because of the second property.

Every inner face has exactly one vertex and one edge of each colour. We will refer to these triangular faces as *chambers*.

In the original paper [2] only operations that preserve 3-connectivity of polyhedra were discussed, so the result of the operation also had to have only vertices of degree at least 3. In [13] operations were also discussed that produce maps with 1- or 2-cuts, but the restriction that vertices in the result should have degree at least 3 was kept. Our definition of lsp-operations is even more general. With this definition, the result of applying an lsp-operation may have vertices of degree 1 or 2.

Application of an lsp-operation:

Let O be an lsp-operation and let M be a map. The operation is applied to M by first replacing for $i \in \{0, 1, 2\}$ the i-edges of B_M by copies of the part of the boundary of the outer face of O between v_j and v_k with $i \neq j, k$. The copy of v_j is identified with the j-vertex and the copy of v_k with the k-vertex. Then — depending on the orientation — either a copy of O or a copy of the mirror image of O — which has the same underlying graph as O but the rotation system is the inverse of that of O — is glued into every face of the modified B_M. Note that chambers of B_M sharing an edge have different orientations. The boundary vertices are identified with their copies. This results in a 3-coloured triangulation. An example of the gluing — restricted to a single face — is given in Figure 1. With Lemma 2.1 and Definition 2.2 it follows that this triangulation is the barycentric subdivision of a map $O(M)$, the *result* of applying O to M.

As any symmetry group acts on the chamber system, lsp-operations preserve all the symmetries of a map. New symmetries can also occur. However, all known examples of 3-connected maps where lsp-operations can increase symmetry are maps of genus at least 1 or they are self-dual. It is an open question whether lsp-operations can increase symmetry in plane 3-connected maps (polyhedra) that are not self-dual.

There are also interesting operations such as gyro and snub that are only guaranteed to preserve the orientation-preserving symmetries of maps. These cannot be described by lsp-operations. In the supplementary material of [2] and in [14], local orientation-preserving symmetry-preserving operations (lopsp-operations) are defined similarly to lsp-operations. The most important difference is that here the decoration is glued into double chambers instead of chambers. As with lsp-operations, we will give a more explicit definition of lopsp-operations that is not directly based on tilings.

There are some problems that arise in the original definition of lopsp-operations that do not appear for lsp-operations. With the original definition, it is possible to cut different patches out of a tiling that describe the same operation and must be shown to have the same result. That is why we define a lopsp-operation as a plane triangulation, similar to [14], and not as a quadrangle that we can glue directly into double chambers. Although this simplifies the definition of a lopsp-operation, the same problem comes back when it is described how the operation is applied.

Definition 2.3. Let O be a 2-connected plane map with vertex set V, together with a colouring $c\colon V \to \{0, 1, 2\}$ and three special vertices marked as v_0, v_1, and v_2. We say

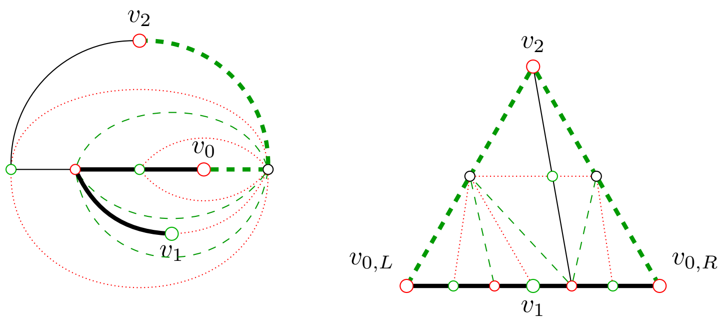


Figure 4: On the left, the lopsp-operation gyro is shown. The thick edges are the edges of the path P. On the right the corresponding double chamber patch O_P is drawn.

that a vertex is of *colour* i if $c(v) = i$. The map O is a *local orientation-preserving symmetry-preserving operation*, lopsp-operation for short, if the following properties hold:

(1) Every face is a triangle.

(2) There are no edges between vertices of the same colour, i.e. the colouring is proper.

(3) For each vertex v different from v_0, v_1, and v_2:

$$c(v) = 1 \Rightarrow deg(v) = 4$$

and

$$c(v_0), c(v_2) \neq 1$$

$$c(v_1) = 1 \Rightarrow deg(v_1) = 2$$

Again we say that an edge has *colour* i if it is not incident to a vertex of colour i and this is well-defined because of the second property. Note that the edges incident with a vertex have two different colours, and as every face is a triangle, these colours appear alternatingly in the cyclic order around the vertex. The requirement that O is 2-connected is mentioned in the beginning, but would in fact also follow from the other conditions. Again every face has exactly one vertex and one edge of each colour and will be referred to as a *chamber*. The dual of O will be referred to as the *flag structure* of O.

Application of a lopsp-operation:

For vertices v, v' in a path P we write $P_{v,v'}$ for the subpath of P from v to v'.

As lopsp-operations are 2-connected, due to Menger's theorem there are two paths, one from v_0 to v_1 and one from v_0 to v_2 that have only v_0 in common. These paths together form a longer path P from v_1 to v_2 through v_0. As a submap of O, P has a single face. In this facial walk only v_1 and v_2 occur once and all other vertices of P occur twice. We say that such a path P is a *cut-path* of O. Consider the internal component of the only face of

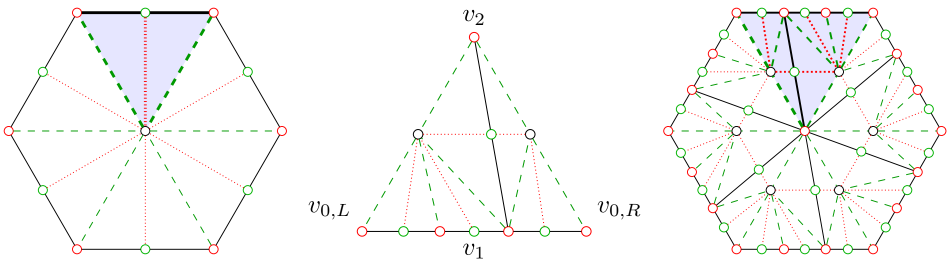


Figure 5: On the left the barycentric subdivision of a hexagonal face is shown. In the middle, a double chamber patch O_P of the operation gyro is drawn, and the right image shows the part of $BO_P(M)$ corresponding to the hexagonal face. The blue shaded area shows one double chamber.

submap P. This is the *double chamber patch* O_P. It can be drawn in the plane, so that the two copies of P form the boundary of the outer face. Figure 4 shows this for the operation gyro. The result of the cutting is a 4-gon with corner vertices v_1, v_2, and two copies of v_0, which we will denote as $v_{0,L}$ and $v_{0,R}$. The *flag structure* of O_P is the flag structure of O where the edges corresponding to edges of P are removed.

The lopsp-operation is now applied by first replacing the edges of a double chamber map D_M to form the map $D_{M,P}$. An edge of colour 2 is replaced by a copy of P_{v_0,v_1} and an edge of colour 1 is replaced by a copy of P_{v_0,v_2} in a way that for $i \in \{0,1,2\}$ a copy of v_i is identified with a vertex of colour i.

Gluing copies of the double chamber patch O_P into the faces of $D_{M,P}$ — identifying corresponding vertices in $D_{M,P}$ and the copies of double chamber patches — gives a coloured map $BO_P(M)$. Note that the orientation inside a double chamber fixes how the different copies of v_0 have to be identified. Unlike with lsp-operations, we do not use mirrored copies of O. Figure 5 gives an example — restricted to one face — of this gluing. A *side* of a double chamber is a path in the boundary of the corresponding face of $D_{M,P}$ that is a copy of the path in O_P between v_2 and $v_{0,L}$, between v_2 and $v_{0,R}$, or between $v_{0,L}$ and $v_{0,R}$. A side is a 1-side if it is between copies of v_0 and v_2 and it is a 2-side if it is between two copies of v_0.

Lemma 2.4. *Let M be a map and let O be a lopsp-operation with a cut-path P. The 3-coloured map $BO_P(M)$ is the barycentric subdivision of a connected map.*

Proof. This follows immediately from Lemma 2.1. □

As lsp-operations preserve all symmetries of a map, they also preserve the orientation-preserving symmetries, so one would expect that for every lsp-operation, there is a lopsp-operation that has the same result when applied to any map. This observation allows to prove some properties of the result of applying lsp- or lopsp-operations only for lopsp-operations. The result for lsp-operations can then be deduced from the corresponding lopsp-operation. Such an equivalent lopsp-operation can be obtained in the following way:

Let O be an lsp-operation, and let c be the boundary of the outer face of O. Let O_{lopsp} be the map obtained by gluing a mirrored copy of the inner face of c into the outer face, identifying the vertices on c with their copies. The vertices v_0, v_1, and v_2 of O are also the vertices v_0, v_1, and v_2 of O_{lopsp}.

Lemma 2.5. *If O is an lsp-operation, then O_{lopsp} is a lopsp-operation, and $O(M) = O_{lopsp}(M)$ for any map M.*

Proof. O_{lopsp} is obviously a triangulation of the disc and there are no edges between vertices with the same colour. As O_{lopsp} consists of two copies of O, glued along the boundary c, we can associate a unique vertex $o(x)$ of O with every vertex x of O_{lopsp}. The degree of x in O_{lopsp} is given by

$$deg(x) = \begin{cases} deg(o(x)) & \text{if } o(x) \text{ is not in } c \\ 2deg(o(x)) - 2 & \text{if } o(x) \text{ is in } c \end{cases}.$$

From the degree restrictions for lsp-operations we can now deduce the degree restrictions in the definition of lopsp-operations for O_{lopsp}. It follows that O_{lopsp} is a lopsp-operation.

Choosing the cut-path in O_{lopsp} that corresponds to the path from v_1 to v_2 through v_0 in c for the application of O_{lopsp} shows immediately that the results of applying O and O_{lopsp} are isomorphic: a double chamber is filled in the same way by O_{lopsp} as two adjacent chambers are filled by O. □

3 The path invariance of lopsp-operations

The cut-path chosen to apply an operation is far from unique, so there are many ways to apply a single lopsp-operation. In this section it is proved that although the ways in which the operation is applied differ, the result of applying a lopsp-operation to a map is independent of the chosen path. An essential tool in proving this are *chamber flips*, which simulate homotopic deformations.

Definition 3.1. Let P be a directed walk in a barycentric subdivision or lopsp-operation. For any two different vertices of a chamber C, there are two different simple paths P_0, P_1 between these vertices in the boundary of C. If for $i \in \{0, 1\}$ path P_i occurs at a certain position in P, then a *chamber flip* of C (at this position) is the operation of replacing P_i by P_{1-i}.

As a first tool we will discuss transformations of one path into another:

Lemma 3.2. *Let P, P' be two directed paths of the form $P = P_s R$, $P' = P_s R'$ from x to y in a lopsp-operation T, so that $R' R^{-1}$ is the facial walk of an internally plane face f in the submap of T consisting of the vertices and edges of P and P'.*

Then there is a sequence of paths $P = P_0, P_1, \ldots, P_k = P'$ so that for $1 \leq i \leq k$ path P_i is obtained from P_{i-1} by a chamber flip and every vertex of P_i is in P_s or in the boundary or the interior of f. As chamber flips can be reversed, the same is true with the role of P and P' interchanged.

Proof. We will prove this by induction on the number $|\mathscr{C}|$ with $\mathscr{C}$ the set of chambers of T inside f. If $|\mathscr{C}| = 1$, then R and R' are the two paths along the boundary of a chamber, so one can be transformed into the other by one chamber flip and we are done. Now assume that $|\mathscr{C}| \geq 2$. We prove that there are at least two chambers in $\mathscr{C}$ that have a connected intersection with ∂f that contains at least one edge: Let $\mathscr{F}_f$ be the dual of T restricted to

$\mathscr{C}$ and without edges that correspond to edges in ∂f. If T is the barycentric subdivision of a map then $\mathscr{F}_f$ is part of the flag graph of that map. There are at least two chambers in $\mathscr{C}$ that contain an edge of ∂f. Assume that there is a chamber C such that $C \cap \partial f$ is disconnected. This chamber C splits the set $\mathscr{C}$ into two parts, i.e. the vertex corresponding to C is a cut-vertex of $\mathscr{F}_f$. In each component of $\mathscr{F}_f \setminus C$ there is at least one chamber that shares an edge with ∂f. Let C_0 be a chamber that contains an edge of ∂f that has the largest distance d_{max} to C along a path in $\mathscr{F}_f$. If this chamber has a disconnected intersection with ∂f, then its corresponding vertex is a cut-vertex of $\mathscr{F}_f$. This implies that there is a chamber that shares an edge with ∂f and has a larger distance to C than d_{max}, which is in contradiction with the maximality of d_{max}. Repeating this argument for the other component of $\mathscr{F}_f \setminus C$, it follows that in each of the two components there is a chamber that has a connected intersection with the facial walk ∂f that contains at least one edge.

Assume that one of these two chambers intersects ∂f in a single edge or in two edges of P or of P'. Then we can do a chamber flip to obtain either a path P_1 or P_{k-1}, so that we can apply induction to P_1, P' or P, P_{k-1} and use that each chamber flip can be undone by a reverse chamber flip.

If the intersection of neither of the two chambers with ∂f is one or two edges of P or P', then both intersections consist of one edge of P and one edge of P'. For one of the chambers, the shared vertex of those edges is the first vertex of R and R'. Applying a chamber flip replacing the edge of P, we get a path P_1 to which we can apply induction. □

Lemma 3.3. *Let Q, Q' be two directed paths from x to y in a lopsp-operation, and z a vertex not contained in either of the paths.*

Then there is a sequence of paths $Q = Q_0, Q_1, \ldots, Q_k = Q'$ from x to y so that for $1 \leq i \leq k$ the path Q_i is obtained from Q_{i-1} by a chamber flip and none of the paths contain z.

Proof. We will prove this by backwards induction on the number n of edges in the beginning of Q that Q and Q' have in common. Remember that for vertices v, v' in a path Q we write $Q_{v,v'}$ for the subpath of Q from v to v'.

If $n = |Q'|$, then $Q = Q'$, so assume that $n < |Q'|$ and that the assumption is true for $n' > n$. Then there is a first vertex a in Q that is incident with an edge that is in Q' but not in Q. Let b be the next vertex after a in Q' that Q' shares with Q. We will show that Q can be transformed to $Q'_{x,a}Q'_{a,b}Q_{b,y}$ in the described way, so that we can apply induction to transform $Q'_{x,a}Q'_{a,b}Q_{b,y}$ into Q'.

Let c be the cycle $Q_{a,b} \cup Q'_{a,b}$. We call the face of c containing z the exterior. Note that neither $Q'_{x,a} = Q_{x,a}$ nor $Q_{b,y}$ intersects c in a vertex other than a or b.

There are four possibilities for the position of $Q_{x,a}$ and $Q_{b,y}$. These are depicted in Figure 6. If $Q_{x,a}$ or $Q_{b,y}$ are in the interior of c, we use them as part of the face boundary when applying Lemma 3.2, otherwise we do not. As Lemma 3.2 already allows to consider paths that start with a common part outside the face, we can choose P, P' from Lemma 3.2 in the following way:

$Q_{b,y}$ outside: Choose $P = Q_{x,a}Q_{a,b}$, $P' = Q'_{x,a}Q'_{a,b}$.
$Q_{b,y}$ inside: Choose $P = Q_{x,a}Q_{a,b}Q_{b,y}$, $P' = Q'_{x,a}Q'_{a,b}Q_{b,y}$.

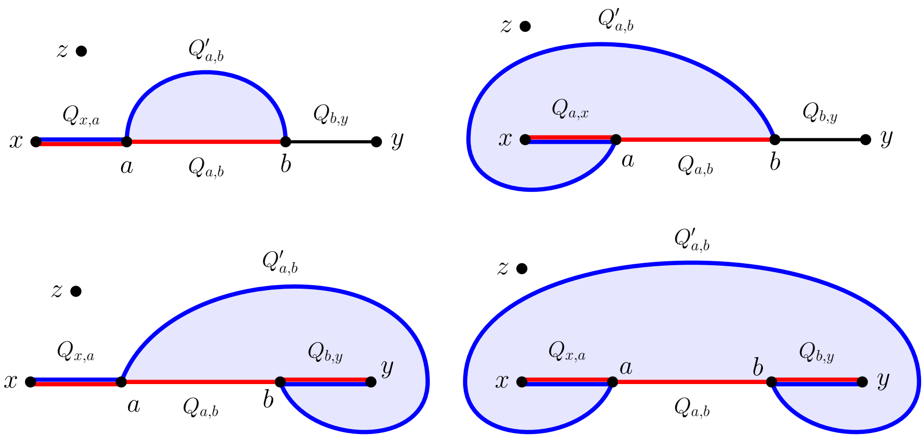


Figure 6: The four different cases in the proof of Lemma 3.3 are shown here. The shaded area represents the interior.

Note that in case $Q_{x,a}$ is outside c it forms the P_s from Lemma 3.2, otherwise P_s consists of a single vertex. In each case Lemma 3.2 can be applied to prove that Q can be transformed to $Q'_{x,a}Q'_{a,b}Q_{b,y}$ in the described way, and as the beginning of $Q'_{x,a}Q'_{a,b}Q_{b,y}$ has more than n edges in common with Q', we can apply reverse induction. □

Let M be a map, O a lopsp-operation with cut-path P and O_P the corresponding double chamber patch. Recall that $BO_P(M)$ is obtained by gluing copies of O_P into D_M. Therefore every vertex v in $BO_P(M)$ is in at least one copy of O_P. If v is in more than one copy, v corresponds to the same vertex of O in each of these copies. Similarly, every edge or face of $BO_P(M)$ also corresponds to exactly one edge or face of O respectively. This allows us to define a surjective mapping π_P, that maps every vertex, edge, and face of $BO_P(M)$ to its corresponding vertex, edge, or face of O.

The mapping π_P is not a bijection, but we can define a kind of inverse function π_P^{-1}. It maps a set X of vertices, edges or faces in O to the set of all the vertices, edges or faces in $BO_P(M)$ whose image under π_P is in X. If we apply π_P^{-1} to a single vertex, edge or face x of O, we will often write $\pi_P^{-1}(x)$ instead of $\pi_P^{-1}(\{x\})$. For submaps M' of O the image $\pi_P^{-1}(M')$ is a subset of vertices and edges of $BO_P(M)$. If these form a connected graph, we interpret it as a map with the embedding induced by $BO_P(M)$.

The definition of $BO_P(M)$ depends on P. We will now prove that the result of an operation is independent of P, so that we can define $O(M)$ for a lopsp-operation O.

Theorem 3.4. *Let O be a lopsp-operation and let P and Q be two cut-paths in O. Let M be a map. Then $BO_P(M) \cong BO_Q(M)$.*

Proof. The idea of this proof is as follows. We define a submap $BO_P(M)|_Q$ of $BO_P(M)$ and prove that the underlying graph of this map is isomorphic as a graph to $D_{M,Q}$. Then we prove that they are also isomorphic as maps, and that the internal component of each face of $BO_P(M)|_Q$ is isomorphic to O_Q. It follows that $BO_P(M)$ is isomorphic to $BO_Q(M)$.

Let e be an edge of D_M, and let j be 1 if e has colour 2 and 2 if e has colour 1. Let P^e be the copy of P_{v_j,v_0} in $BO_P(M)$ that replaced e. By Lemma 3.3 there is a series of paths $P_{v_j,v_0} = P_0, \dots, P_k = Q_{v_j,v_0}$ from v_j to v_0 in O, so that the path P_{i+1} is obtained from P_i by a chamber flip of a chamber C_i and none of v_0, v_1, v_2 occur as interior points of any of the paths. We define a sequence of paths $P^e = P^e_0, \dots, P^e_k$ in $BO_P(M)$ with $\pi_P(P^e_i) = P_i$ for $0 \leq i \leq k$. The path P^e_{i+1} will be obtained from P^e_i by applying a chamber flip to a chamber $C \in \pi_P^{-1}(C_i)$. The chamber flips in O on the paths P_i replace subpaths of one or two edges. In case of one edge it is clear that a corresponding chamber flip can be performed on P^e_i in $BO_P(M)$. In case of two edges, we have to prove that the two corresponding edges of P^e_i are also contained in the same chamber. As P^e_i is a path, the two edges share one of their vertices, say v. By definition of the paths P_i we get that $\pi_P(v) \notin \{v_0, v_1, v_2\}$. For such a vertex v it is true that if $e_1, e_2, \dots, e_k$ is the cyclic order of edges around v, then $\pi_P(e_1), \pi_P(e_2), \dots, \pi_P(e_k)$ is the cyclic order of the edges around the vertex $\pi_P(v)$. If a chamber flip is applied to the edges $\pi_P(e_j)$ and $\pi_P(e_{j+1})$ to go from P_i to P_{i+1}, then we can apply a chamber flip to the edges e_j and e_{j+1} to go from P^e_i to a new path P^e_{i+1}. Thus our sequence of paths $P^e = P^e_0, \dots, P^e_k$ in $BO_P(M)$ with $\pi_P(P^e_i) = P_i$ is defined for $0 \leq i \leq k$ and $\pi_P(P^e_k) = Q_{v_j,v_0}$. We denote P^e_k as Q^e. Note that Q^e is isomorphic to Q_{v_j,v_0}, not to Q. Let $BO_P(M)|_Q$ be the map consisting of all the vertices and edges of $BO_P(M)$ contained in Q^e for some edge e. With the rotational orders induced by $BO_P(M)$ we have that $BO_P(M)|_Q$ is a submap of $BO_P(M)$. First we prove that as (non-embedded) graphs, $BO_P(M)|_Q$ and $D_{M,Q}$ are isomorphic.

Two paths Q^e and $Q^{e'}$ can only intersect in their endpoints: Every other vertex v of Q^e and $Q^{e'}$ satisfies $\pi_P(v) \notin \{v_0, v_1, v_2\}$, which implies that v has only two incident edges that are mapped to edges in Q by π_P. It follows that two paths of the form Q^e are either disjoint — except possibly for their endpoints — or identical. We prove by induction that P^e_i and $P^{e'}_i$ are disjoint (except for their endpoints) for all $0 \leq i \leq k$ and edges e and e' in D_M. If e and e' are edges of a different colour this is trivial as at least one of their endpoints is different. Assume that e and e' have the same colour. By our previous argument it suffices to show that their first edge is different. For $i = 0$ this is clear. Assume that it is true for $i-1$. Let ε_i and ε'_i be the first edges of P^e_i and $P^{e'}_i$ respectively. We can assume that they are both incident with the same vertex $x \in \pi_P^{-1}(\{v_0, v_1, v_2\})$. The paths P^e_i and $P^{e'}_i$ are obtained from P^e_{i-1} and $P^{e'}_{i-1}$ by one chamber flip for each path. Either $\varepsilon_i = \varepsilon_{i-1}$ and $\varepsilon'_i = \varepsilon'_{i-1}$, or the chamber flips replace ε_{i-1} and ε'_{i-1} by both their previous edges or both their next edges in the rotational order around x. As ε_{i-1} and ε'_{i-1} are different edges, ε_i and ε'_i are also different edges, which proves our statement.

It follows that $BO_P(M)|_Q$ and $D_{M,Q}$ are isomorphic as graphs. Next we prove that they are also isomorphic as maps.

Let m denote the total number of chamber flips necessary to transform first P_{v_1,v_0} to Q_{v_1,v_0} and then P_{v_2,v_0} to Q_{v_2,v_0}. With every face (that is: double chamber) D of D_M and $0 \leq i \leq m$ we can now associate a closed walk W_i that consists of the four paths $P^{e_1}_i, P^{e_2}_i, P^{e_3}_i, P^{e_4}_i$ in $BO_P(M)$ where $e_1, \dots, e_4$ are the four edges of D, in the same order as they appear in D.

Claim: $BO_P(M)|_Q$ is a submap of $BO_P(M)$ that is isomorphic as a map to $D_{M,Q}$ and the internal component of each face is isomorphic to O_Q.

Let $\mathscr{C}$ be the set of all chambers in $BO_P(M)$, and let n be the number of chambers in O. We will define functions $\alpha_i \colon \mathscr{C} \to \mathbb{Z}$ $(0 \leq i \leq m)$ with the following properties:

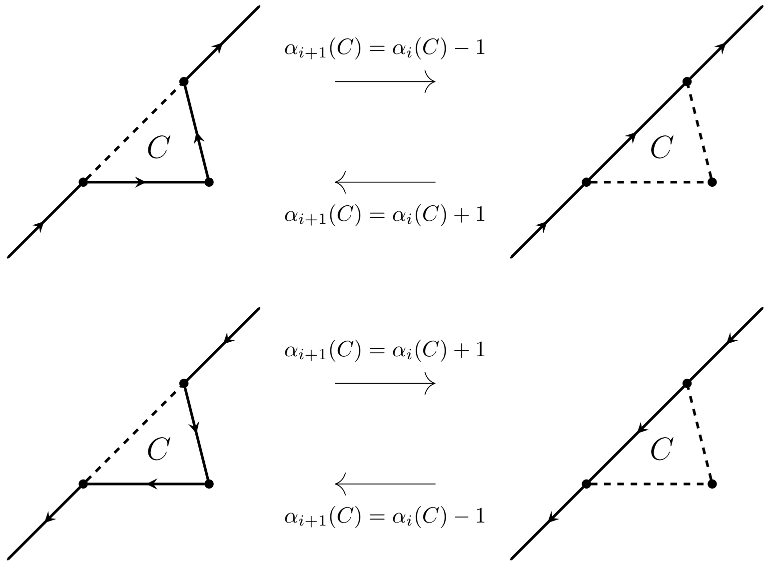


Figure 7: The evolution of α after chamber flips. The bold paths with arrows are W_i and W_{i+1}.

(i) Let C, C' in $BO_P(M)$ be two adjacent chambers sharing the directed edges e and e^{-1}, so that C is on the left of e. For $e' \in \{e, e^{-1}\}$ we define $n_i(e')$ as the number of times e' occurs in the cyclic walk W_i. Then $\alpha_i(C) - \alpha_i(C') = n_i(e) - n_i(e^{-1})$.

(ii) For every chamber C in O: $\sum_{C' \in \pi_P^{-1}(C)} \alpha_i(C') = 1$

As a consequence of (ii) we have $\sum_{C \in \mathscr{C}} \alpha_i(C) = n$.

The walk W_0 is an internally plane facial walk of $D_{M,P}$ with an internal component that is isomorphic to O_P. We define $\alpha_0(C) = 1$ if C is a chamber on the inside of W_0 and $\alpha_0(C) = 0$ if C is on the outside. As W_0 has exactly one copy of each chamber in O inside we get (ii) for α_0. As α_0 only differs for neighbouring chambers if they share an edge of W_0, and then in the way described by (i), we also get (i).

For $i > 0$ we define α_i inductively. Let C be the chamber of O to which a chamber flip is applied when changing W_{i-1} to W_i. These chamber flips occur in two places of W_{i-1}, and in fact in different directions. Two chambers C^-, C^+ with $\pi_P(C^-) = \pi_P(C^+) = C$ are involved, C^- on the left of the cyclic walk and C^+ on the right. We now define $\alpha_i(C^-) = \alpha_{i-1}(C^-) - 1$ and $\alpha_i(C^+) = \alpha_{i-1}(C^+) + 1$. This is illustrated in Figure 7. As we once add one and once subtract one for two chambers with the same image under π_P, (ii) is immediate. Property (i) can be checked easily by looking at α_i for C^-, C^+, and the neighbouring chambers sharing an edge with them.

For $i = 0$, The function α_i describes whether a chamber is inside or outside W_i. For other i this is not always the case. If W_i self-intersects the intuitive meaning of α_i is less obvious.

For $j = 1$ or $j = 2$, the two edges of W_i incident to the j-vertex x of D are always moved in the same direction by the chamber flips. This implies that $\{\alpha_i(C) \mid x \in C\}$ is the same set for every $0 \leq i \leq m$. As $\{\alpha_0(C) \mid x \in C\} \subseteq \{0, 1\}$ — it can be $\{1\}$ if

M has a loop — every chamber that contains x is mapped to 0 or 1 by α_m. The degree in W_m of every vertex that is not in $\pi_P^{-1}(\{v_0, v_1, v_2\})$ is two, so we can follow W_m from v_1 and from v_2 to the copies of v_0 to conclude that for each edge of W_m, the two chambers C and C' containing it have $\alpha_m(C) \in \{0, 1\}$ and $\alpha_m(C') \in \{0, 1\}$. The α_m values of two adjacent chambers can only differ if their shared edge is in W_m, so 0 and 1 are the only values of α_m. Note that this argument only works because every vertex of W_m that is not in $\pi_P^{-1}(\{v_0, v_1, v_2\})$ has degree 2 in W_m. For $0 < i < m$ this is not necessarily the case, and for those values of i the mapping α_i may have values different from 0 and 1.

Consider the submap $\mathscr{F}_m$ of the dual of $BO_P(M)$ — i.e. the flag graph of the map N such that $B_N = BO_P(M)$ — induced by the chambers C with $\alpha_m(C) = 1$, where edges in that map corresponding to edges in W_m are removed. By (ii) it follows that for every chamber C_O in O there is exactly one vertex in $\mathscr{F}_m$. For every edge in the flag structure of O_Q there is exactly one edge in $\mathscr{F}_m$, as by (i) adjacent chambers have the same value under α_m if their shared edge is not in W_m. In fact, these are all the edges of $\mathscr{F}_m$: The maximum degree of a vertex in $\mathscr{F}_m$ is 3, as a chamber is adjacent to three others. Let k be the number of edges in Q. As there are $2k$ edges in W_m, we get $2 \cdot |E_{\mathscr{F}_m}| = \sum_{v \in \mathscr{F}_m} deg(v) \leq 3n - 2k$. We also have $2|E_{\mathscr{F}_m}| \geq 2|E_{O_Q}| = 3n - 2k$ and thus $|E_{\mathscr{F}_m}| = |E_{O_Q}|$. It follows that the flag structure of O_Q is isomorphic to $\mathscr{F}_m$. As there are no edges in the flag structure of O_Q that correspond to edges of Q, the walk W_m is the facial walk of a face of $BO_P(M)|_Q$. It follows that $BO_P(M)|_Q$ and $D_{M,Q}$ are isomorphic maps and the internal component of each face of $BO_P(M)|_Q$ is isomorphic to O_Q. □

Definition 3.5. Let O be a lopsp-operation and let M be a map. Choose any cut-path P in O. The *result* $O(M)$ of applying O to M is the map with barycentric subdivision $BO_P(M)$.

By Lemma 2.4 and Theorem 3.4, $O(M)$ is well-defined and independent of the chosen path. We can also define the map $\pi := \pi_P$ as it is independent of the chosen path.

4 The effect of lsp- and lopsp-operations on polyhedrality

Polyhedral maps are simple maps that are 3-connected and have 'face-width' at least three. The face-width (or representativity) of a map is a measure of 'local planarity'. Embeddings of high face-width share certain properties with plane maps. We will define face-width in a combinatorial way, using barycentric subdivisions. It is not difficult to prove that the definition given here is equivalent to the definition in e.g. [20].

A cycle in a map M is *contractible* if – as a submap of M – it has a simple internally plane face. The *face-width* of a map M, denoted $fw(M)$, is the minimal length of a non-contractible cycle in B_M, divided by two. If M has no non-contractible cycles, i.e. M is plane, then we define $fw(M) = \infty$.

Definition 4.1. For $k \geq 1$ we define a map M to be **c**k if:

- M has no cut-sets with fewer than k vertices
- $fw(M) \geq k$
- The size of every face of M is at least k
- The degree of every vertex of M is at least k

The condition that neither cuts with fewer than k vertices nor vertices with degree smaller than k may be present instead of just requiring the map to be k-connected is chosen in order to deal with small boundary cases. For example, a cycle is 2-connected, but its dual is a map with only two vertices so it is not 2-connected. Both a cycle and its dual are $\mathbf{c}2$.

A *polyhedral map* is a simple, 3-connected map that has face-width at least three.

Lemma 4.2. *A map is* $\mathbf{c}3$ *if and only if it is polyhedral.*

Proof. It suffices to prove that every $\mathbf{c}3$-map is simple and has at least four vertices. The rest of the statement is trivial when the definitions are written out. Let M be a $\mathbf{c}3$-map. Facial loops and facial 2-cycles are excluded by the restrictions on face sizes and non-facial loops and non-facial 2-cycles imply either smaller cuts or a smaller face-width. Therefore M is simple. It has at least 4 vertices as it has minimum degree at least 3 and it is simple. □

The reason why the term $\mathbf{c}3$ is used instead of polyhedral in this article is that many results are proven for $\mathbf{c}k$ maps for general $k \in \{1, 2, 3\}$.

The following lemma characterises $\mathbf{c}2$- and $\mathbf{c}3$-maps by a condition based on the chamber system. A 4-cycle in a barycentric subdivision is called *trivial* if it has a face that has no vertex or only a single colour-1 vertex in its interior.

Lemma 4.3. *Let M be a map.*

(i) *M is $\mathbf{c}2$ if and only if B_M has no cycles of length 2.*

(ii) *M is $\mathbf{c}3$ if and only if M is $\mathbf{c}2$, and B_M has no nontrivial cycles of length 4.*

Proof. (i): Let M be a map and assume that M is not $\mathbf{c}2$. There are four possible reasons for not being $\mathbf{c}2$: the existence of a cutvertex, the existence of a facial loop (a face of size 1), the existence of a vertex of degree 1, or the existence of a non-contractible 2-cycle in B_M. The last three immediately imply the existence of a 2-cycle in B_M, so assume that M has a cutvertex v. If there is a loop in M, then there is a 2-cycle in B_M, so assume that M has no loops. Then vertex v has neighbours x and y in different components such that y follows x in the rotational order around v. The facial walk $(x, v), (v, y), (y, w_1), \ldots, (w_k, x)$ of the face f containing this angle must also contain v as one of the w_i as otherwise part of the facial walk would be a path from x to y in $M \setminus \{v\}$. This implies that in the barycentric subdivision there are 2 edges between v and the vertex corresponding to f — a 2-cycle.

Conversely, assume that there is a 2-cycle c in B_M. If there is a 0- or 2-vertex of degree two in B_M then there is a vertex of degree 1 or a face of size 1 in M, so we can assume that every vertex of B_M has degree at least four. We can also assume that every cycle of length 2 in B_M is contractible, as otherwise $fw(M) = 1$ and we are done. This implies that every 2-cycle has two well-defined sides.

Assume w.l.o.g. that c is *innermost*, that is: it contains no 2-cycle in its simple, plane face f_c. Let v and w be the vertices of c. Assume that v has colour 1. Then the two neighbours of v that are not w are on different sides of c. Every face has only three edges and there are no vertices of degree 2, so there are two edges between each of these neighbours of v and w. This is not possible as c is innermost. It follows that the two vertices of c have colours 0 and 2 respectively. There is at least one vertex e_f of colour 1 in the interior of f_c. This vertex e_f has degree 4. If there are no 0-vertices in the interior of f_c then there must

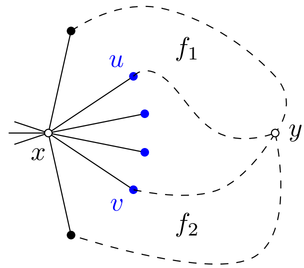


Figure 8: This figure clarifies the proof of (ii) of Lemma 4.3. The blue vertices are all in the same component of $M \setminus \{x, y\}$, and the black vertices are not in that component.

be two edges between e_f and the 0-vertex of c. This is a contradiction with the assumption that c is innermost. It follows that f_c has a 0-vertex in its interior. As every 2-cycle has two well-defined sides, there is an innermost 2-cycle in the other face g_c of c. Using the same arguments as for f_c on that cycle we get that g_c also has a 0-vertex in its interior. Every path in B_M between the 0-vertices in the two faces using only 2-edges must pass through the 0-vertex of c. It follows that this vertex is a cutvertex of M, so that M is not **c**2.

(ii): Let M be a map and assume that M is not **c**3. If it is also not **c**2 we are done, so assume that M is **c**2. There are four possible reasons for not being **c**3: the existence of a cutset $\{x, y\}$ of size two, the existence of a face of size two, the existence of a vertex with degree 2, or the existence of a non-contractible 4-cycle in B_M. Again the last three, as well as double edges forming a non-facial 2-cycle in M, immediately give a nontrivial 4-cycle in B_M, so assume that there are no double edges or loops in M, but there is a 2-cut $\{x, y\}$.

Both x and y have neighbours in different components. Let $u \neq y$ be a neighbour of x, so that the previous vertex in the rotational order around x is not in the same component of $M \setminus \{x, y\}$ as u. Let v be the last vertex, as seen from u, in the rotational order around x such that v and all vertices in the rotational order between u and v are in the same component as u, as shown in Figure 8. Note that $u = v$ is possible. If the edges (x, u) and (v, x) would belong to the same face then there would be a colour-1 cycle of length 2 in B_M, so they are in different faces f_1 and f_2 of M. Both f_1 and f_2 must also contain y as otherwise the next, resp. previous neighbour of x would belong to the same component as u and v. The cycle x, f_1, y, f_2 is a nontrivial cycle of length 4 in B_M.

Conversely, assume that M is **c**3 and that M is not **c**2 or B_M has a nontrivial cycle of length 4. As M is **c**3 it is also **c**2, so B_M has a nontrivial cycle c of length 4. Note that there are no double edges in B_M as M is **c**2, and M is simple and 3-connected by Lemma 4.2.

The cycle c is contractible because $fw(M) \geq 3$. It therefore has two well-defined sides. Assume first that c has no vertices of colour 0 on one side. Then there must be a 2-vertex on that side, as there cannot only be vertices of colour 1. This 2-vertex can have degree at most 4 as it can be adjacent to at most two 0-vertices in c and there are no double edges in B_M. This implies a facial 2-cycle in M, a contradiction. It follows that there is at least one 0-vertex on each side of c. Every colour-2 path between 0-vertices on different sides passes through c. This implies that the vertices and edges of M corresponding to vertices of c form a cut of M. Ignoring edges if one of their incident vertices is also in c, this cut consists of 2 vertices, a vertex and an edge or two edges. For each of the edges we can choose one of its incident vertices such that we find a cut-set consisting of 2 vertices, which is a contradiction with the 3-connectivity of M. □

Lemma 4.3 is very useful to determine whether a map is **c**2 or **c**3. It will often be used in the following lemmas and theorems. The main theorem of this last section is Theorem 4.9, which shows the equivalence of different definitions of **c**k-lopsp-operations and states that when applying **c**k-lopsp-operations with $k \in \{1, 2, 3\}$ to certain maps, the result is **c**k. The most difficult part of its proof is captured in Theorem 4.5 for **c**2-maps and Theorem 4.7 for **c**3-maps.

Lemma 4.4. *Let O be a lopsp-operation with a cut-path P of minimal length.*

(i) *If the vertices of an edge e in O are both in P_{v_0,v_i} for an $i \in \{1, 2\}$, then e or an edge with the same vertices as e is also in P_{v_0,v_i}.*

(ii) *If the vertices of an edge in O_P are in different copies of P_{v_0,v_1}, then there is a nontrivial 4-cycle in two copies of O_P sharing their copies of P_{v_0,v_1}.*

Proof. (i): This follows immediately from the minimality of the length of P.
(ii): If P_{v_0,v_1} is $v_0 = t_1, \dots, t_k = v_1$ and we denote one copy with $t_1, \dots, t_k$ and the other with $t'_1, \dots, t'_k$, then — again due to minimality and as O has no loops — such an edge connects w.l.o.g. t_i with t'_{i+1} for some $1 \leq i < (k-1)$. Considering two copies of O_P sharing the copies of P_{v_0,v_1}, this gives a 4-cycle $c = t_i, t'_{i+1}, t'_i, t_{i+1}$ with v_1 in the interior. If c was trivial, then v_1 would be a 1-vertex adjacent to all 4 vertices on c — also t_i and t'_i — which contradicts the minimality of P. □

Theorem 4.5. *Let M be a* **c**2*-map and let O be a lopsp-operation. Then $O(M)$ is* **c**2 *if and only if for each cut path P in O we have that there is no 2-cycle in O_P.*

Note that we must consider 2-cycles in O_P and not in O. It is possible that there are 2-cycles in O that do not induce 2-cycles in O_P for any cut-path. For example, the operation gyro, shown in Figure 4*, has several 2-cycles but none in O_P for any cut-path P.*

Proof. If there is a 2-cycle in O_P for a cut-path P then each copy of O_P inserted into $D_{M,P}$ contains a copy of this 2-cycle. Lemma 4.3 now implies that $O(M)$ is not **c**2.

Conversely, assume that $O(M)$ is not **c**2. Then there is a 2-cycle c in $B_{O(M)}$. Let x and y be the vertices of c and let e_1 and e_2 be its edges. Let P be a cut-path in O of minimal length. Note that by Lemma 4.3 applied to M, every double chamber has two different 0-vertices and therefore the boundary of each double chamber is simple. It follows that if there exists a double chamber that contains both edges of c in its interior or on its boundary, then c induces a cycle in a copy of O_P and we are done. Assume that e_1 and e_2 are in different double chambers D_1 and D_2 respectively.

Both D_1 and D_2 contain x and y, so those vertices are on the boundary of both double chambers. Assume first that x and y are not both on copies of P_{v_0,v_1} or both on copies of P_{v_0,v_2}. Then D_1 and D_2 share their 1-vertex and their 2-vertex which implies a 2-cycle in B_M, a contradiction. It follows that x and y are both on copies of P_{v_0,v_1} or both on copies of P_{v_0,v_2}. If they are in the same copy of P_{v_0,v_1} or P_{v_0,v_2}, then by Lemma 4.4(i) an edge e_0 with the same vertices is also in that copy of P_{v_0,v_1} or P_{v_0,v_2}, so that O_P contains a 2-cycle.

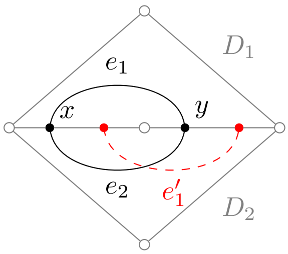


Figure 9: This figure clarifies a step in the proof of Theorem 4.5. It shows that a 2-cycle in $B_{O(M)}$ with its vertices on different copies of the same P_{v_0,v_i} cannot exist. The middle vertex can be the 1- or the 2-vertex of the double chambers. In either case, the left- and rightmost vertices are the 0-vertices.

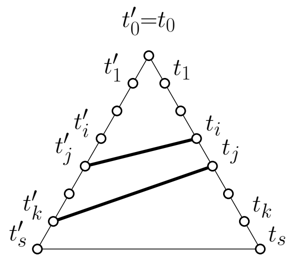


Figure 10: The double chamber patch in the proof of Lemma 4.6. There can be no edge $\{t'_l, t_m\}$ with $l < j$ and $m > i$.

The last possibility is that x and y are in different copies of P_{v_0,v_1} or in different copies of P_{v_0,v_2}. As O does not contain loops we have $\pi(x) \neq \pi(y)$. Applying Jordan's curve theorem to c we get that the edge e'_1 in D_2 with $\pi(e_1) = \pi(e'_1)$ cannot exist — a contradiction (see Figure 9). □

Lemma 4.6. *Let M be a* c3*-map and let O be a lopsp-operation with a cut-path P of minimal length. Let f be a 2-vertex of D_M, and consider the submap S_f of $B_{O(M)}$ consisting of all the edges and vertices in the double chambers with 2-vertex f. If there is a nontrivial 4-cycle c in S_f, then there is a 2-cycle in O_P or c is contained in either only one of these double chambers, or in two adjacent double chambers.*

Proof. As M is c3, the submap of all vertices and edges of $O(M)$ belonging to one of the double chambers containing f is plane. The map formed by the vertices and edges on the 2-sides of the double chambers in S_f is a simple cycle, that we consider to be the boundary of the outer face of the map S_f. There are at least three double chambers in S_f. If c contains only edges on edges of D_M, then c is the boundary of one double chamber, so assume that c contains at least one edge in the interior of a double chamber.

The boundary ∂D of every double chamber D in S_f is a cycle. As S_f is plane, it follows with the Jordan curve theorem that c must cross ∂D an even number of times. By *cross* we mean that there is a subpath of c whose first and last vertex are on different sides of c, and whose other vertices are all in ∂D. Let D be a double chamber in S_f that contains an edge of c in its interior. If c crosses ∂D 0 times, then c is contained in one copy of O_P and we are done. If c crosses ∂D 4 times then there must be an edge outside of D that has both its vertices on ∂D. In this case Lemma 4.4 implies that there is a 2-cycle in O_P and we are done. We can therefore assume that c crosses ∂D exactly twice. Note that every crossing is on a 1-side. Assume that the vertices of these crossings are on the same 1-side of D. If there would be only one edge of c in D this again leads to a 2-cycle in O_P with Lemma 4.4. If there is no crossing in f it is clear that a chamber D' adjacent to D also has two crossings with c on the same 1-side. If f is in c that follows from the fact that f is on every 1-side and there must be at least two edges of c in a double chamber that has 2 crossings with c on the same 1-side. It follows that c is completely contained in D and D', i.e. in two adjacent double chambers.

We can now assume that c has two crossings with ∂D that are on different 1-sides and not in f. As c crosses into the double chambers adjacent to D those must also have two crossings on different 1-sides. Repeating this argument we get that every double chamber in S_f has two crossings with c on different 1-sides.

It follows that every double chamber in S_f contains a subpath of c connecting vertices different from f on their two 1-sides. As there are at least three double chambers in S_f and there must be at least one edge of c in each one, there are exactly three or four double chambers in S_f. In each case there are at least two adjacent double chambers that contain only one edge of c. Let e_1 and e_2 be the only edges of c in two adjacent double chambers. As the vertices of e_1 and e_2 are on different 1-sides of O_P the edges e_1 and e_2 are not in P. Therefore the edges $\pi(e_1)$ and $\pi(e_2)$ induce unique edges, that we will also denote with $\pi(e_1)$ and $\pi(e_2)$, in O_P. If the vertices of P_{v_2,v_0} in O are — in this order — $t_0, t_1, \ldots, t_s$, then we will denote the vertices on the different 1-sides of O_P with $t_0, t_1, \ldots, t_s$, resp. $t'_0, t'_1, \ldots, t'_s$. We have $\pi(e_1) = \{t_i, t'_j\}$ and $\pi(e_2) = \{t_j, t'_k\}$ with w.l.o.g. $0 < i < j \le s$. Note that $i \ne j$ as there are no loops in O. Due to the Jordan curve theorem applied to the cycle $t_0 = t'_0, t'_1, \ldots, t'_j, t_i, t_{i-1}, \ldots, t_0$ there is no edge $\{t_m, t'_l\}$ in O_P with $m > i$ and $l < j$. As $j > i$ and $\pi(e_2) = \{t_j, t'_k\}$ is in O_P, it follows that $k > j$. This situation is shown in Figure 10.

If there are four double chambers in S_f we can repeat this argument on every pair of adjacent double chambers. With x_1, x_2, x_3, x_4 the vertices of c in cyclic order and $\pi(x_a) = t_{i_a}$ we get that $i_a < i_{a+1}$ for all $1 \le a \le 3$ and $i_4 < i_1$, so that by transitivity $i_1 < i_1$, a contradiction. If there are only three double chambers in S_f, then there must be two edges e_3 and e_4 of c in the same double chamber. The edges $\pi(e_3)$ and $\pi(e_4)$ form a path from t'_i to t_k. Such a path would have to cross both the edges $\pi(e_1)$ and $\pi(e_2)$, which is only possible if the path has at least three edges — it must contain t_i or t'_j and t_j or t'_k — which is a contradiction. □

Theorem 4.7. *Let M be a* **c**3*-map and let O be a lopsp-operation. Then $O(M)$ is* **c**3 *if and only if it is* **c**2 *and for each cut-path P in O we have that there is no nontrivial 4-cycle in a patch of two adjacent copies of O_P sharing one of their sides.*

Proof. The implication that $O(M)$ is not **c**3 if there is a 2- or nontrivial 4-cycle for some cut-path P is obvious, as corresponding pairs of two adjacent copies of O_P in $B_{O(M)}$ would contain such cycles.

For the other implication we assume that $O(M)$ is not **c**3 but it is **c**2, and that there is no nontrivial 4-cycle in a patch of two adjacent copies of O_P for a cut-path P. We will come to a contradiction by constructing such a 4-cycle. Let P be a cut-path in O of minimal length. We will refer to the copies of v_2 or v_0 in the double chamber patches as the *corners* of the double chamber patches. By Lemma 4.3 there is a nontrivial 4-cycle c in $B_{O(M)}$. Let X be a set of double chambers in D_M of minimal size, so that the union of all double chamber patches for double chambers in X contains c. For simplicity we will also refer to the set of those double chamber patches as X. The fact that c has four edges implies that $1 \leq |X| \leq 4$. If $|X| = 1$ then c can be thought of as a 4-cycle in O_P, which we assumed does not exist, so $|X| > 1$. We make the following observations:

(i) Every double chamber in X shares at least two of its corners with other elements of X: As $|X| > 1$ the cycle c 'enters' and 'leaves' any double chamber $D \in X$ in two different vertices. Both of these vertices must be in the intersection of D with the other double chambers of X.

(ii) If two double chambers share two corners they also share the side containing those two corners. This follows immediately from the fact that there are no double edges in M and B_M.

(iii) If the intersection of a double chamber $D \in X$ with the other double chambers of X is exactly one side, then there are at least two edges of c in D: Assume that there is only one edge e of c in D, and let D' be the double chamber of X sharing the side with D. Both vertices of e are on the side shared by D and D', but e itself is not, as otherwise D could be removed from X and X would not be minimal. If the vertices of e are on the same edge of D_M, then Lemma 4.4 implies that there is a 2-cycle in O_P, a contradiction with Theorem 4.5. If the vertices of e are on different copies of P_{v_0,v_1} then Lemma 4.4 implies a nontrivial 4-cycle in two adjacent copies of O_P, a contradiction.

It follows from (i) and (ii) that if $|X| = 2$ then c is a 4-cycle in two adjacent copies of O_P, so $|X| > 2$. We will now prove that there is a cycle of length 4 in D_M that contains at least one edge of each double chamber in X. We call such a cycle a *saturating* 4-cycle.

Assume first that every double chamber in X shares the same 2-vertex. With (i), (ii), and (iii), it follows easily that X consists of all the three or four double chambers corresponding to one face of M. Lemma 4.6 now implies that $O(M)$ is not **c**2 or that there is a 4-cycle in two adjacent copies of O_P. Both are contradictions.

Now assume that there is a 2-vertex f of D_M such that there is only one edge e of c in the union of all the double chamber patches of X with 2-vertex f. The edge e has both its vertices on the same 2-side. As at least one of its vertices is not a corner, both double chambers sharing that 2-side are in X. This is a contradiction with (iii).

It follows that there are two 2-vertices f and g in D_M such that the unions X_f and X_g of double chambers patches in X with 2-vertex f and g respectively each contain exactly two edges of c. With (i), (ii), and (iii) it follows that there are 0-vertices v and w of double chambers in X_f and X_g such that v, f, w, g is a saturating 4-cycle.

As M is **c**3, Lemma 4.3 implies that the saturating 4-cycle is the boundary of a double chamber, or two double chambers sharing the 1-vertex.

If e is the 1-vertex in these one or two double chambers, then c is contained in the set N_e consisting of all six double chambers that share a side with a double chamber containing e. We say that the two double chambers with 1-vertex e are the central double chambers, and the four other double chambers in N_e are the extremal double chambers. The three possible configurations of N_e with respect to shared sides are shown in Figure 11. Using the fact

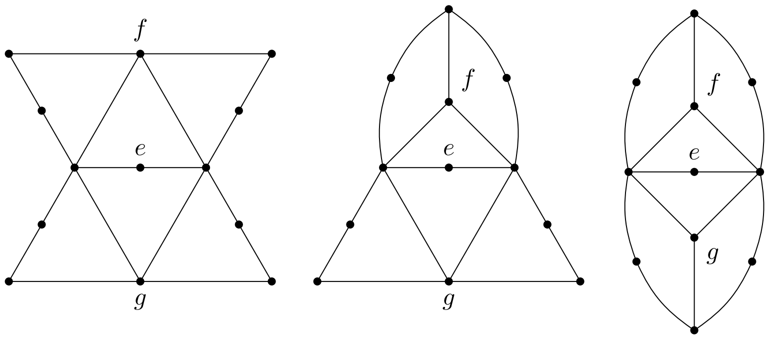


Figure 11: The three possible configurations of the six double chambers in the set N_e are shown here. Different vertices in the drawing represent different vertices of D_M.

that there are no nontrivial 4-cycles in B_M it can easily be verified that no two vertices in Figure 11 represent the same vertex of D_M.

We already proved that $|X| \geq 3$ and that there are two edges of c in X_f and two in X_g. Therefore we can assume w.l.o.g. that X_f consists of two double chambers. It follows from (iii) that these two double chambers are extremal double chambers. The two vertices of the edge e are in c. If the third vertex of c in X_f is f, then with Lemma 4.4 and the fact that there are no 2-cycles in O_P, it follows that the edges of c in X_f are 1-edges of D_M. In that case we can replace the two extremal double chambers in X by the central double chamber so X was not of minimal size, a contradiction. If the third vertex x of c in X_f is not f, then the extremal double chambers share a side and x is on this side. Let e_1 and e_2 be the edges of c in these double chambers. The edge e_1 corresponds to an edge in O_P from w.l.o.g. $v_{0,R}$ to an internal vertex of $P_{(v_{0,L}),v_2}$ and e_2 corresponds to an edge from $v_{0,L}$ to $P_{(v_{0,R}),v_2}$. This would imply crossing edges in the plane map O_P, a contradiction.

It follows that X_f and X_g each consist of only one double chamber, which by (i) must be the central double chamber. Then c is a nontrivial 4-cycle in a patch of two adjacent copies of O_P, a contradiction. □

For a lopsp-operation O, let T_O be the tiling of the Euclidean plane obtained by applying O to the regular hexagonal tiling of the plane. We say T_O is the *associated tiling* of O. With the definition for lopsp-operations from [2] this is the tiling from which O is defined. In Section 5 we will further explore the fundamental connection between lopsp-operations and tilings.

We will use the connectivity of the associated tiling of a lopsp-operation to define when an operation is ck. With Theorem 4.5 and Theorem 4.7 we will then prove an equivalent characterisation in Theorem 4.9 which does not depend on the associated tiling.

Definition 4.8. For $k \in \{1, 2, 3\}$ a lopsp-operation O is **c**k if the associated tiling T_O is k-connected and all faces have size at least k. An lsp-operation is **c**k if the equivalent lopsp-operation O_{lopsp} is **c**k.

For a **c**k-lopsp-operation O and a map M with minimum face size at least k and minimum degree at least k that is not necessarily **c**k, the map $O(M)$ also has minimum face size at least k and minimum degree at least k. For the vertices of $B_{O(M)}$ of colour 0 or 2 that are not 0-vertices or 2-vertices of D_M, the fact that they have degree at least $2k$ follows from O being a $\mathbf{c}k$-lopsp-operation, as these degrees also occur in the tiling T_O. For the others it follows from the degrees of 0-vertices and 2-vertices in D_M. In case M is also $\mathbf{c}k$, we have a stronger result:

Theorem 4.9. *The following statements are equivalent for $k \in \{1,2,3\}$ and a lopsp-operation O:*

(a) *O is a $\mathbf{c}k$-lopsp-operation*

(b) *For all $\mathbf{c}k$-maps M, $O(M)$ is $\mathbf{c}k$.*

(c) *There exists a $\mathbf{c}k$-map M such that $O(M)$ is $\mathbf{c}k$.*

Proof. **(a) ⇒ (b)**

For $k = 1$ this is trivial. Assume that there is a $\mathbf{c}2$-map M such that $O(M)$ is not $\mathbf{c}2$. By Theorem 4.5 there is a 2-cycle in O_P for some cut-path P in O. This cycle induces a cycle in B_{T_O}, which implies a 1-cut or a face of size 1 in the tiling T_O. It follows that O is not a $\mathbf{c}2$-operation — a contradiction. Similarly, if there is a $\mathbf{c}3$-map M such that $O(M)$ is not $\mathbf{c}3$, we find a 2-cut in T_O using the cycle from Theorem 4.7.

(b) ⇒ (a) For $k = 1$ this is trivial.

The tiling T_O is obtained by inserting copies of O_P (for some P) into the double chamber map of the hexagonal tiling. Let us call the map formed by the subdivided 2-edges of the double chamber map of the hexagonal tiling the hexagonal skeleton.

Assume now that O is not a $\mathbf{c}2$-lopsp operation. Then there is a face f of size 1 or a 1-cut $\{x\}$ in T_O. In case of a 1-cut, at least one of the components of $T_O \setminus \{x\}$, say C_0, is finite. Let C denote a finite submap of the hexagonal skeleton that contains f, resp. C_0 together with the cut vertex x. Using Goldberg-Coxeter operations (see [2] or [4]) with sufficiently large parameters to construct large icosahedral fullerenes, we get a fullerene F, that is $\mathbf{c}3$ and contains an isomorphic copy of C with the paths between the 0-vertices replaced by edges. Applying O to that fullerene, we get a submap S of $O(F)$ that has a face f' of size 1 or that is isomorphic to C_0 and where all vertices corresponding to vertices of C_0 have — except for the vertex x' corresponding to x — only neighbours in S. So f' or the vertex x', which is a cut-vertex of $O(F)$, are contradictions to the assumption.

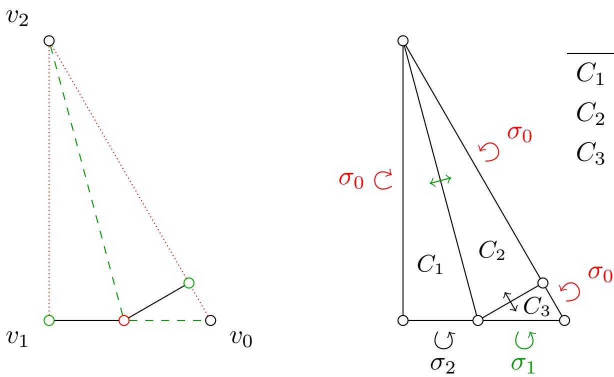
The case $k = 3$ is completely analogous, with also a 2-face and a 2-cut in the argument.

(b) ⇒ (c) Trivial.

(c) ⇒ (b) Note that the conditions in Theorems 4.5 and 4.7 are — except for M being $\mathbf{c}k$ — independent of M, as O_P and the union of two copies of O_P sharing a side are the same for all these M. This implies that if $O(M)$ is $\mathbf{c}k$ for some $\mathbf{c}k$-map M, then $O(N)$ is $\mathbf{c}k$ for any $\mathbf{c}k$-map N. □

One of the main results of this paper, Theorem 4.10, now follows from Theorem 4.9 and Lemma 2.5.

Theorem 4.10. *If M is a polyhedral map and O is a $\mathbf{c}3$-lsp- or $\mathbf{c}3$-lopsp-operation, then $O(M)$ is also a polyhedral map.*



	m_{01}	m_{12}
C_1	12	3
C_2	12	3
C_3	3	3

Figure 12: The operation truncation and the Delaney-Dress symbol encoding a tiling from which the operation can be obtained when the original definition is applied.

5 Connection to tilings

In a series of papers [8, 9, 10], Andreas Dress (in later papers together with coauthors) developed a finite symbol encoding the topology as well as the symmetry of periodic tilings. He attributed the idea to Matthew Delaney and called these symbols *Delaney symbols*. In later papers by other authors, these symbols are called *Delaney-Dress symbols*. In [6] and [10] Delaney-Dress symbols of periodic tilings of the Euclidean plane and the hyperbolic plane are characterized.

In this section we show that there is a very fundamental connection between l(op)sp-operations and Delaney-Dress symbols and therefore to tilings. Recall that we defined the associated tiling T_O of a lopsp-operation O as the tiling that is the result of applying O to the hexagonal tiling of the plane, i.e. the tiling with Schläfli symbol $\{6, 3\}$. We will find the same tiling in a different way using Delaney-Dress symbols, and we will see that from a mathematical point of view the choice of the tiling $\{6, 3\}$ is quite arbitrary. The hexagonal tiling was chosen because it was also used in the original definition of lsp-operations in [2], where in turn it was chosen as a tribute to a paper by Goldberg [15]. By proving this connection it follows that our abstract combinatorial definitions are equivalent – in the 3-connected case – to the definitions of lsp- and lopsp-operations in [2].

As a topological definition of tilings falls outside the scope of this article, we will directly start with the combinatorial characterization described in [6, 9, 10]. We will sketch the connection to tilings, but for a detailed description we refer the reader to [6] or [10].

Theorem 5.1 (A.W.M. Dress [9])**.** *Let $\mathscr{D}$ be a set together with an action (from the right) of the Coxeter group $\Sigma = \langle \sigma_0, \sigma_1, \sigma_2 \mid \sigma_i^2 = 1 \rangle$ on $\mathscr{D}$, and for $(i, j) \in \{(0, 1), (0, 2), (1, 2)\}$ let $m_{ij} \colon \mathscr{D} \to \mathbb{N}$ be maps with $m_{02}(C) = 2$ for all $C \in \mathscr{D}$. The tuple $(\mathscr{D}, m_{01}, m_{02}, m_{12})$ is the* Delaney-Dress symbol *of a tiling of the Euclidean plane if and only if the following properties hold:*

(1) *$\mathscr{D}$ has finitely many elements*

(2) *Σ acts transitively on $\mathscr{D}$*

(3) *For* $i, j \in \{0, 1, 2\}, i < j$, m_{ij} *is constant on* $\langle \sigma_i, \sigma_j \rangle$*-orbits and* $C(\sigma_i \sigma_j)^{m_{ij}(C)} = C$ *for all* $C \in \mathscr{D}$

(4) *We have*

$$\mathscr{C}(\mathscr{D}, m_{01}, m_{02}, m_{12}) = \sum_{C \in \mathscr{D}} \left(\frac{1}{m_{01}(C)} + \frac{1}{m_{12}(C)} - \frac{1}{m_{02}(C)} \right) = 0$$

Such Delaney-Dress symbols encode the combinatorial structure of periodic tilings of the Euclidean plane, together with a symmetry group acting on the tiling. If $\mathscr{C}(\mathscr{D}, m_{01}, m_{02}, m_{12}) \neq 0$, the tuple can also be a Delaney-Dress symbol, but then it encodes a periodic tiling of the hyperbolic plane ($\mathscr{C} < 0$) or — in case additional divisibility rules are fulfilled — the sphere ($\mathscr{C} > 0$) [6]. The elements of $\mathscr{D}$ are the orbits of chambers of the tiling under the symmetry group. An element $C \in \mathscr{D}$ with $C\sigma_i = C$ represents an orbit of chambers with mirror symmetries of the tiling stabilizing the edges of colour i. If there are no $C \in \mathscr{D}$ with $C\sigma_i = C$, the symmetry group contains no pure reflections, but maybe sliding reflections. If there are no odd cycles, that is $C\sigma_{i_1} \dots \sigma_{i_k} \neq C$ for odd k, all symmetries are orientation preserving. The maps m_{01} and m_{12} give information about the symmetry group of the tiling. Let $\{i, j, k\} = \{0, 1, 2\}$, $i < j$ and for $C \in \mathscr{D}$ let $r_{ij}(C) = \min\{r \mid C(\sigma_i\sigma_j)^r = C\}$. Note that r_{ij} is constant on $\langle \sigma_i, \sigma_j \rangle$-orbits. If a $\langle \sigma_i, \sigma_j \rangle$-orbit $C^{\langle \sigma_i, \sigma_j \rangle}$ contains no C' with $C'\sigma_i = C'$ or $C'\sigma_j = C'$, then the vertices of colour k of the corresponding chambers in the tiling are centers of an f_r-fold rotation with $f_r = m_{ij}(C)/r_{ij}(C)$. If an orbit $C^{\langle \sigma_i, \sigma_j \rangle}$ contains a C' with $C'\sigma_i = C'$ or $C'\sigma_j = C'$, then with $f_m = 2m_{ij}(C)/r_{ij}(C)$ for $f_m > 1$ the vertices of colour k of the chambers in orbit C are intersections of mirror axes with an angle of $360/f_m$ degrees.

We will now associate a tuple $(\mathscr{D}_O, m_{01}, m_{02}, m_{12})$ with an lsp- or lopsp-operation O and prove that it is a Delaney-Dress symbol. In fact, it will be a Delaney-Dress symbol of the tiling $O(T)$ where T is the tiling with Schläfli symbol $\{6, 3\}$, i.e. the hexagonal tiling of the plane where every vertex has degree 3 and every face has 6 edges. Due to the relation between Delaney-Dress symbols and tilings as described in [6] and [10], this also shows the equivalence of the combinatorial definitions of lsp- and lopsp-operations defined here and the geometric ones given in [2]. There a l(op)sp-operation is described as a 'triangle' cut out of a tiling in such a way that certain conditions on the symmetry are satisfied.

One could replace the values 3 and 6 we will use for defining the mappings m_{ij} by, for example, 4 and 4, and Theorem 5.3 would still be true. It would however be the Delaney-Dress symbol of the tiling that can be obtained by applying O to the square tiling of the plane, which is 4-regular and every face has 4 edges. By using other numbers, other tilings — even spherical or hyperbolic ones — could be used as source tilings. All of those tilings can be used to define l(op)sp-operations in the geometric way that was described in [2] for the hexagonal tiling.

Let O be an lsp-operation and let $\mathscr{D}_O$ be the set of chambers of O. We define the action of Σ on $\mathscr{D}_O$ by letting $C\sigma_i = C'$ if C and C' share their i-edge, and $C\sigma_i = C$ if the i-edge of C is in the outer face of O. For $(i, j) \in \{(0, 1), (0, 2), (1, 2)\}$, let $v^{ij}(C)$ be the vertex of chamber C that is not of colour i or j. We get:

$$r_{ij}(C) = \min\{r \mid C(\sigma_i\sigma_j)^r = C\} = \begin{cases} \frac{\left|C^{\langle\sigma_i,\sigma_j\rangle}\right|}{2} = \frac{deg(v^{ij}(C))}{2} & \text{if } v^{ij}(C) \text{ is not in the outer face} \\ \left|C^{\langle\sigma_i,\sigma_j\rangle}\right| = deg(v^{ij}(C)) - 1 & \text{if } v^{ij}(C) \text{ is in the outer face} \end{cases}$$

To find the Delaney-Dress symbol of the tiling obtained by applying O to $\{6,3\}$ we define $m_{ij} \colon \mathscr{D}_O \to \mathbb{N}$ as follows:

$$m_{ij}(C) = \begin{cases} r_{ij}(C)\cdot 2 & \text{if } v^{ij}(C) = v_1 \\ r_{ij}(C)\cdot 3 & \text{if } v^{ij}(C) = v_0 \\ r_{ij}(C)\cdot 6 & \text{if } v^{ij}(C) = v_2 \\ r_{ij}(C) & \text{if } v^{ij}(C) \notin \{v_0, v_1, v_2\} \end{cases}$$

Note that the requirements for the vertex degrees in an lsp-operation imply that for all $C \in \mathscr{D}_O$, the value $m_{02}(C)$ is 2.

We define $\mathscr{D}(O) = (\mathscr{D}_O, m_{01}, m_{02}, m_{12})$ and call it the Delaney-Dress symbol corresponding to the lsp-operation O. This correspondence is illustrated for the operation truncation in Figure 12. Theorem 5.2 states that it is in fact a Delaney-Dress symbol of a tiling of the Euclidean plane.

By our previous remarks there is a 2-fold rotation around each copy of v_1 in that tiling, a 3-fold rotation around each copy of v_0, and a 6-fold rotation around each copy of v_2. There are also intersections of mirror axes with $90°$, $60°$, and $30°$ angles at v_1, v_0, and v_2 respectively. This is the symmetry we expect when applying an lsp-operation to tiling $\{6,3\}$. This is also the symmetry that is required to define an lsp-operation from a tiling with the geometric definition.

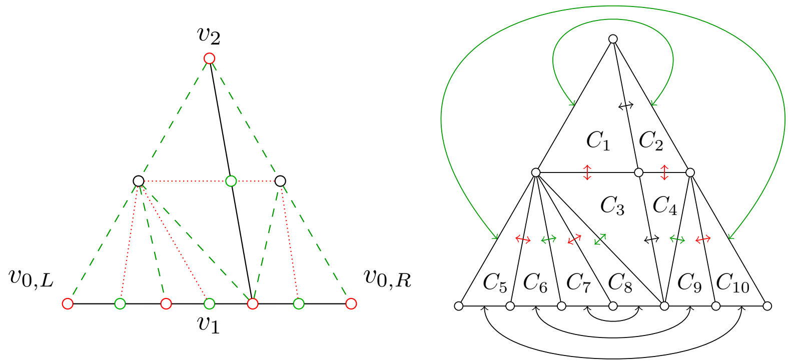
Theorem 5.2. *If O is an lsp-operation, then $\mathscr{D}(O) = (\mathscr{D}_O, m_{01}, m_{02}, m_{12})$ is the* Delaney-Dress symbol *of a tiling of the Euclidean plane.*

Proof. We have to prove the properties in Theorem 5.1. The first two properties are obvious, so we will focus on the other two.

(3): Let $(i,j) \in \{(0,1),(0,2),(1,2)\}$. A $\langle\sigma_i,\sigma_j\rangle$-orbit consists of all the chambers sharing the same vertex $v^{ij}(C)$, so that by definition m_{ij} is constant on $\langle\sigma_i,\sigma_j\rangle$-orbits. It is clear that $C(\sigma_i\sigma_j)^{m_{ij}(C)} = C(\sigma_i\sigma_j)^{r_{ij}(C)\cdot k} = C$

(4): Let $\{i,j,k\} = \{0,1,2\}$. For a vertex v of colour k and $i < j$ we define $\alpha(v) = \sum\limits_{\substack{C\in\mathscr{D}_O \\ v\in C}} \left(\frac{1}{m_{ij}(C)}\right)$.

Counting the number of chambers with a certain vertex and using the definition of m_{ij}, we get that



	m_{01}	m_{12}
C_1	5	6
C_2	5	6
C_3	5	3
C_4	5	3
C_5	5	3
C_6	5	3
C_7	5	3
C_8	5	3
C_9	5	3
C_{10}	5	3

Figure 13: On the left, the double chamber patch of the lopsp-operation gyro is shown and on the right the corresponding Delaney-Dress symbol.

$$\alpha(v) = \begin{cases} 2 & \text{if } v \text{ is an inner vertex} \\ 1 & \text{if } v \text{ is an outer vertex different from } v_i \text{ for } i = 0,1,2 \\ 1/2 & \text{if } v = v_1 \\ 1/3 & \text{if } v = v_0 \\ 1/6 & \text{if } v = v_2 \end{cases} .$$

Let n be the number of vertices (and equivalently edges) in the outer face. As every vertex of O has exactly one colour we get that:

$$\begin{aligned} \mathscr{C}(\mathscr{D}_O, m_{01}, m_{02}, m_{12}) &= \sum_{C\in\mathscr{D}_O} \left(\frac{1}{m_{01}(C)} + \frac{1}{m_{12}(C)} - \frac{1}{m_{02}(C)} \right) \\ &= \sum_{C\in\mathscr{D}_O} \left(\frac{1}{m_{01}(C)} + \frac{1}{m_{12}(C)} + \frac{1}{m_{02}(C)} \right) - \sum_{C\in\mathscr{D}_O} (1) \\ &= \sum_{v\in V_O} \alpha(v) - (|F_O| - 1) \\ &= (|V_O| - n)\cdot 2 + (n-3)\cdot 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} - (|F_O| - 1) \\ &= 2|V_O| - |F_O| - n - 1 \end{aligned}$$

By counting the number of directed edges associated with edges in the triangulated disk O in two ways, we get that $2|E_O| = 3(|F_O| - 1) + n$ or equivalently $|F_O| = 2|E_O| - 2|F_O| + 3 - n$. We also know that O is plane, so $|V_O| - |E_O| + |F_O| = 2$. It follows that:

$$\mathscr{C}(\mathscr{D}_O, m_{01}, m_{02}, m_{12}) = 2|V_O| - 2|E_O| + 2|F_O| - 3 + n - n - 1 = 0 \qquad \square$$

We will now prove the corresponding result for lopsp-operations. Let O be a lopsp-operation and let $\mathscr{D}_O$ be the set of chambers of O. We define the action of Σ on $\mathscr{D}_O$ by letting $C\sigma_i = C'$ if C and C' share their i-edge. For lopsp-operations there is no outer face, so $r_{ij}(C)$ is always $\frac{deg(v^{ij})}{2}$. We define $m_{01}, m_{02}, m_{12} : \mathscr{D}_O \to \mathbb{N}$ exactly as before:

$$m_{ij}(C) = \begin{cases} r_{ij}(C)\cdot 2 & \text{if } v^{ij}(C) = v_1 \\ r_{ij}(C)\cdot 3 & \text{if } v^{ij}(C) = v_0 \\ r_{ij}(C)\cdot 6 & \text{if } v^{ij}(C) = v_2 \\ r_{ij}(C) & \text{if } v^{ij}(C) \notin \{v_0, v_1, v_2\} \end{cases}$$

Again $m_{02}(C) = 2$ for all $C \in \mathscr{D}_O$. We define $\mathscr{D}(O) = (\mathscr{D}_O, m_{01}, m_{02}, m_{12})$ and in Theorem 5.3 we prove that it is a Delaney-Dress symbol. The operation gyro and its corresponding Delaney-Dress symbol are shown as an example in Figure 13. Once again, the tiling described by the Delaney-Dress symbol is the result of applying the operation to the hexagonal tiling of the plane. In Section 4 we named this tiling the associated tiling T_O of O. There are 2-, 3-, and 6-fold rotations at the copies of v_1, v_0, and v_2 respectively. In lopsp-operations there is no chamber C such that $C\sigma_i = C$ so there are no pure reflections encoded in the Delaney-Dress symbol. This is the symmetry required in the geometric definition of lopsp-operations.

Theorem 5.3. *If O is a lopsp-operation, then $\mathscr{D}(O) = (\mathscr{D}_O, m_{01}, m_{02}, m_{12})$ is the* Delaney-Dress symbol *of a tiling of the Euclidean plane.*

Proof. We prove the properties in Theorem 5.1. Again, the first two are obvious.

(3): As in the proof of Theorem 5.2.

(4): Let $\{i, j, k\} = \{0, 1, 2\}$. For a vertex $v \in V_O$ of colour k and $i < j$ we again define $\alpha(v) = \sum_{\substack{C \in \mathscr{D}_O \\ v \in C}} \left(\frac{1}{m_{ij}(C)}\right)$.

Counting the number of chambers with a given vertex v and using the definition of m_{ij}, we get that

$$\alpha(v) = \begin{cases} 2 & \text{if } v \notin \{v_0, v_1, v_2\} \\ 1 & \text{if } v = v_1 \\ 2/3 & \text{if } v = v_0 \\ 1/3 & \text{if } v = v_2 \end{cases}.$$

We can now compute $\mathscr{C}(\mathscr{D}_O, m_{01}, m_{02}, m_{12})$:

$$\begin{aligned}\mathscr{C}(\mathscr{D}_O, m_{01}, m_{02}, m_{12}) &= \sum_{C \in \mathscr{D}_O} \left(\frac{1}{m_{01}(C)} + \frac{1}{m_{12}(C)} - \frac{1}{m_{02}} \right) \\ &= \sum_{C \in \mathscr{D}_O} \left(\frac{1}{m_{01}(C)} + \frac{1}{m_{12}(C)} + \frac{1}{m_{02}} - 1 \right) \\ &= \sum_{v \in V_O} \alpha(v) - |\mathscr{D}_O| \\ &= (|V_O| - 3) \cdot 2 + \frac{2}{3} + 1 + \frac{1}{3} - |F_O| \\ &= 2|V_O| - |F_O| - 4\end{aligned}$$

As O is a triangulation, we get that $2|E_O| = 3|F_O|$ and as O is plane, we have $|V_O| - |E_O| + |F_O| = 2$. It follows that:

$$\mathscr{C}(\mathscr{D}_O, m_{01}, m_{02}, m_{12}) = 2|V_O| - 2|E_O| + 2|F_O| - 4 = 0 \qquad \square$$

In Lemma 2.5 we proved that for every lsp-operation there is an equivalent lopsp-operation O_{lopsp}. Lemma 5.4 proves formally that the Delaney-Dress symbols of the lsp-operation and its corresponding lopsp-operation in fact encode isomorphic tilings.

Lemma 5.4. *The Delaney-Dress symbols $\mathscr{D}(O)$ and $\mathscr{D}(O_{lopsp})$ are Delaney-Dress symbols of combinatorially isomorphic tilings.*

Proof. Mapping each chamber C_{lopsp} of $\mathscr{D}(O_{lopsp})$ onto the corresponding chamber C of $\mathscr{D}(O)$, we have (in the notation of [10]) a morphism between the symbols and in the notation of [6] a Delaney map f, that is: For all $k \in \{0, 1, 2\}$, $(i, j) \in \{(0, 1), (0, 2), (1, 2)\}$, and chambers C of $\mathscr{D}(O_{lopsp})$ we have $f(C\sigma_k) = (f(C))\sigma_k$ and $m_{ij}(C) = m_{ij}(f(C))$.

The existence of such a morphism guarantees (see [6, 10]) that $\mathscr{D}(O)$ and $\mathscr{D}(O_{lopsp})$ code combinatorially isomorphic tilings and that the tiling coded by $\mathscr{D}(O_{lopsp})$ can be obtained from the tiling coded by $\mathscr{D}(O)$ by *symmetry breaking* — That is: modifying the tiling, so that the combinatorial structure is preserved, but some metric symmetries of the tiling are destroyed. $\square$

6 Future work

In the last section of [2] many open problems are described. They are sometimes just formulated for lsp-operations, but are often as relevant and interesting for lopsp-operations, so we refer the reader to [2]. A very interesting question is whether ambo is 'essentially' the only lsp-operation that can increase the symmetry of polyhedra, i.e. plane 3-connected maps. More specifically: Assume that for an lsp-operation O and a polyhedron M, the polyhedron $O(M)$ has more symmetries than M. Is M self-dual and can O be written as the product of ambo and other lsp-operations? For lopsp-operations this is certainly not true. For example, applying gyro to the tetrahedron gives the dodecahedron, which has a much larger symmetry group. Classifying lopsp-operations that can introduce new symmetries would be an interesting problem, but maybe even more difficult than solving the problem for lsp-operations.

We know that there is at least one lsp-operation (dual) that does not always preserve 3-connectivity for maps, if the face-width is at most two [1], so an obvious question is which other operations do not always preserve 3-connectivity. This was answered for lsp-operations in [24], where the class of such operations, called *edge-breaking operations*, was characterized. Recently, these results have been extended to lopsp-operations. An article with the new results has been submitted [25].

Another problem mentioned in [2] — the generation of lsp-operations for a given inflation factor — has been solved [13]. Such an algorithm not only allows the generation of lsp-operations, but also the generation of polyhedra and other maps with some specific symmetry groups of the embedding. For generating lopsp-operations a program has been written very recently, but it has not been published yet.

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